“Ocean Turbulence from SPACE”

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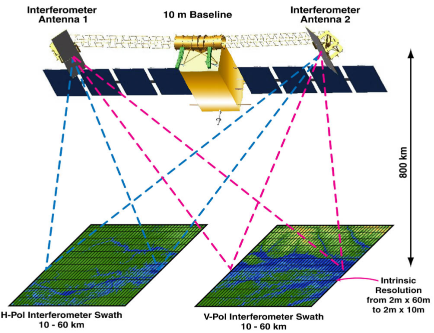
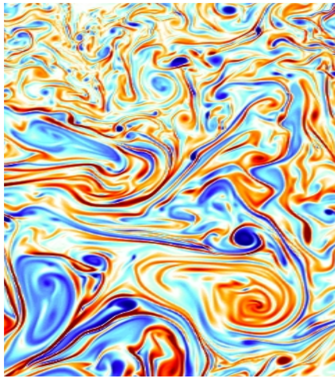
(XV) – Geostrophic turbulence (b):
Vertical normal modes and 3-D direct cascade





Quasi-geostrophic equations

Momentum eqs. [see 2nd class and 8th class]

$R_o = \frac{U}{fL} < 1. \qquad \frac{U}{\beta L^2} > 1.$

$$\frac{\partial \vec{v}}{\partial t} + \vec{v}\cdot\nabla\vec{v} + f\,\vec{k}\times\vec{v} = -\frac{\nabla p}{\rho_0} + [\text{mixing}] \qquad \vec{v} = (u,v).$$

Any motions can be decompsed into a rotational (non-divergent) part and an irrotational (divergent part).

$$\boxed{\vec{v} = k \times \nabla\psi + \nabla\chi}$$

we assume that the non-divergent part is entirely captured by the $O(1)$ dynamics. So the $O(R_o)$ dynamics need only to contain the divergent part.

$\Rightarrow \vec{v} = \vec{v}_0 + R_o\vec{v}_1 \qquad$ with $\vec{v}_0 = k \times \psi$

and $\boxed{\text{Curl}\,\vec{v}_1 = 0 \qquad \nabla\cdot\vec{v}_1 = -\frac{\partial w_1}{\partial z} = \Delta\chi_1} \quad \vec{v}_1 = (u_1, v_1).$

We also define an ageostrophic pressure (as for 2-D flows)

$$\boxed{p_1 = \frac{p}{\rho_0} - (f_0 + \beta y)\psi.}$$

χ_1 and p_1 are related to the ageostrophic circulation

The resulting eqs [retaining only $O(R_o)$ terms] are:

$$\boxed{\begin{aligned} &\frac{\partial U_0}{\partial t} + U_0 \cdot \nabla U_0 = -\nabla p_1 - f_0 k \times U_1 - \beta \psi \vec{j} \quad (1) \\ &\frac{d\psi_z}{dt} = -\frac{N^3}{f} w_1 \quad (2). \quad \text{with } N^2 = -\frac{g}{\rho_0}\frac{d\bar{\rho}}{dz} \end{aligned}}$$

Curl of (1) leads to [using $\zeta = V_{0x} - U_{0y} = \Delta\psi$]

$$\boxed{\frac{d\zeta}{dt} + \beta V_0 = f \cdot w_{1z}} \quad (3).$$

From (2) and using the thermal wind balance we get:

$$\boxed{f w_{1z} = -\frac{\partial}{\partial z}\left[\frac{f^2}{N^2}\frac{d\psi_z}{dt}\right] = -\frac{d}{dt}\frac{\partial}{\partial z}\left[\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right]} \quad (4).$$

(3) and (4) lead to:

$$\boxed{\frac{dq}{dt} + \beta\psi_x = 0 \quad (5) \quad \text{with } q = \Delta\psi + \frac{\partial}{\partial z}\left[\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right]} \quad (6)$$

q is the QG potential vorticity.

ψ is obtained from q by inverting a 3-D elliptic operator (6). using appropriate boundary conditions.

