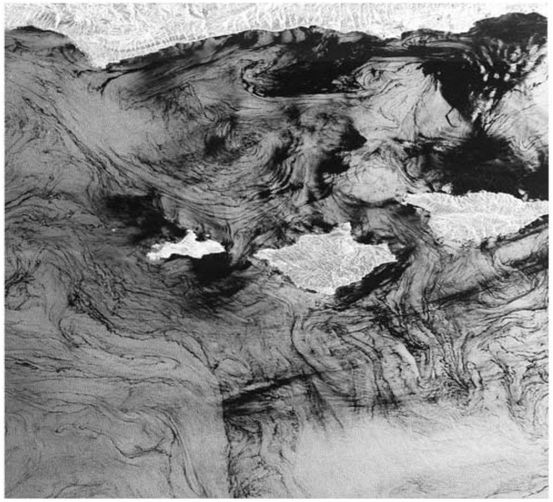
“Wave-Turbulence Interactions in the Oceans”

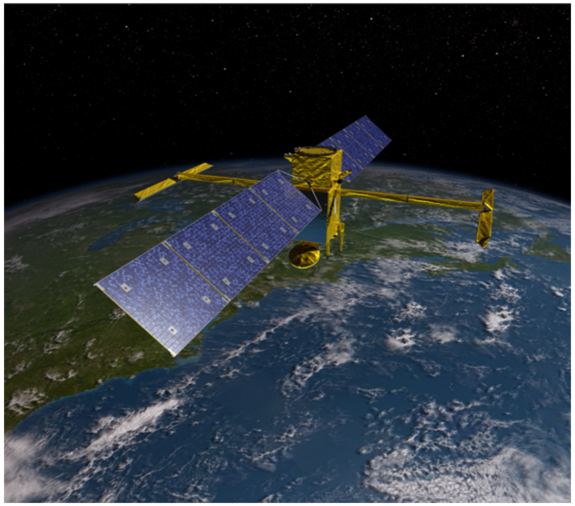
https://oceanturbulence.github.io

Patrice Klein (Caltech/JPL/Ifremer)

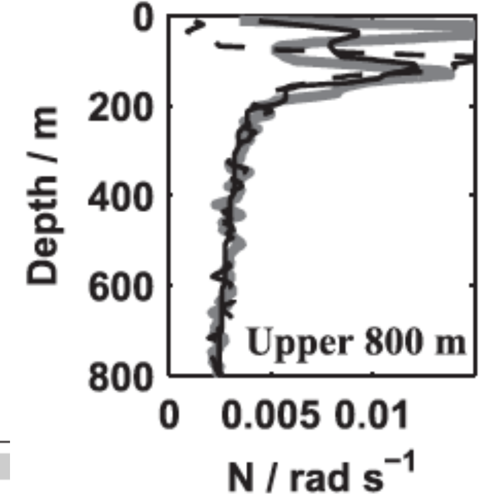
(IX) Seasonality of waves (NIWs and Tidal motions)

Interactions waves-balanced motions (a)





LAST CLASS … Interpreting NIWs observations using WKB approximation to take into account *N= N(z)*



WKB SCALED MEANS SCALED BY $N(z)$: $\dfrac{P_m^2(z)}{N(z)} = \text{cst}$

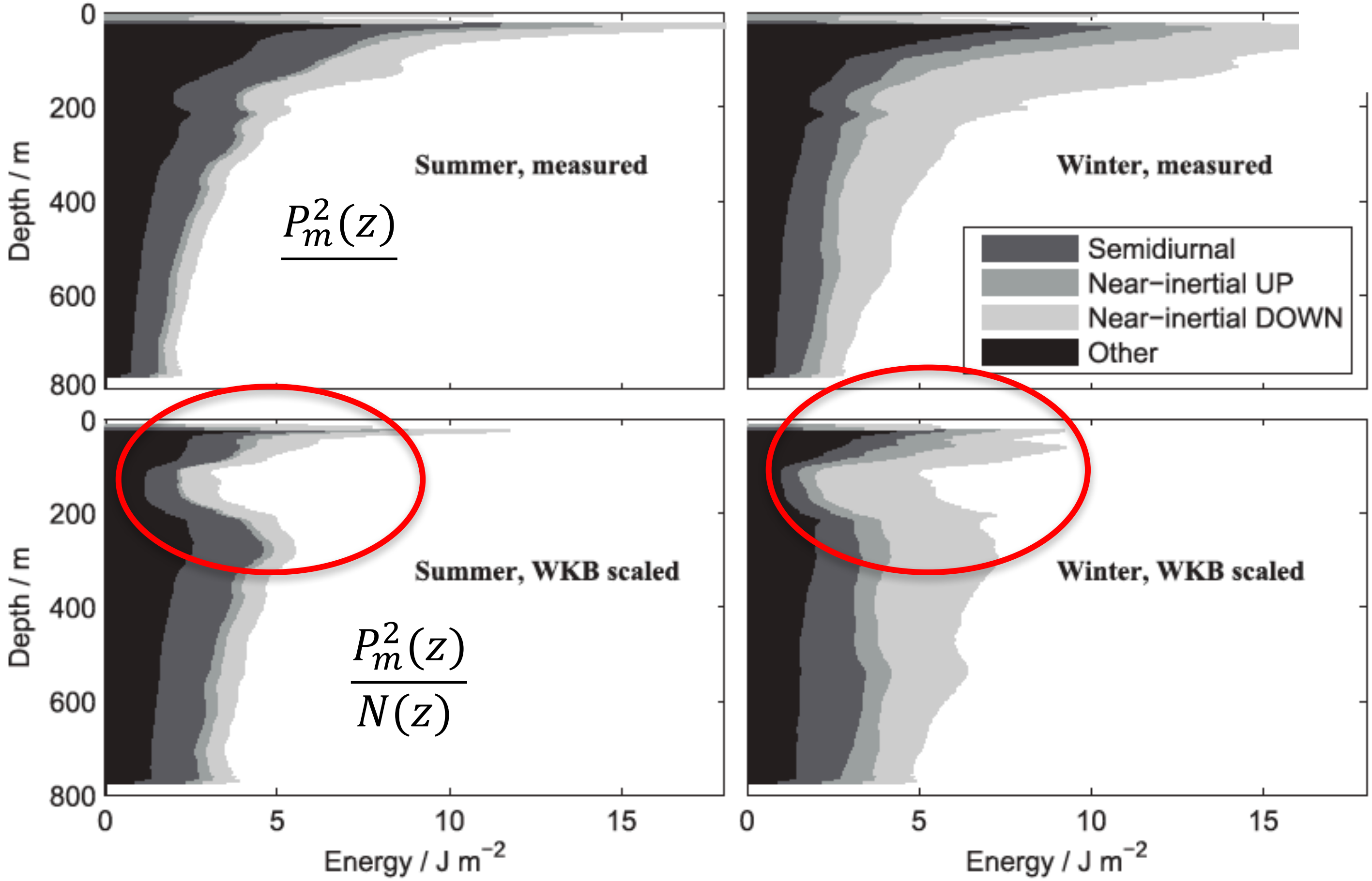
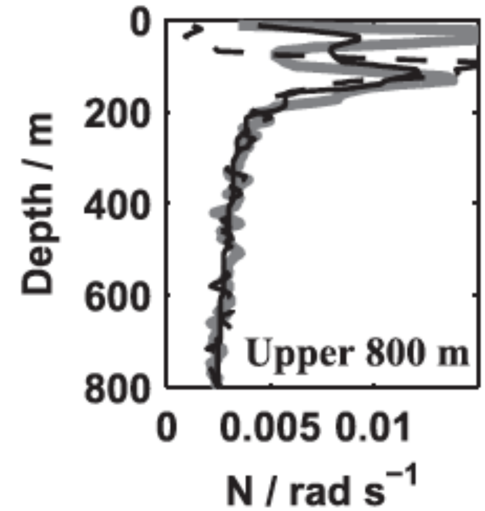


