Propagation of waves in an inhomogeneous medium: Young and Ben-Jelloul approach

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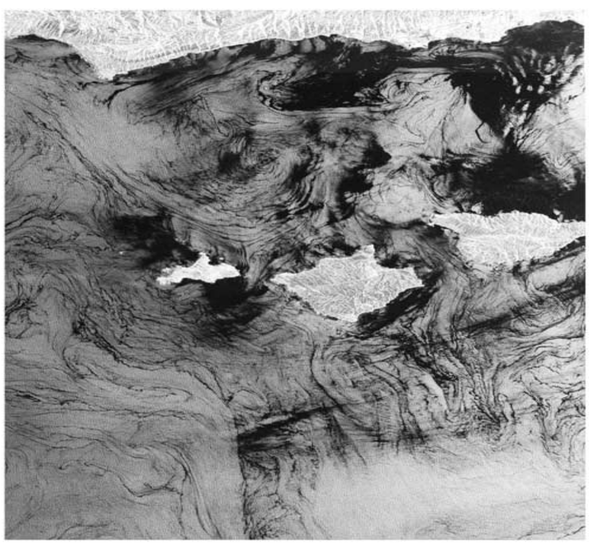
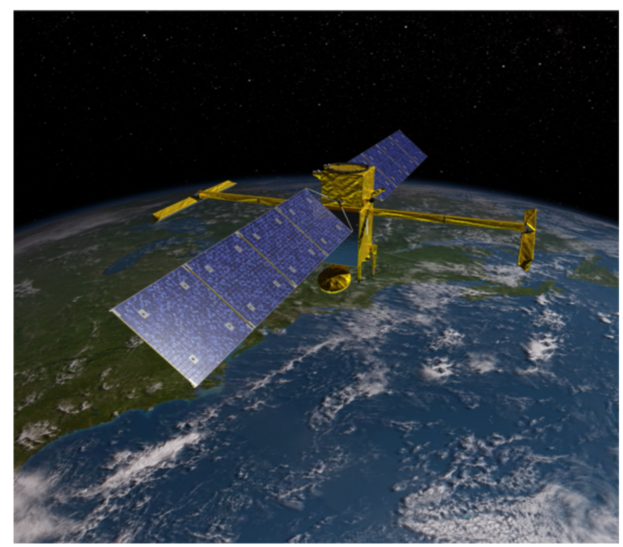




Introduction: Solving waves in a stratified and rotating flow

- From primitive equation (hydrostatic, Bousinessq), assuming waves are small perturbation, so linearize it

- if NOT considering wave-turbulence interaction

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{g}{\rho_0}\rho'$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} + w\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z} = 0.$$

no nonlinear terms

5 eq, 5 variables, linear

- manipulate to get the equation for *p'*

$$\frac{\partial}{\partial t}\left[\left(\frac{p'_z}{N^2}\right)_{ztt} + f^2\left(\frac{p'_z}{N^2}\right)_z + \Delta p'\right] = 0$$

$$\frac{\partial p'}{\partial z} = 0 \text{ at } z=0,-H$$

- **Textbook solution, by assuming N^2 is constant**

Fourier series decomposion:

$$P' = P_0\, e^{-i(kx+ly+mz-wt)}$$

=>

$$\omega^2 = f^2 + N^2 \frac{k^2 + l^2}{m^2}$$

i.e.

$$m^2 = N^2 \frac{(k^2+l^2)}{w^2-f^2}$$

• **In reality, N^2 is not constant with depth, what is the solution?**

• First approach: WKB approximation (scale separation)

if $N^2(z)$ has a vertical varying lengthscale >> wave vertical wavelength, then locally N^2 can be treated as constant

