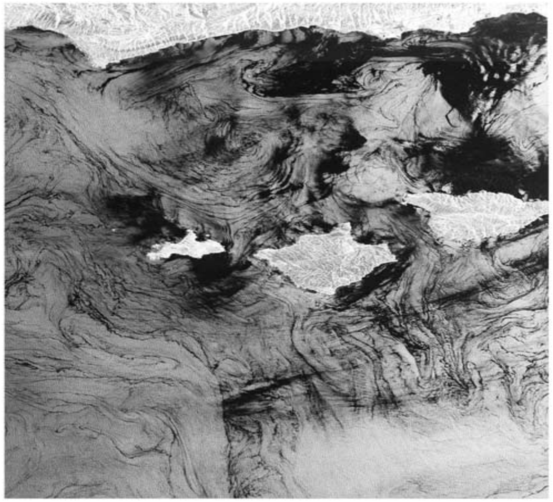
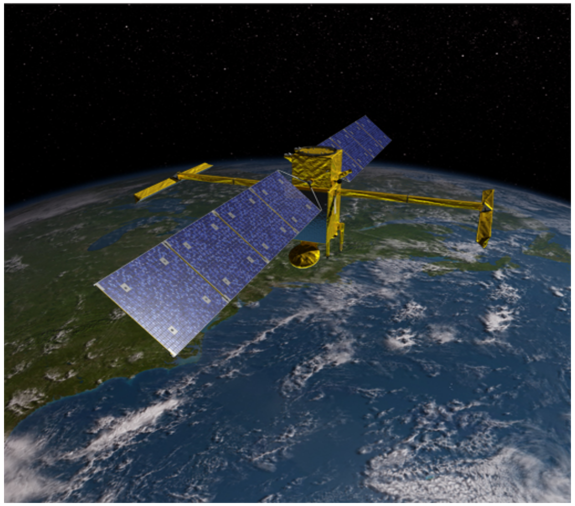
“Wave-Turbulence Interactions in the Oceans”

https://oceanturbulence.github.io

Patrice Klein (Caltech/JPL/Ifremer)

(VII) Vertical Propagation of NIWs





NEAR-INERTIAL (f) AND TIDAL WAVES (M2) EXPLAIN A LARGE PART OF THE WAVE SPECTRUM

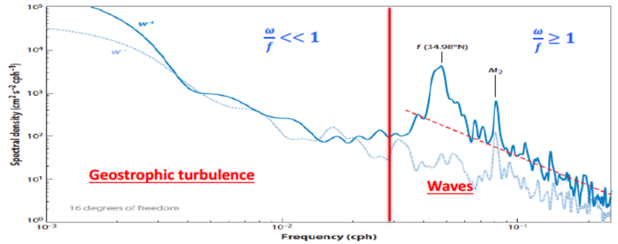
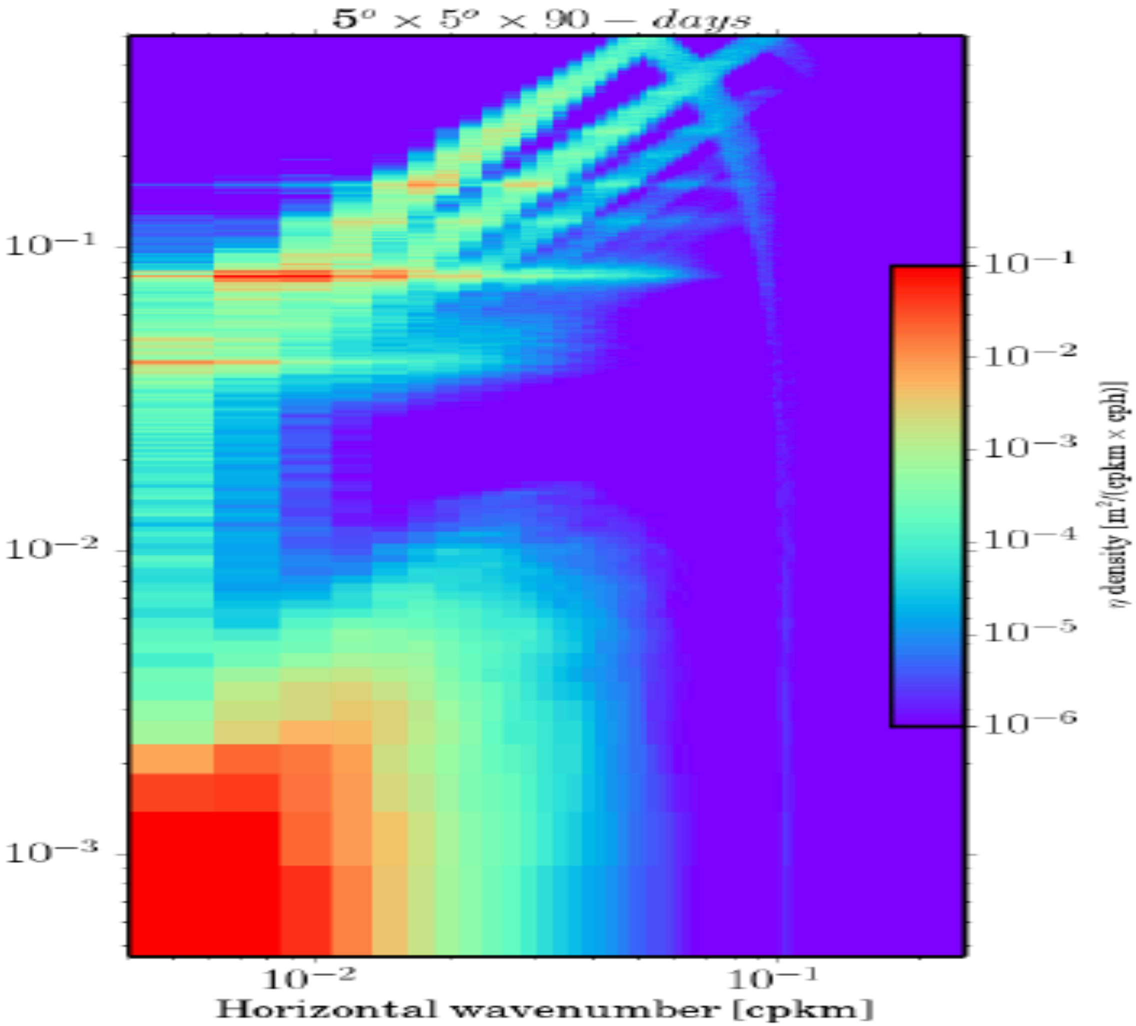
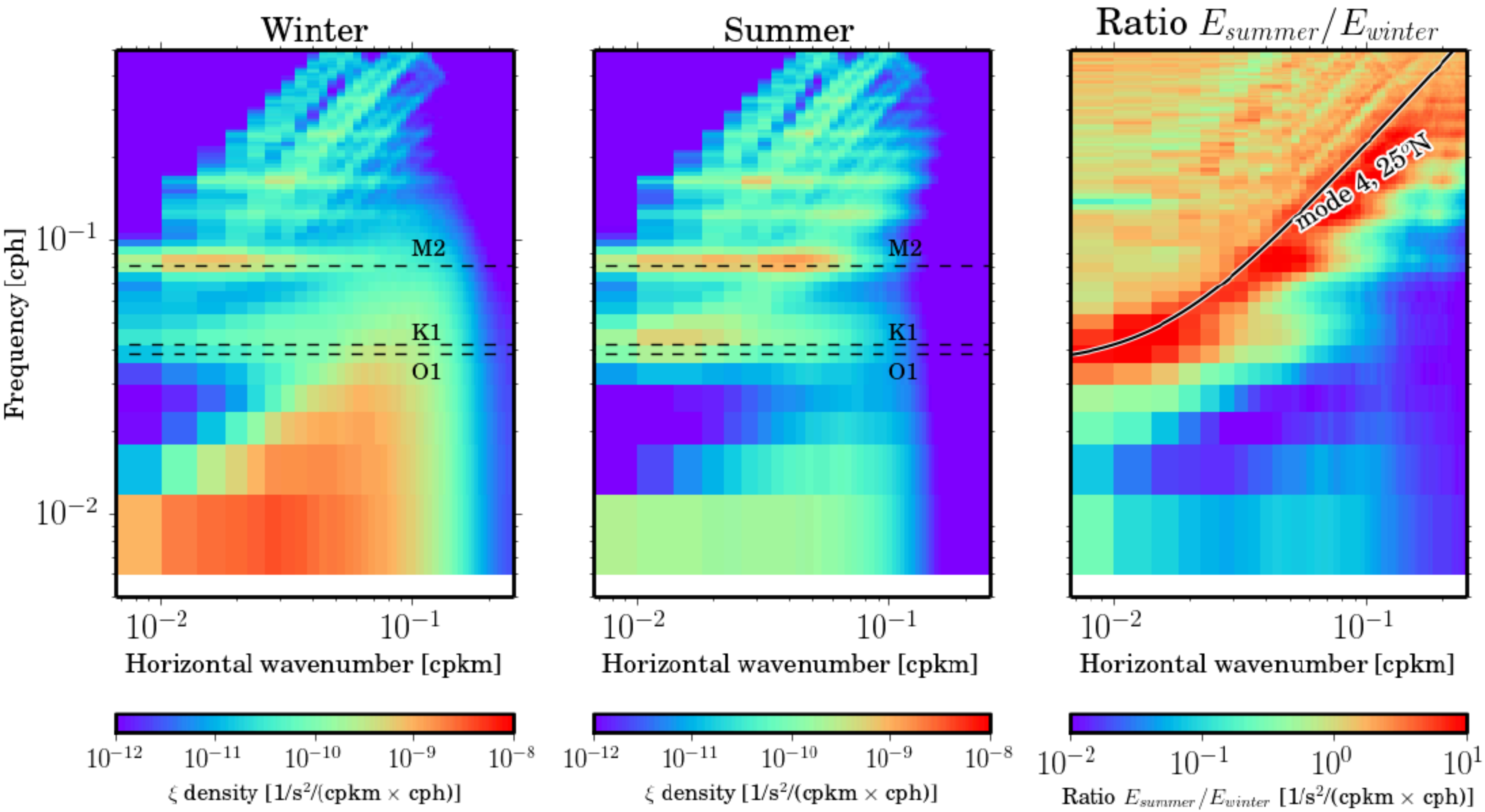


Figure 1

Rotary velocity spectrum at 261-m depth from current-meter data from the WHOI699 mooring gathered during the WESTPAC1 experiment (mooring at 6,149-m depth.) The solid blue line (w^+) is clockwise motion, and the dashed blue line (w^-) is counterclockwise motion; the differences between these emphasize the downward energy propagation that often dominates the near-inertial band. The dashed red line is the line $E_0 N\omega^{-p}$ with $N = 2.0$ cycles per hour (cph), $E_0 = 0.096$ cm^2 s^{-2} cph^{-2}, and $p = 2.25$, which is quantitatively similar to levels in the Cartesian spectra presented by Fu (1981) for station 5 of the Polygon Mid-Ocean Experiment (POLYMODE) II array.



The high frequency part of the wave spectrum is characterized by discrete bands at high wavenumbers!



The high frequency part of the wave spectrum is characterized by a strong seasonality !

WORK IN PROGRESS: HECTOR TORRES JPL 2017

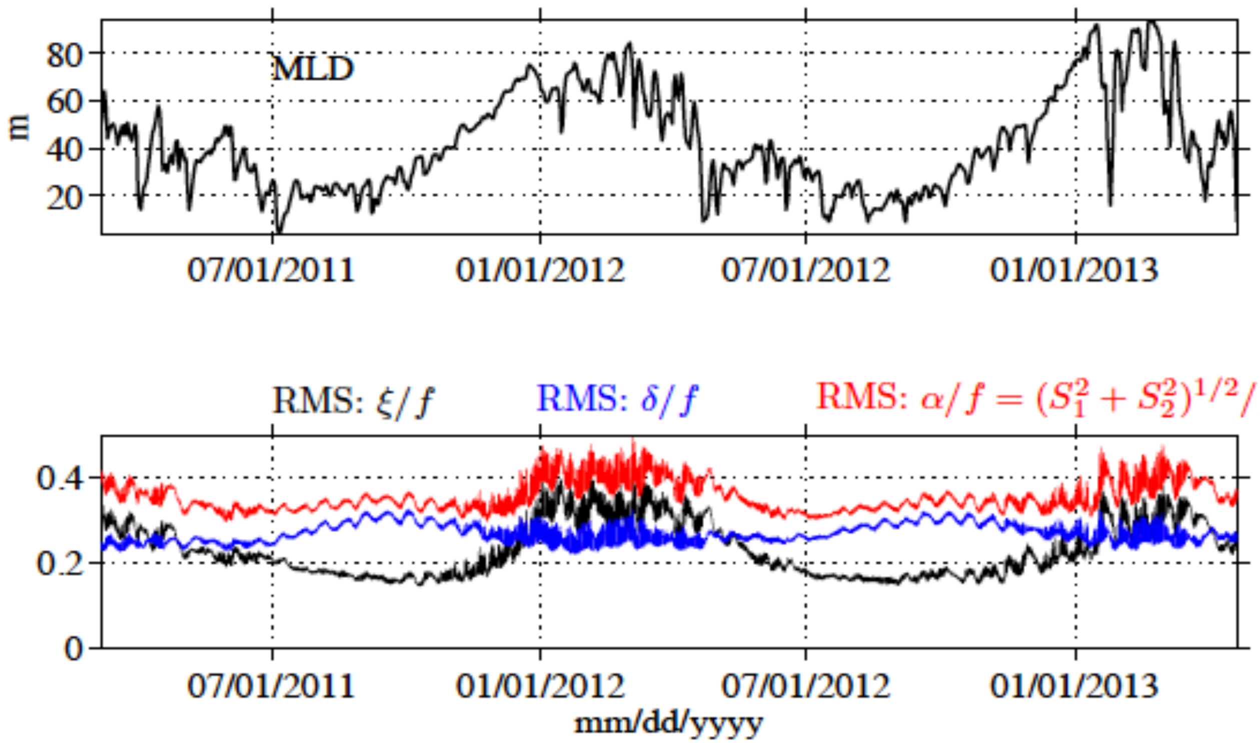
Dynamical fields display a strong seasonality!