FIG. 12. Energy profiles for (left) summer and (right) winter, (top) measured and (bottom) WKB scaled.

… $N^2=N^2(Z)$



THIS AFFECTS THE PROPAGATION OF WAVE KE. THIS REQUIRES TO SOLVE:

$$\frac{\partial}{\partial t}\left[\left(\frac{p'_z}{N^2}\right)_{ztt} + f^2\left(\frac{p'_z}{N^2}\right)_z + \Delta p'\right] = 0$$

Using p'(x,y,z,t) = P(z).$e^{i(k.x+l.y-\omega t)}$, leads to:

$$\left(\frac{P_z}{N^2}\right)_z - \frac{k^2+l^2}{f^2-\omega^2}P = 0 \quad (1)$$

- ONE SOLUTION IS TO USE THE **WKB APPROXIMATION** (POSSIBLE IF N^2 IS SMOOTHLY VARYING ON THE VERTICAL = N^2 is constant locally).
  USING **P(Z)= P_m(z).cos**$[m(z).z]$ IN (1) LEADS TO:

$$m^2(z) = \frac{N^2(z)\left(k^2+l^2\right)}{\omega^2 - f^2}$$

- ANOTHER SOLUTION IS TO USE THE **VERTICAL NORMAL MODES** FOR P(z): SIMILAR TO WKB WHEN **N^2(Z)** IS SMOOTHLY VARYING, BUT MUCH BETTER WHEN **N^2(Z)** IS STRONGLY VARYING.

VERTICAL NORMAL MODES

This leads to solve:

$$\frac{\partial}{\partial t}\left[\left(\frac{p'_z}{N^2}\right)_{ztt} + f^2\left(\frac{p'_z}{N^2}\right)_z + \Delta p'\right] = 0$$

Using p' **(x,y,z,t) = P(z).**$e^{i(k.x+l.y-\omega t)}$, leads to

$$\left(\frac{P_z}{N^2}\right)_z - \frac{k^2+l^2}{f^2-\omega^2}P = 0$$

We need to find new vertical normal modes for P(z) **without** using the WKB approximation

These vertical normal modes, $F_m(z)$, should be such that $\int_{-H}^{0} F_m . F_n \, dz = \delta_{mn}$.
Then we can use:

$$P(z) = \sum_{m=1}^{M} P_m . F_m(z)$$

This decomposition is also valid for *U(z)* and *V(z)*!

Vertical dimension using normal modes:

Normal modes are obtained by solving the Sturm-Liouville equation
(after J. Sturm (1803–1855) and J. Liouville (1809–1882)):

$$\frac{d}{dz}\frac{f^2}{N^2}\frac{dF_m}{dz} = -\lambda_m^2 F_m .$$

with $dF_m/dz = 0$ at $z = 0, -H$.

F_m is the eigenfunction and λ_m^2 the eigenvalue (or the vertical wavenumber) associated with mode m.

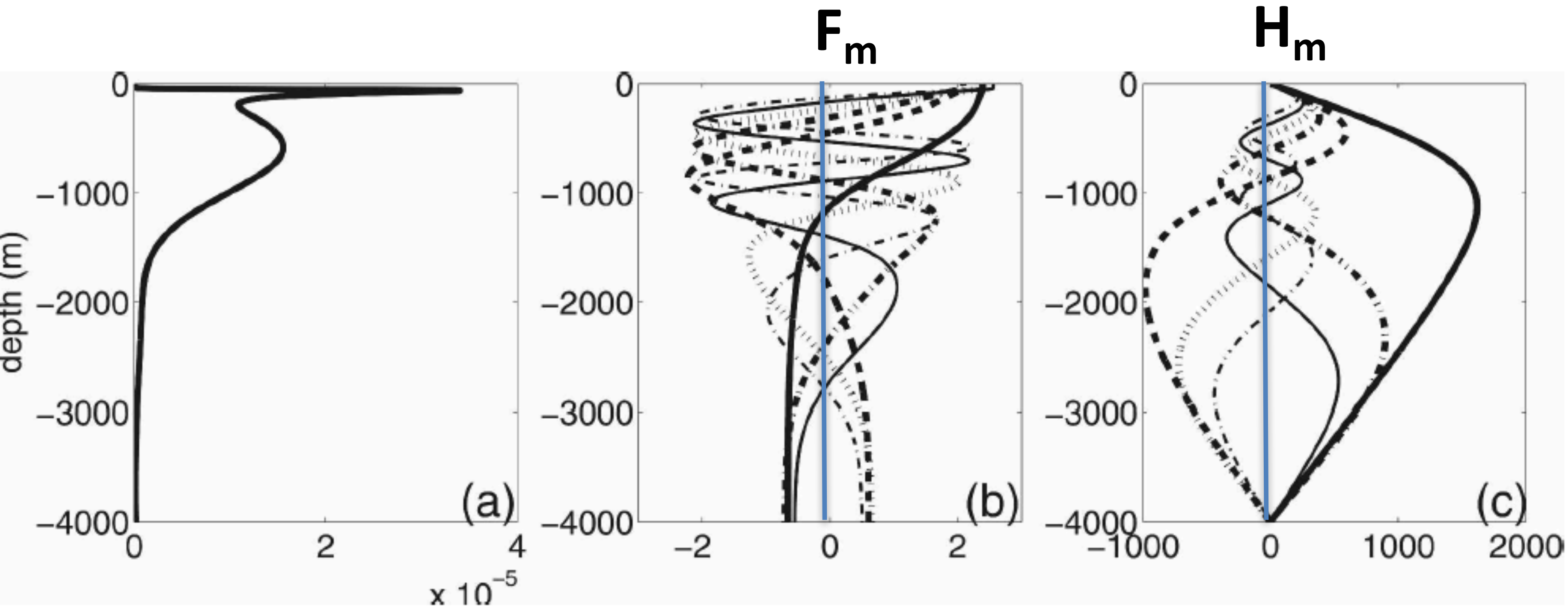
Modes are orthonormal:

$$\int_{-H}^{0} F_m \cdot F_n \, dz = \delta_{mn}$$

with $\lambda_m = 1/r_m$. r_m is also called the Rossby radius of deformation of mode