The problem to solve is:

$$\left.\begin{aligned} &\boxed{\begin{aligned} &\Delta\psi + \frac{\partial}{\partial z}\left[\frac{f^2}{N^2}\frac{\partial\psi}{\partial z}\right] = q. \\ &\text{with } \left.\frac{\partial\psi}{\partial z}\right|_{z=0} = -\frac{g}{\rho_0 f_0}\,\rho'|_{z=0} \text{ and } \left.\frac{\partial\psi}{\partial z}\right|_{z=-H} = 0. \end{aligned}} \end{aligned}\right\} \quad (7).$$

<u>Only the time evolution of $q(t)$ [eq. 5] and $\rho'|_{z=0}(t)$ [eq. 2 with $W=0$] are needed!</u>

<u>Today we assume $\rho'(t, z = 0) = 0$!</u>

Ageostrophic circulation

p_1 is given by the divergence of (1).

$$\boxed{\nabla\cdot\left[U_0\cdot\nabla U_0\right] = -\Delta p_1 + \beta u_0}$$

Note that $\nabla\cdot\left[U_0\cdot\nabla U_0\right] = \frac{1}{2}\,O:W = \frac{1}{2}\left[S_1^2+S_2^2-\zeta^2\right]$,

χ_1 is given by the Omega equation using $\frac{\partial w_1}{\partial z} = -\Delta\chi_1$:

$$\boxed{\Delta w_1 + \frac{f^2}{N^2}\,w_{1zz} = \frac{2f}{N^2}\,\nabla\cdot Q}$$

with:

$$\boxed{Q = \left[\nabla U_0\right]^T\cdot\nabla\rho = -\left[\nabla U_0^T\right]\cdot\left[\frac{f\rho_0}{g}\,k\times U_{0z}\right]}$$

Understanding the non-linear scale interactions requires to move to the spectral (3-D) space

1) Horizontal dimensions:

It is convenient to express variables (ψ) in terms of double Fourier integral:

$$\boxed{\psi(x,y) = \frac{1}{2\pi}\iint_{k\,l} \hat{\psi}(k,l)\, e^{i(kx+ly)}\, dk\, dl.}$$

with k and l respectively the zonal and meridional wavenumbers.
If ψ is the stream function $[u = -\psi_y,\ v = \psi_x]$ we get:

$$|\hat{u}(k,l)|^2 + |\hat{v}(k,l)|^2 = (k^2+l^2)\,|\hat{\psi}(k,l)|^2$$

$$|\hat{\zeta}(k,l)|^2 = -(k^2+l^2)^2\,|\hat{\psi}(k,l)|^2 \quad \text{with } \zeta = v_x - u_y.$$

We will use:

$$\boxed{\begin{aligned} E(K) &= \int_k \int_l \big[|\hat{u}(k,l)|^2 + |\hat{v}(k,l)|^2\big]\, dk\, dl. \quad k^2+l^2=K^2 \\ Z(K) &= \int_k \int_l |\hat{\zeta}(k,l)|^2\, dk\, dl. \quad k^2+l^2=K^2 \end{aligned}}$$

The resulting PV equation is:

$$\frac{\partial}{\partial t}\hat{q}_{kl} + \hat{J}_{kl}(\psi, q) + ik\beta\psi_{kl} = -\nu(k^2+l^2)\hat{q}_{kl}.$$

with $$\hat{q}_{kl}(z) = -\left[k^2+l^2+\frac{d}{dz}\frac{f^2}{N^2}\frac{d}{dz}\right]\hat{\psi}_{kl}(z)$$

Note that $J(a,b) = a_x b_y - b_x a_y.$

$$\Rightarrow J(\psi, q) = u q_x + v q_y.$$

$\hat{J}_{kl}(\psi, q)$ is the Fourier transform of $J(\psi, q)$.

2) Vertical dimension using normal modes:

Normal modes are obtained by solving the Sturm-Liouville equation (after J. Sturm (1803–1855) and J. Liouville (1809–1882)):

$$\frac{d}{dz}\frac{f^2}{N^2}\frac{dF_m}{dz} = -\lambda_m^2 F_m.$$

with $dF_m/dz = 0$ at $z = 0, -H$.

F_m is the eigenfunction and λ_m^2 the eigenvalue (or the vertical wavenumber) associated with mode m.

