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Homework 1

Question 3

Let $\vec{q}_1, \dots, \vec{q}_k \in \mathbb{R}^k$ k units and pairwise orth. Show that:

$$\vec{q}_1 \vec{q}_1^{\,T} + \dots + \vec{q}_k \vec{q}_k^{\,T} = Id$$

Solution

$$\vec{q}_1 \vec{q}_1^{\,T} + \dots + \vec{q}_k \cdot \vec{q}_k^{\,T} = (\vec{q}_1 \dots \vec{q}_k) \begin{pmatrix} \vec{q}_1^{\,T} \\ | \\ \vec{q}_k^{\,T} \end{pmatrix}$$

Naming $P = (\vec{q}_1 \dots \vec{q}_k)$, since $\vec{q}_1, \dots, \vec{q}_k$ are k-units and pairwise orth. $\Rightarrow P$ is orthogonal $\Rightarrow P^{-1} = P^T$, and $P^T = \begin{pmatrix} \vec{q}_1^{\,T} \\ | \\ \vec{q}_k^{\,T} \end{pmatrix}$.

$$\underline{\vec{q}_1 \vec{q}_1^{\,T} + \dots + \vec{q}_k \vec{q}_k^{\,T}} = P \circ P^T = P \circ P^{-1} = \underline{Id.}$$

Question 4

Let $A = \begin{pmatrix} 2 & 2 \\ -3 & 5 \\ 5 & -3 \\ -4 & -4 \end{pmatrix}$

a) Calculate $A^T A$ and it's spectral dec.

b) Find $(A^T A)^{-1}$ and $(A^T A)^{-1/2}$.

Solution

a) $$A^T A = \begin{pmatrix} 2 & -3 & 5 & -4 \\ 2 & 5 & -3 & -4 \end{pmatrix} \begin{pmatrix} 2 & 2 \\ -3 & 5 \\ 5 & -3 \\ -4 & -4 \end{pmatrix} = \underline{\begin{pmatrix} 54 & -10 \\ -10 & 54 \end{pmatrix}}$$

Find the eigen values: $|A-\lambda I| = 0 \Leftrightarrow \begin{vmatrix} 54-\lambda & -10 \\ -10 & 54-\lambda \end{vmatrix} = 0$

$\Leftrightarrow \begin{cases} \lambda_1 = 64 \\ \lambda_2 = 44. \end{cases}$ Find the eigen vectors:

(V1) $\begin{pmatrix} 54-64 & -10 \\ -10 & 54-64 \end{pmatrix} v_1 = 0 \Leftrightarrow \begin{pmatrix} -10 & -10 \\ -10 & -10 \end{pmatrix} v_1 = 0 \Leftrightarrow$

$\Leftrightarrow v_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t$, $t \in \mathbb{R}$, we select the unit vector $\underline{v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$

(V2) $\begin{pmatrix} 54-44 & -10 \\ -10 & 54-44 \end{pmatrix} v_2 = 0 \Leftrightarrow \begin{pmatrix} 10 & -10 \\ -10 & 10 \end{pmatrix} v_2 = 0 \Leftrightarrow v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} t$, $t \in \mathbb{R}$.

We select the unit vecter $\underline{v_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$

Checking v_1 and v_2 are orthogonal (although we already know that due to the dimension of each subspace): $v_1^T v_2 = (1\ -1)\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cdot \frac{1}{2} = 0$.

Then $P = (v_1\ v_2) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is orthogonal.

Using $\Lambda = \begin{pmatrix} 64 & 0 \\ 0 & 44 \end{pmatrix}$, the spectral decomposition of $A^T A$ is:

$$A^T A = P \cdot \Lambda \cdot P^{-1} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 44 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \frac{1}{\sqrt{2}} =$$

$$\underline{= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 64 & 0 \\ 0 & 44 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$$

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(b) Find $(A^TA)^{-1} = (P\cdot\Lambda\cdot P^T)^{-1} = P\cdot\Lambda^{-1}\cdot P^T =$

$$= \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1/64&0\\0&1/44\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} =$$

$$= \frac{1}{1408}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}11&0\\0&16\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} =$$

$$= \frac{1}{1408}\begin{pmatrix}11&16\\-11&16\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} = \underline{\frac{1}{1408}\begin{pmatrix}27&5\\5&27\end{pmatrix}}$$

$(A^TA)^{-1/2}$ is a matriz such that $\left((A^TA)^{-1/2}\right)^2 = (A^TA)^{-1}$

Let $B = P\cdot\underbrace{\begin{pmatrix}1/\sqrt{64}&0\\0&1/\sqrt{44}\end{pmatrix}}_{\Lambda_2}P^T$, then $B^2 = P\cdot\Lambda_2\cdot P^T\cdot P\cdot\Lambda_2\cdot P^T =$

$$= P\cdot\Lambda_2\cdot\Lambda_2\cdot P^T = P\cdot\begin{pmatrix}1/64&0\\0&1/44\end{pmatrix}\cdot P^T = (A^TA)^{-1}.$$