Hence, the vertical velocity field can be decomposed (from the continuity equation) as:

$$\mathbf{W}(z) = \sum_{m=1}^{M} W_m \cdot H_m(z) \quad \text{with} \quad H_n(z) = \int_z^0 F_n(z')dz'$$



1. (a) Vertical profiles of N^2, (b) the first six eigenfunctions F_n given by Eq. (3), and (c) the first six functions H_n. Units in (a) are s^{-2}.

$$\mathcal{L}(\cdot) = \frac{\partial}{\partial z}\left[\frac{f^2}{N^2}\frac{\partial}{\partial z}(\cdot)\right] \qquad \mathcal{L}F_n = -\frac{1}{r_n^2}F_n \qquad H_n(z) = \int_z^0 F_n(z')dz'$$

Vertical profiles of $F_m(z)$ strongly depend on the strong variatons of $N^2(z)$: vertical scales may differ from a cosine function …

$p(x,y,z,t) = \sum_m p_m(x,y).F_m(z)$ for *m=1,M*

$$\frac{\partial u_m}{\partial t} - f.v_m = -\frac{\partial p_m}{\partial x}$$
$$\frac{\partial v_m}{\partial t} + f.u_m = -\frac{\partial p_m}{\partial y}$$
$$\frac{\partial p_m}{\partial t} + f^2 r_m^2.\left(u_{mx} + v_{my}\right) = 0$$

Using $p_m(x,y,t)$ $[or\ um(x,y,t), vm(x,y,t)] \approx e^{i.(k.x+l.y-\omega t)}$ leads to:

$$\omega_m^2 = f^2[1 + r_m^2.\left(k^2 + l^2\right)]$$

We have **M « Shallow Water » systems!**

How vertical normal modes can help to understand the impact of waves on the ocean dynamics: an example

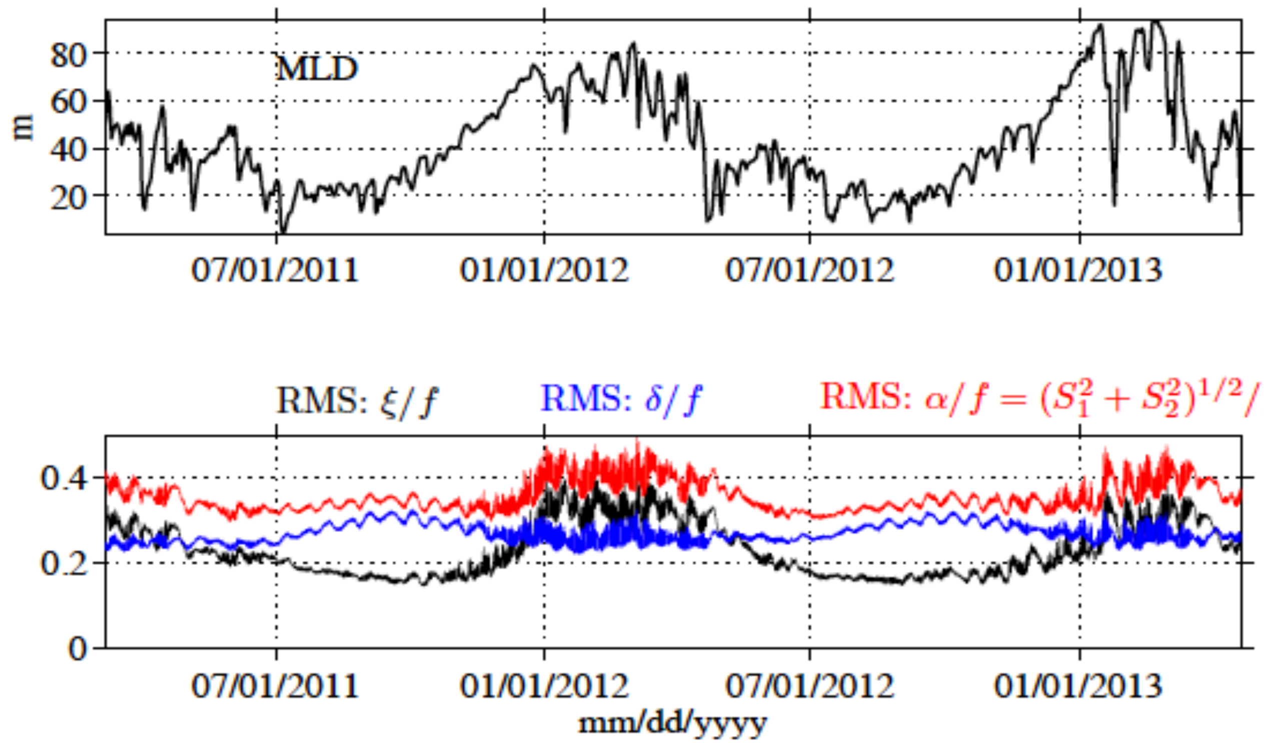
Dynamical fields (waves + balanced motions) display a strong seasonality!

Impacts the waves ($\omega > f$)

Square of the relative vorticity:
$\xi^2 = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})^2$

Square of the horizontal flow divergence:
$\delta^2 = (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2$

Square of the horizontal strain:
$S^2 = S_1^2 + S_2^2 = (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2$



Relative vorticity is larger in winter (0.4) and divergence larger in summer (0.3)!

