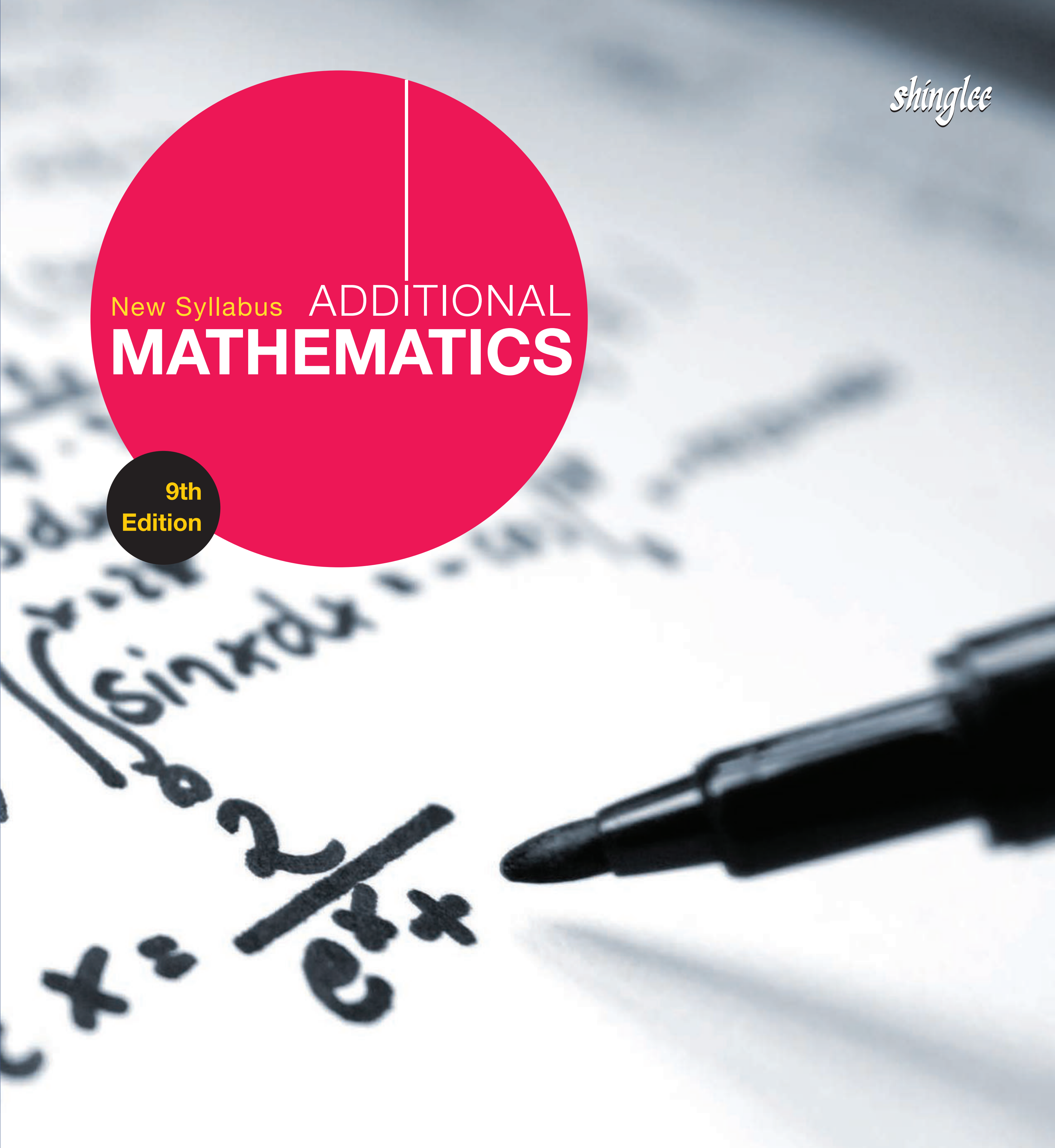

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New Syllabus

ADDITIONAL MATHEMATICS

9th Edition

Consultant • Dr Yeap Ban Har

Authors • Joseph Yeo
Teh Keng Seng
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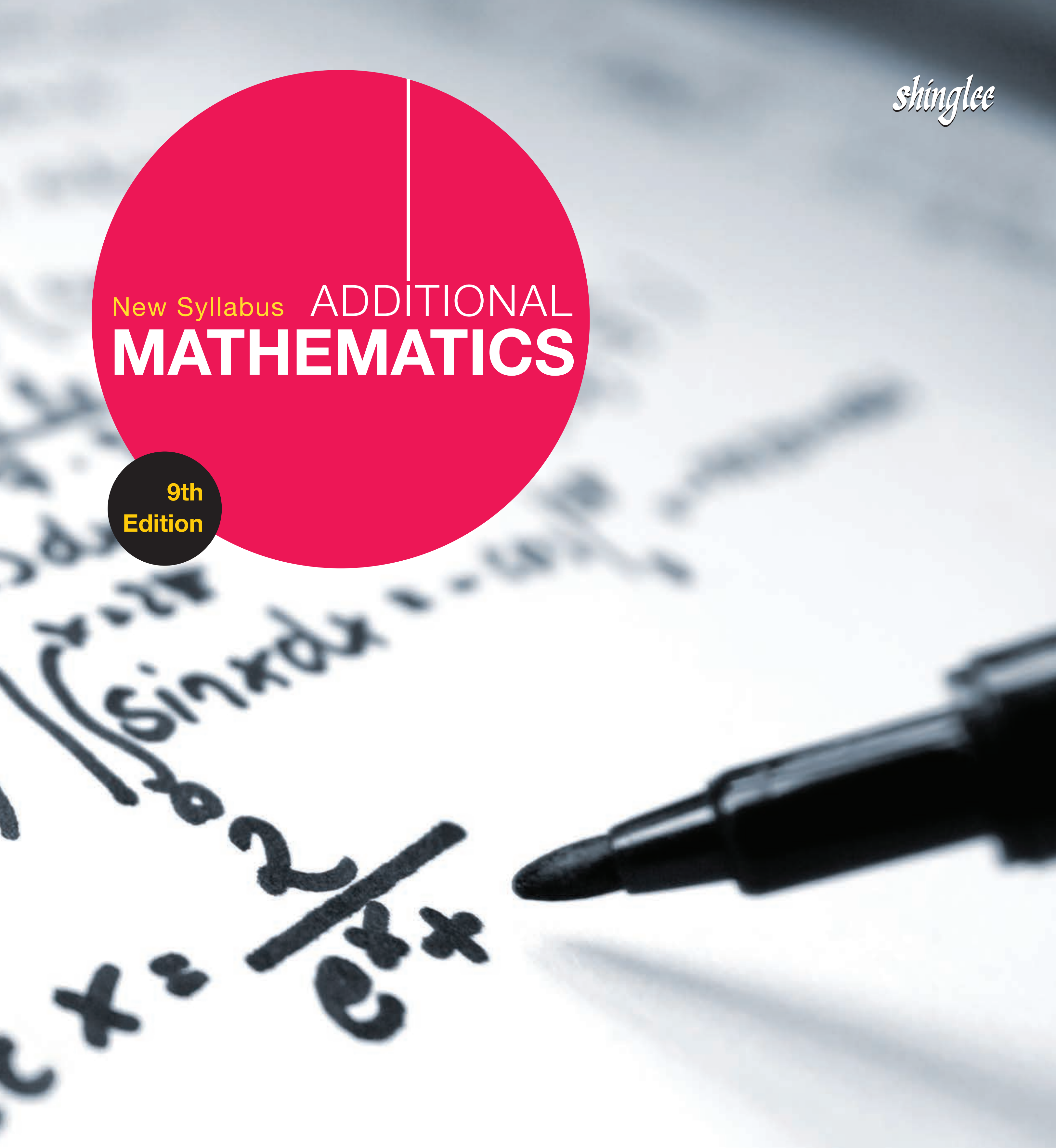

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SHINGLEE PUBLISHERS PTE LTD
120 Hillview Avenue #05-06/07
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PREFACE

New Syllabus
Additional
Mathematics
NSAM

New Syllabus Additional Mathematics (NSAM) is an MOE-approved textbook specially designed to provide valuable learning experiences to engage the hearts and minds of students sitting for the GCE O-level examination in Additional Mathematics. Included in the textbook are **Investigation**, **Class Discussion**, **Thinking Time** and Alternative Assessment such as **Journal Writing** to support the teaching and learning of Mathematics.

Every chapter begins with a chapter opener which motivates students in learning the topic. Interesting stories about mathematicians, real-life examples and applications are used to arouse students' interest and curiosity so that they can appreciate the beauty of Mathematics in their surroundings and in the sciences.

The use of ICT helps students to visualise and manipulate mathematical objects more easily, thus making the learning of Mathematics more interactive. Ready-to-use interactive ICT templates are available at http://www.shinglee.com.sg/StudentResources/

The chapters in the textbook have been organised into three strands — **Algebra**, **Geometry and Trigonometry** and **Calculus**. The colours purple, green and red at the bottom of each page indicate these. Chapters and sections which have been excluded from the Normal (Academic) syllabus are clearly indicated with a ★.

KEY FEATURES

CHAPTER OPENER

Each chapter begins with a chapter opener to arouse students' interest and curiosity in learning the topic.

LEARNING OBJECTIVES

Learning objectives help students to be more aware of what they are about to study so that they can monitor their own progress.

RECAP

Relevant prerequisites will be revisited at the beginning of the chapter or at appropriate junctures so that students can build upon their prior knowledge, thus creating meaningful links to their existing schema.

WORKED EXAMPLE

This shows students how to apply what they have learnt to solve related problems and how to present their working clearly. A suitable heading is included in brackets to distinguish between the different Worked Examples.

PRACTISE NOW

At the end of each Worked Example, a similar question will be provided for immediate practice. Where appropriate, this includes further questions of progressive difficulty.

SIMILAR QUESTIONS

A list of similar questions in the Exercise is given here to help teachers choose questions that their students can do on their own.

EXERCISE

The questions are classified into three levels of difficulty – Basic, Intermediate and Advanced.

SUMMARY

At the end of each chapter, a succinct summary of the key concepts is provided to help students consolidate what they have learnt.

CHALLENGE YOURSELF

Optional problems are included at the end of each chapter to challenge and stretch high-ability students to their fullest potential.

REVISION EXERCISE

This is included after every few chapters to help students assess their learning.

Learning experiences have been infused into **Investigation**, **Thinking Time**, **Class Discussion**, and **Journal Writing**.

Investigation

Activities are included to guide students to investigate and discover important mathematical concepts so that they can construct their own knowledge meaningfully.

Thinking Time

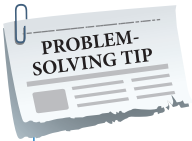
Key questions are also included at appropriate junctures to check if students have grasped various concepts and to create opportunities for them to further develop their thinking.

Questions are provided for students to discuss in class, with the teacher acting as the facilitator. The questions will assist students to learn new knowledge, think mathematically, and enhance their reasoning and oral communication skills.

Journal Writing

Opportunities are provided for students to reflect on their learning and to communicate mathematically. It can also be used as a formative assessment to provide feedback to students to improve on their learning.

MARGINAL NOTES



This guides students on how to approach a problem.

INTERNET RESOURCES

This guides students to search on the Internet for valuable information or interesting online games for their independent and self-directed learning.

Just For Fun

This contains puzzles, fascinating facts and interesting stories about Mathematics as enrichment for students.

ATTENTION

This contains important information that students should know.

INFORMATION

This includes information that may be of interest to students.

RECALL

This contains certain mathematical concepts or rules that students have learnt previously.

Contents

New Syllabus Additional Mathematics **NSAM**

★ excluded from the N(A) syllabus

New Syllabus Additional Mathematics **NSAM**

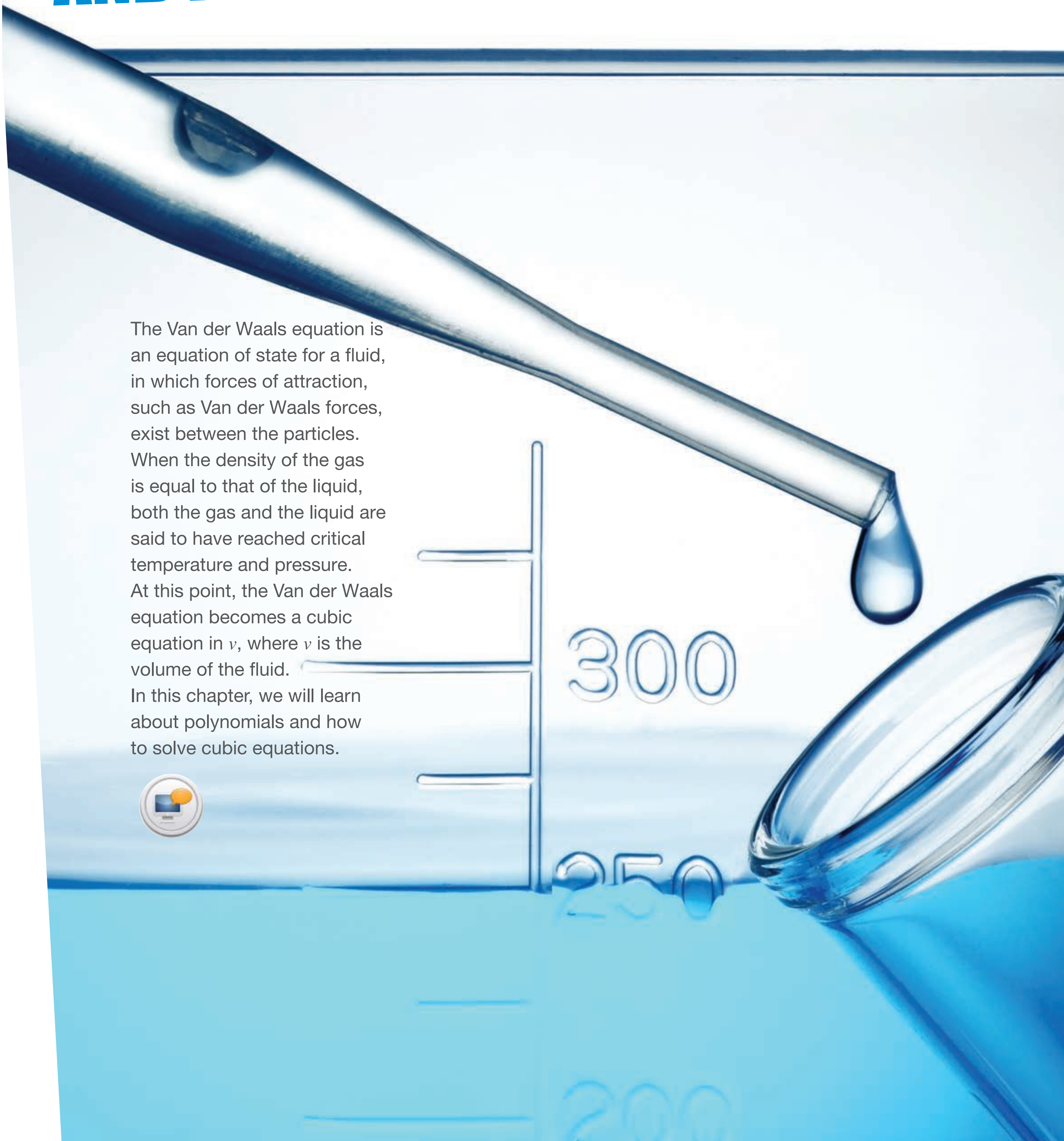
excluded from the N(A) syllabus ★

New Syllabus Additional Mathematics **NSAM**

excluded from the N(A) syllabus ★

excluded from the N(A) syllabus

SIMULTANEOUS EQUATIONS, POLYNOMIALS AND PARTIAL FRACTIONS



The Van der Waals equation is an equation of state for a fluid, in which forces of attraction, such as Van der Waals forces, exist between the particles. When the density of the gas is equal to that of the liquid, both the gas and the liquid are said to have reached critical temperature and pressure. At this point, the Van der Waals equation becomes a cubic equation in v, where v is the volume of the fluid. In this chapter, we will learn about polynomials and how to solve cubic equations.

CHAPTER 1

Learning Objectives

At the end of this chapter, you should be able to:

- solve two simultaneous equations in two variables, where at least one is a non-linear equation,
- use the Remainder Theorem to find the remainder when an expression is divided by a linear factor,
- factorise cubic expressions or polynomials of higher degree using the Factor Theorem,
- solve cubic equations and problems involving cubic equations,
- express improper fractions as the sum of a polynomial and a proper fraction,
- perform partial fraction decomposition of an algebraic fraction.

1.1 LINEAR AND NON-LINEAR SIMULTANEOUS EQUATIONS

Recap

We have learnt how to solve a pair of simultaneous linear equations using three methods: elimination, substitution and graphical methods. In this section, we will learn how to solve a pair of simultaneous equations in which one is linear while the other is **non-linear**.

Thinking Time

1. Use a graphing software to plot the graphs of the equations $2y - 3x = 1$ and $y + 9x^2 = 8$ on the same graph.
   (i) What are the coordinates of the points of intersection of the two graphs?
   (ii) We say that $x = -1$, $y = -1$ and $x = \frac{5}{6}$, $y = 1\frac{3}{4}$ are the solutions of the simultaneous equations $2y - 3x = 1$ and $y + 9x^2 = 8$.
2. Plot the graphs of the equations $x + 2y = 5$ and $x^2 - 3y + x = 9$ on the same graph.
   (i) What are the coordinates of the points of intersection of $x + 2y = 5$ and $x^2 - 3y + x = 9$?
   (ii) What can we say about the solutions of the simultaneous equations $x + 2y = 5$ and $x^2 - 3y + x = 9$?

What can we conclude about the solutions of simultaneous equations and the points of intersection of the two graphs of the equations?

Worked Example 1

(Solving Linear and Non-Linear Simultaneous Equations)

Solve the simultaneous equations

$$2y - 3x = 1,$$
$$y + 9x^2 = 8.$$

Hence, state the coordinates of the points of intersection of the line $2y - 3x = 1$ and the curve $y + 9x^2 = 8$.

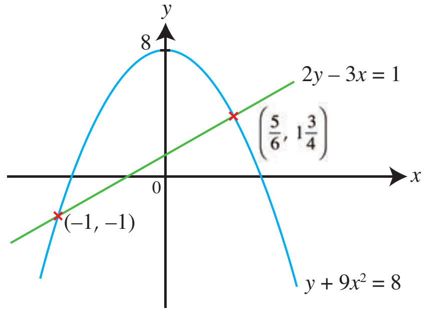
Solution

$$2y - 3x = 1 \text{ ---------- (1)}$$
$$y + 9x^2 = 8 \text{ ---------- (2)}$$

From (1), $$y = \frac{3x+1}{2} \text{ ---- (3)}$$

Substitute (3) into (2): $$\frac{3x+1}{2} + 9x^2 = 8$$
$$3x + 1 + 18x^2 = 16$$
$$18x^2 + 3x - 15 = 0$$
$$6x^2 + x - 5 = 0$$
$$(6x - 5)(x + 1) = 0$$

$\therefore \quad x = \frac{5}{6}$ or $x = -1$



Substitute $x = \frac{5}{6}$ into (3): $y = \frac{3\left(\frac{5}{6}\right)+1}{2} = 1\frac{3}{4}$

Substitute $x = -1$ into (3): $y = \frac{3(-1)+1}{2} = -1$

From the sketch, we see that the line $2y - 3x = 1$ intersects the curve $y + 9x^2 = 8$ at two points.

$\therefore$ The coordinates of the points of intersection are $\left(\frac{5}{6}, 1\frac{3}{4}\right)$ and $(-1, -1)$.

Practise Now 1

Similar Questions:
Exercise 1A
Questions 1(a)-(h), 2(a)-(b), 3, 8

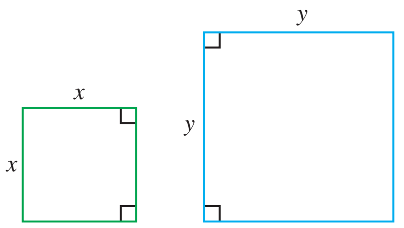
1. Solve the simultaneous equations $x + 2y = 5$ and $x^2 - 3y + x = 9$. Hence, state the coordinates of the points of intersection of the line $x + 2y = 5$ and the curve $x^2 - 3y + x = 9$.
2. Find the coordinates of the points of intersection of the line $x + y = 7$ and the curve $x^2 + xy - 2y^2 = 10$.
3. Solve the simultaneous equations $\frac{1}{x} + \frac{1}{y} = \frac{7}{12}$ and $xy = 12$.

Worked Example 2

(Application of Simultaneous Equations)
The sum of the perimeters of two squares is 40 cm while the sum of their areas is 58 cm^2. Find the lengths of the sides of the squares.

Solution

Let the lengths of the sides of the squares be x cm and y cm.



We have

$$4x + 4y = 40 \quad \text{------} \quad (1)$$
$$x^2 + y^2 = 58 \quad \text{------} \quad (2)$$

From (1),

$$y = 10 - x \quad \text{------} \quad (3)$$

Substitute (3) into (2):

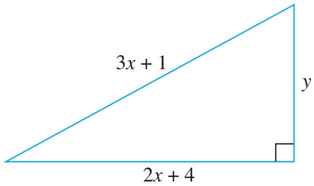
$$x^2 + (10 - x)^2 = 58$$
$$x^2 + 100 - 20x + x^2 = 58$$
$$2x^2 - 20x + 42 = 0$$
$$x^2 - 10x + 21 = 0$$
$$(x - 3)(x - 7) = 0$$
$$\therefore \quad x = 3 \text{ or } x = 7$$

When $x = 3$, $y = 10 - 3 = 7$.
When $x = 7$, $y = 10 - 7 = 3$.
$\therefore$ The lengths of the sides of the squares are 3 cm and 7 cm.

Practise Now 2

Similar Questions:
Exercise 1A
Questions 4-7

The perimeter of the right-angled triangle is 30 cm. Find the lengths of the sides of the triangle.



Basic Level | Intermediate Level | Advanced Level

Exercise 1A

1 Solve each of the following pairs of simultaneous equations.

(a) $2x + y = 5$
$x^2 + y^2 = 5$

(b) $y = 2 - x$
$x(x + y) = 5 - 3y^2$

(c) $2x = 1 - 3y$
$3y^2 - x^2 = 2$

(d) $x^2 + y^2 = 34$
$y + 3x = 14$

(e) $x - y = 3$
$3y^2 = x^2 + 2xy + 1$

(f) $4x + y = 7$
$4x^2 - 4xy + y^2 = 1$

(g) $2x + 3y = 13$
$2xy + 5y^2 - 4x^2 = 41$

(h) $x - 7y = 2$
$x^2 = 34y^2 + 7xy - 16$

2 Solve each of the following pairs of simultaneous equations.

(a) $2y = 3x - 1$
$\frac{4x}{y} + \frac{9y}{x} = 15$

(b) $\frac{2}{x} + \frac{3}{y} = 13$
$2x + 3y = 2$

3 The values of the expression $ax^2 + bx + 1$ are 1 and 4 when x takes the values of 2 and 3 respectively. Find the value of the expression when $x = 4$.

4 The sum of the circumference of two circles is 38π cm and the sum of their areas is 193π cm^2. Calculate the radii of the circles.

5 The sum and product of two numbers x and y are 3 and 1.25 respectively. Form two equations involving x and y and solve them to obtain the possible values of x and y.

6 If $x = 1$ and $y = 2$ is a solution to the simultaneous equations $ax + by = 2$ and $bx + a^2y = 10$, find the possible values of a and b.

7 Two positive numbers differ by $1\frac{1}{2}$ and the sum of their squares is $9\frac{1}{8}$. Find the numbers.

8 The line $2x + 3y = 8$ meets the curve $2x^2 + 3y^2 = 110$ at the points A and B. Find the coordinates of A and of B.

1.2 POLYNOMIALS

What is a polynomial?

Recall that we have learnt about algebraic expressions such as $2a + b$ (linear in two variables), $x^2 - 4x + 3$ (quadratic in one variable) and $\frac{1}{5 - x}$ (algebraic fraction). In this section, we will study a certain type of algebraic expression called **polynomials**.

Investigation

Polynomial and Non-polynomials

The table below shows some examples and non-examples of polynomials in one variable.

Polynomials	Non-polynomials
$x^2 - 4x + 3$	$2x^2 - 7x + x^{-1}$
$2x^5 + 7x^4$	$-x^{\frac{1}{2}} + x^{-3}$
$5x^3 + 6 - \frac{5}{7}x$	$x^2 - \sqrt{x}$
8	$\frac{1}{x} - \frac{2}{x^4} - \sqrt[3]{x}$

1. Compare the powers of x in polynomials with those in non-polynomials. What do you notice?
2. Polynomials cannot contain terms such as $\sqrt{x} = x^{\frac{1}{2}}$. Can you think of other terms that polynomials cannot include?
3. Explain to your classmates some features of a polynomial.

From the above investigation, a **polynomial** in x is an *algebraic expression* consisting of terms with *non-negative powers* of x only. Its general form is:

$$\boxed{a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_2x^2 + a_1x + a_0,}$$

where n is a non-negative integer, the coefficients $a_n, a_{n-1}, a_{n-2}, \ldots a_2, a_1, a_0$ are constants and x is a variable.

Linear and quadratic expressions are examples of polynomials.
Polynomials **cannot** contain terms involving fractional powers or negative powers of the variable. For example, polynomials in x cannot contain $\sqrt{x}\left(=x^{\frac{1}{2}}\right)$, $\sqrt[3]{x}\left(=x^{\frac{1}{3}}\right)$, $\frac{1}{x}\ (=x^{-1})$ and $\frac{1}{x^2}\ (=x^{-2})$, because the powers of x must be non-negative integers.

We usually arrange the terms of a polynomial in *decreasing powers* of the variable, e.g. $2x^3 + 7x^2 - 4x + 3$. Sometimes, the terms can be arranged in *ascending powers* of the variable, especially when the leading coefficient is negative, e.g. $2 + 3x - x^4$.

INFORMATION

A **zero polynomial** is denoted by 0 (i.e. all the coefficients are 0) and has no degree.

INFORMATION

The term with the highest power is called the **leading term** and its coefficient is called the **leading coefficient**.

Degree of a Polynomial

If $\boldsymbol{a_n \neq 0}$, then the highest power of x in the polynomial is n and it is called the **degree** of the polynomial. For example, $x^3 + 5$ is a polynomial of degree 3.

What is the degree of each polynomial in the investigation on the previous page?
In particular, state the degree of a
(i) **linear** expression,
(ii) **quadratic** expression,
(iii) non-zero **constant** polynomial.

Thinking Time

Can any or all of the coefficients a_n, a_{n-1}, a_{n-2}, ... a_2, a_1, a_0 of a polynomial be equal to 0? Explain your answer.

Evaluating Polynomials

We can also denote a polynomial in x by P(x), Q(x), g(x) and so on.
To find the value of $P(x) = 2x^5 + 7x^4$ when $x = 3$, we substitute $x = 3$ into P(x):

$$P(3) = 2(3)^5 + 7(3)^4 = 1053.$$

What is the value of the polynomial function $P(x) = 2x^5 + 7x^4$ when $x = -2$?
For simplicity, we will just call $P(x) = 2x^5 + 7x^4$ a ‘polynomial’.

Addition and Subtraction of Polynomials

To find the sum and the difference of polynomials, we add and subtract **like terms**.

Worked Example 3

(Addition and Subtraction of Polynomials)

If $P(x) = 2x^3 + x^2 - 4x + 5$ and $Q(x) = x^2 - 6x + 5$, find an expression for

(i) $P(x) + Q(x)$, **(ii)** $P(x) - Q(x)$.

State the degree of each expression.

Solution

(i) $P(x) + Q(x) = (2x^3 + x^2 - 4x + 5) + (x^2 - 6x + 5)$

$= 2x^3 + 2x^2 - 10x + 10$

Degree of $P(x) + Q(x) = 3$

ATTENTION

The sum or the difference of polynomials is a polynomial.

(ii) $P(x) - Q(x) = (2x^3 + x^2 - 4x + 5) - (x^2 - 6x + 5)$

$= 2x^3 + x^2 - 4x + 5 - x^2 + 6x - 5$

$= 2x^3 + 2x$

Degree of $P(x) - Q(x) = 3$

RECALL

$-(a + b - c) = -a - b + c$

Practise Now 3

If $P(x) = 4x^3 + 3x^2 - 5x + 5$ and $Q(x) = 4x^3 - 2x + 6$, find an expression for

(i) $P(x) + Q(x)$, **(ii)** $P(x) - Q(x)$.

State the degree of each expression.

Similar Questions:
Exercise 1B
Questions 1(a)-(d), 2(a)-(d)

Thinking Time

$P(x)$ and $Q(x)$ are polynomials.

1. How is the degree of $P(x) + Q(x)$ related to the degree of $P(x)$ and of $Q(x)$?
2. Can the degree of $P(x) - Q(x)$ be less than the degrees of both $P(x)$ and $Q(x)$? Explain your answer.

Multiplication of Polynomials

We have learnt how to find the product of algebraic expressions using the distributive law.

Worked Example 4

(Multiplication of Polynomials)

If $P(x) = 2x^4 - 4x$ and $Q(x) = x^3 + 3x^2 - 5$, find

(i) $P(x) \times Q(x)$, **(ii)** $P(2) \times Q(2)$,

(iii) the relationship between the degrees of $P(x)$, $Q(x)$ and $P(x) \times Q(x)$.

Solution

(i) $P(x) \times Q(x) = (2x^4 - 4x)(x^3 + 3x^2 - 5)$
$= 2x^7 + 6x^6 - 10x^4 - 4x^4 - 12x^3 + 20x$
$= 2x^7 + 6x^6 - 14x^4 - 12x^3 + 20x$

(ii) $P(2) \times Q(2) = 2(2)^7 + 6(2)^6 - 14(2)^4 - 12(2)^3 + 20(2)$
$= 360$

(iii) Degree of $P(x) = 4$
Degree of $Q(x) = 3$
Degree of $P(x) \times Q(x) = 7$
$\therefore$ Degree of $P(x) \times Q(x)$ = Degree of $P(x)$ + Degree of $Q(x)$

ATTENTION

The product of polynomials is a polynomial.

Practise Now 4

Similar Questions:
Exercise 1B
Questions 3(a)-(e), 4(a)-(e)

If $P(x) = 5x^3 + 2x - 3$ and $Q(x) = x^2 - 3x + 6$, find

(i) $P(x) \times Q(x)$, **(ii)** $P(-1) \times Q(-1)$,

(iii) the relationship between the degrees of $P(x)$, $Q(x)$ and $P(x) \times Q(x)$.

In general, if $P(x)$ and $Q(x)$ are ***non-zero*** polynomials, then

$$\boxed{\text{Degree of } P(x) \times Q(x) = \text{Degree of } P(x) + \text{Degree of } Q(x)}$$

Equality of Polynomials

Two polynomials $P(x)$ and $Q(x)$ are **equal** if and only if the coefficients of the terms with same powers of x are equal.

Given that $Ax^3 + Bx^2 + Cx + D = 2x^3 + 3x^2 - x + 4$ for all values of x, where $A = 2$ and $B = 3$, what is the value of C and of D?

Worked Example 5

(Finding Unknown Coefficients)

Given that $5x^2 - 7x + 3 = A(x-1)(x-2) + B(x-1) + C$ for all values of x, find the values of A, B and C.

Solution

Method 1: Substitute suitable values of x

Since $5x^2 - 7x + 3 = A(x-1)(x-2) + B(x-1) + C$ for **all** values of x, we can select any value of x to find the unknowns A, B and C.

Let $x = 1$:

$$5(1)^2 - 7(1) + 3 = A(1-1)(1-2) + B(1-1) + C$$
$$C = 1$$

Let $x = 2$:

$$5(2)^2 - 7(2) + 3 = A(2-1)(2-2) + B(2-1) + 1$$
$$B = 8$$

Let $x = 0$:

$$3 = A(0-1)(0-2) + 8(0-1) + 1$$
$$2A - 8 + 1 = 3$$
$$A = 5$$

$\therefore\ A = 5,\ B = 8,\ C = 1$

> **PROBLEM-SOLVING TIP**
>
> We choose $x = 1$ to eliminate the terms involving A and B. Why do we choose $x = 2$ and then $x = 0$? Can we choose other values of x such as -1, $\frac{1}{2}$ or 4?

Method 2: Equate coefficients

$$\begin{aligned} 5x^2 - 7x + 3 &= A(x^2 - 3x + 2) + B(x-1) + C \\ &= Ax^2 + (-3A + B)x + 2A - B + C \end{aligned}$$

Equating coefficients of x^2: $5 = A$

Equating coefficients of x:
$$\begin{aligned} -7 &= -3A + B \\ -7 &= -3(5) + B \\ -7 &= -15 + B \\ B &= 8 \end{aligned}$$

Equating coefficients of x^0:
$$\begin{aligned} 3 &= 2A - B + C \\ 3 &= 2(5) - 8 + C \\ 3 &= 10 - 8 + C \\ C &= 1 \end{aligned}$$

$\therefore\ A = 5,\ B = 8,\ C = 1$

Practise Now 5

If $3x^2 + 2x - 9 = A(x-2)(x+1) + B(x-2) + C$ for all values of x, find the values of A, B and C using both of the above methods.

Similar Questions:
Exercise 1B
Questions 7(a)-(c), 9

For Worked Example 5 and Practise Now 5, discuss with your classmates which method you prefer, stating your reason(s) clearly.

Worked Example 6

(Equating Coefficients)
If $2x^3 - 11x - 6 = (x + 2)(ax^2 + bx + c)$, find the values of a, b and c.

Solution

$2x^3 - 11x - 6 = (x + 2)(ax^2 + bx + c)$

By observation,

$a = 2$ (because only $x(ax)^2$ will give the term in x^3)

$c = -3$ (because only $2(c)$ will give the constant term)

$\therefore\ 2x^3 - 11x - 6 = (x + 2)(2x^2 + bx - 3)$

Equating coefficients of x^2: $0 = b + 4$ ($0x^2$ on LHS)

$b = -4$

$\therefore\ a = 2,\ b = -4,\ c = -3$

PROBLEM-SOLVING TIP

If we substitute $x = -2$, we will get 0 on both sides. If we substitute $x = 0$, we can obtain c, but c and a can be found more easily by observation. To find b, we can equate the coefficients of x^2 or of x.

Practise Now 6

Similar Questions:
Exercise 1B
Questions 8(a)-(d), 10

1. For all values of x, $3x^3 + 4x^2 - 17x - 11 = (Ax + 1)(x + B)(x - 2) + C$. Find the values of A, B and C.

2. For all values of x, $6x^3 + 7x^2 - x + 5 = (Ax + B)(2x + 1)(x - 1) + C$. Find the values of A, B and C.

Division of Polynomials

Before we learn how to divide a polynomial by another polynomial, let us recall the long division of numbers, e.g. divide 7 by 3:

$$\begin{array}{r|l} & \;\;2 \leftarrow \text{quotient} \\ \text{divisor} \rightarrow 3 & \;\;7 \leftarrow \text{dividend} \\ & -6 \\ \hline & \;\;1 \leftarrow \text{remainder} \end{array}$$

We can express 7 in terms of the divisor, the quotient and the remainder as follows:

$$\underset{\text{dividend}}{7} = \underset{\text{divisor}}{3} \times \underset{\text{quotient}}{2} + \underset{\text{remainder}}{1}$$

This is called the **Division Algorithm for Positive Integers**.

Class Discussion

Discuss with your classmates:
When you divide one positive integer by another,
(a) is the dividend always larger than or equal to the divisor? Explain your answer.
(b) is the remainder always smaller than the divisor? Explain your answer.

Similarly, we can divide a polynomial by another polynomial using long division and then express the relationship between the two polynomials using the corresponding division algorithm for polynomials.

Worked Example 7

(Long Division of Polynomials)
Divide $x^3 + 2x - 7$ by $x - 2$ and state the remainder.

Solution

$$
\begin{array}{rl}
 & x^2 + 2x + 6 \quad \longleftarrow \text{quotient} \\
\text{divisor} \longrightarrow x - 2 \big) & x^3 + 0x^2 + 2x - 7 \quad \longleftarrow \text{dividend} \\
 & -(x^3 - 2x^2) \\
\hline
 & 2x^2 + 2x \\
 & -(2x^2 - 4x) \\
\hline
 & 6x - 7 \\
 & -(6x - 12) \\
\hline
 & 5 \quad \longleftarrow \text{remainder}
\end{array}
$$

$\therefore$ The remainder is 5.

PROBLEM-SOLVING TIP

The dividend does not have a term in x^2, so we have to put $+ 0x^2$.

Thinking Time

1. For Worked Example 7, express $x^3 + 2x - 7$ in terms of $x - 2$ using the division algorithm.
2. When we divide a polynomial by another polynomial,
   (a) the degree of the dividend must be greater than or equal to the degree of the divisor. Why is this so?
   (b) the degree of the remainder must be smaller than the degree of the divisor. Why is this so?
   (c) what is the relationship between the degrees of the dividend, divisor and quotient?

Practise Now 7

Similar Questions:
Exercise 1B
Questions 5(a)-(k), 6

1. Divide $4x^3 + 7x - 26$ by $2x - 3$ and state the remainder.
2. Divide $4x^3 + 3x^2 - 16x - 12$ by $x + 2$. Hence, express $4x^3 + 3x^2 - 16x - 12$ in terms of $x + 2$.

In general, if a polynomial P(x) is divided by another polynomial D(x), the resulting quotient is Q(x) and the remainder is R(x), then the **Division Algorithm for Polynomials** states that

$$\mathrm{P}(x) = \mathrm{D}(x) \times \mathrm{Q}(x) + \mathrm{R}(x)$$

$$\boxed{\textbf{dividend} = \textbf{divisor} \times \textbf{quotient} + \textbf{remainder}}$$

and **the degree of R(x) < the degree of D(x)**.

Basic Level | Intermediate Level | Advanced Level

Exercise 1B

1 Find the sum of each of the following polynomials.

(a) $x^3 + 5x^2 + 2,\ 3x^2 - 4x + 7$

(b) $2x + 5x^3 - 2x^2 + 7,\ 9 + 4x - 5x^2 - 2x^3$

(c) $5x^3 + 2x^2 + 4,\ 2x^2 + 7x - 5,\ 7 - 5x^2 + 3x - 4x^3$

(d) $2x^4 - 3x^2 + 4x^3 - 5,\ 9x^2 - 7x^3 - 2x^4,\ 7 - x + 3x^4$

2 Subtract

(a) $3x^2 - x - 1$ from $5x^2 + 3x - 7$,

(b) $2x^2 - 7x + 5$ from $5 + 8x - 2x^3$,

(c) $4x^3 + 2x - 9$ from $11 - 3x - 4x^2 + 5x^3$,

(d) $3x^2 + 5x^4 - 6x + 7$ from $8x + 5x^3 + x^5 - 2x^4$.

3 Find each of the following products.

(a) $(2x + 3)(7x - 4)$

(b) $(5x + 7)(6x - 1)$

(c) $(2x - 5)(2x^2 + 5x - 1)$

(d) $(x + 3y)(2x^2 + xy + y)$

(e) $(3x + 5y)(3x^2 - 2xy + y^2)$

4 Simplify each of the following.

(a) $(2x - 1)(3x + 5) + (x + 5)(3x - 7)$

(b) $(5x - 9)(x + 4) - (2x - 3)(7 - x)$

(c) $(x + 3)(x^2 - 5x + 2) + (3x^2 - 6x + 7)(2x - 1)$

(d) $(x + y)(x^2 - xy + y^2) - (2x - y)(x^2 + 5xy - 3y^2)$

(e) $(x + 3)(x^3 - 2x + 4) - (2x - 7)(3 + 4x - 2x^3)$

5 Find the quotient and remainder for each of the following.

(a) $(5x^2 + 7x - 8) \div (x - 1)$

(b) $(-4x^2 + 7x + 160) \div (7 - x)$

(c) $(3x^3 + 4x - 5) \div (x + 1)$

(d) $(x^3 + 7x^2 - 4x) \div (x + 3)$

(e) $(5x^3 + 11x^2 - 5x + 1) \div (x + 2)$

(f) $(9x^3 + 6x^2 + 4x + 5) \div (3x + 1)$

(g) $(12x^3 + 12x^2 + 7x + 1) \div (2x + 1)$

(h) $(3x^3 - 7) \div (x - 3)$

(i) $(x^3 + 4x^2 + 3x + 7) \div (x^2 + 2x + 1)$

(j) $(2x^3 + 5x^2 + 9) \div (x^2 + 3x - 4)$

(k) $(5x^4 + 2x^2 + 7x - 4) \div (x^2 - 5x + 3)$

6 Divide $-x^3 + 7x^2 - 4x - 12$ by $2x - 1$. Hence, express $-x^3 + 7x^2 - 4x - 12$ in terms of $2x - 1$.

7 Find the value of A and of B in each of the following.

(a) $A(3x - 2) + B(5 - x) = 11x - 16$

(b) $A(x - 3)(x + 1) + B(x - 3) = 7x^2 - 17x - 12$

(c) $3x^3 + 5x^2 + 13x + 1 = (3x - 1)(x^2 + 2x + A) + B$

8 Find the values of A, B and C in each of the following.

(a) $4x^2 + 3x - 7 = A(x - 1)(x + 3) + B(x - 1) + C$

(b) $2x^3 + 6x^2 - 2x + 8 = (Ax + 2)(x - 1)(x - B) + C$

(c) $6x^3 - 11x^2 + 6x + 5 = (Ax - 1)(Bx - 1)(x - 1) + C$

(d) $2x^4 - 13x^3 + 19x^2 + 5x + 1 = (x - 4)(Ax + 1)(x^2 + Bx + 1) + C$

9 Find the values of A, B and C, given that $3x^2 + 5x + C = A(x + 1)^2 + B(x + 1) + 4$ for all values of x.

10 The expression $(ax + b)(x - 1) + c(x^2 + 5)$ is equal to 18 for all real values of x. By substituting values of x or otherwise, find the values of a, b and c.

1.3 REMAINDER THEOREM

In the previous section, we have learnt how to use long division to find the remainder when a polynomial is divided by another polynomial. We will now learn another method to find the remainder when a polynomial is divided by a linear divisor.

Investigation

Remainder Theorem

1. Copy and complete the following table. The remainder R can be found by long division. For **(a)**, the long division has been done in Worked Example 7.

No.	Dividend P(x)	Linear Divisor $x - c$	Remainder R	P(c)
(a)	$x^3 + 2x - 7$	$x - 2$		P(2) =
(b)	$4x^3 + 7x + 15$	$x + 1$		P(−1) =
(c)	$3 - 8x + 4x^3 + x^4$	$x + 3$		
(d)	$2x^3 - 4x^2 + 2$	$x - 1$		

2. What is the degree of the remainder? Why?
3. What is the relationship between the remainder and the value of P(c)?
4. What can you say about the divisor when the remainder is zero?

