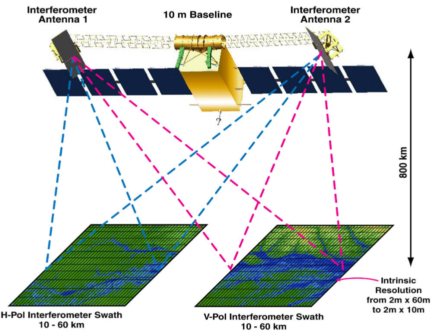
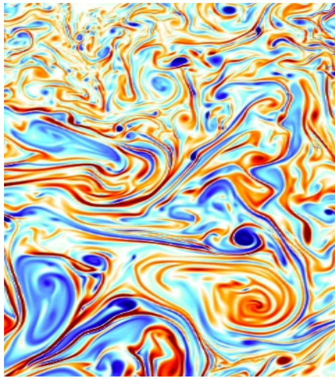
"Ocean Turbulence from SPACE"

Patrice Klein (Caltech/JPL/Ifremer)

(XII) – SQG and 2D turbulence: major differences





SQG turbulence is driven by the time evolution of surface density .

2D turbulence is driven by the time evolution of the relative vorticity.

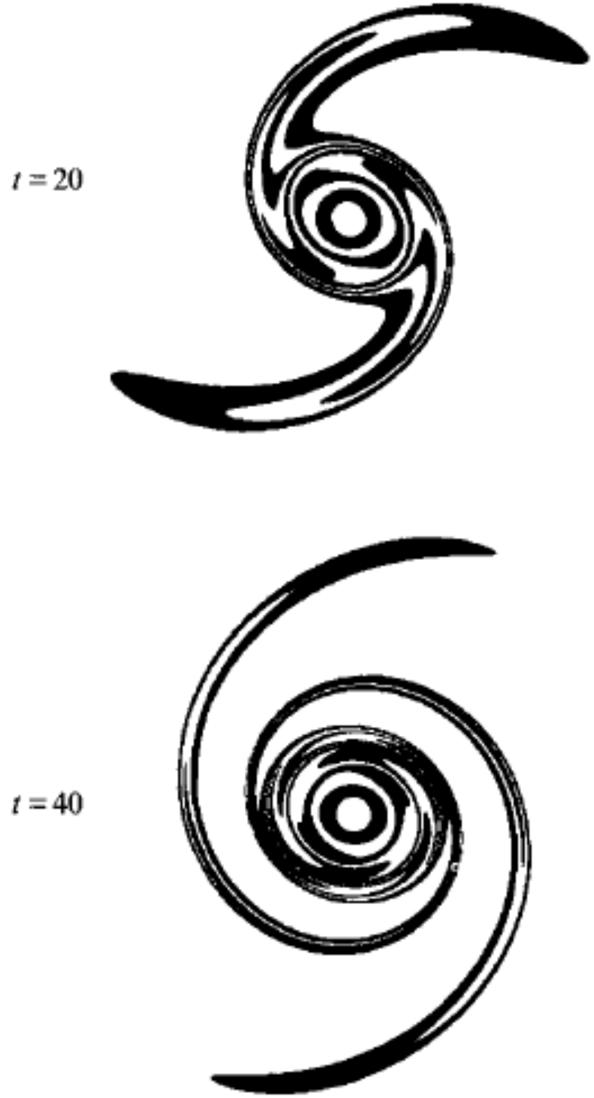
Both scalars are stirred by mesoscale eddies, leading to a direct cascade of the variance of ζ and ρ_s.

But an inverse KE cascade from the smallest scales to larger scales is found in SQG flow and NOT in 2D flow in the absence of a specific forcing.

What are the mechanisms that drive this inverse KE cascade in SQG flows?

Submesoscales (filaments) are unstable in SQG flows, not in 2D flows!

From Held et al.’95



Axisymmetrisation of an elliptic vortex in a 2D flow

Filaments are stable and ultimately dissipated ! (Dritschel JFM’89, Dritschel et al. JFM’91)

From Held et al.'95

Axisymmetrisation of an elliptic vortex in SQG flow

Filaments in SQG flows are unstable leading to the production of small-scale eddies!

