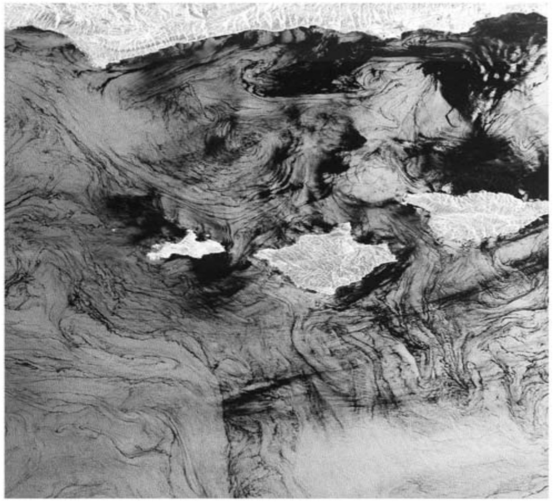
“Wave-Turbulence Interactions in the Oceans”

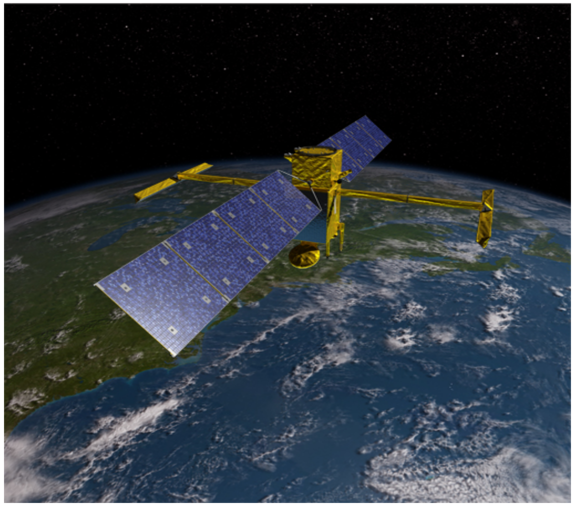
https://oceanturbulence.github.io

Patrice Klein (Caltech/JPL/Ifremer)

(XIV) Interactions between waves and balanced motions

Propagation of waves in an inhomogeneous medium: ray tracing approach





Wave propagation in a stationary barotropic jet: *V(x)*

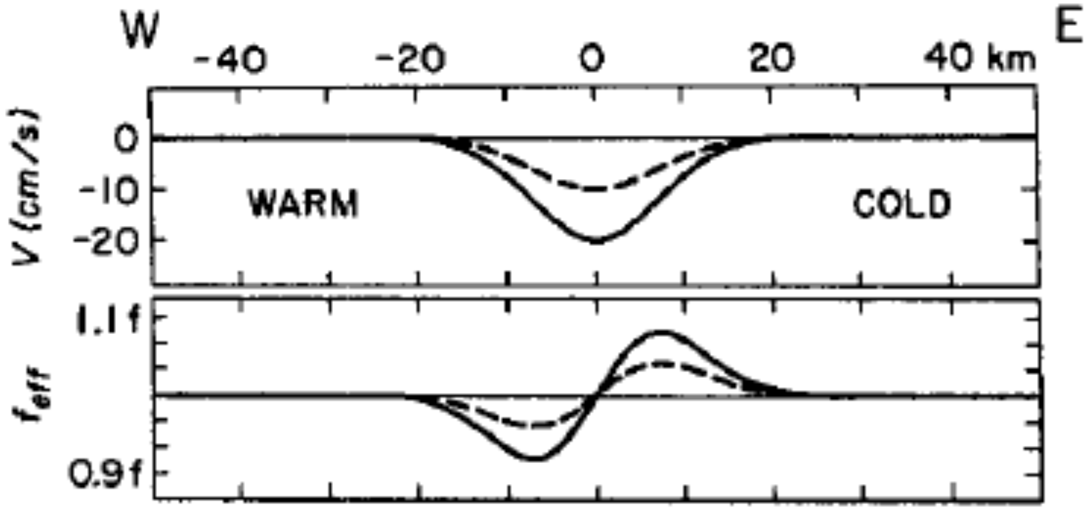


FIG. 4. A model southward baroclinic jet (upper frame) analogous to the North Pacific Subtropical Front, and the associated effective Coriolis frequency $f_{eff} = f + \zeta/2$ (lower frame). The solid curves represent values at the surface, and dashed curves values at a depth of 100 m. Internal waves propagate freely only for frequencies lying above the f_{eff}-curve.

$$\partial V/\partial x = 0.1f/10km$$

Kunze JPO 1985

$$p(x,y,z,t) = \sum_m p_m(x,y).F_m(z)$$

For baroclinic mode m:

$$\frac{\partial u_m}{\partial t} - \mathrm{f}v_m = -\frac{\partial p_m}{\partial x}$$

$$\frac{\partial v_m}{\partial t} + \boldsymbol{u_m V_x} + \mathrm{f}u_m = -\frac{\partial p_m}{\partial y}$$

$$\frac{\partial p_m}{\partial t} + f^2.r_m^2\left(u_{mx} + v_{my}\right) = 0$$

Doppler terms are not considered for the sake of simplicity. Searching for plane wave solutions as, $p_m(x,y,t), u_m(x,y,t), v_m(x,y,t) \approx e^{i.(k.x+l.y-\omega t)}$, leads to:

$$\omega_r^2 \approx f^2 + \boldsymbol{f.V_x} + f^2 r_m^2.k^2$$

When N^2 is constant, $F_m(z) = cos(m.z)$ and $r_m^2 = N^2/(f^2.m^2)$

Physics involved in the dispersion of waves by balanced motions

Let us consider: $S_1=S_2=0$, $\zeta = V_x \neq 0$

$$\frac{\partial}{\partial t}\begin{bmatrix} u \\ v \end{bmatrix} + \begin{bmatrix} \mathbf{0} & -(f+\boldsymbol{\zeta/2}) \\ (f+\boldsymbol{\zeta/2}) & \mathbf{0} \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = -\begin{bmatrix} p_x \\ p_y \end{bmatrix}$$

$$\frac{\partial p}{\partial t} + f^2 . r^2 (u_x + v_y) = 0$$

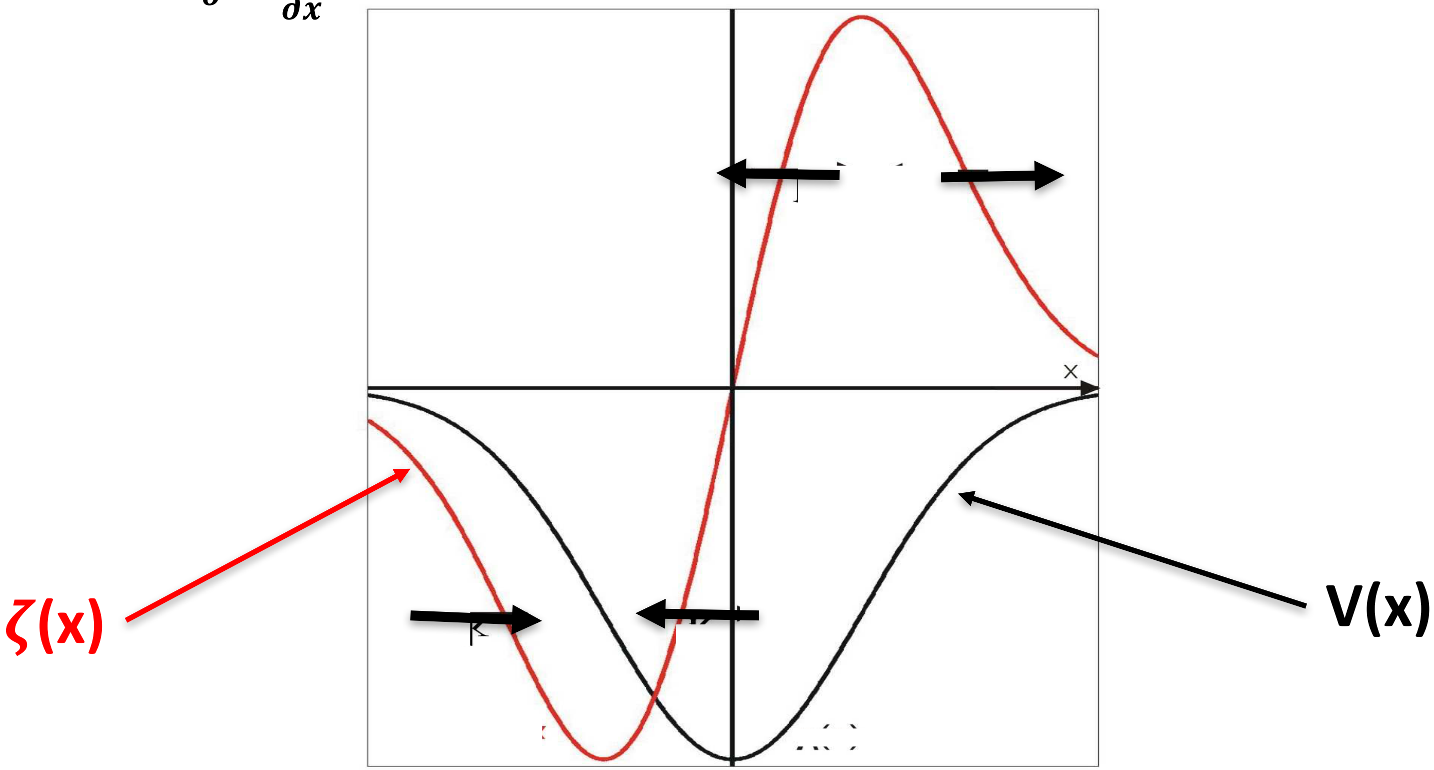
$$\omega_r^2 = f^2 + \boldsymbol{f.V_x} + f^2 r_m^2 . k^2$$

$$\Rightarrow \quad \boldsymbol{u(x,t) = u_o . \cos[(k_o . x - (f+\zeta)t]}$$
$$\boldsymbol{\approx u_o . \cos[(k . x - (f+\zeta_o)t]}$$

with $\boldsymbol{k = k_o - \frac{\partial \zeta}{\partial x} t}$

To continue to satisfy the dispersion relation, as the relative vorticity changes when the wave packet propagates, the wave vector adjusts …

This leads to: $k = k_o - \frac{\partial \zeta}{\partial x} t$.



