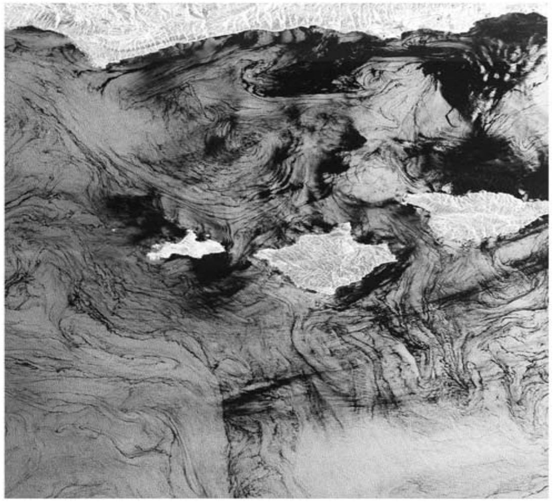
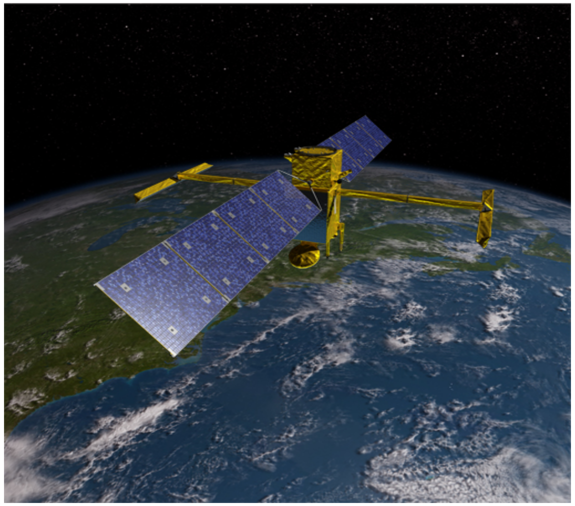
“Wave-Turbulence Interactions in the Oceans”

Patrice Klein (Caltech/JPL/Ifremer)