Impacts the waves ($\omega > f$)

Square of the relative vorticity:
$\xi^2 = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})^2$

Square of the horizontal flow divergence:
$\delta^2 = (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2$

Square of the horizontal strain:
$S^2 = S_1^2 + S_2^2 = (\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y})^2 + (\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y})^2$



Relative vorticity is larger in winter (0.4) and divergence larger in summer (0.3)!

Hector Torres: JPL 2016 … **in the Eastern North Pacific** (10°N-30°N, -130°W—110°W)

Seasonality is different with the wave part is removed!

and kinematic properties: $\boxed{\text{daily-averaged}}$

Square of the relative vorticity:
$\xi^2 = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})^2$

Square of the horizontal flow divergence:
$\delta^2 = (\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y})^2$

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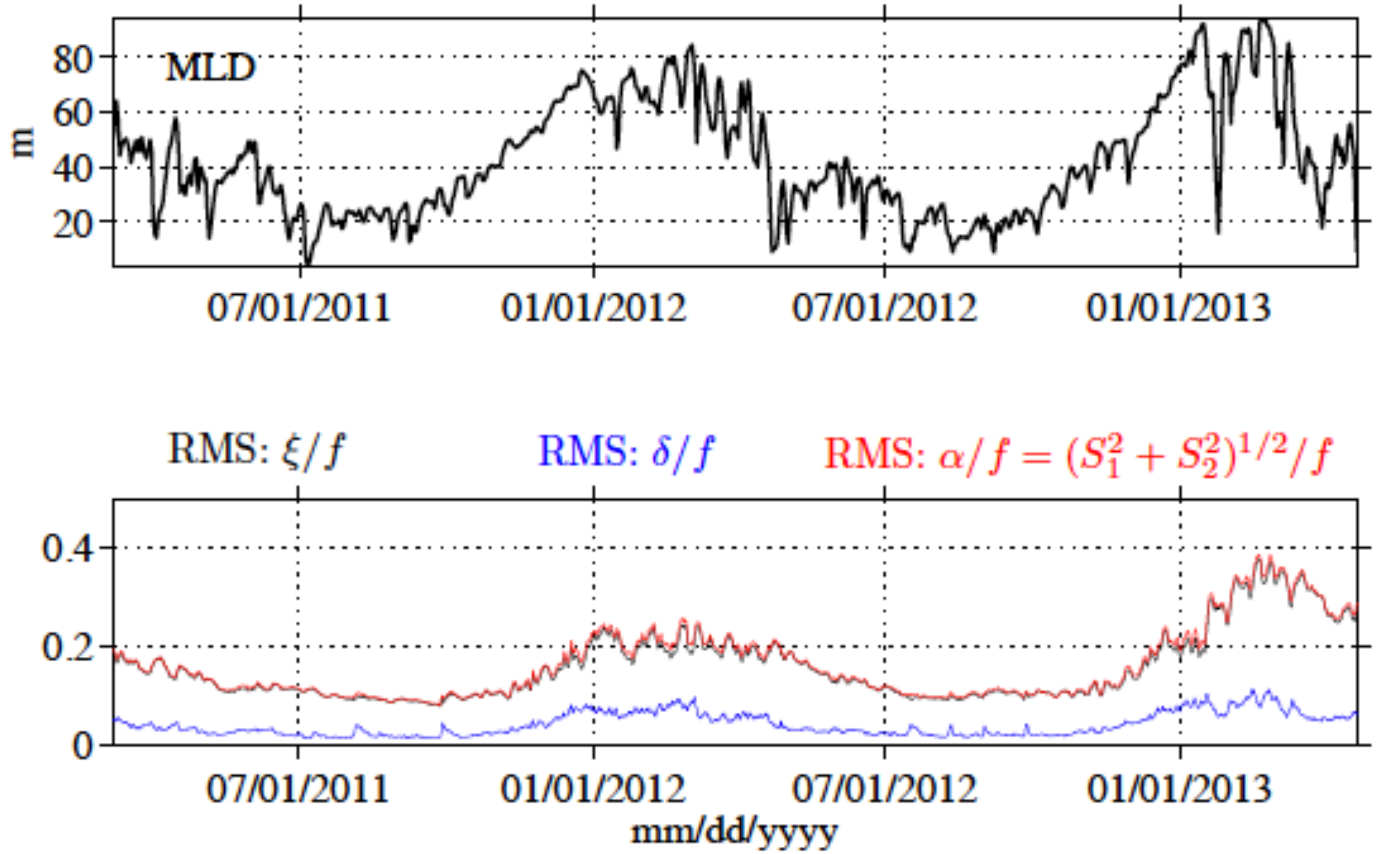
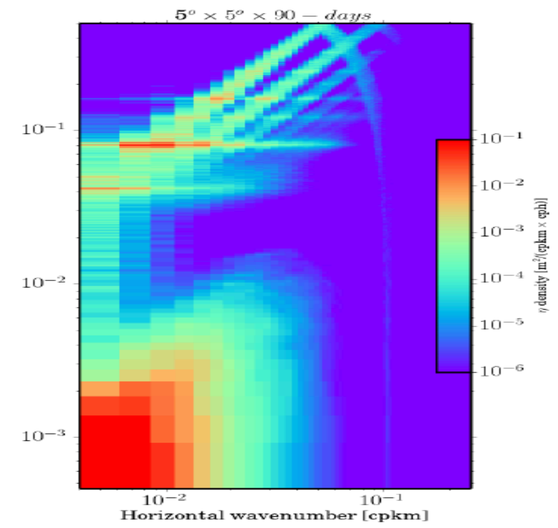


Hector Torres: JPL 2016 … **in the Eastern North Pacific** (10°N-30°N, -130°W—110°W)

WAVE SPECTRUM:



- WHY THESE DISCRETE KE BANDS ?
- WHY A SEASONALITY ENHANCED IN SUMMER ?
- SINCE WAVE FORCINGS ARE LARGE-SCALE, WHAT PRODUCES WAVES AT SMALL SCALES?

LET US CONSIDER THE DYNAMICS OF THESE WAVES …

Waves in a stratified and rotating flow

We assume the flow is hydrostatic at first order, and R_{ol}<<1 (nonlinear terms are negligeable), and 1/fT=O(1). The resulting equations are:

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho_0}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho_0}\frac{\partial p'}{\partial y}$$

$$0 = -\frac{1}{\rho_0}\frac{\partial p'}{\partial z} - \frac{g}{\rho_0}\rho'$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial \rho'}{\partial t} + w\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z} = 0.$$

Let us find the equation for *p'* …

We use $N^2(z) = -\frac{g}{\rho_0}\frac{\mathrm{d}\bar{\rho}}{\mathrm{d}z} > f^2 > 0.$

This leads to:

$$\frac{\partial}{\partial t}\left[\left(\frac{p'_z}{N^2}\right)_{ztt} + f^2\left(\frac{p'_z}{N^2}\right)_z + \Delta p'\right] = 0$$

<u>1 – let us assume N^2= cst</u>

=> $$\frac{\partial}{\partial t}\left[p'_{zztt} + f^2 p'_{zz} + N^2 \Delta p'\right] = 0$$

Boundary conditions (vertical velocity, W, is zero at the boundaries):

W= 0 , at z=0,-H

which leads (from the density eq.) to: $\frac{\partial \rho'}{\partial t} = 0$

and therefore (using the hydrostatic approximation) to $\frac{\partial p'}{\partial z} = 0$ at z=0,-H

Using p'(x,y,z,t) = P(z).$e^{-i(k.x+l.y-\omega t)}$ leads to

$$P_{zz} - \frac{k^2+l^2}{f^2-\omega^2} N^2 . P = 0$$

P can be decomposed in a Fourier series that satisfies the BCs [$P_z = 0$ AT Z=0,-H];

This leads to P(z) = $P_m . \cos mz$. It satisfies the BCs if m=$\frac{n\pi}{H}$, $n = 1, 2, 3, \ldots$ M

Then we get the following dispersion relation(for each m):

$$\omega^2 = f^2 + N^2 \frac{k^2+l^2}{m^2}$$

- **Strong similitudes between the SW system and the equations for p' and ω^2 ... [C_o^2 =gH is replaced by N^2/m^2. But m can vary since several vertical scales are allowed]. M vertical normal modes and M dispersion relations!**

- **Note that N^2/f^2m^2 is the equivalent of L_d^2 (see before) as gH/f^2 was called R^2**

- **ω^2 depends on Bu= N^2K^2/f^2m^2 = L_d^2/L^2 is also called the Burger number**

$$=> \omega^2 = f^2.(1+Bu)$$

GROUP VELOCITY: $\overrightarrow{C_g} = \nabla_k \omega$

$$C_{gx} = \frac{\partial \omega}{\partial k} = \frac{N^2 k}{\omega m^2} = \frac{\omega^2 - f^2}{\omega} \frac{k}{k^2 + l^2}$$

$$C_{gy} = \frac{\partial \omega}{\partial l} = \frac{N^2 l}{\omega m^2} = \frac{\omega^2 - f^2}{\omega} \frac{l}{k^2 + l^2}$$

$$C_{gz} = \frac{\partial \omega}{\partial m} = -\frac{N^2 (k^2 + l^2)}{\omega m^3} = -\frac{\omega^2 - f^2}{\omega m}$$ (downward propagation if $\omega > 0$)

Note that: $\vec{C}_g . \vec{K}$=0 !

Example:

$m = \frac{\pi}{200}$ m^{-1}, $\omega = 1.02 f$, leads to $C_{gz} \sim -22\, m.d^{-1}$

These calculations are valid for near-inertial motions, tidal motions and, higher frequency motions!

Let us use these calculations to understand the propagation of near-inertial waves in the ocean interior where their energy is available for mixing

Near-inertial waves are mostly generated by fast moving winter storms. They are initially large-scale.

Initially, NIW are mostly within the surface mixed-layer.

Then they propagate in the ocean interior

Let us analyse the characteristics of their energy propagation using the previous analytical results and data from Ocean Station Papa.

WHAT FRACTION OF THE WIND WORK FLUXES IN THE OCEAN INTERIOR?