From the investigation, we observe that

> if a polynomial $P(x)$ is divided by a **linear divisor** $x - c$, the remainder is $P(c)$.

By considering the Division Algorithm for Polynomials, how do we prove the above result?

Thinking Time

When a polynomial $P(x)$ is divided by a linear factor $x - c$, then $Q(x)$ and $R(x)$ represent the quotient and remainder respectively. Using the Division Algorithm for Polynomials, express $P(x)$ in terms of $x - c$.

1. State the degree of the polynomial $R(x)$.
   To find $R(x)$, we need to eliminate $Q(x)$. What value of x can be used to eliminate the term involving $Q(x)$? Hence, evaluate the remainder.
2. If $x - c$ is a factor of $P(x)$, what can you say about the remainder? Using your result in Question 1, state the value of $P(c)$.

In general, the **Remainder Theorem** states that

> if a polynomial $P(x)$ is divided by a **linear divisor** $ax + b$,
> the remainder is $P\left(-\frac{b}{a}\right)$.

Worked Example 8

(Using the Remainder Theorem to find the Remainder)

Find the remainder when $4x^3 - 8x^2 + 9x - 5$ is divided by

(a) $x + 2$, **(b)** $2x - 3$.

Solution

Let $P(x) = 4x^3 - 8x^2 + 9x - 5$.

(a)
$$\begin{aligned} R &= P(-2) \\ &= 4(-2)^3 - 8(-2)^2 + 9(-2) - 5 \\ &= -87 \end{aligned}$$

(b)
$$\begin{aligned} R &= P\left(\frac{3}{2}\right) \\ &= 4\left(\frac{3}{2}\right)^3 - 8\left(\frac{3}{2}\right)^2 + 9\left(\frac{3}{2}\right) - 5 \\ &= 4 \end{aligned}$$

Practise Now 8

Find the remainder when $3x^3 - 4x^2 - 5$ is divided by

(a) $x - 2$, **(b)** $x + 5$, **(c)** $2x - 3$.

Similar Questions:
Exercise 1C
Questions 1(a)-(f)

Worked Example 9

(Finding an Unknown using the Remainder Theorem)

Find the value of k if $4x^7 + 5x^3 - 2kx^2 + 7k - 4$ has a remainder of 12 when divided by $x + 1$.

Solution

Let $f(x) = 4x^7 + 5x^3 - 2kx^2 + 7k - 4$.

Using the Remainder Theorem, $f(-1) = 12$.

i.e. $4(-1)^7 + 5(-1)^3 - 2k(-1)^2 + 7k - 4 = 12$

$$-4 - 5 - 2k + 7k - 4 = 12$$

$$5k = 25$$

$$\therefore \quad k = 5$$

Practise Now 9

Similar Questions:
Exercise 1C
Questions 2-10

1. Find the value of k if $5x^5 + 2kx^3 - 6kx^2 + 9$ has a remainder of 22 when divided by $x - 1$.
2. The expression $4x^3 + Ax^2 + Bx + 3$ leaves a remainder of 9 when divided by $x - 1$ and a remainder of -27 when divided by $x + 3$. Find the value of A and of B.

Basic Level | Intermediate Level | Advanced Level

Exercise 1C

1 Find the remainder of each of the following.

(a) $x^3 - 5x^2 + 3x + 7$ is divided by $x - 1$
(b) $-2x^3 - 7x^2 + 6x - 9$ is divided by $x + 1$
(c) $x^3 + 2x - 5$ is divided by $x + 3$
(d) $-5x^2 + 3x - 6$ is divided by $2x + 1$
(e) $5x^3 + 4x^2 - 6x + 7$ is divided by $x - 2$
(f) $2x^3 + 4x^2 - 6x + 7$ is divided by $2x - 1$

2 Find the value of k in each of the following.

(a) $x^3 - 3x^2 + 8kx + 5$ has a remainder of 17 when divided by $x - 3$.
(b) $x^3 + x^2 - kx + 4$ has a remainder of 8 when divided by $x - 2$.
(c) $7x^9 + kx^8 + 3x^5 + 5x^4 + 7$ has a remainder of -3 when divided by $x + 1$.
(d) $5x^3 - 4x^2 + (k + 1)x - 5k$ has a remainder of -2 when divided by $x + 2$.

3 The remainder obtained when $5x^3 - 6x^2 + kx - 3$ is divided by $x - 1$ is equal to the remainder obtained when the same expression is divided by $x - 2$. Find the value of k.

4 The expression $2x^3 + hx^2 - 6x + 1$ leaves a remainder of $2k$ when divided by $x + 2$ and a remainder of k when the same expression is divided by $x - 1$. Find the value of h and of k.

5 Given that $y = 3x^3 + 7x^2 - 48x + 49$ and that y has the same remainder when it is divided by $x + k$ or $x - k$, find the possible values of k.

6 When $x^6 + 5x^3 - px - q$ and $px^2 - qx - 1$ are each divided by $x + 1$, the remainders obtained are 7 and -6 respectively. Find the value of p and of q.

Basic Level | Intermediate Level | Advanced Level

Exercise 1C

7. When the expression $2x^4 + px^3 + 5x^2 + 7$ is divided by $(x + 1)(x - 2)$, the remainder is $qx + 18$. This result may be expressed as $2x^4 + px^3 + 5x^2 + 7 = (x + 1)(x - 2)Q(x) + (qx + 18)$, where $Q(x)$ is the quotient. By substituting suitable values of x, find the value of p and of q.

8. Find the value of k if $(3x + k)^3 + (4x - 7)^2$ has a remainder of 33 when divided by $x - 3$.

9. When the expression $3x^3 + px^2 + qx + 8$ is divided by $x^2 - 3x + 2$, the remainder is $5x + 6$. Find the value of p and of q.

10. When $x^2 + ax + b$ and $x^2 + hx + k$ are divided by $x + p$, their remainders are equal. Express p in terms of a, b, h and k.

1.4 FACTOR THEOREM

From the previous section, we have learnt that if a polynomial $P(x)$ is divided by a linear divisor $x - c$, then the Division Algorithm for Polynomials states that

$$P(x) = (x - c)Q(x) + R$$

and the Remainder Theorem states that the remainder $R = P(c)$.
What if the remainder $R = 0$?
Then $P(c) = 0$ and $P(x) = (x - c)Q(x)$, i.e. $x - c$ is a **factor** of $P(x)$.

In other words,

> $x - c$ is a linear factor of the polynomial $P(x)$ if and only if $P(c) = 0$.

In general, the **Factor Theorem** states that

> $ax + b$ is a linear factor of the polynomial $P(x)$
>
> if and only if $P\left(-\frac{b}{a}\right) = 0$.

The Factor Theorem is a **special case** of the Remainder Theorem when the remainder = 0.

Worked Example 10

(Finding an Unknown using the Factor Theorem)
Find the value of k for which $x - 2$ is a factor of $f(x) = 3x^3 - 2x^2 + 5x + k$.
Hence, find the remainder when $f(x)$ is divided by $2x + 3$.

Solution

$f(x) = 3x^3 - 2x^2 + 5x + k$

$x - 2$ is a factor of $f(x)$, so $f(2) = 0$.

i.e. $\quad 3(2)^3 - 2(2)^2 + 5(2) + k = 0$

$k = -26$

$\therefore \quad f(x) = 3x^3 - 2x^2 + 5x - 26$

When $f(x)$ is divided by $2x + 3$, remainder $R = f\left(-\frac{3}{2}\right)$.

i.e. $$R = 3\left(-\frac{3}{2}\right)^3 - 2\left(-\frac{3}{2}\right)^2 + 5\left(-\frac{3}{2}\right) - 26$$

$$= -48\frac{1}{8}$$

Practise Now 10

Similar Questions:
Exercise 1D
Questions 1-5

Find the value of k for which $x - 3$ is a factor of $f(x) = 4x^3 + 6x^2 - 9x + 2k$. Hence, find the remainder when $f(x)$ is divided by $x + 2$.

Worked Example 11

(Finding an Unknown using the Factor Theorem)

Given that $3x^2 - x - 2$ is a factor of the polynomial $3x^3 + hx^2 - 5x + k$, find the values of h and of k. Hence, factorise the polynomial completely.

Solution

Let $f(x) = 3x^3 + hx^2 - 5x + k$.

Since $3x^2 - x - 2 = (3x + 2)(x - 1)$ is a factor of $f(x)$, then $3x + 2$ and $x - 1$ are also factors of $f(x)$.

By the Factor Theorem, $f\left(-\frac{2}{3}\right) = 0$ and $f(1) = 0$.

i.e. $f\left(-\frac{2}{3}\right) = 3\left(-\frac{2}{3}\right)^3 + h\left(-\frac{2}{3}\right)^2 - 5\left(-\frac{2}{3}\right) + k = 0$

$$-\frac{8}{9} + \frac{4}{9}h + \frac{10}{3} + k = 0$$

$$-8 + 4h + 30 + 9k = 0$$

$$4h + 9k = -22 \text{ --- (1)}$$

$f(1) = 3(1)^3 + h(1)^2 - 5(1) + k = 0$

$$3 + h - 5 + k = 0$$

$$h + k = 2 \text{ --- (2)}$$

(2) × 4: $\quad 4h + 4k = 8$ --- (3)

(1) – (3): $\quad 5k = -30$

$$k = -6$$

Subst. $k = -6$ into (2):

$$h - 6 = 2$$

$$h = 8$$

$\therefore h = 8$ and $k = -6$

Hence, $f(x) = 3x^3 + 8x^2 - 5x - 6 = (3x^2 - x - 2)(ax + b)$

By observation, $a = 1$ and $b = 3$.

$\therefore 3x^3 + 8x^2 - 5x - 6 = (3x^2 - x - 2)(x + 3)$

$= (3x + 2)(x - 1)(x + 3)$

ATTENTION

Since $3x^2 - x - 2$ is a factor of $f(x)$, then
$f(x) = (3x^2 - x - 2)Q(x)$
$= (3x + 2)(x - 1)Q(x)$.
This shows that $3x + 2$ and $x - 1$ are also factors of $f(x)$.

RECALL

Refer to Worked Example 6 on how to find a and b by observation. You can also use long division.

Practise Now 11

Similar Questions:
Exercise 1D
Questions 6-10

If $x^2 - 3x + 2$ is a factor of $f(x) = 2x^4 + px^3 + x^2 + qx - 12$, find the value of p and of q. Hence, factorise $f(x)$ completely.

Basic Level | Intermediate Level | Advanced Level

Exercise 1D

1. Find the value of k in each of the following.
   (a) $x - 1$ is a factor of $3x^3 + 7x^2 - kx + 5$
   (b) $x + 2$ is a factor of $2x^3 + kx^2 - 8x - 18$
   (c) $x + 4$ is a factor of $x^3 + 5x^2 - kx + 48$
   (d) $2x - 1$ is a factor of $4x^4 + 3x^3 - 7x^2 + 14x + k$
   (e) $x - 1$ is a factor of $k^2x^4 - 3kx^2 + 2$
   (f) $x + 2$ is a factor of $(x + 1)^5 + (5x + k)^3$

2. Given that $x - 1$ is a factor of the expression $x^3 - kx + 2$, find the value of k and the remainder when the expression is divided by $x - 2$.

3. Given that $x^2 - x - 2$ and $x^3 + kx^2 - 10x + 6$ have a common factor, find the possible values of k.

4. Find the value of p and of q for which $x - 3$ is a common factor of the expressions $x^2 + (p + q)x - q$ and $2x^2 + (p - 1)x + (p + 2q)$.

5. Find the value of p and of q for which $x + 1$ and $2x - 1$ are factors of $4x^4 + 2x^3 + (q - 1)x - q - p$. Hence, explain why $x - 3$ is not a factor of the expression.

6. Given that $x^3 + 3x^2 + hx + k$, where h and k are constants, is exactly divisible by $x - 2$ and leaves a remainder of 30 when divided by $x + 1$, find the value of h and of k. Hence, factorise the expression completely.

7. If $x - k$ is a factor of the expression $kx^3 + 5x^2 - 7kx - 8$, where k is a positive integer, find the value of k. Hence, find the other factors of the expression.

8. If $x^2 - 2x - 3$ is a factor of the expression $x^4 + px^3 + qx - 81$, find the value of p and of q. Hence, factorise the expression completely.

9. It is given that $x + 2$ is a factor of $f(x) = a(x - 1)^2 + b(x - 1) + c$. The remainders when $f(x)$ is divided by $x + 1$ and $x - 1$ are -11 and 9 respectively. Find the values of a, b and c.

10. Given that $x + 1$ and $x - 3$ are factors of the expression $x^4 + px^3 + 5x^2 + 5x + q$, find the value of p and of q. Hence, find the other two factors of the expression.

1.5 CUBIC EXPRESSIONS AND EQUATIONS

Cubic Expressions

Worked Example 12

(Factorisation of Cubic Expressions)

Factorise $x^3 - x^2 - 4x + 4$ completely.

Solution

Let $f(x) = x^3 - x^2 - 4x + 4$.

Since the polynomial in x is of degree 3, it will not have more than three linear factors. Suppose a, b and c are integers such that $f(x) = (x - a)(x - b)(x - c)$, then the product of the constant term of each factor is abc and this should be equal to 4, i.e. a, b and c are factors of 4. Thus, the possible factors of $f(x)$ are $x \pm 1$, $x \pm 2$ and $x \pm 4$.

Try $x - 1$, $f(1) = 1 - 1 - 4 + 4 = 0$.

$\therefore$ $x - 1$ is a factor of $f(x)$.

The other factors can be found by any of the 3 methods shown below.

Method 1: Trial and error

Try $x + 2$, $f(-2) = -8 - 4 + 8 + 4 = 0$.

$\therefore$ $x + 2$ is a factor of $f(x)$.

Try $x - 2$, $f(2) = 8 - 4 - 8 + 4 = 0$.

$\therefore$ $x - 2$ is a factor of $f(x)$.

$\therefore$ $f(x) = (x - 1)(x + 2)(x - 2)$

Method 2: Long division

$$
\begin{array}{r@{}l}
 & x^2 \qquad\quad - 4 \\
x-1\,\big) & \overline{x^3 - x^2 - 4x + 4} \\
 & -(x^3 - x^2) \\
 & \overline{\qquad\qquad -4x + 4} \\
 & \qquad\quad -(-4x + 4) \\
 & \overline{\qquad\qquad\qquad 0}
\end{array}
$$

$\therefore$ $f(x) = (x - 1)(x^2 - 4)$
$= (x - 1)(x + 2)(x - 2)$

Method 3: Comparing coefficients

Since $f(x)$ is of degree 3 and $x - 1$ is one of the factors, the other factor must be of degree 2 and in the form $ax^2 + bx + c$.

$$x^3 - x^2 - 4x + 4 = (x - 1)(ax^2 + bx + c)$$
$$= ax^3 + (b - a)x^2 + (c - b)x - c$$

By observation, $a = 1$ and $c = -4$.

Equating the coefficients of x : $c - b = -4$
$b = 0$

$\therefore f(x) = (x - 1)(x^2 - 4) = (x - 1)(x + 2)(x - 2)$

Practise Now 12

Similar Questions:
Exercise 1E
Questions 1(a)-(g)

1. Factorise $x^3 + 2x^2 - 29x + 42$ completely.

2. Factorise $2x^3 + 9x^2 + 7x - 6$ completely.

Investigation

Number of Real Roots of a Cubic Equation

Use a suitable graphing software to plot the graph of $y = x^3 - 5x^2 + 2x + 8$.

(i) What are the x-coordinates of the points of intersection of the graph of $y = x^3 - 5x^2 + 2x + 8$ and the x-axis?

(ii) Solve the equation $x^3 - 5x^2 + 2x + 8 = 0$.

(iii) What is the relationship between the solutions of a cubic equation and the x-coordinates of the points of intersection of the curve with the x-axis?

(iv) What are the linear factors of the polynomial? Hence, factorise the polynomial completely. Check your answer using methods you have learnt earlier.

Repeat the above by drawing the graphs of $y = x^3 + 4x^2 - 3x - 18$, $y = x^3 + 4x^2 - 23x + 6$, $y = x^3 - 8x^2 + 13x - 6$ and $y = x^3 - x^2 + 4x - 12$.

Using the graph, can you state a linear factor of each cubic polynomial?

What can you conclude about the number of real roots of a cubic polynomial and the number of points of intersection of the graph with the x-axis?

Cubic Equations

To solve the quadratic equation $ax^2 + bx + c = 0$, we can use the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this section, we will learn how to solve cubic equations of the form $ax^3 + bx^2 + cx + d = 0$ $(a \neq 0)$, with the help of the Factor Theorem to find a linear factor first and by factorisation subsequently.

Worked Example 13

(Solving a Cubic Equation)

Solve the cubic equation $2x^3 + 3x^2 - 5x - 6 = 0$.

Solution

Let $P(x) = 2x^3 + 3x^2 - 5x - 6$.

The positive and negative factors of -6 are ±1, ±2, ±3 and ±6.

By trial and error, $P(-1) = -2 + 3 + 5 - 6 = 0$.

By the Factor Theorem, $x - (-1)$, i.e. $x + 1$, is a factor of $P(x)$.

$\therefore$ $P(x) = 2x^3 + 3x^2 - 5x - 6 = (x + 1)(ax^2 + bx + c)$

By observation, $a = 2$ and $c = -6$.

$\therefore$ $P(x) = 2x^3 + 3x^2 - 5x - 6 = (x + 1)(2x^2 + bx - 6)$

Equating coefficients of x^2: $3 = b + 2$

$b = 1$

$\therefore$ $P(x) = 2x^3 + 3x^2 - 5x - 6 = (x + 1)(2x^2 + x - 6)$

$= (x + 1)(2x - 3)(x + 2)$

Hence, $2x^3 + 3x^2 - 5x - 6 = 0$

$(x + 1)(2x - 3)(x + 2) = 0$

$x = -2, -1$ or $\frac{3}{2}$

INFORMATION

Since $P(1) = 2 + 3 - 5 - 6 \neq 0$, so $x - 1$ is not a factor of $P(x)$. By trial and error, you may have obtained $P(-2) = 0$, which implies that $x + 2$ is a factor of $P(x)$. Then $P(x) = (x + 2)(ax^2 + bx + c)$.

RECALL

We have learnt how to equate coefficients in Worked Example 6.

Practise Now 13

Similar Questions:
Exercise 1E
Questions 3(a)-(e)

1. Solve the equation $2x^3 + 11x^2 - 7x - 6 = 0$.

2. Solve the equation $3x^3 + 4x^2 - 49x + 30 = 0$.

With reference to Worked Example 13, discuss with your classmates:

1. We can also find the values of a, b and c by dividing $2x^3 + 3x^2 - 5x - 6$ by $x + 1$ using long division. Which method do you prefer? Explain your choice.
2. What value of x must you substitute into $P(x)$ so that you can use the Factor Theorem to conclude that $2x - 3$ is a factor? How is this value of x linked to the factors of the constant term -6 and the coefficient of x^3?
3. Can you use the Factor Theorem to obtain **all** the factors of the cubic polynomial? Explain your answer.

From the discussion, we observe that it may not be possible to find all the factors using the Factor Theorem if there is only one linear factor for the cubic polynomial, e.g.

$$f(x) = 8x^3 - 10x^2 + x - 6 = (2x - 3)(4x^2 + x + 2),$$

where the second factor is quadratic and *cannot be factorised* into two linear factors.

Moreover, to find the linear factor, it is not enough to substitute only the positive and negative factors of the constant term -6 into $f(x)$ because in this case $f\left(\frac{3}{2}\right) = 0$.

If the coefficient of x^3 is not 1, then we have to consider the factors of the coefficient of x^3 (i.e. 8) and the constant term (i.e. -6) to find the linear factor.

From the linear factor $2x - 3$ given above, we observe that

- the numerator 3 of $\frac{3}{2}$ is a factor of -6, the constant term,
- the denominator 2 of $\frac{3}{2}$ is a factor of 8, the coefficient of x^3.

From the above, how do you determine the values of x that can be substituted into $f(x) = ax^3 + bx^2 + cx + d$, so as to find a linear factor of $f(x)$?

Worked Example 14

(Solving a Cubic Equation)

Solve the cubic equation $8x^3 + 10x^2 - 5x - 6 = 0$.

Solution

Let $f(x) = 8x^3 + 10x^2 - 5x - 6$.

The positive and negative factors of -6 are ± 1, ± 2, ± 3 and ± 6 and those of 8 are ± 1, ± 2, ± 4 and ± 8.

By trial and error,

$$f\left(-\frac{3}{4}\right) = 8\left(-\frac{3}{4}\right)^3 + 10\left(-\frac{3}{4}\right)^2 - 5\left(-\frac{3}{4}\right) - 6 = 0.$$

By the Factor Theorem, $x + \frac{3}{4}$, i.e. $4x + 3$, is a factor of $f(x)$.

$\therefore\ f(x) = 8x^3 + 10x^2 - 5x - 6 = (4x + 3)(ax^2 + bx + c)$

By observation, $a = 2$ and $c = -2$.

$\therefore\ f(x) = 8x^3 + 10x^2 - 5x - 6 = (4x + 3)(2x^2 + bx - 2)$

Equating coefficients of x: $-5 = 3b - 8$

$$b = 1$$

$\therefore\ f(x) = 8x^3 + 10x^2 - 5x - 6 = (4x + 3)(2x^2 + x - 2)$

Hence,

$$\begin{aligned} 8x^3 + 10x^2 - 5x - 6 &= 0 \\ (4x + 3)(2x^2 + x - 2) &= 0 \\ x &= -\frac{3}{4} \text{ or } \frac{-1 \pm \sqrt{1^2 - 4(2)(-2)}}{2(2)} \\ &= -\frac{3}{4} \text{ or } \frac{-1 \pm \sqrt{17}}{4} \\ &= -0.75, -1.28 \text{ or } 0.781 \\ &\quad \text{(to 3 s.f.)} \end{aligned}$$

PROBLEM-SOLVING TIP

Notice that $-\frac{3}{4}$ is the same as $\frac{-3}{4} = \frac{3}{-4} = \frac{-6}{8} = \frac{6}{-8}$.

RECALL

If $ax^2 + bx + c$ cannot be factorised, then for the equation $ax^2 + bx + c = 0$, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

Practise Now 14

Similar Questions:
Exercise 1E
Questions 4(a)-(f)

1. Solve the equation $x^3 + 4x^2 - 23x + 6 = 0$, giving your answers correct to 2 decimal places where necessary.

2. Solve the equation $x^3 + 5x^2 - 19x + 10 = 0$, giving your answers correct to 2 decimal places where necessary.

Worked Example 15

(Finding an Unknown Polynomial)

The term containing the highest power of x in the polynomial $f(x)$ is $6x^3$. One of the roots of the equation $f(x) = 0$ is $\frac{1}{2}$. Given that $x^2 - 2x + 3$ is a quadratic factor of $f(x)$, find

(i) an expression for $f(x)$ in ascending powers of x,

(ii) the number of real roots of the equation $f(x) = 0$.

Solution

(i) Since $\frac{1}{2}$ is a root of the equation $f(x) = 0$, by Factor Theorem, $x - \frac{1}{2}$, i.e. $2x - 1$ is a factor of $f(x)$.

Since $2x - 1$ and $x^2 - 2x + 3$ are factors of $f(x)$, hence $f(x)$ can be written as $f(x) = a(2x - 1)(x^2 - 2x + 3)$.

Since the term containing the highest power of x in $f(x)$ is $6x^3$, by observation (or equating coefficients of x^3), $a = 3$.

$$\therefore\ f(x) = 3(2x - 1)(x^2 - 2x + 3)$$
$$= 6x^3 - 15x^2 + 24x - 9$$

(ii) $f(x) = 3(2x - 1)(x^2 - 2x + 3) = 0$

$2x - 1 = 0$ or $x^2 - 2x + 3 = 0$

$$x = \frac{1}{2} \quad \text{or} \quad x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(3)}}{2(1)}$$

$$= \frac{1}{2} \quad \text{or} \quad x = \frac{2 \pm \sqrt{-8}}{2} \text{ (no real solution)}$$

$\therefore$ $f(x) = 0$ has only one real root.

Practise Now 15

Similar Questions:
Exercise 1E
Questions 5, 6

1. The term containing the highest power of x in the polynomial $f(x)$ is $6x^4$. Given that $x^2 - 5x + 1$ is a quadratic factor of $f(x)$ and two of the solutions of $f(x) = 0$ are $\frac{1}{2}$ and 2, find an expression for $f(x)$ in descending powers of x.

2. It is given that $x^2 - 5x + 3$ is a quadratic factor of $f(x)$ and that two of the roots of the equation $f(x) = 0$ are $x = -2$ and $x = 3$. If the term containing the highest power of x in $f(x)$ is $2x^4$, find an expression for $f(x)$ in ascending powers of x.

Thinking Time

How many real roots can a cubic equation have? Can it have 2 real roots? Explain your reasons clearly.
Hint: See Worked Examples 13-15 and the investigation earlier in this section 1.5.

Recall the following three algebraic identities:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$
$$a^2 - b^2 = (a + b)(a - b)$$

ATTENTION

$(a + b)^2 \neq a^2 + b^2$
$(a - b)^2 \neq a^2 - b^2$
$(a + b)^3 \neq a^3 + b^3$
$(a - b)^3 \neq a^3 - b^3$

We have also learnt that $a^2 + b^2$ cannot be factorised into two linear factors.

In this section, we will learn how to factorise $a^3 + b^3$ and $a^3 - b^3$.

Investigation

Factorisation of $a^3 \pm b^3$

1. Factorise each of the following expressions completely.
(i) $a^3 + a^2b + ab^2$,
(ii) $a^2b + ab^2 + b^3$.
By subtracting your answer in **(ii)** from that in **(i)**, factorise $a^3 - b^3$.

2. Factorise each of the following expressions completely.
(i) $a^3 - a^2b + ab^2$,
(ii) $a^2b - ab^2 + b^3$.
By adding your answers in **(i)** and **(ii)**, factorise $a^3 + b^3$.

Use the Factor Theorem to find the factors of $a^3 - b^3$ and $a^3 + b^3$.
Do you obtain the same answers as above?

From the above investigation, we have two special algebraic identities:

Sum of Cubes:	$a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
Difference of Cubes:	$a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Thinking Time

Using Factor Theorem, explain *why*
(i) $a + b$ is a factor of $a^3 + b^3$,
(ii) $a - b$ is a factor of $a^3 - b^3$.

Worked Example 16

(Factorisation using the Sum and Difference of Two Cubes)

Factorise **(a)** $8x^3 + 125y^3$ **(b)** $(3x + 1)^3 - 8$

Solution

(a) $$\begin{aligned} 8x^3 + 125y^3 &= (2x)^3 + (5y)^3 \\ &= (2x + 5y)[(2x)^2 - (2x)(5y) + (5y)^2] \\ &= (2x + 5y)(4x^2 - 10xy + 25y^2) \end{aligned}$$

(b) $$\begin{aligned} (3x + 1)^3 - 8 &= (3x + 1)^3 - (2)^3 \\ &= [(3x + 1) - 2][(3x + 1)^2 + (3x + 1)(2) + 2^2] \\ &= (3x - 1)[9x^2 + 6x + 1 + 6x + 2 + 4] \\ &= (3x - 1)(9x^2 + 12x + 7) \end{aligned}$$

Practise Now 16

Factorise **(a)** $216p^3 + 343q^6$ **(b)** $64 - (2x + 3)^3$

Similar Questions:
Exercise 1E
Questions 2(a)-(e)

Applications of Cubic Equations

Cubic equations can be found in Physics and Chemistry problems involving electrical resistance or equations of state for real gases. Break-even points in economics and just-in-time manufacturing also involve the solving of cubic equations. You can search the Internet for more information.

Worked Example 17

(Van der Waals Equation)

The Van der Waals equation is an equation that relates several properties of real gases, which may be written as

$$v^3 - \frac{1}{3}\left(1 + \frac{8T}{p}\right)v^2 + \frac{3}{p}v - \frac{1}{p} = 0,$$

where v, T and p are the volume, temperature and pressure of the gas respectively.

At the critical temperature, i.e. $T = 1$ K and $p = 1$ atm, the equation becomes $v^3 - 3v^2 + 3v - 1 = 0$, where v is measured in dm^3.

Solve the equation $v^3 - 3v^2 + 3v - 1 = 0$ to find the value of v at the critical temperature.

Solution

Let $f(v) = v^3 - 3v^2 + 3v - 1$.
The positive and negative factors of -1 are ± 1.
By trial and error, $f(1) = (1)^3 - 3(1)^2 + 3(1) - 1 = 0$.
By the Factor Theorem, $v - 1$ is a factor of $f(v)$.
$\therefore\ f(v) = v^3 - 3v^2 + 3v - 1 = (v - 1)(av^2 + bv + c)$
By observation, $a = 1$ and $c = 1$.
$\therefore\ f(v) = (v - 1)(v^2 + bv + 1)$
Equating coefficients of v: $3 = -b + 1$
$b = -2$
$\therefore\ f(v) = (v - 1)(v^2 - 2v + 1) = (v - 1)^3$
Hence, $v^3 - 3v^2 + 3v - 1 = 0$
$(v - 1)^3 = 0$
$v = 1$
$\therefore$ At the critical temperature, the volume is 1 dm^3.

Practise Now 17

Similar Questions:
Exercise 1E
Question 7

The price of an object, $P(x)$, is related to the supply quantity, x, by the polynomial $P(x) = x^3 - 4x^2 + hx$, where h is a constant and $x \geqslant 3$. Given that $P(5) = 29$, find the value of h.

Basic Level | Intermediate Level | Advanced Level

Exercise 1E

1 Factorise each of the following.

(a) $x^3 - 5x^2 - x + 5$ **(b)** $2x^3 + x^2 - 22x + 24$ **(c)** $2x^3 - 11x^2 - x + 30$
(d) $x^3 - 9x^2 + 26x - 24$ **(e)** $2x^3 + x^2 - 13x + 6$ **(f)** $x^3 - 6x^2 - x + 30$
(g) $12x^3 - 8x^2 - 3x + 2$

2 Factorise each of the following.

(a) $x^3 - 8$ **(b)** $125x^3 - 64y^3$ **(c)** $27x^6 + 8y^9$
(d) $343a^3 + 216x^6$ **(e)** $216 - (3x - 4)^3$

3 Solve each of the following equations.

(a) $x^3 + 2x^2 - x - 2 = 0$

(b) $x^3 - 3x^2 - 4x + 12 = 0$

(c) $4x^3 + x^2 - 27x + 18 = 0$

(d) $3x^3 - 16x^2 - 37x + 14 = 0$

(e) $5x^3 - 26x^2 + 35x - 6 = 0$

4 Solve each of the following equations, giving your answers correct to 2 decimal places where necessary.

(a) $x^3 - 9x^2 + 11x + 6 = 0$

(b) $x^3 - 8x^2 + 16x - 3 = 0$

(c) $x^3 - 10x^2 + 28x - 16 = 0$

(d) $2x^3 - 3x^2 - 13x + 7 = 0$

(e) $3x^3 - 23x^2 + 7x + 5 = 0$

(f) $4x^3 - 16x^2 + 15x - 4 = 0$

(g) $x^4 + 2x^3 - 7x^2 - 8x + 12 = 0$

(h) $2x^4 - 19x^3 + 61x^2 - 74x + 24 = 0$

5 The term containing the highest power of x in the polynomial f(x) is $3x^4$. Given that two of the roots of the equation f(x) = 0 are -2 and 4 and that $x^2 - 7x + 2$ is a quadratic factor of f(x), find an expression for f(x) in descending powers of x.

6 It is given that $2x^2 - 7x + 1$ is a quadratic factor of f(x) and the term containing the highest power of x in f(x) is $4x^4$. If two of the roots of f(x) = 0 are -0.5 and 3, find an expression for f(x) in ascending powers of x.

7 The sum of the radii of two spherical balls is 22 cm and the difference of their volumes is $3050\frac{2}{3}$ cm^3. Find the radii of the balls. (Take π to be $\frac{22}{7}$.)

8 The expression $x^{2n} - k$ has $x + 3$ as a factor and leaves a remainder of -80 when divided by $x + 1$. Calculate the value of n and of k. With these values of n and k, factorise $x^{2n} - k$ completely.

9 Given that $f(x) = x^{2n} - (p + 1)x^2 + p$, where n and p are positive integers, show that $x - 1$ is a factor of f(x) for all values of p. When $p = 4$, find the value of n for which $x + 2$ is a factor of f(x). Factorise f(x) completely.

10 Given that $81x^4 - 6x^3 - 9a^2x^2 - 11x - 3\frac{8}{9}$ is divisible by $3x + a$, show that $2a^3 + 33a - 35 = 0$. Hence, solve the equation for all real values of a.

11 The data in the table lists the annual water consumption of each person in a city.

Year	2000	2002	2004	2006	2008	2010
Water used (m^3)	57.6	59.8	60.0	60.6	64.0	72.6

(i) Using ICT, generate a scatter plot for the data provided.

(ii) Suggest a polynomial function that may be used to model the data.

(iii) It is believed that a polynomial of the form $y = ax^3 + bx^2 + cx + d$ models the data. Hence, estimate the water consumption in 2012.

1.6 PARTIAL FRACTIONS

excluded from the N(A) syllabus

Algebraic Fractions (Recap)

Three examples of algebraic fractions are

$$\frac{1}{5-x},\ -\frac{x^3+8}{4x^2-x+6} \text{ and } \frac{7}{x+\sqrt{x}}.$$

The numerator and denominator of the first two algebraic fractions are *polynomials* while the denominator of the last algebraic fraction is not a polynomial.

In this section, we will study only algebraic fractions that are ratios of two polynomials P(x) and Q(x), i.e. $\frac{\text{P}(x)}{\text{Q}(x)}$, where $\text{Q}(x) \neq 0$.

We have learnt about proper and improper fractions.

For example, $\frac{2}{3}$ is a *proper* fraction because the numerator < the denominator; while $\frac{7}{3}$ and $\frac{3}{3}$ are *improper* fractions because the numerator ⩾ the denominator.

So how do we define a proper and an improper algebraic fraction?

INFORMATION

An algebraic fraction $\frac{\text{P}(x)}{\text{Q}(x)}$, where P($x$) and Q($x$) are polynomials and $\text{Q}(x) \neq 0$, is also called a rational expression.

Investigation

Proper and Improper Algebraic Fractions

The table below shows some examples of proper and improper algebraic fractions $\frac{\text{P}(x)}{\text{Q}(x)}$.

No.	Proper Algebraic Fractions	Improper Algebraic Fractions
(a)	$\frac{1}{5-x}$	$-\frac{x^3+8}{4x^2-x+6}$
(b)	$\frac{8-x}{x^2-2x+3}$	$\frac{x^4-x^3+2x}{9-5x}$
(c)	$-\frac{4-2x+3x^4}{2x^5+7}$	$\frac{x}{x+2}$

Compare the degrees of P(x) and Q(x) in both the proper and improper algebraic fractions.

(i) What do you call an algebraic fraction if the degree of P(x) < the degree of Q(x)?

(ii) What do you call an algebraic fraction if the degree of P(x) ⩾ the degree of Q(x)?

We have learnt in Section 1.2 that we can use the Division Algorithm for Positive Integers to express an *improper* fraction, such as $\frac{7}{3}$, as:

$$\underset{\text{dividend}}{7} = \underset{\text{divisor}}{3} \times \underset{\text{quotient}}{2} + \underset{\text{remainder}}{1}$$

We can also divide the above equation by the divisor 3, so another way to express $\frac{7}{3}$ is:

$$\frac{7}{3} = 2 + \frac{1}{3}$$

which is the sum of a positive integer and a *proper* fraction.

Similarly, we can use the Division Algorithm for Polynomials to express an **improper** algebraic fraction as the sum of a **polynomial** and a **proper** algebraic fraction.

Thinking Time

(i) Is $\frac{x^3+3}{x-1}$ a proper or an improper algebraic fraction?

(ii) When $x^3 + 3$ is divided by $x - 1$, the quotient is $x^2 + x + 1$ and the remainder is 4.

Use the Division Algorithm for Polynomials to express $x^3 + 3$ in terms of $x - 1$. Hence write $\frac{x^3+3}{x-1}$ as the sum of a polynomial and a proper fraction.

In general, another way to write the Division Algorithm is:

$$\boxed{\frac{\text{Dividend}}{\text{Divisor}} = \text{Quotient} + \frac{\text{Remainder}}{\text{Divisor}}}$$

Worked Example 18

(Expressing an Improper Algebraic Fraction as the Sum of a Polynomial and a Proper Algebraic Fraction)

Express $\frac{x^3+2x^2-3}{x+1}$ as the sum of a polynomial and a proper fraction.

Solution

By long division, we have

$$\begin{array}{r l}
 & x^2 + x - 1 \\
x+1 \,\big) & x^3 + 2x^2 \qquad - 3 \\
 & -(x^3 + x^2) \\
\hline
 & x^2 + 0x \\
 & -(x^2 + x) \\
\hline
 & -x - 3 \\
 & -(-x - 1) \\
\hline
 & -2
\end{array}$$

(leave a space for the term in x)

(write $0x$ in the space)

$$\therefore \frac{x^3+2x^2-3}{x+1} = x^2 + x - 1 - \frac{2}{x+1}$$

Practise Now 18

Similar Questions:
Exercise 1F
Questions 1-8

Express each of the following improper fractions as the sum of a polynomial and a proper fraction.

(a) $\dfrac{2x^3+7x+5}{x-2}$ **(b)** $\dfrac{6x^2-8x+9}{(3x-1)(x+2)}$

Basic Level | Intermediate Level | Advanced Level

Exercise 1F

excluded from the N(A) syllabus

1 Express each of the following improper fractions as the sum of a polynomial and a proper fraction.

(a) $\dfrac{8x+3}{2x-1}$ **(b)** $\dfrac{15x^3+3}{3x^2-1}$

(c) $\dfrac{14x^2+3}{2x^2+7}$

2 Express each of the following improper fractions as the sum of a polynomial and a proper fraction.

(a) $\dfrac{15x^2-7x+3}{x^2-5x-2}$

(b) $\dfrac{4x^3-4x^2+2x-1}{(2x-3)(x+1)}$

(c) $\dfrac{3x^3-x^2-24x+6}{x^2-9}$

3 By performing long division, express $\dfrac{5x^3+3x^2-13}{x^2-2x+5}$ as the sum of a polynomial and a proper fraction.

4 It is given that $\dfrac{9x^3+3x^2-13}{3x^2+5}$ can be expressed in the form $ax+b+\dfrac{cx+d}{3x^2+5}$. Find the values of a, b, c and d.

5 Find the values of a, b, c and d for which $\dfrac{6x^3+4x^2-3x}{3x^3+2}=a+\dfrac{bx^2+cx+d}{3x^3+2}$.

6 Find the values of a, b, c and d for which $\dfrac{6x^3+13x^2-2x-31}{x^2+x-2}=ax+b+\dfrac{cx+d}{x^2+x-2}$.

7 Find the values of a, b, c and d for which $\dfrac{9x^4+12x^3+3x^2+5x-1}{3x^3+x}=ax+b+\dfrac{cx+d}{3x^3+x}$.

8 Given that $\dfrac{15x^3-11x^2-33x}{5x-2}$ can be expressed in the form $ax^2+bx+c+\dfrac{d}{5x-2}$, find the values of a, b, c and d.

Partial Fractions (Recap)

Let us recall how to add and subtract algebraic fractions. For example,

(a) 2 distinct linear factors

$$\begin{aligned}\frac{1}{x+1}+\frac{3}{x-2} &= \frac{x-2}{(x+1)(x-2)}+\frac{3(x+1)}{(x+1)(x-2)}\\ &= \frac{(x-2)+3(x+1)}{(x+1)(x-2)}\\ &= \frac{4x+1}{(x+1)(x-2)}\end{aligned}$$

(b) 1 linear factor and 1 repeated linear factor

$$\begin{aligned}\frac{2}{x+1}+\frac{1}{(x+1)^2} &= \frac{2(x+1)+1}{(x+1)^2}\\ &= \frac{2x+2+1}{(x+1)^2}\\ &= \frac{2x+3}{(x+1)^2}\end{aligned}$$

(c) 1 linear factor and 1 quadratic factor

$$\frac{4}{x-6}+\frac{2x-1}{x^2+3} = \frac{4(x^2+3)+(x-6)(2x-1)}{(x-6)(x^2+3)} = \frac{6x^2-13x+18}{(x-6)(x^2+3)}$$

In this section, we will learn how to do the *reverse*, i.e. we will split or **decompose** a single algebraic fraction into two or more **partial fractions**.

Case 1: Proper Fraction with Distinct Linear Factors in the Denominator

Thinking Time

The following shows two examples of combining two algebraic fractions into a single algebraic fraction:

(a) $\dfrac{1}{x+1}+\dfrac{3}{x-2}=\dfrac{4x+1}{(x+1)(x-2)}$

(b) $\dfrac{2}{x+3}-\dfrac{4}{2x-5}=-\dfrac{22}{(x+3)(2x-5)}$

1. What is the relationship between the denominator of the fraction on the RHS and the denominators of the two fractions on the LHS?
2. What are the degrees of the numerator and the denominator of each of the two fractions on the LHS? In other words, are the fractions on the LHS proper or improper fractions?
3. What are the degrees of the numerator and the denominator of the fraction on the RHS? In other words, is the fraction on the RHS a proper or an improper fraction?
4. From the above observations, to do the reverse, we have $\dfrac{px+q}{(ax+b)(cx+d)}=\dfrac{A}{ax+b}+\dfrac{B}{\boxed{}}$, where A and B are constants.

Worked Example 19

(Distinct Linear Factors in the Denominator)

Express $\frac{10x-11}{(x+1)(2x-5)}$ in partial fractions.

Solution

Let $\frac{10x-11}{(x+1)(2x-5)} = \frac{A}{x+1} + \frac{B}{2x-5}$, where A and B are constants.

Multiply throughout by $(x + 1)(2x - 5)$, we have

$$10x - 11 = A(2x - 5) + B(x + 1)$$

Let $x = -1$:

$$10(-1) - 11 = A[2(-1) - 5] + B(-1 + 1)$$
$$-21 = -7A$$
$$\therefore A = 3$$

Let $x = \frac{5}{2}$:

$$10\left(\frac{5}{2}\right) - 11 = A\left[2\left(\frac{5}{2}\right) - 5\right] + B\left(\frac{5}{2} + 1\right)$$
$$14 = \frac{7}{2}B$$
$$\therefore B = 4$$

$$\therefore \frac{10x-11}{(x+1)(2x-5)} = \frac{3}{x+1} + \frac{4}{2x-5}$$

INFORMATION

We can also solve for A and B by comparing coefficients.

Practise Now 19

Similar Questions:
Exercise 1F
Questions 1(a)-(d), 7, 8

1. Express $\frac{9x-5}{(x-3)(2x+5)}$ in partial fractions.

2. Factorise $2x^3 + 3x^2 - 17x + 12$ completely. Hence, express $\frac{7x^2-25x+8}{2x^3+3x^2-17x+12}$ in partial fractions.