(V) Channel modes and Kelvin waves





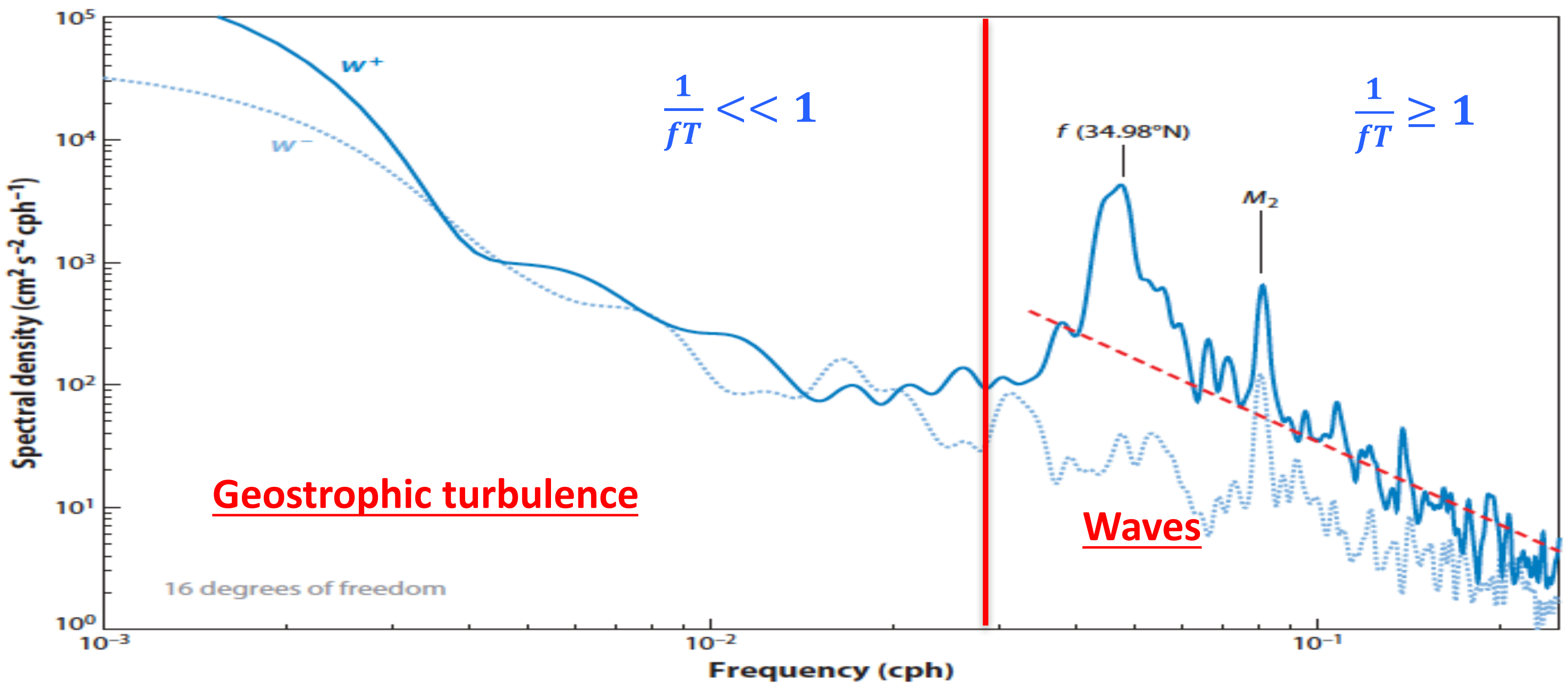


Figure 1

Rotary velocity spectrum at 261-m depth from current-meter data from the WHOI699 mooring gathered during the WESTPAC1 experiment (mooring at 6,149-m depth.) The solid blue line (w^+) is clockwise motion, and the dashed blue line (w^-) is counterclockwise motion; the differences between these emphasize the downward energy propagation that often dominates the near-inertial band. The dashed red line is the line $E_0 N \omega^{-p}$ with $N = 2.0$ cycles per hour (cph), $E_0 = 0.096$ cm^2 s^{-2} cph^{-2}, and $p = 2.25$, which is quantitatively similar to levels in the Cartesian spectra presented by Fu (1981) for station 5 of the Polygon Mid-Ocean Experiment (POLYMODE) II array.

A frequency spectrum displays different properties between fast and slow motions

SHALLOW WATER MODEL

$$\frac{\partial u}{\partial t} - fv = -g\frac{\partial \eta}{\partial x}$$
$$\frac{\partial v}{\partial t} + fu = -g\frac{\partial \eta}{\partial y}$$
$$\frac{\partial \eta}{\partial t} + H\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) = 0$$

Wave equation: $\frac{\partial}{\partial t}\left[\frac{\partial^2 \eta}{\partial t^2} + \ + f^2\eta - c_o^2 \Delta\eta\right] = 0 \ with \ c_o^2 = gH$

Using: $\begin{pmatrix} \eta \\ u \\ v \end{pmatrix} = \Re\begin{pmatrix} A \\ U \\ V \end{pmatrix} \mathrm{e}^{\mathrm{i}(k_x x + k_y y - \omega t)}$

leads to:

$$\omega^2 = f^2 + c_o^2[k_x^2 + k_y^2]$$

with $c_o^2 = gH$ and R=$\sqrt{gH}/f$ R is a Rossby radius

Two limits for the Poincaré waves:

Short wave limit:

$(k_x^2 + k_y^2) \gg f^2$/gH=1/R^2 : gravity waves solutions: $\omega \sim \pm\sqrt{gH(k_x^2 + k_y^2)}$

=> Length scale of the wave disturbance is not large enough to feel the Earth rotation

Long wave limit:

$(k_x^2 + k_y^2) \ll f^2$/gH= 1/R^2 : inertial oscillation solutions: $\omega \sim \pm f$

$\Rightarrow$ Length scale of the wave disturbance is large such that rotation effects dominate gravity effects.

The criterium is R/L.

The **Rossby radius of deformation** is the horizontal length scale over which the height field adjusts during approach to the geostrophic equilibrium (<u>or the GW frequency approaches f</u>).