(ALFORD ET AL. JPO 2012)

DEPTH PENETRATION OF WIND-GENERATED NEAR-INERTIAL WAVES USING THE ANALYSIS OF TWO YEARS OF ADCP (154m, 227m, 800m) DATA AT OCEAN STATION PAPA (Northern end of the atmospheric storm track) TO BETTER UNDERSTAND THEIR ROLE IN MIXING THE DEEP OCEAN

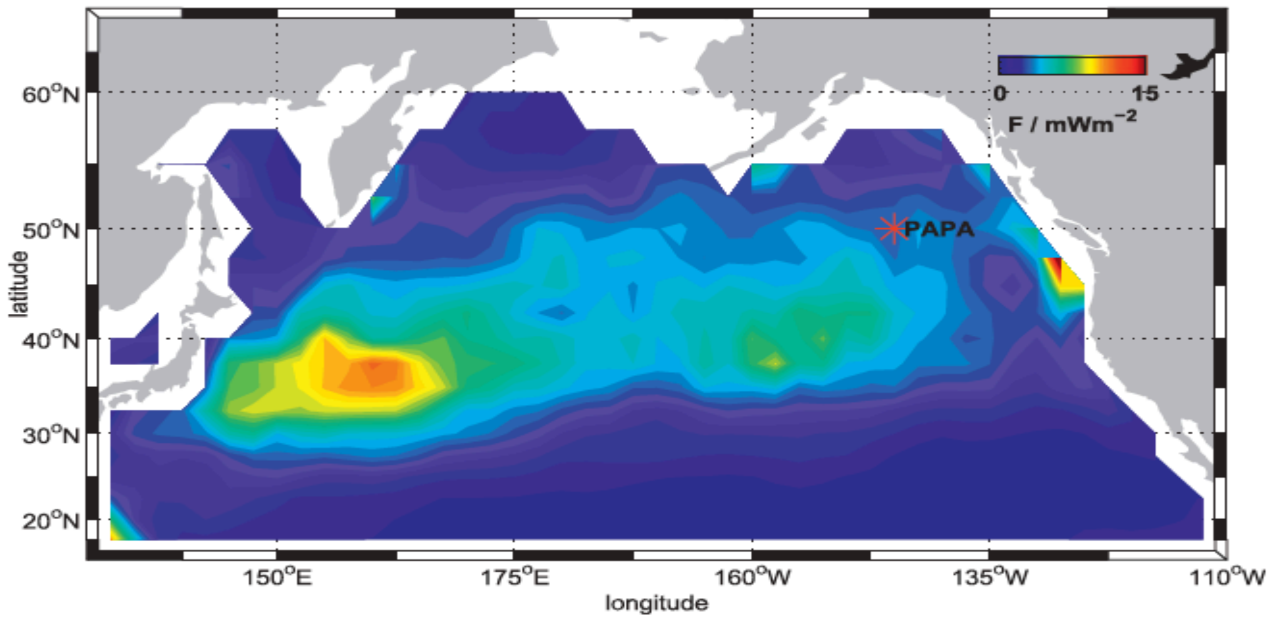


FIG. 1. Map of the north Pacific showing the location of the Papa moorings. Colors give annual-mean energy flux computed from NCEP winds and the Pollard–Millard slab model (Alford 2003b).

INTERNAL WAVE ENERGY IS PROPAGATING DOWNWARD

TWO YEARS OF ADCP DATA (June 2008 to June 2010) AT OCEAN STATION PAPA FROM SURFACE DOWN TO 800M (ALFORD ET AL. JPO'12)

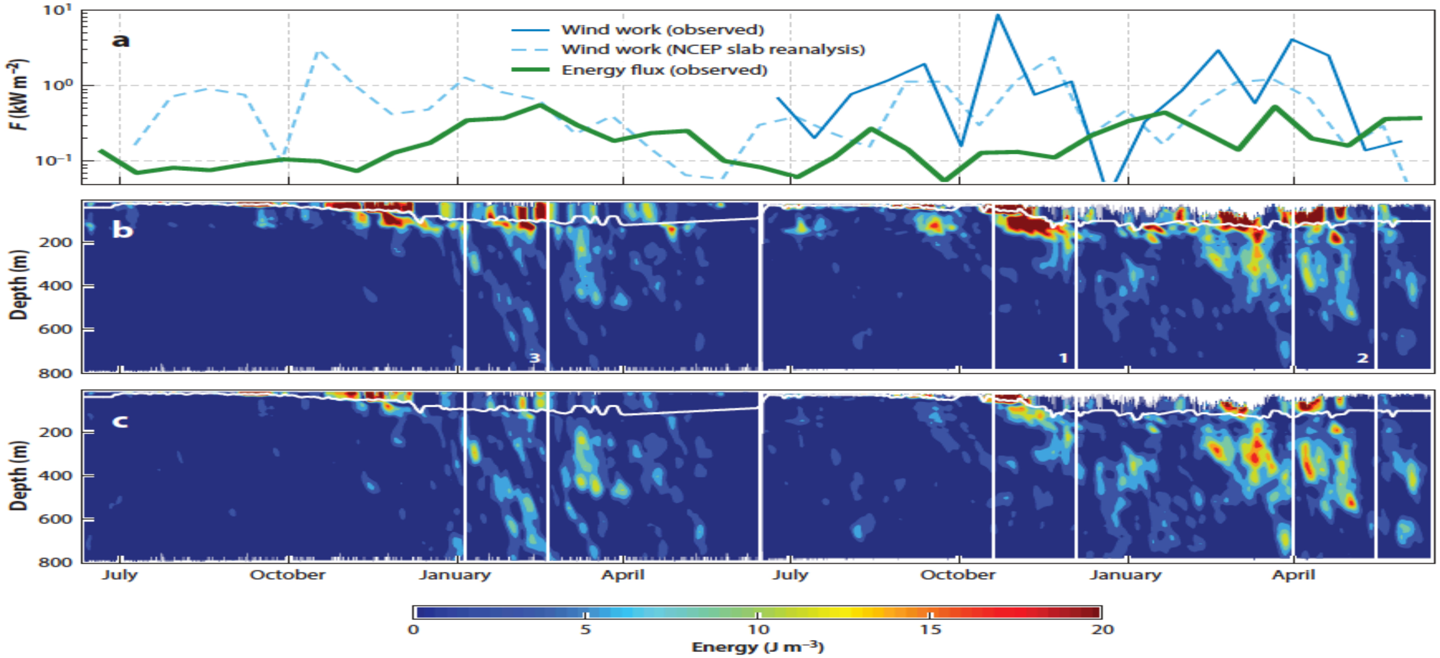


Figure 5

Near-inertial waves at Ocean Station Papa. (*a*) Wind work from observations (*solid blue line*) and from Equation 2 forced with reanalysis winds (*dashed blue line*), along with observed energy flux computed as the mean of energy from 600 to 800 m multiplied by $c_{gz} = 1.03 \times 10^{-4}$ m s^{-1} (9 m d^{-1}; *thick green line*). All three lines have been smoothed over 20 days. (*b*) Near-inertial kinetic energy for the whole two-year record. (*c*) The same as panel *b* but additionally accounting for WKB refraction. In panels *b* and *c*, the mixed-layer depth is overplotted in white. Abbreviations: NCEP, National Centers for Environmental Prediction; WKB, Wentzel, Kramers, Brillouin. Modified from Alford et al. (2012).

THE DOWNWARD KE FLUX CAN BE ESTIMATED AS : **$F(z)=C_{GZ}(z).KE(z)$**,

WITH $C_{gz} = -\frac{\omega^2 - f^2}{\omega m}$ [requires to know ω and m]

FREQUENCY SPECTRUM AVERAGED BETWEEN 120 M AND 800 M FOR THE TWO YEAR PERIOD.

WAVES REPRESENT 87% OF THE TOTAL KE (47% FOR THE NIWs)

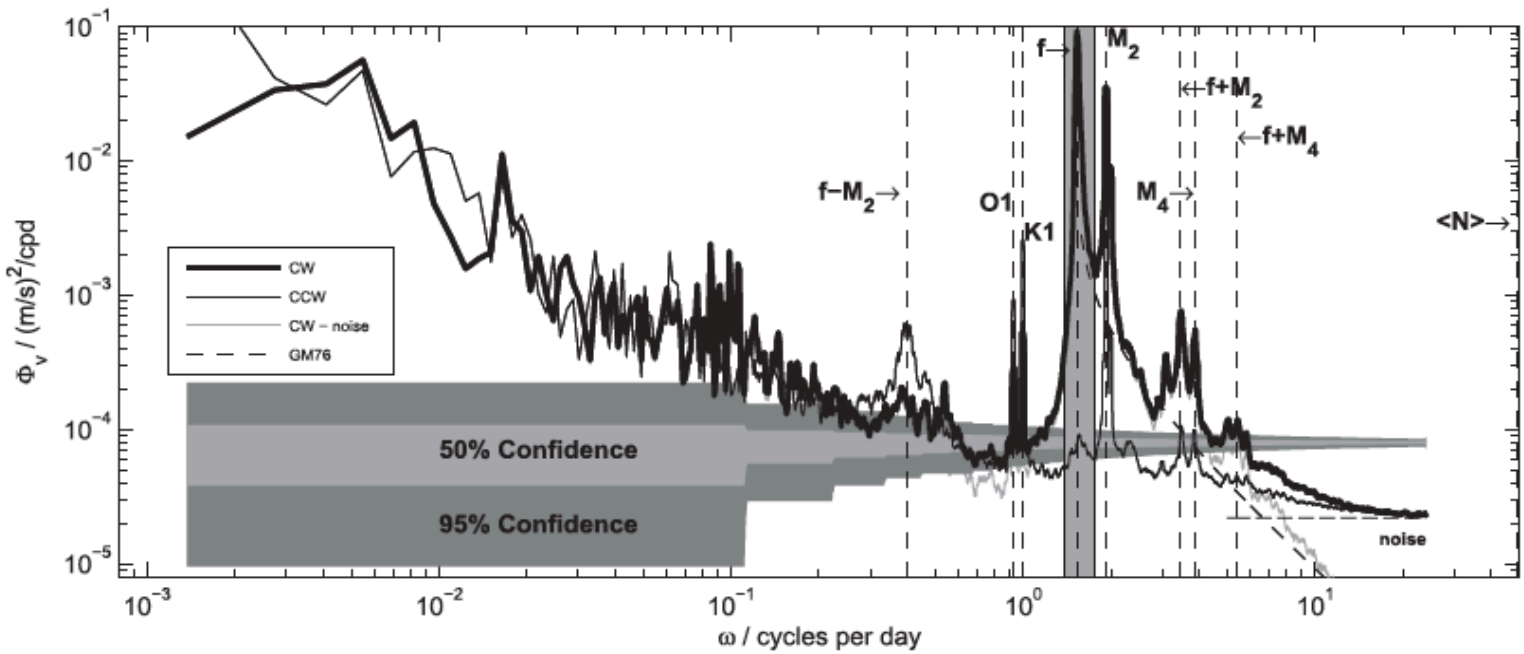
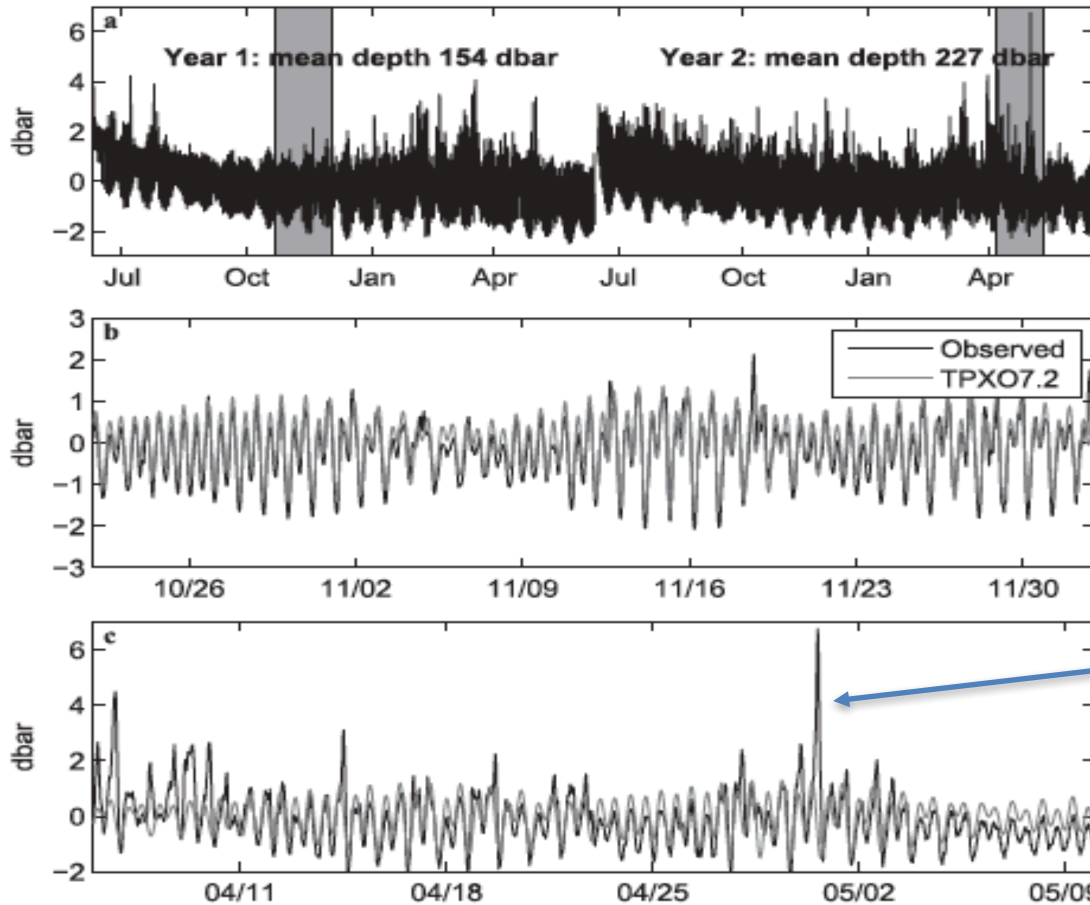


FIG. 8. Rotary frequency spectrum of WKB-scaled velocity. Multitaper spectral estimates using three tapers from 120- to 800-m depths are averaged together and then smoothed over even intervals in log frequency (see text); 95% and 50% confidence intervals are indicated. The near-inertial band used in bandpass filters and in later spectral closeup plots is shown in light gray. The GM76 model spectrum is indicated with a dashed curve. The noise floor of the ADCP measurements is indicated at high frequency with a horizontal dashed line. The light gray line is the measured CW spectrum with this noise floor subtracted.