Case 2: Proper Fraction with Repeated Linear Factors in Denominator

For Case 1 where the two linear factors in the denominator are distinct,

$$\frac{px+q}{(ax+b)(cx+d)} = \frac{A}{ax+b} + \frac{B}{cx+d}.$$

We shall now consider the case where the linear factor in the denominator is repeated, such as in $\frac{px+q}{(ax+b)^2}$.

Thinking Time

The following shows two examples of combining two algebraic fractions into a single algebraic fraction:

(a) $\dfrac{2}{x+1}+\dfrac{1}{(x+1)^2}=\dfrac{2x+3}{(x+1)^2}$

(b) $\dfrac{3}{2x-5}-\dfrac{2}{(2x-5)^2}=\dfrac{6x-17}{(2x-5)^2}$

1. What is the relationship between the denominator of the fraction on the RHS and the denominators of the two fractions on the LHS?

2. From the above observation, to do the reverse, we have

   $\dfrac{px+q}{(ax+b)^2}=\dfrac{A}{ax+b}+\dfrac{B}{\square}$, where A and B are constants.

Worked Example 20

(Repeated Linear Factors in the Denominator)

Express $\dfrac{3x-2}{(x-1)^2}$ in partial fractions.

Solution

Let $\dfrac{3x-2}{(x-1)^2}=\dfrac{A}{x-1}+\dfrac{B}{(x-1)^2}$, where A and B are constants.

Multiply throughout by $(x-1)^2$, we have

$$3x-2=A(x-1)+B$$

Let $x=1$: $\quad 3(1)-2=A(1-1)+B$

$$B=1$$

Let $x=0$: $\quad 3(0)-2=A(0-1)+B$

$$A=3$$

$$\therefore \dfrac{3x-2}{(x-1)^2}=\dfrac{3}{x-1}+\dfrac{1}{(x-1)^2}$$

Practise Now 20

Similar Questions:
Exercise 1F
Questions 2(a), 2(b)

1. Express $\dfrac{5-15x}{(2x+1)^2}$ in partial fractions.

2. Express $\dfrac{5x-32}{x^2+4x+4}$ in partial fractions.

Worked Example 21

(Repeated Linear Factors in the Denominator)

Express $\dfrac{10-3x-2x^2}{(2x+3)(x-1)^2}$ in partial fractions.

Solution

Let $\dfrac{10-3x-2x^2}{(2x+3)(x-1)^2} = \dfrac{A}{2x+3} + \dfrac{B}{x-1} + \dfrac{C}{(x-1)^2}$.

Multiply throughout by $(2x + 3)(x - 1)^2$,

$$10 - 3x - 2x^2 = A(x - 1)^2 + B(x - 1)(2x + 3) + C(2x + 3)$$

Let $x = 1$: $\quad 5 = 5C$

$\quad C = 1$

Let $x = -\dfrac{3}{2}$: $\quad 10 = \dfrac{25}{4}A$

$\quad A = \dfrac{8}{5}$

Let $x = 0$: $\quad 10 = A - 3B + 3C$

$\quad B = -\dfrac{9}{5}$

$$\therefore \dfrac{10-3x-2x^2}{(2x+3)(x-1)^2} = \dfrac{8}{5(2x+3)} - \dfrac{9}{5(x-1)} + \dfrac{1}{(x-1)^2}$$

Practise Now 21

Similar Questions:
Exercise 1F
Questions 2(c)-(f)

1. Express $\dfrac{4x^2+5x-32}{(x+2)^2(x-11)}$ in partial fractions.
2. Factorise $x^3 - 2x^2 - 4x + 8$ completely. Hence, express $\dfrac{x^2-10x+36}{x^3-2x^2-4x+8}$ in partial fractions.

Case 3: Proper Fraction with a Quadratic Factor in the Denominator that Cannot be Factorised

We will only consider the case where the quadratic factor in the denominator of a proper algebraic fraction is in the form $x^2 + c^2$, where c is a constant.

What happens if the quadratic factor in the denominator can be factorised?

Worked Example 22

(Quadratic Factor in the Denominator that cannot be Factorised)

Express $\dfrac{7x^2+x+15}{x^3+3x}$ in partial fractions.

Solution

The denominator $x^3 + 3x = x(x^2 + 3)$. (factorise the denominator)

Let $\dfrac{7x^2+x+15}{x^3+3x} = \dfrac{7x^2+x+15}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$.

Multiply throughout by $x(x^2 + 3)$,

$$7x^2 + x + 15 = A(x^2 + 3) + x(Bx + C)$$

Let $x = 0$: $\quad 15 = A(0 + 3) + 0$

$\quad A = 5$

Let $x = 1$: $\quad 7 + 1 + 15 = 5(1 + 3) + 1[B(1) + C]$

$\quad B + C = 3$ ------------- (1)

Let $x = -1$: $\quad 7 - 1 + 15 = 5(1 + 3) - 1[B(-1) + C]$

$\quad B - C = 1$------------- (2)

Solving (1) and (2) simultaneously, we have $B = 2$ and $C = 1$.

$$\therefore \frac{7x^2+x+15}{x^3+3x} = \frac{5}{x} + \frac{2x+1}{x^2+3}$$

ATTENTION

When the quadratic factor in the denominator cannot be factorised, the numerator of this partial fraction is linear, i.e. $Bx + C$, where B and C are constants.

Practise Now 22a

Similar Questions:
Exercise 1F
Questions 3(a)-(d)

1. Express $\dfrac{5x^2+12x+9}{(x+1)(x^2+5)}$ in partial fractions.

2. Express $\dfrac{1}{4x(x^2+4)}$ in partial fractions.

Class Discussion

Discuss the following with your classmates.

(a) In Worked Example 22, the second partial fraction is $\dfrac{Bx+C}{x^2+c^2}$, where the denominator is quadratic and the numerator is linear. Is it possible for the numerator to be a constant? If yes, why do we write $\dfrac{Bx+C}{x^2+c^2}$ and not $\dfrac{C}{x^2+c^2}$?

(b) In Worked Example 21, the last partial fraction is $\dfrac{C}{(x-1)^2}$, where the denominator is quadratic but the numerator is a constant. Why is the numerator not linear?

Hint: See Worked Example 20.

Case 4: Improper Fraction

If an algebraic fraction is improper, we have to first express it as the sum of a polynomial and a proper fraction (see Worked Example 18), and then decompose the proper fraction into partial fractions.

For example, to express $\dfrac{4x^3+8x^2-50x-132}{2x^2+x-28}$ in partial fractions, we use long division to divide $4x^3 + 8x^2 - 50x - 132$ by $2x^2 + x - 28$ to obtain the quotient $2x + 3$ and the remainder $3x - 48$. Therefore,

$$\frac{4x^3+8x^2-50x-132}{2x^2+x-28} = 2x+3+\frac{3x-48}{2x^2+x-28}.$$

Next, we decompose $\dfrac{3x-48}{2x^2+x-28}$ into partial fractions using Case 1 since the denominator can be factorised into two distinct linear factors:

$$\frac{3x-48}{2x^2+x-28} = \frac{3x-48}{(x+4)(2x-7)}$$

Let $$\frac{3x-48}{(x+4)(2x-7)} = \frac{A}{x+4}+\frac{B}{2x-7}.$$

Multiply throughout by $(x + 4)(2x - 7)$,

$$3x-48 = A(2x-7)+B(x+4)$$

Let $x = -4$: $$3(-4)-48 = A[2(-4)-7]+B(-4+4)$$
$$-60 = -15A$$
$$A = 4$$

Let $x = \frac{7}{2}$: $$3\left(\frac{7}{2}\right)-48 = A\left[2\left(\frac{7}{2}\right)-7\right]+B\left(\frac{7}{2}+4\right)$$
$$-\frac{75}{2} = \frac{15}{2}B$$
$$B = -5$$

$$\frac{3x-48}{2x^2+x-28} = \frac{4}{x+4}-\frac{5}{2x-7}$$

Hence, $$\frac{4x^3+8x^2-50x-132}{2x^2+x-28} = 2x+3+\frac{4}{x+4}-\frac{5}{2x-7}.$$

Practise Now 22b

Similar Questions:
Exercise 1F
Questions 4(a)-(d), 5, 6

1. Express $\dfrac{3x^3-5}{x^2-1}$ in partial fractions.

2. Express $\dfrac{6x^3-37x^2+62x-14}{2x^2-11x+15}$ in partial fractions.

Application of Partial Fractions

One reason for decomposing an algebraic fraction into partial fractions is the original algebraic fraction is too complicated to manipulate. An application of partial fractions is dealt with later in Chapter 14 on Integration. In this section, we will look at another example.

Worked Example 23

(Application of Partial Fractions)

(i) Express $\frac{1}{m(m+1)}$ in partial fractions.

(ii) Hence, evaluate

$$\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{2011\times 2012}+\frac{1}{2012\times 2013}.$$

Solution

(i) Let $\frac{1}{m(m+1)} = \frac{A}{m}+\frac{B}{m+1}$.

Multiply throughout by $m(m + 1)$,

$$1 = A(m+1) + Bm$$

Let $m = 0$: $\quad 1 = A(0 + 1)$

$A = 1$

Let $m = -1$: $\quad 1 = A(-1 + 1) + B(-1)$

$B = -1$

$$\therefore \frac{1}{m(m+1)} = \frac{1}{m}-\frac{1}{m+1}$$

(ii) $\frac{1}{1\times 2}+\frac{1}{2\times 3}+\frac{1}{3\times 4}+\ldots+\frac{1}{2011\times 2012}+\frac{1}{2012\times 2013}$

$$= \left(\frac{1}{1}-\cancel{\frac{1}{2}}\right)+\left(\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}\right)+\left(\cancel{\frac{1}{3}}-\cancel{\frac{1}{4}}\right)+\ldots+\left(\cancel{\frac{1}{2011}}-\cancel{\frac{1}{2012}}\right)+\left(\cancel{\frac{1}{2012}}-\frac{1}{2013}\right)$$

$$= 1-\frac{1}{2013}$$

$$= \frac{2012}{2013}$$

Practise Now 23

Similar Question:
Exercise 1F
Question 10

(i) Express $\frac{2}{n(n+1)(n+2)}$ in partial fractions.

(ii) Hence, evaluate

$$\frac{2}{1\times 2\times 3}+\frac{2}{2\times 3\times 4}+\frac{2}{3\times 4\times 5}+\ldots+\frac{2}{2010\times 2011\times 2012}+\frac{2}{2011\times 2012\times 2013}.$$

Exercise 1G

excluded from the N(A) syllabus

1 Express each of the following in partial fractions.

(a) $\frac{7x+6}{x(x+2)}$ (b) $\frac{11x-2}{3x(x-1)}$

(c) $\frac{7x+11}{(x+3)(x-2)}$ (d) $\frac{15x+13}{(x-1)(3x+1)}$

2 Express each of the following in partial fractions.

(a) $\frac{x}{(x-1)^2}$ (b) $\frac{x+3}{(x+4)^2}$

(c) $\frac{5x^2-3x+2}{x(x-1)^2}$ (d) $\frac{8x^2+15x+12}{x(x+2)^2}$

(e) $\frac{-20x-2}{(x+1)(x-2)^2}$ (f) $\frac{39x^2-35x+11}{(2x+3)(3x-1)^2}$

3 Express each of the following in partial fractions.

(a) $\frac{5x^2+3x+20}{x(x^2+4)}$ (b) $\frac{5x^2+x+4}{(x+1)(x^2+3)}$

(c) $\frac{2x^2-3x+27}{(x+3)(x^2+7)}$ (d) $\frac{-8x^2-7x+28}{(3x-2)(2x^2+9)}$

4 Express each of the following improper fractions in partial fractions.

(a) $\frac{2x^2-11x+12}{x^2-5x+6}$

(b) $\frac{2x^3-5x^2-14x-17}{x^2-3x-4}$

(c) $\frac{4x^3+x^2-15x+21}{(x+2)(x-1)^2}$

(d) $\frac{6x^4-5x^3+15x^2-4x+5}{(2x-3)(x^2+5)}$

5 It is given that $\frac{2x^3-3x^2-2x-2}{x^2-4x+3}$ can be expressed in the form $ax+b+\frac{c}{x-1}+\frac{d}{x-3}$. Find the values of a, b, c and d.

6 Divide $18x^3-15x^2-42x+3$ by $6x^2-x-15$.

Hence, express $\frac{18x^3-15x^2-42x+3}{6x^2-x-15}$ in partial fractions.

7 Factorise $2x^3+3x^2-50x-75$ completely.

Hence, express $\frac{3x^2+19x+90}{2x^3+3x^2-50x-75}$ in partial fractions.

8 Factorise $8x^3-22x^2-9x+9$ completely.

Hence, express $\frac{25x+75}{8x^3-22x^2-9x+9}$ in partial fractions.

9 **(i)** The cubic polynomial $f(x)=3x^3+hx^2+kx-8$ is exactly divisible by $3x-2$ and leaves a remainder of 5 when it is divided by $x-1$. Calculate the value of h and of k. Hence, factorise $f(x)$ completely.

(ii) Using the results in **(i)**, express $\frac{4x+24}{f(x)}$ in partial fractions.

10 **(i)** Express $\frac{20}{n(n+2)}$ in partial fractions.

(ii) Hence, simplify

$$\frac{20}{1\times 3}+\frac{20}{2\times 4}+\frac{20}{3\times 5}+\ldots+\frac{20}{n^2-1}+\frac{20}{n^2+2n}.$$

SUMMARY

1. To solve a pair of linear and non-linear simultaneous equations, express a variable of the linear equation in terms of the other variable, then substitute it into the non-linear equation to solve.

2. The **Division Algorithm for Polynomials** states that:

$$\begin{array}{ccccccccc} \mathrm{P}(x) & = & \mathrm{D}(x) & \times & \mathrm{Q}(x) & + & \mathrm{R}(x) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \text{dividend} & = & \text{divisor} & \times & \text{quotient} & + & \text{remainder} \end{array}$$

   and **the degree of R(x) < the degree of D(x)**.

3. The **Remainder Theorem** states that:
   If a polynomial P(x) is divided by a linear divisor $x - c$, the remainder is P(c).
   If a polynomial P(x) is divided by a linear divisor $ax + b$, the remainder is $\mathrm{P}\left(-\frac{b}{a}\right)$.

4. The **Factor Theorem** states that:
   $ax + b$ is a linear factor of the polynomial P(x) if and only if $\mathrm{P}\left(-\frac{b}{a}\right) = 0$.
   The Factor Theorem can be used to *solve certain cubic equations*.

5. Two special algebraic identities:
   Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$
   Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

6. To decompose $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$, where P($x$) and Q($x$) are polynomials and Q($x$) $\neq 0$, into partial fractions:

   Step 1: If $\frac{\mathrm{P}(x)}{\mathrm{Q}(x)}$ is improper, express it as the sum of a polynomial and a proper algebraic fraction first.

   Step 2: Factorise the denominator of the proper algebraic fraction if possible.

   Step 3: Express in partial fractions according to Case 1, 2 or 3.

Case	Denominator contains	Algebraic Fraction	Partial Fractions
1	Distinct linear factors	$\frac{px+q}{(ax+b)(cx+d)}$	$\frac{A}{ax+b} + \frac{B}{cx+d}$
2	Repeated linear factors	$\frac{px+q}{(ax+b)^2}$	$\frac{A}{ax+b} + \frac{B}{(ax+b)^2}$
3	Quadratic factor $x^2 + c^2$ (cannot be factorised)	$\frac{px+q}{(ax+b)(x^2+c^2)}$	$\frac{A}{ax+b} + \frac{Bx+C}{x^2+c^2}$

   Step 4: Solve for unknown constants by substituting suitable values of x.

Review Exercise 1

1. Solve the following simultaneous equations.

   (a) $x + 2y = 5,\ 5x^2 + 4y^2 + 12x = 29$ (b) $5x = 3y + 12,\ \dfrac{3x}{y} - \dfrac{2y}{x} = 1$

   (c) $xy = 20,\ 2x - 11 = 2y$ (d) $x - y + \dfrac{1}{12} = 0,\ \dfrac{1}{x} - \dfrac{1}{y} - 2 = 0$

2. The line $3y = 4x - 15$ intersects the curve $8x^2 = 45 + 27y^2$ at the points A and B. Find the coordinates of A and of B.

3. Find the values of A, B and C in each of the following.

   (a) $4x^2 - 13x + 5 = A(x - 2)(x - 3) + B(x - 2) + C$ (b) $7x^2 - 14x + 13 = A(x - 1)(x + 3) + B(x - 1) + C$

   (c) $2x^3 + 3x^2 - 14x - 5 = (Ax + B)(x + 3)(x + 1) + C$ (d) $x^4 - 10x^2 - 13x + 1 = (x^2 + Ax + 3)(x - 1)(x - 3) + Bx + C$

4. Find the value of k in each of the following.

   (a) $x + 2$ is a factor of $2x^3 - 3x^2 + kx + 18$ (b) $2x - 1$ is a factor of $2x^3 + x^2 - kx + 30$

   (c) $x - 2$ is a factor of $2x^3 + kx^2 + 11x - 2$ (d) $x + 3$ is a factor of $(2x + 5)^9 + (3x + k)^3$

   (e) $(3x^k - 7x^2 + 15) \div (x - 2)$ has a remainder of 11 (f) $(2kx^2 - k^2x - 11) \div (x - 3)$ has a remainder of 4

5. Factorise each of the following.

   (a) $2x^3 - 7x^2 + 9$ (b) $x^3 - 2x^2 - 15x + 36$
   (c) $6x^3 + 19x^2 - 24x - 16$ (d) $12x^3 + 28x^2 - 7x - 5$
   (e) $8x^6 - 343$ (f) $27a^3 + 64y^6$
   (g) $(2x + 3)^3 - (4x - 5)^3$ (h) $(3x + 5)^3 + (x + 1)^3$

6. Solve each of the following equations, giving your answers correct to 2 decimal places where necessary.

   (a) $2x^3 - 3x^2 - 17x + 30 = 0$ (b) $6x^3 + 11x^2 - 4x - 4 = 0$
   (c) $2x^3 - 7x^2 - 10x + 24 = 0$ (d) $8x^4 - 42x^3 + 29x^2 + 42x + 8 = 0$
   (e) $x^3 - 6x^2 + 2x + 12 = 0$ (f) $2x^3 - 31x^2 + 29x - 7 = 0$
   (g) $4x^3 - 37x^2 + 37x - 7 = 0$ (h) $6x^3 + 5x^2 - 26x + 11 = 0$

7. When the expression $7x^{21} - 5x^{15} + ax^6$ is divided by $x + 1$, the remainder is 2. Find the value of a. Hence, find the remainder when the expression is divided by $x - 1$.

8. Find the value of p and of q for which the expression $12x^4 + 16x^3 + px^2 + qx - 1$ is divisible by $4x^2 - 1$. Hence, find the other factors of the expression.

9. The expression $(px + q)(x - 1) + r(x^2 + 2)$ is equal to 12 for all values of x. By substituting suitable values of x, or otherwise, find the values of p, q and r.

10. Find the value of p and of q if $4x^2 - 4x - 3$ is a factor of the expression $8x^4 + px^3 + qx^2 + x + 3$. Hence, factorise the expression completely.

11. Given that $x + 5$ is a common factor of $x^3 + kx^2 - Ax + 15$ and $x^3 - x^2 - (A + 5)x + 40$, find the value of k and of A. Hence, factorise $x^3 + kx^2 - Ax + 15$ completely.

12. Express each of the following in partial fractions.

(a) $\dfrac{10x-4}{(2x-1)(x+2)}$ (b) $\dfrac{x^2+16}{(3-x)(2x-1)^2}$ (c) $\dfrac{5x}{(2x+1)(x^2+1)}$

(d) $\dfrac{9x^2+2x+14}{(2x+3)(x-1)^2}$ (e) $\dfrac{x^2+4x-7}{x^2+x-12}$ (f) $\dfrac{4x-1}{x^3-4x}$

13. It is given that $\dfrac{2x^3-9x^2+7x+8}{(2x+1)(x-2)}$ can be expressed in the form $ax+b+\dfrac{c}{(2x+1)(x-2)}$.
Find the values of a, b and c.

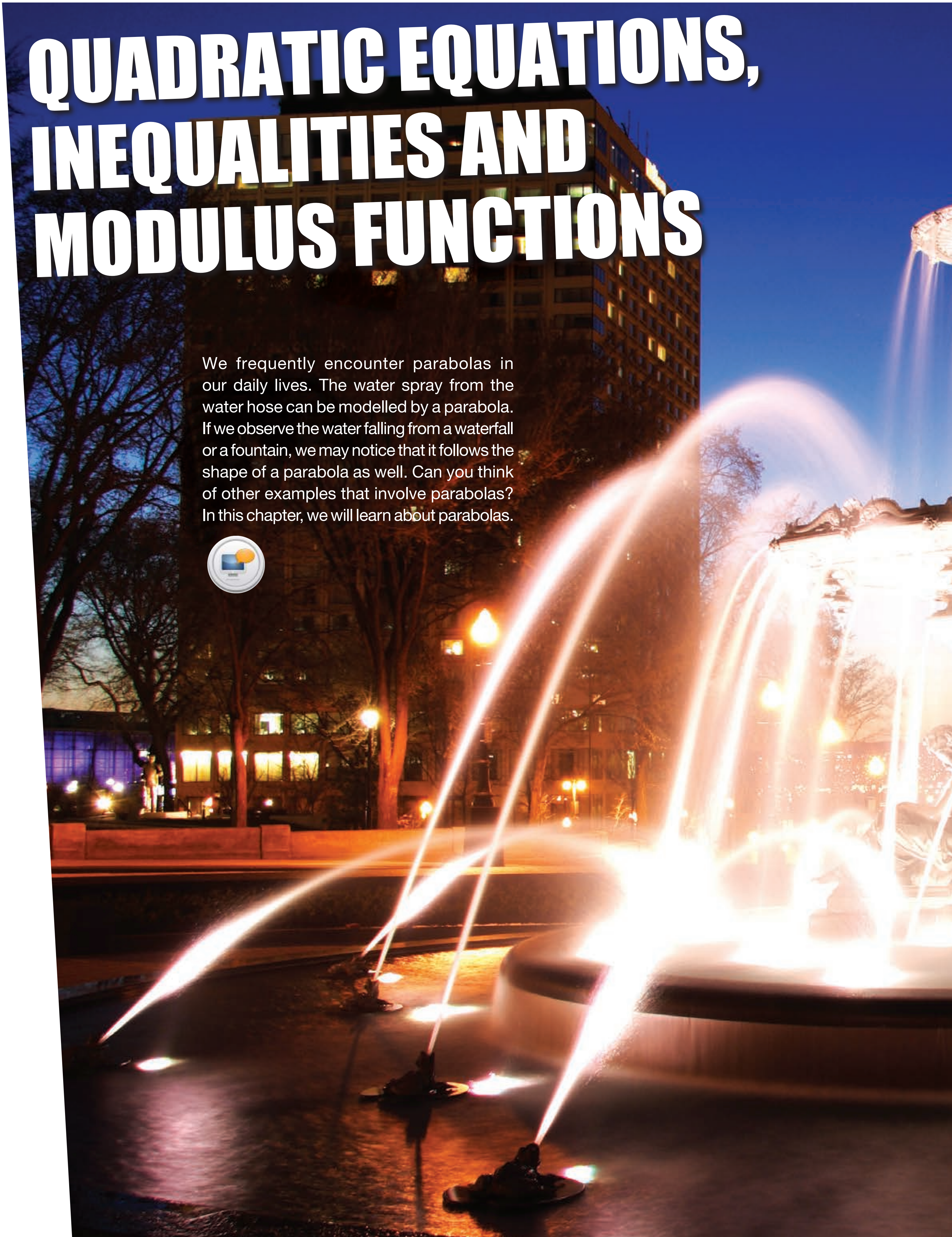
14. It is given that $\dfrac{x^3-x^2-4x+1}{(x-2)(x+2)} = ax+b+\dfrac{c}{(x-2)(x+2)}$. Find the values of a, b and c.

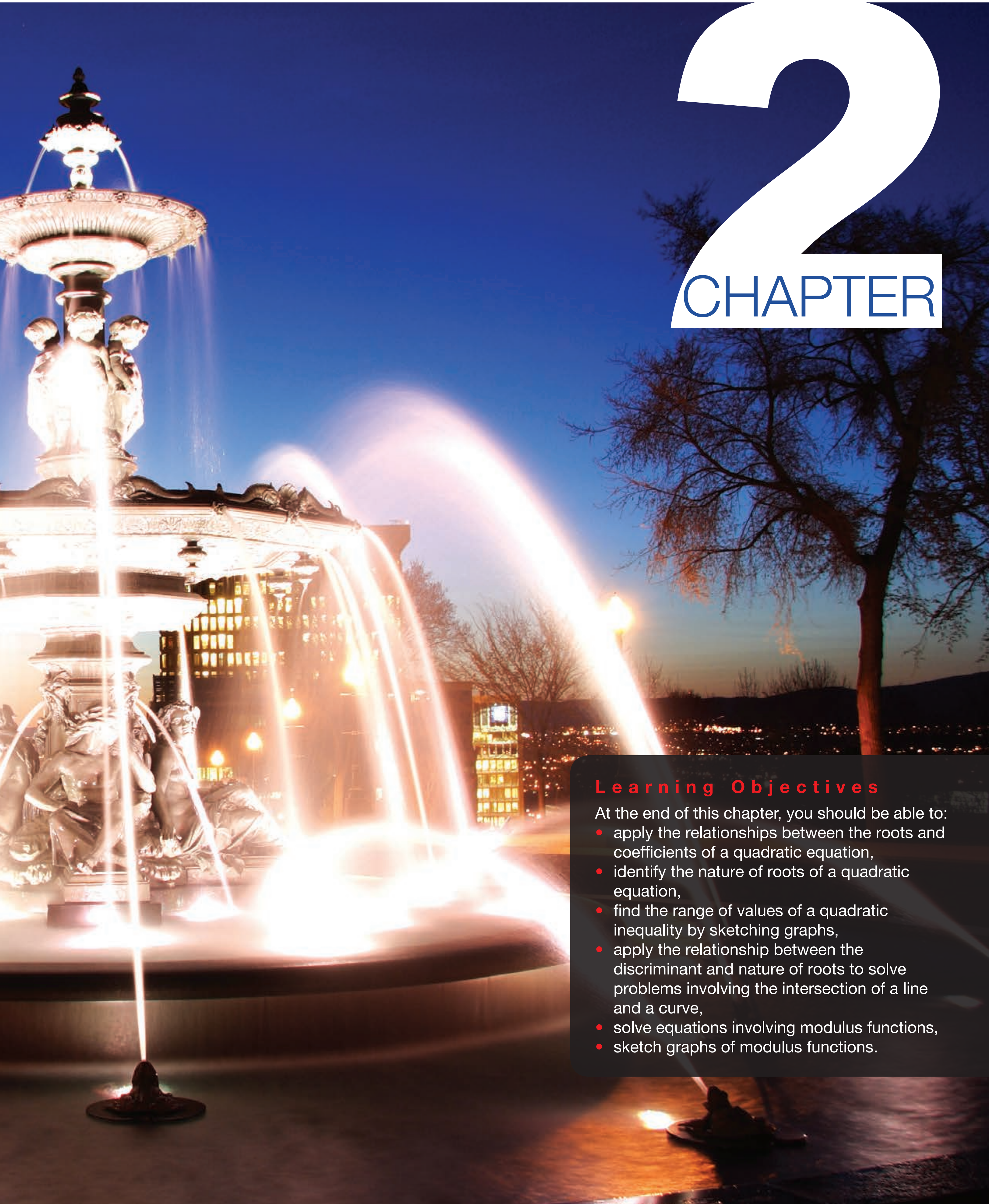
1. The general formula to solve a quadratic equation is $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$. There is also a general formula for solving a cubic equation. Find out more about this general formula and use it to solve $2x^3 + 3x^2 - 5x - 6 = 0$.

2. Decomposing an algebraic fraction, whose denominator contains only distinct linear factors, into partial fractions can be done easily using Heaviside's Cover-Up Rule. Find out more about this rule and use it to express $\dfrac{3x-1}{(x-2)(x+3)}$ in partial fractions.

QUADRATIC EQUATIONS, INEQUALITIES AND MODULUS FUNCTIONS

We frequently encounter parabolas in our daily lives. The water spray from the water hose can be modelled by a parabola. If we observe the water falling from a waterfall or a fountain, we may notice that it follows the shape of a parabola as well. Can you think of other examples that involve parabolas? In this chapter, we will learn about parabolas.





2 CHAPTER

Learning Objectives

At the end of this chapter, you should be able to:

- apply the relationships between the roots and coefficients of a quadratic equation,
- identify the nature of roots of a quadratic equation,
- find the range of values of a quadratic inequality by sketching graphs,
- apply the relationship between the discriminant and nature of roots to solve problems involving the intersection of a line and a curve,
- solve equations involving modulus functions,
- sketch graphs of modulus functions.

2.1 SUM AND PRODUCT OF ROOTS

Recap

We have learnt in O-level mathematics that the roots of a quadratic equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In this section, we shall examine the relationships between the roots and the coefficients of a quadratic equation.

Properties of the Roots of a Quadratic Equation

The quadratic equation $ax^2 + bx + c = 0$ can be written as $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$ ------ (1).

If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, its equation can be written as $(x - \alpha)(x - \beta) = 0$.
i.e. $x^2 - (\alpha + \beta)x + \alpha\beta = 0$ ------ (2)

Comparing (1) and (2), we have

$$-(\alpha + \beta) = \frac{b}{a}$$

$$\alpha + \beta = -\frac{b}{a}$$

i.e.

$$\boxed{\begin{array}{l} \text{Sum of roots, } \alpha + \beta = -\dfrac{b}{a} \\ \text{Product of roots, } \alpha\beta = \dfrac{c}{a} \end{array}}$$

Therefore, (1) can be written as

$$\boxed{x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0.}$$

Note that the above result is true for all quadratic equations, including those having no real roots.

Worked Example 1

(Sum and Product of Roots)
If α and β are the roots of the equation $2x^2 - 4x - 3 = 0$, where $\alpha > \beta$, find the value of each of the following.

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(ii)** $\alpha^2 + \beta^2$ **(iii)** $\alpha - \beta$ **(iv)** $\alpha^3 - \beta^3$

Solution

$$\alpha + \beta = -\frac{b}{a} = -\frac{(-4)}{2} = 2$$

$$\alpha\beta = \frac{c}{a} = -\frac{3}{2}$$

PROBLEM-SOLVING TIP

Always write down the value of $\alpha + \beta$ and of $\alpha\beta$, before solving the rest of the problem.

(i) $\dfrac{1}{\alpha}+\dfrac{1}{\beta} = \dfrac{\beta+\alpha}{\alpha\beta}$

$= \dfrac{2}{-\frac{3}{2}}$

$= -\dfrac{4}{3}$

(ii) $\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 2^2 - 2\left(-\dfrac{3}{2}\right) = 7$

RECALL

$a^2 + b^2 = a^2 + b^2 + 2ab - 2ab$
$= (a+b)^2 - 2ab$

(iii) $(\alpha-\beta)^2 = \alpha^2+\beta^2-2\alpha\beta$

$= 7 - 2\left(-\dfrac{3}{2}\right) = 10$

Since $\alpha > \beta$,

$\alpha - \beta = \sqrt{10}$

RECALL

$a^3 - b^3$
$= (a-b)(a^2 + ab + b^2)$

(iv) $\alpha^3-\beta^3 = (\alpha-\beta)(\alpha^2+\alpha\beta+\beta^2)$

$= \sqrt{10}\left[7+\left(-\dfrac{3}{2}\right)\right]$

$= \dfrac{11\sqrt{10}}{2}$

Practise Now 1

Similar Questions:
Exercise 2A
Questions 1(a)-(j), 4(i)-(v), 18

1. If α and β are the roots of the equation $2x^2 + x - 6 = 0$, where $\alpha > \beta$, find the value of each of the following.

(i) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$ **(ii)** $\alpha^2+\beta^2$

(iii) $\alpha-\beta$ **(iv)** $\alpha^3-\beta^3$

2. If α and β are the roots of the equation $4x^2 - x + 7 = 0$, where $\alpha > \beta$, find the value of each of the following.

(i) $4\alpha + 4\beta$ **(ii)** $\dfrac{\alpha}{\beta}+\dfrac{\beta}{\alpha}$

(iii) $\alpha^3+\beta^3$ **(iv)** $\alpha^4+\beta^4$

Worked Example 2

(Forming a Quadratic Equation when the Roots are Given)

If α and β are the roots of the equation $2x^2 - 5x = 1$, form equations whose roots are

(i) 3α, 3β, **(ii)** $\dfrac{\alpha}{\beta}$, $\dfrac{\beta}{\alpha}$.

Solution

Rewriting $2x^2 - 5x = 1$, we have $2x^2 - 5x - 1 = 0$.

$$\alpha+\beta = -\frac{b}{a} = -\frac{(-5)}{2} = \frac{5}{2}$$

$$\alpha\beta = \frac{c}{a} = -\frac{1}{2}$$

PROBLEM-SOLVING TIP

Always rearrange an equation in the form $ax^2 + bx + c = 0$ before writing the values of a, b and c.

(i) Sum of new roots, $3\alpha + 3\beta = 3(\alpha + \beta) = \frac{15}{2}$

Product of new roots, $3\alpha \times 3\beta = 9\alpha\beta = -\frac{9}{2}$

New equation is x^2 – (sum of roots)x + (product of roots) = 0.

$$x^2 - \left(\frac{15}{2}\right)x + \left(-\frac{9}{2}\right) = 0$$

i.e. $$2x^2 - 15x - 9 = 0$$

(ii) Sum of new roots, $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = \frac{\alpha^2 + \beta^2}{\alpha\beta} = \frac{(\alpha + \beta)^2 - 2\alpha\beta}{\alpha\beta}$

$$= \frac{\left(\frac{5}{2}\right)^2 - 2\left(-\frac{1}{2}\right)}{-\frac{1}{2}}$$

$$= -\frac{29}{2}$$

Product of new roots, $\frac{\alpha}{\beta} \times \frac{\beta}{\alpha} = 1$

New equation is x^2 – (sum of roots)x + (product of roots) = 0.

$$x^2 - \left(-\frac{29}{2}\right)x + 1 = 0$$

i.e. $$2x^2 + 29x + 2 = 0$$

Practise Now 2

Similar Questions:
Exercise 2A
Questions 2, 3, 5, 10

1. If α and β are the roots of the equation $5x^2 - 4x = -9$, form equations whose roots are
(i) 2α, 2β, **(ii)** $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$.

2. If α and β are the roots of the equation $3x^2 - x + 5 = 0$, form equations whose roots are
(i) α^2, β^2, **(ii)** $\frac{1}{\alpha}$, $\frac{1}{\beta}$, **(iii)** α^3, β^3.

Worked Example 3

(Finding an Unknown in a Quadratic Equation)
The roots of the equation $3x^2 - 2kx + k + 4 = 0$ are α and β. If $\alpha^2 + \beta^2 = \frac{16}{9}$, find the possible values of k.

Solution

$3x^2 - 2kx + k + 4 = 0$

$\alpha + \beta = -\frac{b}{a} = -\frac{(-2k)}{3} = \frac{2k}{3}$

$\alpha\beta = \frac{c}{a} = \frac{k+4}{3}$

Given that $\alpha^2 + \beta^2 = \frac{16}{9}$, i.e. $(\alpha+\beta)^2 - 2\alpha\beta = \frac{16}{9}$

$$\left(\frac{2k}{3}\right)^2 - 2\left(\frac{k+4}{3}\right) = \frac{16}{9}$$
$$\frac{4k^2}{9} - \frac{2k+8}{3} = \frac{16}{9}$$
$$4k^2 - 6k - 24 = 16 \qquad \text{(multiply throughout by 9)}$$
$$4k^2 - 6k - 40 = 0$$
$$2k^2 - 3k - 20 = 0$$
$$(2k+5)(k-4) = 0$$
$$k = -\frac{5}{2} \text{ or } k = 4$$

Practise Now 3

Similar Questions:
Exercise 2A
Questions 6-9, 11-13, 16, 17, 19

1. The roots of the equation $5x^2 - 2hx + h = 0$ are α and β. If $\alpha^2 + \beta^2 = \frac{24}{25}$, find the possible values of h.

2. The roots of the quadratic equation $2x^2 - ax + 3 = 0$ (where $a > 0$) are α and β, while those of the equation $9x^2 - 52x + 4 = 0$ are $\frac{1}{\alpha^2}$ and $\frac{1}{\beta^2}$. Find the value of a.

Worked Example 4

(Multiplication of Polynomials)

Given that α is a root of the equation $x^2 = x - 3$, show that

(i) $\alpha^3 + 2\alpha + 3 = 0$, **(ii)** $\alpha^4 + 5\alpha^2 + 9 = 0$.

Solution

(i) Since α is a root of the equation $x^2 = x - 3$, it will satisfy the equation.

$\therefore$ $\alpha^2 = \alpha - 3$ ----- (1)

(1) $\times\ \alpha$: $\alpha^3 = \alpha^2 - 3\alpha$

Replace α^2 with $\alpha - 3$: $\alpha^3 = \alpha - 3 - 3\alpha$

$$\alpha^3 + 2\alpha + 3 = 0 \quad \text{(shown)}$$

(ii) Since $\alpha^2 = \alpha - 3$,

$\therefore$ $\alpha^4 = (\alpha - 3)^2$

$$\alpha^4 = \alpha^2 - 6\alpha + 9$$
$$\alpha^4 = \alpha^2 - 6(\alpha^2 + 3) + 9 \qquad \text{(use (1) to make } \alpha \text{ the subject)}$$
$$\alpha^4 = \alpha^2 - 6\alpha^2 - 18 + 9$$
$$\alpha^4 + 5\alpha^2 + 9 = 0 \quad \text{(shown)}$$

Practise Now 4

Similar Questions:
Exercise 2A
Questions 14, 15, 20

1. Given that α is a root of the equation $2x^2 = 3x - 7$, show that
(i) $4\alpha^3 + 5\alpha + 21 = 0$, **(ii)** $4\alpha^4 + 19\alpha^2 + 49 = 0$.

2. Given that α is a root of the equation $5x^2 = x + 1$, show that $125\alpha^6 + 2 = \alpha^3 + 18\alpha^2$.

Exercise 2A

1 Find the sum and product of the roots.

(a) $3x^2 + 5x - 7 = 0$ **(b)** $2x^2 - 5x + 2 = 0$

(c) $2t^2 - 5 = 0$ **(d)** $3p^2 + 7p = 0$

(e) $kx^2 - 2x + 1 = 0$ **(f)** $x^2 = 3x + 7$

(g) $x(x - 2) = 5$ **(h)** $2x(x + 3) = 4x + 7$

(i) $2x + \frac{2}{x} = 3$ **(j)** $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{4}$

2 If α and β are the roots of the equation $2x^2 - 3x + 4 = 0$, form the equation with roots $\alpha + 2$ and $\beta + 2$.

3 If α and β are the roots of the equation $2x^2 + 5x - 7 = 0$, form the equation whose roots are

(i) 2α, 2β, **(ii)** $\frac{1}{\alpha}$, $\frac{1}{\beta}$,

(iii) α^2, β^2, **(iv)** α^3, β^3.

4 The roots of the equation $3x^2 - 8x + 2 = 0$ are α and β. Find the value of each of the following.

(i) $\frac{1}{\alpha} + \frac{1}{\beta}$ **(ii)** $\alpha^2 + \beta^2$

(iii) $\frac{1}{\alpha^2} + \frac{1}{\beta^2}$ **(iv)** $\frac{\alpha}{\beta} + \frac{\beta}{\alpha}$

(v) $\alpha^4 + \beta^4$

5 If the roots of the equation $3x^2 + 2x - 3 = 0$ are α and β, form an equation whose roots are $\frac{\alpha}{\beta}$ and $\frac{\beta}{\alpha}$.

6 The roots of the equation $x^2 + (p - 10)x = 10p$ are α and $\alpha + 4$. Find the possible values of p.

7 The roots of the equation $3x^2 + hx + 4k = 0$ are α and $\frac{1}{\alpha}$.

(i) Find the value of k.

(ii) Given further that $\alpha^2 + \frac{1}{\alpha^2} = 14$, find the possible values of h.

8 Given that α and 2α are the roots of the equation $8x^2 + kx + 9 = 0$, find the value of α and of k.

9 The roots of the equation $3x^2 + 5x + 1 = 0$ are α and β while the roots of the equation $hx^2 - 4x + k = 0$ are $\alpha + 3$ and $\beta + 3$. Find the value of h and of k.

10 Given that α and β are the roots of the equation $2x^2 - 4x - 13 = 0$, find

(i) the value of $\frac{\alpha}{\beta+2} + \frac{\beta}{\alpha+2}$,

(ii) an equation whose roots are $\frac{\alpha}{\beta+2}$ and $\frac{\beta}{\alpha+2}$.

11 **(a)** The roots of the equation $x^2 - (2k + 4)x + (k^2 + 3k + 2) = 0$ are non-zero and one root is twice the other. Find the value of k.

(b) If one root of $x^2 - 5kx + 3k^2 + 2k - 1 = 0$ is four times the other, find the value of k.

12 Given that α is a common root of the equations $x^2 - 7x + k = 0$ and $x^2 + 8x - 2k = 0$, where $k \neq 0$, find the value of k and of α.

13 If α and β are the roots of the equation $x^2 - 3x + k = 0$ while α^2 and β^2 are the roots of the equation $9x^2 + hx + 1 = 0$, find the value of h and of k.

14 Given that α is a root of the equation $2x^2 = 3x - 4$, show that

(i) $4\alpha^3 + 12 = \alpha$,

(ii) $4\alpha^4 + 7\alpha^2 + 16 = 0$.

15 Given that α is a root of the equation $x^2 - 2x + 5 = 0$, show that

(i) $2\alpha^3 + \alpha^2 + 25 = 0$,

(ii) $\alpha^4 = 5 - 12\alpha$.

16 The roots of the equation $x^2 - 8x + h = 0$ are α and $\alpha + 3k$. Express h in terms of k.

17 The equation $3x^2 - 6x + 1 = 0$ has roots α and β.

(i) Write down the value of $\alpha + \beta$ and of $\alpha\beta$.

A second equation $x^2 + px + q = 0$ has roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$.

(ii) Find the value of p and of q.

18 The equation $x^2 + px + q = 0$ has positive roots α and β. Given further that $\alpha - \beta = 7$ and $\alpha^2 + \beta^2 = 85$, find the value of p and of q.

19 The roots of the equation $3x^2 - 5x + 1 = 0$ are α and β, while the roots of the equation $27x^2 + hx + k = 0$ are $\alpha + \beta$ and $\alpha^2 + \beta^2$. Find the value of h and of k.

20 The roots of the equation $x^2 - 7x + h = 0$ are α^2 and $\frac{6}{\alpha}$.

(i) Given that the roots of the equation are unequal, express h in terms of α and show that $\alpha^3 - 7\alpha + 6 = 0$.

