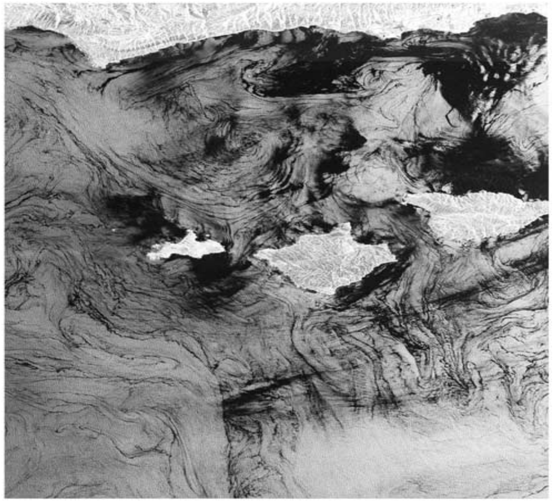
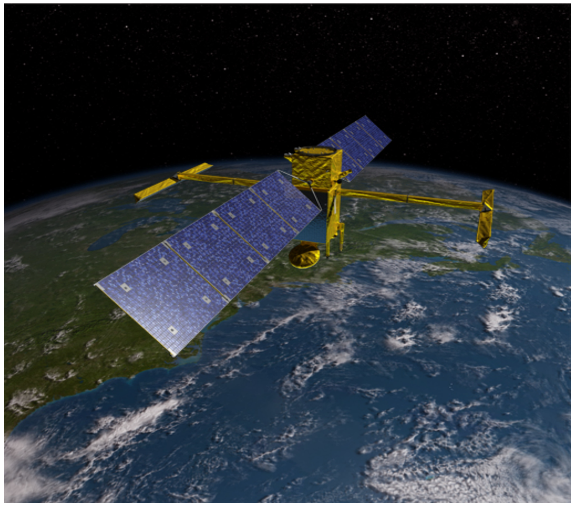
“Wave-Turbulence Interactions in the Oceans”

Patrice Klein (Caltech/JPL/Ifremer)

(IV) Basic concepts on Waves (a)