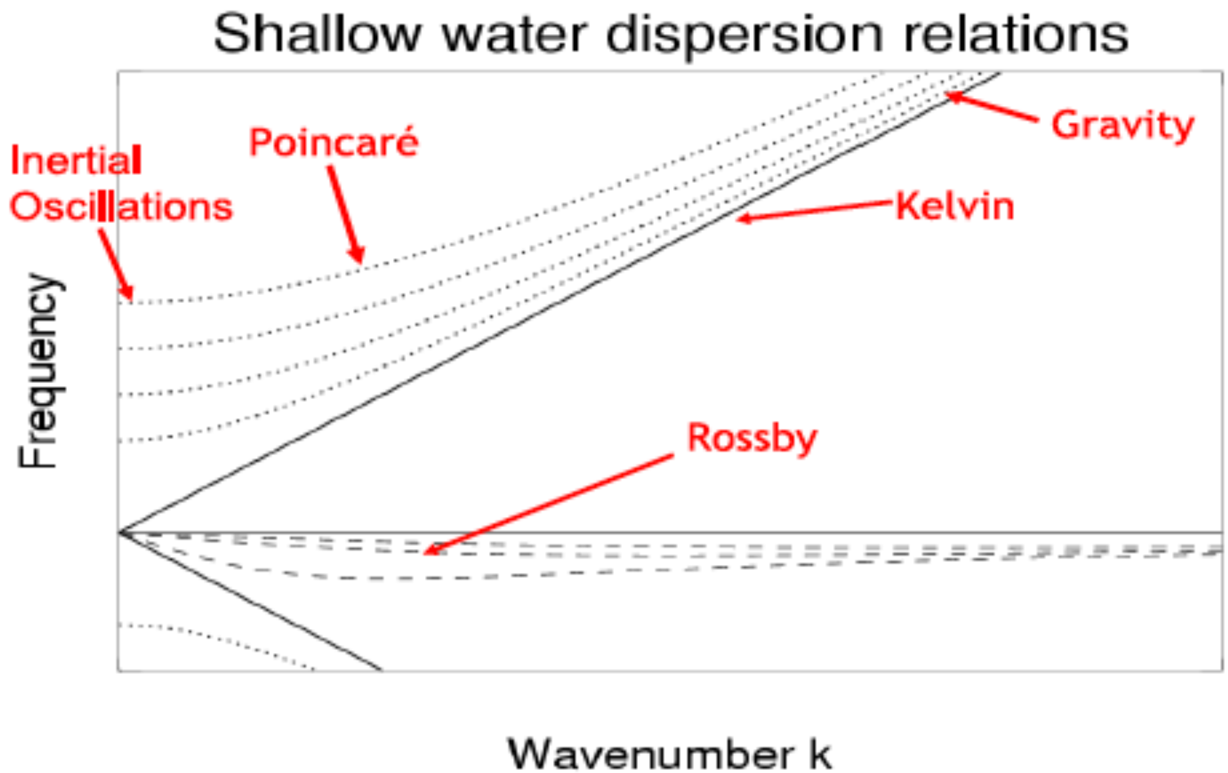
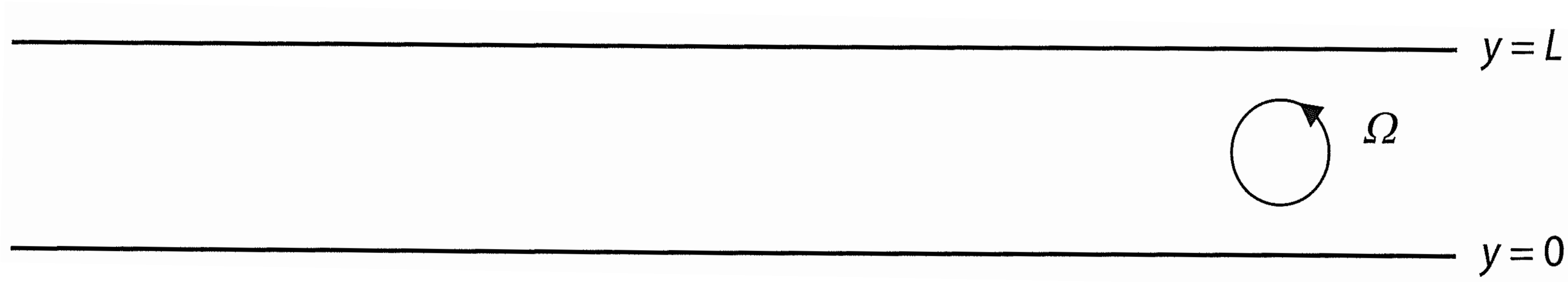