(ii) Solve the equation $\alpha^3 - 7\alpha + 6 = 0$.

2.2 NATURE OF ROOTS OF A QUADRATIC EQUATION

Recall that the general solution of a quadratic equation $ax^2 + bx + c = 0$ is

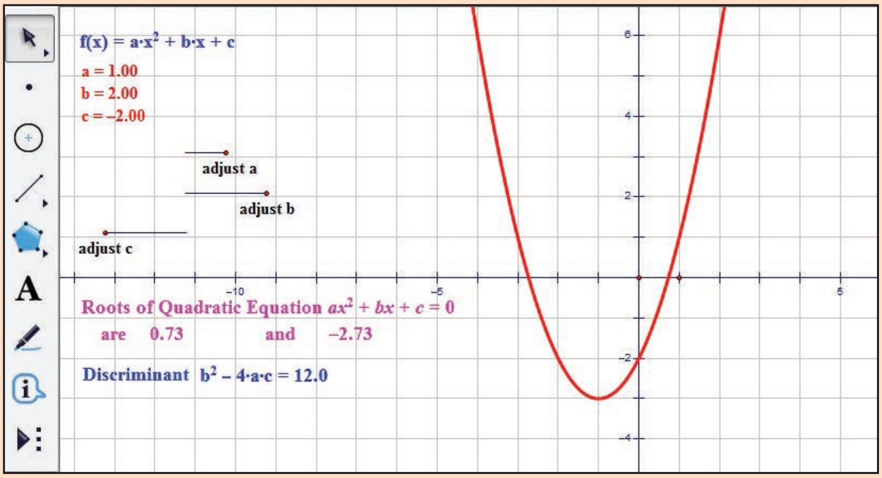
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$ is known as the **discriminant**.
In this section, we shall examine how the discriminant of a quadratic equation determines the nature of the roots of the equation.

Investigation

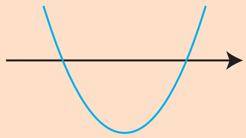
Roots of Quadratic Equations

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template Roots of Quadratic Equations as shown below.



We will make use of this template to find out how the values of a, b and c affect the position of the curve $y = ax^2 + bx + c$ and the roots of the quadratic equation $ax^2 + bx + c = 0$.

1. Change the values of a, b and c to obtain each of the following equations. Copy the following table and record the values of $b^2 - 4ac$ and the roots. Then determine the nature of the roots and sketch the curve. The first one has been done for you.

No.	Quadratic Equation	$b^2 - 4ac$	Roots	Nature of Roots	Sketch of Curve
(a)	$x^2 + 2x - 2 = 0$	12	–2.73, 0.732	2 distinct real roots	
(b)	$-2x^2 - x + 1 = 0$				
(c)	$4x^2 - 4x + 1 = 0$				

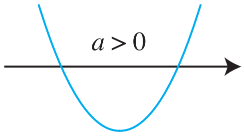
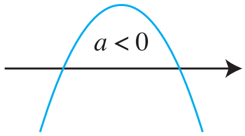
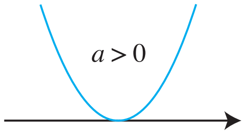
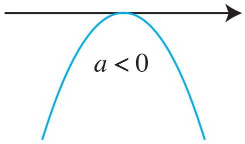
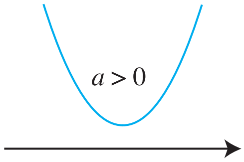
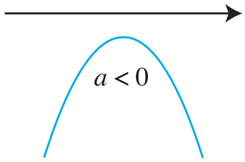
No.	Quadratic Equation	$b^2 - 4ac$	Roots	Nature of Roots	Sketch of Curve
(d)	$-x^2 + 4x - 4 = 0$				
(e)	$x^2 + 1 = 0$				
(f)	$-2x^2 + x - 2 = 0$				

2. Observe the value of $b^2 - 4ac$ and the corresponding nature of roots in the table above. How do you identify the nature of roots from the value of $b^2 - 4ac$?
3. Explain why the nature of the roots of the equation $ax^2 + bx + c = 0$ is related to the sign of $b^2 - 4ac$.
4. Observe the nature of the roots and the curve in the table.
   What can you say about the nature of the roots and the points of intersection of the curve with the x-axis?
5. Using what you have learnt above, copy and complete the table below without using any software or solving the equations.

No.	Quadratic Equation	$b^2 - 4ac$	Nature of Roots	Sketch of Curve
(a)	$2x^2 + 3x - 2 = 0$			
(b)	$9x^2 + 6x + 1 = 0$			
(c)	$-3x^2 + x - 4 = 0$			

6. By observing the tables in this investigation, describe the position of the curve $y = ax^2 + bx + c$ when $b^2 - 4ac < 0$.

From the investigation on the previous pages, we are able to conclude the following for a quadratic equation $ax^2 + bx + c = 0$ ($a \neq 0$) and its corresponding curve $y = ax^2 + bx + c$.

$b^2 - 4ac$	Nature of Roots of a Quadratic Equation	Shapes of Curve $y = ax^2 + bx + c$	Characteristics of Curve
> 0	2 real and distinct roots	 or 	$y = ax^2 + bx + c$ cuts the x-axis at 2 distinct points
$= 0$	2 real and equal roots	 or 	$y = ax^2 + bx + c$ touches the x-axis at 1 point
< 0	no real roots	 or 	$y = ax^2 + bx + c$ lies entirely above (for $a > 0$) or entirely below (for $a < 0$) the x-axis

From the above, we can also conclude that

$$\boxed{b^2 - 4ac \geqslant 0 \Leftrightarrow \text{the roots are real.}}$$

Worked Example 5

(Quadratic Equation with Equal Roots)

The equation $x^2 - 4x - 1 = 2k(x - 5)$, where k is a constant, has two equal roots. Find the possible values of k.

Solution

$$x^2 - 4x - 1 = 2k(x - 5)$$
$$x^2 - 4x - 1 = 2kx - 10k$$

Rearranging, we have $x^2 - (4 + 2k)x + (10k - 1) = 0$.

i.e. $a = 1$, $b = -(4 + 2k)$, $c = 10k - 1$

For equal roots, $b^2 - 4ac = 0$

$$[-(4 + 2k)]^2 - 4(1)(10k - 1) = 0$$
$$16 + 16k + 4k^2 - 40k + 4 = 0$$
$$4k^2 - 24k + 20 = 0$$
$$k^2 - 6k + 5 = 0$$
$$(k - 1)(k - 5) = 0$$
$$k = 1 \text{ or } k = 5$$

ATTENTION

Equal roots may also be referred to as **repeated roots** or **coincident roots**.

Practise Now 5

Similar Questions:
Exercise 2B
Questions 2, 3, 5

1. The equation $x^2 + 3x + k = 5x - 3$ has repeated roots. Find the value of the constant k.
2. Given that the equation $5px^2 + 18 = 6px$ has coincident roots, find the value of the constant p.

Worked Example 6

(Quadratic Equation with Real and Distinct Roots)

Find the range of values of k for which the equation $2x^2 + 5x + 3 - k = 0$ has two real and distinct roots.

Solution

$$2x^2 + 5x + 3 - k = 0$$

i.e. $a = 2$, $b = 5$, $c = 3 - k$

For real and distinct roots,

$$\begin{aligned} b^2 - 4ac &> 0 \\ 5^2 - 4(2)(3 - k) &> 0 \\ 25 - 24 + 8k &> 0 \\ 8k &> -1 \\ k &> -\frac{1}{8} \end{aligned}$$

Practise Now 6

Similar Questions:
Exercise 2B
Questions 6, 8, 13

1. Find the range of values of p for which the equation $-x^2 - 4x + 2p = 7$ has two real and distinct roots.

2. Find the range of values of k for which the equation $3x^2 - 4x = 2x(x + 1) + 2k$ has two real and distinct roots.

Worked Example 7

(Quadratic Equation with No Real Roots)

Find the range of values of k for which the equation $2x^2 + 7x + 3k = 0$ has no real roots.

Solution

$$2x^2 + 7x + 3k = 0$$

i.e. $a = 2$, $b = 7$, $c = 3k$

For no real roots,

$$\begin{aligned} b^2 - 4ac &< 0 \\ 7^2 - 4(2)(3k) &< 0 \\ 49 - 24k &< 0 \\ -24k &< -49 \\ k &> \frac{49}{24} \end{aligned}$$

Practise Now 7

Similar Questions:
Exercise 2B
Questions 4, 7, 9, 11

1. Given that the equation $hx^2 + 5x + 2 = 0$ has no real roots, find the range of values of h.

2. Find the range of values of p for which the equation $x^2 - 2px + p^2 + 5p - 6 = 0$ has no real roots.

Worked Example 8

(Quadratic Equation with Real Roots)

Show that the equation $x^2 + kx = 4 - 2k$ has real roots for all real values of k.

Solution

$$x^2 + kx = 4 - 2k$$
$$x^2 + kx + 2k - 4 = 0$$

i.e. $a = 1, b = k, c = 2k - 4$

$$\begin{aligned} b^2 - 4ac &= k^2 - 4(1)(2k - 4) \\ &= k^2 - 8k + 16 \\ &= (k - 4)^2 \geqslant 0 \text{ for all real values of } k \end{aligned}$$

Since $b^2 - 4ac \geqslant 0$, the equation has real roots for all real values of k.

Practise Now 8

Similar Questions:
Exercise 2B
Question 10, 14

1. Show that the equation $3x^2 + px = 3 - p$ has real roots for all real values of p.

2. Show that the roots of the equation $ax^2 + (3a + b)x + 3b = 0$ are real for all real values of a and b.

Basic Level | Intermediate Level | Advanced Level

Exercise 2B

1. Determine the nature of the roots of each of the following equations.
   (a) $3x^2 + 5x + 4 = 0$
   (b) $2x^2 - 7x + 9 = 0$
   (c) $x^2 - 8x + 16 = 0$
   (d) $2x^2 + 7x - 1 = 0$
   (e) $3 - 4x - x^2 = 0$
   (f) $4x - 2 - 3x^2 = 0$
   (g) $(x - 2)^2 = 3$
   (h) $\frac{1}{x} - \frac{1}{x+1} = \frac{1}{5}$

2. The equation $x^2 + x(2x + p) + 3 = 0$ has equal roots. Find the values of p.

3. Find the values of k for which the equation $(k+1)x^2 + 4(k-2)x + 2k = 0$ has repeated roots.

4. The equation $kx^2 + 2kx - 4 + k = 0$ has no real roots. Find the range of values of k.

5. Find the values of p for which the equation $(2p + 3)x^2 + (4p - 14)x + 16p + 1 = 0$ has coincident roots.

6. Given that the roots of the equation $ax^2 - 4ax + 4a - 5 = 0$ are real and distinct, find the range of values of a.

7. Find the range of values of k for which the equation $3x^2 - 7x + k = 0$ has no real roots.

8. Given that the equation $px^2 + 5x + 1 = 0$ has real and distinct roots, find the range of values of p.

9. Given that the equation $5x^2 + 7x - 3k = 6$ has no real roots, find the range of values of k.

10 Find the range of values of k for which the equation $x^2 + 2kx + k^2 - 5k + 7 = 0$ has real roots.

11 Given that the roots of the equation $(p + 2)x^2 - 2px = 5 - p$ are not real, find the range of values of p.

12 Show that the equation $x + 5 = \dfrac{p-3}{x}$ has coincident roots when $p = -\dfrac{13}{4}$.

13 Show that the equation $2x^2 + 5hx = 3k$ has real and distinct roots for all positive values of k.

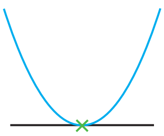
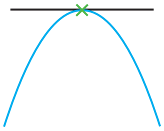
14 Show that the roots of the equation $x^2 + (1 - k)x = k$ are real for all real values of k.

15 Show that the roots of the equation $px^2 - 3x - p = 0$ are real and distinct for all real values of p.

2.3 MAXIMUM AND MINIMUM VALUES OF GENERAL QUADRATIC FUNCTIONS

Recap

The equation of a quadratic function is $f(x) = ax^2 + bx + c$, where $a \neq 0$. The graph of the function $y = ax^2 + bx + c$ is called a parabola. The shape of the parabola depends on the value of a, the coefficient of x^2.

	
If $a > 0$, the curve has a minimum point. The **turning point** of this curve is a **minimum point** and it occurs at the lowest point of the curve.	If $a < 0$, the curve has a maximum point. The **turning point** of this curve is a **maximum point** and it occurs at the highest point of the curve.

Quadratic graphs have turning points, which are either maximum or minimum, depending on the coefficient of x^2, which is a.

In O-level mathematics, we have learnt how to **complete the square** for a quadratic function, where the coefficient of x^2 is 1.

Worked Example 9

(Completing the Square for $a = 1$)

Express $x^2 - 3x + 5$ in the form $(x - p)^2 + q$, where p and q are constants. Hence, state

(i) the minimum value of $x^2 - 3x + 5$,

(ii) the value of x at which the minimum value occurs.

Sketch the curve $y = x^2 - 3x + 5$.

Solution

adding and subtracting (half the coefficient of x)^2

$$\begin{aligned} x^2 - 3x + 5 &= x^2 - 3x + 5 + \left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2 \\ &= x^2 - 3x + \left(-\frac{3}{2}\right)^2 + 5 - \left(-\frac{3}{2}\right)^2 \\ &= \left(x - \frac{3}{2}\right)^2 + \frac{11}{4} \end{aligned}$$

ATTENTION

As the coefficient of x is -3, half the coefficient of x is $-\frac{3}{2}$. Hence, we add and subtract $\left(-\frac{3}{2}\right)^2$.

(i) Minimum value of $x^2 - 3x + 5$ is $\frac{11}{4} = 2\frac{3}{4}$

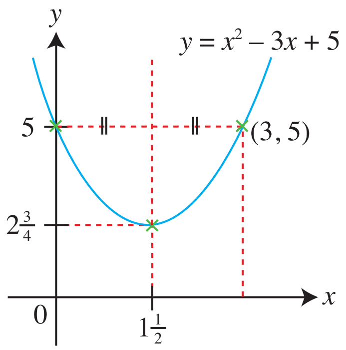
(ii) Minimum value occurs when $\left(x - \frac{3}{2}\right)^2 = 0$,

i.e. $x = \frac{3}{2}$

$\therefore$ The coordinates of the minimum point are $\left(\frac{3}{2}, 2\frac{3}{4}\right)$.

When $x = 0$, $y = 5$.

By symmetry, the third point is $(3, 5)$.



PROBLEM-SOLVING TIP

To ensure that the shape of the curve is not distorted, we should use a scale for both axes.

Practise Now 9

Similar Question:
Exercise 2C
Question 4

Express $x^2 + 5x - 3$ in the form $(x + h)^2 + k$, where h and k are constants. Hence, state

(i) the minimum value of $x^2 + 5x - 3$,

(ii) the value of x at which the minimum value occurs.

Sketch the curve $y = x^2 + 5x - 3$.

Completing the Square ($a \neq 1$)

In Worked Example 9, we have learnt that $x^2 - 3x + 5$ can be written in the form $(x - p)^2 + q$, as $\left(x - \frac{3}{2}\right)^2 + \frac{11}{4}$. What if the coefficient of x^2 is not 1? How then do we complete the square?

Worked Example 10

(Completing the Square for $a \neq 1$)

Express $2x^2 - 6x - 10$ in the form $a(x - h)^2 + k$, where a, h and k are constants. Hence, state

(i) the minimum value of $2x^2 - 6x - 10$,

(ii) the value of x at which the minimum value occurs.

Solution

Consider $2x^2 - 6x - 10$.

Ensure that the coefficient of x^2 is 1 before completing the square.

$$\begin{aligned} 2x^2 - 6x - 10 &= 2(x^2 - 3x - 5) \\ &= 2\left[x^2 - 3x - 5 + \left(-\frac{3}{2}\right)^2 - \left(-\frac{3}{2}\right)^2\right] \\ &= 2\left[\left(x - \frac{3}{2}\right)^2 - 5 - \frac{9}{4}\right] \\ &= 2\left[\left(x - \frac{3}{2}\right)^2 - \frac{29}{4}\right] \\ &= 2\left(x - \frac{3}{2}\right)^2 - \frac{29}{2} \end{aligned}$$

ATTENTION

To complete the square for $x^2 - 3x - 5$, we add and subtract $(\frac{1}{2} \times$ coefficient of $x)^2$, i.e. $\left(-\frac{3}{2}\right)^2$.

(i) Minimum value of $2x^2 - 6x - 10$ is $-\frac{29}{2} = -14\frac{1}{2}$

(ii) Minimum value occurs when $\left(x - \frac{3}{2}\right)^2 = 0$,

i.e. $x = \frac{3}{2}$

Practise Now 10

Similar Questions:
Exercise 2C
Questions 1(a)-(d), 3(a)-(e), 5, 6

(i) Express each of the following in the form $a(x - h)^2 + k$.

(a) $3x^2 - 6x - 3$ **(b)** $5x^2 + 8x - 2$

(c) $5x - 2x^2$ **(d)** $-4x^2 + 4x - 1$

(e) $-4x^2 + 4x + 3$ **(f)** $3x^2 + 2x + \frac{1}{3}$

(ii) Write down the maximum or minimum value for each expression and the value of x at which this occurs.

Worked Example 11

(Completing the Square for $a \neq 1$)

Given that $f(x) = 10 + 8x - 2x^2$, find the values of the constants a, h and k for which $f(x) = a(x - h)^2 + k$. Hence, state the maximum value of $f(x)$ and the corresponding value of x when this occurs. Sketch the curve $y = f(x)$.

Solution

By completing the square,

$$\begin{aligned} f(x) = 10 + 8x - 2x^2 &= -2(x^2 - 4x - 5) \\ &= -2[x^2 - 4x + (-2)^2 - 5 - (-2)^2] \\ &= -2[(x-2)^2 - 9] \\ &= -2(x-2)^2 + 18 \\ &= 18 - 2(x-2)^2 \end{aligned}$$

$\therefore a = -2,\ h = 2,\ k = 18$

Since $(x - 2)^2 \geqslant 0$ for all real values of x,

then $-2(x - 2)^2 \leqslant 0$,

i.e. $18 - 2(x - 2)^2 \leqslant 18$.

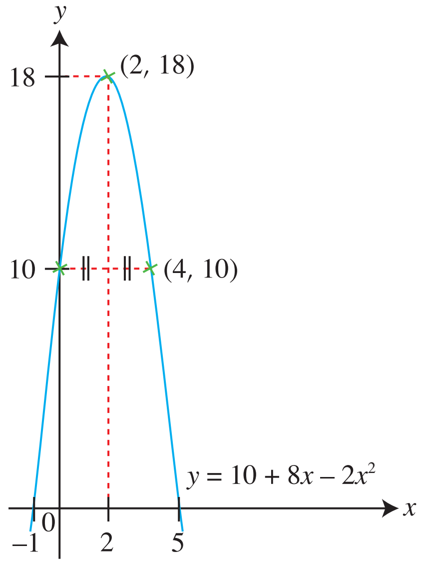
$\therefore$ The maximum value of $f(x)$ is 18 when $(x - 2)^2 = 0$, i.e. when $x = 2$.

When $x = 0$, $y = 10$.

By symmetry, the third point is $(4, 10)$.

When $y = 0$,

$$\begin{aligned} 18 - 2(x-2)^2 &= 0 \\ 2(x-2)^2 &= 18 \\ (x-2)^2 &= 9 \\ x - 2 &= \pm 3 \\ x = -1 \text{ or } x &= 5 \end{aligned}$$



Practise Now 11

Similar Questions:
Exercise 2C
Questions 7-9

1. Express $y = 15 + 3x - 2x^2$ in the form $y = a(x - h)^2 + k$, where h and k are constants. Hence, state the maximum value of y and the corresponding value of x. Sketch the curve $y = 15 + 3x - 2x^2$.
2. Given that $f(x) = 17 + 6x - 3x^2 = a(x - b)^2 + c$, where a, b and c are constants, find the values of a, b and c. Hence, state the maximum value of $f(x)$ and the corresponding value of x when this occurs. Sketch the curve $y = f(x)$.

Journal Writing

By completing the square, transform the expression $ax^2 + bx + c$ to the form $a(x - h)^2 + k$. Deduce the quadratic formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ by equating both $a(x - h)^2 + k$ and $ax^2 + bx + c$ to zero. By considering positive and negative values of a, h and k, sketch possible graphs of $y = a(x - h)^2 + k$.

Worked Example 12

(Completing the Square for $a \neq 1$)

By expressing $y = \frac{1}{2}x^2 - 4x + 6$ in the form $y = a(x + b)^2 + c$, where a, b and c are constants, state the minimum value of y and the value of x at which this occurs. Hence, sketch the curve $y = \frac{1}{2}x^2 - 4x + 6$.

Solution

By completing the square,

$$\begin{aligned} y &= \frac{1}{2}x^2 - 4x + 6 \\ &= \frac{1}{2}(x^2 - 8x + 12) \\ &= \frac{1}{2}[(x-4)^2 - 16 + 12] \\ &= \frac{1}{2}[(x-4)^2 - 4] \\ &= \frac{1}{2}(x-4)^2 - 2 \end{aligned}$$

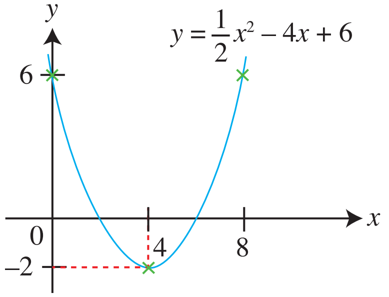
Minimum value of $y = -2$

Minimum value occurs when $(x - 4)^2 = 0$, i.e. $x = 4$

The coordinates of the minimum point are $(4, -2)$.

When $x = 0$, $y = 6$.

By symmetry, the third point is $(8, 6)$.



Practise Now 12

Similar Questions:
Exercise 2C
Questions 2(a)-(f)

Express each of the following in the form $y = a(x + b)^2 + c$, where a, b and c are constants. State the maximum or the minimum value of y and the corresponding value of x. Hence, sketch each of the following curves.

(a) $y = 2x^2 + x - 6$ **(b)** $y = 7x + 2x^2$ **(c)** $y = \frac{1}{2}x^2 - 3x + 5$

To sum up, for the curve $y = a(x - h)^2 + k$, we note the following:

- The shape of the curve depends on the value of a.
- The coordinates of the turning point are (h, k).
- The equation of the line of symmetry is $x = h$.
- The maximum/minimum value of y is k and the corresponding value of x is h.

Exercise 2C

1 State the maximum or minimum value of y for each of the following functions and its corresponding value of x.

(a) $y = (5x - 2)^2 - 4$ **(b)** $y = 2 - (2x - 1)^2$

(c) $y = -2 - (2x + 5)^2$ **(d)** $y = -3 + (3x + 4)^2$

2 Sketch, on separate diagrams, the graphs of each of the following, showing the x- and y-intercepts. State clearly the coordinates of the maximum or minimum points.

(a) $y = (x - 1)(x + 7)$

(b) $y = (2x - 1)(2x + 7)$

(c) $y = (x + 2)(x - 5)$ **(d)** $y = (x + 3)(2 - x)$

(e) $y = 3x - 4x^2$ **(f)** $y = 3x^2 - 8x$

3 By completing the square, find the maximum or minimum value of each of the following functions. In each case, state the value of x at which the function is a maximum or a minimum.

(a) $y = 2x^2 - 4x + 7$ **(b)** $y = 2x^2 - 5x - 1$

(c) $y = 3x^2 + 7x + 9$ **(d)** $y = 4x - 1 - x^2$

(e) $y = 13 - 6x - 3x^2$

4 Given that $x^2 - 5x + 13 = (x - p)^2 + q$ for all real values of x, find the value of p and of q. Hence, state

(i) the minimum value of $x^2 - 5x + 13$,

(ii) the value of x at which the minimum value occurs.

Sketch the curve $y = x^2 - 5x + 13$.

5 Express $y = 4x^2 - 6x + 14$ in the form $y = 4(x - p)^2 + q$, where p and q are constants. Hence, state the minimum value of y.

6 The value of a stock portfolio is given by the function $y = 3x^2 - 4x + 5$, where y is the value of the portfolio in thousands of dollars and x is the time in years. Find x when the value of the portfolio is at its lowest.

7 By expressing $y = 2x^2 + 4x + 17$ in the form $y = a(x + b)^2 + c$, where a, b and c are constants, state the minimum value of y and the value of x at which this occurs. Hence, sketch the curve $y = 2x^2 + 4x + 17$.

8 Express $y = 14 - 3x - x^2$ in the form $y = p - (x + q)^2$, where p and q are constants. Hence, state the maximum value of y and the value of x at which the maximum value occurs. Sketch the curve $y = 14 - 3x - x^2$.

9 The height, y metres, of the water sprayed from a hose, is given as $y = -x^2 + 4x - 1$, where x is the horizontal distance travelled by the water. Find the greatest height of the water sprayed and the horizontal distance from the hose when this occurs.

10 Show that $3x^2 - 4x + 2$ is always positive for all real values of x.

11 Show that $5 - 4x - x^2$ can never be greater than 9.

2.4 QUADRATIC INEQUALITIES

Recap

Listed below are some properties of linear inequalities that we have learnt in O-level mathematics.

If a and b are real numbers,

(i)	$a > b$	implies	$a + c$	$>$	$b + c$	for any real number c
(ii)	$a > b$	implies	$a - c$	$>$	$b - c$	for any real number c
(iii)	$a > b$	implies	ac	$>$	bc	for any positive number c
		but	ac	$<$	bc	for any negative number c
(iv)	$a > b$	implies	$\frac{a}{c}$	$>$	$\frac{b}{c}$	for any positive number c
		but	$\frac{a}{c}$	$<$	$\frac{b}{c}$	for any negative number c.

Quadratic Inequalities

Consider $2x^2 - 3x + 1 \leqslant 0$ and $(x - 3)(x + 1) > 0$. These are called quadratic inequalities. In this section, we will learn how to solve quadratic inequalities.

Thinking Time

To solve the quadratic **equation** $(x - 3)(x + 1) = 0$, we have

$$x - 3 = 0 \quad \text{or} \quad x + 1 = 0$$

i.e. $$x = 3 \quad \text{or} \quad x = -1.$$

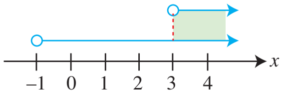
To solve the quadratic **inequality** $(x - 3)(x + 1) > 0$, can we do this:

$$x - 3 > 0 \quad \text{or} \quad x + 1 > 0$$

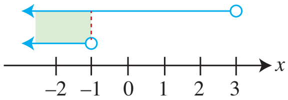
i.e. $$x > 3 \quad \text{or} \quad x > -1?$$

Choose a few values of x that satisfy $x > 3$ *or* $x > -1$ and substitute each value into $(x - 3)(x + 1)$ to see if $(x - 3)(x + 1)$ is always greater than 0. Are these the only possible values? What can you conclude?

To sum up the Thinking Time on the previous page, there are two cases to consider when we solve $(x-3)(x+1)>0$.



Case 1: $x-3>0$ *and* $x+1>0$ (i.e. both factors must be positive)
i.e. $x>3$ and $x>-1$
For x to satisfy *both* inequalities, $x>3$.



Case 2: $x-3<0$ *and* $x+1<0$ (i.e. both factors must be negative)
i.e. $x<3$ and $x<-1$
For x to satisfy *both* inequalities, $x<-1$.

Since either **Case 1 or Case 2** must be satisfied, the solution is $x<-1$ **or** $x>3$.

Check: Choose a few values of x that satisfy $x<-1$ or $x>3$ and substitute each value into $(x-3)(x+1)$ to verify that $(x-3)(x+1)$ is greater than 0.

Class Discussion

We have observed that the solution for the quadratic inequality $(x-3)(x+1)>0$ is $x<-1$ or $x>3$.

(i) Solve the quadratic equation $(x-3)(x+1)=0$.

(ii) By considering the graph of $y=(x-3)(x+1)$, what can you say about the solution of the quadratic inequality $(x-3)(x+1)>0$?

(iii) Discuss with your classmate what you can observe about the relationship between the solutions of the quadratic inequality and the related quadratic equation.

(iv) What do you think the answer will be if the quadratic inequality is $(x-3)(x+1)<0$? Explain your answer.

Repeat the above steps **(i)** to **(iv)** for the quadratic inequality $(x-a)(x-b)>0$, where $a<b$, and its related quadratic equation $(x-a)(x-b)=0$.

From the above class discussion, we can devise another method of solving a quadratic inequality by making use of the solutions of its corresponding quadratic equation, as shown in Worked Example 13.

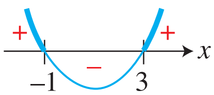
Worked Example 13

(Solving Quadratic Inequalities)

(a) Solve $(x-3)(x+1)>0$.
(b) Solve $(x-3)(x+1)<0$.

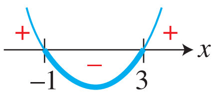
Solution

(a) Let $y=(x-3)(x+1)$.
When $y=0$, $x=3$ or $x=-1$.
For $(x-3)(x+1)>0$,



$\therefore\ x<-1$ or $x>3$

(b) For $(x-3)(x+1)<0$,



$\therefore -1<x<3$

PROBLEM-SOLVING TIP

Sketch the graph of $y=(x-3)(x+1)$, showing the x-intercepts clearly. For $(x-3)(x+1)>0$, we need to find the range of values of x for which the curve is above the x-axis (so that $y>0$).

Practise Now 13

Similar Questions:
Exercise 2D
Questions 1(a)-(l), 3(a)-(d), 4-7

1. Solve $(4x - 5)(3x + 1) > 0$.
2. Find the range of values of x for which $3x^2 < x^2 - x + 3$.
3. Find the range of values of x for which $x(5x - 3) < 2x(x - 4) + 2$.

Worked Example 14

(Quadratic Inequality involving Roots of an Equation)
Find the range of values of p for which $x^2 - px + p + 3 = 0$ has real and distinct roots. State the values of p such that the equation has coincident roots.

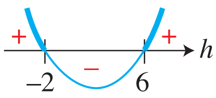
Solution

$$x^2 - px + p + 3 = 0$$

i.e. $a = 1,\ b = -p,\ c = p + 3$

For real and distinct roots,

$$\begin{aligned} b^2 - 4ac &> 0 \\ (-p)^2 - 4(1)(p + 3) &> 0 \\ p^2 - 4p - 12 &> 0 \\ (p + 2)(p - 6) &> 0 \\ \therefore\ p < -2 \text{ or } p &> 6 \end{aligned}$$



If the equation has coincident roots,

$$\begin{aligned} b^2 - 4ac &= 0 \\ (p + 2)(p - 6) &= 0 \\ \therefore\ p = -2 \text{ or } p &= 6 \end{aligned}$$

Practise Now 14

Similar Questions:
Exercise 2D
Questions 8-10

1. Find the range of values of k for which the equation $x^2 + 2k + 10 = x - 3kx$ has real roots.
2. Given that the equation $(a + 1)x^2 + 4ax = 8x - 2a$ has no real roots, find the range of values of a.

Cases When $ax^2 + bx + c$ is Positive or Negative

From the investigation in Section 2.2, we are able to observe and understand how the sign of $b^2 - 4ac$ affects the position of the graph of $y = ax^2 + bx + c$.

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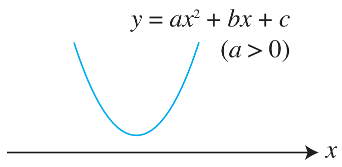
<tr>
<td>
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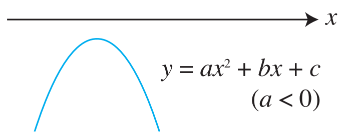
<td>
</td>
</tr>
<tr>
<td>When $a > 0$ and $b^2 - 4ac < 0$, $y = ax^2 + bx + c$ lies entirely above the x-axis, i.e. $y = ax^2 + bx + c > 0$ for all real values of x.</td>
<td>When $a < 0$ and $b^2 - 4ac < 0$, $y = ax^2 + bx + c$ lies entirely below the x-axis, i.e. $y = ax^2 + bx + c < 0$ for all real values of x.</td>
</tr>
</table>

Worked Example 15

(Quadratic Expression which is Always Positive)

Find the smallest value of the integer k for which $kx^2 + 7x + 3$ is always positive for all real values of x.

Solution

For $kx^2 + 7x + 3$ to be always positive for all real values of x, the curve $y = kx^2 + 7x + 3$ must lie entirely above the x-axis.

i.e.
$$\begin{aligned} b^2 - 4ac &< 0 \\ 7^2 - 4(k)(3) &< 0 \\ 49 - 12k &< 0 \\ -12k &< -49 \\ k &> \frac{49}{12} \end{aligned}$$

$\therefore$ The smallest integer value of k is 5.

Practise Now 15

Similar Questions:
Exercise 2D
Questions 11, 13, 17

1. Find the smallest integer value of k for which $3x^2 + 4x + k$ is always positive for all real values of x.
2. Find the range of values of k for which $2x^2 + 5x + k$ is always positive for all real values of x.

Worked Example 16

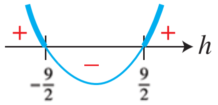
(Quadratic Expression which is Always Negative)

Find the largest value of the integer h for which $hx^2 + 9x + h$ is always negative for all real values of x.

Solution

For $hx^2 + 9x + h$ to be always negative for all real values of x, the curve $y = hx^2 + 9x + h$ must lie entirely below the x-axis.

i.e.
$$\begin{aligned} b^2 - 4ac &< 0 \\ 9^2 - 4(h)(h) &< 0 \\ 81 - 4h^2 &< 0 \\ 4h^2 - 81 &> 0 \\ (2h + 9)(2h - 9) &> 0 \\ h < -\frac{9}{2} \text{ or } h &> \frac{9}{2} \end{aligned}$$



Since $hx^2 + 9x + h$ is always negative, $h < 0$.
$\therefore$ The largest integer value of h is –5.

Practise Now 16

Similar Questions:
Exercise 2D
Questions 2, 12, 14

1. Find the smallest integer value of k for which $-5x^2 + 7x - k$ is always negative for all real values of x.
2. Find the range of values of p for which $-3x^2 + 4x - p$ is always negative for all real values of x.

Exercise 2D

1 Find the range of values of x which satisfy each of the following inequalities, showing the answer on a number line.

(a) $(x + 2)(x - 3) > 0$ **(b)** $(x - 4)(x + 7) < 0$

(c) $12x^2 \geqslant 10 - 7x$ **(d)** $2x^2 + 7x \geqslant 4$

(e) $x^2 - 7x + 10 < 0$ **(f)** $x^2 > 13x - 42$

(g) $2x^2 > 3x + 54$ **(h)** $2x^2 + x > 28$

(i) $2 + 3x > 5x^2$ **(j)** $(x + 2)^2 > 2x + 7$

(k) $4(x + 1)(x - 4) + 25 \geqslant 0$

(l) $(2x + 1)(3x - 1) < 14$

2 Determine the range of values of k for which the function $y = 2k + 5x - 2x^2$ is always negative for all real values of x.

3 Find the range of values of x which satisfy each of the following inequalities.

(a) $4(2x - 3)^2 \geqslant x^2$ **(b)** $2 - x - x^2 < 0$

(c) $3x^2 \leqslant x^2 - x + 3$ **(d)** $2x(2 - x) < 3(x - 2)$

4 Given that x^2 is smaller than $\frac{7}{2}x + 36$, find the range of values of x.

5 Solve the inequality $(3x - 5)^2 - \left(\frac{3}{2}\right)^2 \geqslant 0$.

6 Given that $x^2 - 5x + 1$ lies between -5 and 15, find the range of values of x.

7 Find the range of values of x for which $1 - x < (x - 1)(5 - x) < 3$.

8 If the equation $(k + 1)x^2 + 4kx - 8x + 2k = 0$ has real roots, find the range of values of k.

9 Find the range of values of k for which the equation $x^2 + 3kx + 2k = x - 10$ has real and distinct roots. State the values of k for which the equation has equal roots.

10 Given that the quadratic equation $x^2 + (2a - 1)x + a^2 = 0$ has no real roots, find the range of values of a.

11 Find the smallest integer value of k for which $2x^2 + 5x + 3k$ is always positive for all real values of x.

12 Find the largest integer value of k for which $-7x^2 + 9x + k$ is always negative for all real values of x.

13 The operating cost of a company can be approximately modelled as $C(x) = 17 - 8x - 2x^2$, where x is the time in years. Given that its revenue function is $R(x) = 5 - 3x$, find the number of years during which the company is not making a profit.

14 Find the range of values of k for which the expression $3 - 4k - (k + 3)x - x^2$ is negative for all real values of x.

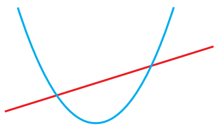
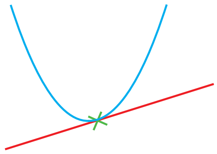
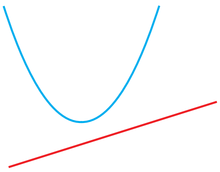
15 Find the range of values of x for which $\frac{5}{6x^2 - 11x - 35} < 0$.

16 Find the range of values of x for which $\frac{7x^2 + 7 - 14x}{3x^2 + x - 10} > 0$.

17 Find the conditions for the function $f(x) = ax^2 + bx + c$ to be always positive for all real values of x.

2.5 INTERSECTION OF A LINE AND A CURVE

Given a line and a curve, there are three possibilities as shown in the diagrams below. To find the points of intersection, the two equations can be solved algebraically or graphically.

Case 1	Case 2	Case 3
		
The line cuts the curve at 2 real and distinct points.	The line touches the curve at 1 real point, i.e. the line is a **tangent** to the curve.	The line does not intersect the curve.

In this section, we will learn the conditions for a given line to intersect a given curve, be a tangent to a given curve and not intersect a given curve.

Investigation

Intersection of a Line and a Curve

1. Use a graphing software to plot the graph of $y = x^2 - 2x - 3$.

2. Now consider the straight line $y = x + k$. Use the values of k given in the table to plot the straight line. Copy and complete the table below.

No.	k	Equation of Line	No. of Points of Intersection	Nature of Solutions	Characteristics of Line and Curve
(a)	2	$y = x + 2$	2	Real and distinct	Line cuts the curve at 2 distinct points
(b)	-1				
(c)	$-\frac{21}{4}$				
(d)	-6				
(e)	-10				

3. How is the number of points of intersection of the line and the curve related to the nature of solutions?

Extending the results of the investigation done in Section 2.2, we can further conclude the following: To find out whether a straight line intersects a quadratic curve, we solve these two equations simultaneously to obtain an equation of the form $ax^2 + bx + c = 0$.

$b^2 - 4ac$	Nature of Solutions	Characteristics of Line and Curve
> 0	2 real and distinct roots	Line cuts the curve at 2 distinct points
$= 0$	2 real and equal roots	Line is a tangent to the curve
< 0	no real roots	Line does not intersect the curve

Worked Example 17

(Line Intersects Curve at 2 Distinct Points)

Find the range of values of m for which the line $y = mx + 5$ intersects the curve $x^2 + y^2 = 9$ at two distinct points.

Solution

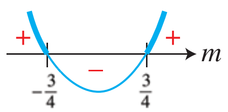
$$y = mx + 5 \text{ ------- } (1)$$
$$x^2 + y^2 = 9 \text{ -------------- } (2)$$

Subst. (1) into (2):

$$x^2 + (mx + 5)^2 = 9$$
$$x^2 + m^2x^2 + 10mx + 25 = 9$$
$$(1 + m^2)x^2 + 10mx + 16 = 0$$

Since the line intersects the curve at two distinct points,

$$b^2 - 4ac > 0$$
$$(10m)^2 - 4(1 + m^2)(16) > 0$$
$$100m^2 - 64 - 64m^2 > 0$$
$$36m^2 - 64 > 0$$
$$9m^2 - 16 > 0$$
$$(3m + 4)(3m - 4) > 0$$
$$\therefore m < -\frac{3}{4} \text{ or } m > \frac{3}{4}$$



INFORMATION

The equation in the form $x^2 + y^2 = r^2$ represents a circle with centre $(0, 0)$. We will learn more about the equation of a circle in Chapter 6.

Practise Now 17

Similar Questions:
Exercise 2E
Questions 2, 4, 9

1. Find the range of values of k for which the line $y = kx + 2$ will intersect the curve $x^2 + y^2 = 2$ at two real and distinct points.
2. Find the range of values of k for which the line $y = (k + 5)x - 2$ will intersect the curve $y = 11x - x^2 - 3$ at two real and distinct points.

Worked Example 18

(Line does not Intersect Curve)

Find the range of values of k for which the line $y = kx + 4$ does not intersect the curve $y = x^2 - 4x + 5$. State the values of k for which $y = kx + 4$ will be a tangent to the curve $y = x^2 - 4x + 5$.

Solution

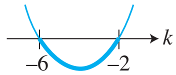
$$y = kx + 4 \text{ ------------- } (1)$$
$$y = x^2 - 4x + 5 \text{ -------- } (2)$$

Subst. (1) into (2):

$$\begin{aligned} x^2 - 4x + 5 &= kx + 4 \\ x^2 - (4 + k)x + 1 &= 0 \end{aligned}$$

Since the line does not intersect the curve,

$$\begin{aligned} b^2 - 4ac &< 0 \\ [-(4 + k)]^2 - 4(1)(1) &< 0 \\ 16 + 8k + k^2 - 4 &< 0 \\ k^2 + 8k + 12 &< 0 \\ (k + 6)(k + 2) &< 0 \\ \therefore -6 < k &< -2 \end{aligned}$$



If $y = kx + 4$ will be a tangent to the curve $y = x^2 - 4x + 5$,

$$\begin{aligned} b^2 - 4ac &= 0 \\ (k + 6)(k + 2) &= 0 \\ \therefore k &= -6 \text{ or } k = -2 \end{aligned}$$

Practise Now 18

Similar Questions:
Exercise 2E
Questions 1, 3, 5-8, 10

1. Find the range of values of k for which the line $y = kx - 2$ does not intersect the curve $y = x^2 + 3x + 7$. State the values of k for which $y = kx - 2$ will be a tangent to the curve $y = x^2 + 3x + 7$.
2. The line $y = 3x + k$ does not intersect the curve $y = 5 - 3x - 2x^2$. Find the range of values of k. State the value of k for which $y = 3x + k$ is a tangent to the curve $y = 5 - 3x - 2x^2$.

Basic Level | Intermediate Level | Advanced Level

Exercise 2E

1. Find the values of k for which $y = x + k$ is a tangent to the curve $y = 7x - kx^2$.
2. The line $y = kx + 6$ cuts the curve $2x^2 = xy + 3$ at two real and distinct points. Find the range of values of k.
3. Find the range of values of k for which the line $kx - y = 2$ and the curve $x^2 = 4x - y^2$ do not intersect.
4. The line $y = 2x + k$ cuts the curve $y = x^2 - 3$ at two distinct points. Find the range of values of k.