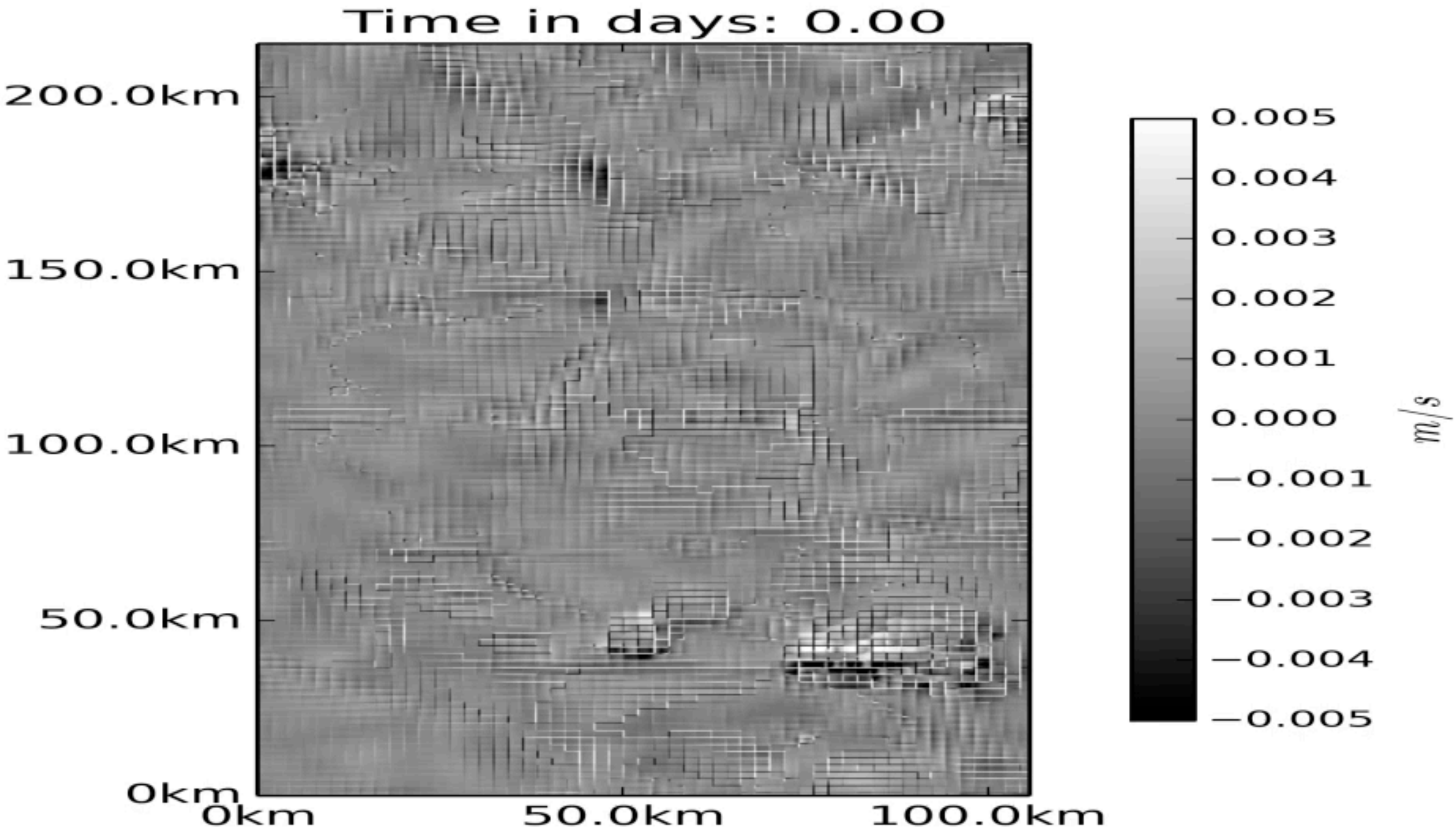
Momentum equations

- **L/U: time it takes for a fluid parcel to travel on a distance L with the velocity U**

- **T: the time for the information to propagate on a distance L.**

- **T can be close or significantly differ from L/U:**
- ❑ **ripples in water,**
- ❑ **waves in a sport stadium (signal moves laterally to much greater distance than the motion of individuals (move up and down))**

$\omega = 2\pi/T$ is the frequency of the motions



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Momentum equations

Rossby number: R_{oL}= U/fL < 1 [U=10^{-1} m/s, L=50 km, f=10^{-4} s^{-1} => R_{oL}=2.10^{-2}]

Slow propagating motions: $R_{oT}=\frac{1}{fT} << 1$

$$\cancel{\frac{\partial U}{\partial t}} + \cancel{U.\nabla U} - fk\times U = -\frac{\nabla p}{\rho_o}$$

$\Rightarrow$ **Geostrophic balance at zero order** $[c = \frac{\omega}{k} \ll U]$

Fast propagating motions: $R_{oT}=\frac{1}{fT} \geq 1$

$$\frac{\partial U}{\partial t} + \cancel{U.\nabla U} - fk\times U = -\frac{\nabla p}{\rho_o}$$