Pressure measured at the top ADCP (154 m and 227 m).

The gray lines in (b) and (c) show the prediction of tidal (barotropic) elevation (see Jinbo's presentation next week ...)



Mooring dived 6.5m below its usual depth

FIG. 3. Pressure measured at the top ADCP (a) for the whole record and (b),(c) for the two sample periods indicated in (a) in gray. The time-mean for each year, indicated in (a), is subtracted before plotting. In (b),(c), the tidal elevation prediction from TPXO7.2 is also plotted in gray.

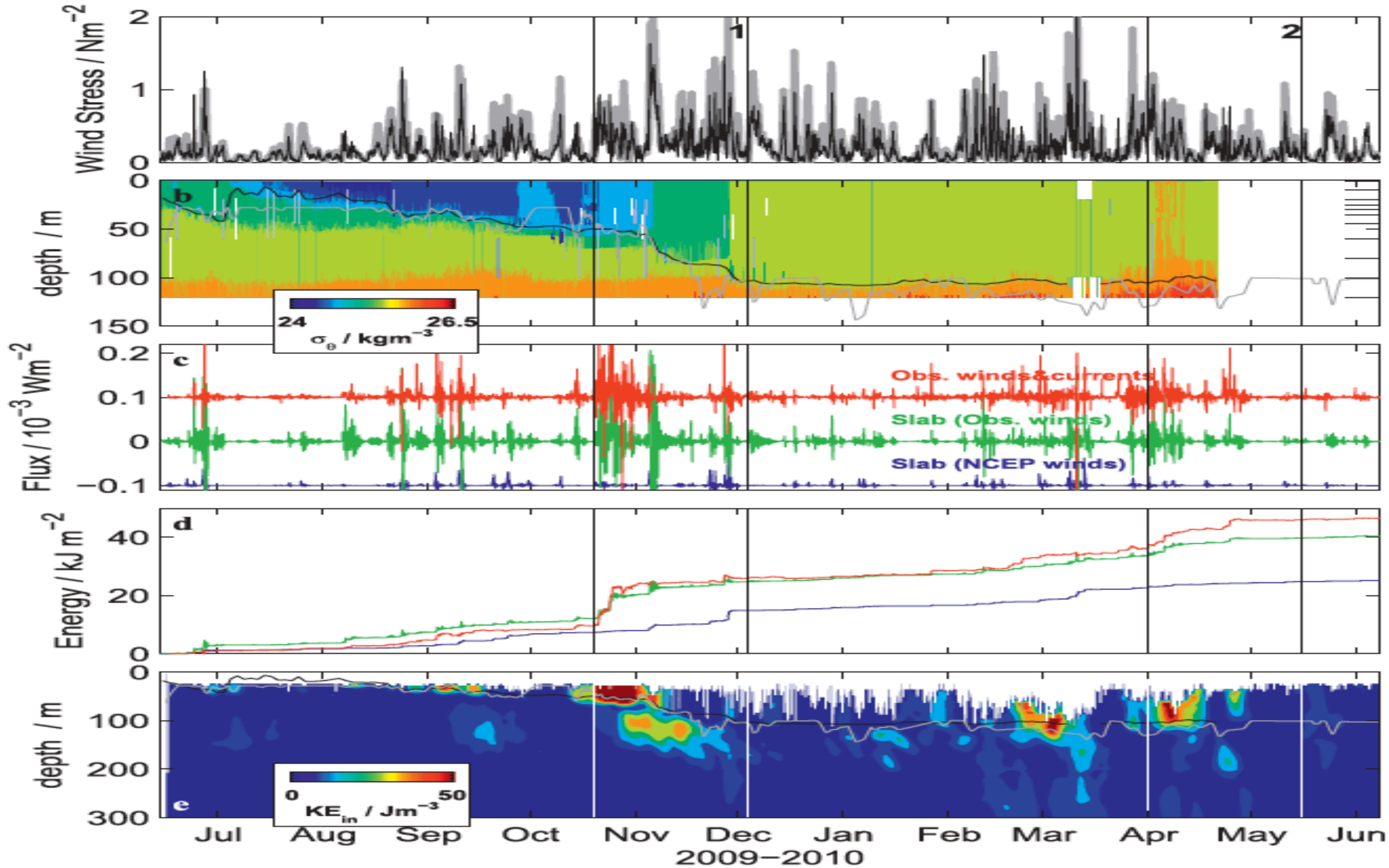
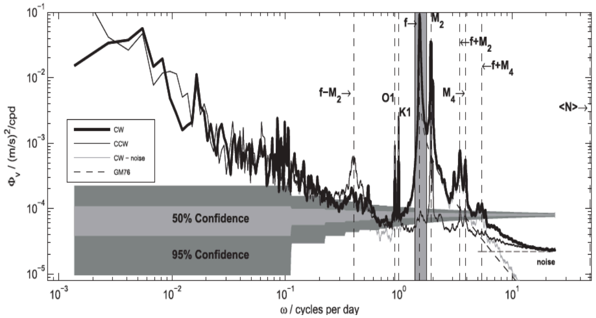


FIG. 4. (a) Wind stress computed using the Large and Pond (1981) wind stress parameterization from NCEP reanalysis winds (gray) and measured at the surface buoy (black). (b) Potential density measured from Sea-Bird Electronics MicroCATs on the surface buoy (depths indicated at right). Time series of MLD determined from density (black) and shear (gray) are overplotted; see text. (c) Wind work computed from the slab model driven with NCEP and measured winds (blue and green) and from measured winds and inertial currents (red). Successive traces are offset by 0.1. (d) The time integral of (c) showing the cumulative energy input to the mixed layer from each flux estimate. (e) Inertial KE in the upper 300 m. The two MLD estimates from (b) are replotted. Periods examined in detail are indicated with vertical bars.

- Wind work is strongly intermittent. Better estimated using observed winds than NCEP winds (see (c) and (d))!
- Downward KE propagation takes one month to go from 50m to 100m!

NIW FREQUENCIES AT A FUNCTION OF DEPTH:

1) PEAK VALUE IS SLIGHTLY SMALLER THAN F (F_{eff});
2) THERE IS SHIFT TO HIGHER FREQUENCIES AT DEPTH.



A Butterworth filter is applied (forward and backward) to get NIWs in the frequency band {0.9, 1.15}f.

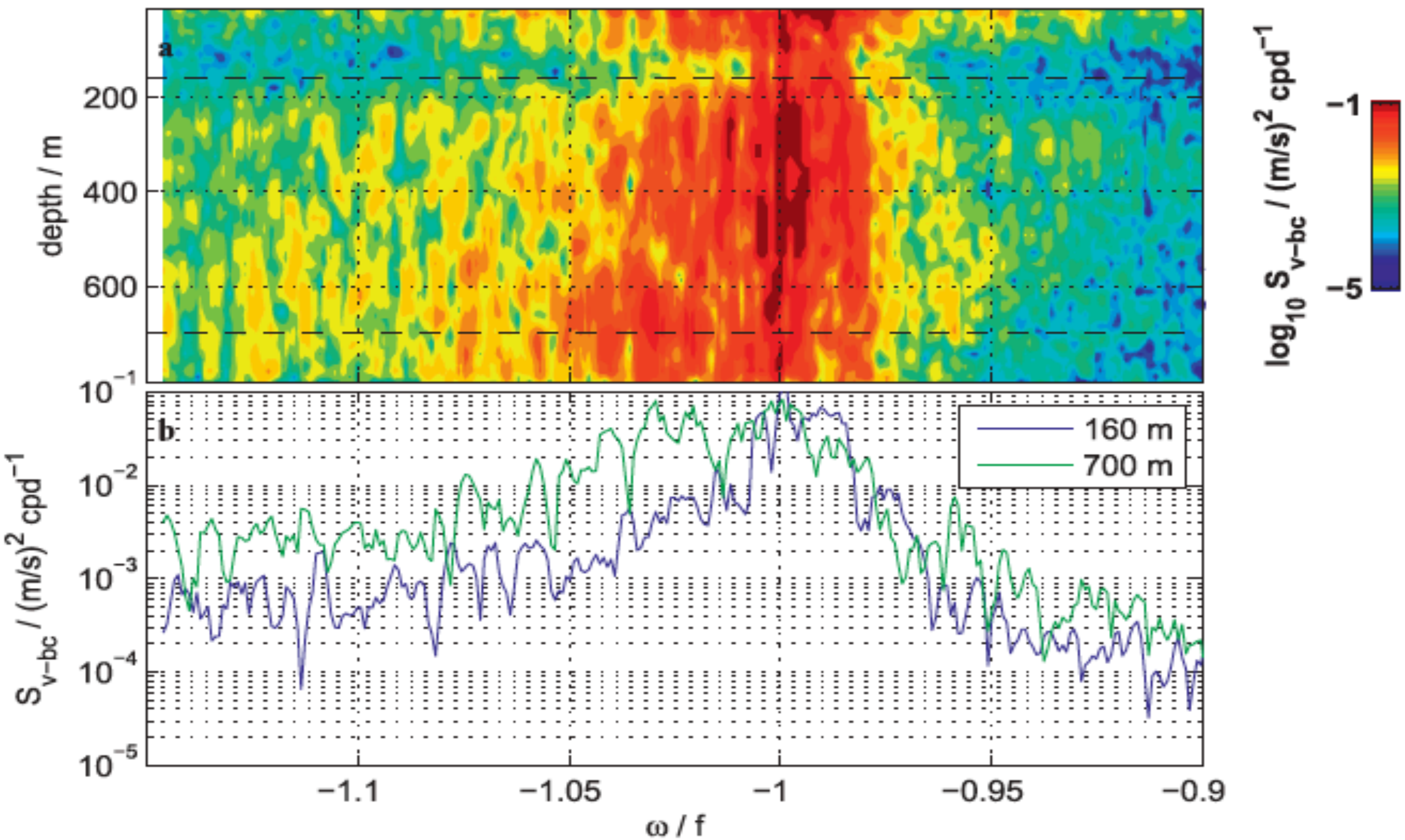


FIG. 9. (a) Rotary frequency spectrum of WKB-scaled velocity vs depth, zoomed in around the CW near-inertial frequency. The horizontal axis is normalized by *f*. Gray lines are the instrument depths in years 1 and 2. (b) Line plots of the spectrum at the two depths shown in (a). Dashed lines in (a) indicate depths of the spectra plotted in (b).

VERTICAL WAVENUMBER: SHEAR IS DOMINANT FOR VERTICAL SCALES OF $\sim 230\ m$

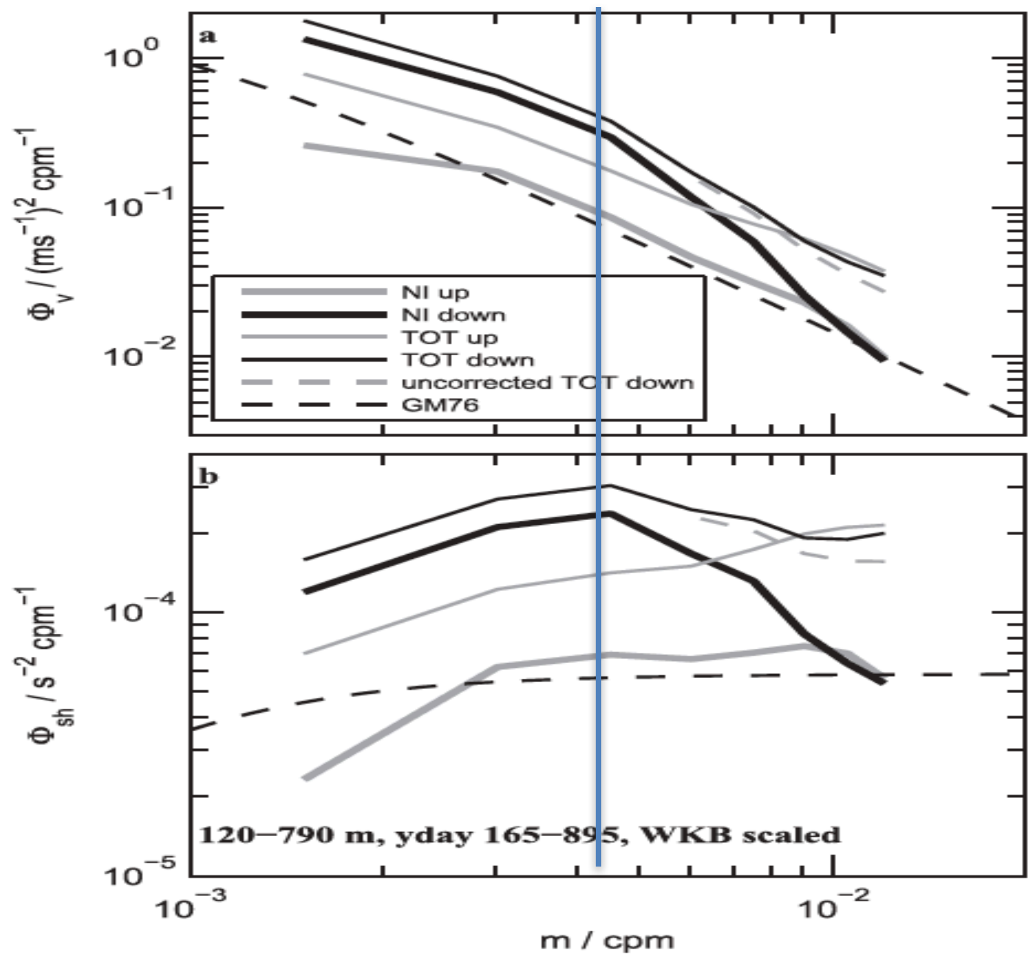


FIG. 10. Vertical wavenumber spectrum of (a) velocity and (b) shear for all frequencies (thin lines) and near-inertial frequencies (thick lines). Data have been WKB stretched and scaled. The GM76 spectrum is indicated in each panel with dashed lines. The effect of correcting the spectrum for the finite wavenumber response of the ADCP is shown (dashed line).