Hector Torres: JPL 2016 … **in the Eastern North Pacific** (10°N-30°N, -130°W—110°W)

Seasonality is different when the wave part is removed!

and kinematic properties: $\boxed{\text{daily-averaged}}$

Square of the relative vorticity:
$\xi^2 = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})^2$

Square of the horizontal flow divergence:
$\delta^2 = (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2$

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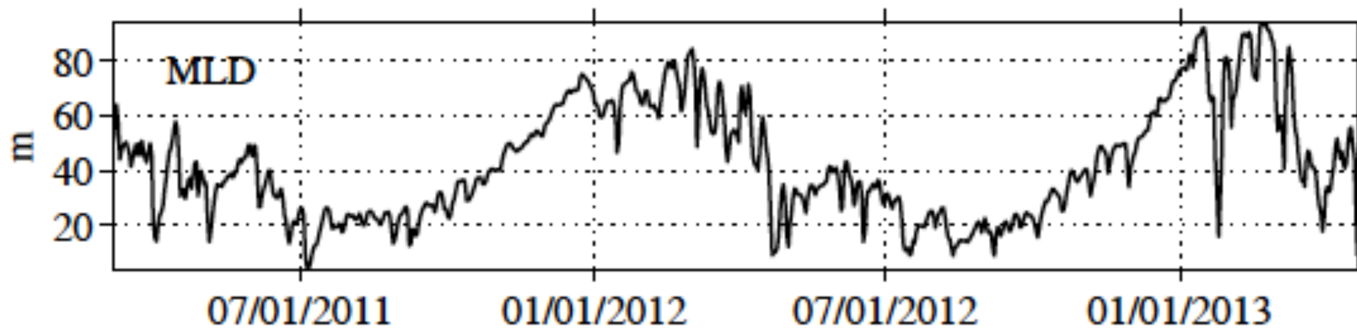

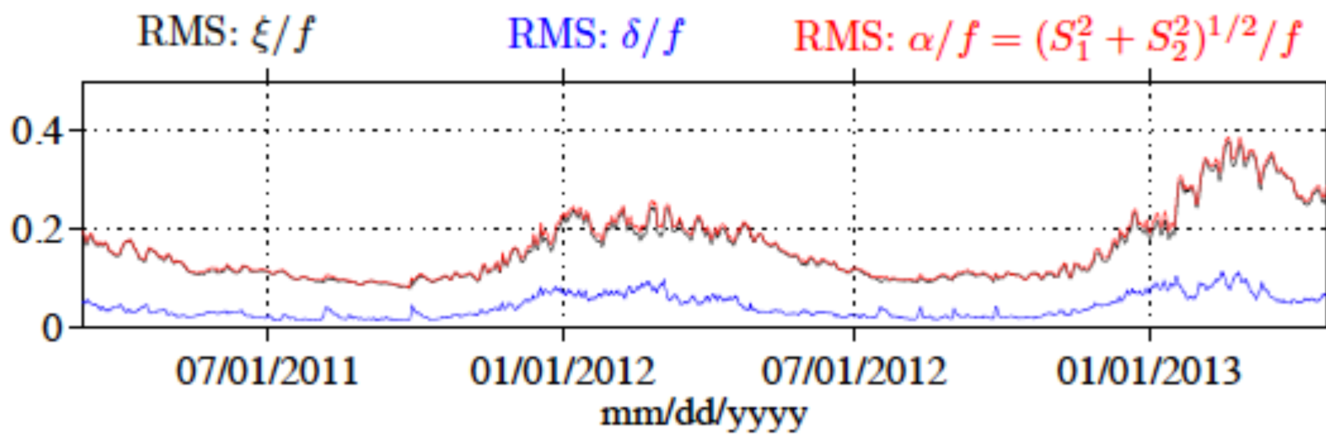

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Rocha et al. GRL 2016: in the Kuroshio Extension

Seasonality of waves and balanced motions

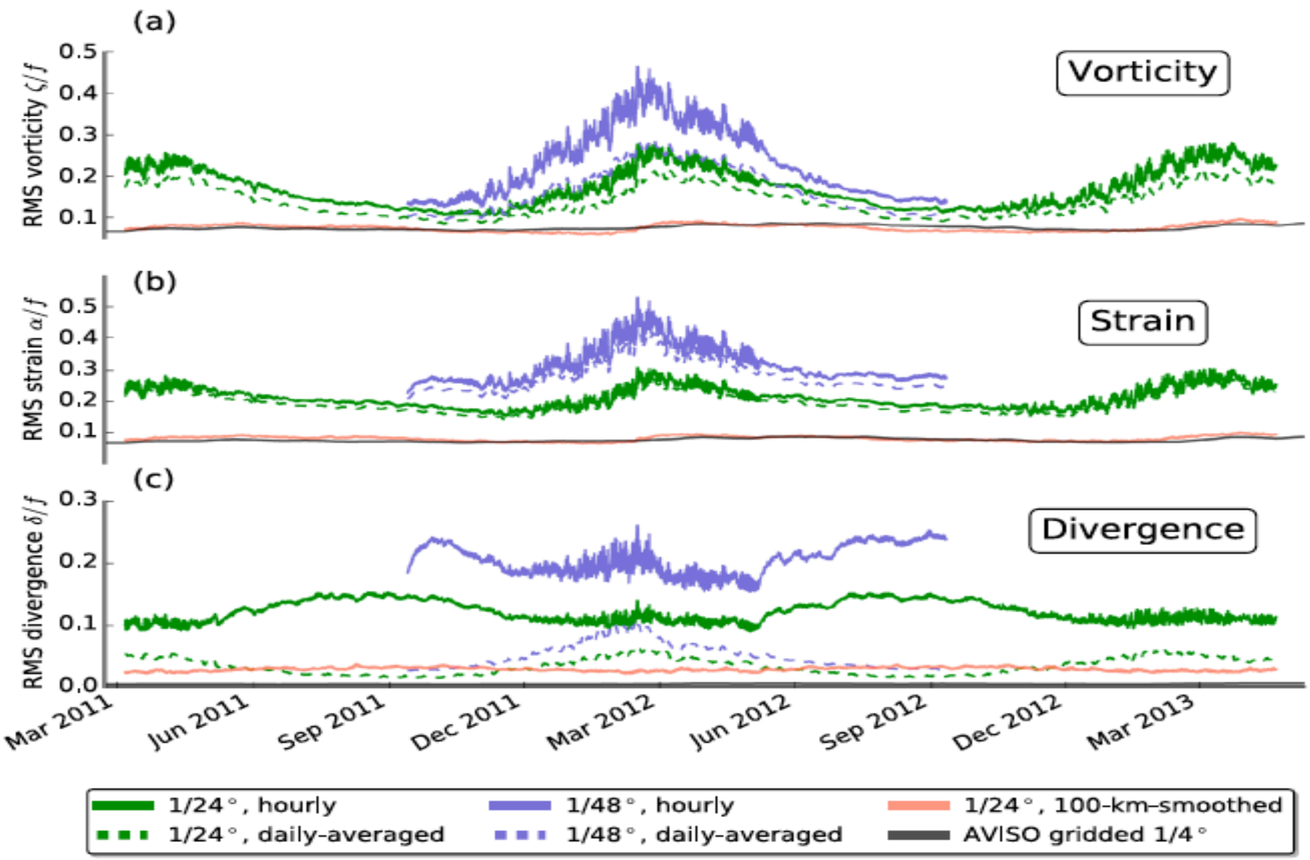
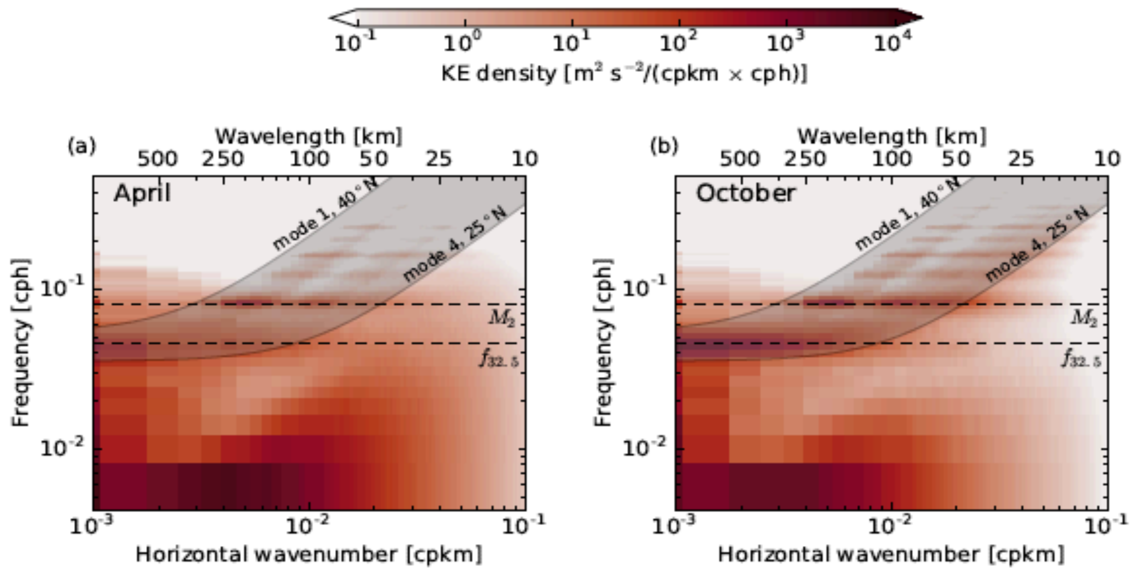