Schematic representation of the wave propagation induced by a geostrophic jet V(x). The velocity, V(x), (in black) and the relative vorticity, ζ(x) (in red) are shown. We have:

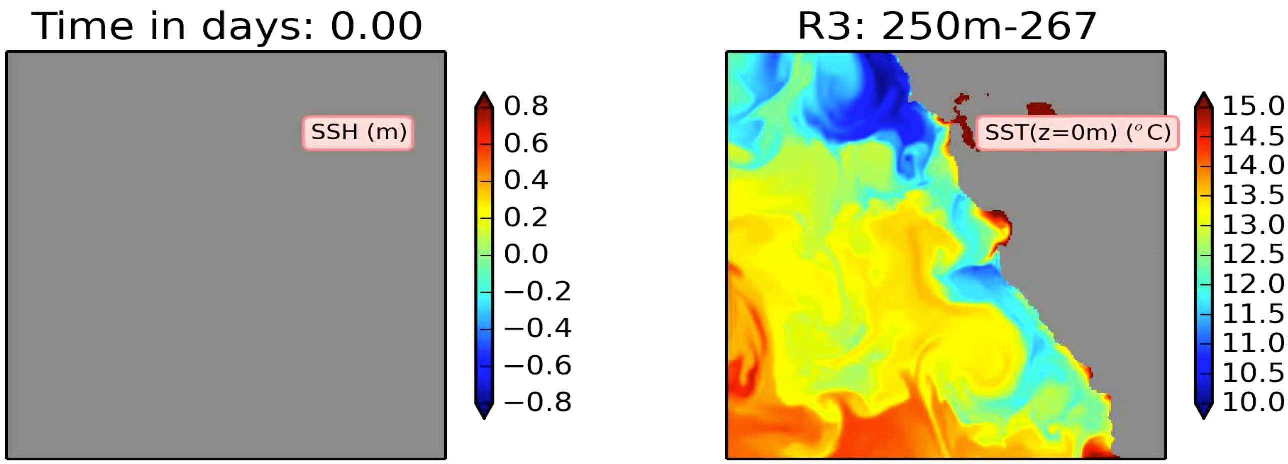
$$k \approx -\frac{\partial \zeta}{\partial x} t$$

The wavenumbers (black arrows) are proportional to $d\zeta/dx$**.**

Waves propagate in the direction of the wavevector!

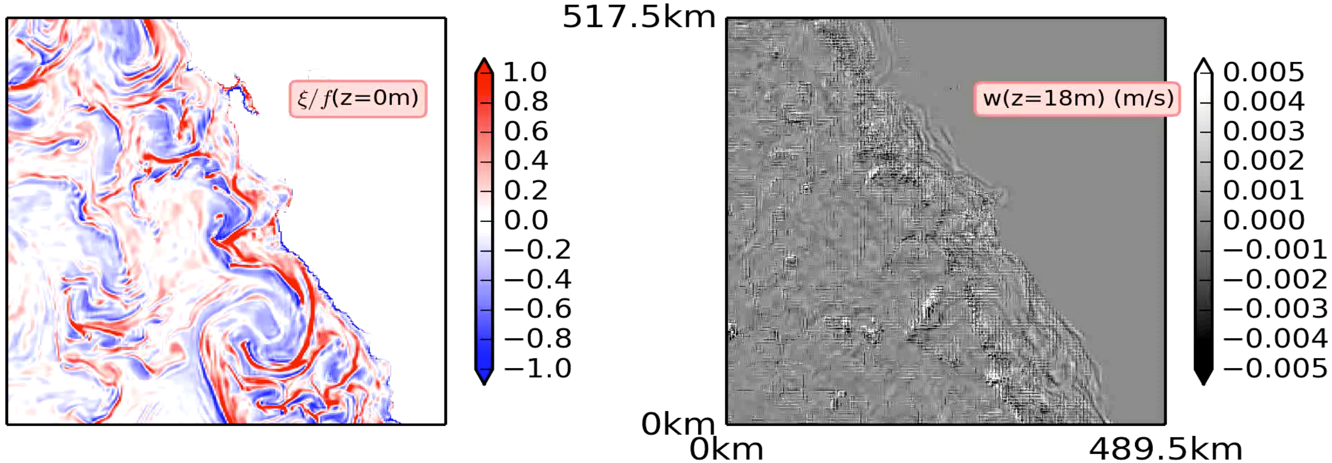
Altimeter observations

Infrared images



U,V observations

SAR images



Hector T., Dimitris M. JPL 2017

Waves and geostrophic turbulence have different signatures on SSH, SST, … ?

BAROTROPIC JET

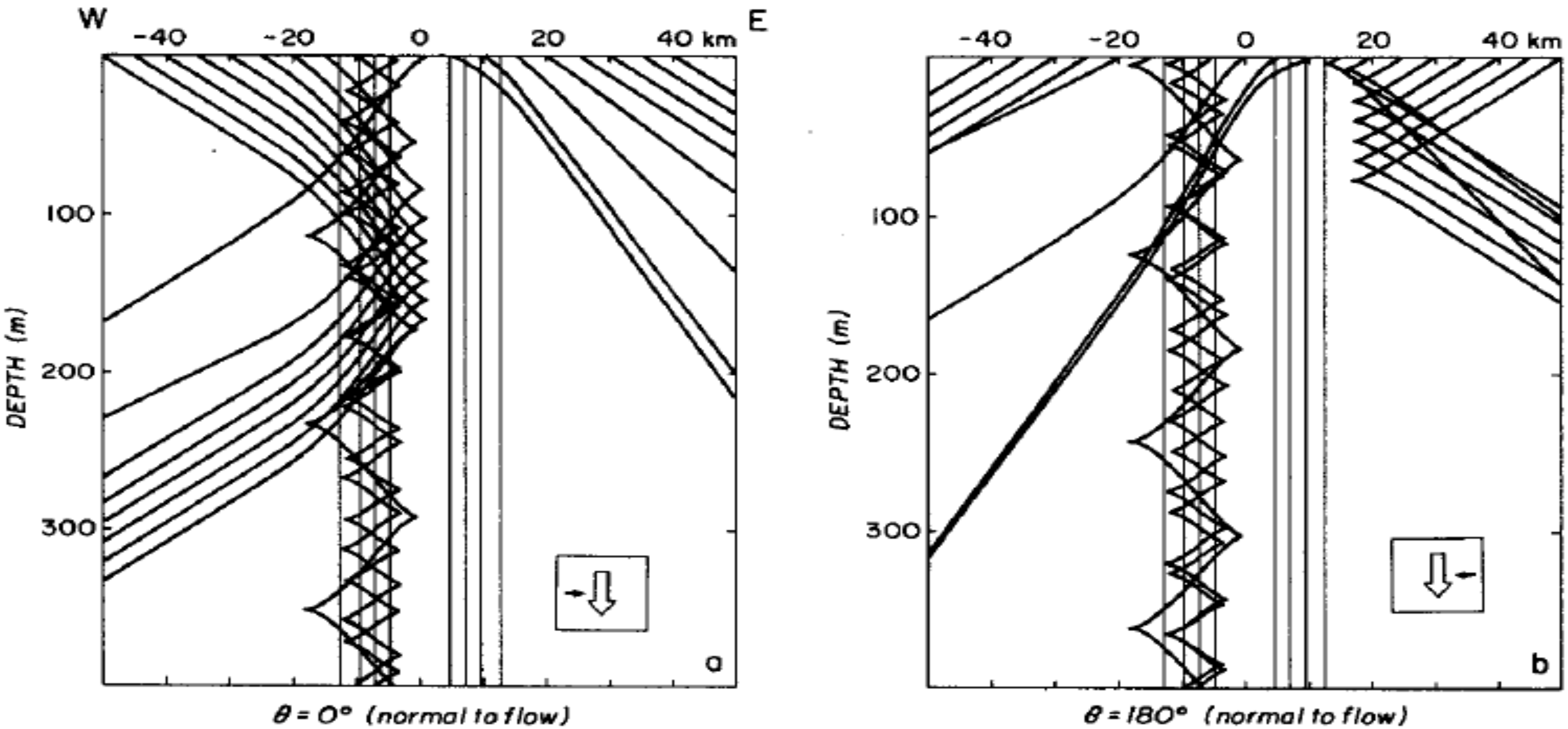


FIG. 5. Ray paths (thick solid lines) for near-inertial waves propagating downward and into a barotropic jet. The thin vertical lines are isotachs. Flow is out of the page. In this and Figs. 7–9, the horizontal wavevector $(k_x, k_y) = k_H(\cos\theta, \sin\theta)$ where k_x is the acrossfront and k_y the alongfront wavenumber. The legend insert in the lower right indicates the orientation of the horizontal wavevector with respect to the mean flow in each case. The two cases above are for normal incidence. Waves originating outside the jet are deflected away from the positive vorticity ridge on the eastern side of the jet (Fig. 4). Waves originating in the negative vorticity trough on the western side of the jet are trapped.

How to estimate these ray paths? **Ray tracing method**

Ray tracing method

If $\overrightarrow{X}_w$ is the wave position, then $\frac{d\overrightarrow{X}_w}{dt} = \overrightarrow{C}_g$

$$\frac{dX_w}{dt} = ?$$

$$\frac{dZ_w}{dt} = ?$$

If a$= A.\cos(k.x + m.z - \omega.t)$

$$\frac{dk}{dt} = ?$$

$$\frac{dm}{dt} = ?,$$

$$\frac{d\omega}{dt} = ?$$

WAVE KINEMATICS

Any physical variable (pressure, velocity, or whatever) can be written as Fourier series whose components are like:

$$a = A\cos(\underbrace{k_x x + k_y y - \omega t + \phi}_{\alpha}). \qquad \alpha = k_x x + k_y y - \omega t + \phi$$

- **A is the wave amplitude ($-A \leq a \leq A$)**
- **α is the phase (ϕ is a constant (=0 in this course))**
- **k_x, k_y are the wavenumbers in x and y [$k_x = 2\pi/\lambda_x$, $k_y = 2\pi/\lambda_y$ with λ_x, λ_x the wavelengths in the x and y directions]**
- **ω is the frequency [$\omega = 2\pi/T$ with T the period]**

WAVENUMBER AND FREQUENCY

Plane wave:

$$a = A\cos(k_x x + k_y y - \omega t + \phi).$$

Phase:

$$\alpha = k_x x + k_y y - \omega t + \phi$$

- **The wavenumber, $\vec{k}$ = [k_x, k_y] is perpendicular to the phase lines:**

$$\vec{k} = \nabla\alpha$$

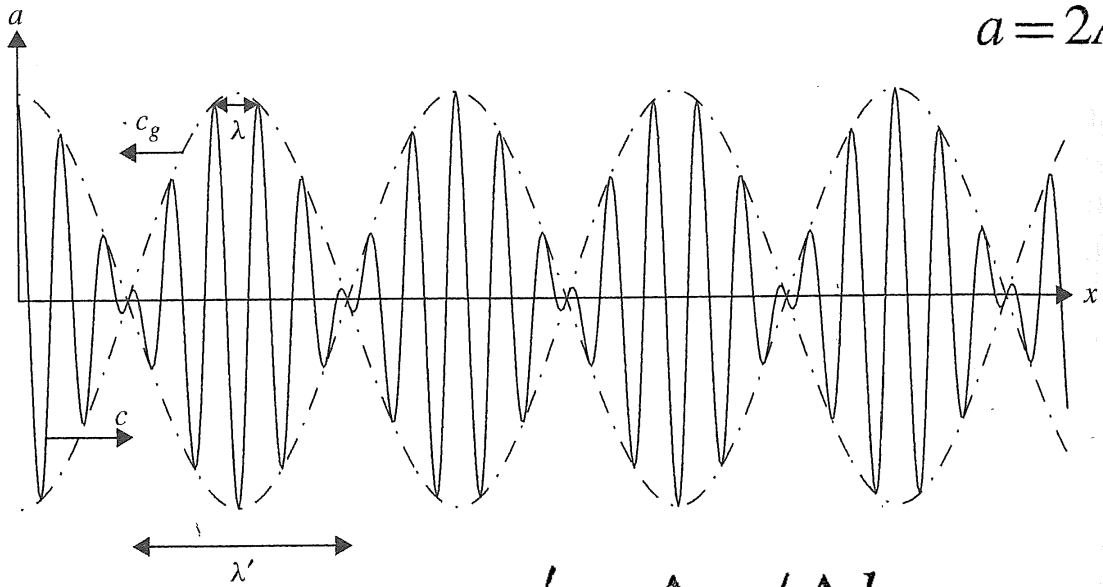
- **Frequency is defined as:**

$$\omega = -\,\partial\alpha/\partial t$$

GROUP VELOCITY AND ENERGY PROPAGATION

$$a = A\cos(k_1 x - \omega_1 t) + A\cos(k_2 x - \omega_2 t),$$

$$a = 2A\cos\left(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t\right)\cos(kx - \omega t)$$