5 Find the range of values of k for which the line $2y = k - x$ does not intersect the curve $y^2 = 20 - 4x$.

6 If the line $x + ky = 10$ is a tangent to the curve $x^2 + y^2 = 10$, find the possible values of k.

7 Find the possible values of k for which the line $y = \frac{1}{2}x + k$ is a tangent to the curve $x^2 + y^2 = 8k$.

8 Find the range of values of k for which the line $y = 2x + k$ does not intersect the curve $x^2 + 2y^2 = 8$ at any point.

9 Given that the line $x + 2y = 3$ intersects the curve $x^2 + y^2 = p$ at two real and distinct points, find the range of values of p.

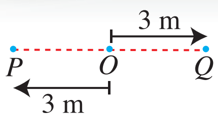
10 The line $y = mx + c$ is a tangent to the curve $x^2 + y^2 = 4$. Prove that $4m^2 = c^2 - 4$.

11 The line $y = mx + c$ is a tangent to the curve $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where a, b, m and c are constants. Show that $c^2 - b^2 = a^2m^2$.

12 The curve $y = x^2 - 2(a + b)x + 2(b - c)$ does not intersect the line $y = 0$ for all real values of x. Find an inequality connecting a, b and c.

2.6 MODULUS FUNCTIONS

excluded from the N(A) syllabus



If you start from the origin, O, and walk 3 m to your right to a point Q, we say that Q is at a distance of +3 m from the origin. If you walk 3 m to the left of the origin to the point P, we say that P is at a distance of –3 m from the origin. However, the numerical distance from the origin in both situations is 3 m. We say that 3 m is the **absolute value**.

The absolute value of x, or the **modulus** of x, denoted by $|x|$, is defined by

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Hence, when $x = -5$, $|x| = |-5| = 5$,

when $x = 5$, $|x| = |5| = 5$.

Equations Involving Absolute Values

From the definition of absolute values, we have the following property:

If $|x| = k$, where $k \geqslant 0$, then $x = k$ or $x = -k$.

We shall use this property to solve equations involving absolute values.

Worked Example 19

(Equations involving Absolute Valued Functions)

Solve each of the following equations.

(a) $|2x - 3| = 11$ **(b)** $|x^2 - 8| = 4$

(c) $|2x^2 - 3x| = x$

Solution

(a) $|2x - 3| = 11$

$$2x - 3 = 11 \quad \text{or} \quad 2x - 3 = -11$$
$$2x = 14 \qquad\qquad 2x = -8$$
$$x = 7 \qquad\qquad x = -4$$
$$x = -4 \text{ or } 7$$

(b) $|x^2 - 8| = 4$

$$x^2 - 8 = 4 \quad \text{or} \quad x^2 - 8 = -4$$
$$x^2 = 12 \qquad\qquad x^2 = 4$$
$$x = \pm\sqrt{12} \qquad\qquad x = \pm 2$$
$$x = \pm 2\sqrt{3}$$
$$x = -2\sqrt{3}, -2, 2 \text{ or } 2\sqrt{3}$$

(c) $|2x^2 - 3x| = x$

$$2x^2 - 3x = x \quad \text{or} \quad 2x^2 - 3x = -x$$
$$2x^2 - 4x = 0 \qquad\qquad 2x^2 - 2x = 0$$
$$2x(x - 2) = 0 \qquad\qquad 2x(x - 1) = 0$$
$$x = 0 \text{ or } x = 2 \qquad\qquad x = 0 \text{ or } x = 1$$
$$x = 0, 1 \text{ or } 2$$

INFORMATION

Another method of solving Worked Example 19(a) is to square both sides of each equation to obtain a quadratic equation. Always check your answers because taking the square on both sides of the equation may introduce 'extra answers' that are not applicable.

Practise Now 19

Similar Questions:
Exercise 2F
Questions 1-4

Solve each of the following equations.

(a) $|3x - 7| = 13$ **(b)** $|x^2 - 3| = 6$

(c) $|3x^2 - 5x| = 7x$

Basic Level | Intermediate Level | Advanced Level

excluded from the N(A) syllabus

Exercise 2F

1 Solve each of the following equations.

(a) $|x-5| = 17$ **(b)** $|2x-7| = 19$

(c) $|-3x+7| = 5$

2 Solve each of the following equations.

(a) $\left|\dfrac{3x+1}{5}\right| = 8$ **(b)** $\left|\dfrac{2x-7}{3}\right| = 6$

(c) $|3x-2| = x$ **(d)** $|x+3| = 2x$

3 Solve each of the following equations.

(a) $\left|\dfrac{2x-3}{x+4}\right| = 5$ **(b)** $\left|\dfrac{3x-8}{x+2}\right| = 13$

(c) $|2x-1| = x+7$ **(d)** $|3x-7| = 5x+6$

4 Solve each of the following equations.

(a) $|x^2-3| = 1$ **(b)** $|x^2-3x| = x-3$

(c) $|2x^2-5| = 3x$ **(d)** $|2x^2+3| = 7x$

(e) $|15x^2+13x| = 20$

5 Simplify

(a) $4|\pi-9|-3|7-2\pi|$, giving your answer in terms of π,

(b) $2|8-3\pi|+3|\pi-13|$, giving your answer in terms of π,

(c) $3|15-8e|-2|9-4e|$, giving your answer in terms of e.

6 Solve each of the following pairs of simultaneous equations.

(a) $y = x+3$
$y = |x^2+2x-3|$

(b) $2y = 6-x$
$y = 6-|x-3|$

(c) $y = x+13$
$y = |2x^2-8|$

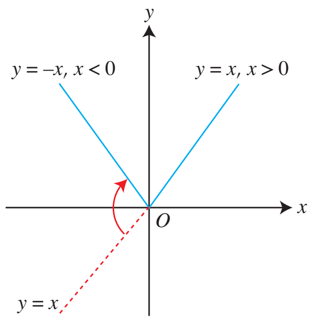
excluded from the N(A) syllabus

2.7 GRAPHS OF MODULUS FUNCTIONS

Graphs of $y = |f(x)|$ where f(x) is linear

Consider the modulus function $y = |f(x)|$, where f(x) is linear, such as $y = |x|$. To plot the graph of $y = |x|$, we can set up a table of values as shown below, before plotting the points.

x	–3	–2	–1	0	1	2	3
$y = \|x\|$	3	2	1	0	1	2	3



The graph of $y = |f(x)|$ is made up of two straight lines which meet at the **vertex** (0, 0) to form a **V-shape**. The graph is **symmetrical** about the vertical line passing through the vertex.

Instead of setting up a table of values, we can also **sketch** the graph of $y = x$ first, then reflect in the x-axis the part of the line $y = x$ which lies below the x-axis.

Worked Example 20

(Graph of $y = |f(x)|$ where $f(x)$ is linear)

Sketch the graph of $y = |2x + 1|$.

Solution

First, sketch $y = 2x + 1$:

When $x = 0$, $y = 1$.

When $y = 0$, $2x + 1 = 0$

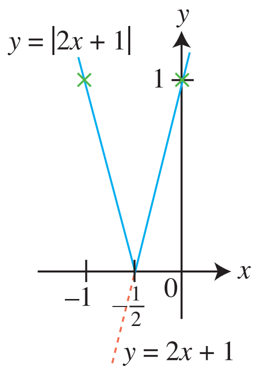
$$2x = -1$$
$$x = -\frac{1}{2}$$

Next, reflect in the x-axis the part of the line $y = 2x + 1$ that is below the x-axis.

To get the third point, we reflect (0, 1) in the line of symmetry $x = -\frac{1}{2}$ to obtain (–1, 1).

PROBLEM-SOLVING TIP

Although this is a sketch, we should still use a scale so that we can label all the 3 critical points (x- and y-intercepts and the vertex) clearly.



Practise Now 20

Similar Questions:
Exercise 2G
Questions 1(a), (b), 3(a), (b)

Sketch the graph of each of the following.

(a) $y = |2x - 1|$ **(b)** $y = |3 - x|$

Graphs of $y = |f(x)|$ where f(x) is quadratic

Worked Example 21

(Graph of $y = |f(x)|$ where f(x) is Quadratic)

Sketch the graph of $y = |x^2 + 2x - 3|$.

Solution

First, sketch $y = x^2 + 2x - 3$:

When $x = 0$, $y = -3$.

When $y = 0$, $x^2 + 2x - 3 = 0$

$(x - 1)(x + 3) = 0$

$x = 1$ or $x = -3$

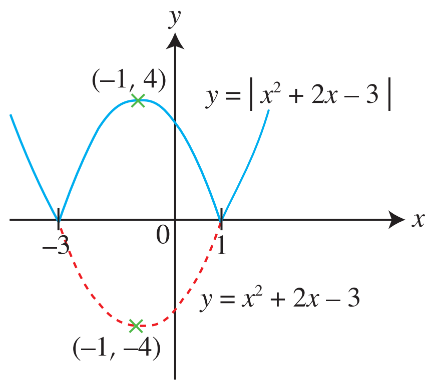
$\therefore$ The x-intercepts are -3 and 1.

$$x\text{-coordinate of minimum point} = \frac{-3+1}{2} = -1$$

$$y\text{-coordinate of minimum point} = (-1)^2 + 2(-1) - 3 = 1 - 2 - 3 = -4$$

$\therefore$ The coordinates of the minimum point are $(-1, -4)$.

Next, reflect in the x-axis the part of the curve $y = x^2 + 2x - 3$ that is below the x-axis.



PROBLEM-SOLVING TIP

Although this is a sketch, we should still use a scale so that we can label all the 4 critical points clearly.

Practise Now 21

Similar Questions:
Exercise 2G
Questions 2(a)-(e), 4(a), (b)

Sketch the graph of each of the following.

(a) $y = |x^2 + x - 6|$ **(b)** $y = |9 - x^2|$

Basic Level | Intermediate Level | Advanced Level

excluded from the N(A) syllabus

Exercise 2G

1 Sketch the graph of each of the following.

(a) $y = |x - 2|$ **(b)** $y = |3x - 2|$

2 Sketch the graph of each of the following.

(a) $y = |x^2 - 3x + 2|$ **(b)** $y = |x^2 - 5x - 6|$

(c) $y = |2x^2 + 5x - 3|$ **(d)** $y = |x^2 - 16|$

(e) $y = |x^2 + 5x - 6|$

3 Sketch the graph of each of the following.

(a) $y = |3x - 4|$ for $-2 \leqslant x \leqslant 3$

(b) $y = |11 - 3x|$ for $2 \leqslant x \leqslant 6$

4 Sketch the graph of each of the following.

(a) $y = |(x - 1)(x - 5)|$ for $0 \leqslant x \leqslant 6$

(b) $y = |x^2 - x - 6|$ for $-4 \leqslant x \leqslant 4$

SUMMARY

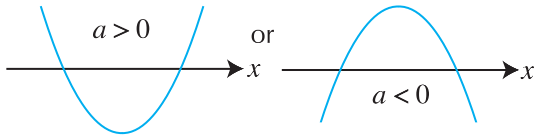
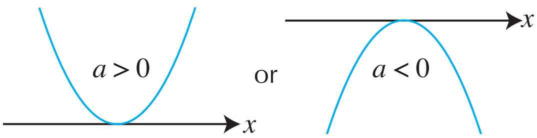
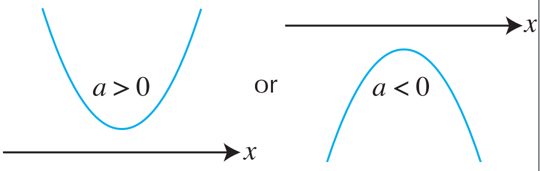
1. Let α and β be the roots of a quadratic equation $ax^2 + bx + c = 0$.

$$\text{Sum of roots, } \alpha + \beta = -\frac{b}{a}$$

$$\text{Product of roots, } \alpha\beta = \frac{c}{a}$$

2. $x^2 - (\text{sum of roots})x + (\text{product of roots}) = 0$

3. For a quadratic equation $ax^2 + bx + c = 0$ and its corresponding curve $y = ax^2 + bx + c$,

$b^2 - 4ac$	Nature of Roots	Shapes of Curve $y = ax^2 + bx + c$	Characteristics of Curve
> 0	2 real and distinct roots		$y = ax^2 + bx + c$ cuts the x-axis at 2 distinct points
$= 0$	2 real and equal roots		$y = ax^2 + bx + c$ touches the x-axis at 1 point
< 0	no real roots		$y = ax^2 + bx + c$ lies entirely above (for $a > 0$) or entirely below (for $a < 0$) the x-axis

4. If $b^2 - 4ac \geqslant 0 \Leftrightarrow$ the roots are real.

5. For a quadratic curve $y = \text{f}(x)$ and a straight line $y = mx + c$, solving the two equations simultaneously will give us a quadratic equation of the form $ax^2 + bx + c = 0$.

$b^2 - 4ac$	Nature of Solutions	Characteristics of Line and Curve
> 0	2 real and distinct roots	Line cuts the curve at 2 distinct points
$= 0$	2 real and equal roots	Line is a tangent to the curve
< 0	no real roots	Line does not intersect the curve

6. For a quadratic function $f(x) = ax^2 + bx + c$ that is written as $f(x) = a(x - h)^2 + k$,
   (i) if $a > 0$, the minimum point is (h, k),
   (ii) if $a < 0$, the maximum point is (h, k).

7. The modulus function $|x|$ is defined by:

$$|x| = \begin{cases} x, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Review Exercise 2

1. Given that α and β are the roots of the equation $2x^2 - 3x - 4 = 0$, form a quadratic equation with integer coefficients whose roots are
   (a) $3\alpha, 3\beta$, **(b)** $\alpha + 3, \beta + 3$, **(c)** $2\alpha + 3, 2\beta + 3$, **(d)** $\frac{3}{\alpha}, \frac{3}{\beta}$, **(e)** $\alpha + 2\beta, \beta + 2\alpha$.

2. The equation $2x^2 - 5x + 3 = 0$ has roots α and β. Find a quadratic equation with integer coefficients whose roots are
   (a) $-2\alpha, -2\beta$, **(b)** $\alpha - 5, \beta - 5$, **(c)** $\frac{1}{\alpha^2}, \frac{1}{\beta^2}$, **(d)** $\alpha - \frac{2}{\beta}, \beta - \frac{2}{\alpha}$, **(e)** α^2, β^2.

3. The roots of the equation $2x^2 + 7x + 9 = 0$ are α and β, while the roots of the equation $ax^2 + 5x + c = 0$ are $\alpha + 2$ and $\beta + 2$. Find the value of a and of c.

4. Given that α and β are the roots of the equation $x^2 + 5x = 1$,
   (i) show that $\alpha^3 = 26\alpha - 5$,
   (ii) form an equation whose roots are α^3 and β^3.

5. Given that α is one of the roots of the equation $x^2 = 2x + 5$, show that
   (i) $\frac{1}{\alpha} = \frac{\alpha - 2}{5}$, **(ii)** $\alpha^3 = 9\alpha + 10$.

6. If the roots of the equation $a^2x^2 - (a + 2)x + 1 = 0$ are equal, find the possible values of a.

7. The equation $(p - 3)x^2 - 2(2p - 3)x + 4p + 1 = 0$ has real and equal roots. Find the value of p.

8. The equation $kx^2 - 2kx + k = 3$ has real and distinct roots. Find the range of values of k.

9. The equation $(1 - a)x^2 + 9 = 4ax$ has no real roots. Find the range of values of a.

10. **(a)** Find the range of values of x for which $x(10 - x) \geqslant 24$.
(b) Find the value of k for which $2y + x = k$ is a tangent to the curve $y^2 + 4x = 20$.

11. Given the curve $y = 4x^2 + mx + m - 3$, find
(i) the possible values of m for which the x-axis is a tangent to the curve,
(ii) the range of values of m for which the curve has a positive y-intercept.

12. Find the possible values of k for which the x-axis is a tangent to the curve $y = x^2 + k(x - 1) + 4 - x$. Hence, find the coordinates of the points at which the curve touches the x-axis.

13. **(a)** Find the range of values of x for which $(x + 1)(x - 3) < 3x - 7$.
(b) Find the range of values of k for which the line $y = 2x + 3$ will not cut the curve $y = x(1 + 2k - x) + 2$.

14. Find the range of values of k for which the line $y = x - 2k - 5$ will meet the curve $y = x^2 + 3kx + 5$ at two distinct points.

15. Find the least integer value of k for which the expression $2x(x + 5) + k$ is always positive for all real values of x.

16. Find the range of values of k for which the expression $x^2 - 8kx + 2x + 15k^2 - 2k - 7$ is never negative for all real values of x.

17. Find the condition for the line $y = mx + c$ to cut the curve $y^2 = 4ax$ at two distinct points, where a, m and c are constants and $a > 0$.

18. Find the condition for the line $y = x - q$ to never meet the curve $2x^2 - xy = 2y + 1$.

19. Given that $a + 5bx - x^2$ is negative for all real values of x, find an inequality connecting a and b.

20. Given that the line $y = mx + c$ is a tangent to the curve $b^2x^2 + a^2y^2 = (ab)^2$, where a, b, c and m are constants, show that $b^2 + a^2m^2 = c^2$.

21. Solve each of the following equations.
(a) $|x - 9| = 13$
(b) $|9x - 1| = 2x + 7$

22. Sketch the graph of each of the following.
(a) $y = |3x + 1|$
(b) $y = |4 - 3x|$
(c) $y = |(x - 1)(2x + 1)|$

1. Find all possible integer values of n if $n^2 - 3n + 2$ is prime.

2. Given that a, b and c are real numbers, prove that the roots of the equation $(x - a)(x - b) = c^2$ are real. State the conditions for the roots to be equal.

3. Find the range of values of a if the inequality $\frac{x^2 + ax - 2}{x^2 - x + 1} < 2$ is satisfied for all real values of x.

4. Given that x is real, find the range of values of x for which the function $f(x) = \frac{x+1}{x^2 + x + 1}$ will lie.

5. Find the range of values of x for which $\frac{(2x-3)(x+1)}{(3-5x)} > 0$.

6. Which is correct: $\sqrt{a^2} = a$ or $\sqrt{a^2} = |a|$?

7. Solve the equation $|2x - y - 3| + \sqrt{7x - 3y - 13} = 0$.

8. Solve the equation $x^2 - 2|x| - 3 = 0$.
Use a graphing software to draw the graph of $y = x^2 - 2|x| - 3$. What is the equation of the line of symmetry of the curve? Explain the geometrical meaning of the two branches of the curve about this line of symmetry.

REVISION EXERCISE A1

1. Find the coordinates of the points of intersection of the line $2x + 5y = 1$ and the curve $x^2 + xy = 4 + 2y^2$.

2. Given that $x + 2$ is a factor of $F(x) = A(x - 1)^2 + B(x - 1) + C$ and that when $F(x)$ is divided by $x - 1$ and $x + 1$, the remainders are 9 and -11 respectively, find the values of the constants A, B and C.

3. When $x^2 - bx + 3$ and $2b - x$ are divided by $x - a$, the remainders are 1 and 4 respectively. Find the value of a and of b.

4. Solve the equation $3x^3 + x^2 - 15x + 2 = 0$, giving your answers correct to two decimal places where necessary.

5. Express each of the following in partial fractions.

(a) $\dfrac{8x - 27}{(x - 3)(x - 4)}$ **(b)** $\dfrac{5x + 9}{(x + 3)^2}$

6. Given that α and β are the roots of the equation $2x^2 - 4x + 5 = 0$, write down the value of $\alpha + \beta$ and of $\alpha\beta$. Find an equation whose roots are $\alpha + \dfrac{1}{\alpha}$ and $\beta + \dfrac{1}{\beta}$.

7. Express $y = 2x^2 - 8x - 5$ in the form $y = a(x + p)^2 + q$, where p and q are constants. Hence, state the least value of y and its corresponding value of x.

8. Find the range of values of k for which the line $y = 2x + k$ will cut the curve $y = \dfrac{2}{1 - x}$ at two real and distinct points.

9. Find the range of values of k for which $x^2 + (k + 3)x + 2k + 3 > 0$ for all real values of x.

10. Sketch the graph of $y = |2x + 5|$, indicating the coordinates of the intercepts with the coordinate axes and the vertex.

REVISION EXERCISE A2

1. Solve the simultaneous equations

$$2y = x + 4,$$
$$2x^2 + 4y^2 = 38 + xy.$$

2. Given that $3x^2 - 11x + 7 = Ax(x - 2) + B(x - 2) + C$ for all values of x, find the values of A, B and C.

3. When $x^4 + ax^3 + bx^2 + 6$ is divided by $x - 1$ and $x - 2$, the remainders are 12 and 54 respectively. Find the value of a and of b. Hence, find the remainder when the expression is divided by $x + 1$.

4. If $x - 3$ is a factor of $x^3 - x^2 + ax + 9$, find the value of a. Hence, factorise the expression completely.

5. Express each of the following in partial fractions.

   (a) $\dfrac{2x^2 + 6x + 15}{(x-2)(x^2+3)}$ **(b)** $\dfrac{2x^2 - 4}{x(x-4)}$

6. Given that α and β are the roots of the equation $5x^2 - 9x + 2 = 0$, write down the value of $\alpha + \beta$ and of $\alpha\beta$. Find an equation whose roots are $\alpha + 3\beta$ and $3\alpha + \beta$.

7. If the equation $4x^2 + (k - 2)x + 1 = 0$ has no real roots, find the range of values of k.

8. Given that the line $y = 3x + 2$ meets the curve $y = x^2 + 5x + q$ at two real and distinct points, find the range of values of q.

9. If the line $y = 3x + k$ is a tangent to the curve $x^2 + y^2 = 14$, find the possible values of k.

10. Sketch the graph of $y = |x^2 + 5x - 6|$, indicating the coordinates of the intercepts with the coordinate axes and the turning point.

BINOMIAL THEOREM

CHAPTER 3

How do we expand $(a + b)^n$, where n is a positive integer? An illustration in the book 'Precious Mirror' written by Zhu Shijie in 1303 shows the coefficients of the binomial expansion of $(a + b)^n$ for $n = 0$ to 8. In China, this is known as Yang Hui's (1238-1298) Triangle even though the Chinese mathematician Jia Xian (1010-1070) discovered this triangle two centuries earlier.

Nevertheless, the earliest discovery was made by an Indian, Pingala, who published the book 'Chandas Shastra' between 5th and 2nd century B.C. However, in much of the Western world, this triangle is called Pascal's Triangle, named after the French mathematician Blaise Pascal (1623-1662).

In this chapter, we will learn how to expand $(a + b)^n$ using Pascal's Triangle and the Binomial Theorem.

Learning Objectives

At the end of this chapter, you should be able to:

- use Pascal's Triangle to expand $(a + b)^n$, where n is a small positive integer,
- use the Binomial Theorem to expand $(a + b)^n$, where n is a positive integer,
- use the notations $n!$ and $\binom{n}{r}$,
- find a specific term in the expansion of $(a + b)^n$ using the Binomial Theorem.
- apply the Binomial Theorem to solve problems.

3.1 BINOMIAL EXPANSION OF $(1 + b)^n$

We have learnt about polynomials in Chapter 1. A polynomial consisting of only two unlike terms is called a **binomial**, e.g. $2x + y$ and $a - 3b$. In Secondary 2 mathematics, we have also learnt how to expand the square of a binomial using identities such as:

$$(a + b)^2 = a^2 + 2ab + b^2$$
$$(a - b)^2 = a^2 - 2ab + b^2$$

In this chapter, we will learn how to expand $(a + b)^n$, where n is a positive integer. The expansion of $(a + b)^n$ is called the **binomial expansion**.

Pascal's Triangle

Investigation

Pascal's Triangle

1. Copy and complete the following. Expand $(1 + b)^n$ for $n = 0, 1, 2, 3$ and 4, by using the above identities or by multiplying out the terms where appropriate.

$n = 0$: $(1 + b)^0 = 1$

$n = 1$: $(1 + b)^1 = 1 + b$

$n = 2$: $(1 + b)^2 = 1 + \_\_\_\_ b + b^2$

$n = 3$: $(1 + b)^3 = (1 + b)^1(1 + b)^2$

$= (1 + b)(1 + \_\_\_\_ b + b^2)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

$= 1 + 3b + \_\_\_\_ b^2 + b^3$

$n = 4$: $(1 + b)^4 = (1 + b)^1(1 + b)^3$

$= (1 + b)(1 + 3b + \_\_\_\_ b^2 + b^3)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

$= 1 + 4b + \_\_\_\_ b^2 + \_\_\_\_ b^3 + b^4$

2. First, we arrange the terms in each of the above expansions so that the powers of b are in *ascending order* from 0 (since $b^0 = 1$) to n.

Copy and complete the following. Write down the *coefficients* of each term as follows (leave $n = 5$ blank for now):

$n = 0$:					1					
$n = 1$:				1	+	1				
$n = 2$:			1	+	2	+	1			
$n = 3$:		1	+	3		____		____		
$n = 4$:	1		4		____		____		____	
$n = 5$:										

3. The triangle in Question 2 is called **Pascal's Triangle**. We will observe some patterns in Pascal's Triangle.
   (i) What is the total number of terms in the n^{th} row?
   (ii) What is the first number in each row?
   (iii) What is the last number in each row?
   (iv) Describe the symmetry in Pascal's Triangle.
   (v) How can you find the numbers in each row using numbers in the previous row?

4. Using the patterns discovered above, write down the coefficients of each term for $n = 5$ and 6 in Question 2.

5. Hence, expand $(1 + b)^5$.
   $(1 + b)^5 = 1 + 5b +$ ______________________ $+ b^5$

6. Can Pascal's Triangle be used to expand $(1 + b)^{20}$?
   Explain your answer.

Worked Example 1

(Binomial Expansion using Pascal's Triangle)
Using Pascal's Triangle, write the terms in the expansion of $(1 + b)^6$. Hence, deduce the expansion of $(1 - x)^6$.

Solution

Using Pascal's Triangle,
$(1 + b)^6 = 1 + 6b + 15b^2 + 20b^3 + 15b^4 + 6b^5 + b^6$
Let $b = -x$.
Then $(1 - x)^6$
$$= 1 + 6(-x) + 15(-x)^2 + 20(-x)^3 + 15(-x)^4 + 6(-x)^5 + (-x)^6$$
$$= 1 - 6x + 15x^2 - 20x^3 + 15x^4 - 6x^5 + x^6$$

ATTENTION
For the expansion of $(1 - x)^6$, the signs of the terms alternate between + and −.

Practise Now 1

Using Pascal's Triangle, write the terms in the expansion of $(1 + b)^7$. Hence, deduce the expansion of
(a) $(1 - x)^7$, **(b)** $(1 + 2y)^7$.

Similar Questions:
Exercise 3A
Questions 1(a)-(d), 6(a)-(d)

Journal Writing

(a) The Fibonacci sequence is a set of numbers that begins with 1 and 1. Each subsequent term can be obtained by finding the sum of the two numbers before it,
i.e. 1, 1, 2, 3, 5, 8, 13, 21, …
Try to identify the Fibonacci sequence in Pascal's Triangle.
(b) Pascal's Triangle has many other patterns. Write some of the patterns that you observe.

3.2 BINOMIAL COEFFICIENTS

From the previous investigation, we observe that Pascal's Triangle can be used to expand $(1 + b)^n$ if n is small because we need the numbers in the previous row in order to obtain the numbers in the next row. Therefore, a more efficient method to find the coefficients of terms in the binomial expansions is needed. These coefficients are also known as **binomial coefficients**.

Investigation

Binomial Coefficients

Let us consider $(1 + b)^4 = 1 + 4b + 6b^2 + 4b^3 + b^4$, where $n = 4$.

The terms are: $T_1 = 1$, $T_2 = 4b$, $T_3 = 6b^2$, $T_4 = 4b^3$, $T_5 = b^4$.

The coefficients are: 1, 4, 6, 4, 1.

Find this function on your calculator: nCr (or ${}^nC_r$ or $\binom{n}{r}$, depending on the model).

In this textbook, we will use the notation $\binom{n}{r}$.

To find $\binom{4}{0}$, key the following into your calculator: $\boxed{4}\ \boxed{n\text{C}r}\ \boxed{0}$.

1. Evaluate the following using a calculator:
   $\binom{4}{0}$; $\binom{4}{1}$; $\binom{4}{2}$; $\binom{4}{3}$; $\binom{4}{4}$.

2. Compare $\binom{4}{r}$, where $0 \leqslant r \leqslant 4$, with the coefficients in the expansion of $(1 + b)^4$.

   What do you notice?

3. Notice that the coefficient of the first term T_1 is $\binom{4}{0}$ and not $\binom{4}{1}$.

   Thus, in the expansion of $(1 + b)^4$, what is the $(r + 1)^{\text{th}}$ term?

4. Evaluate each of the following using a calculator and compare with the coefficients in the expansion of $(1 + b)^5$. What do you notice?
   $\binom{5}{0}$; $\binom{5}{1}$; $\binom{5}{2}$; $\binom{5}{3}$; $\binom{5}{4}$; $\binom{5}{5}$.

In general, the Binomial Theorem for $(1 + b)^n$, where n is a positive integer, states that:

$$\boxed{(1+b)^n = \binom{n}{0} + \binom{n}{1}b + \binom{n}{2}b^2 + \ldots + \binom{n}{n}b^n}.$$

In the expansion of $(1 + b)^n$, the $(r + 1)^{\text{th}}$ term is: $\boxed{T_{r+1} = \binom{n}{r}b^r}$.

INFORMATION

The binomial coefficient $\binom{n}{r}$ is different from a 2×1 matrix or column vector.

Worked Example 2

(Binomial Expansion using a Calculator)

Find, in ascending powers of b, the first four terms in the expansion of

(a) $(1 + b)^{20}$, **(b)** $(1 - 3b)^{20}$.

Solution

(a) $$(1+b)^{20} = \binom{20}{0} + \binom{20}{1}b + \binom{20}{2}b^2 + \binom{20}{3}b^3 + \ldots$$
$$= 1 + 20b + 190b^2 + 1140b^3 + \ldots$$

(b) $$(1-3b)^{20} = \binom{20}{0} + \binom{20}{1}(-3b) + \binom{20}{2}(-3b)^2 + \binom{20}{3}(-3b)^3 + \ldots$$
$$= \binom{20}{0} - \binom{20}{1}(3b) + \binom{20}{2}(3b)^2 - \binom{20}{3}(3b)^3 + \ldots$$
$$= 1 - 60b + 1710b^2 - 30\ 780b^3 + \ldots$$

Practise Now 2

Similar Questions:
Exercise 3A
Questions 2(a)-(d), 7(a)-(d)

Find, in ascending powers of x, the first four terms in the expansion of

(a) $(1 + x)^{10}$, **(b)** $\left(1 - \frac{x}{2}\right)^8$.

Thinking Time

By comparing $\binom{n}{r}$ with Pascal's Triangle,

(i) simplify $\binom{n}{0}$, $\binom{n}{n}$, $\binom{n}{1}$ and $\binom{n}{n-1}$;

(ii) determine the relationship between $\binom{n}{r}$ and $\binom{n}{n-r}$.

Formula for $\binom{n}{r}$

If n is an unknown (see Worked Example 4 later), we cannot evaluate $\binom{n}{r}$ with a calculator, so we need to learn a formula to find $\binom{n}{r}$ in terms of n and r.
First, we observe that:

$$\binom{4}{1} = 4 = \frac{4}{1};\ \binom{4}{2} = 6 = \frac{4\times 3}{2\times 1};\ \binom{4}{3} = 4 = \frac{4\times 3\times 2}{3\times 2\times 1};\ \binom{4}{4} = 1 = \frac{4\times 3\times 2\times 1}{4\times 3\times 2\times 1}.$$

There is a pattern for $\binom{4}{r}$ if $r > 0$, but what if $r = 0$?

To match with Pascal's Triangle, we have to define $\binom{4}{0} = 1$.

In general, if r is a positive integer less than n, then

$$\boxed{\binom{n}{0} = 1 \text{ and } \binom{n}{r} = \frac{n(n-1)(n-2)\ \ldots\ (n-r+1)}{r\times(r-1)\times(r-2)\times\ \ldots\ \times 3\times 2\times 1}.}$$

Worked Example 3

(Evaluation of a Binomial Coefficient)

Without using a calculator,

(a) evaluate $\binom{7}{3}$, **(b)** find an expression for $\binom{n}{3}$.

Solution

(a) $\binom{7}{3} = \frac{7 \times 6 \times 5}{3 \times 2 \times 1} = 35$

(b) $\binom{n}{3} = \frac{n(n-1)(n-2)}{3 \times 2 \times 1}$

$= \frac{n(n-1)(n-2)}{6}$

PROBLEM-SOLVING TIP

Since $r = 3$, the numerator and the denominator has 3 factors each. For the numerator, start with $n = 7$.

Practise Now 3

Similar Questions:
Exercise 3A
Questions 3(a)-(d), 8(a)-(d)

1. Evaluate each of the following without using a calculator.

(a) $\binom{8}{3}$ **(b)** $\binom{6}{4}$ **(c)** $\binom{15}{2}$

2. Express each of the following in terms of n.

(a) $\binom{n}{2}$ **(b)** $\binom{n}{n-1}$

3. Verify that the formula for $\binom{n}{r}$ works for $\binom{n}{n}$.

Worked Example 4

(Finding an Unknown Index)

In the expansion of $(1 + 2b)^n$ in ascending powers of b, the coefficient of the third term is 84. Find the value of n.

Solution

In the expansion of $(1 + b)^n$,

the $(r + 1)^{\text{th}}$ term is $\binom{n}{r} b^r$.

Hence, in the expansion of $(1 + 2b)^n$,
the third term is

$$\binom{n}{2}(2b)^2 = \frac{n(n-1)}{2 \times 1} \times 4b^2$$
$$= 2n(n-1)b^2$$

$\therefore$ The coefficient of the third term is

$$2n(n - 1) = 84$$
$$n(n - 1) = 42$$
$$n^2 - n - 42 = 0$$
$$(n - 7)(n + 6) = 0$$
$$n = 7 \text{ or } n = -6$$
$$\therefore \quad n = 7$$

PROBLEM-SOLVING TIP

- For the third term, $r = 2$.
- Alternatively, for $n(n - 1) = 42$, the solution can be obtained by observation, i.e. $7 \times 6 = 42$.

Practise Now 4

Similar Questions:
Exercise 3A
Questions 12, 13

In the expansion of $(1 - 3b)^n$ in ascending powers of b, the coefficient of the third term is 324. Find the value of n.

Worked Example 5

(Coefficients of a Binomial Expansion)

In the expansion of $(1 + kx)^8$, where $k > 0$, the coefficient of x^3 is four times the coefficient of x^2. Find the value of k.

Solution

Using the formula of the $(r + 1)^{\text{th}}$ term,

$$T_{r+1} = \binom{n}{r} b^r$$

Term containing x^3: $T_4 = \binom{8}{3}(kx)^3 = 56k^3x^3$

Term containing x^2: $T_3 = \binom{8}{2}(kx)^2 = 28k^2x^2$

Since the coefficient of x^3 is four times the coefficient of x^2, we have

$$56k^3 = 4(28k^2)$$

$$56k^3 - 112k^2 = 0$$

$$56k^2(k - 2) = 0$$

i.e. $k = 0$ (rejected) or $k = 2$

Practise Now 5

Similar Questions:
Exercise 3A
Questions 9(a)-(d), 10

In the expansion of $(1 + ax)^5$, the coefficient of x^5 is three times the coefficient of x^4. Find the value of a.

Alternative Formula for $\binom{n}{r}$

How do you evaluate $\binom{20}{8}$ by using the formula for $\binom{n}{r}$?

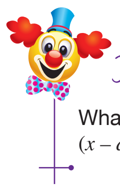
$\binom{20}{8} = \dfrac{20 \times 19 \times 18 \times 17 \times 16 \times 15 \times 14 \times 13}{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}$ is not easy to evaluate, even with the help of a calculator.

Therefore, we have to learn an alternative formula for $\binom{n}{r}$.

We write the denominator $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$ as $8!$ (read as '8 **factorial**').
Use the function $\boxed{\mathbf{n!}}$ in a calculator to find its value.

Therefore, if n is a positive integer, then

$$\boxed{n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1.}$$



Just For Fun

What is $(x - a)(x - b)(x - c)(x - d) \ldots (x - z)$?

The factorials of the first few numbers are given as follows:

$$4 \times 3 \times 2 \times 1 = 4! = 24$$
$$3 \times 2 \times 1 = 3! = 6$$
$$2 \times 1 = 2! = 2$$
$$1 = 1! = 1$$

Now $\binom{20}{8} = \frac{20 \times 19 \times 18 \times \ldots \times 13}{8 \times 7 \times 6 \times \ldots \times 1} = \frac{20 \times 19 \times 18 \times \ldots \times 13}{8!}$ is still not easy to evaluate using a calculator because the numerator contains many terms.

We observe that
$$\binom{20}{8} = \frac{20 \times 19 \times 18 \times \ldots \times 13}{8!}$$
$$= \frac{(20 \times 19 \times 18 \times \ldots \times 13) \times 12!}{8! \times 12!}$$
$$= \frac{20!}{8! \times 12!}$$

Now 8 + 12 = 20, or 12 = 20 – 8 (i.e. $n - r$).

In general, $\boxed{\binom{n}{r} = \frac{n!}{r!(n-r)!}, \text{ where } r \text{ is a non-negative integer less than or equal to } n.}$

Since 0! = 1, this formula works even when $r = 0$ and $r = n$:

$$\boxed{\binom{n}{0} = \frac{n!}{0!(n-0)!} = \frac{n!}{0!\,n!} = 1 \quad \text{and} \quad \binom{n}{n} = \frac{n!}{n!(n-n)!} = \frac{n!}{n!\,0!} = 1.}$$

Worked Example 6

(Finding Unknown Coefficients)

Evaluate $\binom{7}{3}$ using the alternative formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

PROBLEM-SOLVING TIP

We can write $\binom{7}{3} = \frac{7!}{3!\,4!}$ straightaway as we observe that $3 + 4 = 7$.

Solution

$$\binom{7}{3} = \frac{7!}{3!(7-3)!} = \frac{7!}{3!\,4!} = 35$$

Practise Now 6

Similar Questions:
Exercise 3A
Questions 4(a)-(d), 5(a)-(d), 11

Evaluate each of the following using the alternative formula.

(a) $\binom{9}{5}$ **(b)** $\binom{20}{7}$

Exercise 3A

1 Using Pascal's Triangle, write the terms in the expansion of each of the following.

(a) $(1 + x)^6$ **(b)** $(1 - x)^5$

(c) $(1 + 3x)^4$ **(d)** $(1 - 2x)^3$

2 Find, in ascending powers of x, the first four terms in each of the following expansions.

(a) $(1 + x)^8$ **(b)** $(1 - x)^9$

(c) $(1 - 3x)^{10}$ **(d)** $(1 + 2x)^{11}$

3 Without using a calculator, evaluate each of the following.

(a) $\binom{9}{3}$ **(b)** $\binom{5}{4}$

(c) $\binom{14}{2}$ **(d)** $\binom{10}{5}$

4 Use a calculator to evaluate each of the following.

(a) $7!$ **(b)** $9!$

(c) $4! \times 5!$ **(d)** $\frac{8!}{3!}$

5 Evaluate each of the following using the formula $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

(a) $\binom{8}{4}$ **(b)** $\binom{9}{6}$

(c) $\binom{10}{5}$ **(d)** $\binom{16}{14}$

6 Using Pascal's Triangle, write the terms in the expansion of each of the following.

(a) $\left(1+\frac{x}{2}\right)^6$ **(b)** $\left(1-\frac{x}{3}\right)^5$

(c) $(1 - ky)^4$ **(d)** $(1 + x^2)^3$

7 Find, in ascending powers of x, the first four terms in each of the following expansions.

(a) $\left(1-\frac{x}{2}\right)^7$ **(b)** $\left(1+\frac{x}{4}\right)^8$

(c) $(1 + px)^9$ **(d)** $(1 - x^2)^{10}$

8 Express each of the following in terms of n.

(a) $\binom{n}{4}$ **(b)** $\binom{n-1}{2}$

(c) $\binom{n}{1}$ **(d)** $\binom{n}{n}$

9 Find the coefficient of x^3 and x^5 in the expansion of each of the following.

(a) $(1 + 6x)^{10}$ **(b)** $(1 - 3x)^8$

(c) $\left(1+\frac{x}{2}\right)^7$ **(d)** $\left(1-\frac{x}{4}\right)^5$

10 In the expansion of $(1 + kx)^9$, the coefficient of x^5 is three times that of x^4. Find the value of k.

11 **(i)** Use a calculator to evaluate each of the following.

(a) $2! \times 4!$ **(b)** $8!$

(ii) Is $2! \times 4! = 8!$?

(iii) Is $a! \times b! = (a \times b)!$? Explain your answer.

12 In the expansion of $(1 - 2b)^n$ in ascending powers of b, the coefficient of the third term is 112. Find the value of n.

13 In the expansion of $(1 + 3x)^n$ in ascending powers of x, the coefficient of the third term is 135. Find the value of n.

3.3 BINOMIAL THEOREM

In previous sections, we have learnt how to find the expansion of $(1 + b)^n$. What about the binomial expansion of $(a + b)^n$?

Investigation

Binomial Theorem

By multiplying out the terms in each of the following, we have:

$n = 0$: $(a + b)^0 = 1$

$n = 1$: $(a + b)^1 = a + b$

$n = 2$: $(a + b)^2 = a^2 + 2ab + b^2$

$n = 3$: $(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$

$n = 4$: $(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$

Compare this with the expansion of $(1 + b)^n$ for $n = 1$ to 4 in Section 3.1.

1. What do you notice about the corresponding coefficients in the expansion of $(a + b)^n$ and $(1 + b)^n$ for $n = 1$ to 4?
2. In the expansion of $(1 + b)^n$, the terms can be arranged such that the powers of b are in ascending order from 0 (since $b^0 = 1$) to n.

   In the expansion of $(a + b)^n$, does each term contain a?
   What happens to the powers of a?
3. What do you notice if you add the power of a and the power of b in each term? What is the sum equal to?
4. In the expansion of $(1 + b)^n$, the $(r + 1)^{\text{th}}$ term is: $T_{r+1} = \binom{n}{r} b^r$.

   In the expansion of $(a + b)^n$, what is the $(r + 1)^{\text{th}}$ term?

In general, the **Binomial Theorem** for $(a + b)^n$, where n is a positive integer, states that:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n-1}ab^{n-1} + \binom{n}{n}b^n.$$

In the expansion of $(a + b)^n$, the $(r + 1)^{\text{th}}$ term is: $T_{r+1} = \binom{n}{r}a^{n-r}b^r$.

Worked Example 7

(Expansion using the Binomial Theorem)

Expand and simplify each of the following.