Figure 2: Dispersion relations for a rotating shallow water system. The Poincaré wave solutions are produced in the presence of a height perturbation in a rotating shallow water system. The Kelvin waves require the presence of a boundary (or the equator) and the Rossby waves require the presence of a gradient in potential vorticity.

CHANNEL MODES (J. Pedlosky 2003)

Let us consider a channel of width L:



The equation of motion for the wave is (from SW equations):

$$\nabla^2\eta - \frac{1}{c_0^2}\frac{\partial^2\eta}{\partial t^2} - \frac{f^2}{c_0^2}\eta = 0, \qquad \text{with } c_o = \sqrt{gH}$$

With the y-component of the velocity, v, given by:

$$\frac{\partial^2 v}{\partial t^2} + f^2 v = -g\frac{\partial^2\eta}{\partial t\partial y} + gf\frac{\partial\eta}{\partial x}$$

v must vanish at the boundaries (y=0,L)

CHANNEL MODES

We can look for solutions of the form: $\eta = \bar{\eta}(y)e^{i(kx-\omega t)}$

$\bar{\eta}$ (y) satisfies the ordinary differential equation:

$$\frac{\mathrm{d}^2\overline{\eta}}{\mathrm{d}y^2}+\left\{\frac{\omega^2-f^2}{c_0^2}-k^2\right\}\overline{\eta}=0$$

With the boundary conditions (at y=0,L):

$$\frac{\mathrm{d}\overline{\eta}}{\mathrm{d}y}+k\frac{f}{\omega}\overline{\eta}=0$$

Let us define ℓ such that:

$$\boxed{\ell^2=\frac{\omega^2-f^2}{c_0^2}-k^2}$$

Solution becomes:

$$\overline{\eta}(y) = A\sin\ell y + B\cos\ell y$$

With the boundary conditions: $A\ell+B\dfrac{kf}{\omega}=0$ at y=0 ,

and $A\ell\cos\ell L-B\ell\sin\ell L+\dfrac{kf}{\omega}\{A\sin\ell L+B\cos\ell L\}=0$ at y=L

Combining these two BCs leads to (used later to get *l*):

$$\sin\ell L\left[\omega^2\ell^2+k^2f^2\right]=\sin\ell L\left[\omega^2\left(\frac{\omega^2-f^2}{c_0^2}\right)-k^2\omega^2+k^2f^2\right]$$

$$=\sin\ell L\left[\frac{\omega^2}{c_0^2}-k^2\right]\left(\omega^2-f^2\right)=0$$

CHANNEL MODES

$$\sin \ell L\left[\frac{\omega^2}{c_0^2}-k^2\right]\left(\omega^2-f^2\right)=0$$

Solutions for ω (for any ℓ):

1. $\omega = \pm f$: limit **for long wavelengths**: why this solution for any ℓ ?

2. $\omega = \pm k c_0$: limit **for short gravity waves (y-independent):** why this solution for any ℓ ?

3. $\sin \ell L = 0$ **: solutions are ℓL=nπ, n=1,2,3, …**

Let us consider the third solution …

Using : $$\ell^2 = \frac{\omega^2 - f^2}{c_0^2} - k^2$$

Leads to the dispersion relation: $$\omega_n^2 = f^2 + c_0^2\left[k^2 + n^2\pi^2/L^2\right]$$

This is exactly **the dispersion relation for Poincaré waves** except that the wavenumber ℓ is quantized in multiple of $\frac{\pi}{L}$: $\ell L = n\pi$, $n = 1,2,3\ldots$