$$c' = \Delta\omega / \Delta k.$$

$$c_g = \frac{d\omega}{dk}.$$

$$\boldsymbol{C_g = \nabla_k \omega}$$

BAROTROPIC JET: V(x) with N^2 is constant

$$\omega \approx f + \frac{\zeta(x)}{2} + \frac{N^2 k^2}{2 f m^2}$$

$$\vec{k} = \nabla\alpha\,, \quad \omega = -\frac{\partial\alpha}{\partial t}$$

$$\vec{C}_g = \nabla_{\vec{k}}\omega,$$

$$\frac{\partial\vec{k}}{\partial t} = \nabla\left[\frac{\partial\alpha}{\partial t}\right] = -\nabla\omega$$

$$\nabla\omega = (\nabla\omega)_{\vec{k}} + \nabla_{\vec{k}}\omega.\nabla\vec{k} = (\nabla\omega)_{\vec{k}} + \vec{C}_g.\nabla\vec{k}$$

$$\Rightarrow \quad \boxed{\frac{\partial\vec{k}}{\partial t} + \vec{C}_g.\nabla\vec{k} = -(\nabla\omega)_{\vec{k}}}$$

$$\frac{\partial\omega}{\partial t} = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}} + \nabla_{\vec{k}}\omega.\frac{\partial\vec{k}}{\partial t} = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}} - \vec{C}_g.\nabla\omega$$

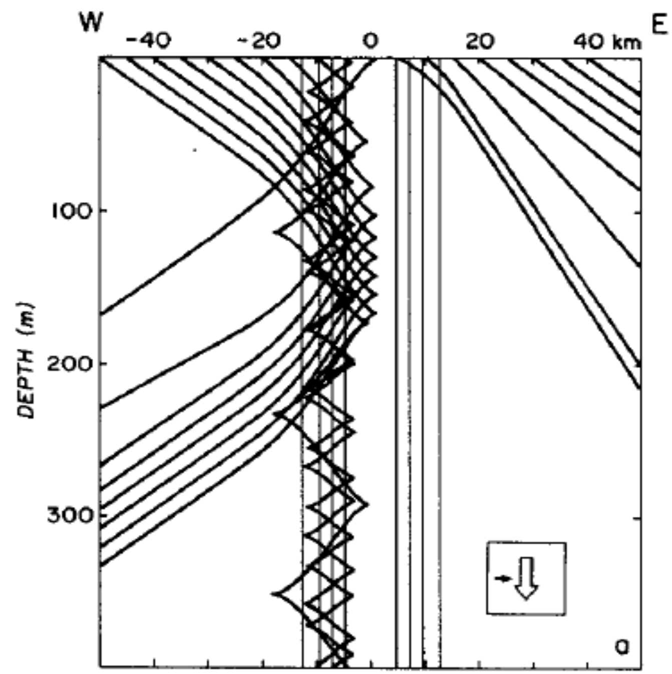
$$\Rightarrow \quad \boxed{\frac{\partial\omega}{\partial t} + \vec{C}_g.\nabla\omega = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}}}$$

Let us consider a stationary BAROTROPIC JET: V(x)
N^2 is constant

$$\omega \approx f + \frac{\zeta(x)}{2} + \frac{N^2 k^2}{2 f m^2}$$

$$\frac{dk}{dt} = -\frac{\partial \omega}{\partial x} = -\frac{1}{2}\frac{\partial \zeta}{\partial x}$$

$$\frac{dm}{dt} = -\frac{\partial \omega}{\partial z} = 0, \qquad \frac{d\omega}{dt} = 0$$

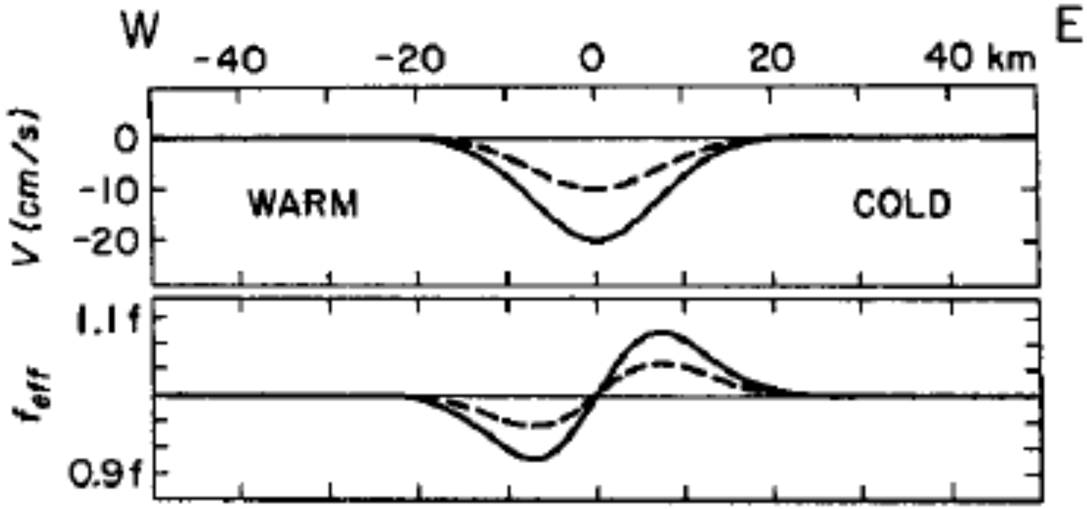


Wave position:

$$\frac{dX_w}{dt} = \frac{N^2 k}{f m^2}, \frac{d^2 X_w}{dt^2} = -\frac{1}{2}\frac{\partial \zeta}{\partial x} \cdot \frac{N^2}{f m^2}$$

$$\frac{dZ_w}{dt} = \frac{N^2 k^2}{f m^3}, \frac{d^2 Z_w}{dt^2} = -\frac{\partial \zeta}{\partial x} \cdot \frac{N^2 k}{f m^3}$$

Let us consider a stationary barotropic jet: *V(x)*



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$$\partial V/\partial x = 0.1f/10km$$

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Propagation of a wave packet in a stationary barotropic jet

group velocity is: $\overrightarrow{C_g} = \nabla_k \omega$

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$$C_{gx} = \frac{\partial \omega}{\partial k} = \frac{N^2 k}{\omega m^2} = \frac{\omega^2 - f^2}{\omega} \frac{k}{k^2 + l^2}$$

$$C_{gy} = \frac{\partial \omega}{\partial l} = \frac{N^2 l}{\omega m^2} = \frac{\omega^2 - f^2}{\omega} \frac{l}{k^2 + l^2}$$

$$C_{gz} = \frac{\partial \omega}{\partial m} = -\frac{N^2 (k^2 + l^2)}{\omega m^3} = -\frac{\omega^2 - f^2}{\omega m}$$ (downward propagation if $\omega > 0$)

Barotropic jet:

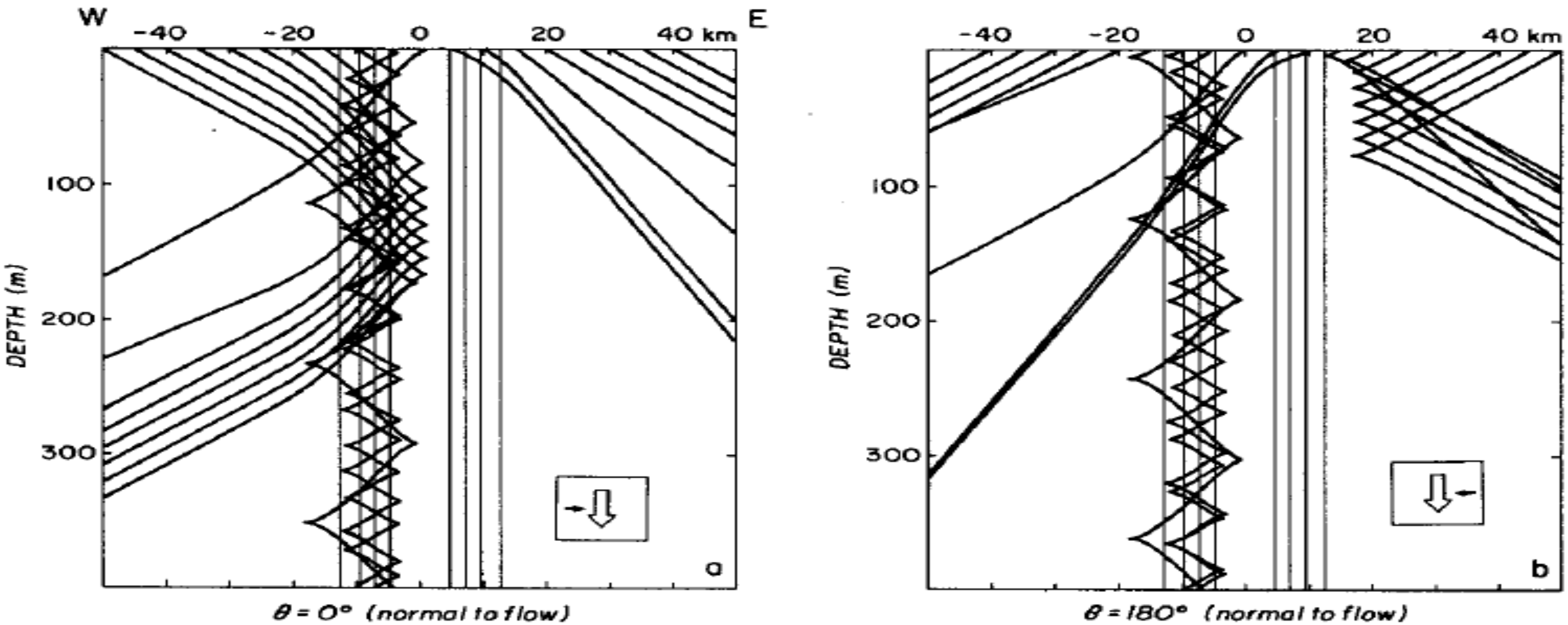
$m = \frac{2.\pi}{100}$ m^{-1}, $k_o = \frac{2.\pi}{40km}$, $\omega = 1.06f$

To continue to satisfy the dispersion relation, as the relative vorticity changes when the wave packet propagates, the wavector adjusts …

This leads to: $k = k_o - \frac{\partial \zeta}{\partial x} t$.