ESTIMATION OF THE VERTICAL GROUP VELOCITY

$$C_{gz} = \frac{\partial \omega}{\partial m} = -\frac{N^2(k^2 + l^2)}{\omega m^3} = -\frac{\omega^2 - f^2}{\omega m}$$

Using: $m = \frac{2\pi}{250}$ m^{-1}, $\omega = 1.02f$,

leads to: $C_{gz} \sim -14\, m.d^{-1}$

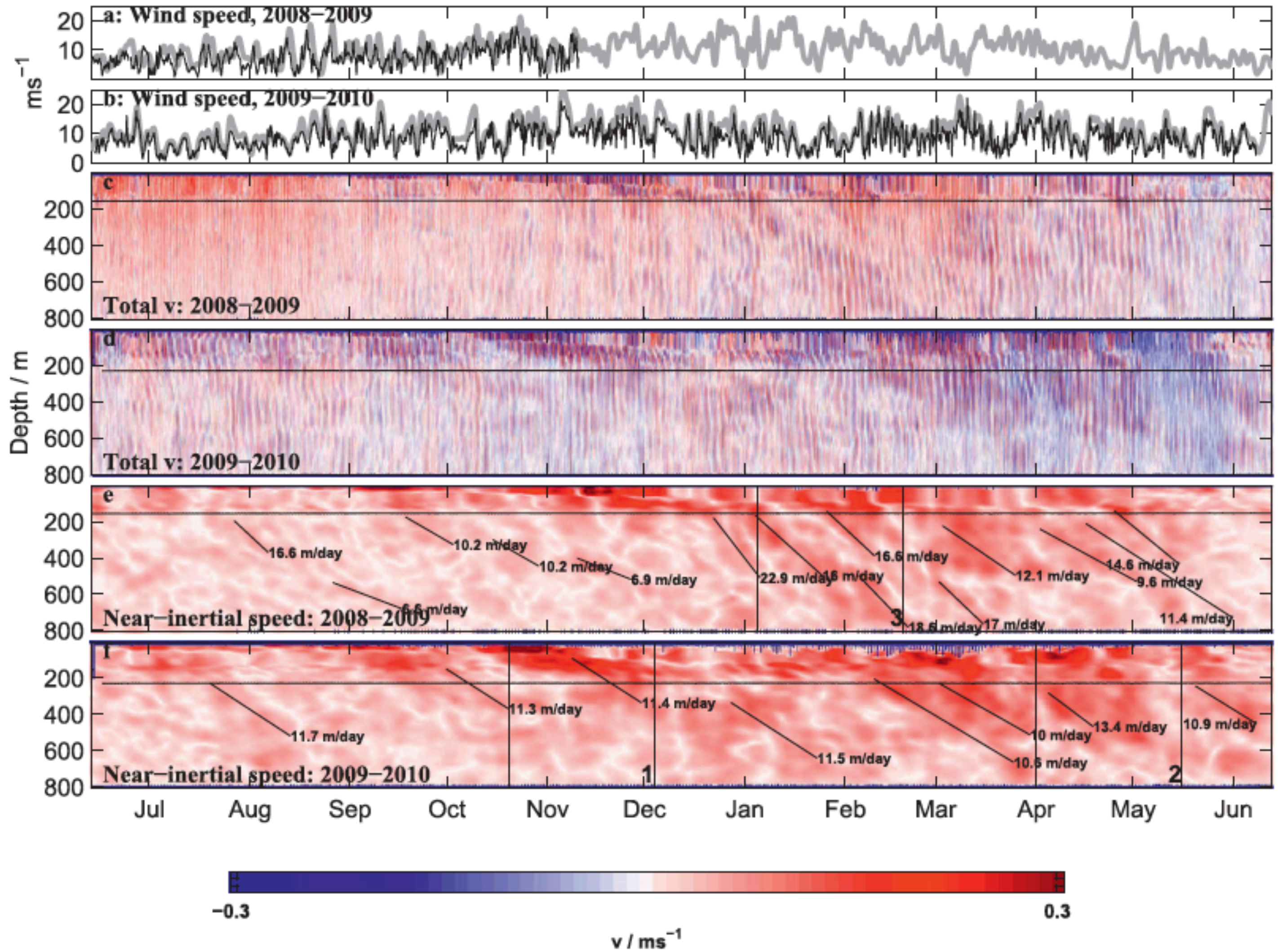
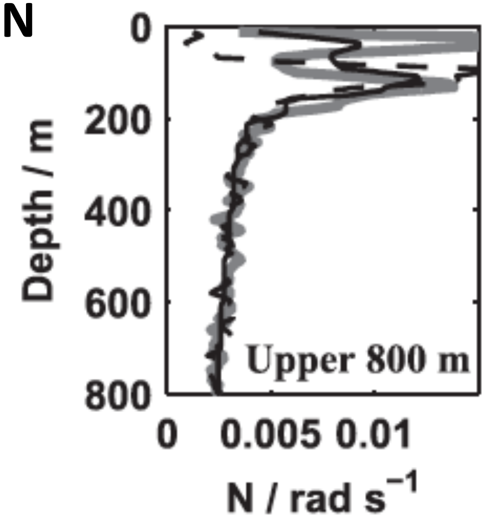


FIG. 2. Wind speed, measured meridional velocity, and near-inertial speed for the entire 2-yr record for (a),(c),(e) the first year and (b),(d),(f) second year. (a),(b) Wind speed measured at the surface buoy (black) and from NCEP reanalysis (gray); (c),(d) meridional velocity; and (e),(f) near-inertial speed obtained by bandpassing (see text). The measured depth of the upper ADCP is indicated in each panel. In (e),(f), downward migration of each identified event (see text) is indicated with downward-sloping lines. The implied downward group velocity of each is indicated. Closeup periods plotted in Figs. 13–15 are shown.

Wentzel-Kramer-Brillouin **APPROXIMATION**



N^2 USUALLY DEPENDS ON Z ($N^2=N^2(Z)$). THIS AFFECTS THE PROPAGATION OF NIW KE. IN THAT CASE WE NEED TO SOLVE:

$$\frac{\partial}{\partial t}\left[\left(\frac{p'_z}{N^2}\right)_{ztt} + f^2\left(\frac{p'_z}{N^2}\right)_z + \Delta p'\right] =$$

Using p'(x,y,z,t) = P(z).$e^{-i(k.x+l.y-\omega t)}$, leads to:

$$\left(\frac{P_z}{N^2}\right)_z - \frac{k^2+l^2}{f^2-\omega^2}P = 0 \quad (1)$$

- ONE SOLUTION IS TO USE THE **NORMAL VERTICAL MODES** FOR P(z) **(SEE NEXT CLASS).** THIS IS POSSIBLE WHEN THE NIW KE IS KNOWN FROM THE SURFACE DOWN TO THE BOTTOM.

- ONE SOLUTION IS TO USE THE **WKB APPROXIMATION** (POSSIBLE IF N^2 IS SMOOTHLY VARYING ON THE VERTICAL = N^2 is constant locally).

USING **P(Z)= P_m(z).cos**$[m(z).z]$ IN (1) LEADS TO:

$$m^2(z) = \frac{N^2(z)\left(k^2+l^2\right)}{\omega^2 - f^2}$$

WKB APPROXIMATION

$$m^2(z) = \frac{N^2(z)(k^2 + l^2)}{\omega^2 - f^2}$$

So, $\boldsymbol{m}\ (\boldsymbol{z})$ **is roughly proportional to N(z).** This means that the vertical scales become small for large N and large for small N.

Note that the vertical velocity group is still equal to:

$$C_{gz} = \frac{\partial \omega}{\partial m} = -\frac{\omega^2 - f^2}{\omega m\ (z)}$$

with frequency, ω unchanged with z (see next week).

On the other hand, the vertical energy flux has to be constant (there is no energy accumulation at a given level). So

$$F(z) = P_m^2(z).\, C_{gz}(z) = -P_m^2(z).\frac{\omega^2 - f^2}{\omega m(z)} = cst.$$

Using **P(Z)= P$_m$(z).cos**$[\boldsymbol{m}(\boldsymbol{z}).\, \boldsymbol{z}]$ (

$\boldsymbol{P_m^2}\ (\boldsymbol{z})$**and** $\boldsymbol{m}(\boldsymbol{z})$ **are roughly proportional to N(z). Thus, for each mode** $\boldsymbol{m}$**, where N(z) is large,** $\boldsymbol{P_m^2}\ (\boldsymbol{z})$**and** $\boldsymbol{m}(\boldsymbol{z})$ **should be large and the opposite when where N(z) is small (see next figures…)**

NEAR-INERTIAL WAVES: STRONG VERTICAL SHEAR (HIGH BAROCLINIC MODES)

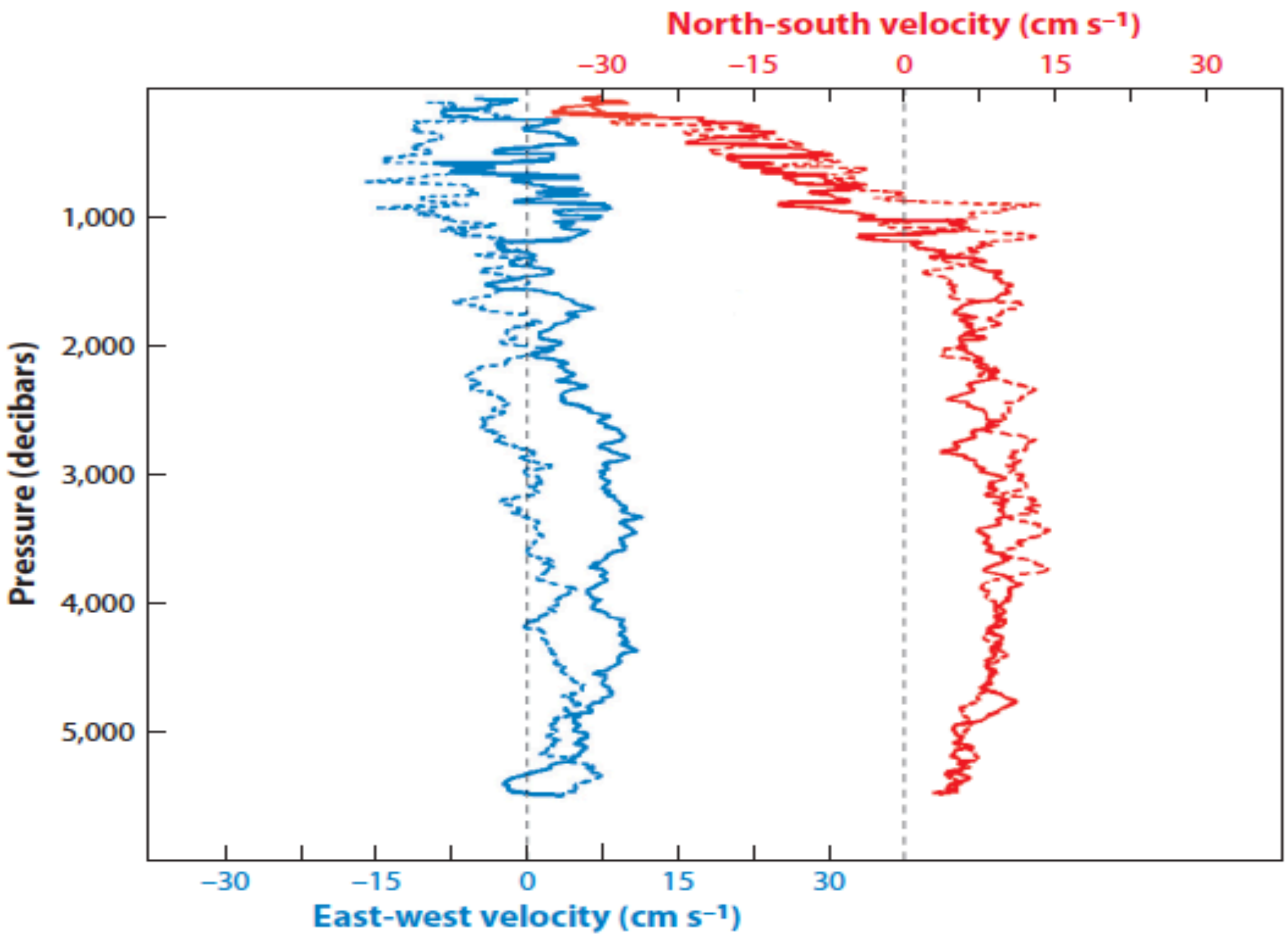
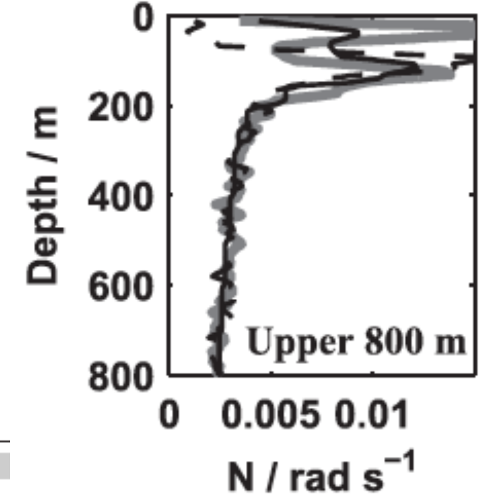