$\Rightarrow \underline{B = (A^TA)^{-1/2}}$. Let's compute it:

$$\underline{B} = \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1/\sqrt{64}&0\\0&1/\sqrt{44}\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} = \frac{1}{2}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}1/8&0\\0&\frac{1}{2\sqrt{11}}\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} =$$

$$= \frac{1}{16\sqrt{11}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\begin{pmatrix}\sqrt{11}&0\\0&4\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} = \frac{1}{16\sqrt{11}}\begin{pmatrix}\sqrt{11}&4\\-\sqrt{11}&4\end{pmatrix}\begin{pmatrix}1&-1\\1&1\end{pmatrix} = \cancel{\frac{1}{16\sqrt{11}}}$$

$$= \frac{1}{16\sqrt{11}}\begin{pmatrix}4+\sqrt{11}&4-\sqrt{11}\\4-\sqrt{11}&4+\sqrt{11}\end{pmatrix} = \underline{\begin{pmatrix}\frac{1}{4\sqrt{11}}+\frac{1}{16}&\frac{1}{4\sqrt{11}}-\frac{1}{16}\\\frac{1}{4\sqrt{11}}-\frac{1}{16}&\frac{1}{4\sqrt{11}}+\frac{1}{16}\end{pmatrix}} = \underline{\frac{1}{4\sqrt{11}}\begin{pmatrix}1&1\\1&1\end{pmatrix} + \frac{1}{16}\begin{pmatrix}1&-1\\-1&1\end{pmatrix}}.$$

Question 5 ⓐ Let $S, D, C \in \mathbb{R}^{k \times k}$ invertible and $\vec{x}, \vec{y} \in \mathbb{R}^k$.

Show: $(D\vec{x})^T (CSD^T)^{-1} (C\vec{y})^T = \vec{x}^T S^{-1} \vec{y}$.

ⓑ Set $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $S = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$.

Compute the expression in ⓐ. Does this contradict the statement in ⓐ?

__Solution__

ⓐ $(D\vec{x})^T (CSD^T)^{-1} (C\vec{y})^{\cancel{T}} =$

$$= \underset{(1\times k)}{\vec{x}^T} \cdot \underset{(k\times k)}{D^T} \cdot \underset{(k\times k)}{(D^T)^{-1}} \underset{(k\times k)}{S^{-1}} \underset{(k\times k)}{C^{-1}} \underset{(k\times k)}{C} \underset{(k\times 1)}{\vec{y}} =$$

$$= \underset{(1\times k)}{\vec{x}^T} \cdot \underset{(k\times k)}{S^{-1}} \cdot \underset{(k\times 1)}{\vec{y}}.$$

Where all multiplications are aplicable and D^T is inverible because D is too.

ⓑ $(C\vec{x})^T (CSC^T)^{-1} (C\vec{x}) \overset{?}{=} \vec{x}^T S^{-1} \vec{x}$

$C\vec{x} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $(C\vec{x})^T = (3\ 2)$.

$CSC^T = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 3 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$

$(C\vec{x})^T (CSC^T)(C\vec{x}) = (3\ 2) \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 3 \\ 2 \end{pmatrix} = (14\ \ 7) \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \underline{56}$

$\neq$

$\vec{x}^T S^{-1} \vec{x} = (2\ 1\ 1) \begin{pmatrix} 1/3 & & \\ & 1 & \\ & & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = (2/3\ \ 1\ \ 1) \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix} = \frac{4}{3} + 2 = \underline{\frac{10}{3}}$

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This doesn't contradict the claim in part (a) since C is nor square nor invertible.

Question 2 Let $C = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_n \end{pmatrix} \in \mathbb{R}^{n \times n}$, $D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{pmatrix} \in \mathbb{R}^{p \times p}$

and $A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1p} \\ a_{21} & a_{22} & \cdots & a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{np} \end{pmatrix} \in \mathbb{R}^{n \times p}$. Calculate ~~$n\times p$~~ CAD.

Solution

We can simply compute the product:

$$CAD = \begin{pmatrix} c_1 \cdot a_{11} & c_1 \cdot a_{12} & \cdots & c_1 \cdot a_{1p} \\ c_2 \cdot a_{21} & c_2 \cdot a_{22} & \cdots & c_2 \cdot a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_n \cdot a_{n1} & c_n a_{n2} & \cdots & c_n \cdot a_{np} \end{pmatrix} \cdot D =$$

$$= \begin{pmatrix} c_1 d_1 \cdot a_{11} & c_1 d_2 a_{12} & \cdots & c_1 d_p a_{1p} \\ c_2 d_1 a_{21} & c_2 d_2 a_{22} & \cdots & c_2 d_p a_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ c_n d_1 a_{n1} & c_n d_2 a_{n2} & \cdots & c_n d_p a_{np} \end{pmatrix}$$