=> Wave equations $[c = \frac{\omega}{k} >> U$, **with** $k = 2\pi/L]$

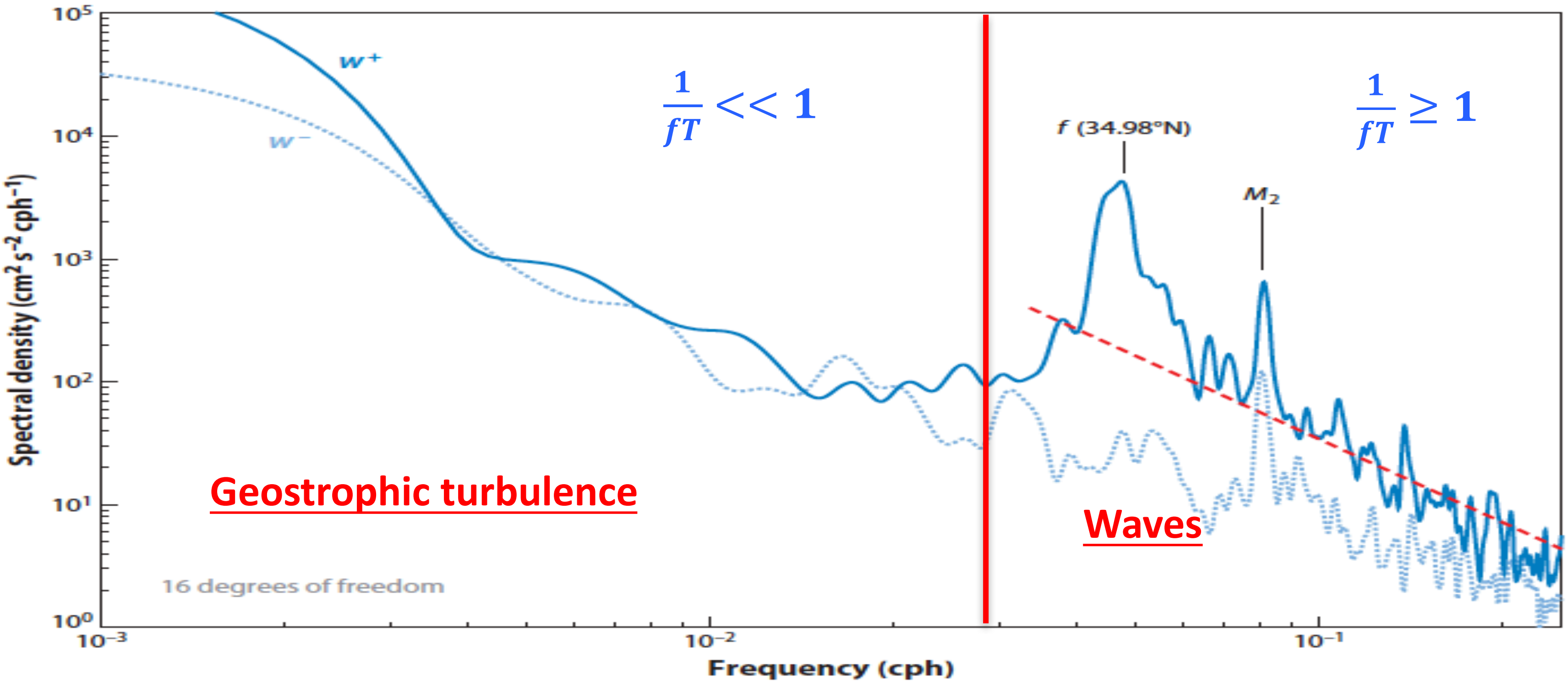


Figure 1

Rotary velocity spectrum at 261-m depth from current-meter data from the WHOI699 mooring gathered during the WESTPAC1 experiment (mooring at 6,149-m depth.) The solid blue line (w^+) is clockwise motion, and the dashed blue line (w^-) is counterclockwise motion; the differences between these emphasize the downward energy propagation that often dominates the near-inertial band. The dashed red line is the line $E_0 N \omega^{-p}$ with $N = 2.0$ cycles per hour (cph), $E_0 = 0.096$ cm^2 s^{-2} cph^{-2}, and $p = 2.25$, which is quantitatively similar to levels in the Cartesian spectra presented by Fu (1981) for station 5 of the Polygon Mid-Ocean Experiment (POLYMODE) II array.

A frequency spectrum displays different properties between fast and slow motions

WAVE KINEMATICS

Any physical variable (pressure, velocity, or whatever) can be written as Fourier series whose components are like:

$$a = A\cos(\underbrace{k_x x + k_y y - \omega t + \phi}_{\alpha}). \qquad \alpha = k_x x + k_y y - \omega t + \phi$$

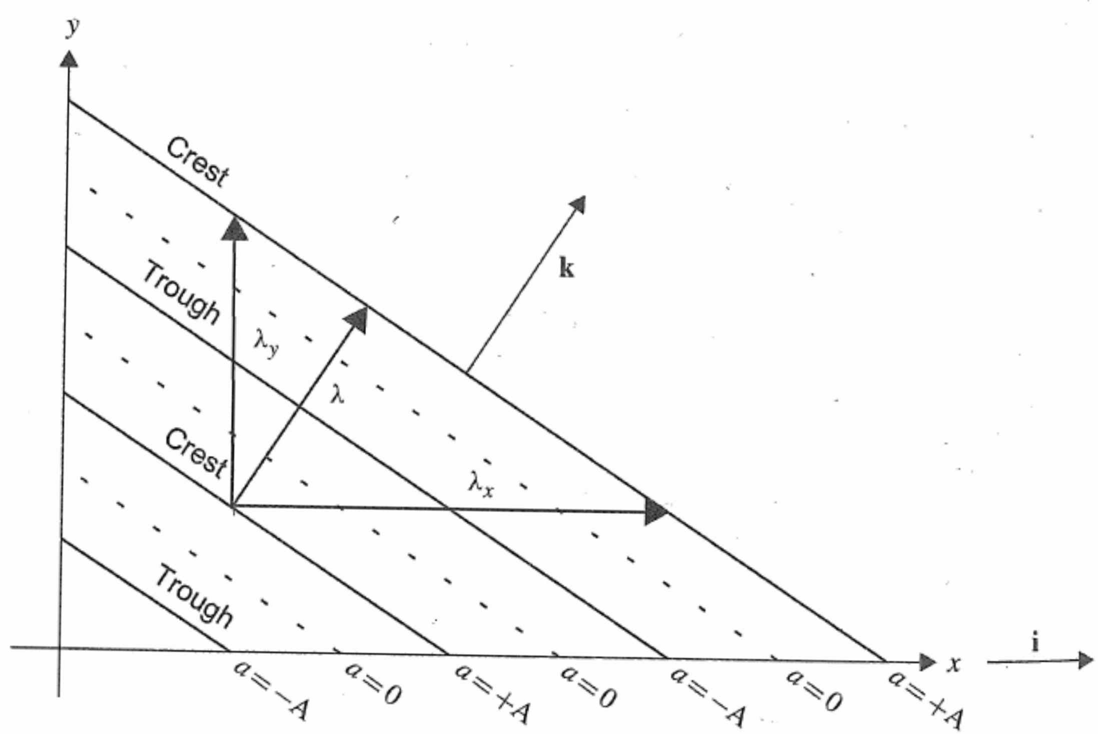
- **A is the wave amplitude ($-A \leq a \leq A$)**
- **α is the phase (ϕ is a constant (=0 in this course))**
- **k_x, k_y are the wavenumbers in x and y [$k_x = 2\pi/\lambda_x$, $k_y = 2\pi/\lambda_y$ with $\lambda_{x,}$, λ_x the wavelengths in the x and y directions]**
- **ω is the frequency [$\omega = 2\pi/T$ with T the period]**