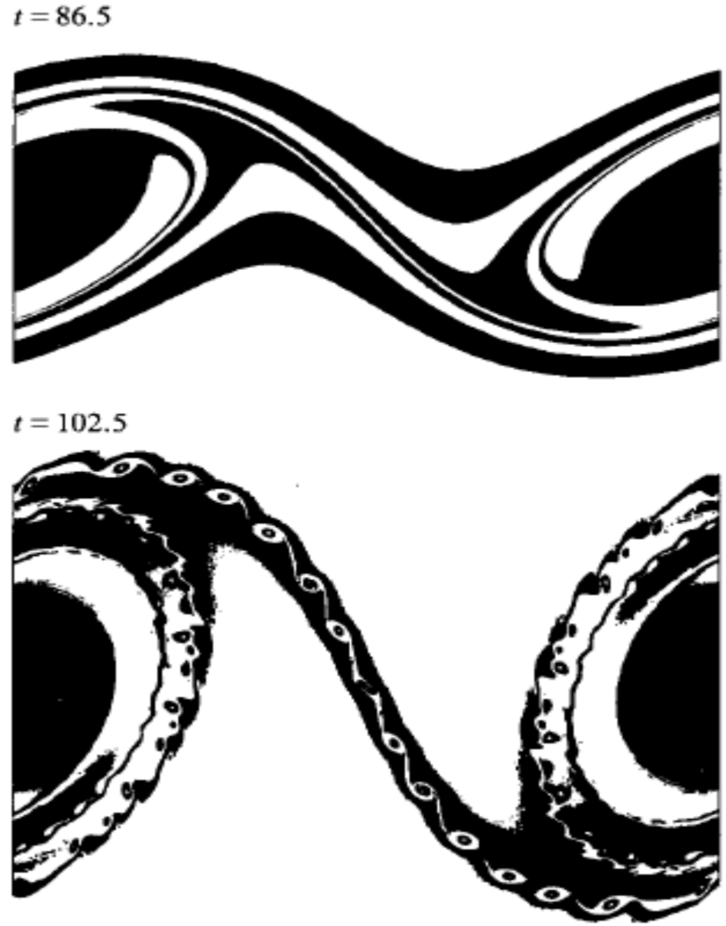


FIGURE 5. An unstable temperature filament, with the initial condition (21); temperatures shown at $t = 86.5$ and 102.5.

From Held et al.'95

Filament instability is a generic feature in SQG flows. Why ?

Instability of parallel flows using Rayleigh equation

(from Euler equations)

The linearized vorticity eq. for 2D flows is:

$$\frac{\partial \zeta'}{\partial t} + U \frac{\partial \zeta'}{\partial x} + v' \frac{\partial Z}{\partial y} = 0 \qquad (1)$$

with $Z = -\frac{\partial U}{\partial y}$.

Using a streamfunction ψ' such that

$$u' = -\frac{\partial \psi'}{\partial x} \quad v' = \frac{\partial \psi'}{\partial y} \text{ and } \zeta' = \Delta \psi'$$

the linear vorticity eq. is:

$$\frac{\partial \Delta \psi'}{\partial t} + U \frac{\partial \Delta \psi'}{\partial x} + \frac{\partial Z}{\partial y} \frac{\partial \psi'}{\partial x} = 0 \qquad (2)$$

Using $\psi' = \vec{\varphi}(y)\, e^{ik(x-ct)}$ leads to:

$$(U - c)(\vec{\varphi}_{yy} - k^2 \vec{\varphi}) - U_{yy} \vec{\varphi} = 0, \qquad (3).$$

(3) is known as the Rayleigh's equation !

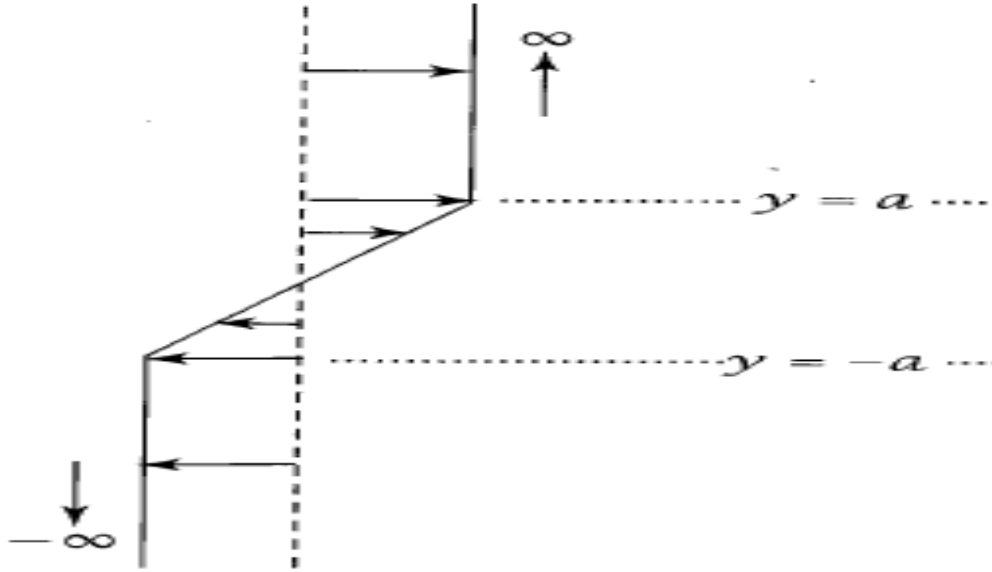
Filaments in 2D flows are unstable but can be stabilized by stirring mechanisms

Rayleigh's eq. is difficult to solve analytically.
The formalism used in papers by Dritschel et al. to investigate the stability/instability of filaments is to consider piecewise linear flows [same as for the classical Kelvin-Helmholtz instability]:
U_y is constant over some intervals with U (and U_y) changing at the line of discontinuity.

Filaments in 2D flows are unstable **but can be stabilized by stirring mechanisms**

Piecewise linear flows

Vallis 2006



The curvature of the flow, U_{yy}, is taken into account through some matching conditions at the lines of discontinuity.