(a) $(2a + 3)^5$ **(b)** $\left(3x - \dfrac{y}{4}\right)^4$

Solution

(a) $(2a + 3)^5$

$= (2a)^5 + \binom{5}{1}(2a)^4(3) + \binom{5}{2}(2a)^3(3)^2 + \binom{5}{3}(2a)^2(3)^3 + \binom{5}{4}(2a)(3)^4 + (3)^5$

$= 32a^5 + 240a^4 + 720a^3 + 1080a^2 + 810a + 243$

(b) $\left(3x - \dfrac{y}{4}\right)^4$

$= \left[3x + \left(-\dfrac{y}{4}\right)\right]^4$

$= (3x)^4 + \binom{4}{1}(3x)^3\left(-\dfrac{y}{4}\right) + \binom{4}{2}(3x)^2\left(-\dfrac{y}{4}\right)^2 + \binom{4}{3}(3x)\left(-\dfrac{y}{4}\right)^3 + \left(-\dfrac{y}{4}\right)^4$

$= 81x^4 - 27x^3y + \dfrac{27}{8}x^2y^2 - \dfrac{3}{16}xy^3 + \dfrac{y^4}{256}$

Practise Now 7

Similar Questions:
Exercise 3B
Questions 1(a)-(c)

1. Expand each of the following using the Binomial Theorem.

(a) $(x + 2y)^4$ **(b)** $\left(3a - \dfrac{b}{2}\right)^5$

2. Find, in ascending powers of x, the first four terms in the expansion of $\left(2 - \dfrac{x}{3}\right)^6$.

3. Write down and simplify, in ascending powers of x, the first four terms in each of the following expansions.

(a) $\left(2x^2 - \dfrac{1}{x}\right)^8$ **(b)** $\left(2 + \dfrac{1}{4x}\right)^{16}$

Worked Example 8

(Finding Coefficients by Expansion)

Find, in ascending powers of x, the first three terms in the expansion of $\left(2-\frac{x}{4}\right)^{10}$.

Hence, obtain the coefficient of x^2 in the expansion of $(1+2x-3x^2)\left(2-\frac{x}{4}\right)^{10}$.

Solution

$$\left(2-\frac{x}{4}\right)^{10} = \left[2+\left(-\frac{x}{4}\right)\right]^{10}$$

$$= 2^{10} + \binom{10}{1}2^9\left(-\frac{x}{4}\right) + \binom{10}{2}2^8\left(-\frac{x}{4}\right)^2 + \ldots$$

$$= 1024 - 1280x + 720x^2 - \ldots$$

$$(1 + 2x - 3x^2)\left(2-\frac{x}{4}\right)^{10}$$

$$= (1 + 2x - 3x^2)(1024 - 1280x + 720x^2 - \ldots)$$

$\therefore$ Coefficient of $x^2 = 1(720) + 2(-1280) - 3(1024)$

$= 720 - 2560 - 3072$

$= -4912$

PROBLEM-SOLVING TIP

In the expansion of $(x - y)^n$, it is usual to consider the negative quantity $(-y)$ as $+ (-y)$. In other words, $(x - y)^n = [x + (-y)]^n$. The Binomial Theorem can then be applied.

Practise Now 8

Similar Questions:
Exercise 3B
Questions 8-10, 18

1. Obtain and simplify the first four terms in the expansion of $\left(2+\frac{x^2}{4}\right)^8$ in ascending powers of x. Hence, evaluate the coefficient of x^4 in the expansion of $\left(2-\frac{3}{x^2}\right)\left(2+\frac{x^2}{4}\right)^8$.

2. Obtain the first three terms in the expansion of $(2 + 3x)^5(1 - 2x)^6$ in ascending powers of x.

3. Write down and simplify the first three terms in the expansion of each of the following in ascending powers of x.

   (i) $(1 + 3x)^5$ **(ii)** $\left(3-\frac{1}{2}x\right)^5$

   Hence, obtain the coefficient of x^2 in the expansion of $\left(3+\frac{17}{2}x-\frac{3}{2}x^2\right)^5$.

Worked Example 9

(Expansion of a Trinomial by using the Binomial Theorem)

Find, in ascending powers of x, the first three terms in the expansion of $(1 + x)^6$. Hence, obtain the coefficient of x^2 in the expansion of $(1 - 3x - 4x^2)^6$.

Solution

$$(1 + x)^6 = 1 + \binom{6}{1}x + \binom{6}{2}x^2 + \ldots$$
$$= 1 + 6x + 15x^2 + \ldots$$

Method 1:

$$\begin{aligned}(1 - 3x - 4x^2)^6 &= [(1 + x)(1 - 4x)]^6\\ &= (1 + x)^6(1 - 4x)^6\\ &= (1 + 6x + 15x^2 + \ldots)\left[1 + \binom{6}{1}(-4x) + \binom{6}{2}(-4x)^2 + \ldots\right]\\ &= (1 + 6x + 15x^2 + \ldots)(1 - 24x + 240x^2 + \ldots)\end{aligned}$$

$$\begin{aligned}\text{Coefficient of } x^2 &= 1(240) + 6(-24) + 15(1)\\ &= 111\end{aligned}$$

Hence, the coefficient of x^2 is 111.

Method 2:

$$\begin{aligned}(1 - 3x - 4x^2)^6 &= [1 + (-3x - 4x^2)]^6\\ &= 1 + 6(-3x - 4x^2) + 15(-3x - 4x^2)^2 + \ldots\\ &= 1 - 18x - 24x^2 + 15(9x^2 + \ldots) + \ldots\\ &= 1 - 18x - 24x^2 + 135x^2 + \ldots\\ &= 1 - 18x + 111x^2 + \ldots\end{aligned}$$

Hence, the coefficient of x^2 is 111.

Thinking Time

In Worked Example 9, will you obtain the same answer if you group the terms as $[(1 - 3x) - 4x^2]^6$, before applying the Binomial Theorem? Which method do you prefer and why?

Practise Now 9

Similar Questions:
Exercise 3B
Questions 12, 13, 17

1. Find the coefficient of x^2 in the expansion of $(1 - 3x + 2x^2)^{10}$.

2. Expand $(1 + 2x + x^2)^8$ in ascending powers of x up to and including the term in x^3. Hence, find the constant term in the expansion of
$\left(3 - \frac{1}{2x}\right)^2(1 + 2x + x^2)^8$.

Worked Example 10

(Use of the General Term Formula)

Find the 8th term in the expansion of $\left(4+\frac{x}{2}\right)^{13}$ in ascending powers of x.

Solution

$$8^{\text{th}}\text{ term} = \binom{13}{7}4^{13-7}\left(\frac{x}{2}\right)^7$$
$$= 1716(4^6)\left(\frac{1}{2}\right)^7 x^7$$
$$= 54\ 912x^7$$

RECALL

$$T_{r+1} = \binom{n}{r}a^{n-r}b^r$$

Practise Now 10

Similar Questions:
Exercise 3B
Questions 2(a)-(c)

Find the 5th term in the expansion of each of the following.

(a) $\left(3-\frac{1}{3x}\right)^{16}$ **(b)** $\left(\frac{y}{x}+\frac{2x^2}{y}\right)^{12}$

Worked Example 11

(Finding a Specific Term)

In the expansion of $\left(2x+\frac{1}{4x}\right)^{20}$, find

(i) the term containing x^6, **(ii)** the term independent of x.

Solution

The general term or $(r+1)^{\text{th}}$ term of the expansion is

$$T_{r+1} = \binom{20}{r}(2x)^{20-r}\left(\frac{1}{4x}\right)^r$$
$$= \binom{20}{r}2^{20-r}(x)^{20-r}\left(\frac{1}{4}\right)^r\left(\frac{1}{x}\right)^r$$
$$= \binom{20}{r}2^{20-r}\left(\frac{1}{4}\right)^r x^{20-2r}$$

(i) For the term containing x^6, we equate $x^{20-2r} = x^6$

i.e. $20 - 2r = 6$; $r = 7$

$\therefore$ Term containing $x^6 = \binom{20}{7}2^{13}\left(\frac{1}{4}\right)^7 x^6$

$= 38\ 760x^6$

(ii) The term independent of x refers to the constant term or the term containing x^0.

For the constant term, we equate $x^{20-2r} = x^0$

i.e. $20 - 2r = 0$

$\therefore$ $r = 10$

Term independent of $x = \binom{20}{10}2^{10}\left(\frac{1}{4}\right)^{10}$

$= \frac{46\ 189}{256}$

$= 180\frac{109}{256}$

Practise Now 11

Similar Questions:
Exercise 3B
Questions 3(a)-(d), 4(a),(b), 5-7, 11, 16

1. In the expansion of $\left(x^2 + \dfrac{1}{x}\right)^{12}$, find

(i) the term containing x^9,

(ii) the term independent of x.

2. In the expansion of $(3 + kx)^{20}$, the coefficients of x^4 and x^5 are in the ratio 3 : 16. Find the value of k.

3. Find the coefficient of x in the expansion of $\left(2x + \dfrac{1}{4x^2}\right)^{16}$.

3.4 APPLICATIONS OF BINOMIAL THEOREM

Estimation of Powers of Certain Numbers

Before calculators were invented, how did mathematicians find the values of numbers such as 1.01^7?

Try to use multiplication to find the value of 1.01^7 without the use of a calculator. Is it very tedious?

The Binomial Theorem can be used to estimate the value of numbers such as 1.01^7.

Worked Example 12

(Estimation by using the Binomial Theorem)
Find the first four terms in the expansion of $(1 + x)^9$ in ascending powers of x. Use your result to estimate the value of 1.01^9.

Solution

$$(1 + x)^9 = 1 + \binom{9}{1}x + \binom{9}{2}x^2 + \binom{9}{3}x^3 + \ldots$$
$$= 1 + 9x + 36x^2 + 84x^3 + \ldots$$

Let $(1 + x)^9 = 1.01^9$. Then $x = 0.01$.

$$\therefore 1.01^9 = 1 + 9(0.01) + 36(0.01)^2 + 84(0.01)^3 + \ldots$$
$$\approx 1.093\ 684$$

ATTENTION

Use your calculator to evaluate 1.01^9. When we compare this value to the one obtained in Worked Example 12, which digit is incorrect? How accurate is the answer obtained by expansion?

Practise Now 12

Similar Questions:
Exercise 3B
Questions 14, 15

1. Find the first four terms in the expansion of $(1 + x^2)^{10}$ in ascending powers of x. Use your result to estimate the value of 1.01^{10}.

2. Find the first four terms in the expansion of $(1 + 4x)^8$ in ascending powers of x. Hence, calculate the value of 1.02^8 correct to 4 decimal places.

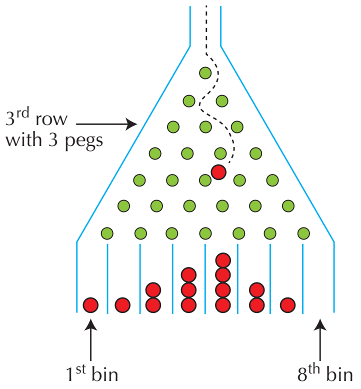
Application of the Binomial Theorem

An important application of the Binomial Theorem is in the field of probability.

Worked Example 13

(Application of the Binomial Theorem)

The figure shows a variant of a quincunx or bean machine invented by Sir Francis Galton in the 17th century. It has 7 rows of pegs, with k pegs in the k^{th} row. When the balls are released from the top of the machine, they bounce to the left or right as they move down the rows of pegs and are collected in separate bins at the bottom.



INFORMATION

Another example of the application of Binomial Theorem in probability can be found in Exercise 3B Question 20.

The probability that a ball lands in the r^{th} bin is given by the r^{th} term in the binomial expansion of $(a + b)^n$, where n is the number of rows of pegs in the machine, and a and b are constants. For symmetrically placed pegs, a ball will bounce to the left or right with equal probability, so $a = b = 0.5$.

(i) Expand $(a + b)^7$.

(ii) Find the probability that a ball lands in the third bin.

Solution

(i)
$$(a+b)^7 = a^7 + \binom{7}{1}a^6b + \binom{7}{2}a^5b^2 + \binom{7}{3}a^4b^3 + \binom{7}{4}a^3b^4 + \binom{7}{5}a^2b^5 + \binom{7}{6}ab^6 + b^7$$
$$= a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$$

(ii) Probability that a ball lands in the third bin $= T_3$
$$= 21a^5b^2$$
$$= 21(0.5)^5(0.5)^2$$
$$= 0.164 \text{ (to 3 s.f.)}$$

Practise Now 13

Similar Question:
Exercise 3B
Questions 19, 20

Calculate the probability that a ball lands in the fifth bin in the above quincunx. Do you think that a ball has a higher chance of falling in a bin nearer to or further away from the centre of the machine? Explain your answer.

Basic Level | Intermediate Level | Advanced Level

Exercise 3B

1 Expand each of the following using the Binomial Theorem.

(a) $(2x + y)^5$ **(b)** $\left(2+\frac{x}{2}\right)^6$

(c) $\left(\frac{a}{2}-\frac{3}{b}\right)^5$

2 Find the term indicated in each of the following expansions.

(a) $\left(2x+\frac{y}{4}\right)^{16}$, 5^{th} term

(b) $\left(\frac{x}{2y}+\frac{y^2}{x}\right)^{12}$, 4^{th} term

(c) $\left(\frac{x^2}{\sqrt{y}}-\frac{\sqrt{y}}{x}\right)^{10}$, 7^{th} term

3 Find the coefficient of the term indicated in each of the following expansions.

(a) $(2+x)^8$, x^5

(b) $\left(x-\frac{3}{x}\right)^{14}$, x^6

(c) $\left(x^2+\frac{2}{x}\right)^8$, x^7

(d) $\left(2-\frac{1}{x^2}\right)^8$, $\frac{1}{x^{10}}$

4 Find the term independent of x in the expansion of each of the following.

(a) $\left(x^2+\frac{2}{x}\right)^6$ **(b)** $\left(\frac{1}{2x^2}-x\right)^9$

5 Find the ratio of the coefficient of x^4 to the coefficient of x^5 in the expansion of $(2+3x)^8$.

6 If the coefficient of x^4 in the expansion of $(3+2x)^6$ is equal to the coefficient of x^4 in the expansion of $(k+3x)^6$, find the values of k.

7 If the coefficients of x^4 and x^5 in the expansion of $(3+kx)^{10}$ are equal, find the value of k.

8 Expand each of the following in ascending powers of x.

(a) $(2-x)^4$ **(b)** $(1+2x)^5$

Hence, or otherwise, find the coefficient of x^3 in the expansion of $(2-x)^4(1+2x)^5$.

9 Write down and simplify, in ascending powers of x, the first three terms in the expansion of

(a) $\left(2+\frac{x}{2}\right)^6$, **(b)** $(3-2x)^6$.

Hence, or otherwise, find the coefficient of x^2 in the expansion of $\left(6-\frac{5}{2}x-x^2\right)^6$.

10 Find, in descending powers of x, the first three terms in the expansion of $\left(2x-\frac{3}{4x}\right)^8$.

Hence, or otherwise, find the coefficient of x^6 in the expansion of $\left(x^2+\frac{2}{x^2}\right)\left(2x-\frac{3}{4x}\right)^8$.

11 Find the term containing x^5y^5 in the expansion of $\left(x+\frac{y}{4}\right)^{10}$.

12 Find, in ascending powers of x, the first three terms in the expansion of $(2+3x-4x^2)^5$.

Exercise 3B

13 Find in ascending powers of x, the first three terms in each of the following expansions.
(a) $(1 + x - 2x^2)^5$ **(b)** $(1 - x - 12x^2)^4$

14 Obtain the first four terms in the expansion of $\left(2 - \frac{x}{2}\right)(1 + x)^7$. Use your result to estimate the value of 1.95×1.1^7.

15 Obtain the first four terms in the expansion of $(1 + 2x - 3x^2)^8$ in ascending powers of x. Hence, by taking $x = 0.01$, find the value of $(1.0197)^8$ correct to 4 decimal places.

16 In the expansion of $\left(x^2 + \frac{a}{x}\right)^8$, $a \neq 0$, the coefficient of x^7 is four times the coefficient of x^{10}. Find the value of a.

17 The expansion of $(1 + px + qx^2)^8$ is $1 + 8x + 52x^2 + kx^3 + \ldots$.
Calculate the values of p, q and k.

18 Write down and simplify, in ascending powers of x, the first three terms in the expansion of $\left(2 - \frac{x}{8}\right)^n$, where n is a positive integer greater than 2. The first two terms in the expansion of $(2 + x)\left(2 - \frac{x}{8}\right)^n$, in ascending powers of x, are $a + bx^2$, where a and b are constants.
(i) Find the value of n.
(ii) Hence, find the value of a and of b.

19 Another quincunx similiar to that in Worked Example 13 has n rows of pegs. The constants a and b both remain equal to 0.5. Given that the probabilities that a ball lands in the third bin and in the fourth bin are equal, find the value of n.

20 40% of a population have blood group O while the other 60% have either blood group A, B or AB. Suppose there are 30 blood donors from your class.
The probability that r of these 30 blood donors have blood group O is given by $\binom{30}{r} \times 0.6^{30-r} \times 0.4^r$. Find
(i) the probability that 12 of these 30 blood donors have blood group O,
(ii) the value of r for which the probability that r of these 30 blood donors have blood group O is at its highest.
Survey all the classmates in your class to find out how many of them have blood group O. Is the percentage of the number of students in your class with blood group O close to 40%?

SUMMARY

1. The Binomial Theorem states that

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + b^n,$$

where n is a positive integer and $0 \leqslant r \leqslant n$.

2. $n! = n \times (n-1) \times (n-2) \times \ldots \times 3 \times 2 \times 1$

3. In the expansion of $(a+b)^n$, the $(r+1)^{\text{th}}$ term is $T_{r+1} = \binom{n}{r}a^{n-r}b^r$,

where $\binom{n}{r} = \dfrac{n(n-1)\ldots(n-r+1)}{r\times(r-1)\times(r-2)\times\ldots\times 3\times 2\times 1} = \dfrac{n!}{r!(n-r)!}$

4. The term independent of x in the expansion of $(a+bx)^n$ refers to the constant term.

Review Exercise 3

1. Write down in descending powers of x, the first three terms in each of the following expansions.
   (a) $(x^2+3y)^{15}$
   (b) $(3x-2y)^4$
   (c) $(x^2+2y^3)^6$
   (d) $\left(\dfrac{x}{a}+\dfrac{y}{b}\right)^8$

2. Find the coefficient of x^3 in the expansion of each of the following.
   (a) $\left(2x-\dfrac{1}{2}\right)^6$
   (b) $(x-10)^5$

3. Find the coefficient of x^2 in the expansion of each of the following.
   (a) $\left(x-\dfrac{1}{2x^2}\right)^{20}$
   (b) $\left(\dfrac{x^2}{4}+\dfrac{2}{x}\right)^{25}$
   (c) $(x^2+5x+2)(1+x)^7$
   (d) $(x+2)^2(1+x)^8$

4. Find the term independent of x in each of the following expansions.

(a) $\left(2x^3 - \frac{1}{4x^5}\right)^8$

(b) $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$

(c) $\left(x^2 + \frac{1}{x}\right)^9$

(d) $\left(x^2 + \frac{1}{2x^6}\right)^{16}$

5. Find, in ascending powers of x, the first four terms in the expansion of $(1 + x + 2x^2)^4$.

6. If the coefficients of x^k and x^{k+1} in the expansion of $(2 + 3x)^{19}$ are equal, find the value of k.

7. In the expansion of $(1 + x)^n$, the coefficient of x^9 is the mean of the coefficient of x^8 and x^{10}. Find the possible values of n where n is a positive integer.

8. In the expansion of $(1 + x)^n$, the coefficient of x^5 is the mean of the coefficients of x^4 and x^6. Calculate the possible values of n.

9. Find, in descending powers of x, the first three terms in the expansion of $\left(x - \frac{2}{x}\right)^6$. Hence, find the coefficient of x^4 in the expansion of $\left(2 + 3x^2\right)\left(x - \frac{2}{x}\right)^6$.

10. The coefficient of x^2 in the expansion of $(2x + k)^6$ is equal to the coefficient of x^5 in the expansion of $(2 + kx)^8$. Find the value of k.

11. Expand $(1 + 2x)(1 - x)^6$ as far as the term in x^4, and use it to evaluate 1.06×0.97^6 correct to 4 decimal places.

12. In the expansion of $(3 + 4x)^n$, the coefficients of x^4 and x^5 are in the ratio $5 : 16$. Find the value of n.

13. Obtain the first four terms in the expansion of $(1 + x + x^2)^{10}$ in ascending powers of x. Hence, find the term independent of x in the expansion of $\left(1 - \frac{1}{2x}\right)^2 (1 + x + x^2)^{10}$.

14. Find the first 4 terms in the expansion of $\left(1 - \frac{x}{4}\right)^{10}$ in ascending powers of x.
Hence, find the coefficient of x^3 in the expansion of
$2\left(1 - \frac{x}{4}\right)^{10} + 3\left(1 - \frac{x}{4}\right)^{11} + 4\left(1 - \frac{x}{4}\right)^{12}$.

15. Expand $(1 - x)^5$ completely. Hence, find k when $k = (1 - x^2)^5 - 5(1 - x^2)^4 + 10(1 - x^2)^3 - 10(1 - x^2)^2 + 5(1 - x^2) - 1$.

16. In the expansion of $\left(4 + \frac{1}{2x}\right)^{12}$, the coefficient of $\frac{1}{x^3}$ is k times the coefficient of $\frac{1}{x^4}$. Calculate the value of k.

17. (a) Write down the first three terms in the binomial expansion of each the following in ascending powers of x.
(i) $(1 + x)^8$ (ii) $(3 - 2x)^4$
Hence, find the coefficient of x^2 in the expansion of $(1 + x)^8(3 - 2x)^4$.

(b) The expansion of $(1 + ax + bx^2)^6$ in ascending powers of x is given by $1 - 12x + 78x^2 + \ldots$. Find the value of a and of b.

(c) Find the first three terms of each of the following expansions in ascending powers of x.
(i) $(1 - 4x)^6$ (ii) $(2 + x)^7$
Hence, find the coefficient of x^2 in the expansion of $(1 - 4x)^6(2 + x)^7$.

18. Given that
$(1 + 2x)^m + (1 + 3x)^n = 2 + 31x + hx^2 + \ldots$,
where m and n are positive integers, and $m + n = 12$, find the value of m and of n. Hence, calculate the value of h.

19. Find, in ascending powers of x, the first three terms in the expansion of $(1 - 2x)^n(2 - 3x)^5$.

(i) If the coefficient of x^2 in the above expansion is 5520, find the value of n.

(ii) Find the value of the coefficient of x.

1. Given that $\left(x^2 + \frac{1}{x}\right)^5 - \left(x^2 - \frac{1}{x}\right)^5 = ax^7 + bx + \frac{c}{x^5}$, find the values of a, b and c. Hence, evaluate $\left(10 + \frac{1}{\sqrt{10}}\right)^5 - \left(10 - \frac{1}{\sqrt{10}}\right)^5$.

2. In the manufacture of light bulbs, 8% of the light bulbs are found to be defective. Using the binomial expansion, find the probability that, out of 10 light bulbs chosen at random, 2 or more are found to be defective.

INDICES, SURDS & LOGARITHMS

On 11 Mar 2011, a 9.0-magnitude undersea earthquake off the coast of Japan set off tsunami waves of up to 38 m high (about 12 storeys), resulting in at least 28 000 people dead or missing.

The magnitude of an earthquake is measured using the Richter scale which involves logarithms. What does 9.0 on the log scale mean? How does it compare to the 7.0-magnitude earthquake that struck New Zealand on 3 Sep 2010?

CHAPTER 4

Learning Objectives

At the end of this chapter, you should be able to:

- simplify expressions and solve equations involving surds,
- apply the laws of logarithms and the change of base formula to solve logarithmic equations,
- solve exponential equations,
- sketch the graphs of exponential and logarithmic functions,
- model and solve problems in the sciences and in the real world using exponential and logarithmic equations.

4.1 INDICES

Recap

In Secondary 3 Mathematics, we have learnt about indices. For example,

power

$2^5 = 2 \times 2 \times 2 \times 2 \times 2,$

base

where 2^5 is read as '2 to the **power** of 5'. In this case, the **base** is 2 and the **index** (plural: indices) is 5. The index is also called the **exponent**.

We have also learnt 9 Laws of Indices as shown below.

If bases a and b are *positive* real numbers, and indices m and n are *real numbers*, then

Law 1: $a^m \times a^n = a^{m+n}$

Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$

} *same base*

Law 3: $(a^m)^n = a^{mn}$ (Special Case of Law 1)

Law 4: $a^n \times b^n = (ab)^n$

Law 5: $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$

} *same index*

INFORMATION

If bases a and b are ***negative***, then indices m and n must be ***integers*** for Laws 1-5 to apply.

If base $a > 0$, and indices m and n are *positive integers*, then

Law 6: $a^0 = 1$ (Special Case of Law 2)

Law 7: $a^{-n} = \dfrac{1}{a^n}$ (Special Case of Law 2)

Law 8: $a^{\frac{1}{n}} = \sqrt[n]{a}$ (Special Case of Law 9)

Law 9: $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$

INFORMATION

Laws 6 and 7 are true even if base $a < 0$, but base $a \neq 0$. Laws 8 and 9 are true even if base $a = 0$, but base a cannot be *negative*.

Exponential Equations

Consider the equation: $2^x = 16$.
This equation involves an *unknown exponent* (or index) and it is called an **exponential equation**. We have learnt how to solve simple exponential equations. In this section, we will learn how to solve more complicated exponential equations.

Worked Example 1

(Simple Exponential Equations)
Solve each of the following equations.

(a) $2^x = 16$ **(b)** $5^{2y} \times 3^y = \frac{1}{75}$

PROBLEM-SOLVING TIP

Convert both sides of the equation to the *same base*.

Solution

(a)
$$2^x = 16$$
$$2^x = 2^4$$
$$\therefore\ x = 4$$

(b)
$$(5^2)^y \times 3^y = \frac{1}{75} \quad \text{(Law 3 of Indices)}$$
$$25^y \times 3^y = 75^{-1} \quad \text{(Law 7 of Indices)}$$
$$(25 \times 3)^y = 75^{-1} \quad \text{(Law 4 of Indices)}$$
$$75^y = 75^{-1}$$
$$\therefore\ y = -1$$

Practise Now 1

Similar Questions:
Exercise 4A
Questions 1(a)-(d), 2(a)-(d)

Solve each of the following equations.

(a) $3^x = 243$ **(b)** $4^y = \frac{1}{8}$

(c) $6^{1-k^2} = 1$ **(d)** $2^{2z} \times 3^z = \frac{1}{144}$

In general, to solve an exponential equation where both sides can be converted to the same base, use the following **Equality of Indices**: equate the unknown index x to the known index n.

If $a^x = a^n$, then $x = n$, where $a \neq -1, 0, 1$.

Class Discussion

1. Discuss with your classmates what happens if $a^x = a^n$ and $a = -1$, 0 or 1.
   Is x still equal to n? Find some counterexamples.

2. Given that $a > 0$, are there values of x for which a^x is 0 or negative? Try some examples such as 2^3, 5^0 and 4^{-2}.

In general,

> if base $a > 0$, then $a^x > 0$ for all real values of x.

Worked Example 2

(Solving Exponential Equations Using Substitution)

Solve the equation $9^{x+1} = 1 - 8(3^x)$.

Solution

$$
\begin{aligned}
9^{x+1} &= 1 - 8(3^x) \\
9^x \times 9^1 + 8(3^x) - 1 &= 0 && \text{(Law 1 of Indices)} \\
9(3^2)^x + 8(3^x) - 1 &= 0 \\
9(3^x)^2 + 8(3^x) - 1 &= 0 && \text{(Law 3 of Indices)}
\end{aligned}
$$

Let $y = 3^x$. Then $9y^2 + 8y - 1 = 0$.

$$
\begin{aligned}
(9y - 1)(y + 1) &= 0 \\
9y - 1 &= 0 \quad \text{or} \quad y + 1 = 0 \\
y &= \frac{1}{9} \quad \text{or} \quad y = -1 \\
\therefore \quad 3^x &= \frac{1}{9} \quad \text{or} \quad 3^x = -1 \text{ (no solution since } 3^x > 0 \\
&\qquad\qquad\qquad\qquad\qquad \text{for all real values of } x) \\
\therefore \quad 3^x &= \frac{1}{3^2} \\
&= 3^{-2} && \text{(Law 7 of Indices)} \\
\therefore \quad x &= -2
\end{aligned}
$$

PROBLEM-SOLVING TIP

Notice that base 9 can be converted to base 3. Then we can use a suitable substitution such as $y = 3^x$.

Practise Now 2

Similar Questions:
Exercise 4A
Questions 3(a)-(c), 7(a), (b)

1. Solve the equation $4^{x+1} = 2 - 7(2^x)$.

2. By using the substitution $u = 3^x$,
(i) express the equation $27^x + 2(3^x) + 1 = 13$ as a cubic equation in u,
(ii) show that $u = 2$ is the only real solution of this equation,
(iii) hence solve the equation $27^x + 2(3^x) + 1 = 13$.

Worked Example 3

(Simultaneous Exponential Equations)

Solve the simultaneous equations

$2^x \times 4^y = \frac{1}{8}$ and $\frac{9^x}{3^{y+1}} = 27$.

Solution

$2^x \times 4^y = \frac{1}{8}$ and $\frac{9^x}{3^{y+1}} = 27$

$2^x \times 2^{2y} = \frac{1}{2^3}$ (Law 3 of Indices)

$2^{x+2y} = 2^{-3}$ (Laws 1 and 7 of Indices)

$x + 2y = -3$ --- (1)

$\frac{3^{2x}}{3^{y+1}} = 3^3$ (Law 3 of Indices)

$3^{2x-(y+1)} = 3^3$ (Law 2 of Indices)

$3^{2x-y-1} = 3^3$

$2x - y - 1 = 3$

$2x - y = 4$ --- (2)

$x + 2y = -3$ --- (1)

(2) × 2: $4x - 2y = 8$ --- (3)

(1) + (3): $5x = 5$

$\therefore x = 1$

Subst. $x = 1$ into (1): $1 + 2y = -3$

$2y = -4$

$y = -2$

$\therefore$ The solution is $x = 1$ and $y = -2$.

Practise Now 3

Similar Questions:
Exercise 4A
Questions 4(a)-(c), 5, 6

1. Solve the simultaneous equations

$3^x \times 9^y = \frac{1}{3}$ and $\frac{8^x}{2^{y+1}} = 8$.

2. The equation of a curve is given by $y = ka^x$, where a and k are constants. Given that the curve passes through (2, 16), (3, 32) and (5, p), find the values of a, k and p.

Basic Level | Intermediate Level | Advanced Level

Exercise 4A

1 Solve each of the following equations.

(a) $5^x = 625$ **(b)** $9^y = \frac{1}{27}$

(c) $7^{p^2-4} = 1$ **(d)** $2^{2x} \times 9^x = \frac{1}{6}$

2 Solve each of the following equations.

(a) $3^y + 3^{y+2} = 90$ **(b)** $2^{t+4} - 2^t = 120$

(c) $4^n + 2^{2n-3} = \frac{3^2}{2^3}$ **(d)** $5^{z^2-4} - 125^z = 0$

3 Solve each of the following equations.

(a) $2(25^x) = 3 - 5^x$

(b) $3^{8y} + 3^2 = 10(3^{4y})$

(c) $7^{2z+1} - 7^z = 7^{z+1} - 22(7^z)$

Exercise 4A

4 Solve each of the following pairs of simultaneous equations.

(a) $49^{2x} \times \left(\frac{1}{7}\right)^{y-4} = \frac{1}{343}$ and

$3^{x+3} \div \frac{1}{27^y} = \frac{1}{9}$

(b) $25^x(125^y) = 1$ and

$27^y \div \left(\sqrt{3}\right)^x = 81\sqrt{3}$

(c) $3^x \times 243^y = 1$ and

$8^x \div 2^{y-12} = \frac{1}{\frac{1}{16^y}}$

5 The equation of a curve is given by $y = ab^x$, where a and b are constants. Given that the curve passes through (1, 21), (2, 63) and (k, 189), find the values of a, b and k.

6 The equation of a curve is given by $y = px^q + 7$, where p and q are constants. If $y = 25$ when $x = 3$, and $y = 57$ when $x = 5$, find the value of p and of q.

7 By using a suitable substitution, solve each of the following equations.

(a) $8^x = 7(2^x) - 6$

(b) $3^{3x+2} + 3^0 = 3^{2x} + 3^{x+2}$

4.2 SURDS

We have learnt from Law 8 of Indices that $a^{\frac{1}{n}} = \sqrt[n]{a}$, e.g. $4^{\frac{1}{2}} = \sqrt{4} = 2$ and $5^{\frac{1}{3}} = \sqrt[3]{5}$. While $\sqrt{4} = 2$ is a rational number, $\sqrt[3]{5}$ is an irrational number. An ***irrational*** number involving the n^{th} root, such as $\sqrt[3]{5}$, is called a **surd**.

Other examples of surds include $\sqrt{7}$, $\sqrt{2}+\sqrt{3}$, $\frac{1+\sqrt{5}}{2}$ and $\sqrt[4]{3}-7$. $\sqrt{4} = 2$ and $\sqrt[3]{125} = 5$ are not surds because they are rational numbers.

Class Discussion

A monkey can only walk on tiles containing surds. Help the monkey find the banana by shading its path. (The monkey can only move along a row or a column; it cannot move diagonally across tiles.)

$\sqrt{2}$	$\sqrt{3}$	$\sqrt[3]{6}$	$\sqrt{5}+2$	2019	$\frac{\sqrt{2}}{3}$	$\frac{3-\sqrt{5}}{2}$
$\sqrt{5}$	2013	$\sqrt{4}$	$\sqrt{10}$	$1-\sqrt[3]{7}$	$7\sqrt[3]{9}$	$\sqrt{100}$
$\sqrt[3]{7}$	$1+\sqrt{2}$	2014	$\sqrt{25}$	$\sqrt[3]{8}$	2017	$\sqrt[3]{12}$
$\sqrt{9}$	$\sqrt{8}$	$\sqrt[3]{27}$		$\sqrt{27}$	$\sqrt[3]{16}$	$6\sqrt{6}$
$\sqrt{7}+\sqrt{4}$	$2\sqrt{5}$	$\sqrt{49}$	2015	$\sqrt[3]{64}$	$\sqrt{\frac{27}{3}}$	$\sqrt[3]{36}$
$\frac{2+\sqrt{3}}{5}$	2018	$\frac{2\times\sqrt{7}}{\sqrt{7}}$	$4\sqrt{8}$	$\frac{1}{2}\left(2-\sqrt{3}\right)$	$\frac{\sqrt[3]{10}}{3}$	$\sqrt{2016}$
$\sqrt{11}$	$\sqrt[4]{10}$	$\sqrt[3]{2}-1$	$\sqrt[5]{25}$	$\sqrt{9}+\sqrt[3]{27}$	$\sqrt{81}$	$\sqrt[7]{1416}$

In this section, we shall deal with surds involving square roots only.

Simplifying Surds

Investigation

Simplifying Surds

1. Are the following statements true? You can use a calculator to evaluate their values.

(a) $\sqrt{16+9} = \sqrt{16} + \sqrt{9}$? **(b)** $\sqrt{16-9} = \sqrt{16} - \sqrt{9}$?

(c) $\sqrt{16 \times 9} = \sqrt{16} \times \sqrt{9}$? **(d)** $\sqrt{\frac{9}{16}} = \frac{\sqrt{9}}{\sqrt{16}}$?

2. For statements that are true, express in general form and prove them.
Hint: Use Laws 4 and 5 of Indices. Take note of the conditions.

3. Evaluate $\sqrt{a} \times \sqrt{a}$.

The above investigation suggests that the following two **Laws of Surds** are true if a and b are *positive*:

Law 1: $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$

Law 2: $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$

Special Case

In particular, if $b = a$ in Law 1 of Surds, then:

Law 3: $\sqrt{a} \times \sqrt{a} = a$ or $\left(\sqrt{a}\right)^2 = \sqrt{a^2} = a$ if $a > 0$

Serious Misconceptions

$$\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$$
$$\sqrt{a-b} \neq \sqrt{a} - \sqrt{b}$$

Worked Example 4

(Simplifying Surds)

Simplify each of the following without using a calculator.

(a) $\sqrt{2} \times \sqrt{8}$ **(b)** $\sqrt{18}$

ATTENTION

For (b), $3\sqrt{2}$ means $3 \times \sqrt{2}$ which is different from $\sqrt[3]{2}$ (cube root of 2). For some calculators, you need to key $3 \times \sqrt{2}$ to evaluate $3\sqrt{2}$.

Solution

(a) $\sqrt{2} \times \sqrt{8}$
$= \sqrt{2 \times 8}$ (Law 1 of Surds)
$= \sqrt{16}$
$= 4$

(b) $\sqrt{18}$
$= \sqrt{9 \times 2}$
$= \sqrt{9} \times \sqrt{2}$ (Law 1 of Surds)
$= 3\sqrt{2}$

Practise Now 4

Similar Questions:
Exercise 4B
Questions 1(a)-(d)

Simplify each of the following without using a calculator.

(a) $\sqrt{27}\times\sqrt{3}$ **(b)** $\dfrac{\sqrt{125}}{\sqrt{5}}$ **(c)** $\sqrt{12}$ **(d)** $\dfrac{\sqrt{80}\times\sqrt{12}}{\left(\sqrt{16}\right)^2}$

Worked Example 5

(Addition of Surds)

Simplify $\sqrt{32}+\sqrt{50}$ without using a calculator.

ATTENTION

$\sqrt{32}+\sqrt{50}\neq\sqrt{32+50}$

Solution

$$\begin{aligned}\sqrt{32}+\sqrt{50} &= \sqrt{16\times 2}+\sqrt{25\times 2}\\ &= \sqrt{16}\times\sqrt{2}+\sqrt{25}\times\sqrt{2} \quad \text{(Law 1 of Surds)}\\ &= 4\sqrt{2}+5\sqrt{2}\\ &= 9\sqrt{2} \quad \text{(similar to '}4x+5x=9x\text{')}\end{aligned}$$

ATTENTION

In general,
$p\sqrt{a}+q\sqrt{a}=(p+q)\sqrt{a}$
$p\sqrt{a}-q\sqrt{a}=(p-q)\sqrt{a}$

Practise Now 5

Similar Questions:
Exercise 4B
Questions 2(a)-(d)

Simplify each of the following without using a calculator.

(a) $\sqrt{75}+\sqrt{108}$ **(b)** $\sqrt{80}-\sqrt{20}$ **(c)** $\sqrt{24}+\sqrt{54}-\sqrt{216}$

Worked Example 6

(Multiplication of Surds)

Simplify $\left(3+5\sqrt{2}\right)\left(4-\sqrt{2}\right)$ without using a calculator.

Solution

$$\begin{aligned}\left(3+5\sqrt{2}\right)\left(4-\sqrt{2}\right) &= 12-3\sqrt{2}+20\sqrt{2}-10 \quad (\text{since } 5\sqrt{2}\sqrt{2}=10)\\ &= 2+17\sqrt{2}\end{aligned}$$

Practise Now 6

Similar Questions:
Exercise 4B
Questions 3(a)-(d)

Simplify each of the following without using a calculator.

(a) $(7+2\sqrt{3})(5-\sqrt{3})$ **(b)** $(4-3\sqrt{2})^2$

(c) $(3+2\sqrt{5})(3-2\sqrt{5})$ **(d)** $(3\sqrt{6}+4\sqrt{2})^2$

RECALL

$(x+y)^2=x^2+2xy+y^2$
$(x-y)^2=x^2-2xy+y^2$

Conjugate Surds

Class Discussion

1. From Practise Now 6 parts (a), (b) and (d), the product of two irrational numbers is also an irrational number, whereas in part (c), the **product** of two **irrational** numbers is a **rational** number. What patterns can you observe about the two irrational numbers in part (c)? Discuss with your classmates why the product is a rational number.

2. Use the patterns in Question 1 to find a pair of irrational numbers whose product is a rational number.

In general, if p, q and a are rational numbers, and $a > 0$, then:

> The *product* of irrational **conjugate surds**, $p + q\sqrt{a}$ and $p - q\sqrt{a}$, is a *rational* number.

Note: To find the product, use the algebraic identity: $(x + y)(x - y) = x^2 - y^2$.
Is the product of $q\sqrt{a} + p$ and $q\sqrt{a} - p$ also a rational number? Show your working.

Rationalising the Denominator

We use the above result to eliminate a surd from the denominator of an expression. The process is called rationalising the denominator.

Worked Example 7

(Rationalisation of Denominator)
Rationalise the denominator of each of the following.

(a) $\dfrac{6}{\sqrt{2}}$ **(b)** $\dfrac{7}{2+\sqrt{3}}$

PROBLEM-SOLVING TIP

To rationalise the denominator, use the product of conjugate surds.

Solution

(a) $\dfrac{6}{\sqrt{2}} = \dfrac{6}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$ (multiplying by 1 will not change its value)

$= \dfrac{6\sqrt{2}}{2}$ $(\sqrt{2} \times \sqrt{2} = 2)$

$= 3\sqrt{2}$

(b) $\dfrac{7}{2+\sqrt{3}} = \dfrac{7}{2+\sqrt{3}} \times \dfrac{2-\sqrt{3}}{2-\sqrt{3}}$ ($2 + \sqrt{3}$ and $2 - \sqrt{3}$ are conjugate surds)

$= \dfrac{7(2-\sqrt{3})}{2^2 - (\sqrt{3})^2}$ (use $(x + y)(x - y) = x^2 - y^2$)

$= \dfrac{14 - 7\sqrt{3}}{4 - 3} = 14 - 7\sqrt{3}$

Practise Now 7

Similar Questions:
Exercise 4B
Questions 4(a)-(d), 5(a)-(d), 7(a)-(d), 10(a)-(d), 11

1. Simplify each of the following without using a calculator.

(a) $\frac{12}{\sqrt{3}}$ **(b)** $\frac{22}{4+\sqrt{5}}$ **(c)** $\frac{5}{2\sqrt{6}-3}$ **(d)** $\frac{5}{4-3\sqrt{3}}+\frac{7}{3\sqrt{3}+4}$

2. Express $(4-\sqrt{6})^2-\frac{6}{3-\sqrt{6}}$ in the form $a+b\sqrt{6}$, where a and b are integers.

3. Given that $\sqrt{h+k\sqrt{5}}=\frac{4}{(3-\sqrt{5})^2}$, where h and k are rational numbers, find the value of h and of k.

Worked Example 8

(Equation involving Surds)

The solution of the equation $x\sqrt{12}=x\sqrt{7}+\sqrt{3}$ is $\frac{p+\sqrt{q}}{5}$. Without using a calculator, find the values of the integers p and q.