WAVENUMBER AND WAVELENGTH

At a time t:

$$a = A\cos(k_x x + k_y y - \omega t + \phi). \qquad \alpha = k_x x + k_y y - \omega t + \phi$$

- **A wave crest (through) is defined as the line in the (x,y) plane along which a=A (-A)**
- **These lines (and in general all lines along which a is constant) are called phase lines**
- **The wavenumber, $\vec{k}$ = [k_x, k_y] is perpendicular to the phase lines: $\vec{k} = \nabla\alpha$**



$$\lambda_x = \frac{2\pi}{k_x}. \qquad \lambda_y = \frac{2\pi}{k_y}.$$

$$\frac{1}{\lambda^2} = \frac{1}{\lambda_x^2} + \frac{1}{\lambda_y^2} = \frac{k_x^2 + k_y^2}{4\pi^2}$$

$$\lambda = \frac{2\pi}{k}, \qquad k = \sqrt{k_x^2 + k_y^2}$$

FREQUENCY, PHASE SPEED

Frequency can be defined as: $\boldsymbol{\omega = \partial\alpha/\partial t}$

Let us follow a crest line (a=A) during the time interval $\Delta t = t_1 - t_2$.

$\alpha = k_x x + k_y y - \omega t \Rightarrow \Delta\alpha = 0$ between t_1 and t_2.

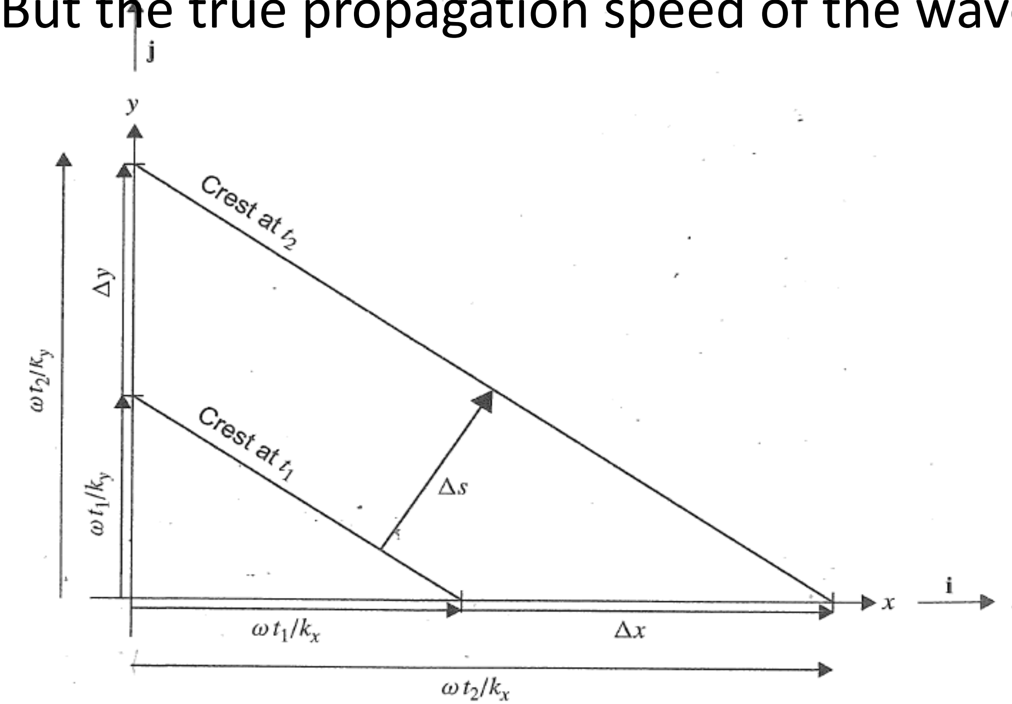
The wave crest has progressed over Δx and Δy with the propagation speed:

$$c_x = \frac{\Delta x}{\Delta t} = \frac{\omega}{k_x}. \qquad c_y = \frac{\Delta y}{\Delta t} = \frac{\omega}{k_y}.$$