Filaments in 2D flows are unstable but can be stabilized by stirring mechanisms

Piecewise linear flows:

1) Rayleigh equation (with U_{yy}=0)

$$(U - c)(\hat{\varphi}_{yy} - k^2\hat{\varphi}) = 0$$

2) Matching conditions at the interface :

$$\frac{\partial u'}{\partial t} + U\frac{\partial u'}{\partial x} + v'\frac{\partial U}{\partial y} = -\frac{\partial p'}{\partial x} \qquad (4).$$

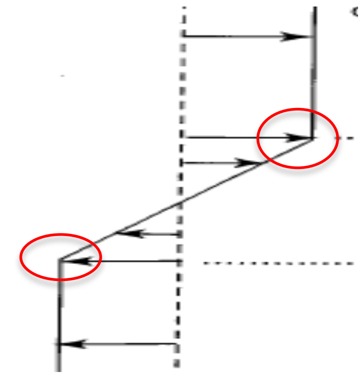
Using normal modes: $\psi'(x,y,t) = \hat{\psi}(y)\, e^{ik(x-ct)}$ and,

$p' = \hat{p}\, e^{ik(x-ct)}$,

(1) becomes:

$$\boxed{ik(U-c)\hat{\psi}_y - ik\hat{\psi}\, U_y = -ik\hat{p}} \qquad (5).$$

Filaments in 2D flows are unstable but can be stabilized by stirring mechanisms



2) Matching conditions at the interface

a - pressure is continuous across the interface.
Using (5) on each side of the interface leads to:

$$\boxed{(U_1 - c)\hat{\varphi}_{1y} - \hat{\varphi}_1 U_{1y} = (U_2 - c)\hat{\varphi}_{2y} - \hat{\varphi}_2 U_{2y}} \quad (6)$$

b - normal velocity on either side of the interface should be consistent with the motion of the interface itself.

$$v = \frac{d\eta}{dt} \Rightarrow \frac{\partial \eta}{\partial t} + U\frac{\partial \eta}{\partial x} = \frac{\partial \varphi'}{\partial x}.$$

Using $\eta = \hat{\eta}\, e^{ik(x-ct)}$.

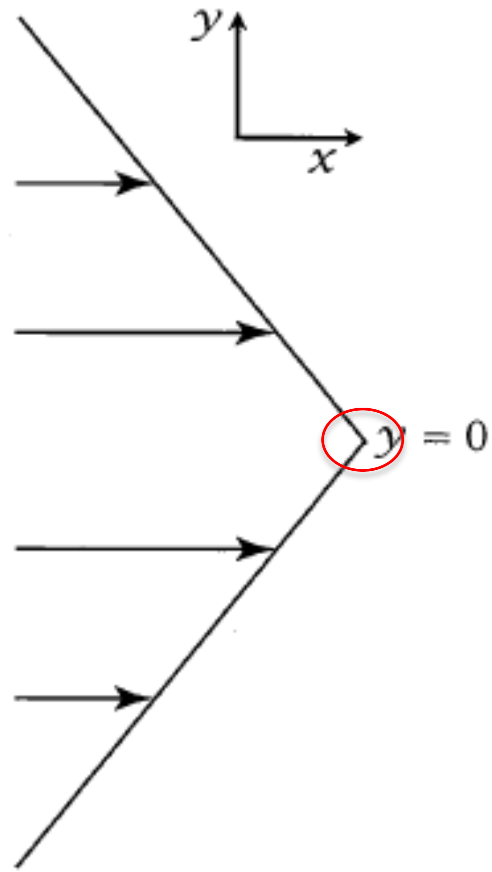
$$\Rightarrow (U_1 - c)\hat{\eta} = \hat{\varphi}_1 \qquad (U_2 - c)\hat{\eta} = \hat{\varphi}_2.$$

$$\rightarrow \boxed{\frac{\hat{\varphi}_1}{U_1 - c} = \frac{\hat{\varphi}_2}{U_2 - c}} \quad (7)$$

(6) and (7) are homogeneous equations that allow to determine c

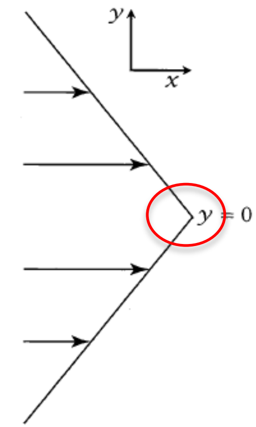
Filaments in 2D flows are unstable but can be stabilized by stirring mechanisms

Application to edge waves …



Filaments in 2D flows are unstable but can be stabilized by stirring mechanisms

Application to edge waves …



Since $U_{yy} = 0$ on either side of the interface, the Rayleigh's eq. is:

$$(U - c)(\hat{\psi}_{yy} - k^2 \hat{\psi}) = 0.$$

Assuming $U - c \neq 0$, solutions are:

$$\hat{\psi} = \begin{cases} \phi_1 e^{-ky} & y > 0 \\ \phi_2 e^{ky} & y < 0 \end{cases}$$

The matching conditions lead to (at $y=0$):