Figure 2. Time series of the root-mean-square (RMS) of surface (a) vorticity, (b) rate of strain, and (c) horizontal divergence in the LLC outputs and gridded AVISO data. The convergence of meridians account for the small RMS divergence in AVISO line in Figure 2c.

Seasonality of waves and balanced motions



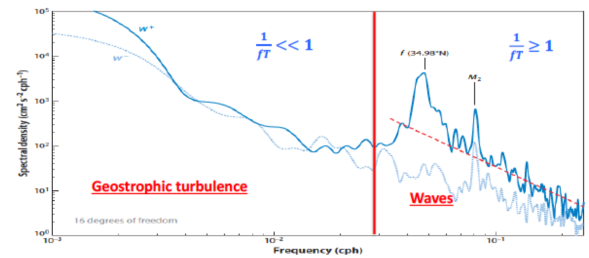


Table 2. The mean-square velocity at submesoscales (10-100 km) estimated from the wavenumber-frequency spectrum of SSH variance in figure S7.

	April	October
sub-inertial ($< 0.8f$)	0.32 m s^{-1}	0.20 m s^{-1}
super-inertial ($> 1.2f$)	0.17 m s^{-1}	0.46 m s^{-1}
near-f ($0.8 - 1.2f$)	0.25 m s^{-1}	0.12 m s^{-1}
near-M_2 ($0.8 - 1.2M_2$)	0.05 m s^{-1}	0.10 m s^{-1}

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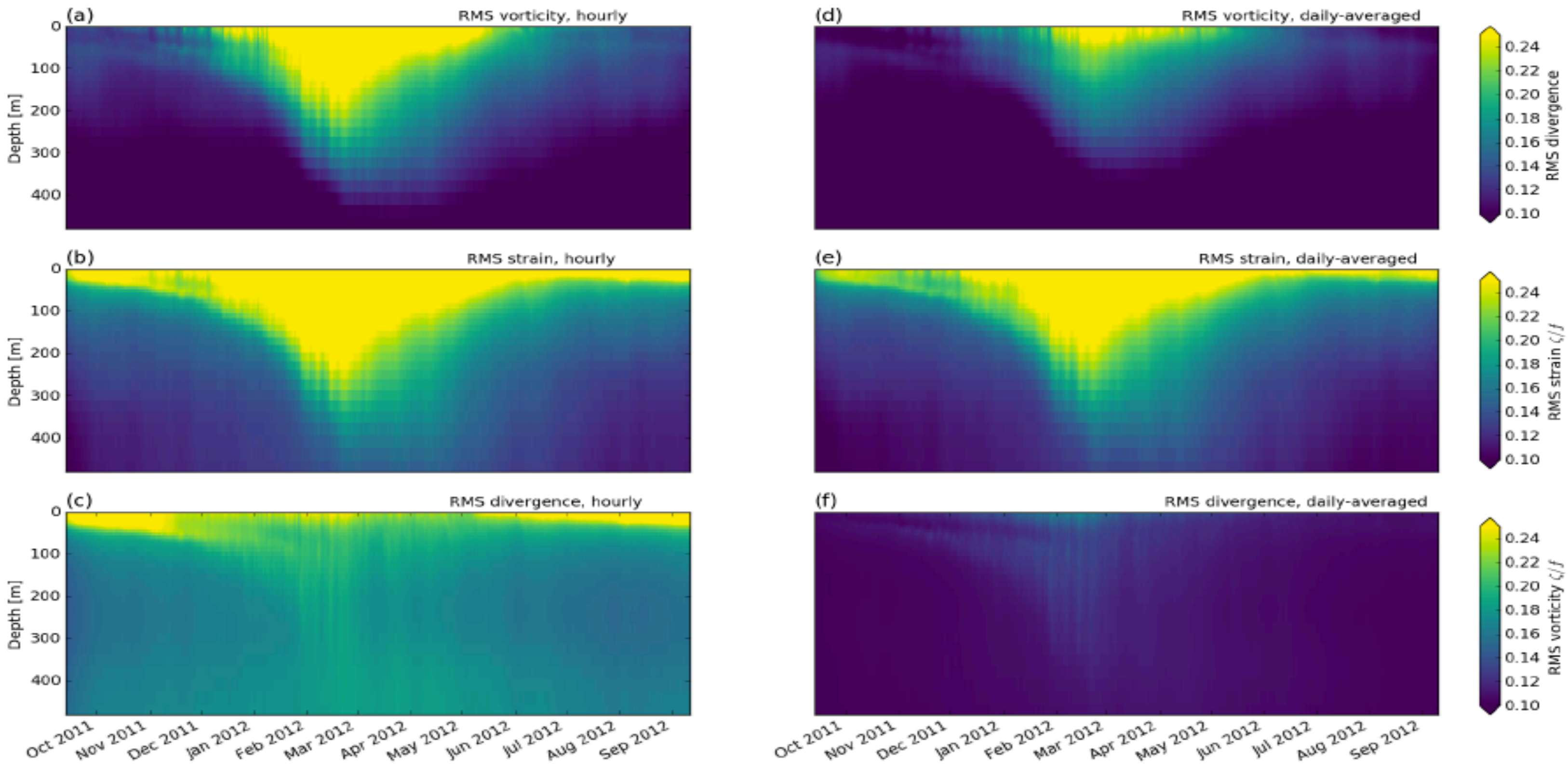