Modes are orthonormal:

$$\int_{-H}^{0} F_m \cdot F_n \, dz = \delta_{mn}$$

$$\Rightarrow \vec{\psi}_{k,l}(z) = \sum_{m=0}^{\infty} \psi_{k,l_m} F_m(z).$$

$$\vec{q}_{kl}(z) = \sum_{m=0}^{\infty} Q_{kl_m} F_m(z).$$

3-D spectral PV equation

$$\hat{\psi}_{k,l}(z) = \sum_{m=0}^{\infty} \psi_{k,l,m} F_m(z), \quad \hat{q}_{kl}(z) = \sum_{m=0}^{\infty} \varphi_{k,l,m} F_m(z).$$

$$\boxed{\frac{\partial}{\partial t} \varphi_{k,l,m} + \sum_{p,q} \varepsilon_{mpq} \hat{J}_{k,l}(\psi_p, \varphi_q) + ik\beta \psi_{k,l,m} = -\nu (k^2+l^2) \varphi_{k,l,m}}$$

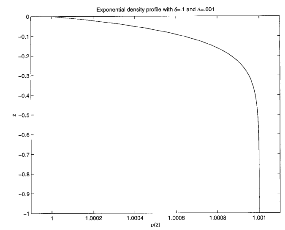
with $\boxed{\varphi_{k,l,m} = -(k^2 + l^2 + \lambda_m^2)\, \psi_{k,l,m}}$

$$\varepsilon_{mpq} = \int_{-H}^{0} F_m \cdot F_p \cdot F_q \, dz$$

The interactions coefficients, ϵ_{mpq}, control the transfer between the different vertical modes.

Normal modes with a depth-dependent stratification $N^2(z)$:

an example (see Smith & Vallis, JPO 2001)



Assume $\bar{\rho}(z) = 1 + \frac{(\rho_{bot} - \rho_{surf})}{\rho_0} \cdot \left[1 - e^{z/a}\right]$

with $a = \delta \cdot H$ $(\delta < 1)$.

$$\Rightarrow N^2(z) = \frac{g(\rho_{bot} - \rho_{surf})}{a \rho_0} e^{z/a}.$$

Using ( ) in ( ) leads to :

$$\lambda_m \simeq (R_d)^{-1} \cdot \frac{m\pi}{2} \quad \text{with} \quad R_d = \frac{1}{f}\sqrt{g(\rho_{bot} - \rho_{surf})/\rho_0}$$

$$F_m(z) \simeq \frac{a}{H} e^{z/2a} \cos\left(m\pi e^{z/2a}\right) \qquad m > 1.$$