$$-k(U_0 - c)\phi_1 - \phi_1 U_{1y} = k(U_0 - c)\phi_2 - \phi_2 U_{2y}$$

$$\phi_1 = \phi_2.$$

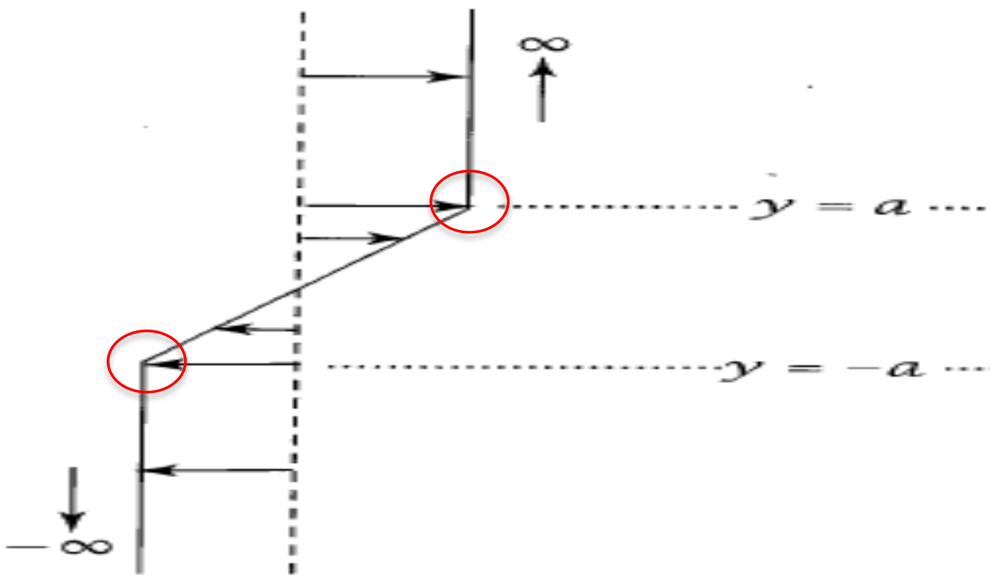
$$\Rightarrow \quad \boxed{c = U_0 + \frac{U_{1y} - U_{2y}}{2k}}$$

C is real and therefore the edge waves are stable!

Filaments in 2D flows are unstable but can be stabilized by stirring mechanisms

Application to a vorticity filament:

Vallis 2006



The filament is characterized by constant vorticity (dU/dy) and U outside the filament is uniform

=> interacting edge waves can lead to instability

Filaments in 2D flows are unstable but can be stabilized by stirring mechanisms

Application to a vorticity filament: interacting edge waves can lead to instability

$\Rightarrow$ $y > a$. $\quad U = U_0$

$-a < y < a \quad U = \frac{U_0}{a} y.$

$y < -a. \quad U = -U_0$

- From the Rayleigh's eq. $[(U - c)\,\hat{\varphi}_{yy} - k^2 \hat{\varphi}]$ solution should be:

$y > a. \quad \hat{\varphi}_1 = A e^{-k(y-a)}.$

$-a < y < a \quad \hat{\varphi}_2 = B e^{k(y-a)} + C e^{-k(y+a)}.$

$y < -a. \quad \hat{\varphi}_3 = D e^{k(y+a)}.$

- We have four unknowns: A, B, C, D.

The matching conditions at the two interfaces provide four homogeneous equations!

$$A[(U_0 - c)k] = B\left[(U_0 - c)k + \frac{U_0}{a}\right] + C e^{2ka}\left[\frac{U_0}{a} - (U_0 - c)k\right]$$

$$A = B + C e^{2ka}.$$

$$D[(U_0 + c)k] = B e^{2ka}\left[-(U_0 + c)k + \frac{U_0}{a}\right] + C\left[\frac{U_0}{a} + (U_0 + c)k\right]$$

$$D = B e^{2ka} + C$$

Filaments in 2D flows are unstable **but can be stabilized by stirring mechanisms**

Application to a vorticity filament: interacting edge waves can lead to instability

The determinant of this system of four homogeneous eqs. leads to:

$$c^2 = \left(\frac{\omega_0}{2ka}\right)^2 \left[(1-2ka)^2 - e^{-4ka}\right]$$

The growth rate (kc_i) is shown on this figure.

kc_i is proportional to the filament vorticity: when $c^2 < 0$

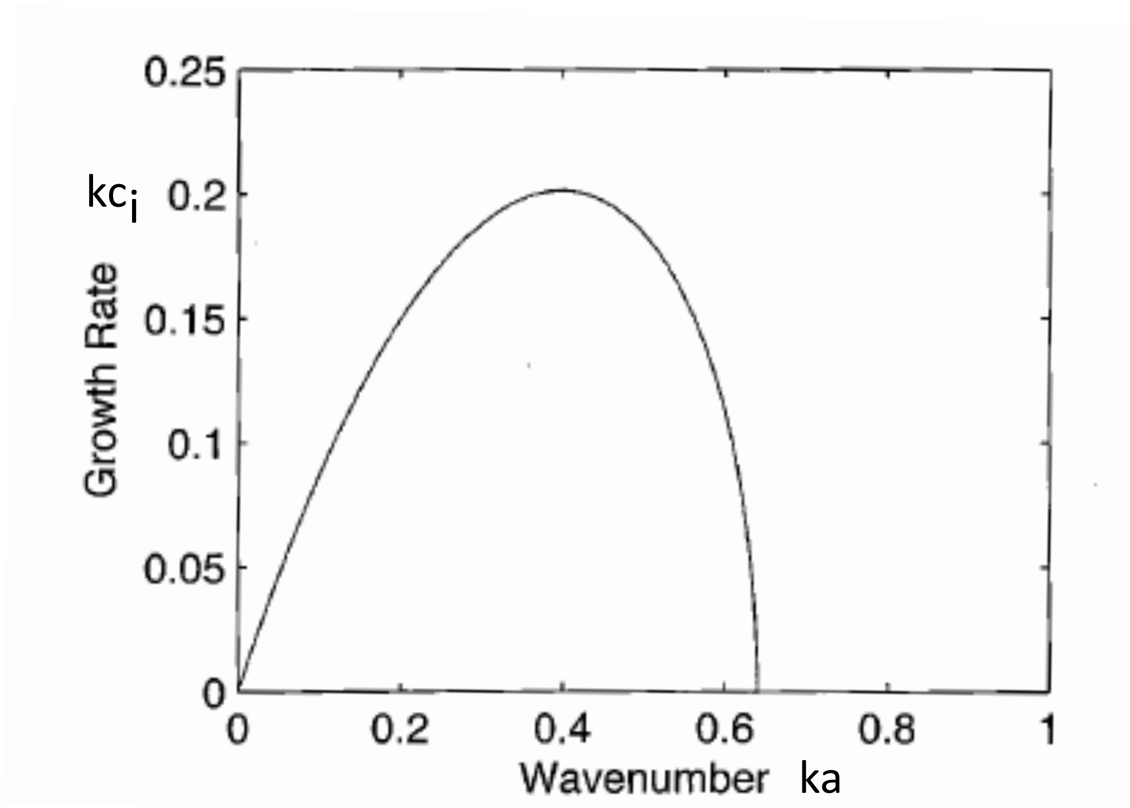
$$\boxed{kc_i = \frac{\omega_0}{2a}\left[(1-2ka)^2 - e^{-4ka}\right]^{1/2}}$$