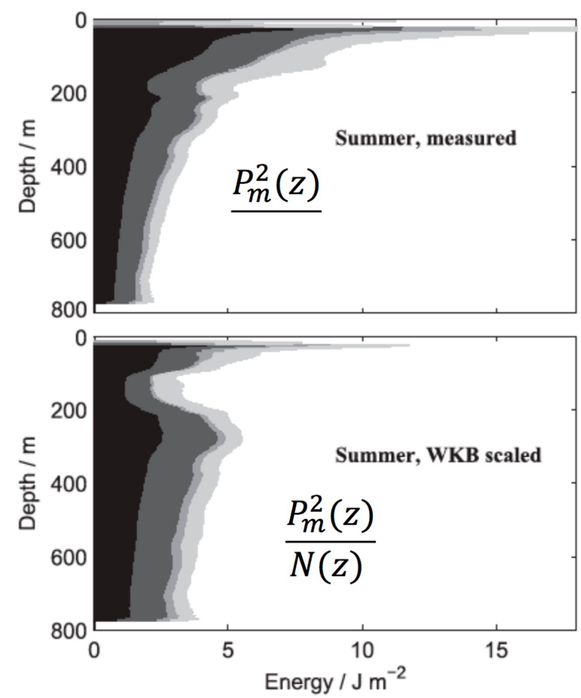
if N^2=const: $$m^2 = N^2 \frac{(k^2+l^2)}{w^2-f^2}$$

now: $$m^2(z) = N^2(z) \frac{(k^2+l^2)}{w^2-f^2}$$

A key result:

wave energy density $P_m^2(z)$ is $\propto N(z)$

$i.e.\quad P_m^2(z)/\,N(z) \sim constant$



- Second approach: vertical normal modes

Fourier series decomposition for x, y:

$$p'(x,y,z,t) = P(z).e^{-i(k.x+l.y-\omega t)}$$

if N^2=const or WKB: $$P(z) = \sum_{m=1}^{M} P_m . \cos mz$$

now: $$P(z) = \sum_{m=1}^{M} P_m . F_m(z)$$

$F_m(z)$ is normal modes different from cos, depending on $N^2(z)$

for certain k, l, w, define $\lambda_m^2 = 1/r_m^2 = \frac{(k^2+l^2)f^2}{\omega^2-f^2}$ dispersion relation

you can solve $F_m(z)$ by: $$\frac{d}{dz}\frac{f^2}{N^2}\frac{dF_m}{dz} = -\lambda_m^2 F_m$$

Problem of the above analysis:

- Vertical group velocity, derived from dispersion relation above, is too small to explain the observation

- Other physics neglected above is important:
Wave-turbulence interaction

Wave-turbulence interaction

Previously linearized primitive equation:

$$\boxed{\begin{aligned}\frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}\\ \frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}\end{aligned}}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{g}{\rho_0}\rho'$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} + w\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z} = 0.$$

Wave-turbulence interaction

previously linearized primitive equation:

$$\boxed{\begin{aligned} \frac{\partial u}{\partial t} - fv &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial x} \\ \frac{\partial v}{\partial t} + fu &= -\frac{1}{\rho_0}\frac{\partial p'}{\partial y} \end{aligned}}$$

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \cdot \nabla u$$

U,V: geostrophic turbulence velocity

in barotropic case: dU/dz=0, dV/dz=0

$$\frac{\partial u}{\partial t} + \boldsymbol{Uu_x} + \boldsymbol{Vu_y} + \boldsymbol{uU_x} + \boldsymbol{vU_y} - fv = -\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + \underbrace{\boldsymbol{Uv_x} + \boldsymbol{Vv_y}}_{\text{Doppler shift}} + \underbrace{\boldsymbol{uV_x} + \boldsymbol{vV_y}}_{\text{Refraction}} + fu = -\frac{\partial p}{\partial y}$$

Vertical normal mode solution

Previously: $\omega_r^2 \approx f^2 + f^2 r_m^2 . (k^2 + l^2)$

Now: $\omega_r^2 \approx f^2 + f^2 r_m^2 . (k^2 + l^2) + \boldsymbol{f} . [\boldsymbol{V_x} - \boldsymbol{U_y}] - \boldsymbol{V_x U_y} - \boldsymbol{U_x^2}$

there is NO energy transfer between wave and turbulence

by relative vorticity

by strain

O(1)

O(Ro)

O(Ro^2)