Solution

$$x\sqrt{12}=x\sqrt{7}+\sqrt{3}$$
$$x\sqrt{12}-x\sqrt{7}=\sqrt{3}$$
$$x(\sqrt{12}-\sqrt{7})=\sqrt{3}$$
$$x=\frac{\sqrt{3}}{\sqrt{12}-\sqrt{7}} \quad \text{-------------- (*)}$$
$$=\frac{\sqrt{3}}{\sqrt{12}-\sqrt{7}}\times\frac{\sqrt{12}+\sqrt{7}}{\sqrt{12}+\sqrt{7}} \quad \text{(rationalise the denominator)}$$
$$=\frac{\sqrt{3}\sqrt{12}+\sqrt{3}\sqrt{7}}{(\sqrt{12})^2-(\sqrt{7})^2} \quad \text{(use } (x+y)(x-y)=x^2-y^2\text{)}$$
$$=\frac{\sqrt{36}+\sqrt{21}}{12-7}$$
$$=\frac{6+\sqrt{21}}{5}$$

Comparing with $\frac{p+\sqrt{q}}{5}$, $p=6$ and $q=21$.

PROBLEM-SOLVING TIP

The following are the first three stages of Pólya's Problem Solving Model. You may search on the Internet to find out more about this Problem Solving Model.

Stage 1 (Understand the problem): Solve the equation to find the value of x which may involve surds, i.e. $x=\frac{p+\sqrt{q}}{5}$.

Stage 2 (Think of a plan): To find x, move all terms involving x on one side of the equation $x\sqrt{12}=x\sqrt{7}+\sqrt{3}$.

Stage 3 (Carry out the plan): See solution on the left until (*). This becomes a routine problem of rationalising the denominator. Continue to solve.

Practise Now 8

Similar Questions:
Exercise 4B
Questions 8, 12

The solution of the equation $x\sqrt{8}=x\sqrt{6}+\sqrt{2}$ is $p+\sqrt{q}$. Without using a calculator, find the values of the integers p and q.

Worked Example 9

(Equation involving Surds)

The area of a triangle is $\left(3-\sqrt{2}\right)$ cm². If the length of its base is $\left(\sqrt{2}-1\right)$ cm, find its height in the form $\left(a+b\sqrt{2}\right)$ cm, where a and b are integers.

Solution

Let the height of the triangle be h cm.

Area of triangle = $\frac{1}{2}$ × base × height

$$3-\sqrt{2} = \frac{1}{2}\times\left(\sqrt{2}-1\right)\times h$$

$$6-2\sqrt{2} = \left(\sqrt{2}-1\right)\times h$$

$$h = \frac{6-2\sqrt{2}}{\sqrt{2}-1}$$

$$= \frac{6-2\sqrt{2}}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1}$$

$$= \frac{6\sqrt{2}+6-2\sqrt{2}\sqrt{2}-2\sqrt{2}}{\left(\sqrt{2}\right)^2-1^2}$$

$$= \frac{6\sqrt{2}+6-4-2\sqrt{2}}{2-1}$$

$$= 2+4\sqrt{2}$$

∴ The height of the triangle is $\left(2+4\sqrt{2}\right)$ cm.

Practise Now 9

Similar Questions:
Exercise 4B
Questions 6, 13

1. The area of a rectangle is $\left(7-\sqrt{3}\right)$ cm². Given that it has a length of $\left(5+\sqrt{3}\right)$ cm, find the breadth of the rectangle in the form $\left(a+b\sqrt{3}\right)$ cm, where a and b are integers.

2. A cuboid has a square base. The length of each side of the base is $(2+\sqrt{3})$ cm and the volume of the cuboid is $(15+6\sqrt{3})$ cm³. Find the height of the cuboid in the form $(p+q\sqrt{3})$ cm, where p and q are integers.

Investigation

Rational and Irrational Roots

Consider the quadratic equation $ax^2 + bx + c = 0$, where a, b and c are **rational** numbers.

1. Solve the following equations using the factorisation method where possible. If the values of the roots are not exact, leave your answers in surd form.

(a) $2x^2 + 3x - 2 = 0$ **(b)** $x^2 + 2x - 1 = 0$

2. **(i)** Which of the above quadratic equations can be solved by the factorisation method? Are its roots rational or irrational?

(ii) Are the roots of the above quadratic equation that cannot be solved by the factorisation method, rational or irrational?

(iii) By looking at the formula for the general solution, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, explain when the roots are rational and when they are irrational.

3. If the roots are irrational, why must they be conjugate surds? Explain this using the formula for the general solution.

INFORMATION

The roots of a quadratic equation, where a, b and c can be *irrational* numbers, are usually not conjugate surds, with some exceptions. Can you explain why this is so? Find some exceptions by considering the formulae for the sum and product of roots.

Worked Example 10

(Equation involving Surds)

Solve the equation $\sqrt{x^2 + 3} = 2x$.

Solution

$$\sqrt{x^2+3} = 2x$$

Squaring both sides,
$$\begin{aligned} x^2 + 3 &= (2x)^2 \\ x^2 + 3 &= 4x^2 \\ 3x^2 &= 3 \\ x^2 &= 1 \\ \therefore \quad x &= \pm\sqrt{1} \\ &= \pm 1 \end{aligned}$$

Check: Subst. $x = -1$ into original equation:
RHS $= 2x = -2$ but LHS $= \sqrt{x^2 + 3}$ cannot be negative.
So $x = -1$ is not a solution.

Check: Subst. $x = 1$ into original equation:
LHS $= \sqrt{4} = 2$ and RHS $= 2x = 2$.

Therefore, the solution is $x = 1$.

ATTENTION

Remember to check your solutions by substituting the values of x into the original equation because **squaring** will introduce an extra 'solution' that is not applicable. E.g. If $x = 3$, then squaring this becomes $x^2 = 9$, but it will introduce an extra 'solution' $x = -3$.

Practise Now 10

Similar Questions:
Exercise 4B
Questions 9(a), (b), 14

Solve each of the following equations.

(a) $\sqrt{40 - x^2} = 3x$ **(b)** $\sqrt{3 - 4x} = 2x$ **(c)** $\sqrt{3x + 6} - 3\sqrt{x - 4} = 0$

Exercise 4B

1 Simplify each of the following without using a calculator.

(a) $\sqrt{2} \times \sqrt{32}$ **(b)** $\dfrac{\sqrt{343}}{\sqrt{7}}$

(c) $\sqrt{63}$ **(d)** $\dfrac{\sqrt{75}\times\sqrt{72}}{\sqrt{24}}$

2 Simplify each of the following without using a calculator.

(a) $\sqrt{112} + \sqrt{28}$

(b) $\sqrt{48} + \sqrt{12} - \sqrt{27}$

(c) $\sqrt{240} - \sqrt{12} \times \sqrt{45}$

(d) $\dfrac{\sqrt{245}-\sqrt{20}}{\sqrt{500}}$

3 Simplify each of the following without using a calculator.

(a) $(5+\sqrt{2})(6-3\sqrt{2})$

(b) $(3+2\sqrt{6})^2$

(c) $(9-2\sqrt{5})(9+2\sqrt{5})$

(d) $(2\sqrt{7}+3\sqrt{5})(2\sqrt{7}-3\sqrt{5})$

4 Rationalise the denominator of each of the following.

(a) $\dfrac{3}{\sqrt{5}}$ **(b)** $\dfrac{4}{3\sqrt{8}}$

(c) $\dfrac{7}{2+\sqrt{3}}$ **(d)** $\dfrac{4}{8-2\sqrt{6}}$

5 Simplify the following without using a calculator.

(a) $\dfrac{5}{4\sqrt{3}-2}$ **(b)** $\dfrac{4}{2\sqrt{6}+7}$

(c) $\dfrac{\sqrt{3}}{2\sqrt{5}+8}$ **(d)** $\dfrac{8}{2\sqrt{5}+3}-\dfrac{4}{2\sqrt{5}-3}$

6 The area of a rectangle is $\sqrt{24}$ cm². If its breadth is $(3-\sqrt{6})$ cm, find its length in the form $(a+b\sqrt{6})$ cm, where a and b are integers.

7 Without using a calculator, express in its simplest surd form,

(a) $\dfrac{3}{\sqrt{8}}+\dfrac{5}{\sqrt{2}}-\dfrac{\sqrt{32}}{3}$, **(b)** $\dfrac{4}{\sqrt{27}}-\dfrac{\sqrt{18}}{4}+\dfrac{4}{\sqrt{3}}$,

(c) $\dfrac{2}{\sqrt{3}}\left(\dfrac{4}{\sqrt{12}}+\dfrac{\sqrt{27}}{3}\right)$, **(d)** $\dfrac{6}{\sqrt{2}}\left(\dfrac{3}{\sqrt{8}}-\dfrac{\sqrt{128}}{3}\right)$.

8 Given that $h=3+\sqrt{2}$, express $\dfrac{h^2+1}{h-2}$ in the form $p+q\sqrt{2}$, where p and q are integers.

9 Solve each of the following equations.

(a) $\sqrt{11x^2+45}=4x$ **(b)** $\sqrt{3x+2}=3x$

(c) $\sqrt{2x-3}-5\sqrt{x+2}=0$

10 Rationalise the denominator of each of the folowing.

(a) $\dfrac{13}{(\sqrt{3}+4)^2}$ **(b)** $\dfrac{3}{(\sqrt{2}+6)^2}+\dfrac{5}{(\sqrt{2}-6)^2}$

(c) $\dfrac{3\sqrt{2}-6}{6+3\sqrt{2}}$ **(d)** $\dfrac{\sqrt{48}-\sqrt{50}}{\sqrt{27}-\sqrt{8}}$

11 If $a=\dfrac{1}{\sqrt{2}}$ and $b=\dfrac{1+a}{1-a}$, express in its simplest surd form,

(i) b, **(ii)** $b-\dfrac{1}{b}$.

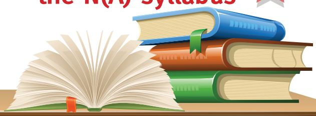
12 The solution of $x\sqrt{7}=x\sqrt{2}+\sqrt{32}$ is $\dfrac{a+b\sqrt{14}}{5}$. Without using a calculator, find the values of the integers a and b.

13 A right circular cylinder has a volume of $(6+2\sqrt{3})\pi$ cm³ and a base radius of $(1+\sqrt{3})$ cm. Find its height in the form of $(a+b\sqrt{3})$ cm, where a and b are integers.

14 A student solved a quadratic equation with rational coefficients and found that the roots are $2-\sqrt{3}$ and $1+\sqrt{3}$. Explain why his answer is wrong.

4.3 INTRODUCTION TO LOGARITHMS

excluded from the N(A) syllabus



Why Study Logarithms?

In Section 4.1, we have learnt how to solve exponential equations where both sides can be converted to the *same base*, e.g.,

$$\begin{aligned} 10^x &= 100 \\ 10^x &= 10^2 \\ \therefore \quad x &= 2 \end{aligned}$$

However, what happens if the equation is $10^x = 50$ where 50 *cannot* be converted to the same base 10? Since $10^1 = 10$ and $10^2 = 100$, then $10^x = 50$ will be somewhere in between:

$$\begin{aligned} 10^1 &= 10 \\ 10^x &= 50 \\ 10^2 &= 100 \end{aligned}$$

From the above, we know $1 < x < 2$. Now the question is: How do we find the index or the **exponent** x?
Let $x = \log_{10} 50$ (read as *log 50 to base 10*). Log is short form for logarithm.

1. On your calculator, press [log] [50]. What is the answer? $x =$ _______
Note that some calculators indicate [lg] instead of [log].
2. Use your calculator to find 10^x. Do you get 50?

The mathematician John Napier (1550-1617) invented the logarithm to find the index. The purpose of the logarithm is to find the index. There are other real-life applications of the logarithms which we will explore in Section 4.7.

We have seen from the above that the index form or exponential form $10^x = 50$ can be converted to the logarithmic form $x = \log_{10} 50$.

John Napier (1550 – 1617) was a Scottish mathematician, physicist and astronomer. As the study of astronomy involved the use of large numbers, Napier invented logarithms to multiply large numbers easily. This has helped Johannes Kepler (1571-1630) to simplify his calculations and thus discovered the three laws of planetary motion, which in turn provided the basis of Isaac Newton's (1643-1727) theory of gravitation. Therefore, we can see how Mathematics has helped in the advancement of the sciences.

Conversion from Exponential Form to Logarithmic Form and Vice Versa

Worked Example 11

(Conversion from Exponential Form to Log Form)
Convert $10^x = 70$ to logarithmic form.

INFORMATION

Indicial form is also known as exponential form because another name for index is exponent (see Section 4.1).

Solution

Step 1: Identify the *base*: 10
Step 2: Identify the *index*: x
Step 3: In the log form, the index is always by itself and the base is at the bottom. So $x = \log_{10}?$
Finally, fill in the last number 70.

index always by itself $\quad$ *last number to fill in*

$$\underset{\downarrow}{} x = \log_{10} 70$$

base must be at the bottom

Practise Now 11

Similar Questions:
Exercise 4C
Questions 1(a)-(d)

Convert each of the following from exponential form to logarithmic form.

(a) $1000 = 10^3$ **(b)** $4^x = 10$ **(c)** $80 = 10^x$ **(d)** $a^k = 2$

Worked Example 12

(Conversion from Log Form to Exponential Form)
Convert $x = \log_2 3$ to exponential form.

Solution

Step 1: Identify the *base*: 2
Step 2: Identify the *index*: x
Hint: Remember 'index is always by itself'.
Step 3: In the exponential form, write the base and the index first.
So $2^x = ?$ Finally, fill in the last number 3.

index $\quad$ *last number to fill in*

$$2^x = 3$$

base must be at the bottom

Practise Now 12

Similar Questions:
Exercise 4B
Questions 2(a)-(d)

1. Convert each of the following from logarithmic form to exponential form.

(a) $x = \log_8 5$ **(b)** $\log_3 x = \frac{1}{7}$ **(c)** $-2 = \log_4 \frac{1}{16}$ **(d)** $x = \log_a y$

2. If $\log_3 a = x$, $\log_{81} b = y$ and $\frac{a}{b} = 3^c$, express c in terms of x and y.

In general,

$$y = a^x \text{ is equivalent to } x = \log_a y \qquad \text{if } a > 0,\ a \neq 1$$

(index: x; base: a)

Conditions for a Logarithm to be Defined

What are the conditions for $x = \log_a y$ to be defined? Let's investigate.
Since $x = \log_a y$ is equivalent to $y = a^x$, then the conditions for $y = a^x$ to be defined will also apply to $x = \log_a y$.

Investigation

Conditions for a Logarithm to be Defined

1. For $y = a^x$, can the base a be positive, negative or 0? Can the base a be 1?
   Hint: Consider what happens if the index x is also negative, 0 or $\frac{1}{2}$.

2. For $y = a^x$, can y be positive, negative or 0 when $a > 0$? These results will also apply to $x = \log_a y$, i.e. can we take the logarithm of 0 or the logarithm of a negative number?

Copy and complete the following.
Therefore, for $x = \log_a y$ to be defined,
(a) base $a >$ ________ and $a \neq$ ________; and **(b)** $y >$ ________.

Common Logarithms

Logarithms with a base of 10 are called **common logarithms**. From now onwards, we will follow the ISO (International Organisation for Standardisation) standard of using **lg x** to represent $\log_{10} x$. The values of common logarithms can be obtained using a calculator.

ATTENTION
ISO uses $\lg x$ to represent $\log_{10} x$. But most calculators use $\log x$ to represent $\log_{10} x$.

Worked Example 13

(Finding lg)
Use a calculator to find the value of lg 7.

Solution
Depending on the model of your calculator, press either [log] [7] [=] or [7] [log] to obtain the answer: lg 7 = 0.845 (to 3 s.f.)

Practise Now 13

Use a calculator to evaluate each of the following common logarithms.

(a) $\lg 3$ **(b)** $\lg 1$ **(c)** $\lg \frac{1}{1000}$ **(d)** $\lg 123\,456$

Similar Questions:
Exercise 4C
Questions 3(a)-(d)

Thinking Time

Based on Practise Now 13, answer the following questions.

1. Can logarithms be positive, negative or 0?
2. Can logarithms be integers or non-integers?
3. What happens if you take the logarithm of a small number or a big number?

Therefore, if $x = \log_a y$, what kind of number is 'x'?

The Mathematical Constant e

There is another type of logarithms to a special base called e. We will look at this constant e first. We have learnt that the mathematical constant π is an irrational number and its value is approximately 3.142. There is another important mathematical constant discovered by the mathematician Jacob Bernoulli (1654-1705). It is also an irrational number and its value, truncated to 50 decimal places, is:

2.71828 18284 59045 23536 02874 71352 66249 77572 47093 69995…

The famous mathematician Leonhard Euler (1707-1783) was the first to use the letter **e** to represent this mathematical constant.

INTERNET RESOURCES

Go to http://www.shinglee.com.sg/StudentResources/ to do an investigative worksheet on the real-life discovery of e.

Worked Example 14

(Finding Powers of e)

Use a calculator to find the value of e^2.

Solution

Depending on the model of your calculator, press either [e^x] [2] [=] or [2] [e^x] to obtain the answer: $e^2 = 7.39$ (to 3 s.f.)

Practise Now 14

Similar Questions:
Exercise 4D
Questions 3(e)-(h)

Use a calculator to evaluate each of the following powers of e.

(a) e^3 **(b)** e^0 **(c)** $e^{\frac{1}{2}}$ **(d)** e^{-5}

Thinking Time

Based on Practise Now 14, answer the following questions.

1. Can e^x be positive, negative or 0?

2. Is the answer in Question 1 consistent with the result of the Investigation in Section 4.3?

Therefore, state the range of values of e^x for all values of x.

Natural Logarithm

Earlier in this section, we have learnt about common logarithms whose base is 10, i.e. $\lg x = \log_{10} x$.
There is another special type of logarithms whose base is the mathematical constant e.
Logarithms with a base of e, $\log_e x$, are called **natural logarithms** or Naperian logarithms after the inventor of logarithms, John Napier (1550-1617).
The short form for $\log_e x$ is $\ln x$ (pronounced as 'lawn x'). The values of natural logarithms can also be obtained using a calculator by pressing [ln].
Similar to common logarithms, we have

index

$y = e^x$ is equivalent to $x = \log_e y = \ln y$

base

ATTENTION

The value of $\ln 7$ is not equal to that of $\lg 7$ in Worked Example 13. Use your calculator to verify this.

Simple Logarithmic Equations

Worked Example 15

(Simple Logarithmic Equations)
Solve each of the following logarithmic equations.

(a) $\ln 2x = 3$ **(b)** $\log_a 16 = 2$

ATTENTION

For (b), always check your answer to see whether the conditions for base a are satisfied.

Solution

(a) $\ln 2x = 3$

$2x = e^3$ (convert log form to exponential form)

$x = \dfrac{e^3}{2}$

$= 10.0$ (to 3 s.f.)

(b) $\log_a 16 = 2$

$a^2 = 16$ (convert log form to exponential form)

$a = \pm 4$

Check: For logarithms to be defined, a must be positive.
Hence the solution is $a = 4$.

Practise Now 15a

Solve each of the following logarithmic equations.

(a) $\ln 3x = 7$ **(b)** $\log_a 36 = 2$ **(c)** $\log_z (2 - z) = 2$

Similar Questions:
Exercise 4C
Questions 5(a)-(i), 7(a)-(f)

Two Special Properties

Class Discussion

Discuss with your classmates:

1. What is the value of $\log_a 1$? Does it remain the same for all values of a? Explain your answer.
2. What is the value of $\log_a a$?

In general, if $a > 0$ and $a \neq 1$, then:

$$\boxed{\begin{array}{ll} \text{Special Property 1:} & \log_a 1 = 0 \\ \text{Special Property 2:} & \log_a a = 1 \end{array}}$$

What happens if the base a is the mathematical constant e?

$$\boxed{\ln 1 = 0 \quad \text{and} \quad \ln \mathrm{e} = 1}$$

Practise Now 15b

Similar Questions:
Exercise 4C
Questions 4(a)-(d)

★ Simplify each of the following without using a calculator.

(a) $\log_2 1 + 3\log_4 4$ **(b)** $5(\log_a a)^7 - 8\ln 1 + 9\ln \mathrm{e}$

Basic Level | Intermediate Level | Advanced Level

excluded from the N(A) syllabus ★

Exercise 4C

1 Convert each of the following from exponential form to logarithmic form.

(a) $10^x = 40$ **(b)** $10 = 5^x$

(c) $4 = 16^{\frac{1}{2}}$ **(d)** $x = a^y$

2 Convert each of the following from logarithmic form to exponential form.

(a) $x = \log_2 7$ **(b)** $\log_4 x = \frac{1}{5}$

(c) $-3 = \log_3 \frac{1}{27}$ **(d)** $\log_{0.25} k = n$

3 Use a calculator to evaluate each of the following.

(a) $\lg \pi$ **(b)** $\lg 10$

(c) $\lg 100$ **(d)** $\lg (4.38 \times 10^9)$

(e) e^4 **(f)** e^{-1}

(g) $\mathrm{e}^{\frac{1}{3}}$ **(h)** $\mathrm{e}^{-2.5}$

4 Simplify each of the following without the use of a calculator.

(a) $\log_3 3 - 4\log_7 1 + 2\lg 10$

(b) $\log_{\frac{1}{2}} \frac{1}{2} + 3\log_{0.4} \frac{2}{5} - 2\ln 1$

(c) $4(\log_b b)^{2013} + 7\log_\pi 1 - 8\ln \mathrm{e}$

(d) $\dfrac{8(\log_b 1)^b - 6\lg 10}{3(\ln \mathrm{e})^2}$

Basic Level | Intermediate Level | Advanced Level

Exercise 4C

5 Solve each of the following equations.

(a) $\ln 8x = 5$ **(b)** $3 \lg 2y = -6$

(c) $\ln 11 \times \lg (1 - 2y) = 8$

(d) $\log_a 64 = 2$ **(e)** $\log_5 y = -4$

(f) $\log_b (3b - 2) = 2$ **(g)** $\log_{2-3k} 125 = 3$

(h) $\log_y \left(\dfrac{y - y^2}{2} \right) = 3$ **(i)** $(\log_3 x)^2 = 4 \log_3 x$

6 If $\log_4 x = p$ and $\log_{\sqrt{2}} y = q$, express each of the following in terms of p and q.

(a) x^3y^2 **(b)** $\dfrac{\sqrt{x}}{y^4}$

7 Solve each of the following equations.

(a) $\log_3 (\log_2 x) = 2$

(b) $\ln (\lg x) = \lg 2$

(c) $\log_{\sqrt{10}} (\ln y) = 3$

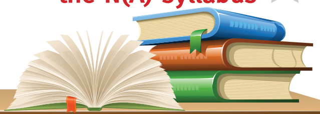
(d) $\log_{\sqrt[3]{2}} (\log_y \frac{1}{64}) = 3$

(e) $\log_{3-4k} 625 = 4$

(f) $\log_{27} 3^{2-5z} = z^2$

4.4 LAWS OF LOGARITHMS AND CHANGE OF BASE FORMULA

excluded from the N(A) syllabus



Product Law of Logarithms

Investigation

Product Law of Logarithms

What do you think lg 20 + lg 30 is equal to? Is it equal to lg 50 or lg 600? To answer this question, let us investigate.

1. Copy and complete the following table using a calculator. Leave non-exact answers to 3 significant figures.

x	y	$x + y$	xy	$\lg x$	$\lg y$	$\lg x + \lg y$	$\lg (x + y)$	$\lg xy$
20	30							
50	40							
2	7							
10	100							

2. Compare the last 3 columns in the table above. What do you notice? This is called the **product law of logarithms**.

ATTENTION

$\lg xy$ means $\lg (xy)$. It does not mean $(\lg x)y$, which is usually written as $y \lg x$.

In general, the **product law of logarithms** is true for any base:

$$\log_a x + \log_a y = \log_a xy \qquad \text{if } x,\ y,\ a > 0,\ a \neq 1$$

Worked Example 16

(Product Law of Logarithms)
Simplify lg 33 + lg 61 without using a calculator.

Solution

$$\begin{aligned} \lg 33 + \lg 61 &= \lg (33 \times 61) \quad \text{(Product Law of Logarithms)} \\ &= \lg 2013 \end{aligned}$$

Practise Now 16a

Similar Questions:
Exercise 4D
Questions 1(a)-(d)

Copy and complete the following without using a calculator.

(a) $\lg 2 + \lg 3 = \lg (\_\_\_\_) = \lg \_\_\_\_$

(b) $\lg 20 + \lg 20 = \lg \_\_\_\_ = \lg \_\_\_\_$

(c) $\lg 5 + \lg \frac{1}{2} = \lg \_\_\_\_ = \lg \_\_\_\_$

(d) $\lg m + \lg n = \lg \_\_\_\_$

Serious Misconceptions

$$\log_a (x + y) \neq (\log_a x)(\log_a y)$$

$$\log_a (x + y) \neq \log_a x + \log_a y$$

ATTENTION

You can only add terms involving logarithms if the bases are the same.

Practise Now 16b

Similar Questions:
Exercise 4D
Questions 1(e)-(h)

Simplify each of the following *if possible*.

(a) $\log_2 3 + \log_2 7$

(b) $\ln 2p + \ln q + \ln 5q$

(c) $\log_5 8 + \log_6 9$

(d) $\log_4 0.5 + \log_4 8$

Thinking Time

Can you prove the product law of logarithms? The following shows a partial proof. Copy and fill in the blanks to complete the proof.

To prove: $\log_a x + \log_a y = \log_a xy$

Let $\log_a x = m$ and $\log_a y = n$.

Then $x = \_\_\_\_\_\_\_$ and $y = \_\_\_\_\_\_$.

(convert log form to exponential form)

So $xy = \_\_\_\_\_\_ \times \_\_\_\_\_\_ = a^{m+n}$.

Thus $\log_a xy = m + n$ (convert exponential from to log form) $= \_\_\_\_\_\_ + \_\_\_\_\_\_$

ATTENTION

Notice that the product law of logarithms follows Law 1 of Indices: $a^m \times a^n = a^{m+n}$

Quotient Law of Logarithms

We have learnt from the investigation in this section that $\lg 20 + \lg 30 = \lg (20 \times 30)$.
What do you think $\lg 30 - \lg 20$ is equal to? Is it equal to $\lg 10$ or $\lg 1.5$?
Make a guess.
Use your calculator to check. Are you correct?

In general, the **quotient law of logarithms** is true for any base:

$$\boxed{\log_a x - \log_a y = \log_a \frac{x}{y} \quad \text{if } x, y, a > 0, a \neq 1}$$

Serious Misconceptions

$$\log_a (x - y) \neq \frac{\log_a x}{\log_a y}$$

$$\log_a (x - y) \neq \log_a x - \log_a y$$

ATTENTION

You can only subtract logarithms if they are of the *same base*.

Thinking Time

Find some counterexamples to show that:

$$\log_a (x - y) \neq \log_a x - \log_a y$$

Worked Example 17

(Quotient Law of Logarithms)
Simplify $\log_3 18 - \log_3 6$.

Solution

$$\begin{aligned} \log_3 18 - \log_3 6 &= \log_3 \frac{18}{6} && \text{(Quotient Law of Logarithms)} \\ &= \log_3 3 \\ &= 1 && \text{(Special Property 2 in Section 4.3)} \end{aligned}$$

Practise Now 17

Similar Questions:
Exercise 4D
Questions 2(a)-(d)

Simplify each of the following *if possible*.

(a) $\log_5 20 - \log_5 4$ **(b)** $\log_a 8m - \log_a 2m + \log_a 9$

(c) $\log_7 3 - \lg 6$

Journal Writing

1. Prove the quotient law of logarithms.
2. Which law of indices does the quotient law of logarithms depend on?
Hint: See the proof for the product law of logarithms.

Power Law of Logarithms

Consider $\lg 2^3$.
Use a calculator to evaluate $\lg 2^3$ and $(\lg 2)^3$.
Are they equal?

$$\begin{aligned}\lg 2^3 &= \lg (2 \times 2 \times 2)\\ &= \lg 2 + \lg 2 + \lg 2 \quad \text{(Product Law of Logarithms)}\\ &= 3 \lg 2\end{aligned}$$

Do the same for $\log_2 7^4$. What do you notice?

ATTENTION

$\lg 2^3$ means $\lg (2^3)$.
It does not mean $(\lg 2)^3$.
$3 \lg 2$ means $3 \times \lg 2$.

In general, the **power law of logarithms** is true for any base:

$$\boxed{\log_a x^r = r \log_a x \qquad \text{if } x, a > 0, a \neq 1}$$

Serious Misconceptions

$$\boxed{\log_a x^r \neq (\log_a x)^r.}$$

Worked Example 18

(Power Law of Logarithms)
Simplify $\log_4 64$.

Solution

$$\begin{aligned}\log_4 64 &= \log_4 4^3\\ &= 3 \log_4 4 \quad \text{(Power Law of Logarithms)}\\ &= 3 \quad \text{(Special Property 2 in Section 4.3)}\end{aligned}$$

Practise Now 18

Similar Questions:
Exercise 4D
Questions 3(a)-(d), 5(a)-(d)

Evaluate each of the following without using a calculator.

(a) $\log_3 81$

(b) $5 \log_3 9 - 2 \log_3 27$

(c) $\log_5 \sqrt{3} + \log_5 \sqrt{10} - \log_5 \sqrt{6}$

(d) $2 \lg 3\frac{1}{3} + 3 \lg 0.4 - 2 \lg \frac{2}{75}$

Worked Example 19

(Power Law of Logarithms)
Express $4 + \log_3 5$ as a single term involving logarithms.

Solution

$$\begin{aligned}4 + \log_3 5 &= 4 \log_3 3 + \log_3 5\\ &= \log_3 3^4 + \log_3 5 \quad \text{(Power Law of Logarithms)}\\ &= \log_3 81 + \log_3 5\\ &= \log_3 (81 \times 5) \quad \text{(Product Law of Logarithms)}\\ &= \log_3 405\end{aligned}$$

PROBLEM-SOLVING TIP

Since the first term does not contain log, we must make log appear. Recall Special Property 2 in Section 4.3: $\log_a a = 1$. Hence we can make log appear in the first term since $4 = 4 \log_3 3$.

In general,

Special Property 3: For any number n,
$$n = n \log_a a \qquad \text{if } a > 0, a \neq 1.$$

Practise Now 19

Similar Questions:
Exercise 4D
Questions 6(a)-(d), 8

Express each of the following as a single logarithm.

(a) $3 + \log_2 7$ **(b)** $2 - \ln \frac{1}{3}$ **(c)** $\log_a a^4 + \log_3 0.2$

Change of Base Formula

How do we find logarithms with a base other than 10 or e since the calculator does not have such a function?

In Worked Example 18, we find the value of $\log_4 64$ by expressing 64 as 4^3 before using the Power Law of Logarithms and Special Property 1: $\log_4 4 = 1$. However, what happens if we want to find the value of $\log_4 24$ using the same method?

The problem is that 24 cannot be expressed as a power of 4.
Hence we need to learn a new formula to find the logarithm of a number to a base other than 10 easily.

For example, let $x = \log_4 24$. Then $4^x = 24$.

Taking lg on both sides, $\lg 4^x = \lg 24$

$x \lg 4 = \lg 24$ (Power Law of Logarithms)

$\therefore \quad x = \dfrac{\lg 24}{\lg 4}$ ← use the calculator to evaluate this

$= 2.29$ (to 3 s.f.)

Thinking Time

Instead of taking lg on both sides, what happens if you take ln on both sides? Will you get the same result? Is there any difference?

In general,

$$\boxed{\log_a b = \frac{\lg b}{\lg a} = \frac{\ln b}{\ln a} \qquad \text{if } a, b > 0,\ a \neq 1}$$

Worked Example 20

(Logarithm with a Base Other Than 10)
Use a calculator to evaluate $\log_5 0.8$.

Solution

$$\log_5 0.8 = \frac{\lg 0.8}{\lg 5} \quad \left(\text{or } \frac{\ln 0.8}{\ln 5}\right)$$
$$= -0.139 \text{ (to 3 s.f.)}$$

Practise Now 20

Similar Questions:
Exercise 4D
Questions 4(a)-(d)

Use a calculator to evaluate each of the following.

(a) $\log_2 7.16$ **(b)** $\log_4 e$ **(c)** $\log_8 \frac{3}{2}$ **(d)** $\log_{\frac{1}{3}} \sqrt{5}$

In general, the above formula works for any base. This is called the **Change of Base Formula**:

$$\log_a b = \frac{\log_c b}{\log_c a} \qquad \text{if } a, b, c > 0;\ a, c \neq 1$$

ATTENTION

You must change to the *same base* c. To remember this formula, notice that b is on 'top' and a is at the 'bottom'.

Special Case

If $c = b$ in the Change of Base Formula, then:

$$\log_a b = \frac{1}{\log_b a} \qquad \text{if } a, b > 0;\ a, b \neq 1$$

Worked Example 21

(Simplifying Logarithms using Change of Base Formula)
Find the value of $\log_3 16 \times \log_4 10 \times \lg 3$ without using a calculator.

Solution

$$\log_3 16 \times \log_4 10 \times \lg 3 = \frac{\lg 16}{\lg 3} \times \frac{\lg 10}{\lg 4} \times \lg 3$$
$$= \lg 4^2 \times \frac{1}{\lg 4} \qquad (\lg 10 = 1)$$
$$= 2 \lg 4 \times \frac{1}{\lg 4}$$
$$= 2$$

Practise Now 21

Similar Questions:
Exercise 4D
Questions 9(a)-(f), 10-13

Find the values of each of the following without using a calculator.

(a) $\log_5 8 \times \log_2 10 \times \lg 5$ **(b)** $\frac{\log_7 4 \times \log_2 9}{\log_{49} \sqrt{3}}$ **(c)** $\log_5 3 \times 4 \log_3 5$

(d) $\ln 8 \times 5 \log_2 e$ **(e)** $\log_2 10 \times \lg \sqrt[3]{2}$ **(f)** $\log_a 9 \times \log_{27} a$

Worked Example 22

(Finding the Value of Another Logarithmic Term given Some Logarithmic Values)

Given that $\log_2 3 = 1.585$ (to 4 s.f.) and $\log_2 5 = 2.322$ (to 4 s.f.), evaluate $\log_2 45$ without using a calculator, leaving your answer to 3 significant figures.

Solution

$\log_2 45 = \log_2 (9 \times 5)$

$= \log_2 9 + \log_2 5$ (Product Law of Logarithms)

$= \log_2 3^2 + \log_2 5$

$= 2 \log_2 3 + \log_2 5$ (Power Law of Logarithms)

$= 2 \times 1.585 + 2.322$

$= 5.49$ (to 3 s.f.)

PROBLEM-SOLVING TIP

Since only the values of $\log_2 3$ and $\log_2 5$ are given, we need to express 45 in terms of 3 and 5.

ATTENTION

You must write $\log_2 (9 \times 5)$, and not $\log_2 9 \times 5$ which means $(\log_2 9) \times 5$.

Practise Now 22

Similar Questions:
Exercise 4D
Question 7

1. Given that $\log_2 3 = 1.585$ (to 4 s.f.) and $\log_2 5 = 2.322$ (to 4 s.f.), evaluate each of the following without using a calculator, leaving your answers to 3 significant figures.

(a) $\log_2 375$ **(b)** $\log_2 0.12$ **(c)** $\log_2 \dfrac{\sqrt{125}}{9}$ **(d)** $\dfrac{\log_2 \sqrt{125}}{\log_2 9}$

2. Given that $\lg 2 = n$, express each of the following in terms of n.

(a) $\log_{\frac{1}{4}} \sqrt{10}$ **(b)** $\log_8 5$ **(c)** $\lg \dfrac{1}{5}$ **(d)** $\lg \sqrt{20}$

Basic Level | Intermediate Level | Advanced Level

Exercise 4D

excluded from the N(A) syllabus

1 Without using a calculator, simplify each of the following *if possible*.

(a) $\lg 6 + \lg 5$ **(b)** $\lg 0.125 + \lg 8$ **(c)** $\lg 5 + \lg 2$ **(d)** $\lg p^2 + \lg \frac{1}{2} q + \lg \dfrac{2}{pq}$

(e) $\log_3 5 + \log_3 8$ **(f)** $\ln 9 + \ln 2$ **(g)** $\lg 6 + \ln 7$ **(h)** $\log_a 2a + \log_a 0.125 + \log_a \dfrac{4}{a}$

2 Without using a calculator, simplify each of the following *if possible*.

(a) $\log_7 30 - \log_7 6$ **(b)** $\log_2 8 - \ln 9$ **(c)** $\lg 7 + \lg 0.2 - \lg 3k$ **(d)** $\log_b 3\sqrt{b} + \log_b b\sqrt{b} - \log_b 3b$

3 Find the values of each of the following without using a calculator.

(a) $\log_5 125$ **(b)** $\lg \frac{1}{100}$

(c) $\log_{64} 8$ **(d)** $\ln \sqrt{e}$

4 Use a calculator to evaluate each of the following.

(a) $\log_3 \pi$ **(b)** $\log_7 1\frac{2}{3}$

(c) $\log_{\frac{1}{5}} \sqrt{10}$ **(d)** $\log_{\sqrt{2}} 3^{\frac{5}{4}}$

5 Simplify each of the following without using a calculator.

(a) $4 \log_3 25 - \log_3 125$

(b) $3 \ln 216 + 4 \ln \frac{1}{36}$

(c) $\log_6 \sqrt{3} + \log_6 \sqrt{24} - \log_6 \sqrt{2}$

(d) $4 \log_2 0.6 + 2 \log_2 \frac{5}{3} - 2 \log_2 0.15$

6 Express each of the following as a single logarithm.

(a) $2 + \log_6 2$ **(b)** $\frac{1}{2} \lg 25 - 3$

(c) $\log_b b^3 + \log_4 3$ **(d)** $\log_c \sqrt{c} + \log_3 \sqrt{12}$

7 Given that $\log_3 2 = 0.6309$ (to 4 s.f.) and $\log_3 5 = 1.465$ (to 4 s.f), evaluate each of the following without the use of a calculator, leaving your answers to 3 significant figures.

(i) $\log_3 80$ **(ii)** $\log_3 1.28$

(iii) $\log_3 \frac{\sqrt{10}}{125}$ **(iv)** $\frac{\log_3 0.125}{\log_3 \sqrt{125}}$

8 Given that $\log_a y = k$, express each of the following in terms of k.

(a) $(\log_a y)^3$ **(b)** $\log_a y^3$

(c) $\log_a (ay)^2$ **(d)** $\log_a \frac{\sqrt{y}}{a^2}$

9 Evaluate each of the following without using a calculator.

(a) $\log_7 81 \times \log_3 100 \times \lg 49$

(b) $\frac{\log_{\sqrt{2}} \frac{1}{5} \times \log_e 16}{\ln \sqrt[3]{25}}$ **(c)** $\log_{11} 9 \times 3 \log_9 11$

(d) $\ln 27 \times 4 \log_3 e$ **(e)** $\log_7 10 \times \lg \sqrt[3]{49}$

(f) $\log_x 16 \times \log_{64} x$

10 Given that $\log_a y = m$, express each of the following in terms of m.

(i) $\log_y a$ **(ii)** $\log_{\sqrt{y}} a$

(iii) $\log_{\frac{1}{a}} y$ **(iv)** $\log_{\sqrt{a}} \frac{1}{y}$

11 Given that $\log_{12} 3 = p$, express each of the following in terms of p.

(i) $\log_{\sqrt{12}} \frac{1}{9}$ **(ii)** $\log_{\frac{1}{12}} 27$

(iii) $\log_{12} \frac{1}{4}$ **(iv)** $\log_{144} 4$

12 If $\log_6 2 = a$ and $\log_5 3 = b$, express $\log_5 2$ in terms of a and b.

13 **(i)** If $a, b, x, y > 0$ and $a, b \neq 1$, show that $\frac{\log_a x}{\log_a y} = \frac{\log_b x}{\log_b y}$.

(ii) Hence, given that $\lg 2 = 0.301$ (to 3 s.f.), find the value of $\frac{\log_3 2}{\log_3 10}$ without using a calculator.

4.5 LOGARITHMIC AND EXPONENTIAL EQUATIONS

excluded from the N(A) syllabus

Logarithmic Equations

In Section 4.3 Worked Example 15, we have learnt to solve simple logarithmic equations involving only one logarithmic term.
In this section, we will learn how to solve more complicated logarithmic equations. There are two types of logarithmic equations, one with logarithmic terms with the same base (see Worked Example 23 and 24) and another with logarithmic terms with *different bases* (see Worked Example 25).

There are *two* main methods to solve logarithmic equations involving more than one logarithmic term.
Method 1 makes use of the **Equality of Logarithms**:

If $\log_a p = \log_a q$, then $p = q$ where $a, p, q > 0$; $a \neq 1$.

Worked Example 23

(Logarithmic Equation with Same Base)
Solve the equation $\log_3 (1 - x) + \log_3 (x + 5) = \log_3 (3x + 11)$.

Solution

$$\begin{aligned}
\log_3 (1 - x) + \log_3 (x + 5) &= \log_3 (3x + 11) \\
\log_3 (1 - x)(x + 5) &= \log_3 (3x + 11) \\
\therefore \quad (1 - x)(x + 5) &= 3x + 11 \quad \text{(Method 1: Equality of Logarithms)} \\
x + 5 - x^2 - 5x &= 3x + 11 \\
x^2 + 7x + 6 &= 0 \\
(x + 1)(x + 6) &= 0 \\
x &= -1 \text{ or } -6
\end{aligned}$$

Check: Subst. $x = -6$ into original equation:
$\log_3 (x + 5) = \log_3 (-1)$ is not defined;
$\log_3 (3x + 11) = \log_3 (-7)$ is not defined.

Subst. $x = -1$ into original equation:
$\log_3 (1 - x) = \log_3 2$ is defined;
$\log_3 (x + 5) = \log_3 4$ is defined;
$\log_3 (3x + 11) = \log_3 8$ is defined.
LHS $= \log_3 2 + \log_3 4 = \log_3 8 =$ RHS

Therefore, the solution is $x = -1$.

ATTENTION

Always check your answers when solving logarithmic equations because applying equality of logarithms may introduce 'extra answers' that are not applicable.

ATTENTION

Do not reject an answer because it is negative. As you can see, $x = -1$ is acceptable.

Practise Now 23

Solve the equation $\log_2 (x + 11) + \log_2 (x + 4) = \log_2 (5x + 23)$.

Similar Questions:
Exercise 4E
Questions 2(a)-(c)

Method 2 for solving logarithmic equations is used to convert the logarithmic form to exponential form (in fact, Section 4.3 Worked Example 15 uses this method).

Worked Example 24

(Logarithmic Equation with Same Base)

Solve the equation $2 \lg x - \lg (x + 20) = 1$.

Solution

$$\begin{aligned}
2 \lg x - \lg (x + 20) &= 1 \\
\lg x^2 - \lg (x + 20) &= 1 && \text{(Power Law of Logarithms)} \\
\lg\left(\frac{x^2}{x+20}\right) &= 1 && \text{(Quotient Law of Logarithms)} \\
\therefore \quad \frac{x^2}{x+20} &= 10^1 && \text{(Method 2: convert log form to exponential form)} \\
x^2 &= 10(x + 20) \\
x^2 - 10x - 200 &= 0 \\
(x + 10)(x - 20) &= 0 \\
x &= -10 \text{ or } 20
\end{aligned}$$

Check: Subst. $x = -10$ into original equation:
$2 \lg x = 2 \lg (-10)$ is not defined.