The instability arises from the interactions between two edge waves when their cross-stream extent is large enough. Their cross-stream decay scale is proportional to ka. The most unstable wavelength corresponds to $ka = 0.63$

Filaments in 2D flows are unstable **but can be stabilized by stirring mechanisms**

Application to a vorticity filament: interacting edge waves can lead to instability

$$kc_i = \frac{U_0}{2a}\left[(1-2ka)^2 - e^{-4ka}\right]^{1/2}$$



Growth is maximum for ka ≈ 0.4

Vallis 2006

Filaments in 2D flows are unstable but can be stabilized by stirring mechanisms

Application to a vorticity filament: interacting edge waves can lead to instability

Vallis 2006

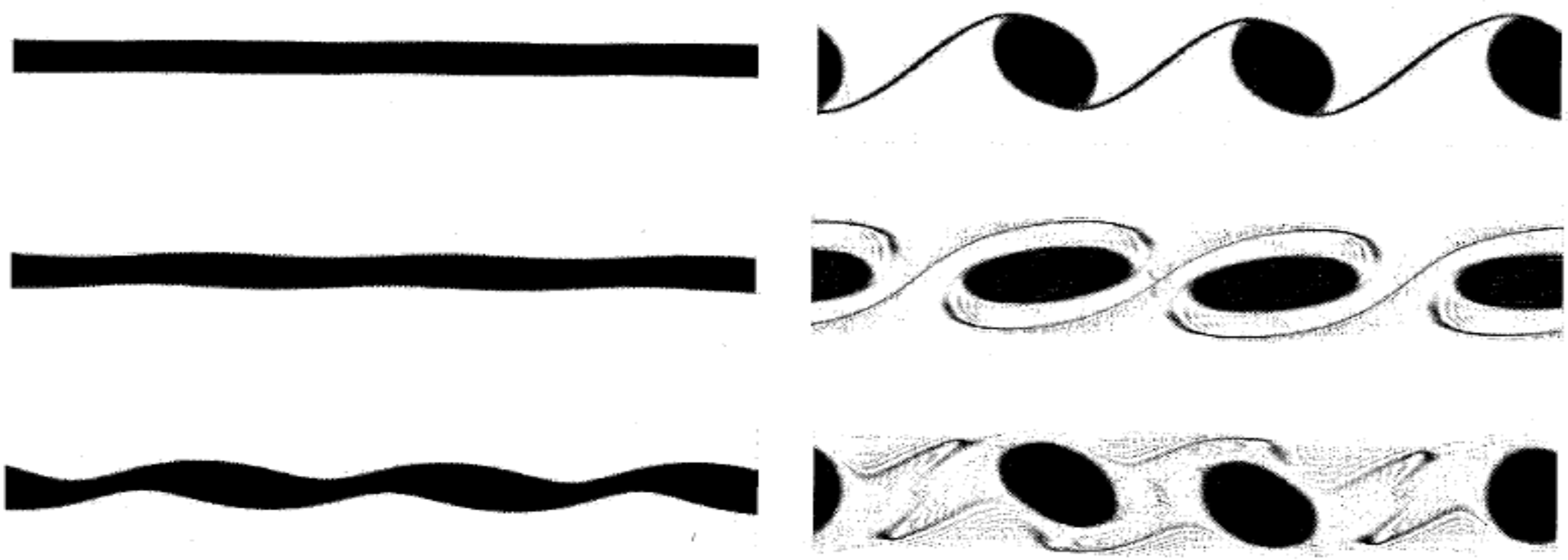


Fig. 6.6 A sequence of plots of the vorticity, at equal time intervals, from a numerical solution of the nonlinear vorticity equation (6.12), with initial conditions as in Fig. 6.4 with $a = 0.1$, plus a very small random perturbation. Time increases first down the left column and then down the right column. The solution is obtained in a rectangular (4×1) domain, with periodic conditions in the x-direction and slippery walls at $y = (0, 1)$. The maximum linear instability occurs for a wavelength of 1.57, which for a domain of length 4 corresponds to a wavenumber of 2.55. Since the periodic domain quantizes the allowable wavenumbers, the maximum instability is at wavenumber 3, and this is what emerges. Only in the first two or three frames is the linear approximation valid.

Filaments in 2D flows are unstable **but are stabilized by stirring mechanisms**

Application to a vorticity filament: interacting edge waves can lead to instability.

But the presence of <u>an adverse shear</u> stabilizes the filament when the shear is more than 0.33 the filament vorticity (Dritschel JFM 1989).