$\omega_i \approx \boldsymbol{i} . [\boldsymbol{k} . \boldsymbol{l} . (\boldsymbol{V_x} + \boldsymbol{U_y}) + (\boldsymbol{k^2} - \boldsymbol{l^2}) . \boldsymbol{U_x}]$

by strain

there is energy transfer between wave and turbulence

Now: $$\omega_r^2 \approx f^2 + f^2 r_m^2.(k^2 + l^2) + \boldsymbol{f}.\overset{\zeta}{\left[\boldsymbol{V_x} - \boldsymbol{U_y}\right]} - \boldsymbol{V_x U_y} - \boldsymbol{U_x^2}$$

$$\frac{dk}{dt} = -\frac{1}{2}\frac{\partial \zeta}{\partial x}$$ Turbulence changes the wavenumber of waves

=> k may increase with time

$$C_{gz} \propto k^2 + l^2$$

=> C_{gz} increases with time, explain the observation!

$C_{gx} \propto k$ also depends on the sign of $\frac{\partial \zeta}{\partial x}$

C_{gx} may change sign within an eddy => waves are trapped within this eddy

WKB solution

- WKB assumpton: NIW wavelength << mesoscale eddy size

Previously:

$$\omega_0 \approx f + \frac{N^2}{2f}\frac{{k_H}^2}{{k_z}^2}$$

Now:

$$\omega_0 \approx f_{\text{eff}} + \frac{N^2_{\text{eff}}}{2f}\frac{{k_H}^2}{{k_z}^2}$$

due to relative vorticity

$$f_{\text{eff}} \approx f + \boxed{\frac{1}{2}\left(\frac{\partial V}{\partial x} - \frac{\partial U}{\partial y}\right)}$$

due to isopycnal slope
in Baroclinic flow

$$N^2_{\text{eff}} = N^2 + \boxed{2{M_x}^2\frac{k_x k_z}{{k_H}^2} + 2{M_y}^2\frac{k_y k_z}{{k_H}^2}}$$

Motivation of Young and Ben-Jelloul 1997

- WKB assumpton: NIW wavelength << mesoscale eddy size

- storm-track-driven NIW has a initial wavelength (~1000km)
>> mesoscale eddy size (~100km),
so WKB assumption can be wrong

- how does small-scale turbulence-wave interaction influence
large-scale wave dispersion?

- Need to isolate the key physics component:
Normal mode method is not obvious to do so,
Asymptotical approach may do so

Using geostrophic velocity and buoyancy fields

$$(U, V, W, B) = (-\Psi_y, \Psi_x, 0, f_0\Psi_z).$$

Primitive equation, linearized,
with wave-turbulence nonlinear terms

$$\frac{Du}{Dt} + \boxed{uU_x + vU_y + wU_z} - fv = -p_x,$$

$$\frac{Dv}{Dt} + \boxed{uV_x + uV_y + wV_z} + fu = -p_y,$$

$$0 = -p_z + b,$$

$$u_x + v_y + w_z = 0,$$

$$\frac{Db}{Dt} + \boxed{uB_x + vB_y} + w(N^2 + B_z) = 0,$$

- To get asymptotical solution

- Need a nondimensional number

$\omega^2 = f^2 + (N^2 K^2)/m^2$

Key assumption: $\omega^2 \sim f^2$

$\omega^2 \sim f^2 >> (N^2 K^2)/m^2$

=> $(N^2 K^2)/(m^2 f^2) << 1$

So define the nondimensional number below which is small:

$\epsilon = (N^2 K^2)/(m^2 f^2) << 1$

$\sim Bu$

- Nodimensionalize the linearized primitive equation, which contain ϵ

$$\frac{Du}{Dt} + \boxed{\epsilon^2} uU_x + \boxed{\epsilon^2} vU_y + \epsilon^{2+q} wU_z - (1 + \epsilon^2 \beta y)v = -\epsilon^2 p_x,$$

$$\frac{Dv}{Dt} + \epsilon^2 uV_x + \epsilon^2 vV_y + \epsilon^{2+q} wV_z + (1 + \epsilon^2 \beta y)u = -\epsilon^2 p_y,$$

$$0 = -p_z + b,$$

$$u_x + v_y + w_z = 0,$$

$$\frac{Db}{Dt} + \epsilon^q uB_x + \epsilon^q vB_y + w(N^2 + \epsilon^{2q} B_z) = 0.$$