Figure 3

Two profiles (*dashed* and *solid lines*) of east-west (*blue*) and north-south (*red*) velocity taken at an interval of half an inertial period, showing high-wavenumber near-inertial motions. Modified from Leaman & Sanford (1975).

$$\frac{P_m^2(z)}{N(z)} = \text{cst}$$

WKB SCALED MEANS SCALED BY $N(z)$



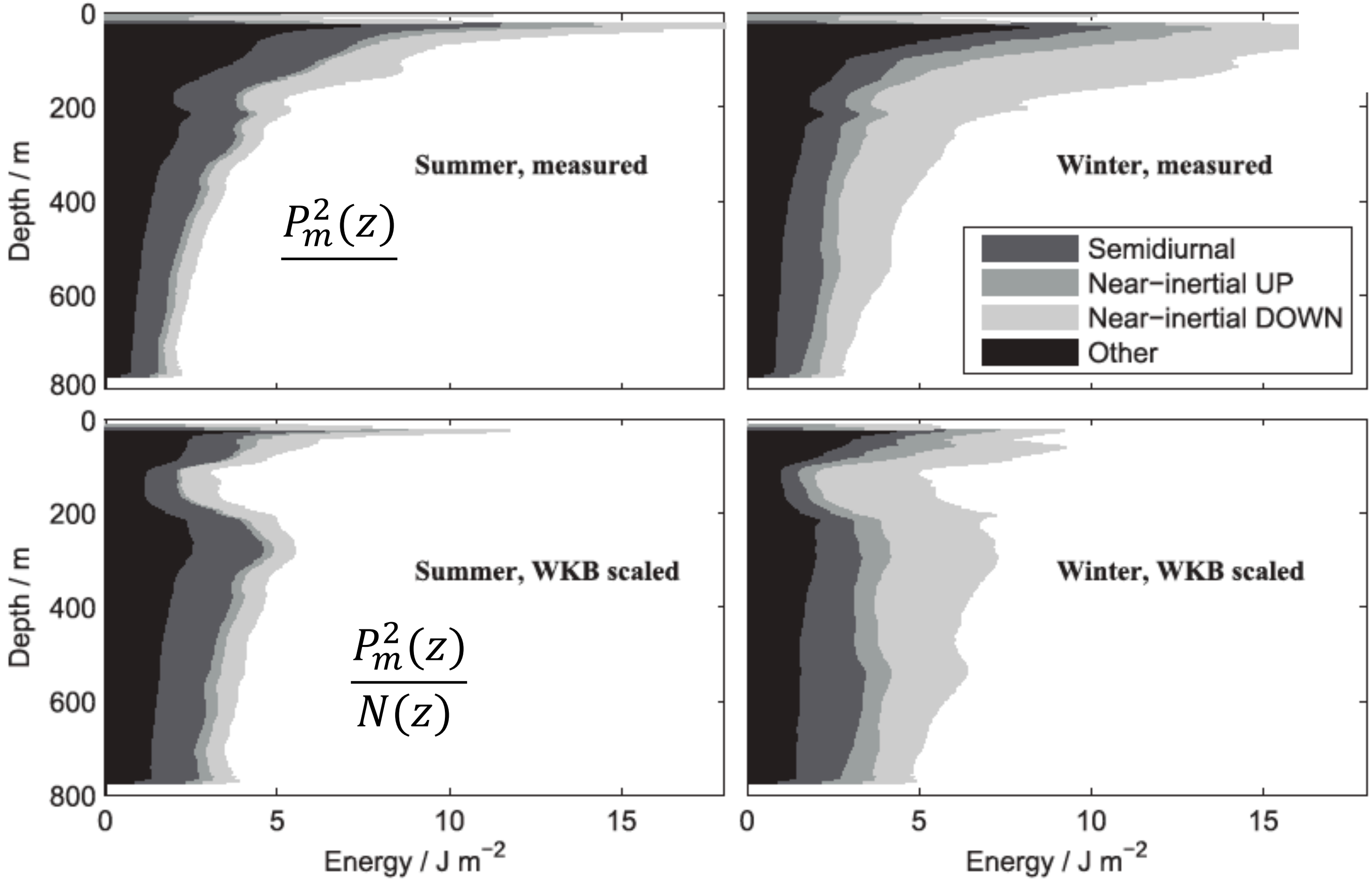


FIG. 12. Energy profiles for (left) summer and (right) winter, (top) measured and (bottom) WKB scaled.

From Alford et al., JPO 2012

TABLE 1. Wind work and observed near-inertial flux estimates ($\times 10^{-3}$ W m^{-2}).

	Year 1	Year 2
Wind work estimates		
Observed currents: observed winds	—	1.53
Slab model: observed winds	—	1.30
Slab model: NCEP winds	1.30	0.93
Observed estimates of downward energy flux		
$\overline{c_{gz}}\mathrm{KE}_{\mathrm{in}}^{\mathrm{WKB}}$ (downward only: 200–600 m)	0.19	0.29
$\overline{c_{gz}}\mathrm{KE}_{\mathrm{in}}^{\mathrm{WKB}}$ (downward only: >600 m)	0.16	0.19
Summed over events: $\Sigma_i c_g^i \mathrm{KE}^i$	0.23	0.21
Wavenumber–frequency spectrum	0.23	0.50

Downward KE flux is estimated using $C_{gz} \sim -1.5\times 10^{-4}\ m.s^{-1}$ and the WKB scaled KE

THE DOWNWARD NIW KE FLUX IS BETWEEN 12% AND 30% OF THE WIND WORK.

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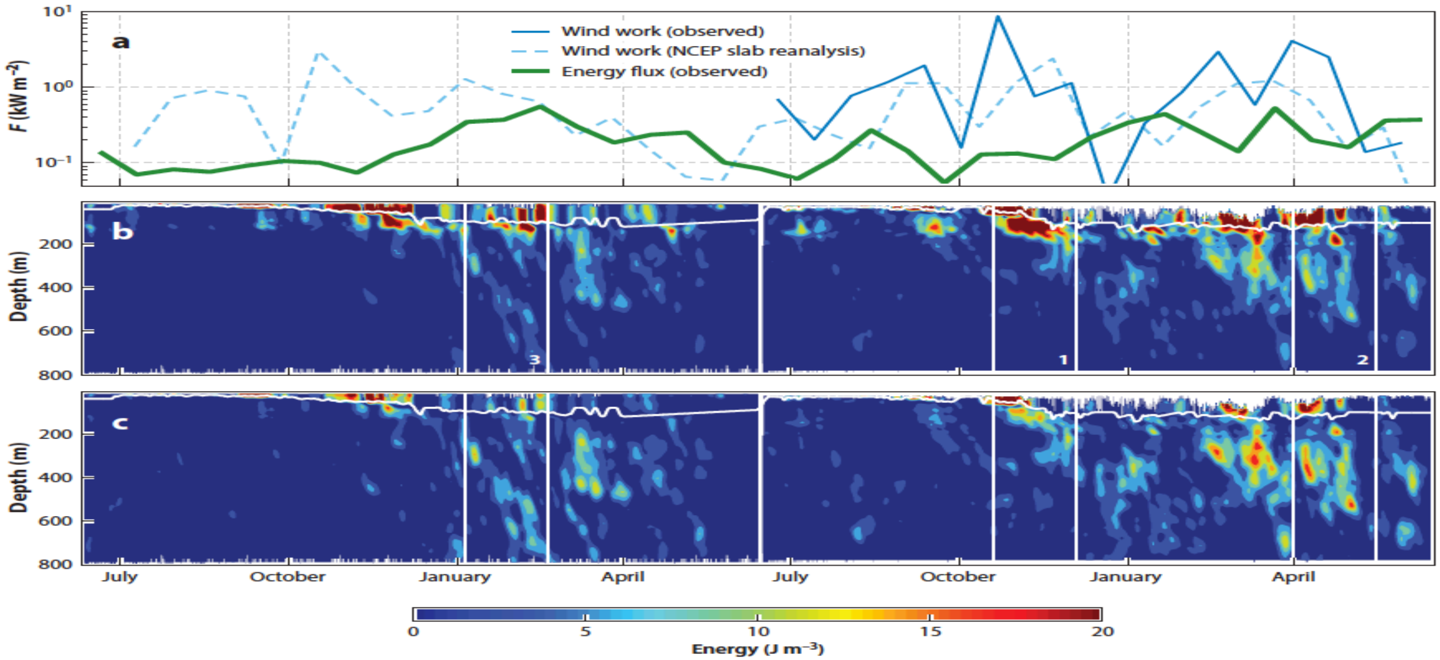

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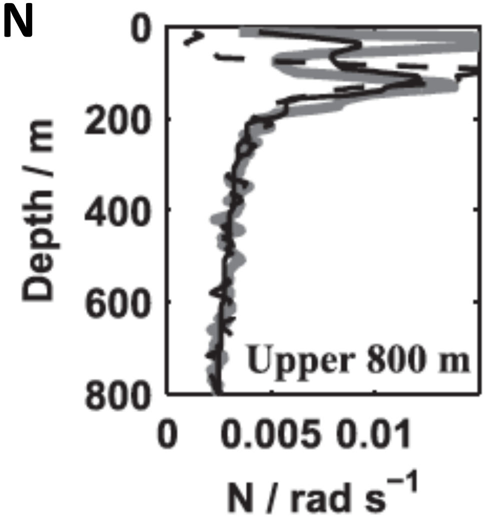

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WKB APPROXIMATION

$$m^2(z) = \frac{N^2(z)(k^2 + l^2)}{\omega^2 - f^2}$$

So, $\boldsymbol{m}\ (\boldsymbol{z})$ **is roughly proportional to N(z).** This means that the vertical scales become small for large N and large for small N.

Note that the vertical velocity group is still equal to:

$$C_{gz} = \frac{\partial \omega}{\partial m} = -\frac{\omega^2 - f^2}{\omega m\ (z)}$$

with frequency, ω unchanged with z (see next week).