Dritschel JFM'89:

- The vorticity filament ($\zeta = U_o$/a) is embedded in an adverse shear flow ($\overline{U}$ = $\zeta\Lambda y$).

- So within the filament the velocity is:

$$U = \zeta[\Lambda - 1]y$$

Filaments in 2D flows are unstable but are stabilized by stirring mechanisms

The vorticity filament ($\zeta = U_o/\text{a}$) is embedded in an adverse shear flow ($\bar{U}$ = $\zeta \Lambda y$).
So within the filament the velocity is:

$$U = \zeta[\Lambda - 1]y$$

Normal analysis similar to the previous one leads to:

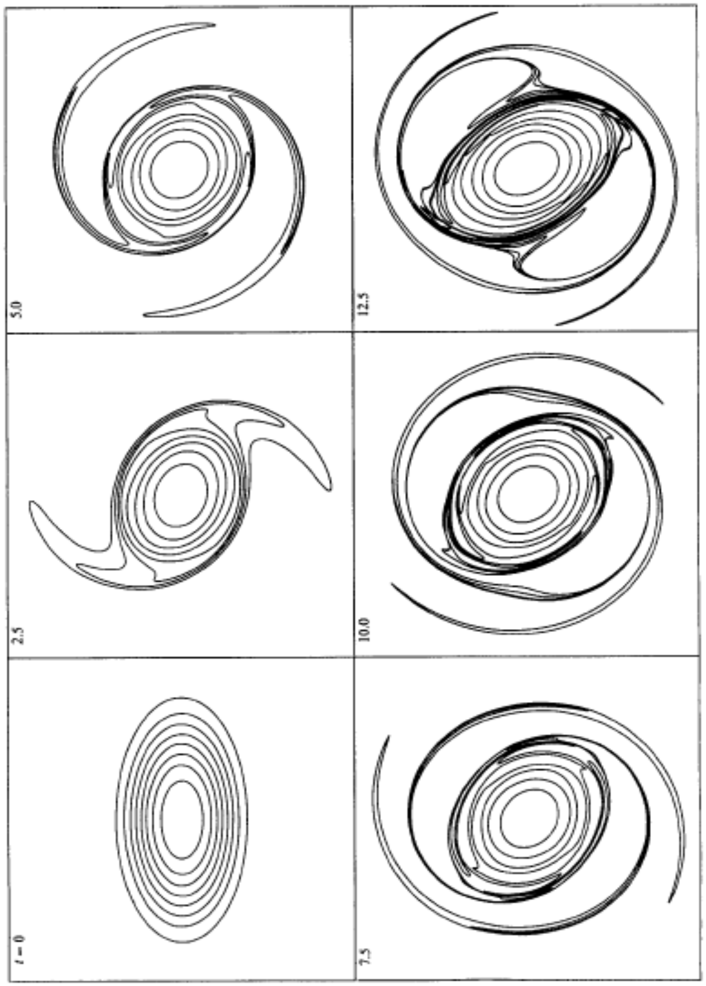
$$\boxed{kc_i = \frac{\zeta}{2}\left[\left(1-2ka[1-\Lambda]\right)^2 - e^{-4ka}\right]^{1/2}}$$

The filament is stabilized by shear when $\Lambda \geq 0.33$

Strain acts to decrease « a » BUT ζ is conserved in a 2D flow!

(Dritschel JFM'89, 91)

Filaments in 2D flows are unstable but are stabilized by stirring mechanisms: **adverse shear is significant close to the vortex**