BAROTROPIC JET: Wave packets propagating east or west … … and downward (taken into account C_{gz}) trapped in $\zeta < 0$ structures

$$\frac{dk}{dt} = -\frac{1}{2}\frac{\partial\zeta}{\partial x}, \quad \frac{dm}{dt} = 0, \quad \frac{d\omega}{dt} = 0$$



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Let us consider a BAROCLINIC JET:

$V(x,z)$

N^2 is constant

BARCLINIC JET: Wave packets propagating east or west …

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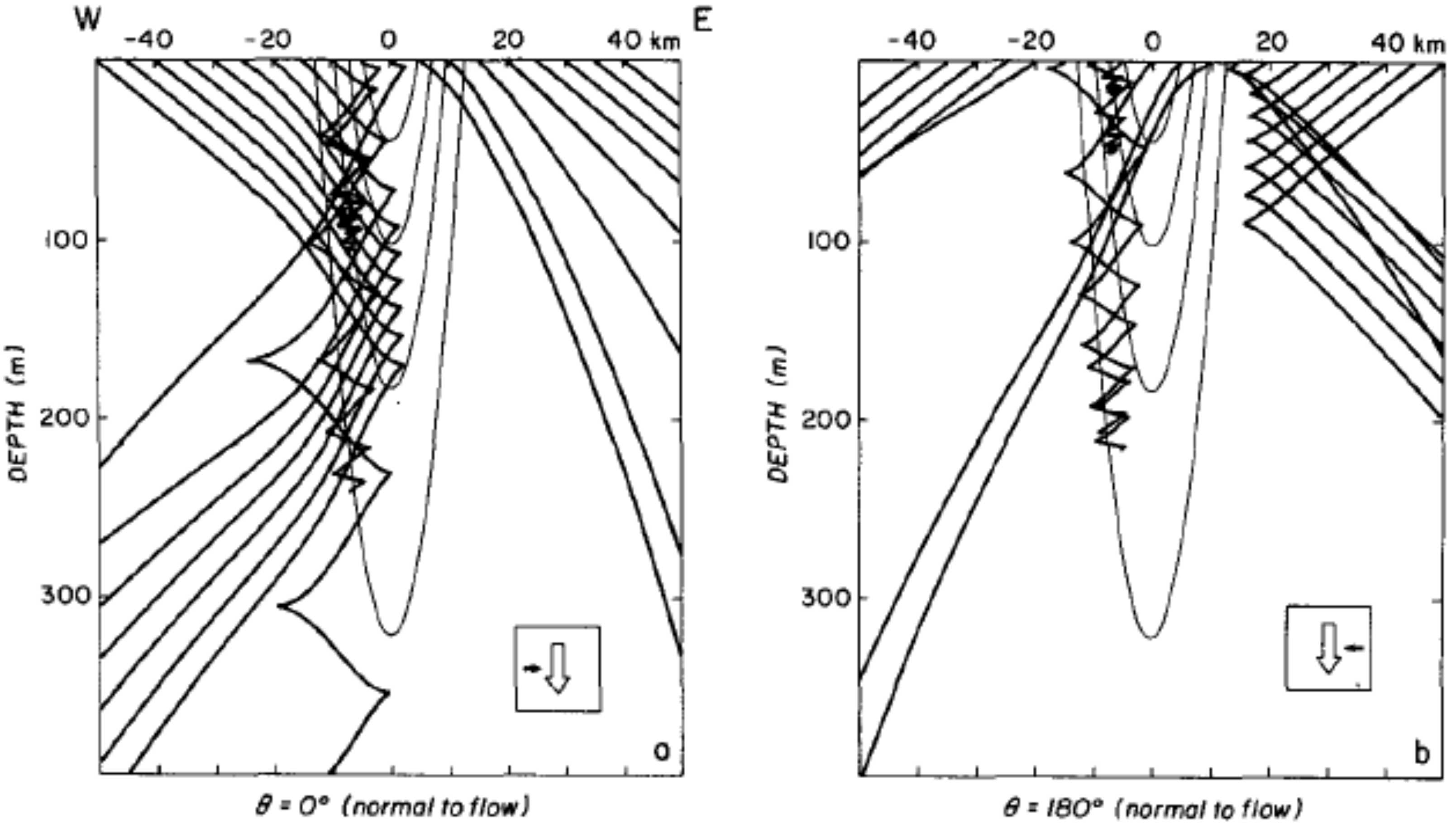


FIG. 7. Normal incidence as in Fig. 5 but for a baroclinic jet. Flow is out of the page. In this case the negative vorticity trough on the western side of the front traps waves vertically as well as horizontally. Near-inertial waves stall at their critical-depth where $f_{\text{eff}} = \omega_0$.

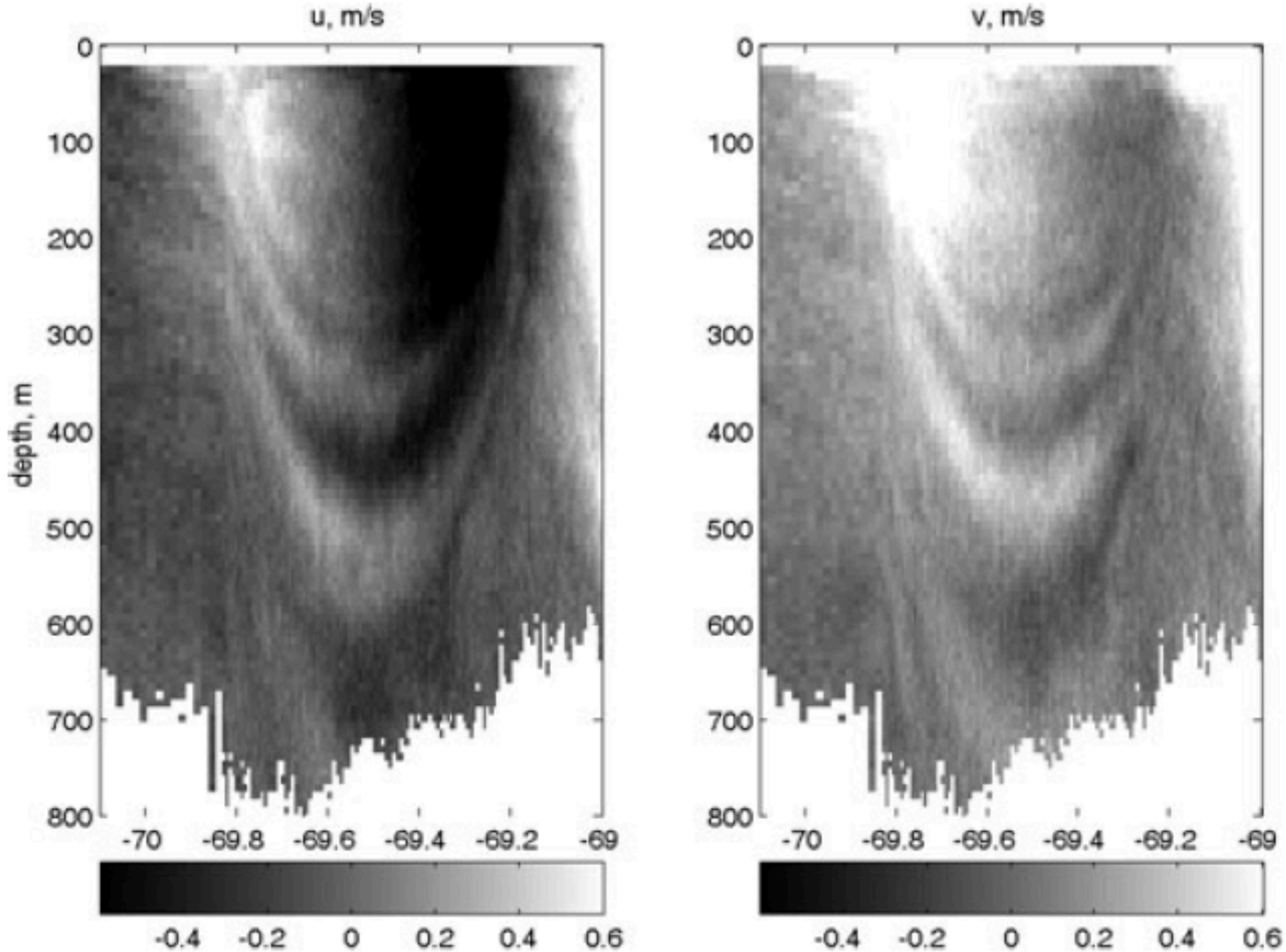


Figure 2. Grey-scale images of SADCP velocity data (zonal component on the left panel and meridional component on the right panel) plotted against longitude showing vertical banding of 100–200 m wavelength within the center of the WCR (Figure 1) with strong WSW flow on the seaward side (69.3°W) and NNW flow on the shoreward side (69.8°W). The phase of the vertical banding slopes up on both flanks of the WCR.