But we can also use basic linear algebra concepts. We know that:

$A = \begin{pmatrix} \vec{a}_1^{\,T} \\ \vdots \\ \vec{a}_k^{\,T} \end{pmatrix} \in \mathbb{R}^{k \times p}$, $C = \begin{pmatrix} c_1 & & \\ & \ddots & \\ & & c_k \end{pmatrix} \in \mathbb{R}^{k \times k}$ then

$$C \cdot A = C \cdot \begin{pmatrix} \vec{a}_1^{\,T} \\ \vdots \\ \vec{a}_k^{\,T} \end{pmatrix} = \begin{pmatrix} c_1 \cdot \vec{a}_1^{\,T} \\ \vdots \\ c_k \cdot \vec{a}_k^{\,T} \end{pmatrix},$$ meaning we multiply ~~each~~ row $\vec{a}_i^{\,T}$ by c_i.

Similarly, $A = (\vec{a_1} \cdots \vec{a_p}) \in \mathbb{R}^{k \times p}$, $D = \begin{pmatrix} d_1 & & \\ & \ddots & \\ & & d_p \end{pmatrix}$

then $AD = (\vec{a_1} \cdots \vec{a_p}) D = (\vec{a_1} \cdot d_1 \cdots \vec{a_p} \cdot d_p)$

meaning we multiply ~~each~~ $\vec{a_i}$ column by d_i.

Using both results is easy to see that CAD will have $(a_{ij} \cdot c_i \cdot d_j)$ in row i and column j.

$\boxed{\text{Question 1}}$ Let $A = (\vec{a_1} \cdots \vec{a_k}) \in \mathbb{R}^{n \times k}$ and $B = \begin{pmatrix} \vec{b_1}^T \\ \vdots \\ \vec{b_k}^T \end{pmatrix} \in \mathbb{R}^{k \times p}$.

Show that $AB = \vec{a_1} \vec{b_1}^T + \cdots + \vec{a_k} \vec{b_k}^T$.

Solution

First, compute AB with ~~on~~ the "classical" form:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nk} \end{pmatrix} \in \mathbb{R}^{n \times k}, \quad B = \begin{pmatrix} b_{11} & \cdots & b_{1p} \\ \vdots & \ddots & \vdots \\ b_{k1} & \cdots & b_{kp} \end{pmatrix} \in \mathbb{R}^{k \times p}.$$

then $$AB = \begin{pmatrix} \sum_{i=1}^{k} a_{1i} b_{i1} & \cdots & \sum a_{1i} \cdot b_{ip} \\ \vdots & \ddots & \vdots \\ \sum_{i=1}^{k} a_{ni} \cdot b_{i1} & \cdots & \sum_{i=1}^{k} a_{ni} \cdot b_{ip} \end{pmatrix}.$$

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$$A = \left(\vec{a_1} \ \text{—} \ \vec{a_k} \right) \in \mathbb{R}^{n \times k} \text{ where } \vec{a_i} = \begin{pmatrix} a_{1i} \\ | \\ a_{ni} \end{pmatrix} \in \mathbb{R}^n \ \forall i \in \{1, \dots, k\}.$$

$$B = \begin{pmatrix} \vec{b_1}^T \\ | \\ \vec{b_k}^T \end{pmatrix} \in \mathbb{R}^{k \times p} \text{ where } \vec{b_i}^T = \left(b_{i1} \ \text{——} \ b_{ip} \right) \ \forall i \in \{1, \dots, k\}.$$

Then

$$\vec{a_i}\vec{b_i}^T = \underset{(n \times 1)}{\begin{pmatrix} a_{1i} \\ | \\ a_{ni} \end{pmatrix}} \underset{(1 \times p)}{\left(b_{i1} \text{—} b_{ip} \right)} = \underset{(n \times p)}{\begin{pmatrix} a_{1i} \cdot b_{i1} & \text{—} & a_{1i} \cdot b_{ip} \\ | & \diagdown & | \\ a_{ni} \cdot b_{i1} & \text{—} & a_{ni} \cdot b_{ip} \end{pmatrix}} \in \mathbb{R}^{n \times p} \quad \forall i \in \{1, \dots, k\}$$

Therefore:

$$\underbrace{\sum_{i=1}^{k} \vec{a_i}\vec{b_i}^T}_{\text{These are all } n \times p \text{ matrices}} = \begin{pmatrix} \sum_{i=1}^{k} a_{1i} \cdot b_{i1} & \text{—} & \sum_{i=1}^{k} a_{1i} \cdot b_{ip} \\ | & \diagdown & | \\ \sum_{i=1}^{k} a_{ni} \cdot b_{i1} & \text{—} & \sum_{i=1}^{k} a_{ni} \cdot b_{ip} \end{pmatrix} = AB.$$