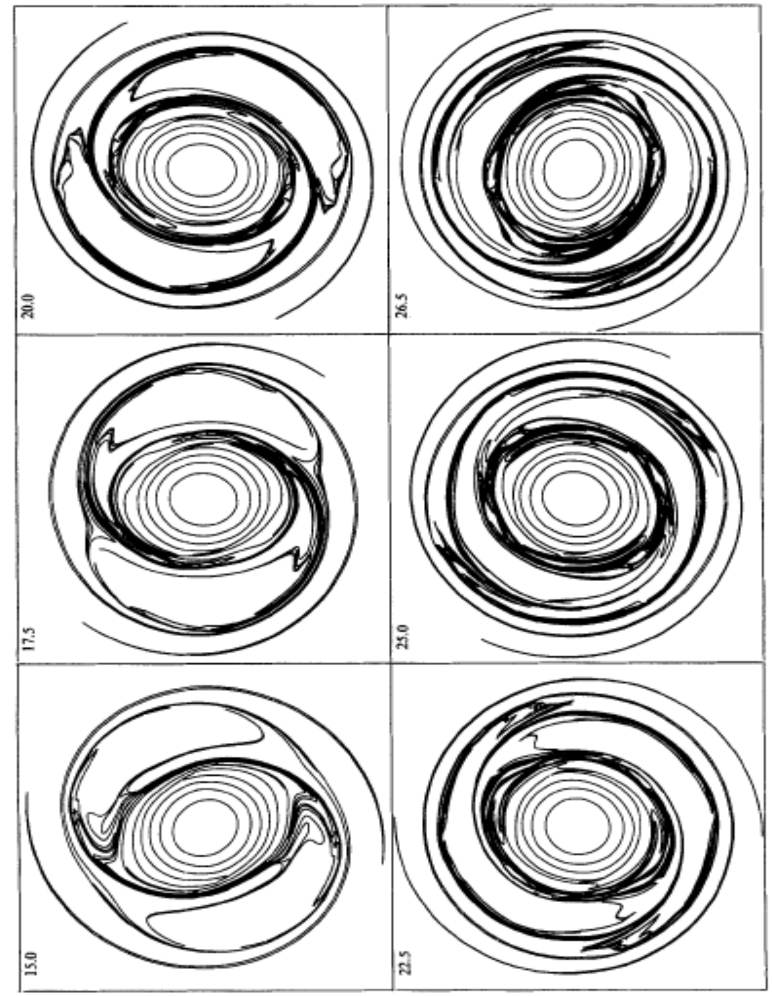
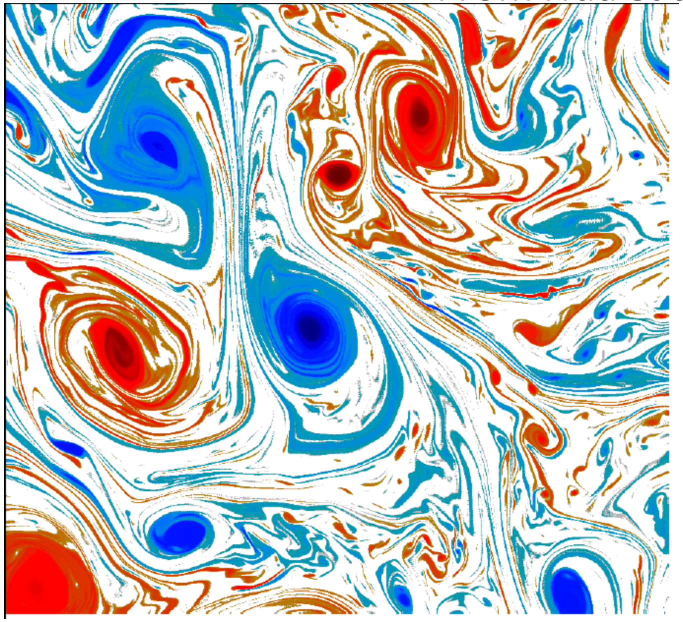


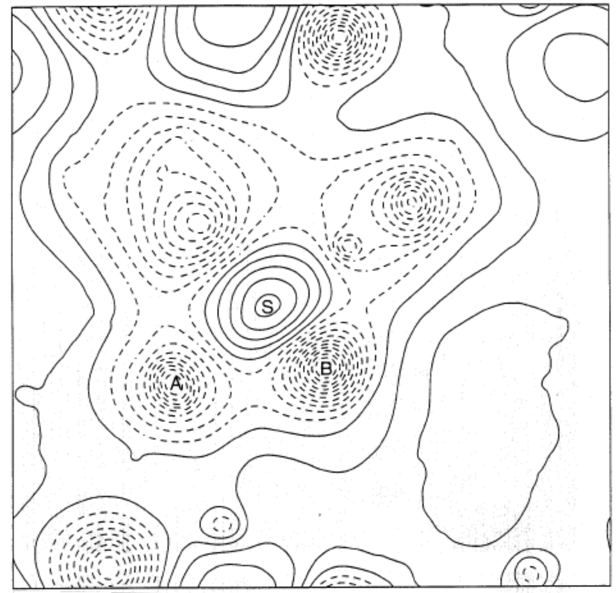
FIGURE 1. For caption see page 198.

Both strain and shear act to stabilize the vorticity filaments since usually $\boldsymbol{\Lambda} \geq \mathbf{0.33}$!

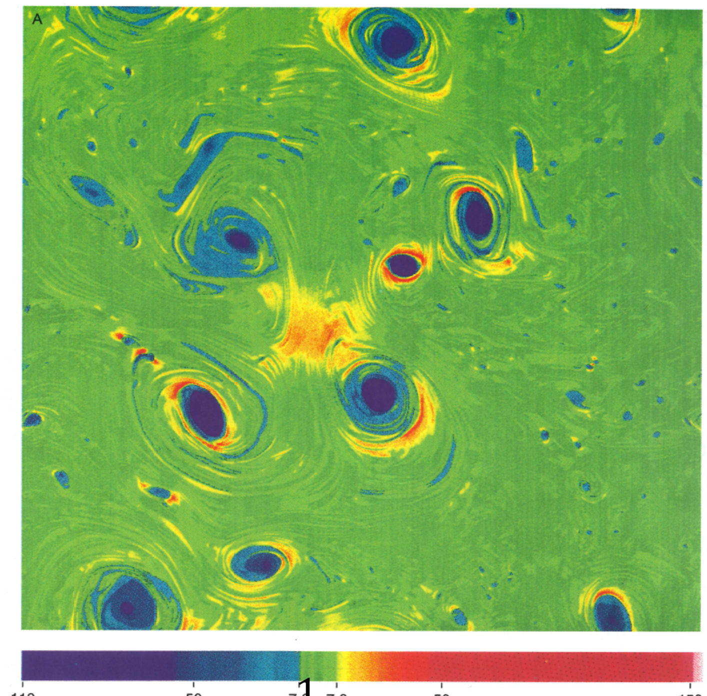
From Hua et al. Physica D 1998 (see also Haller (P.F., 2001))