PHASE LINES FOLLOW THE SLOPE $S=-M^2/N^2$

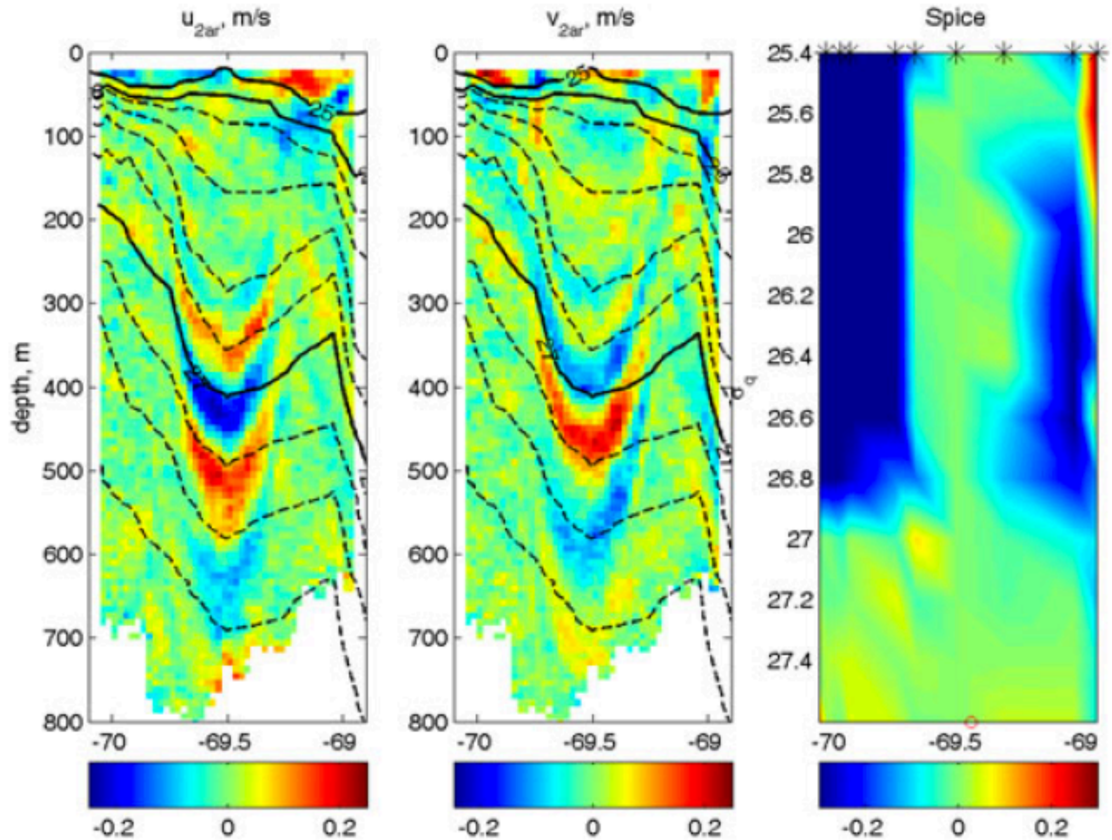


Figure 5. Phase-adjusted anomalies of velocity and spice (relative to ring center; right panel). During the velocity anomaly phase adjustment, the upward phase propagation altered the rotated velocity anomalies, and the constant phase lines (Figure 2) moved slightly upward on the right and downward on the left of ring center. Also plotted are constant σ_θ surfaces at a 0.2 kg m^{-3} contour interval. Solid lines denote the 25, 26, and 27 σ_θ surfaces. These are from the CTD stations taken on the seaward leg and have been shifted westward in longitude an amount of 0.13° to account for translation of the WCR between the two legs of the cruise. The Gulf Stream north wall can be seen intruding at the far right of all panels, consistent with the contact between the Gulf Stream and WCR indicated in Figure 1. The vertical axis is depth for the left and middle panels and σ_θ for the right panel. Longitude of the mooring W2 is indicated by the open circle at the bottom of the right panel.

Dispersion of waves in a 3-D balanced flow field

$$\psi = \psi_0 \exp[i(k_x x + k_y y + k_z z - \omega t)],$$

$$\omega_0 = \omega - (\mathbf{k} \cdot \mathbf{V}) \approx \left[\underset{\text{(i)}}{f_{\text{eff}}} + \underset{\text{(ii)}}{\frac{N^2 k_H^2}{2 f k_z^2}} + \underset{\text{(iii)}}{\underline{\frac{1}{2 k_z} \left(\frac{\partial U}{\partial z} k_y - \frac{\partial V}{\partial z} k_x \right)}} \right]$$

with:

$$f_{\text{eff}} \approx f + \frac{\zeta}{2} = f + \frac{1}{2} \left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y} \right)$$

In a 3-D balanced flow field new terms involve the vertical shear of the horizontal motions, i.e. the horizontal density gradients.
So both the vertical and horizontal density gradients are involved!

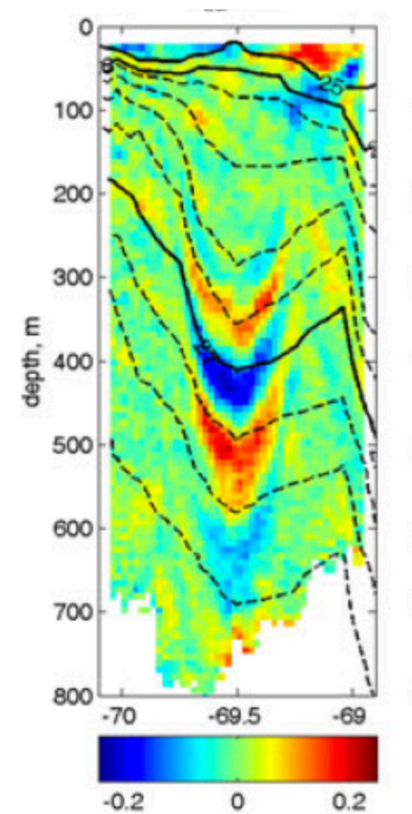
Dispersion relation can be cast in a form similar to that for internal gravity waves propagating in an ocean bereft of balanced motions by replacing the Coriolis and buoyancy frequencies with effective Coriolis and buoyancy frequencies:

$$\boxed{\omega_0 \approx f_{\text{eff}} + \frac{N_{\text{eff}}^2}{2f}\frac{k_H{}^2}{k_z{}^2}}$$

with: $$f_{\text{eff}} \approx f + \frac{\zeta}{2} = f + \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)$$

$$N_{\text{eff}}^2 = N^2 + M_x{}^2\frac{k_x k_z}{k_H{}^2} + M_y{}^2\frac{k_y k_z}{k_H{}^2}$$

where $M_x{}^2 = g(\partial\rho/\partial x)/\rho_0$, $M_y{}^2 = g(\partial\rho/\partial y)/\rho_0$



Interpretation of N_{eff}: slope of the isopycnals is: $s = \frac{M^2}{N^2}$ => $N_{eff}^2 = N^2(1 + s.\frac{m}{k})$

The preceding dispersion relation is obtained using the WKB approximation (balanced flow field is slowly varying in space and time compared with the wave field evolution).

As a consequence, the frequency, ω, is an explicit function of $\vec{k}$ but also of x, y, z :

$$\omega = \omega(\vec{k}, x, y, z)$$

How to use this dispersion relation to infer the propagation of internal gravity waves in a 3-D balanced flow field ?

Let us consider a stationarity BAROCLINIC JET: V(x,z)
N^2 is constant

$\vec{k} = \nabla\alpha$, $\quad \omega = -\dfrac{\partial\alpha}{\partial t}$

$\omega = \omega(k, m\, x, z)$, $\quad \vec{C}_g = \nabla_{\vec{k}}\omega$, $\boxed{\dfrac{d\vec{X}_w}{dt} = \vec{C}_g\ ,}$ $\quad \vec{X}_w$ is the wave position

$$\frac{\partial\vec{k}}{\partial t} = \nabla\left[\frac{\partial\alpha}{\partial t}\right] = -\nabla\omega$$

$$\nabla\omega = (\nabla\omega\,)_{\vec{k}} + \nabla_{\vec{k}}\omega.\nabla\vec{k} = (\nabla\omega\,)_{\vec{k}} + \vec{C}_g.\nabla\vec{k}$$

=> $$\boxed{\frac{\partial\vec{k}}{\partial t} + \vec{C}_g.\nabla\vec{k} = -(\nabla\omega)_{\vec{k}}}$$

$$\frac{\partial\omega}{\partial t} = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}} + \nabla_{\vec{k}}\omega.\frac{\partial\vec{k}}{\partial t} = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}} - \vec{C}_g.\nabla\omega$$

=> $$\boxed{\frac{\partial\omega}{\partial t} + \vec{C}_g.\nabla\omega = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}}}$$

3-D group velocity in a 3-D balanced flow field

$$\omega_0 = \omega - (\mathbf{k} \cdot \mathbf{V}) \approx \left[f_{\text{eff}} + \frac{N^2 k_H{}^2}{2 f k_z{}^2} + \frac{1}{k_z} \left(\frac{\partial U}{\partial z} k_y - \frac{\partial V}{\partial z} k_x \right) \right]$$

$$Cg_x = \frac{\partial \omega_0}{\partial k_x} \approx \frac{N^2 k_x}{f k_z{}^2} - \frac{1}{k_z} \frac{\partial V}{\partial z}$$

$$Cg_y = \frac{\partial \omega_0}{\partial k_y} \approx \frac{N^2 k_y}{f k_z{}^2} + \frac{1}{k_z} \frac{\partial U}{\partial z}$$

$$Cg_z = \frac{\partial \omega_0}{\partial k_z} \approx \frac{-N^2 k_H{}^2}{f k_z{}^3} - \frac{1}{k_z{}^2} \left(\frac{\partial U}{\partial z} k_y - \frac{\partial V}{\partial z} k_x \right)$$

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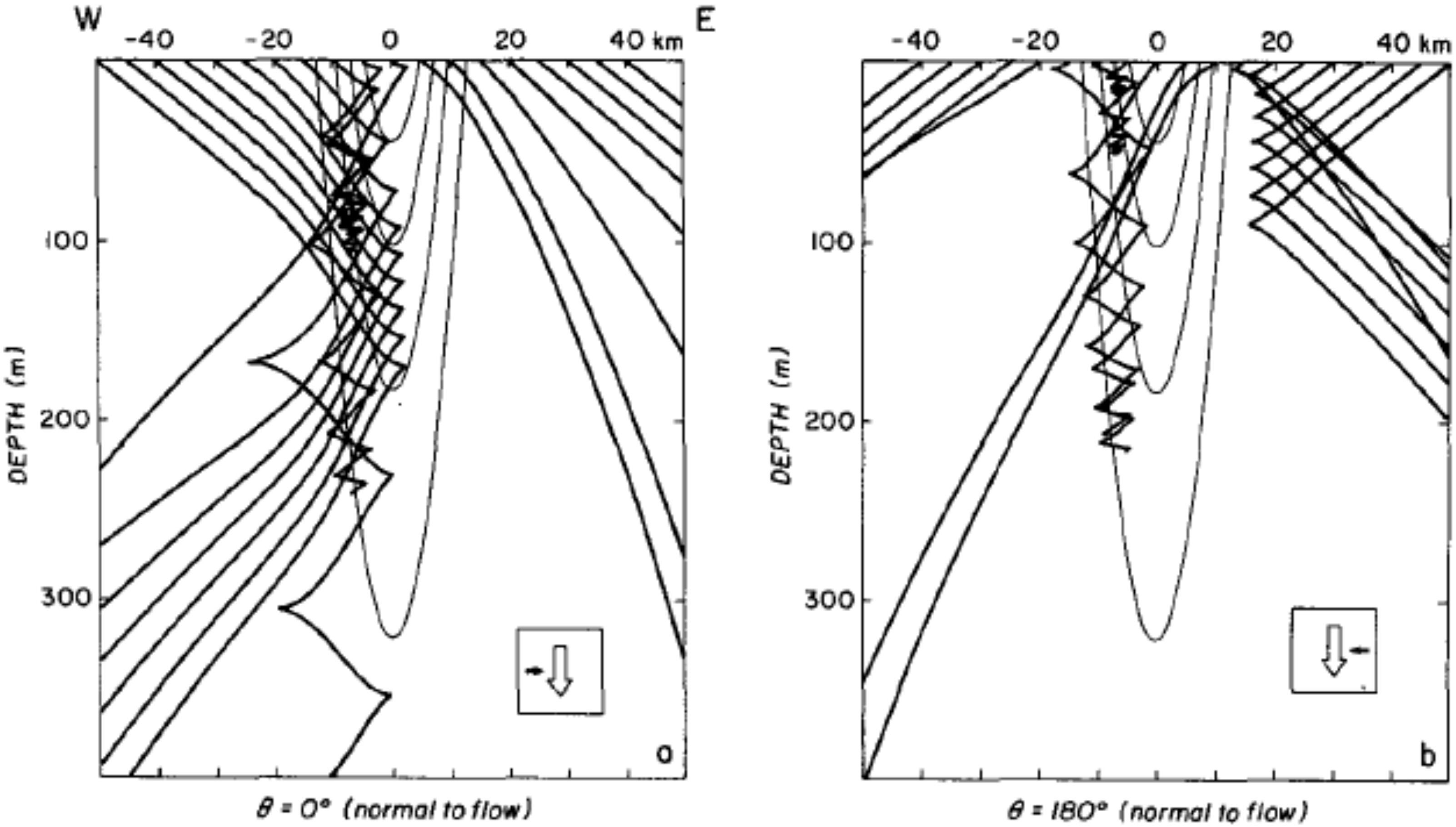