Using $\bar{\eta}(y) = A\sin\ell y + B\cos\ell y$ and the relation between A and B (from the BCs), the solution for η is :

$$\eta = \eta_0\left[\cos(n\pi y/L) - \frac{kfL}{\omega n\pi}\sin(n\pi y/L)\right]\cos(kx - \omega t)$$

and solutions for u and v are:

$$u = \frac{\eta_0}{D}\left[\frac{c_0^2}{(\omega/k)}\cos(n\pi y/L) - \frac{fL}{n\pi}\sin(n\pi y/L)\right]\cos(kx - \omega t)$$

$$v = \frac{-\eta_0}{D}\frac{\left[f^2 + c_0^2 n^2\pi^2/L^2\right]}{\omega n\pi/L}\sin(n\pi y/L)\sin(kx - \omega t)$$

CHANNEL MODES

$$\sin \ell L\left[\frac{\omega^2}{c_0^2}-k^2\right]\left(\omega^2-f^2\right)=0$$

Solutions for ω (for any ℓ):

1. $\omega = \pm f$: limit for long wavelengths: why this solution for any ℓ ?

2. $\omega = \pm k c_0$: **limit for short gravity waves (y-independent):**
   why this solution for $\ell \neq 0$?

3. $\sin \ell L = 0$: solutions are ℓL=nπ, n=1,2,3, …

Let us consider now the second solution …

THE KELVIN MODE

Second solution is: $\omega = \pm k c_0$

This is the dispersion relation for y-independent, nonrotating, short surface waves. But our fluid is rotating and therefore no solution independent of y is a possible solution (see the BCs for η at y=0,L).

Using again: $$\ell^2 = \frac{\omega^2 - f^2}{c_0^2} - k^2$$

we have for this case since $\omega = \pm k c_0$ we hqve: $\ell = \pm i f / c_0$

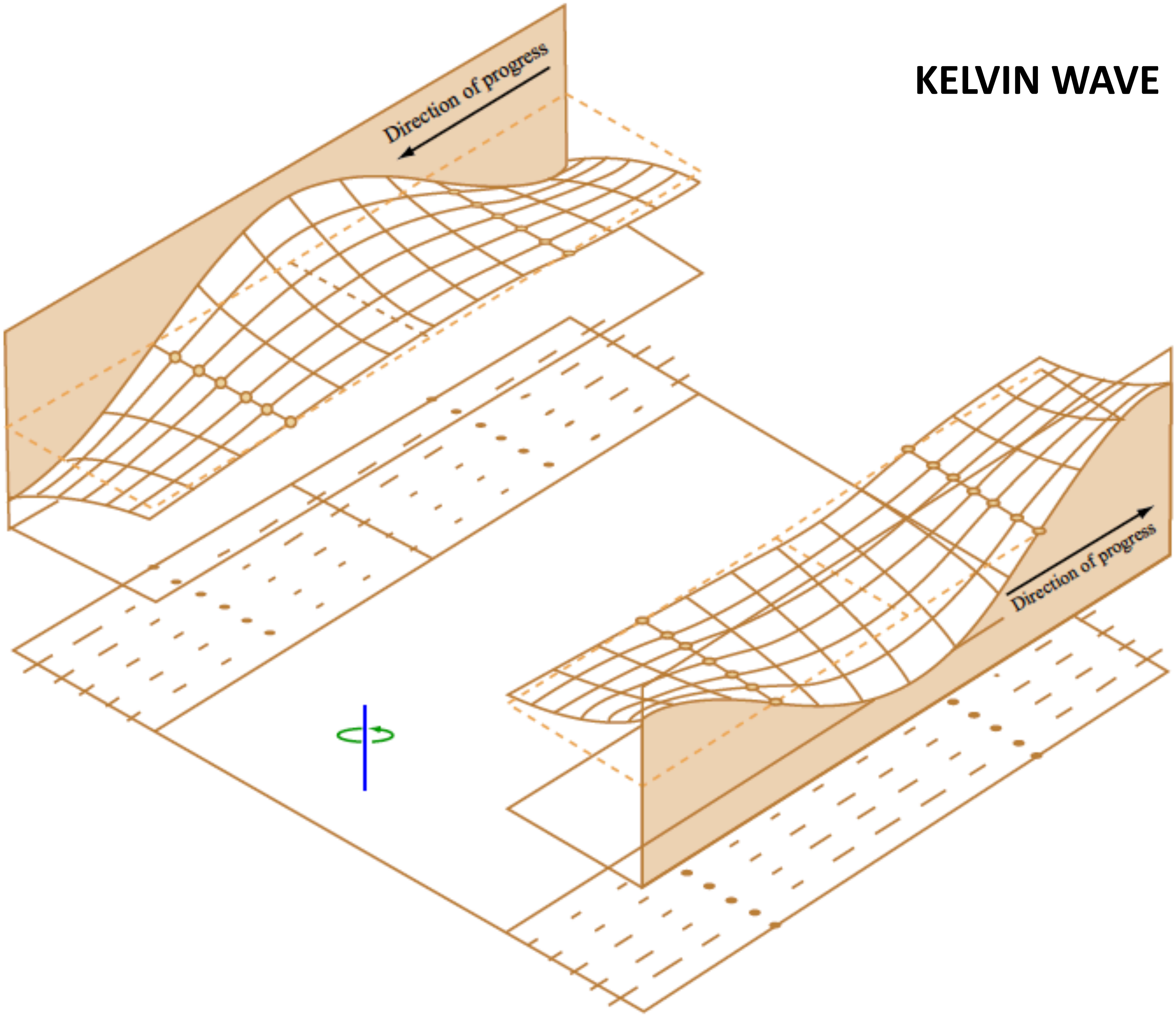
So the cross-channel wavenumber is purely imaginary. Let us look for the solution corresponding to the positive imaginary root.