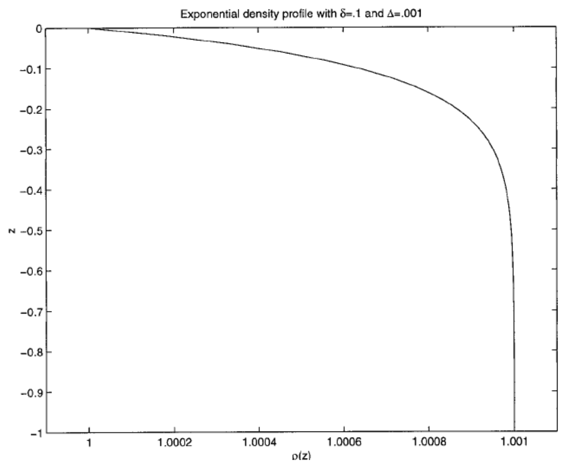
$$F_0(z) = 1$$

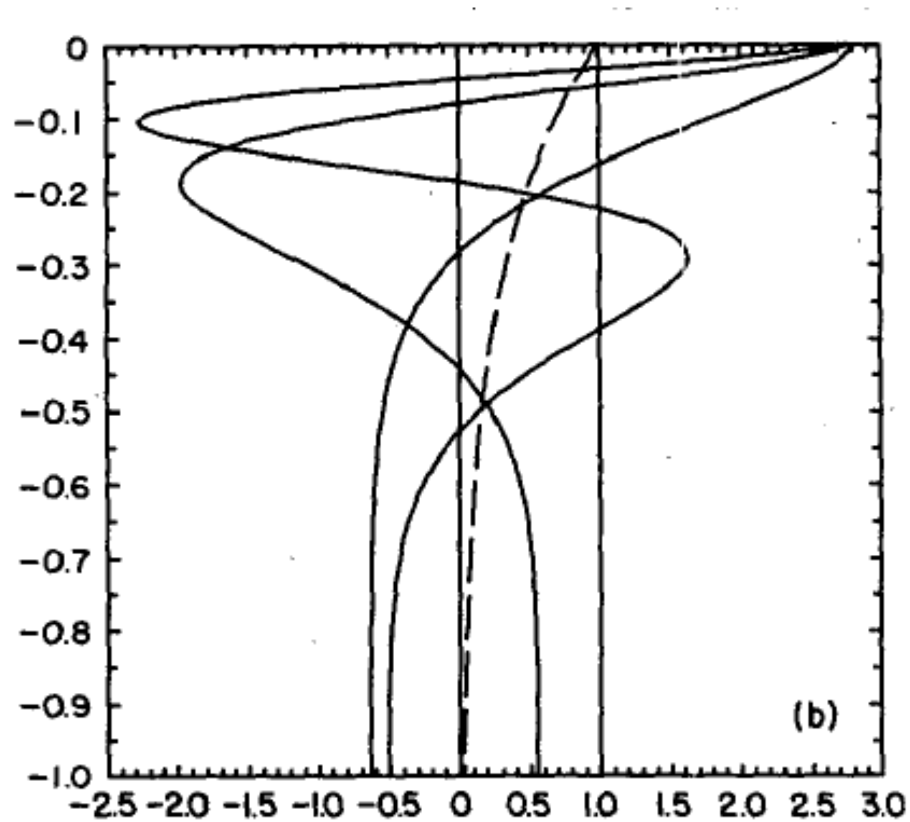
$$\varepsilon_{mpq} = 2\left(\frac{a}{H}\right)^{-1/2} \int_0^1 x^{1/2} \cdot \cos(m\pi x) \cdot \cos(p\pi x) \cdot \cos(q\pi x)\, dx$$

with $x = e^{z/2a}$ and $m, p, q \neq 0$.

$\varepsilon_{mpq} = 1, 0$ if $m \cdot p \cdot q = 0$.

Comments on depth-dependent stratification ($N^2(z)$)





- The vertical wavenumbers are such that: $\lambda_m = m\,.\,\lambda_1$;
- Vertical normal modes are **surface intensified** (see figure);
- The interactions coefficients, ϵ_{mpq}, **control the strength of the nonlinear mode couplings.**

The interactions coefficients, ϵ_{mpq}, control the strength of the nonlinear mode couplings.

$$\varepsilon_{mpq} = 2\left(\frac{a}{H}\right)^{-1/2}\int_0^1 x^{1/2}\cdot\cos(m\pi x)\cdot\cos(p\pi x)\cdot\cos(q\pi x)\,dx \qquad \text{(a)}$$

with $x = e^{3/2a}$ and $m, p, q \neq 0$.

$\varepsilon_{mpq} = 1, 0$ if $m\cdot p\cdot q = 0$.

All of the factors in the integrand in (a) are within [O, 1]. So the integral in (a) is order unity.

Then, all coefficients, ϵ_{mpq} for m.p.q$\neq$ 0, scale as $(a/H)^{-1/2}$
On the other hand coefficients (ϵ_{mpq} for m.p.q = 0) are 0 or 1.

This means that, **when a/H is small ($(a/H)^{-1/2}$ is large), internal transfers between baroclinic modes can occur with much greater efficiency than transfers to (from) barotropic mode.**

3-D stirring mechanisms in geostrophic turbulence

Let us consider a tracer C= C(x,y,z,t) conserved on a Lagrangian trajectory:

At a given depth, C_x and C_y do not depend on C_z.
BUT C_z does depend on C_x and C_y .