Figure S3. Depth-dependence of root-mean-square of vertical vorticity (a,d), rate of lateral strain (b,e) and horizontal divergence (c,f) for hourly (a,b,c) and daily-averaged (d,e,f,) fields. The strong seasonality, as depicted at the surface in Figure 2 of the main manuscript, is confined to the mixed layer, which varies from 50 m in summer to about 350 m in winter.

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Why wave divergence at surface is stronger in summer ?

Vertical divergence ($\partial W/\partial z$) is captured by the Fm(z) functions

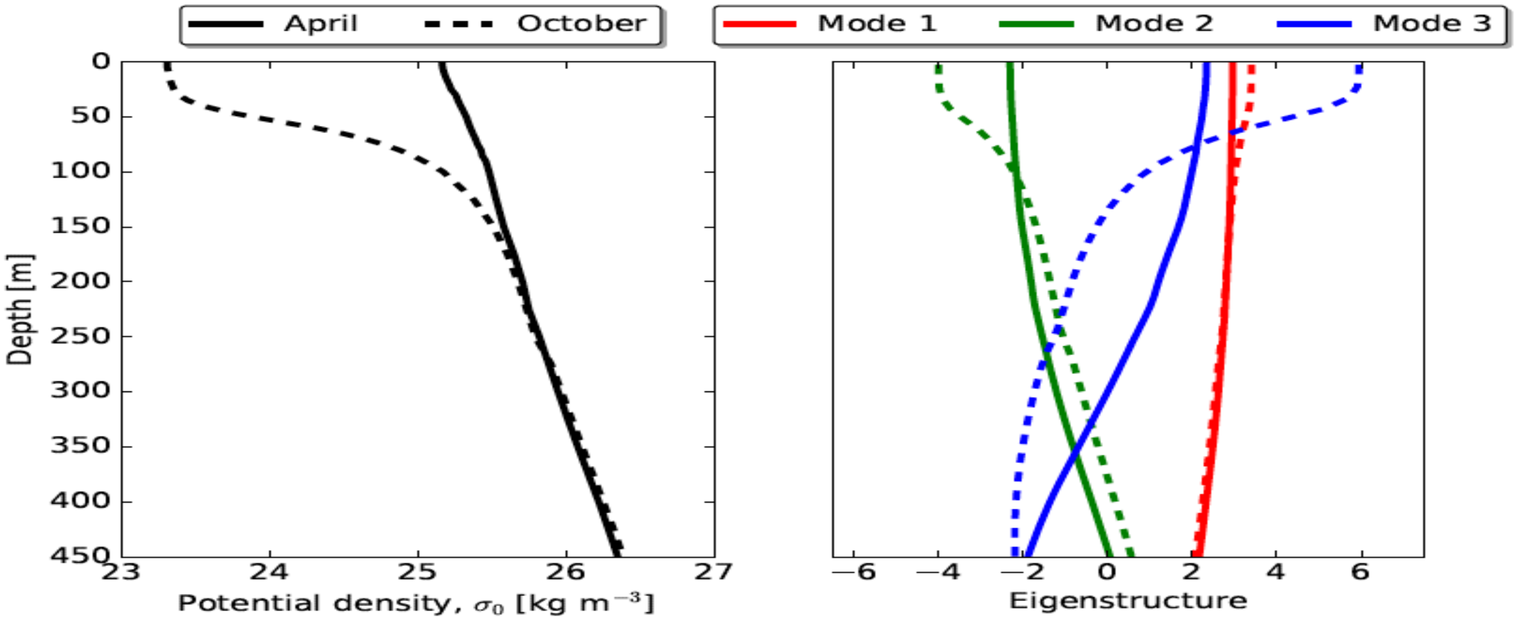

Figure S6. The seasonal variability of the WOA 2013 [*Levitus et al.*, 2013] stratification averaged over the domain and the associated three gravest pressure modes. Only the upper 450 m is shown. Clearly, the changes to the stratification are dramatic, with the formation of a seasonal pycnocline in summer. The strong seasonality of upper-ocean stratification yields a strong seasonality in the near-surface shape and amplitude of baroclinic pressure modes. In particular, the surface amplitude is much larger in summer. Thus the projection of baroclinic tides on the surface may have significant seasonal variations. The WOA data is available online