Using again: $\overline{\eta}(y) = A \sin \ell y + B \cos \ell y$, and the relation between A and B from the BCs (and writing the sine and cosine in their exponential form, we get:

$$\overline{\eta}(y) = \eta_0 \left\{ \mathrm{e}^{-fy/c_0} \left[1 + \omega / k c_0\right] - \mathrm{e}^{fy/c_0} \left[1 - \omega / k c_0\right] \right\}$$

This solution consists of two parts …

KELVIN WAVE



THE KELVIN MODE

Let us consider the wave propagating to the right:

$$\eta = \eta_0 e^{-fy/c_0} \cos(kx - \omega t)$$

This solution would also be valid if only a single wall were present (channel semi-infinite in the +y-direction). Let us calculate v and u. Using the previous relations, we get for v:

$$\begin{aligned} v(f^2-\omega^2) &= gf\eta_x - g\eta_{yt} \\ &= -gfk\sin(kx-kc_0t) + g\frac{f}{c_0}kc_0\sin(kx-kc_0t) \\ &= 0! \end{aligned}$$

So the cross channel velocity is identically zero for all y–values.

We get for u:

$$\begin{aligned} u(f^2-\omega^2) &= -gf\eta_y - g\eta_{xt} \\ &= g\frac{f^2}{c_0}\eta - gk^2c_0\eta \\ &= \frac{g}{c_0}(f^2-k^2c_0^2)\eta = \frac{g}{c_0}(f^2-\omega^2)\eta \end{aligned}$$

THIS LEADS TO:

$$u(f^2-\omega^2)=-gf\eta_y-g\eta_{xt}$$

$$=g\frac{f^2}{c_0}\eta-gk^2c_0\eta$$

$$=\frac{g}{c_0}(f^2-k^2c_0^2)\eta=\frac{g}{c_0}(f^2-\omega^2)\eta$$

$$\boxed{u=-\frac{g}{f}\frac{\partial\eta}{\partial y}}$$

GEOSTROPHIC BALANCE IN THE Y-DIRECTION!