3-D stirring mechanisms in geostrophic turbulence

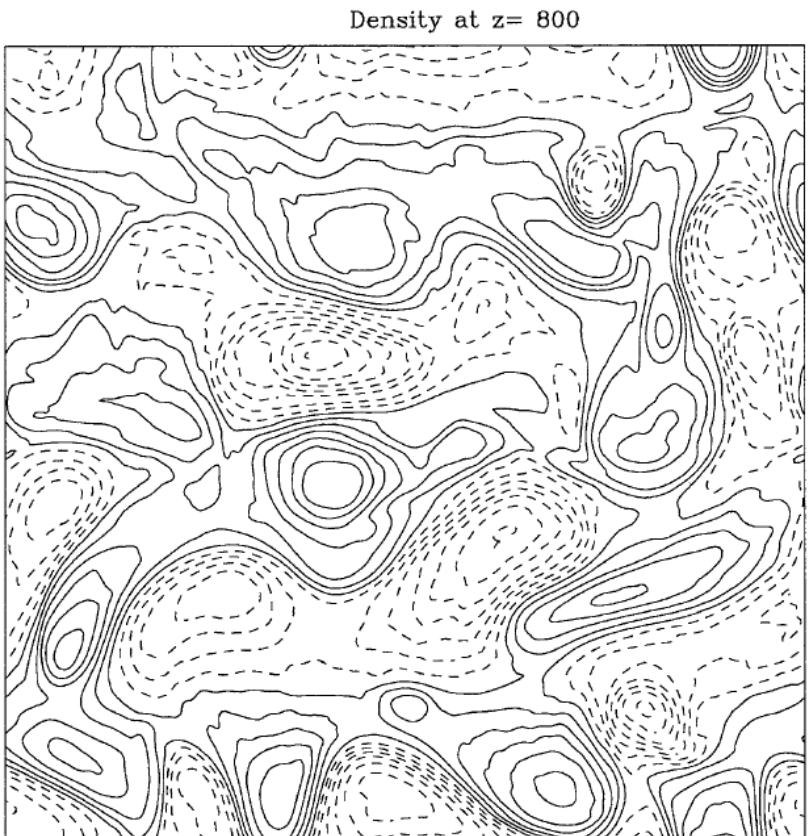
Illustration of the 3-D cascade of the variance of a