in x and y directions

But the true propagation speed of the wave is given by :

$$\Delta s = \frac{\omega \Delta t}{k},$$



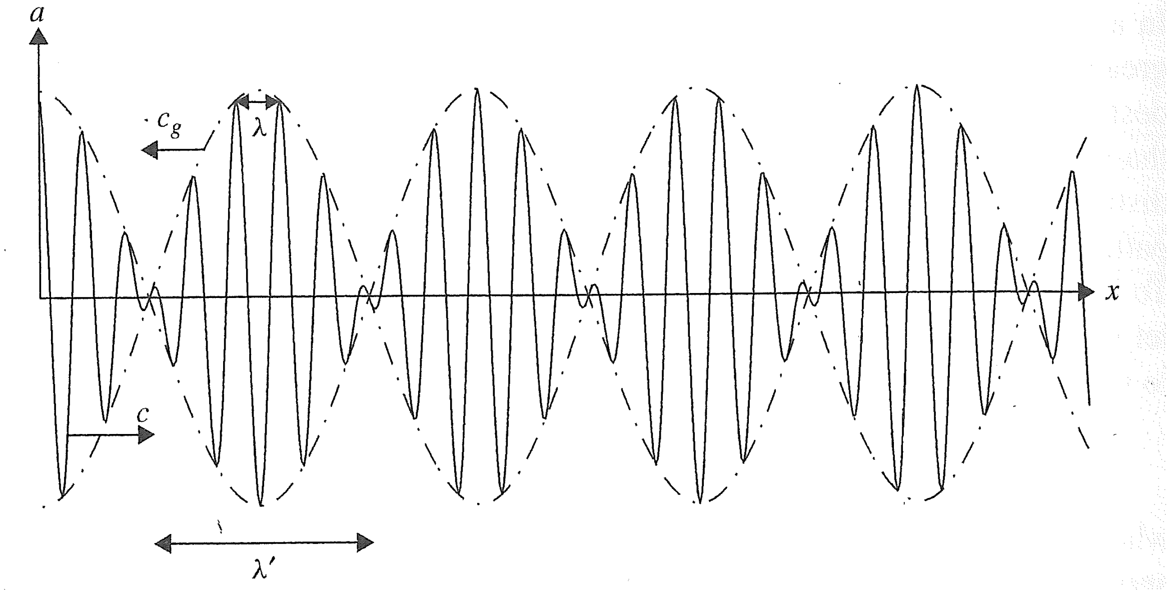
$$\frac{1}{\Delta s^2} = \frac{1}{\Delta x^2} + \frac{1}{\Delta y^2},$$

$$c = \frac{\Delta s}{\Delta t} = \frac{\omega}{k}.$$

GROUP VELOCITY AND ENERGY PROPAGATION

$$a = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t),$$

$$a = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos(kx - \omega t)$$



$$c' = \Delta\omega/\Delta k.$$

$$c_g = \frac{d\omega}{dk}.$$

$$\boxed{\mathbf{c}_g = \nabla_k \omega,}$$

SHALLOW WATER MODEL

THE FLUID IS HOMOGENEOUS, $\frac{\partial p}{\partial z} = 0$ (DENSITY ANOMALIES ARE ZERO), AND FRICTIONLESS.