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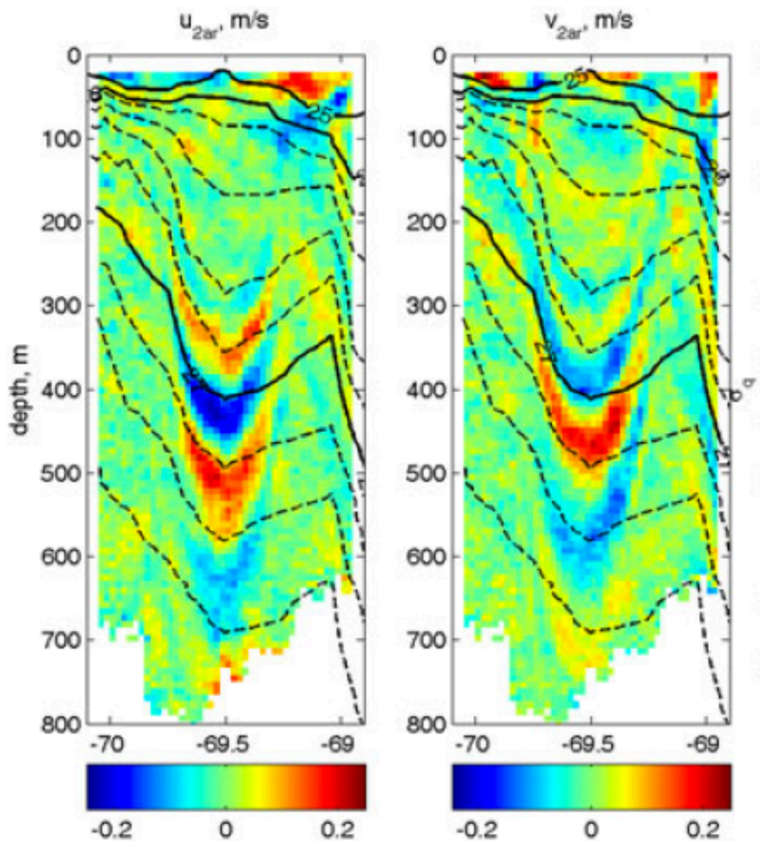
Why the phase lines follow the isopycnals?



Observations

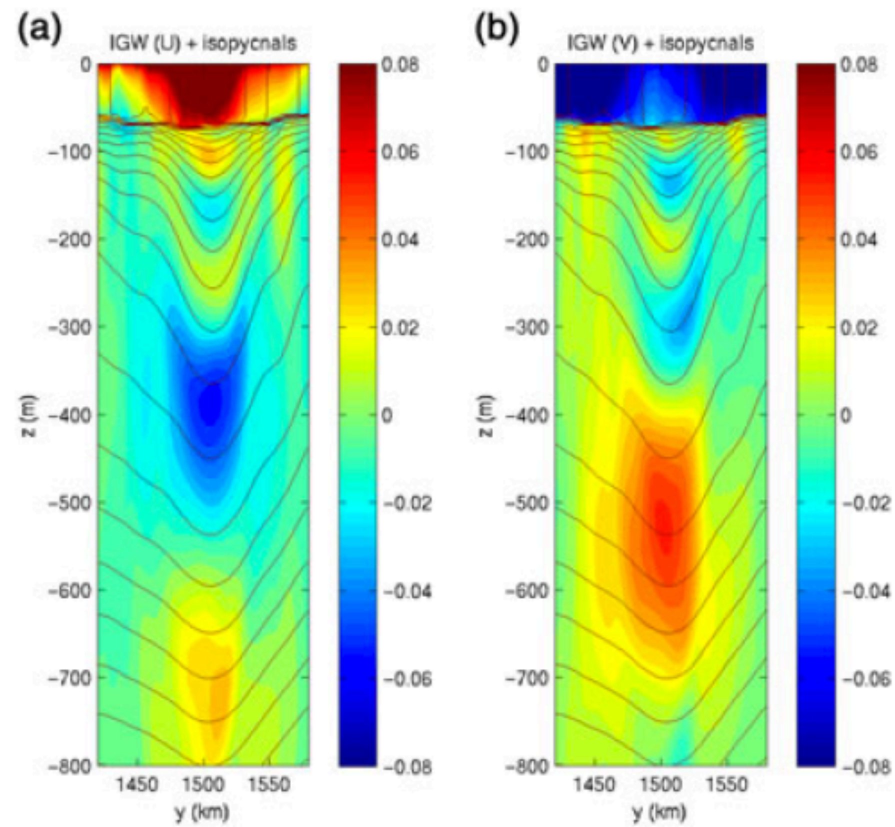


Figure 7. (a) Zonal and (b) meridional components of the NIWs observed in the numerical model. Velocities are in color (ranging from -0.08 to $0.08\,\mathrm{m/s}$) and isopycnals in contours ($\mathrm{ci} = 0.06\,\mathrm{kg/m^3}$).

Numerical simulation

NEAR-INERTIAL (f) AND TIDAL WAVES (M2) EXPLAIN A LARGE PART OF THE WAVE SPECTRUM

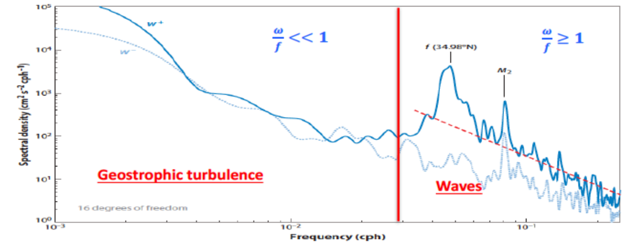
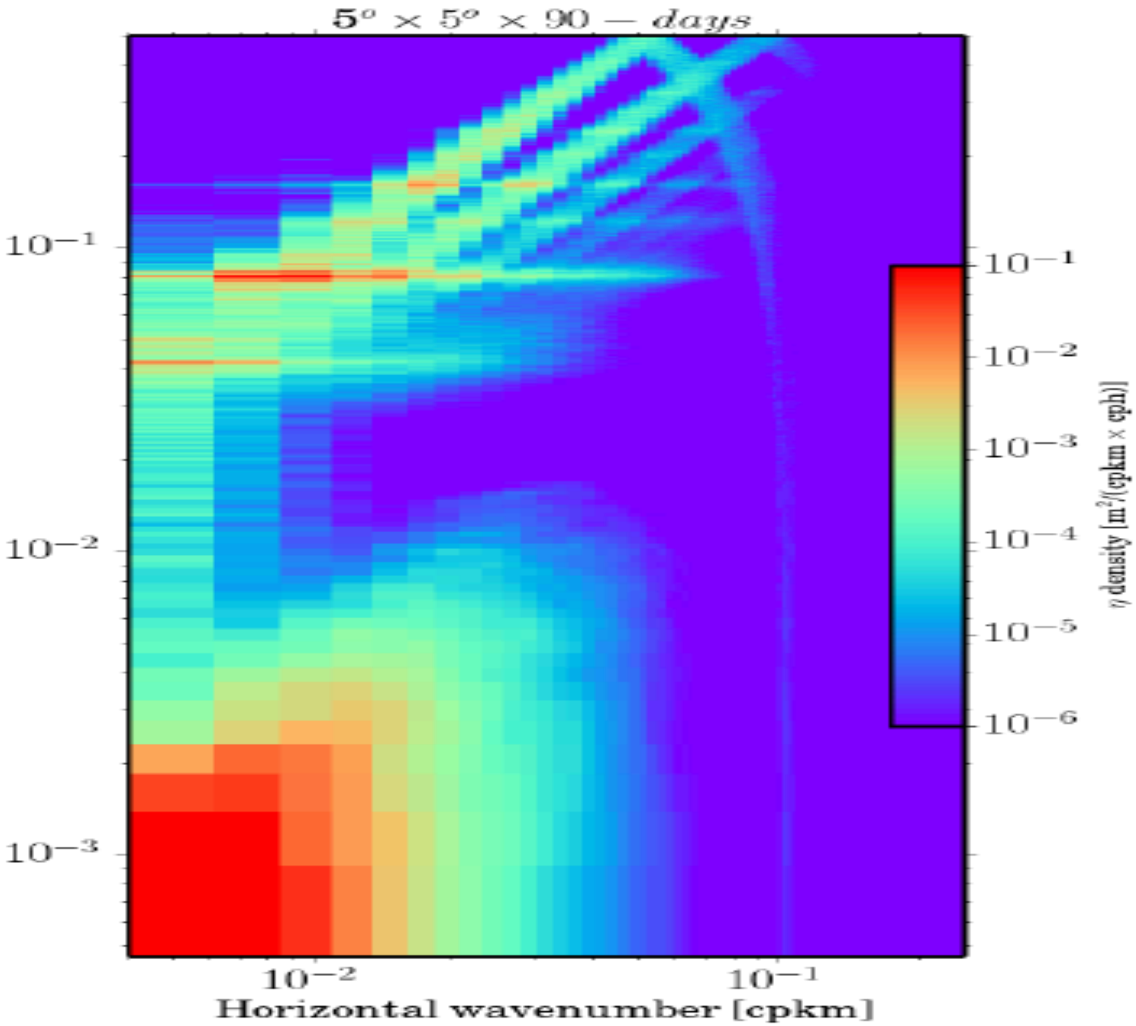


Figure 1

Rotary velocity spectrum at 261-m depth from current-meter data from the WHOI699 mooring gathered during the WESTPAC1 experiment (mooring at 6,149-m depth.) The solid blue line (w^+) is clockwise motion, and the dashed blue line (w^-) is counterclockwise motion; the differences between these emphasize the downward energy propagation that often dominates the near-inertial band. The dashed red line is the line $E_0 N \omega^{-p}$ with $N = 2.0$ cycles per hour (cph), $E_0 = 0.096$ cm² s⁻² cph⁻², and $p = 2.25$, which is quantitatively similar to levels in the Cartesian spectra presented by Fu (1981) for station 5 of the Polygon Mid-Ocean Experiment (POLYMODE) II array.



The high frequency part of the wave spectrum is characterized by discrete bands at high wavenumbers!