- Get asymptotical solution

$u = u_0\epsilon^0 + u_2\epsilon^2 + u_4\epsilon^4 + \ldots..$

u_0 is just inertial oscillation, ω=f

u_2 is the focus of this study

u_4 , u_6 .... are neglected

one conclusion from u_2 solution

- there is no energy transfer between wave and turbulence

ϵ ~ Bu ~ Ro^2 Ri <<1
=> Ro <<1

$$\omega^2 \approx f^2 + f^2 r_m^2 .(k^2 + l^2) + \boldsymbol{f}.[\boldsymbol{V_x} - \boldsymbol{U_y}] - \boldsymbol{V_x U_y} - \boldsymbol{U_x^2}$$

by relative vorticity

by strain

O(1)

O(Ro)

O(Ro^2)

So strain field is zero in the u_2 solution influence of small-scale flux on larger scale

Example

$$\frac{\partial C}{\partial t} + \nabla \cdot (\boldsymbol{u} C) = k \nabla^2 C$$

$$C = \bar{C} + C'$$

$$\frac{\partial \bar{C}}{\partial t} + \nabla \cdot (\overline{\boldsymbol{u}} \bar{C}) + \nabla \cdot \left(\overline{\boldsymbol{u}' C'} \right) = k \nabla^2 \bar{C}$$

• how does small-scale turbulence-wave interaction influence large-scale wave dispersion?

A is a quantity related with wave motion

$$A(x, y, z, t) = \overline{A}(z, t) + A'(x, y, z, t)$$

L is a differential operator:

$$LA \equiv (f_0^2 N^{-2} A_z)_z$$

Wave equation from asymptotical solution

$$\partial_t \iint LA \, dxdy + \frac{i}{2} \iint \nabla^2 \Psi LA \, dxdy = 0$$

Note: eddy size << initial wavelength of waves

size of $\nabla^2\Psi$ << the size of $\bar{A}$

size of $\nabla^2\Psi$ ~ the size of A'

=> $$L\overline{A}_t + \frac{i}{2} \overline{\nabla^2 \Psi LA'} = 0$$

dispersion term zeta term

$$LA'_t + \frac{\partial(\Psi, LA')}{\partial(x,y)} + \frac{i}{2} f_0 \nabla^2 A' + \frac{i}{2} \nabla^2 \Psi LA' - \frac{i}{2} \overline{\nabla^2 \Psi LA'} = -\frac{i}{2} \nabla^2 \Psi L\overline{A}.$$

Initially, A' is zero, wave-turbulence interaction makes A' grows

$$LA'_t = -\frac{i}{2} \nabla^2 \Psi L\overline{A}.$$

Finally, the dispersion term balances the zeta term

$$\frac{i}{2} f_0 \nabla^2 A' = -\frac{i}{2} \nabla^2 \Psi L\overline{A}.$$

the wavelength of A' is similar to that of $\boldsymbol{\psi}$,
so we get the closure:

$$A' \approx -\frac{1}{f_0} \Psi L\overline{A}.$$

Put $A' \approx -\frac{1}{f_0}\Psi L\overline{A}.$ into

$$L\overline{A}_t + \frac{i}{2}\overline{\nabla^2\Psi L A'} = 0$$

=> $$L\overline{A}_t + \frac{i}{2f_0}\overline{\nabla\Psi \cdot \nabla\Psi} L^2\overline{A} = 0$$

$$K \equiv \overline{\nabla\Psi \cdot \nabla\Psi}/2$$ is the EKE of turbulence

=> solve the dispersion relation for larger-scale wave motion $\bar{A}$

$$\omega_n = \frac{1}{2}R_n^2 f_0(k^2 + l^2) - \frac{K}{f_0 R_n^2}$$

- **clearly capture the impact of small-scale wave-turbulence interaction on larger-scale waves**
- **not captured by WKB**
- **explain observation of large C_{gz}**