tracer C = C(x,y,z,t) using a numerical simulation

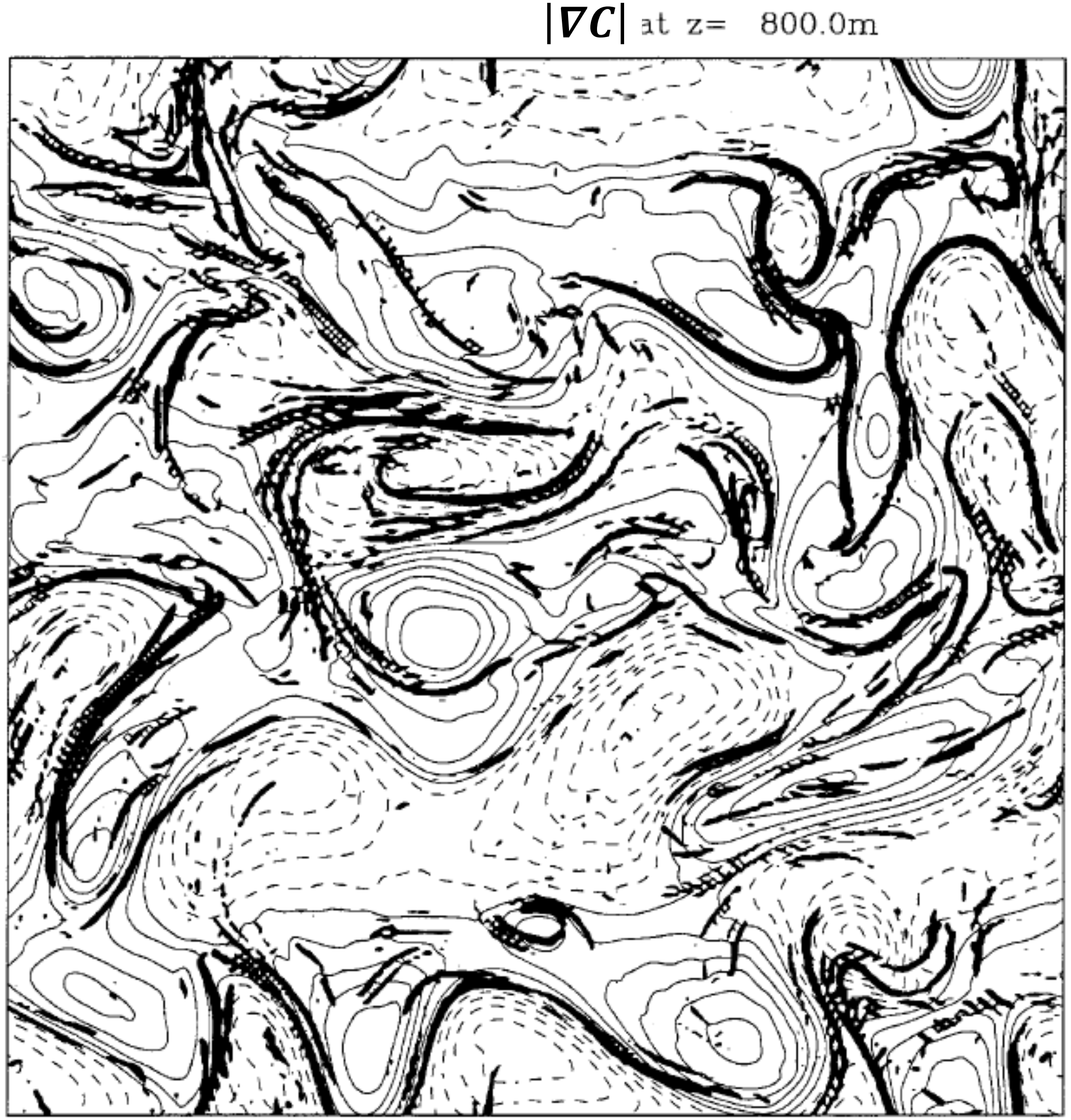
=> $$\frac{d\nabla C}{dt} = -[\nabla U]^T \nabla C$$