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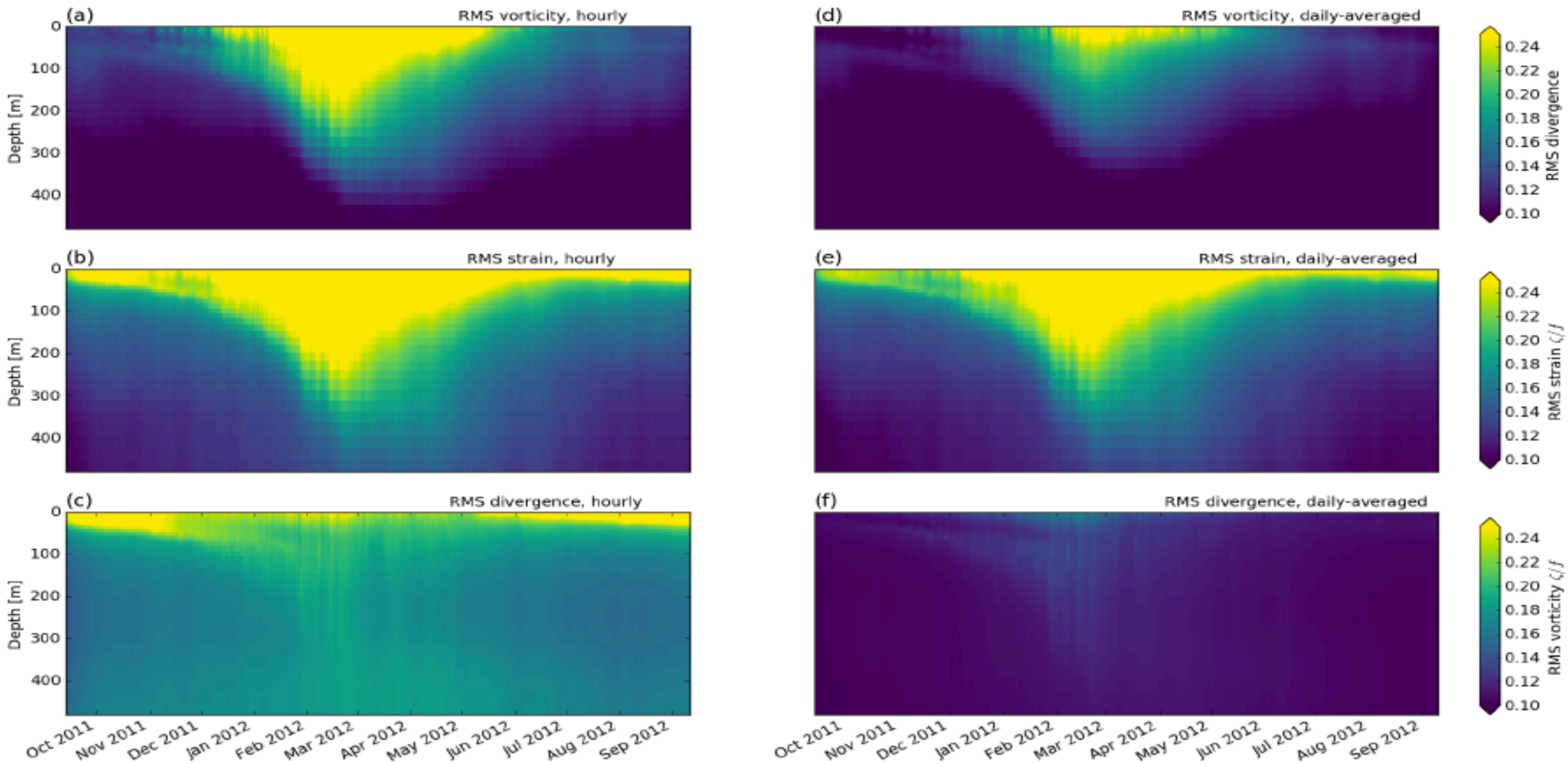

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Seasonality of waves and balanced motions

IMPACTS OF THE **STRONG VARIATIONS OF $N^2(Z)$** :

A part of the seasonality of wave motions $(\omega > 1.02f)$ is due to the impact of the seasonal variations of the mixed-layer depth on tidal motions: they are stronger (smaller) in summer (winter) within (below) the mixed-layer.

FORCINGS:

NIW motions $(0.8f < \omega < 1.02f)$ are stronger in winter than in summer because of the winds stronger in winter.

Submesoscale balanced motions $(\omega < 1.02f)$ are stronger in winter because of the MLIs.

	April	October
sub-inertial ($< 0.8f$)	0.32 m s^{-1}	0.20 m s^{-1}
super-inertial ($> 1.2f$)	0.17 m s^{-1}	0.46 m s^{-1}
near-f ($0.8 - 1.2f$)	0.25 m s^{-1}	0.12 m s^{-1}
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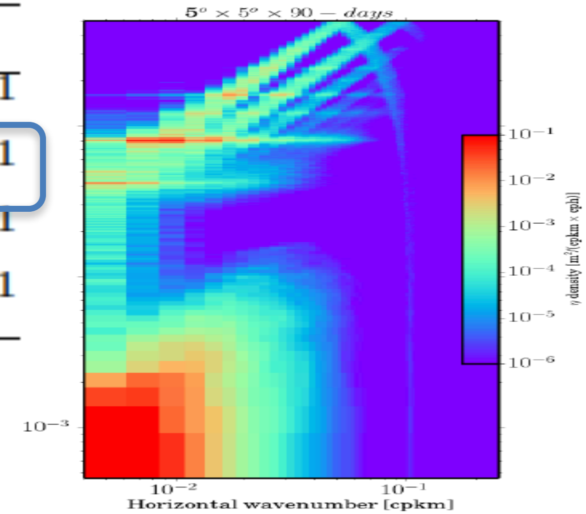


WHY SUPER-INERTIAL MOTIONS NEAR THE SURFACE ARE SO ENERGETIC ?

IMPACT OF BALANCED MOTIONS ON THE WAVE (M2) DYNAMICS

WE COME BACK TO THE WKB APPROXIMATION (FOR THE SAKE OF SIMPLICITY)

Momentum equations

$$\frac{\partial U}{\partial t} + \boldsymbol{UU_x + VU_y} - \mathrm{f}V = -\frac{\partial P}{\partial x}$$
$$\frac{\partial V}{\partial t} + \boldsymbol{UV_x + VV_y} + \mathrm{f}U = -\frac{\partial P}{\partial y}$$

We use: $U=u+U_g$, $V=v+V_g$ $P=p+P_g$ [**u =wave, U_g =balanced motions**] and assume the geostrophy for balanced motions and: $u \ll U_g$, $v \ll V_g$. This leads to

(replacing U_g by U to simplify the notations for the sakeof simplicity):

$$\frac{\partial u}{\partial t} + \boldsymbol{Uu_x} + \boldsymbol{Vu_y} + \boldsymbol{uU_x + vU_y} - fv = -\frac{\partial p}{\partial x}$$
$$\frac{\partial v}{\partial t} + \underbrace{\boldsymbol{Uv_x} + \boldsymbol{Vv_y}}_{\text{Doppler shift}} + \underbrace{\boldsymbol{uV_x + vV_y}}_{\text{Refraction}} + fu = -\frac{\partial p}{\partial y}$$

TRIAD INTERACTIONS

LET US CONSIDER THE SIMPLE EQUATION:

$$\frac{\partial U}{\partial t} = U.U\ ,$$

AND,

$$U(x) = \sum_k \cos(k.x).\widehat{U}_k \qquad => \qquad \widehat{U}_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(k.x).U(x).dx$$

WITH $k = 1, k_{max}$. THE TIME EVOLUTION OF $\widehat{U}_k$ IS:

$$\frac{\partial \widehat{U}_{k_1}}{\partial t} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \cos(k_1.x).U(x).U(x).dx\ ,$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} [\textstyle\sum_{k_2} \cos(k_2.x).\widehat{U}_{k_2}].\sum_{k_3} [\cos(k_3.x).\widehat{U}_{k_3}].\cos(k_1.x).dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \textstyle\sum_{k_2}.\sum_{k_3} [\cos([k_1 \pm k_2.\pm k_3]x).\widehat{U}_{k_2}.\widehat{U}_{k_3}].dx$$

THE R.H.S. IS NON ZERO ONLY IF

$$\boldsymbol{k_1 \pm k_2 \pm k_3 = 0!}$$

THIS IS THE CLASSICAL TRIAD INTERACTION! TRUE FOR WAVENUMBERS AND FREQUENCIES!