3-D group velocity in a 3-D balanced flow field

$$\omega_0 = \omega - (\mathbf{k} \cdot \mathbf{V}) \approx \left[f_{\text{eff}} + \frac{N^2 k_H{}^2}{2 f k_z{}^2} + \frac{1}{k_z} \left(\frac{\partial U}{\partial z} k_y - \frac{\partial V}{\partial z} k_x \right) \right]$$

$$Cg_x = \frac{\partial \omega_0}{\partial k_x} \approx \frac{N^2 k_x}{f k_z{}^2} - \frac{1}{k_z} \frac{\partial V}{\partial z}$$

$$Cg_y = \frac{\partial \omega_0}{\partial k_y} \approx \frac{N^2 k_y}{f k_z{}^2} + \frac{1}{k_z} \frac{\partial U}{\partial z}$$

$$Cg_z = \frac{\partial \omega_0}{\partial k_z} \approx \frac{-N^2 k_H{}^2}{f k_z{}^3} - \frac{1}{k_z{}^2} \left(\frac{\partial U}{\partial z} k_y - \frac{\partial V}{\partial z} k_x \right)$$

How to use the dispersion relation to infer the propagation of internal gravity waves in a 3-D balanced flow field ?

3-D group velocity in a 3-D balanced flow field

$$\omega_0 = \omega - (\mathbf{k}\cdot\mathbf{V}) \approx \left[f_{\text{eff}} + \frac{N^2 k_H{}^2}{2 f k_z{}^2} + \frac{1}{k_z}\left(\frac{\partial U}{\partial z} k_y - \frac{\partial V}{\partial z} k_x\right)\right]$$

$$Cg_x = \frac{\partial \omega_0}{\partial k_x} \approx \frac{N^2 k_x}{f k_z{}^2} - \frac{1}{k_z}\frac{\partial V}{\partial z}$$

$$Cg_y = \frac{\partial \omega_0}{\partial k_y} \approx \frac{N^2 k_y}{f k_z{}^2} + \frac{1}{k_z}\frac{\partial U}{\partial z}$$

$$Cg_z = \frac{\partial \omega_0}{\partial k_z} \approx \frac{-N^2 k_H{}^2}{f k_z{}^3} - \frac{1}{k_z{}^2}\left(\frac{\partial U}{\partial z} k_y - \frac{\partial V}{\partial z} k_x\right)$$

How to use the dispersion relation to infer the propagation of internal gravity waves in a 3-D balanced flow field ?

What is the time evolution of $\vec{k}$ and ω when the wave packet propagates in a field of balanced motions?

$$\vec{k} = \nabla\alpha\,, \quad \omega = -\frac{\partial\alpha}{\partial t}$$

$$\omega = \omega(\vec{k}, x, y, z, t), \quad \vec{C}_g = \nabla_{\vec{k}}\omega,$$

$$\frac{d\vec{X}_w}{dt} = \vec{C}_g\,,$$

$\vec{X}_w$ is the wave position

$$\frac{\partial\vec{k}}{\partial t} = \nabla[\frac{\partial\alpha}{\partial t}] = -\nabla\omega$$

$$\nabla\omega = (\nabla\omega\,)_{\vec{k}} + \nabla_{\vec{k}}\omega.\nabla\vec{k} = (\nabla\omega\,)_{\vec{k}} + \vec{C}_g.\nabla\vec{k}$$

=> $$\frac{\partial\vec{k}}{\partial t} + \vec{C}_g.\nabla\vec{k} = -(\nabla\omega)_{\vec{k}}$$

$$\frac{\partial\omega}{\partial t} = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}} + \nabla_{\vec{k}}\omega.\frac{\partial\vec{k}}{\partial t} = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}} - \vec{C}_g.\nabla\omega$$

=> $$\frac{\partial\omega}{\partial t} + \vec{C}_g.\nabla\omega = \left(\frac{\partial\omega}{\partial t}\right)_{\vec{k}}$$

Dispersion relation in a 3-D balanced flow field

$$\omega_0 = \omega - (\mathbf{k}\cdot\mathbf{V}) \approx \left[\underset{\text{(i)}}{f_{\text{eff}}} + \underset{\text{(ii)}}{\frac{N^2 k_H^2}{2 f k_z^2}} + \underset{\text{(iii)}}{\underline{\frac{1}{2k_z}\left(\frac{\partial U}{\partial z} k_y - \frac{\partial V}{\partial z} k_x\right)}} \right]$$

$$+\, i\left\{ \underset{\text{(iv)}}{\underline{\frac{\zeta}{2 f k_z}\left(\frac{\partial U}{\partial z} k_x + \frac{\partial V}{\partial z} k_y\right)}} + \underset{\text{(v)}}{\frac{N^2}{2 f^2 k_z^2}\left[\frac{\partial U}{\partial x} k_y^2 - \left(\frac{\partial U}{\partial y} + \frac{\partial V}{\partial x}\right) k_x k_y + \frac{\partial V}{\partial y} k_x^2\right]} \right\},$$

with: $$f_{\text{eff}} \approx f + \frac{\zeta}{2} = f + \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)$$

(from E. Kunze JPO 1985)

In a 3-D balanced flow field new terms involve the vertical shear of the horizontal motions, i.e. the horizontal density gradients.
So both the vertical and horizontal density gradients are involved!

Dispersion relation can be cast in a form similar to that for internal gravity waves propagating in an ocean bereft of balanced motions by replacing the Coriolis and buoyancy frequencies with effective Coriolis and buoyancy frequencies:

See Eric Kunze JPO 1985

$$\boxed{\omega_0 \approx f_{\text{eff}} + \frac{N_{\text{eff}}^2}{2f}\frac{{k_H}^2}{{k_z}^2}}$$

with: $$f_{\text{eff}} \approx f + \frac{\zeta}{2} = f + \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)$$

$$N_{\text{eff}}^2 = N^2 + M_x^2 \frac{K_x K_z}{{k_H}^2} + M_y^2 \frac{K_y K_z}{{k_H}^2}$$

where $M_x^2 = g(\partial\rho/\partial x)/\rho_0$, $M_y^2 = g(\partial\rho/\partial y)/\rho_0$

Interpretation:

Slope of the isopycnals is: $s = \frac{M^2}{N^2}$ => $N_{eff}^2 = N^2(1 + s.\frac{m}{k})$

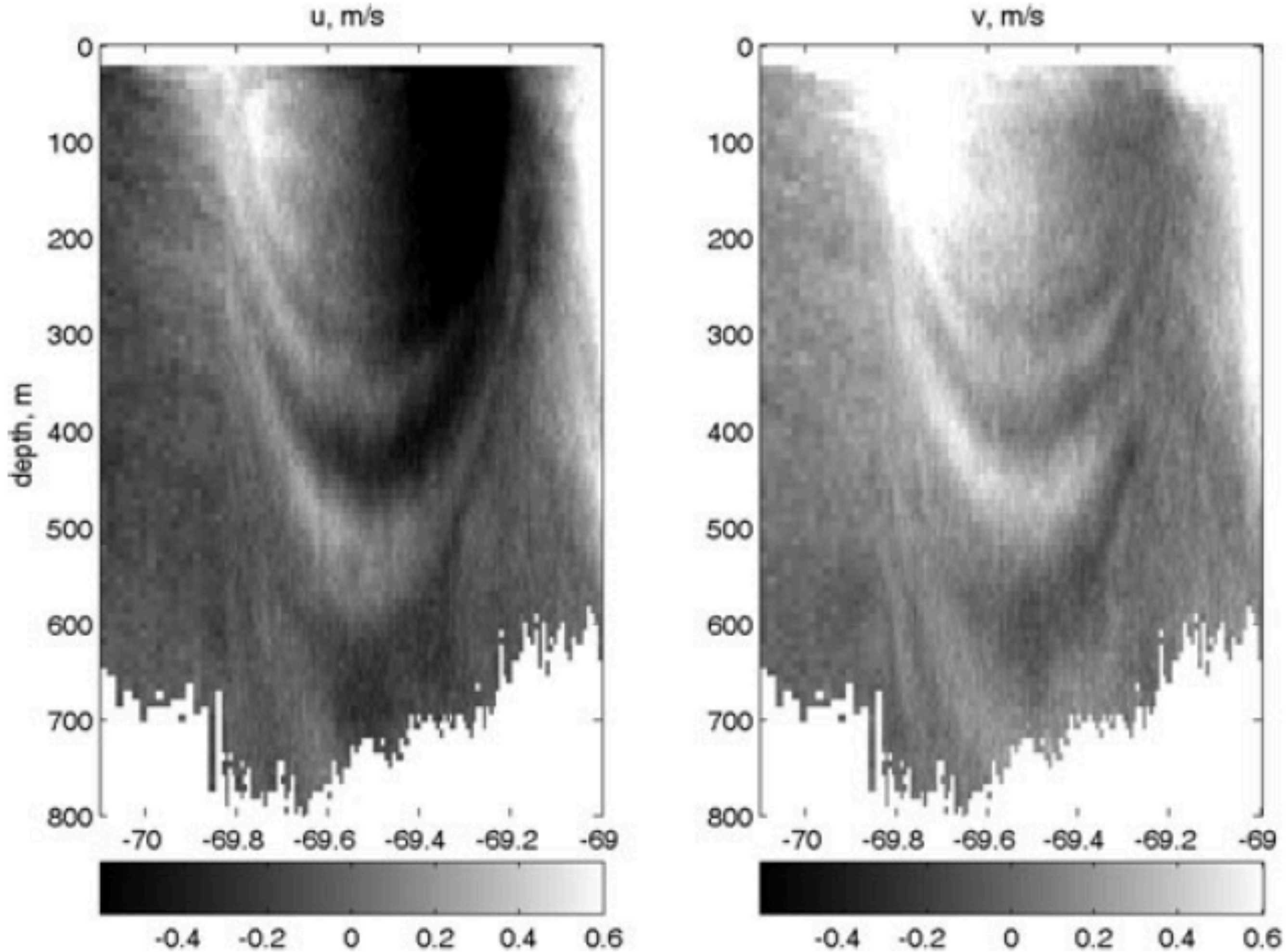


Figure 2. Grey-scale images of SADCP velocity data (zonal component on the left panel and meridional component on the right panel) plotted against longitude showing vertical banding of 100–200 m wavelength within the center of the WCR (Figure 1) with strong WSW flow on the seaward side (69.3°W) and NNW flow on the shoreward side (69.8°W). The phase of the vertical banding slopes up on both flanks of the WCR.