THE RESULTING EQUATIONS ARE:

$$\boxed{\begin{aligned} fu &= -g\eta_y \\ u_t &= -g\eta_x \\ \eta_t &= -Hu_x \end{aligned}}$$

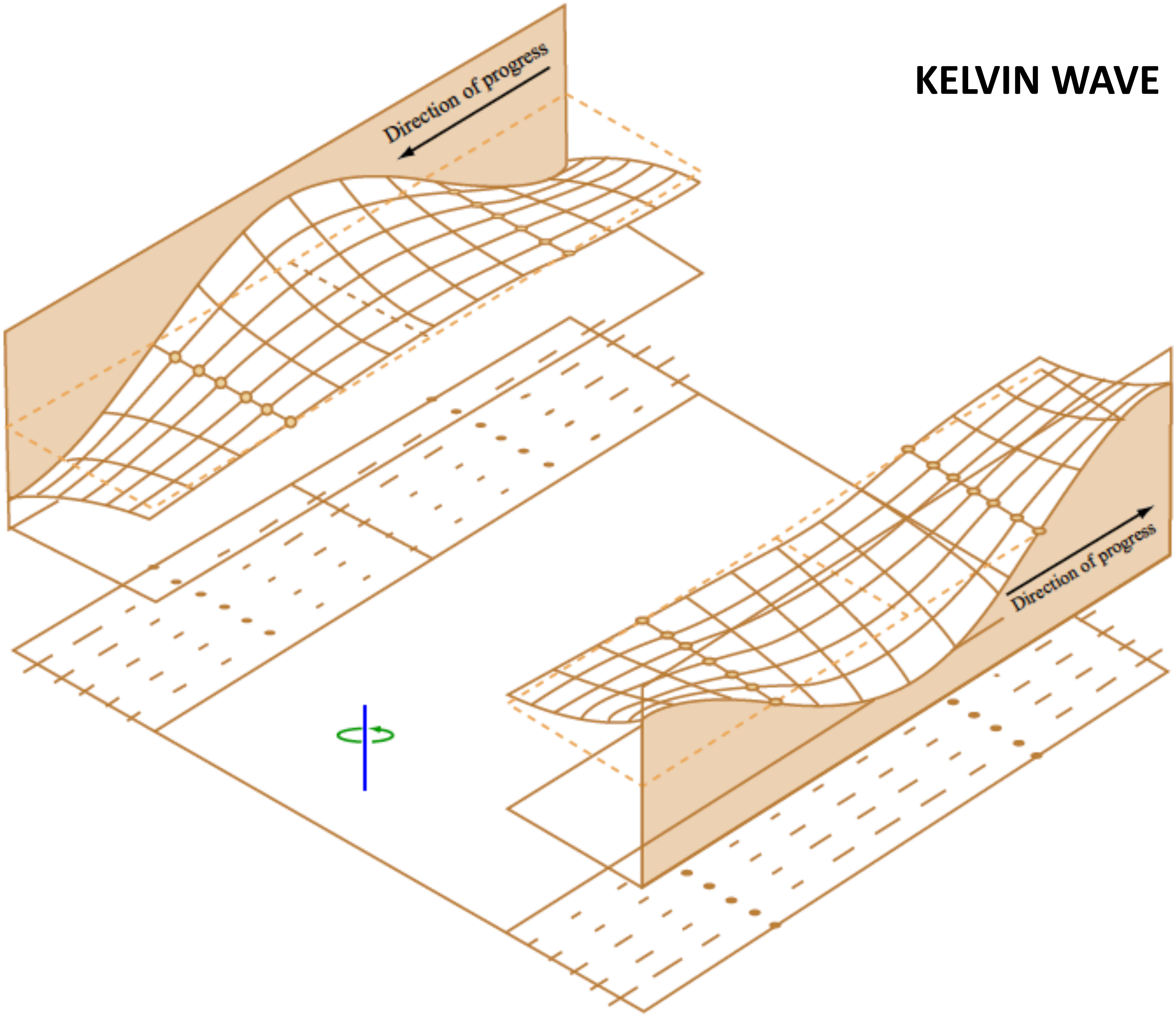
$$\eta_{tt} = c_0^2 \eta_{xx}$$

$$\eta_{yy} - \frac{f^2}{c_o^2}\eta = 0$$

- GRAVITY WAVE EQUATION (Y-INDEPENDENT) AS IN THE NON-ROTATING CASE
- GEOTROPHIC BALANCE IN THE Y-DIRECTION

THIS GW MODE MAINTAINS ITS CHARACTER OF HAVING V=0. IT DOES SO BY INTRODUCING A SLOPING FREE SURFACE ELEVATION THAT BALANCES THE CORIOLIS ACCELERATION OF U.