HORIZONTAL MOTIONS (U AND V) ARE SUPPOSED TO BE Z-INDEPENDENT INITIALLY AND

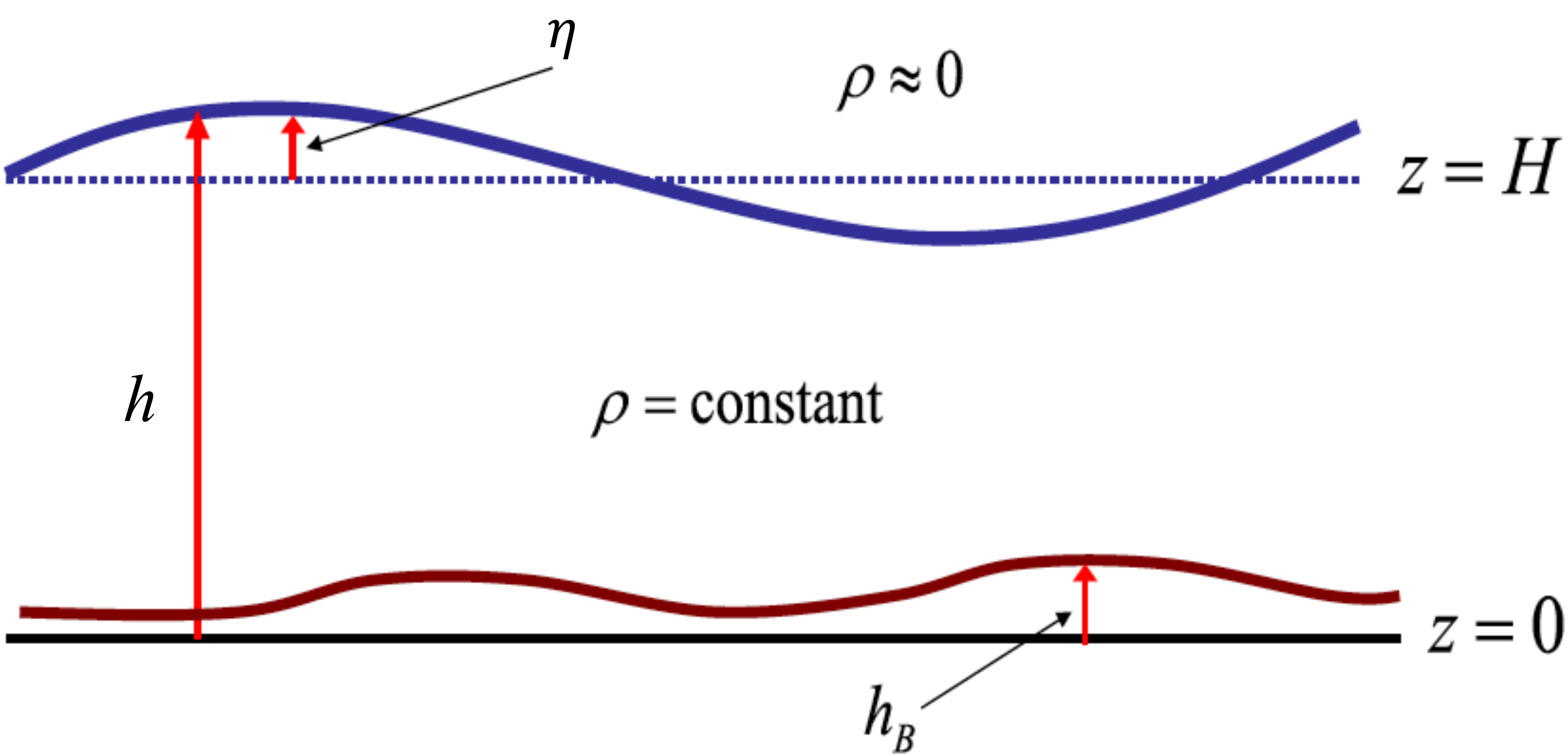
FRICTIONLESS

$$\frac{\partial u}{\partial t} + \cancel{u\frac{\partial u}{\partial x}} + \cancel{v\frac{\partial u}{\partial y}} - f v = -\frac{1}{\rho_0}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \cancel{u\frac{\partial v}{\partial x}} + \cancel{v\frac{\partial v}{\partial y}} + f u = -\frac{1}{\rho_0}\frac{\partial p}{\partial y}.$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

U AND V ARE Z-INDEPENDENT BUT DO NOT NECESSARILY ADD TO ZERO. **W CAN VARY LINEARLY WITH DEPTH**



Schematic of the shallow water system

We assume $h_b=0$

$$w(h+\eta) - 0\ \left[= \frac{d(\eta+H)}{dt}\right] = -\int_{-H}^{\eta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz$$

$$\underbrace{\frac{\partial \eta}{\partial t} + \left(u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y}\right)} + \underbrace{H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right) + \eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)} = 0$$

$$\frac{\Delta H}{T} \qquad U\frac{\Delta H}{L} \qquad H\frac{U}{L} \qquad \Delta H\frac{U}{L}$$

$R_{oT} = \frac{1}{fT} \geq 1$ and $R_{oL} = U/fL < 1$ => $1/T \gg U/L$

$$w(h+\eta)-0 = -\int_{-H}^{\eta}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)dz = \frac{d(\eta+H)}{dt}$$

$$\frac{\partial \eta}{\partial t}+\cancel{\left(u\frac{\partial \eta}{\partial x}+v\frac{\partial \eta}{\partial y}\right)}+H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)+\cancel{\eta\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}=0$$

$$\frac{\Delta H}{T} \qquad U\frac{\Delta H}{L} \qquad H\frac{U}{L} \qquad \Delta H\frac{U}{L}$$

$R_{oT}=\frac{\omega}{f} \geq 1$ and R_{oL}= U/fL < 1 => 1/T >> U/L => UT/L <<1 => 1/T>>U/L

$$\longrightarrow \quad \frac{\partial \eta}{\partial t}+H\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)=0$$

$$\longrightarrow \quad \frac{\Delta H}{H}\sim\frac{UT}{L} \ll 1$$

small variation

SHALLOW WATER MODEL

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$

$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Properties:

Conservation of PV : $\frac{\partial q}{\partial t} = 0 \; with \; q = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} - \frac{f}{H}\eta$ + constant

Wave equation: $\frac{\partial}{\partial t}[\frac{\partial^2 \eta}{\partial t^2} + \; + f^2\eta - c_o^2 \Delta\eta] = 0 \; with \; c_o^2 = gH$

Coming back to the shallow water equations, we use

$$\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = \Re \begin{pmatrix} A \\ U \\ V \end{pmatrix} e^{i(k_x x + k_y y - \omega t)}$$

Which leads to:

$$\begin{aligned} -i\omega U - fV &= -igk_x A \\ -i\omega V + fU &= -igk_y A \\ -i\omega A + H(ik_x U + ik_y V) &= 0. \end{aligned}$$

This is an homogeneous system that admits a solution if its determinant is zero. This leads to

$$\omega[\omega^2 - f^2 - gH\,(k_x^2 + k_y^2)] = 0.$$

This condition is called **the dispersion relation.**

We define $c_o = \sqrt{gH}$ the gravity wave speed and $R = \sqrt{gH/f}$ the Rossby radius of deformation

Coming back to the group velocity: $\frac{\partial \omega}{\partial k_x} = c_o^2 \frac{k_x}{\omega}, \quad \frac{\partial \omega}{\partial k_y} = c_o^2 \frac{k_y}{\omega},$

Thus the group velocity is in the direction of the wave vector

Solutions of the dispersion relation:

$$\omega[\omega^2 - f^2 - gH\ (k_x^2 + k_y^2)] = 0.$$

$\omega = 0$: geostrophic balance

$\omega = \pm\sqrt{f^2 + gH\left(k_x^2 + k_y^2\right)}$: Poincaré waves

<u>Why « dispersion relation »?</u>

A localized burst of activity (which contains waves with many different wavelengths) will be progressively less localized as time goes on (because of their different phase speed and group velocity, both related to the frequency and wavenumbers).

This phenomenon is called dispersion.

Two limits for the Poincaré waves:

Short wave limit:

$\left(k_x^2 + k_y^2\right) \gg\ f^2$/gH=1/R^2 : gravity waves solutions: $\omega\sim\ \pm\sqrt{gH\left(k_x^2 + k_y^2\right)}$

=> Length scale of the wave disturbance is not large enough to feel the Earth rotation

Long wave limit:

$\left(k_x^2 + k_y^2\right) \ll\ f^2$/gH= 1/R^2 : inertial oscillation solutions: $\omega\sim\ \pm f$

$\Rightarrow$ Length scale of the wave disturbance is large such that rotation effects dominate gravity effects.

The criterium is R/L.