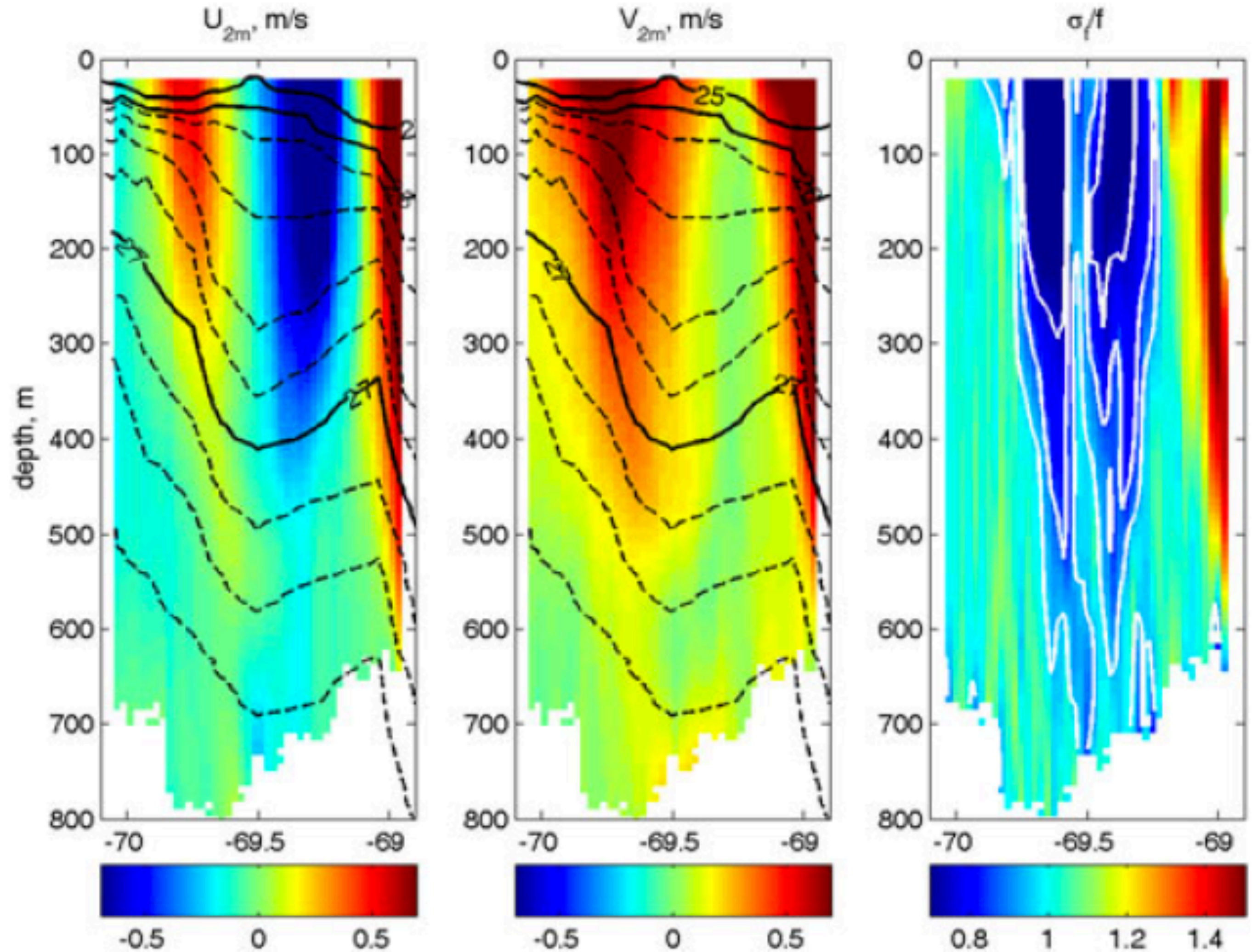


Figure 4. Smoothed velocity components and vortex-modified inertial frequency associated with the circulation of the WCR. In the latter, the white contours illustrate the sub-inertial structure of σ_f/f (equation (B10), contours of 0.75, 0.85, and 0.95), and this can be seen to encompass the region of anomalous velocities and spice (see Figure 5) within the WCR.

PHASE LINES FOLLOW THE SLOPE S=-M²/N²

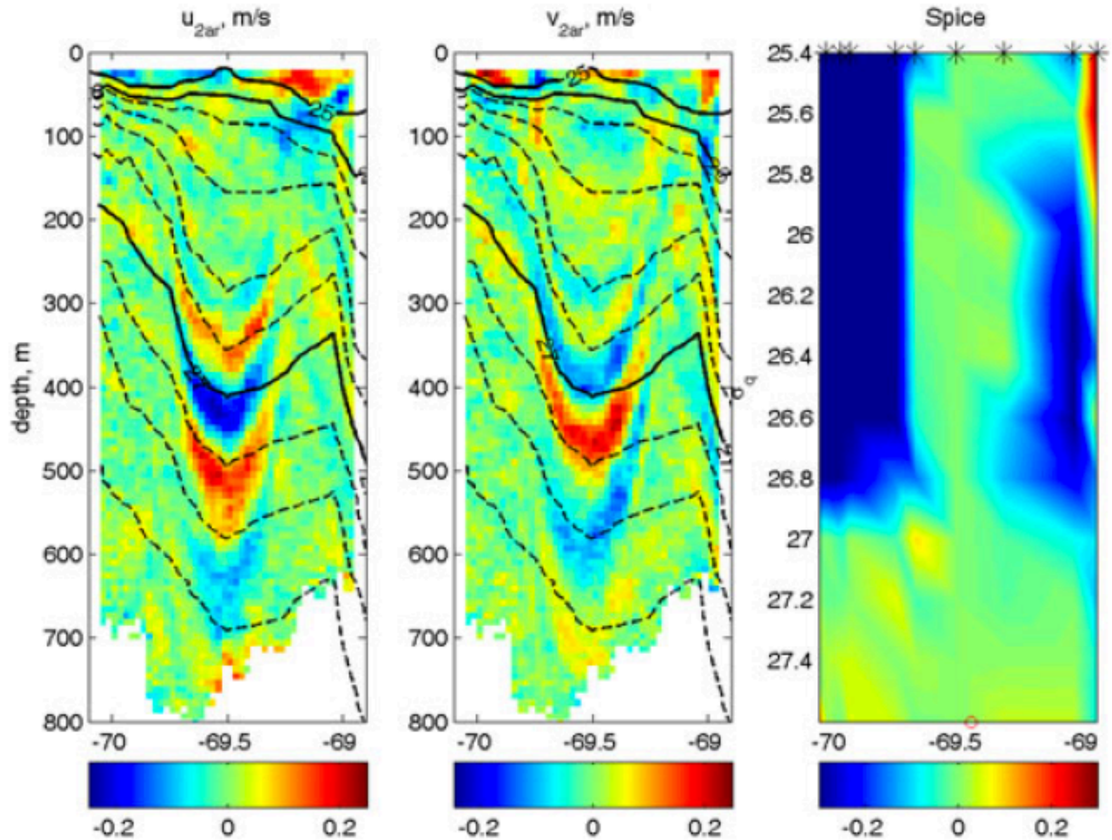


Figure 5. Phase-adjusted anomalies of velocity and spice (relative to ring center; right panel). During the velocity anomaly phase adjustment, the upward phase propagation altered the rotated velocity anomalies, and the constant phase lines (Figure 2) moved slightly upward on the right and downward on the left of ring center. Also plotted are constant σ_θ surfaces at a 0.2 kg m^{-3} contour interval. Solid lines denote the 25, 26, and 27 σ_θ surfaces. These are from the CTD stations taken on the seaward leg and have been shifted westward in longitude an amount of 0.13° to account for translation of the WCR between the two legs of the cruise. The Gulf Stream north wall can be seen intruding at the far right of all panels, consistent with the contact between the Gulf Stream and WCR indicated in Figure 1. The vertical axis is depth for the left and middle panels and σ_θ for the right panel. Longitude of the mooring W2 is indicated by the open circle at the bottom of the right panel.

Dispersion relation can be cast in a form similar to that for internal gravity waves propagating in an ocean bereft of balanced motions by replacing the Coriolis and buoyancy frequencies with effective Coriolis and buoyancy frequencies:

See Eric Kunze JPO 1985

$$\boxed{\omega_0 \approx f_{\text{eff}} + \frac{N_{\text{eff}}^2}{2f}\frac{k_H{}^2}{k_z{}^2}}$$

with:

$$f_{\text{eff}} \approx f + \frac{\zeta}{2} = f + \frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)$$

$$N_{\text{eff}}^2 = N^2 + 2M_x{}^2\frac{\kappa_x\kappa_z}{k_H{}^2} + 2M_y{}^2\frac{\kappa_y\kappa_z}{k_H{}^2}$$

where $M_x{}^2 = g(\partial\rho/\partial x)/\rho_0$, $M_y{}^2 = g(\partial\rho/\partial y)/\rho_0$

Interpretation: Let us focus on the x-z direction.

The slope of the isopycnals is defined as: $s = \frac{M^2}{N^2}$ and the slope of the phase lines as: $-\frac{\mathrm{k}}{\mathrm{m}}$.

This leads to $N_{eff}^2 = N^2\left(1 + s.\frac{m}{k}\right)$

The dispersion relation is:

$$\omega \approx f_{eff} + N^2\left(1 + s.\frac{m}{k}\right)$$

Let us focus on the x-z direction.

The slope of the isopycnals is defined as: $s = \frac{M^2\ (x,z)}{N^2(z)}$ and the slope of the phase lines as: $-\frac{\mathrm{k}}{\mathrm{m}}$.

This leads to $N_{eff}^2 = N^2\left(1 + s.\frac{m}{k}\right)$

The dispersion relation is:

$$\omega \approx f_{eff} + \frac{N^2}{2f}\frac{k^2}{m^2}\left(1 + s.\frac{m}{k}\right)$$

How *k* and *m* adjust to the s changes?

$$\frac{dm}{dt} = -\frac{\partial\omega}{\partial z} = \frac{1}{2f}\frac{k^2}{m^2}\left(1 + s.\frac{m}{k}\right)\frac{dN^2}{dz} - \frac{N^2k}{2fm}.\frac{\partial s}{\partial z}$$

$$\frac{dk}{dt} = -\frac{\partial\omega}{\partial x} = -\frac{1}{2}\frac{\partial\zeta}{\partial x} - \frac{N^2k}{2fm}.\frac{\partial s}{\partial x}$$

$$\frac{d\omega}{dt} = 0$$

$$\frac{dX_w}{dt} = \frac{N^2k}{2fm} + \frac{1}{2m}\frac{g}{\rho_o f}\frac{\partial\rho}{\partial x}$$

$$\frac{dZ_w}{dt} = -\frac{N^2k}{2fm} - \frac{1}{2m^2}\frac{g}{\rho_o f}\left(\frac{\partial\rho}{\partial x}k + \frac{\partial\rho}{\partial z}m\right)$$

Let us focus on the x-z direction.

The slope of the isopycnals is defined as: $s = \frac{M^2\ (x,z)}{N^2(z)}$ and the slope of the phase lines as: $-\frac{\mathrm{k}}{\mathrm{m}}$.

This leads to $N_{eff}^2 = N^2\left(1 + s.\frac{m}{k}\right)$

The dispersion relation is:

$$\omega = f_{eff} + \frac{N^2}{2f}\frac{k^2}{m^2}\left(1 + s.\frac{m}{k}\right) \quad (1)$$

How *k* and *m* adjust to the s changes: solutions for k/m ?

(1) leads to the solutions:

$$-\frac{k}{m} = \frac{-s \pm \sqrt{s^2 - 8\frac{f}{N^2}[f_{eff} - \omega]}}{2}$$

Thus if $\omega \sim f_{eff}$, phase lines of the waves are parallel to the isopycnal slope ($-\frac{k}{m} \approx s$).

Or, if the phase lines of the waves are parallel to the isopycnal slope, waves are almost inertial!