KELVIN WAVE



SSH FROM M_2 BAROTROPIC TIDES (Richman et al. 2012)

SSH FROM M_2 IS BASIN-SCALE! IT BEHAVES AS KELVIN WAVES (6000 KM) PROPAGATING CYCLONICALLY AROUND A BASIN WITH A PERIOD OF ~12 H (see next class)..

USING $\eta = \eta_o . \sin\alpha$ WITH $\alpha = k_x x + k_y y - \omega t$ THE PHASE. THIS FIGURE SHOWS:

η_o in colour and α lines, the white lines

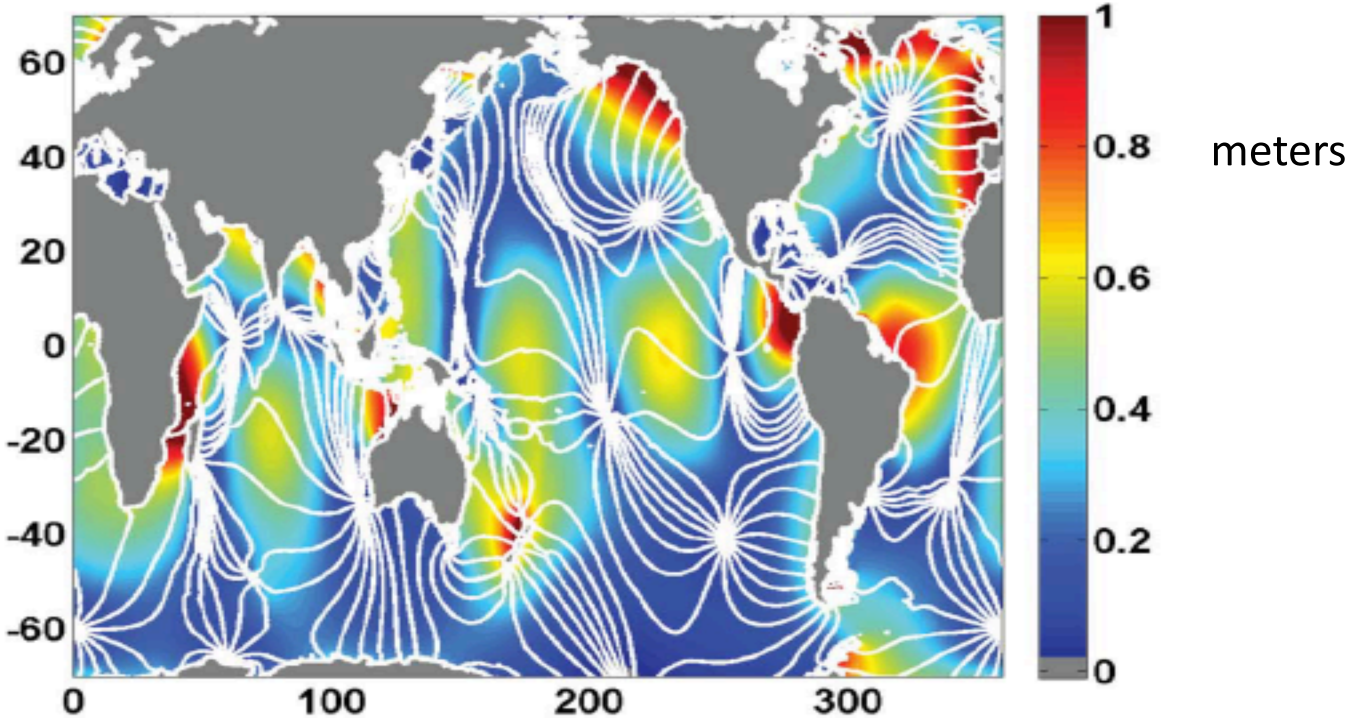


Figure 1. M_2 sea surface elevation amplitudes (m) and phase for (a) HYCOM and (b) TPXO7.2 [*Egbert et al.*, 1994], a highly accurate altimetry-constrained model of the barotropic tides. White lines indicate phase, drawn 20° apart.