On the other hand, the vertical energy flux has to be constant (there is no energy accumulation at a given level). So

$$F(z) = P_m^2(z).\,C_{gz}(z) = -P_m^2(z).\,\frac{\omega^2 - f^2}{\omega m(z)} = cst.$$

Using **P(Z)= $\mathbf{P_m}$(z).cos**$[\boldsymbol{m}(\boldsymbol{z}).\,\boldsymbol{z}]$ (

$\boldsymbol{P_m^2}\ (\boldsymbol{z})$**and** $\boldsymbol{m}(\boldsymbol{z})$ **are roughly proportional to N(z). Thus, for each mode** $\boldsymbol{m}$**, where N(z) is large,** $\boldsymbol{P_m^2}\ (\boldsymbol{z})$**and** $\boldsymbol{m}(\boldsymbol{z})$ **should be large and the opposite when where N(z) is small (see next figures…)**

NEAR-INERTIAL WAVES: STRONG VERTICAL SHEAR (HIGH BAROCLINIC MODES)

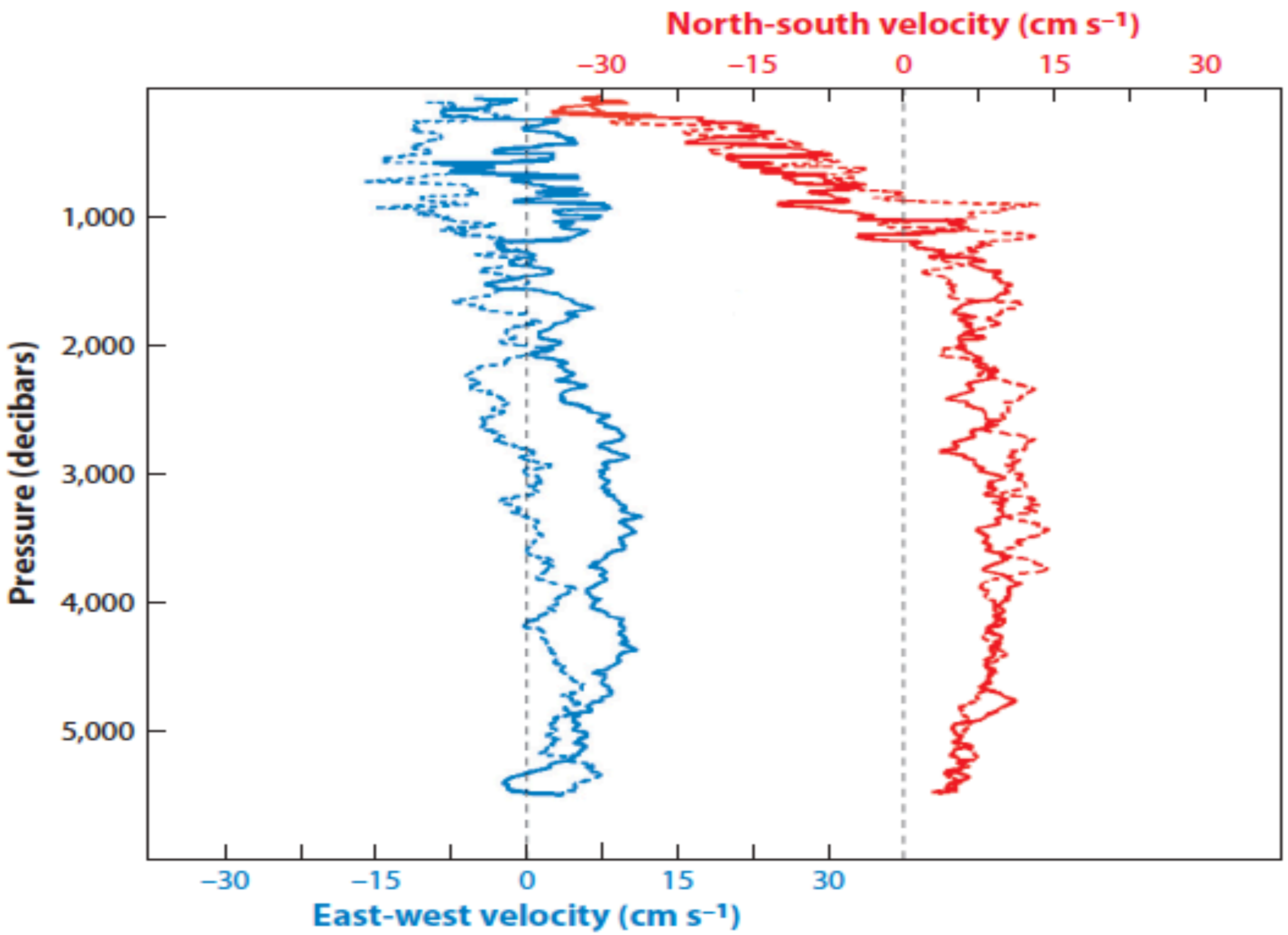
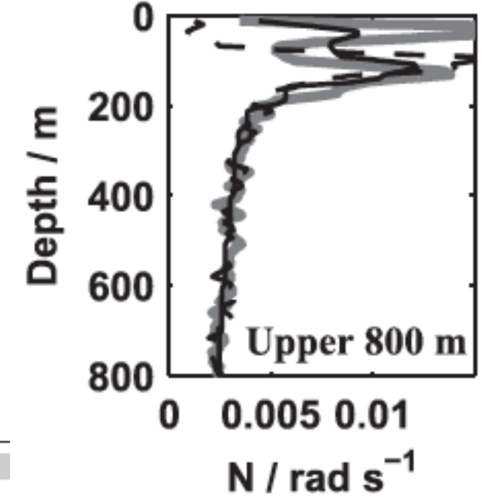


Figure 3

Two profiles (*dashed* and *solid lines*) of east-west (*blue*) and north-south (*red*) velocity taken at an interval of half an inertial period, showing high-wavenumber near-inertial motions. Modified from Leaman & Sanford (1975).

$$\frac{P_m^2(z)}{N(z)} = \text{cst}$$

WKB SCALED MEANS SCALED BY $N(z)$



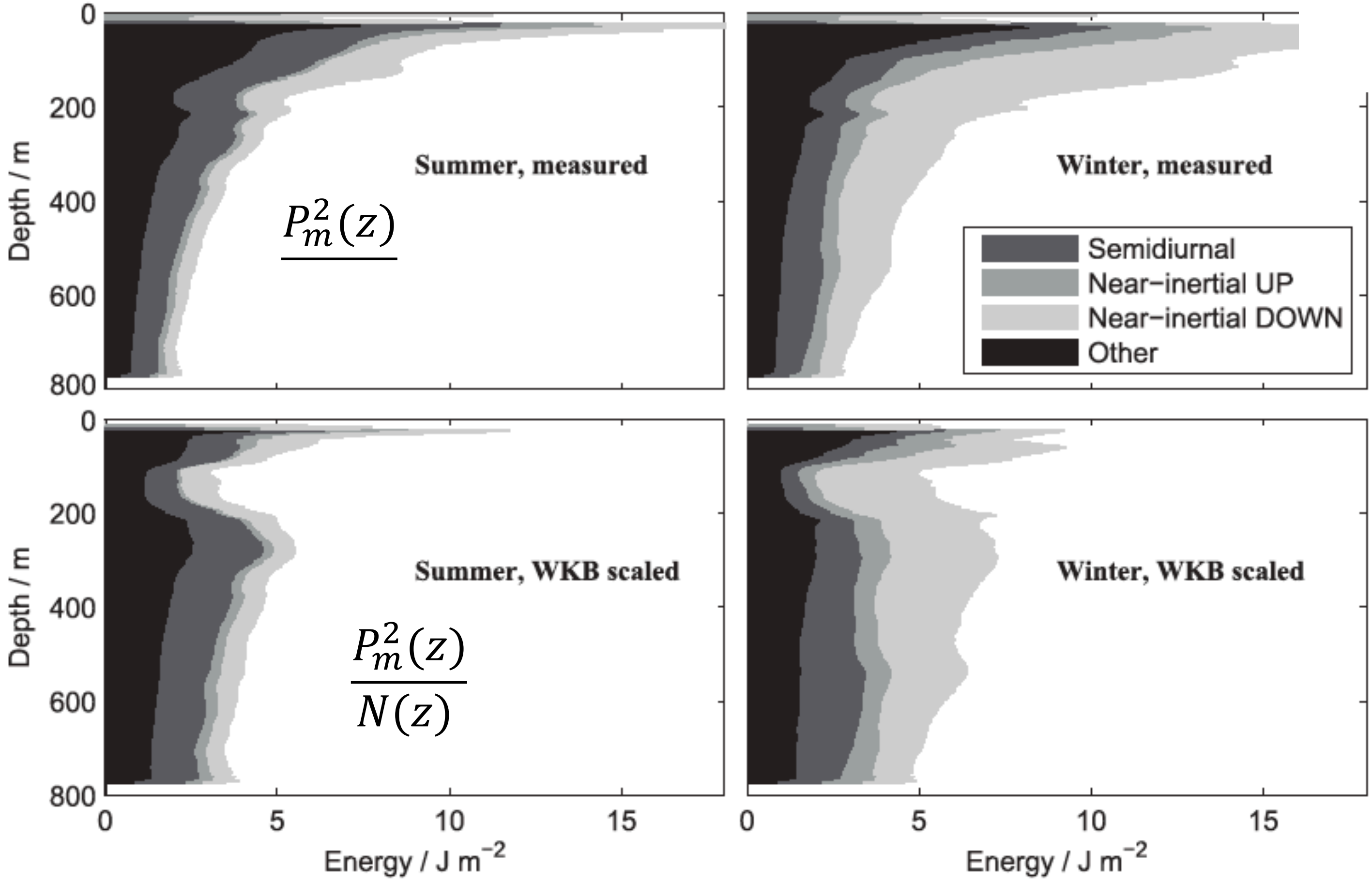
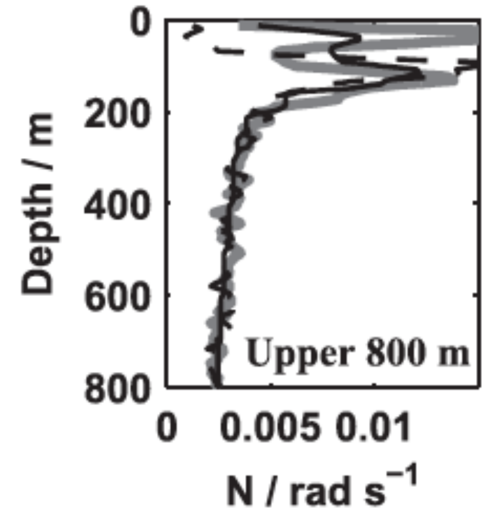


FIG. 12. Energy profiles for (left) summer and (right) winter, (top) measured and (bottom) WKB scaled.

N^2 USUALLY DEPENDS ON Z (**$N^2=N^2(Z)$).** THIS AFFECTS THE PROPAGATION OF NIW KE. IN THAT CASE WE NEED TO SOLVE:

$$\frac{\partial}{\partial t}\left[\left(\frac{p'_z}{N^2}\right)_{ztt} + f^2\left(\frac{p'_z}{N^2}\right)_z + \Delta p'\right] =$$

Using p'(x,y,z,t) = P(z).$e^{-i(k.x+l.y-\omega t)}$, leads to:



$$\left(\frac{P_z}{N^2}\right)_z - \frac{k^2+l^2}{f^2-\omega^2}P = 0 \quad (1)$$

- **A MORE PRECISE SOLUTION IS TO USE THE NORMAL VERTICAL MODES FOR P(z). THIS IS POSSIBLE WHEN THE NIW KE IS KNOWN FROM THE SURFACE DOWN TO THE BOTTOM.**

- ANOTHER SOLUTION IS TO USE THE **WKB APPROXIMATION** (POSSIBLE IF N^2 IS SMOOTHLY VARYING ON THE VERTICAL = N^2 is constant locally).

USING P(Z)= P_m(z).$\cos[m(z).z]$ IN (1) LEADS TO:

$$m^2(z) = \frac{N^2(z)(k^2+l^2)}{\omega^2 - f^2}$$

2 – let us assume $N^2 = N^2(z)$:

This leads to solve:

$$\frac{\partial}{\partial t}\left[(\frac{p'_z}{N^2})_{ztt} + f^2 (\frac{p'_z}{N^2})_z + \Delta p' \right] = 0$$

Using p'(x,y,z,t) = P(z).$e^{-i(k.x+l.y-\omega t)}$, leads to

$$(\frac{P_z}{N^2})_z - \frac{k^2+l^2}{f^2-\omega^2} P = 0$$

We need to find new vertical normal modes for P(z) **without** using the WKB approximation

These vertical normal modes, $F_m(z)$, should be such that $\int_{-H}^{0} F_m . F_n \, dz = \delta_{mn}$.