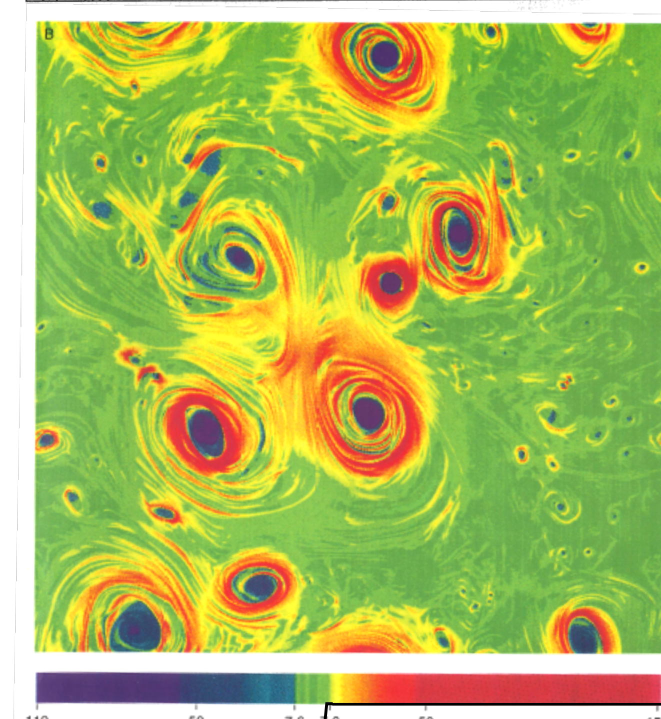
ζ



$\tilde{p}$



$$W/4 = -\frac{1}{2}\Delta\tilde{p}$$



$$\lambda_+ = -\frac{1}{2}\Delta\tilde{p} + \frac{1}{2}\sqrt{\left\{(\tilde{p}_{xx} - \tilde{p}_{yy})^2 + 4\tilde{p}_{xy}^2\right\}}$$

There is a different relationship between the flow ψ and the advected scalar (that comes from zero PV in the interior):

In 2D flow: $\zeta_K = -|K|^2.\psi_K$ **vorticity is conserved**

In SQG flow: $\rho_{s_K} = -|K|.\psi_K$ **surface density is conserved but NOT vorticity**

$$=> \quad \zeta_K = -|K| \; .\rho_{s_K}$$

Because of the density cascade, density gradients and therefore vorticity magnitude exponentially increases within density filaments.

$$\boxed{kc_i = \frac{\zeta}{2}\left[\left(1-2ka\left[1-\Lambda\frac{\zeta o}{\zeta}\right]\right)^2 - e^{-4ka}\right]^{1/2}}$$

The condition $\Lambda\frac{\zeta o}{\zeta} \geq 0.33$ is never attained !

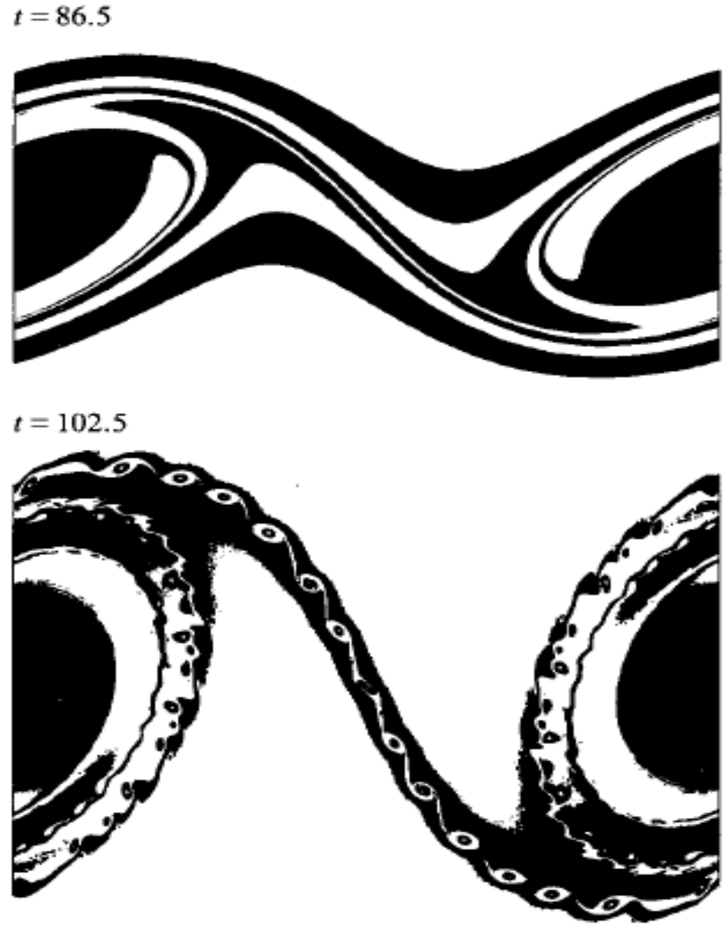


FIGURE 5. An unstable temperature filament, with the initial condition (21); temperatures shown at $t = 86.5$ and 102.5.

Filament instability is a generic feature in SQG flows. The resulting small-scale eddies subsequently cascade to larger scales through the eddy merging processes.

In SQG flows, since w=0 at the surface, the frontogenesis makes the density gradients (and associated relative vorticity) to exponentially increase and therefore the horizontal jets to increase to restore the thermal wind balance. This holds within the surface density filaments.

This corresponds to a transformation of PE into KE (see Georgy presentation).

Whatever the large-scale strain and shear field is, the instability of these fronts (and filaments) in SQG flows is inevitable !

This leads to small-scale eddies that subsequently merge to produce larger eddies ….

… leading to a significant inverse KE cascade.