Wentzel-Kramer-Brillouin **APPROXIMATION**

LET US CONSIDER AGAIN THE SIMPLE EQUATION:

$$\frac{\partial u}{\partial T} = u.U\ ,$$

WITH

$\boldsymbol{u(x)} = \sum_{k_{min}}^{k_{max}} \boldsymbol{cos(k.x).\hat{u}_k}$ and $\mathbf{U}(\boldsymbol{x}) = \boldsymbol{cos(k_3.x).\hat{U}_{k_3}}$ **with $\boldsymbol{k_3} \ll \boldsymbol{k_{min}}$ $(\gg \mathbf{1})$**

THE TIME EVOLUTION OF $\hat{u}_k$ IS:

$$\frac{\partial \hat{u}_{k_1}}{\partial T} = \frac{k_{min}}{2\pi}\int_{x_o-\pi/k_{min}}^{x_{o+}\ \pi/k_{min}} \cos(k_1.x).u(x).U(x).dx\ ,$$
$$= \frac{k_{min}}{2\pi}\int_{x_o-\pi/k_{min}}^{x_o+\pi/k_{min}} \sum_{k_2} [\cos([k_1 \pm k_2]x).\cos(k_3 x_o).\hat{u}_{k_2}.\hat{U}_{k_3}\ ].dx$$

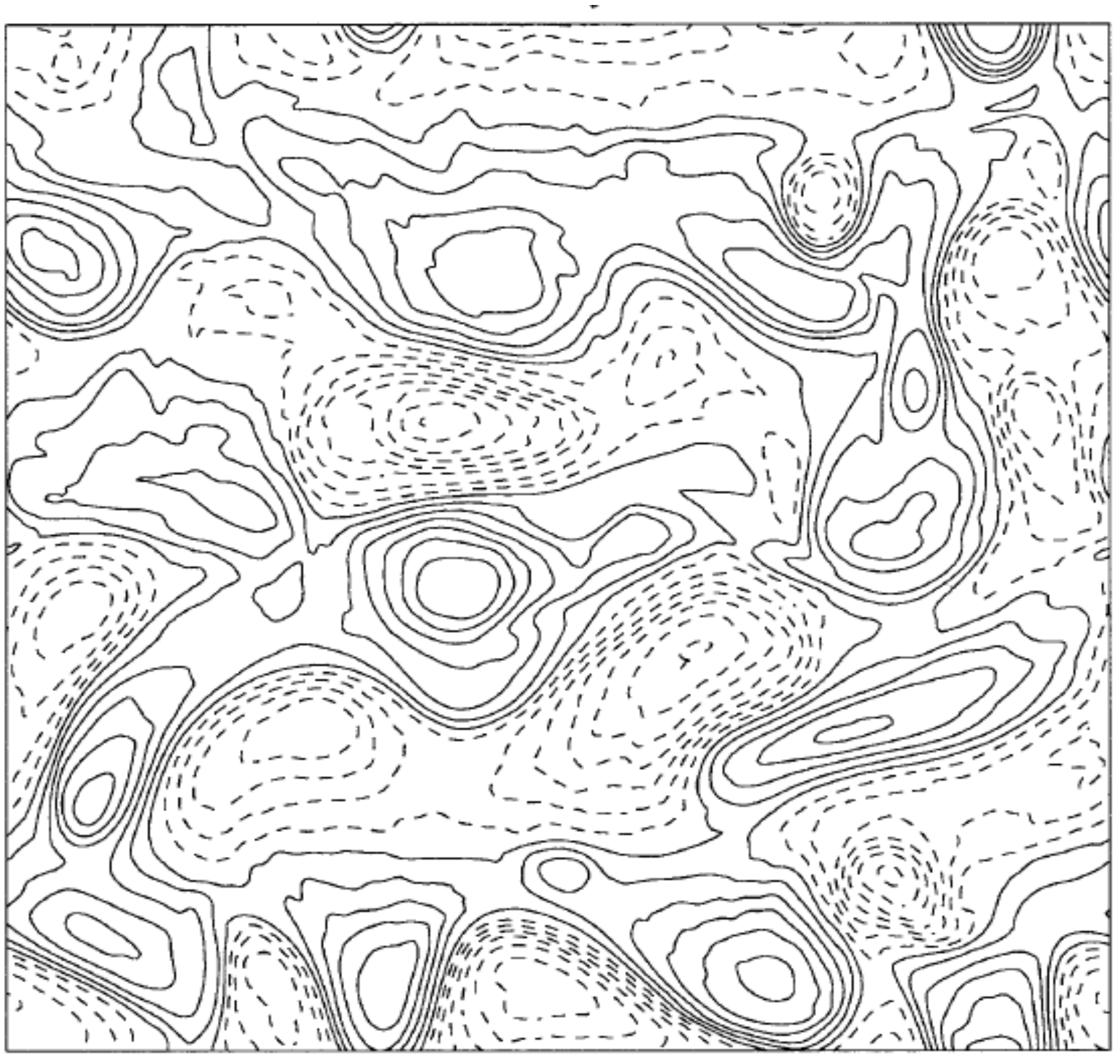
THE ONLY POSSIBILITY IS: $\boldsymbol{k_1 \pm k_2 = 0!}$ AND THEREFORE:

$$\frac{\partial \hat{U}_{k_1}}{\partial T} = \frac{k_{min}}{2\pi}\int_{x_o-\pi/k_{min}}^{x_o+\pi/k_{min}} \cos(k_3.x_o).\hat{u}_{k_1}.\hat{U}_{k_3}.dx$$

$$\boxed{\Rightarrow\ \frac{\partial \hat{u}_{k_1}}{\partial T} = U(x_o).\hat{u}_{k_1}.}$$

THIS IS THE CLASSICAL WKB APPROXIMATION VALID FOR 3-D WAVENUMBERS AND FREQUENCIES!

Let us consider a 2-D (non-divergent) mesoscale eddy field (in geostrophic balance)



$$U = -\psi_y$$
$$V = \psi_x$$

STREAM FUNCTION $\psi(x, y)$

… see next class …