$$\frac{d\nabla \rho}{dt} = \underbrace{-[\nabla U]^T \nabla \rho}_{\text{Q-vector}} + \frac{N^2 \rho_0}{g} \nabla w_1$$

Strong production of $|\nabla C|$ but not of $|\nabla \rho|$

The two fields, ρ and C, are nondimensionalized (same units) and the contour intervals are **identical** on both figures





Strong $|\nabla C|$ are located around mesoscale eddies and in saddle areas.
What is the depth scale of these $|\nabla C|$ patterns ?

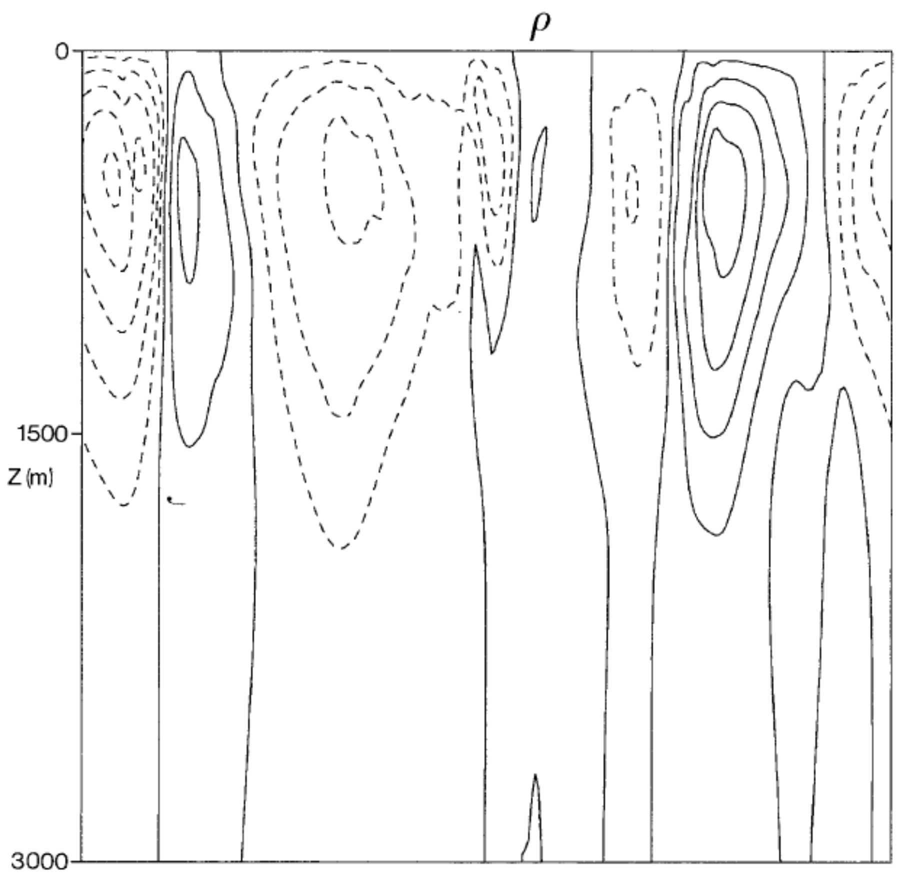
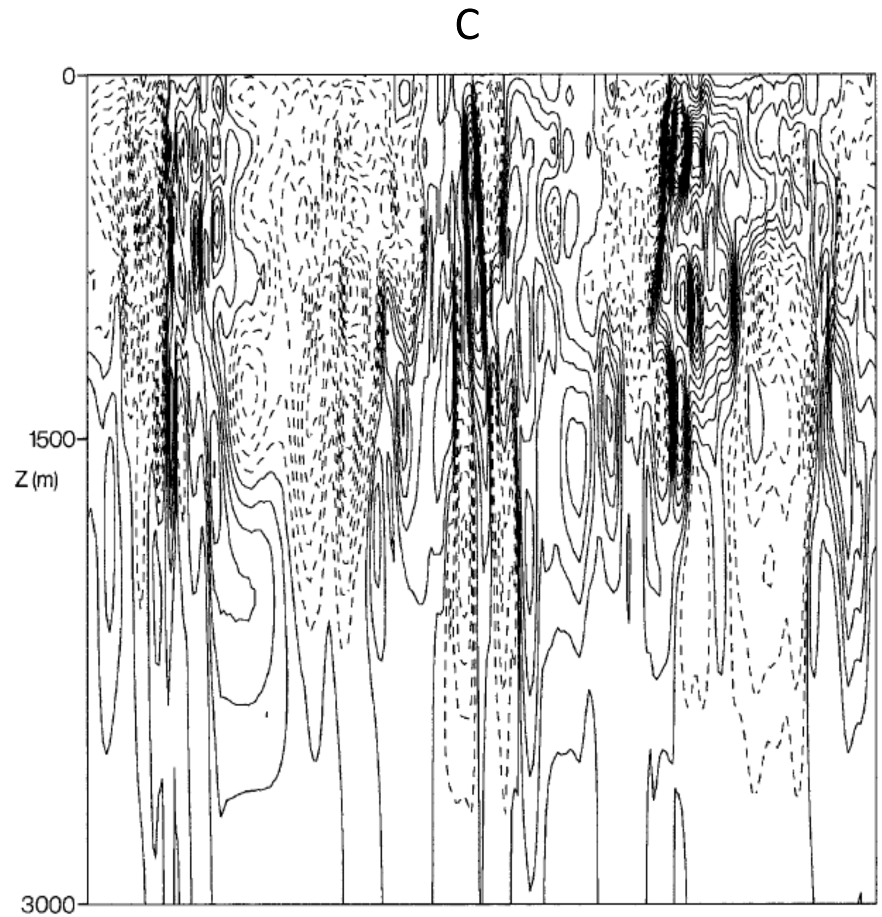
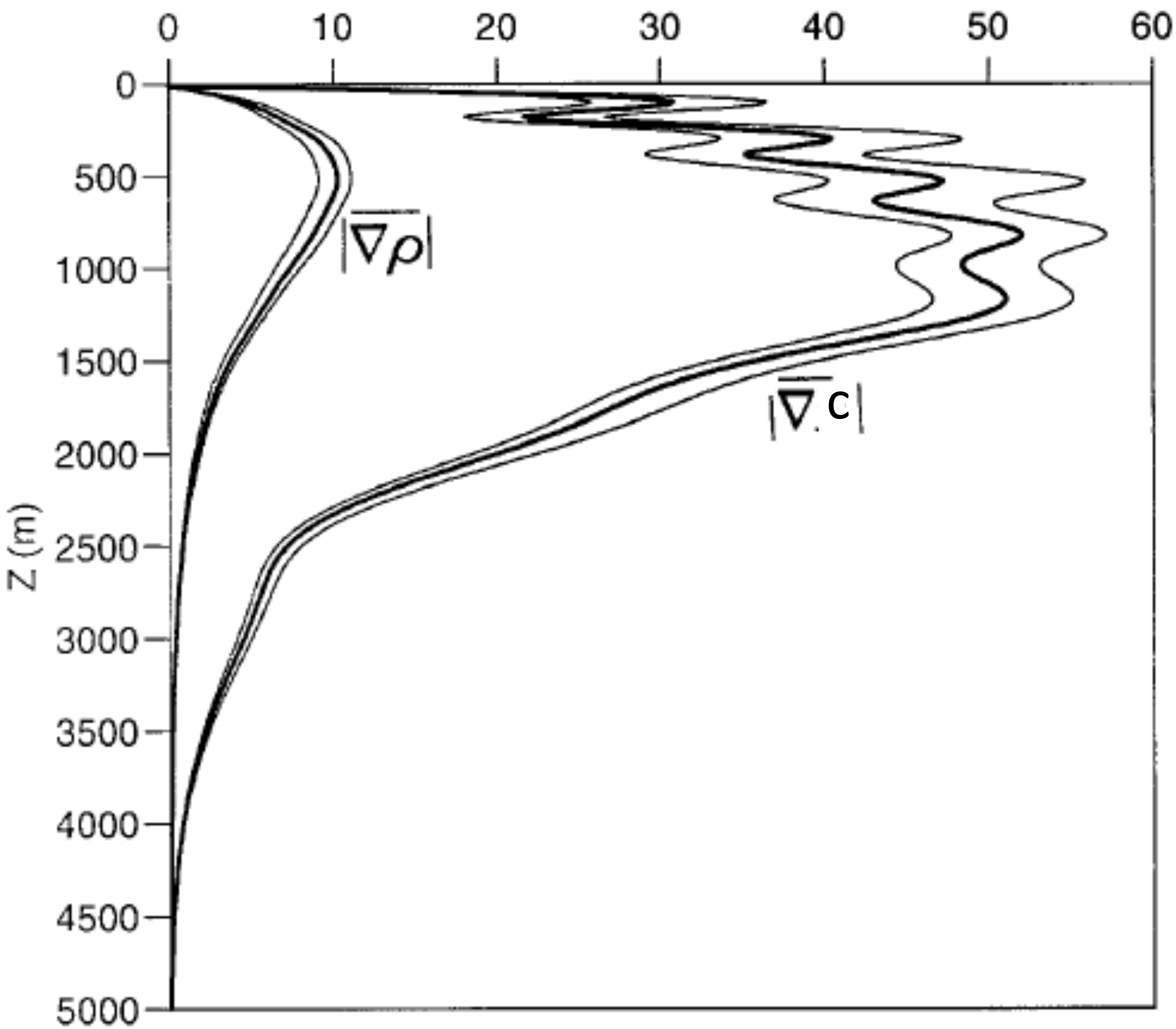


Figure 4. Vertical sections of ρ (a) and χ (b) at the same time as Figure 2. The contour interval in nondimensional units is 1 for both fields. The isocontours range from −8 to 6 for ρ and from −14 to 14 for χ. Solid and dashed lines respectively refer to positive and negative values.



The two fields, ρ and C, are nondimensionalized (same units) and the contour intervals are identical on both figures.

Horizontal gradients, $|\nabla C|$, have a small depth-scale compared with $|\nabla \rho|$ => 3-D cascade



gure 7. Vertical profiles of $|\overline{\nabla\rho}|$ and $|\overline{\nabla\chi}|$ in nondimensional units. The thick lines represent the values averaged over a period of a year while the thin lines represent the extrema over this period.