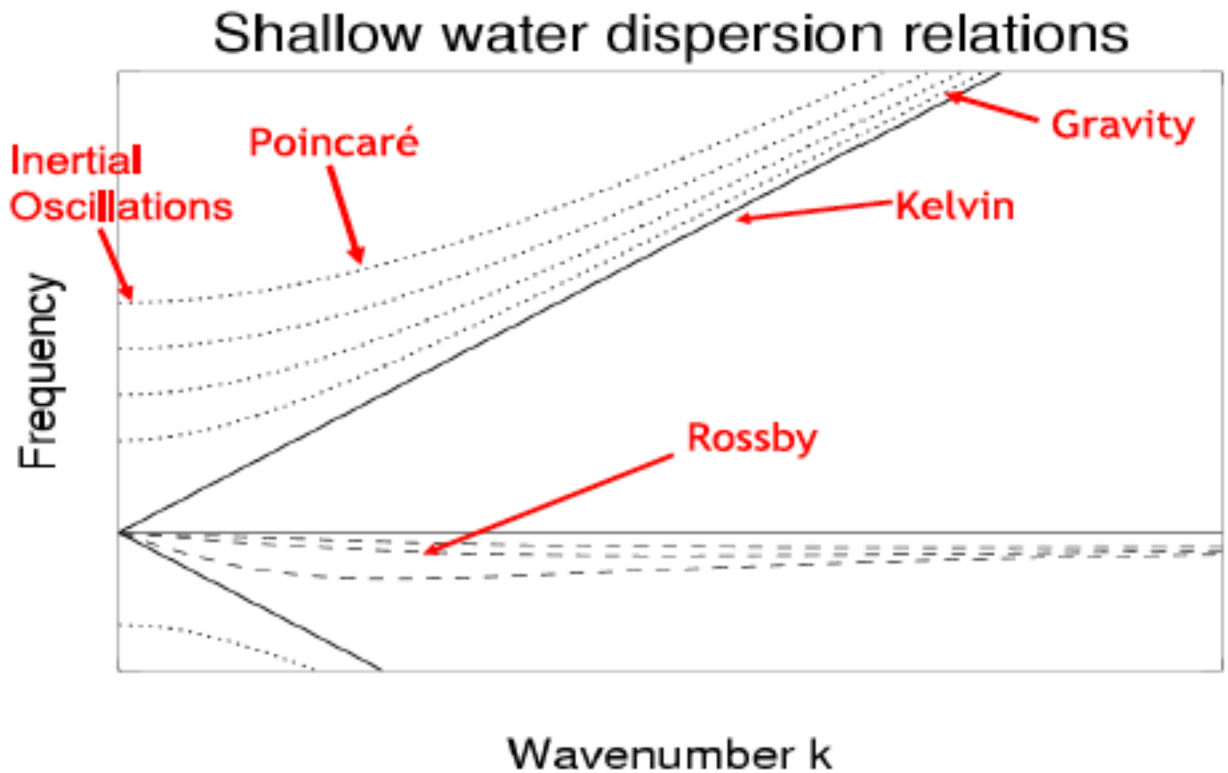
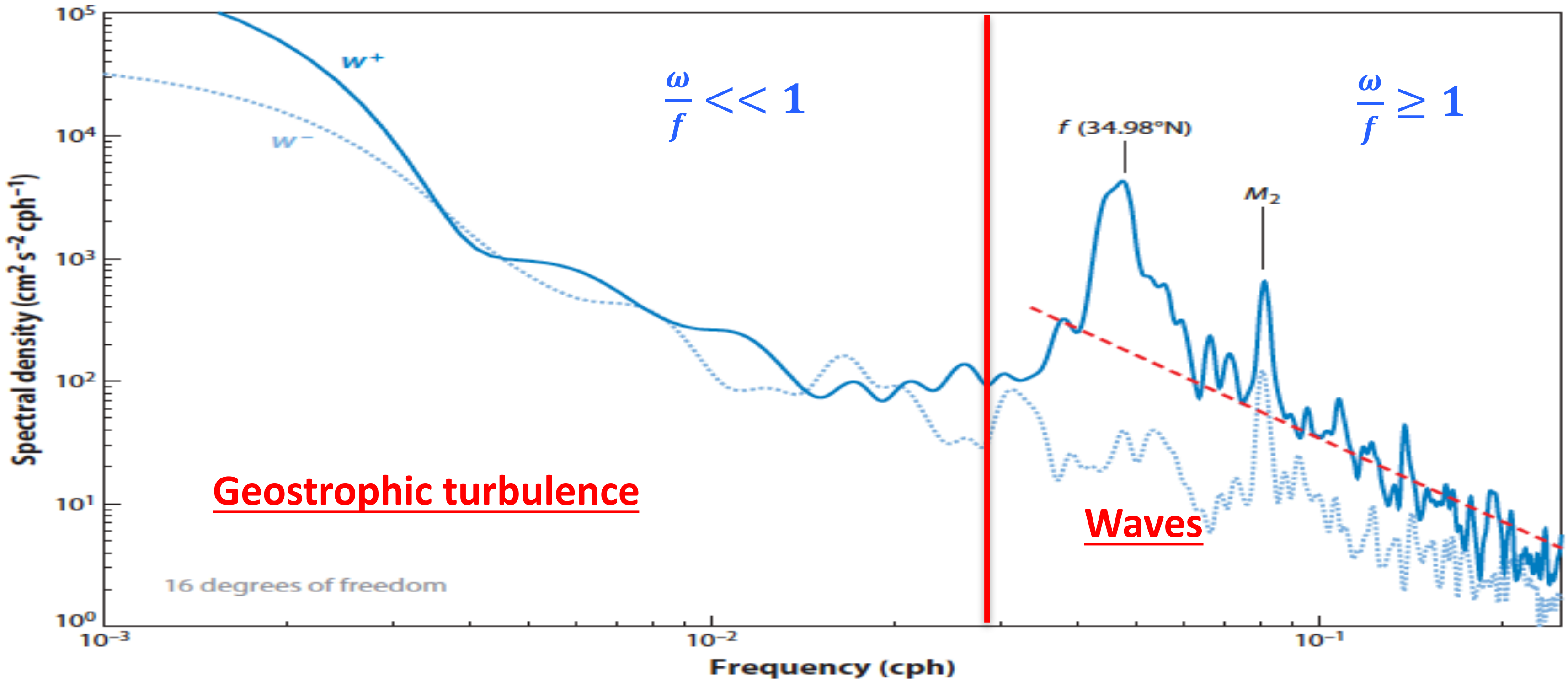


Figure 2: Dispersion relations for a rotating shallow water system. The Poincaré wave solutions are produced in the presence of a height perturbation in a rotating shallow water system. The Kelvin waves require the presence of a boundary (or the equator) and the Rossby waves require the presence of a gradient in potential vorticity.



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