Then: we can use

$$P(z) = \sum_{m=1}^{M} P_m . F_m(z) \text{ [instead of } P(z) = \sum_{m=1}^{M} P_m(z). \cos m(z). z \text{]}$$

Vertical dimension using normal modes:

Normal modes are obtained by solving the Sturm-Liouville equation (after J. Sturm (1803–1855) and J. Liouville (1809–1882)):

$$\frac{d}{dz}\frac{f^2}{N^2}\frac{dF_m}{dz} = -\lambda_m^2 F_m.$$

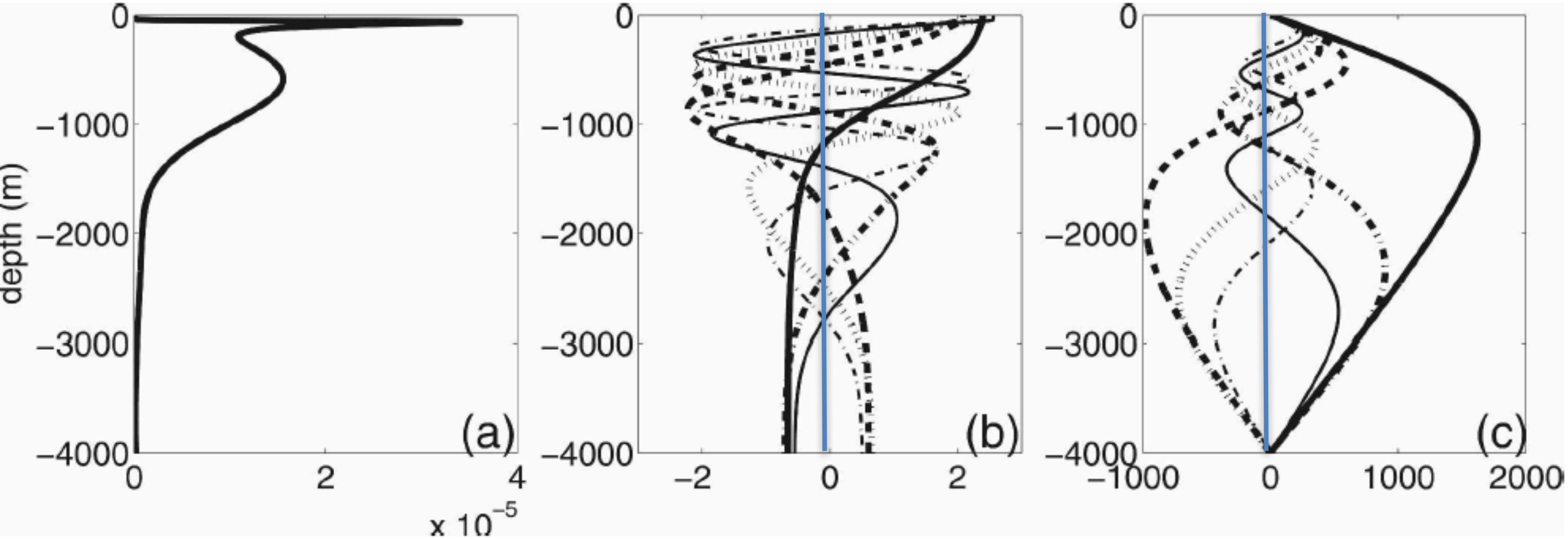
with $dF_m/dz = 0$ at $z = 0, -H$.

F_m is the eigenfunction and λ_m^2 the eigenvalue (or the vertical wavenumber) associated with mode m.

Modes are orthonormal :

$$\int_{-H}^{0} F_m \cdot F_n \, dz = \delta_{mn}$$

with $\lambda_m = 1/r_m$. r_m is also called the Rossby radius of deformation of mode m



1. (a) Vertical profiles of N^2, (b) the first six eigenfunctions F_n given by Eq. (3), and (c) the first six functions H_n. Units in (a) are s^{-2}.

$$\mathcal{L}F_n = -\frac{1}{r_n^2}F_n. \qquad \mathcal{L}(\cdot) = \frac{\partial}{\partial z}\left[\frac{f^2}{N^2}\frac{\partial}{\partial z}(\cdot)\right]$$

$$H_n(z) = \int_z^0 F_n(z')dz'$$

$F_m(z)$ combine $P_m(z)$ and $cos(m(z).z)$ and do assume $N^2(z)$ is smooth!

$$\left(\frac{P_z}{N^2}\right)_z - \frac{k^2 + l^2}{f^2 - \omega^2} P = 0$$

This leads to the following dispersion relation for mode m:

$$\omega^2 = f^2 + f^2 r_m^2 \, [k^2 + l^2]$$

$f.r_m$ is the equivalent of c_o of the SW system and r_m is the Rossby radius of deformation (~ L_d) of mode m (the equivalent of c_o/f=R in the SW system).

Solutions of the linear equations are Poincaré waves, with
- inertial waves as the long wave limit with L>>r_m (Bu<<1) and
- gravity waves as the short limit when L<<r_{rm} (Bu>>1).
- Kelvin waves exist as well.

Since the vertical normal modes have different phase speeds for the same $K^2 = k^2 + l^2$, vertical modes, if initially in phase, quickly become out of phase, leading to the vertical propagation of the kinetic energy (see Gill, JPO'84, Hector).

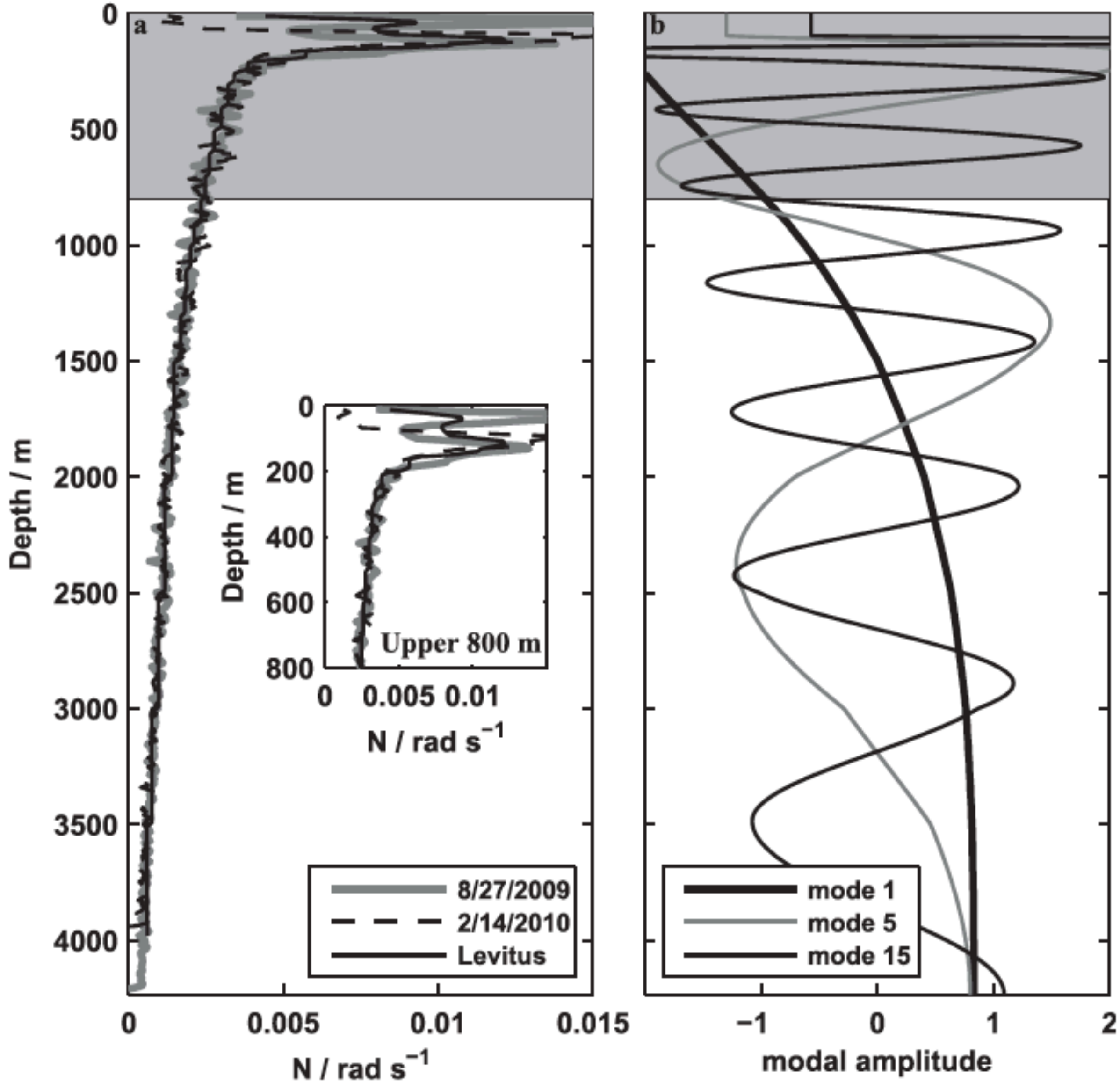


FIG. 5. (a) Buoyancy frequency computed over 8-m intervals from temperature and salinity measured from two line P cruises (gray and dashed lines) and from Levitus (thin black line). The upper 800 m spanned by our measurements is shaded in gray and plotted in the inset. (b) Vertical structure of dynamical modes 1, 5, and 15. Gray is the depth range of our observations.

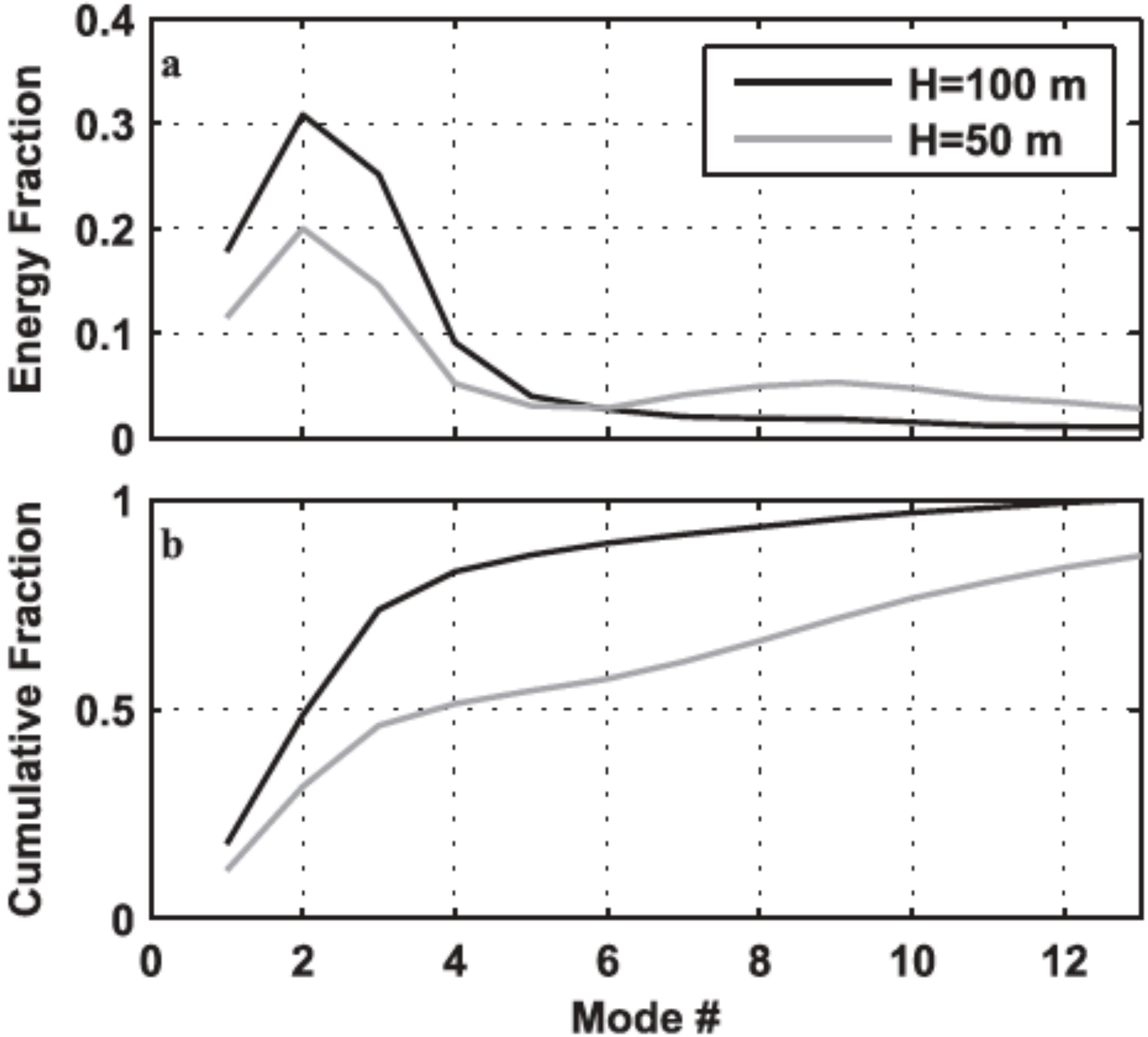


FIG. 7. The fraction of (a) energy and (b) its integral projecting onto each dynamical mode from Gill (1984) for MLDs of 100 and 50 m.