Subst. $x = 20$ into original equation:
$2 \lg x = 2 \lg 20$ is defined;
$\lg (x + 20) = \lg 40$ is defined.

$\text{LHS} = 2 \lg 20 - \lg 40 = \lg \dfrac{400}{40} = \lg 10 = 1 = \text{RHS}$

Therefore, the solution is $x = 20$.

Practise Now 24

Similar Questions:
Exercise 4E
Questions 3(a)-(d), 5, 9, 10

Solve each of the following equations.

(a) $2 \lg x = \lg (2x + 30) + 1$ **(b)** $\log_4 y^2 = \log_4 (y + 2) - 3$

Thinking Time

(i) Solve Worked Example 23 using Method 2.
(ii) Solve Worked Example 24 using Method 1.
(iii) Hence, discuss which method is more suitable for which types of logarithmic equations.

For logarithmic equations involving terms with different bases, use the Change of Base Formula, then use **Method 1** or **2**.

Worked Example 25

(Logarithmic Equation with Different Bases)

Solve the equation $\log_2 x = 25 \log_x 2$.

Solution

$\log_2 x = 25 \log_x 2$

$= \dfrac{25}{\log_2 x}$ (Special Case of Change of Base Formula)

Let $y = \log_2 x$. Then $y = \dfrac{25}{y}$.

$\therefore \quad y^2 = 25$

$y = \pm 5$

$\therefore \quad \log_2 x = \pm 5$

$x = 2^5$ or 2^{-5} (convert from log form to exponential form)

$= 2^5$ or $\dfrac{1}{2^5}$ (Law 7 of Indices)

$= 32$ or $\dfrac{1}{32}$

> **ATTENTION**
>
> Can you convert $\log_2 x$ to $\dfrac{1}{\log_x 2}$ instead? The unknown is now the base x. Try solving it using this method and see if the solution is more tedious.

Practise Now 25

Similar Questions:
Exercise 4E
Questions 6(a)-(d), 12

1. Solve each of the following equations.

(a) $\log_4 x = 9 \log_x 4$

(b) $\log_3 y + \log_9 y = 7$

(c) $\log_{100}\left(\dfrac{4x-3}{3}\right) = \lg 2x + \lg \dfrac{1}{3}$

2. If $\lg r$ and $\lg s$ are the roots of the equation $3x^2 - 2x + 4 = 0$, find the value of each of the following:

(i) $\lg r + \lg s$

(ii) $\lg r \times \lg s$

(iii) $\dfrac{1}{\lg r} + \dfrac{1}{\lg s}$

(iv) $\log_r s + \log_s r$

Solving Exponential Equations Using Logarithms

In Section 4.1 Worked Examples 1 and 2, we have learnt how to solve exponential equations whose terms can be converted to the *same base* and we make use of the **Equality of Indices** **(Method 1)**. In this section, we will look at exponential equations in which the terms cannot be converted to the same base.

Worked Example 26

(Exponential Equations that **cannot** be converted to Same Base)

Solve each of the following equations.

(a) $e^{3x-1} = 8$ **(b)** $5^{y+2} = 7$

Solution

(a) Convert to log form (Method 2)

$e^{3x-1} = 8$

$3x - 1 = \ln 8$ (take ln on both sides)

$x = \dfrac{1+\ln 8}{3}$

$= 1.03$ (to 3 s.f.)

(b) $5^{y+2} = 7$

Taking lg on both sides,

$$\lg 5^{y+2} = \lg 7$$

$$(y + 2)\lg 5 = \lg 7 \quad \text{(Power Law of Logarithms)}$$

$$y + 2 = \frac{\lg 7}{\lg 5}$$

$$y = \frac{\lg 7}{\lg 5} - 2$$

$$= -0.791 \quad \text{(to 3 s.f.)}$$

Practise Now 26

Similar Questions:
Exercise 4E
Questions 1(a)-(d), 7(a)-(f), 8

Solve each of the following equations:

(a) $e^{2x+3} = 6$ **(b)** $10^{1-y} = 14$

(c) $e^{2x} - e^{x} - 6 = 0$ **(d)** $2^{x}(3^{2x}) = 5(7^{x})$

PROBLEM-SOLVING TIP

For (d), the hint is to convert to the same index.

The last stage of Pólya's Problem Solving Model is *Look Back*.

1. By looking back at Worked Examples 15 and 23-25, discuss with your classmates which of the methods can be used for solving different types of problems.
2. By looking back at Worked Examples 1, 2 and 26, discuss the same for exponential equations.

Worked Example 27

(Simultaneous Logarithmic Equations)

Solve the simultaneous equations

$\log_5 (2x - y) = 1$ and $\lg x + \lg y - \lg 3 = 2 \lg 2$.

Solution

$$\log_5 (2x - y) = 1$$
$$2x - y = 5^1$$
$$2x - y = 5$$
$$y = 2x - 5 \text{ --- (1)}$$

and

$$\lg x + \lg y - \lg 3 = 2 \lg 2$$
$$\lg\left(\frac{x \times y}{3}\right) = \lg 2^2$$
$$\frac{xy}{3} = 4$$
$$xy = 12 \text{ --- (2)}$$

Subst. (1) into (2): $x(2x - 5) = 12$

$$2x^2 - 5x - 12 = 0$$
$$(2x + 3)(x - 4) = 0$$

$x = -\frac{3}{2}$ (rejected, $\lg\left(-\frac{3}{2}\right)$ is not defined) or $x = 4$

$y = 2(4) - 5 = 3$

Check: Subst. $x = 4$, $y = 3$ into original equations:

$\log_5 (2x - y) = \log_5 5 = 1$;

$\lg x + \lg y - \lg 3 = \lg 4 + \lg 3 - \lg 3 = \lg 4 = \lg 2^2 = 2 \lg 2$.

Therefore, the solution is $x = 4$ and $y = 3$.

Practise Now 27

Similar Questions:
Exercise 4E
Questions 4(a)-(c), 11

1. Solve the simultaneous equations

$$\log_4 (x - 2y) = 0 \quad \text{and} \quad \lg (x - 1) = 2 \lg y.$$

2. Solve the simultaneous equations

$$\log_3 (3x - y + 5) = 2 \quad \text{and} \quad \ln (11 - y) - \ln x = \ln 2.$$

Basic Level | Intermediate Level | Advanced Level

Exercise 4E

excluded from the N(A) syllabus

1 Solve each of the following equations.

(a) $e^{5-4x} = 3$ **(b)** $10^{y+1} = e^2 \lg e$

(c) $4^{x^2-8} = 7$ **(d)** $e^{2x} - 3e^x - 4 = 0$

2 Solve each of the following equations.

(a) $\log_7 (x + 2) + \log_7 (x + 4) = \log_7 (2x + 5)$

(b) $\ln (3 + 8z - z^2) - \ln (z - 2) = \ln (z + 1)$

(c) $\lg (2y + 6) - \lg (y - 3) = 3 \lg 2$

3 Solve each of the following equations.

(a) $2 \lg x - \lg (x + 60) = 1$

(b) $\ln y^2 = \ln (y + 3) + 2$

(c) $2 + \log_5 (3k - 1) = \log_5 (3k + 11)$

(d) $\log_x 32 = 3 - \log_x 2$

4 Solve each of the following pairs of simultaneous equations.

(a) $\lg (15 - 6x) - \lg y = 2 \lg 3$ and
$\log_2 (2x - y + 15) = 3$

(b) $\log_7 (3a - 5b) = 0$ and
$2 + 2 \log_5 b = \log_5 3(3 - 4a)$

(c) $2^p \times 4^q = 4$ and
$\log_4 (3q + 11) - \log_4 p = 0.5$

5 Solve $\ln (x - 4) = \ln x - 4$, leaving your answer in terms of e.

6 Solve each of the following equations.

(a) $\log_3 x = 64 \log_x 3$

(b) $\log_4 y + \log_{16} y = 8$

(c) $4 \log_5 z - \log_z 5 = 3$

(d) $3 \ln p - 2 \log_p e = 4$

7 Solve each of the following equations.

(a) $3^x(5^{2x}) = 6(2^x)$ **(b)** $5^{y^2-2y} = e$

(c) $3(100^x) - 4(10^x) = 5$ **(d)** $3(2^x) - 2^{-x} = 8$

(e) $16^k + 4^{k-1} = \dfrac{1}{4^3}$ **(f)** $2(7^y) = 3 - 5\sqrt{7^y}$

8 Given that $7^x = 2$ and $2^y = 7$, find the value of xy without using a calculator.

9 If a, $x > 0$ and $a \neq 1$, show that $a^{\log_a x} = x$.

10 Given that $2 \log_2 y = 4 + \log_2 \sqrt{x}$, express y in terms of x.

11 Solve the simultaneous equations

$$2 \log_x y + 2 \log_y x = 5 \quad \text{and} \quad xy = 8.$$

12 If $\ln r$ and $\ln s$ are the roots of the equation $4x^2 - 2x + 5 = 0$, find the value of each of the following.

(a) $\ln r + \ln s$ **(b)** $\ln r \times \ln s$

(c) $\dfrac{1}{\ln r} + \dfrac{1}{\ln s}$ **(d)** $\log_r s + \log_s r$

4.6 GRAPHS OF EXPONENTIAL AND LOGARITHMIC FUNCTIONS

excluded from the N(A) syllabus

Graphs Of Exponential Functions

We have learnt how to solve exponential equations in Sections 4.1 and 4.5.
In this section, we will study the graphs of **exponential functions** $y = a^x$, where a is a positive constant and $a \neq 1$.
What happens if base $a = 1$? Then $y = a^x = 1$ is just a horizontal line.
If base $a = \mathrm{e}$, then $y = \mathrm{e}^x$ is called the **natural exponential function**.

Investigation

Graphs of Exponential Functions

ATTENTION

Search on the Internet: free graphing software.
For $y = 2^x$, type $y = 2^\wedge x$;
for $y = \mathrm{e}^x$, type $y = \exp(x)$.

1. Use a graphing software to plot the following graphs:

(a) Graph A: $y = 2^x$ **(b)** Graph B: $y = 3^x$
(c) Graph C: $y = 4^x$ **(d)** Graph D: $y = \mathrm{e}^x$

2. Answer the following questions about Graphs A to D.

(i) State the coordinates of the point that lies on all the four graphs.
(ii) As x increases, does y increase or decrease? State whether the functions are increasing or decreasing functions.
(iii) What happens to the value of y as x approaches 0? As x continues to decrease, what happens to the value of y? Do the graphs intersect the x-axis?
(iv) In which quadrant(s) do the graphs lie? Can we say that $y = a^x > 0$ for all values of x? Use your graph to explain why the equation $\mathrm{e}^x = 0$ has no solution.
(v) What happens to the position or the shape of the graph as the value of a increases?

3. Use a graphing software to plot the following graphs:

(a) Graph E: $y = \left(\frac{1}{2}\right)^x$ **(b)** Graph F: $y = \left(\frac{1}{3}\right)^x$
(c) Graph G: $y = 2^{-x}$ **(d)** Graph H: $y = 3^{-x}$

4. Answer the following questions about Graphs E and F.

(i) State the coordinates of the point that lies on both the graphs.
(ii) As x increases, does y increase or decrease? State whether the functions are increasing or decreasing functions.
(iii) What happens to the value of y as x approaches 0? As x continues to increase, what happens to the value of y? Do the graphs intersect the x-axis?
(iv) In which quadrant(s) do the graphs lie? Can we say that $y = a^x > 0$ for all values of x? Use your graph to explain why the equation $a^x = 0$ has no solution.
(v) What happens to the position or the shape of the graph as the value of a increases?

5. Examine the relationship between Graph G, Graph E and Graph A.

(i) Which graph is the same as Graph G? Explain your answer using the equations of the graphs and the Laws of Indices.

(ii) Which graph is the reflection of Graph G in the y-axis?

6. Examine the relationship between Graph H, Graph F and Graph B.

(i) Which graph is the same as Graph H? Explain your answer using the equations of the graphs and the Laws of Indices.

(ii) Which graph is the reflection of Graph H in the y-axis?

Worked Example 28

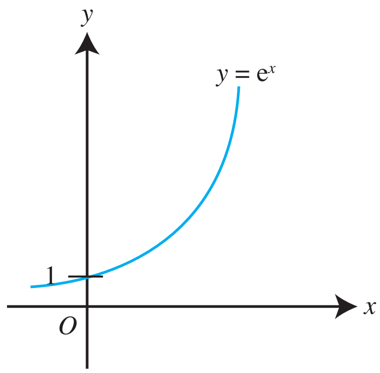
(Sketch Exponential Graph and Solve Equation Graphically)

(i) Sketch the graph of $y = e^x$.

(ii) In order to solve the equation $\frac{1}{2}x = \ln\sqrt{5-x}$, a graph of a suitable straight line has to be drawn on the same set of axes as the graph of $y = e^x$. Find the equation of the straight line and the number of solutions.

Solution

(i)



(ii)

$$\frac{1}{2}x = \ln\sqrt{5-x}$$

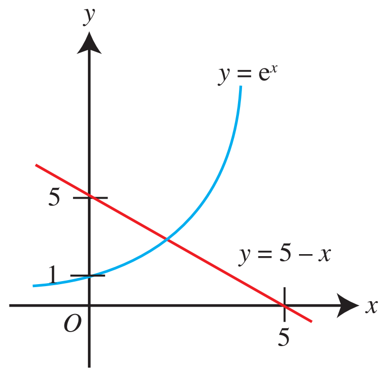
$$\frac{1}{2}x = \ln(5-x)^{\frac{1}{2}}$$

$$\frac{1}{2}x = \frac{1}{2}\ln(5-x)$$

$$x = \ln(5-x)$$

$$e^x = 5 - x$$

$\therefore$ The equation of the straight line is $y = 5 - x$.



From the sketch, the number of solutions of $\frac{1}{2}x = \ln\sqrt{5-x}$ is 1.

Practise Now 28

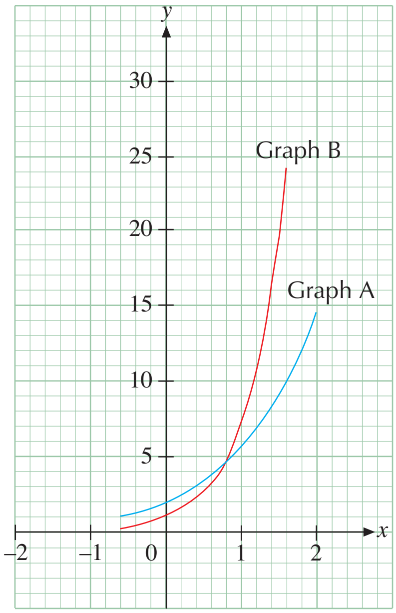
Similar Questions:
Exercise 4F
Questions 1(a)-(d)
2-4

1. (i) Sketch the graph of $y = 4^{-x}$.

(ii) In order to solve the equation $\log_4\left(\frac{1}{\sqrt{x-3}}\right) = \frac{1}{2}x$, a suitable straight line has to be drawn on the same set of axes as the graph of $y = 4^{-x}$. Find the equation of the straight line and the number of solutions.

2. The diagram below shows the graphs of $y = e^{2x}$ and $y = 2e^{x}$.

(i) Identify each graph and explain your answer.



(ii) By drawing a suitable line on the diagram, solve each of the following equations.

(a) $2e^{x} = -1$

(b) $2x - \ln 5 = \ln(2x + 1)$

Graphs Of Logarithmic Functions

We will now learn how to plot the graph of the **logarithmic function** $y = \log_a x$, where base $a > 0$ and base $a \neq 1$.

If base $a = e$, then $y = \ln x$ is called the **natural logarithmic function**.

INFORMATION

We have learnt in Section 4.3 that $y = a^x$ is equivalent to $x = \log_a y$. Thus, if we plot y against x for $x = \log_a y$, we will get the same graph as $y = a^x$.

Investigation

Graphs of Logarithmic Functions

ATTENTION

For $y = \lg x$, you may have to type $y = \log x$.
For $y = \log_2 x$, you need to use the Change of Base Formula and type:
$y = (\ln x) / (\ln 2)$,
or $y = (\log x) / (\log 2)$.

INFORMATION

For base a such that $0 < a < 1$, see Ex 4F Q8.

1. Use a graphing software to plot each of the following graphs.
(a) Graph A: $y = \lg x$ **(b)** Graph B: $y = \log_2 x$ **(c)** Graph C: $y = \ln x$

2. Answer the following questions about Graphs A to C.
(i) State the coordinates of the point that lies on all the three graphs.
(ii) As x increases, does y increase or decrease for all the four graphs? State whether the functions are increasing or decreasing functions.
(iii) What happens to the value of y as x approaches 1? As x continues to decrease, what happens to the value of y? Do the graphs intersect the y-axis?
(iv) In which quadrant(s) do the graphs lie? What can we say about the range of values of x for which $y = \log_a x$ is defined? Use your graph to explain why $y = \log_a x$ is not defined when $x = 0$.
(v) What happens to the position or the shape of the graph as the value of a (in $y = \log_a x$) increases?

3. Use a graphing software to plot each of the following graphs.
(a) Graph C: $y = \ln x$ **(b)** Graph D: $y = e^x$

4. What is the relationship between Graph C and Graph D?

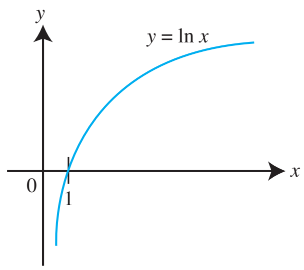
Worked Example 29

(Sketch Logarithmic Graph and Solve Equation Graphically)

(i) Sketch the graph of $y = \ln x$.
(ii) In order to solve the equation $e^{x-3} = x^2$, a graph of a suitable straight line has to be drawn on the same set of axes as the graph of $y = \ln x$. Find the equation of the straight line and the number of solutions.

Solutions

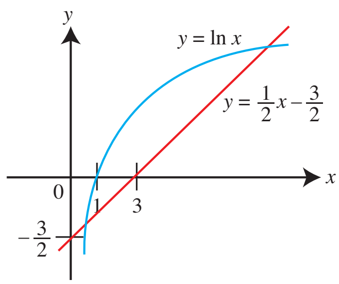
(i)



(ii)

$$e^{x-3} = x^2$$
$$\ln x^2 = x - 3$$
$$2\ln x = x - 3$$
$$\ln x = \frac{1}{2}x - \frac{3}{2}$$

$\therefore$ The equation of the straight line is $y = \frac{1}{2}x - \frac{3}{2}$.



From the sketch, the number of solutions of $e^{x-3} = x^2$ is 2.

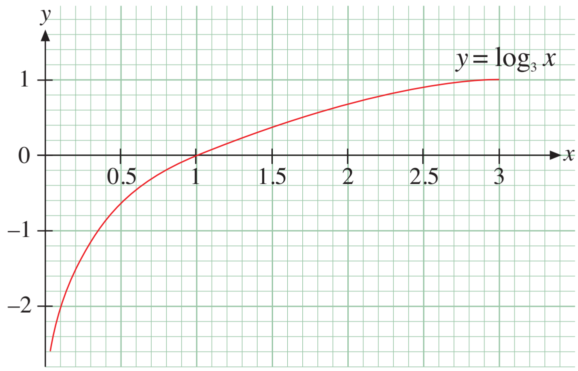
Practise Now 29

Similar Questions:
Exercise 4F
Questions 1(e)-(f), 5-8

★ **1.** **(i)** Sketch the graph of $y = \lg x$.

(ii) In order to solve the equation $10^x = \dfrac{10\,000}{x^3}$, a suitable straight line has to be drawn on the same set of axes as the graph of $y = \lg x$. Find the equation of the straight line and the number of solutions.

★ **2.** The diagram below shows the graph of $y = \log_3 x$. By drawing a suitable line on the diagram, solve the equation $9^x = 3x$.



Basic Level | Intermediate Level | Advanced Level

excluded from the N(A) syllabus ★

Exercise 4F

1 Sketch the graph of each of the following functions on a separate diagram.

(a) $y = 5^x$ **(b)** $y = \left(\dfrac{1}{6}\right)^x$ **(c)** $y = 7^{-x}$ **(d)** $y = 0.25^{-x}$ **(e)** $y = \log_5 x$ **(f)** $y = \log_{1.5} x$

2 **(i)** Sketch each of the following graphs on the same diagram.

(a) $y = e^x$

(b) $y = e^{-x}$

(ii) What is the relationship between the two graphs?

3 **(i)** Sketch the graph of $y = \dfrac{1}{10^x}$.

(ii) In order to solve the equation $x = \lg\left(\dfrac{1}{3-2x}\right)$, a suitable straight line has to be drawn on the same set of axes as the graph of $y = \dfrac{1}{10^x}$. Find the equation of the straight line and the number of solutions.

Basic Level | Intermediate Level | Advanced Level

Exercise 4F

4 Plot the graphs of $y = e^x$ for $-0.5 \leqslant x \leqslant 3$ on the same graph paper using a scale of 5 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis.
By drawing a suitable line on the graph paper, solve each of the following equations.
(i) $3e^x = -2$
(ii) $2 \ln \sqrt{5x+2} = x$

5 **(i)** Sketch each of the following graphs on the same diagram.
(a) $y = 10^x$
(b) $y = \lg x$
(ii) What is the relationship between the two graphs?

6 **(i)** Sketch the graph of $y = \ln x$.
(ii) In order to solve the equation $e^{ex} = xe^4$, a suitable straight line has to be drawn on the same set of axes as the graph of $y = \ln x$. Find the equation of the straight line and the number of solutions.

7 Plot the graph of $y = \log_4 x$ for $0 \leqslant x \leqslant 8$ using a scale of 2 cm to represent 1 unit on the x-axis and 5 cm to represent 1 unit on the y-axis.
By drawing a suitable line on the graph paper, solve the equation $(16^x)x = 4^7$.
Hint: To find the values of $y = \log_4 x$, you have to use the Change of Base Formula.

8 Let $f(x) = \log_a x$ and $g(x) = \log_{\frac{1}{a}} x$, where base $a > 1$, i.e. $0 < \frac{1}{a} < 1$.
(i) Express $g(x)$ in terms of $f(x)$.
(ii) Hence, sketch the graphs of $f(x)$ and $g(x)$ on the same diagram.
(iii) What is the relationship between the two graphs?

4.7 APPLICATIONS OF LOGARITHMS AND EXPONENTIALS

excluded from the N(A) syllabus

In this section, we will look at some examples of how logarithmic and exponential functions have been applied in real-world situations.

Earthquakes

The trigger of this chapter mentions an earthquake in Japan on 11 Mar 2011 that measured 9.0 on the Richter scale and an earthquake in New Zealand on 3 Sep 2010 that measured 7.0 on the Richter scale. We will now try to answer the three questions posed there.

Investigation

Real-life Applications of Logarithms

The Richter scale is a measurement of earthquake magnitudes based on the formula $R = \lg\left(\dfrac{x}{0.001}\right)$, where x is the intensity of the earthquake as registered on a seismograph.

1. Make x the subject of the formula $R = \lg\left(\dfrac{x}{0.001}\right)$.
Hint: Convert to exponential form.

2. Calculate the intensities x for both the 9.0-magnitude earthquake in Japan and the 7.0-magnitude earthquake in New Zealand.

3. How much greater in intensity is the 9.0-magnitude earthquake as compared to the 7.0-magnitude earthquake? Is it twice the magnitude?

4. Suggest an easier way to compare the magnitudes of the two earthquakes without calculating their intensities.

5. The strongest earthquake ever recorded on the Richter scale was 9.5 on 22 May 1960 in Chile. Using the method in Question 4, find out how much greater in intensity this earthquake is as compared to the 9.0-magnitude earthquake in Japan.

Just For Fun

A newspaper article on 16 Jan 2005 reported that most major earthquakes took place on the 26th of the month:
- On 26 Dec 2004, a 9.0-magnitude earthquake occurred in Indonesia.
- On 26 Jan 2001, a 7.9-magnitude earthquake in India killed 20 000 people.
- On 26 Dec 2003, 40 000 people died in the 6.3-magnitude earthquake in Iran.

However, critics said that there were major earthquakes occurring on other days of the month as well. Search the Internet to find out whether there are major earthquakes on other days of the month.

6. The range of the Richter scale is from 1 to 10. Find the range of the intensity x.

7. Why do we use logarithms in the Richter scale? Why can't we use a normal scale based on the intensity x?
Hint: What happens if you take log of a very big or a very small number in Question 6?

In general, the effect of taking the logarithm makes a large number smaller. This is one of the uses of logarithms. Other applications in the real world include pH scale in **Chemistry**, radioactive decay in **Physics**, sound intensity in **Physics**, carbon dating in **Archaeology**, interest compounded continuously in **Banking**, brightness of stars in **Astronomy** and Moore's Law in **Computer Engineering**.

Some scientific or real-world phenomena can be modelled by logarithmic or exponential functions. Can you give examples of such phenomena? Why are these functions used?

Practise Now

Similar Questions:
Exercise 4G
Questions 2, 3, 6, 7

1. **Radioactive Decay.** A radioactive isotope, Krypton-85 (Kr-85), will decay to half its original amount in 10.72 years. The amount of Kr-85 left, A g, is given by the equation $A = A_0 e^{-0.064\,66t}$, where t is the time in years. A sample of Kr-85 has a mass of 43.9 g after 2 years.

(i) Find the initial mass of the sample.

(ii) What percentage of the sample has decayed after 10 years?

(iii) After how many years will there be only one-quarter of the sample left?

INTERNET RESOURCES

The half-life of Kr-85 is 10.72 years. The actual formula for radioactive decay depends on this half-life. Search the Internet to find out more about half-life and the actual formula.

2. **pH value.** pH measures how acidic or alkaline a solution is. Pure water is neutral with a pH of 7. Solutions with a pH of less than 7 are acidic while solutions with a pH of more than 7 are alkaline. The formula for pH is: $\text{pH} = -\lg (a_{\text{H}^+})$, where a_{H^+} is the activity of hydrogen ions in units of moles per litre.

   (i) In a solution, $a_{\text{H}^+} = 0.03$. Find its pH. Is the solution acidic or alkaline?

   (ii) An alkaline solution has a pH of 12.7. Find a_{H^+}.

   (iii) Suppose solution X has twice the value of a_{H^+} as solution Y, what is the difference in their pH values?

Some phenomena in the sciences or in the real world can be described by a mathematical equation. For example, bacteria can multiply at an alarming rate and so their growth can be *modelled* using an exponential equation.

Class Discussion

The data in the table shows the background count rate of a decaying radioactive isotope.

Time, t (hours)	0	0.5	1.0	1.5	2.0	2.5	3.0
Counts, x (per minute)	260	140	75	40	20	11	6

(i) Using ICT, generate a scatter plot for the data provided.

(ii) Suggest a logarithmic or exponential function that may be used to model the data.

(iii) It is believed that an equation of the form $x = ka^{-t}$ models the data. Find possible values of k and a that fit the data above. Hence, write down a function, in terms of x and t, that may be used to model the data.

Worked Example 30

(Exponential Growth)

The growth of some bacteria, N, in a petri dish after t days, can be modelled by the exponential equation $N = N_0 a^t$, where N_0 and a are constants.

(i) It is discovered that the initial number of bacteria is 17. Find the value of N_0.

(ii) After 2 days, there are 68 bacteria. Calculate the value of a.

(iii) Given that the number of bacteria exceeds 300 after k days, where k is an integer, find the value of k.

(iv) Predict the number of bacteria after 6 days.

(v) After 6 days, the number of bacteria is found to be only 986. Suggest two reasons why the actual number of bacteria is different from the number predicted by the equation.

Solution

(i) When $t = 0$, $N = 17$. So $17 = N_0 a^0$

$$\therefore \quad N_0 = 17$$

(ii) When $t = 2$, $N = 68$. So $68 = 17a^2$

$$a^2 = 4$$

$$a = \pm 2$$

If $a = -2$, then N will be negative for odd values of t. However, the number of bacteria, N, cannot be negative.

$\therefore a = 2$

(iii) The equation is now $N = 17(2^t)$.

When $N = 300$, $17(2^t) = 300$

$$2^t = \frac{300}{17}$$

$$\ln 2^t = \ln \frac{300}{17}$$

$$t \ln 2 = \ln \frac{300}{17}$$

$$t = \ln \frac{300}{17} \div \ln 2$$

$$= 4.141 \text{ (to 4 s.f.)}$$

For N to *exceed* 300, $k = 5$.

(iv) When $t = 6$, $N = 17(2^6)$

$$= 1088$$

(v) The actual number of bacteria after 6 days is only 986, which is $1088 - 986 = 102$ fewer than the number of bacteria predicted by the equation.

First reason : Error in data collection (counting of bacteria) but the difference is too big in this case for this to be a good explanation.

Second reason : The modelling equation $N = 17(2^t)$ is not entirely correct. It may be correct when t is small, but in real life, bacteria cannot grow exponentially forever. The data shows that the growth of the bacteria has started to slow down when $t = 6$.

Thinking Time

For Worked Example 30, there is another equation that can fit the same given data:

$$N = \frac{70\,000}{7 + 2^{12-t}}$$

Find the value of N when

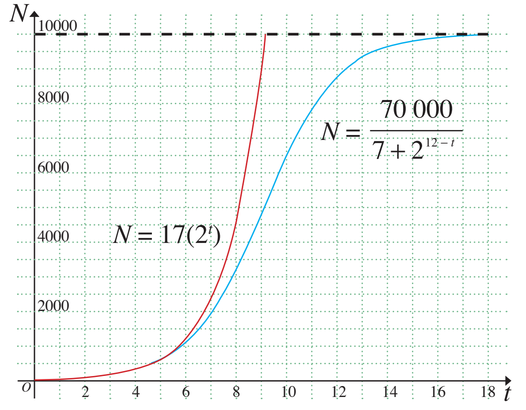
(i) $t = 0$, **(ii)** $t = 2$, **(iii)** $t = 6$.

Do these values of N fit the given data in Worked Example 29?

The figure below shows the two models in the same graph. In the first model, $N = 17(2^t)$, the bacteria grow exponentially forever.

In the second model, $N = \dfrac{70\,000}{7 + 2^{12-t}}$, the bacteria grow exponentially at the beginning, then its growth starts to slow down until the number of bacteria reaches 10 000.

Notice that the two curves appear to overlap very closely from $t = 0$ to 4, and that they start to separate when $t = 6$.



Two Models for Growth of Bacteria

INFORMATION

The curve in the second model is called a **logistic curve**.

In real life, the number of bacteria may start to decrease after some time. Another model (exponential growth followed by decay) is explained in Exercise 4G Question 5.

Practise Now 30

Similar Questions:
Exercise 4G
Questions 1, 4, 5, 8

Depreciation of Value of Asset. A man bought a new car. The value, $\$V$, of the car will depreciate so that after t months, its value can be modelled by $V = 50\,000e^{-kt}$, where k is a constant. The value of the car after 12 months is expected to be \$40 000.

(i) Find the initial value of the car.

(ii) Calculate the expected value of the car after 20 months.

(iii) Given that the value of the car first drops below \$20 000 after n months, where n is an integer, find the value of n.

Exercise 4G

excluded from the N(A) syllabus ★

1 **Exponential Decrease.** The population of a species of insects in a colony, N insects, after t days, can be modelled by $N = 10\,000e^{-0.2t}$.

(i) Find the initial population of the insects.

(ii) Estimate the number of insects after 5 days.

(iii) When will the colony first decrease to 1000 insects?

2 **Sound Intensity.** Loudness of Sound is a subjective measure of sound strength because it varies from person to person. A more objective measure is called the sound intensity **level** $L = 10 \lg \frac{I}{I_0}$, where I is the sound intensity in W/m² (i.e., watts per m²) and I_0 is the minimum sound intensity in W/m² that can be heard by a human. The unit of L is dB (decibel).

(i) What is the smallest value of L that can be heard by a human?

(ii) Given that $L = 90$ when $I = 10^{-3}$, find the minimum sound intensity that can be heard by a human.

(iii) Find the sound intensity level of a refrigerator if its sound intensity is 2×10^{-8} W/m².

(iv) Calculate the sound intensity if the sound intensity level of an airplane taking off is 140 dB.

INTERNET RESOURCES

What are the sound intensity levels of some common phenomena such as a normal conversation or a rock concert? Search the Internet to find out more.

3 **Carbon Dating** is a method used by archaeologists to find out how old a fossil is. When an animal dies, the amount of the radioactive isotope carbon-14 (C-14) present in its body will start to decay. The amount of C-14, A g, that remains in a fossil t years after its death is given by $A = 74e^{-kt}$, where k is a constant. The fossil was found to contain 22.07 g of C-14 when $t = 10\,000$.

(i) What was the original mass of C-14 when the animal died?

(ii) Find the value of k.

(iii) The mass of C-14 is found to be 21 g now. How old is the fossil? Leave your answer correct to the nearest 100 years.

INFORMATION

Archaeologists can determine the original amount of C-14 at the time of death because the percentage of C-14 to carbon-12 (C-12) present in the body is fixed, and the amount of non-radioactive C-12 will remain the same after the organism dies.

4 **Appreciation of Value of Asset.** The value, $\$V$, of a flat can be modelled by $V = V_0e^{pt}$, where t is the time in years since it was built and p is a constant. Calculate

(i) the value of p if, after 5 years, the value of the flat is doubled,

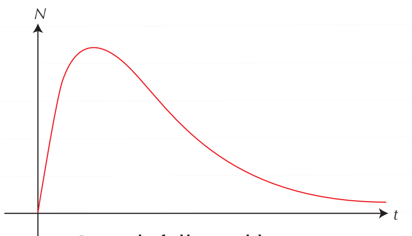
(ii) the age of the flat when the value of the flat is 5 times its original value.

5 **Exponential Growth Followed by Decay.** The number of people, N, living in a town increases to a maximum of 1226 after $3\frac{1}{3}$ years and then starts to decrease. The population growth after t years can be modelled by $N = 1000te^{-at}$, where a is a constant. The figure below shows the graph of this equation.

(i) Find the initial population of the town.

(ii) What is the population of the town after 2 years?

(iii) As the number of years increases, what can you say about the population?



Growth followed by Decay

6 **Compound Interest.** The compound interest formula $A = P\left(1 + \frac{R}{100}\right)^n$ is actually an exponential equation. Instead of compounding interest yearly, monthly or daily, what happens if interest is compounded every hour, minute or second? The general formula for the final amount A, if interest is compounded continuously, is $A = Pe^{\frac{RT}{100}}$, where P is the principal, R % is the annual interest rate and T is the time in years.

Betty wants to put a fixed deposit of \$1000 in a bank for 3 years.

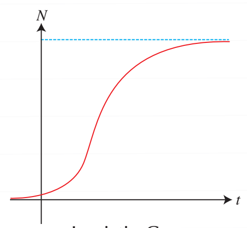
Bank X offers an annual interest of 2% compounded continuously while Bank Y offers an annual simple interest rate of 3%.

(i) Which bank should Betty put her money in? Explain.

(ii) Which has a bigger effect: the interest rate, or whether interest is simple or compounded continuously? Why?

7 **Logistic Curve.** A company has 50 customers initially. The number of customers, N, starts to increase exponentially so that there are 250 customers after 2 years. Then it becomes stable as shown by the logistic curve. The equation of this curve is given by

$N = \dfrac{k}{a + 2^{7-1.5t}}$, where t is the time in years, and a and k are constants.



Logistic Curve

(i) Find the value of a and of k.

(ii) How many customers does the company have after 5 years?

(iii) When does the company have 500 customers?

(iv) What happens to the number of customers in the long run?

Exercise 4G

8 **Brightness of Stars.** There are two measures to describe the brightness of a star. The first measure is the apparent magnitude m (the smaller m is, the brighter the star) as seen by people on Earth. For example, Sirius is the brightest star in the night sky with $m = -1.46$. The second measure is the absolute magnitude M, which is the actual brightness of the star. For example, Polaris (the North Star) has a bigger absolute magnitude than Sirius, but it appears less bright than Sirius because it is further away from the Earth. The absolute magnitude M of a star is given by $M = m + 5 - 5\lg d$, where m is the apparent magnitude of the star and d is the distance of the star from the Earth measured in parsecs (1 parsec $\approx 30.857 \times 10^{12}$ km).

(i) Given that Sirius is at a distance of 2.64 parsecs from the Earth, calculate its absolute magnitude, leaving your answer correct to 2 significant figures.

(ii) If the absolute and apparent magnitudes of Polaris are -3.6 and 1.97 respectively, find its distance from the Earth in km, leaving your answer in standard form. How far is Polaris away from the Earth as compared to Sirius from the Earth?

(iii) Suppose two stars have the same absolute magnitude but one star is twice as far away from the Earth as the other star. Find the difference in their apparent magnitudes, leaving your answer in logarithmic form.

(iv) Star A has an absolute magnitude twice that of Star B, but Star B has an apparent magnitude twice that of Star A. If the sum of the apparent magnitude of Star B and twice the absolute magnitude of Star B is 4, find the ratio of the distance of Star A from the Earth to the distance of Star B from the Earth. Hence, determine which star is further away from the Earth.

INTERNET RESOURCES

Search on the Internet 'distances to astronomical objects' to find out how astronomers calculate the distance of a star from the Earth. There are a few methods. For faraway stars, they use this formula $M = m + 5 - 5\lg d$ since there are other ways to find M and m (see part ii). Besides parsecs, distances in astronomy are also measured in light years and astronomical units. Find out more about these units. Which unit is the biggest and which unit is the smallest?

9 The data in the table lists the population, y thousand, of a small town at 20-week intervals.

Week number, t	0	20	40	60	80	100
Population, y thousand	1.6	4.3	4.9	5.3	5.6	5.8

(i) Using ICT, generate a scatter plot for the data provided.

(ii) Suggest a logarithmic or exponential function that may be used to model the data.

(iii) It is believed that a function of the form $y = \ln(at + b) + c$ models the data. Hence, estimate the population of the town in the 120th week.

SUMMARY

index

$y = a^x$ is equivalent to $x = \log_a y$ if $a > 0, a \neq 1$

base

ATTENTION

$y > 0$; index x can be any real number

Laws of Indices, Surds and Logarithms

The following laws are true for bases a, $b > 0$, and for logarithms, bases a, $b \neq 1$ (although some of these laws are also true under some other conditions).

Laws of Surds		Laws of Indices		Laws of Logarithms
		Law 1: $a^m \times a^n = a^{m+n}$	$\longleftrightarrow$	$\log_a x + \log_a y = \log_a xy$
		Law 2: $\dfrac{a^m}{a^n} = a^{m-n}$	$\longleftrightarrow$	$\log_a x - \log_a y = \log_a \dfrac{x}{y}$
		Law 3: $(a^m)^n = a^{mn}$	$\longleftrightarrow$	$\log_a x^r = r \log_a x$
Law 1: $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$	$\longleftrightarrow$	Law 4: $a^n \times b^n = (ab)^n$		
Law 2: $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$	$\longleftrightarrow$	Law 5: $\dfrac{a^n}{b^n} = \left(\dfrac{a}{b}\right)^n$		
Law 3: $\sqrt{a} \times \sqrt{a} = a$		Law 6: $a^0 = 1$		
		Law 7: $a^{-n} = \dfrac{1}{a^n}$		
		Law 8: $a^{\frac{1}{n}} = \sqrt[n]{a}$		
		Law 9: $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m}$		

Other Properties:

- Product of conjugate surds $p + q\sqrt{a}$ and $p - q\sqrt{a}$ is a rational number
- Change of Base Formula: $\log_a b = \dfrac{\log_c b}{\log_c a}$ (Special Case: $\log_a b = \dfrac{1}{\log_b a}$)
- Special Property 1: $\log_a 1 = 0$ (Special Cases: $\lg 1 = 0$, $\ln 1 = 0$)
- Special Property 2: $\log_a a = 1$ (Special Cases: $\lg 10 = 1$, $\ln e = 1$)
- Special Property 3: $n = n \log_a a$

To Solve Logarithmic Equations:

Method 1: Use Equality of Logarithms

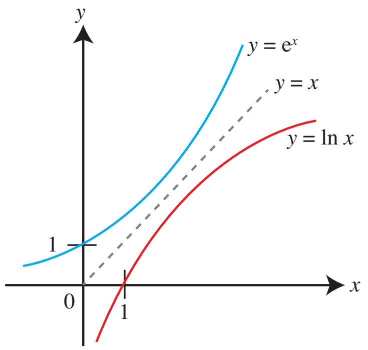
Method 2: Convert log form to exponential form

Method 3: Use Change of Base Formula first

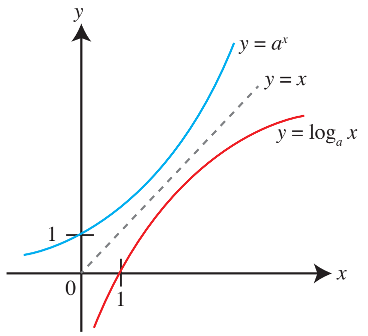
$\longleftrightarrow$ means 'is related to'

To Solve Exponential Equations:
Method 1: Use Equality of Indices
Method 2: Convert exponential form to log form
Method 3: Take log on both sides

Graphs of $y = e^x$ and $y = \ln x$:



Graphs of $y = a^x$ and $y = \log_a x$:



Review Exercise 4

1. If $\log_4 a = x$, $\log_{64} b = y$ and $\dfrac{a}{b} = 2^z$, express z in terms of x and y.

2. Solve each of the following equations.
   (a) $2^{4t+1} + 5(4^t) = 3$
   (b) $6^{2p-3} = 5$
   (c) $2(100^y) - 10^y - 6 = 0$
   (d) $\log_4(x+8) - \log_4(x-4) = 2\log_4 3$
   (e) $\log_2(2x-2) = 3 + \log_2(4x-9)$
   (f) $2\log_3 x - \log_{27} x = 10$

3. Solve the simultaneous equations
   $2^x \times 8^y = \dfrac{1}{2}$ and $\dfrac{9^{x-3}}{3^{1-y}} = 3$.

4. Given that $\sqrt{p + q\sqrt{7}} = \dfrac{5}{(3-\sqrt{7})^2}$, where p and q are rational numbers, find the value of p and of q.

5. The solution of the equation $x\sqrt{5} = \sqrt{27} - x\sqrt{3}$ is $\dfrac{a + b\sqrt{15}}{2}$.
   Without using a calculator, find the values of the integers a and b.

6. A right circular cylinder has a volume of $(8 + 3\sqrt{6})\pi$ cm^3 and a base radius of $(2 + \sqrt{6})$ cm. Find its height in the form $(p + q\sqrt{6})$ cm, where p and q are rational numbers.

7. **(i)** By using the substitution $y = 2^x$, express the equation $2^{3x+1} + 7(2^x) = 4 - 4^x$ as a cubic equation in y.
   (ii) Show that $y = \dfrac{1}{2}$ is the only real solution of this equation.
   (iii) Hence solve the equation $2^{3x+1} + 7(2^x) = 4 - 4^x$.

8. If $\log_{15} 3 = p$ and $\log_7 5 = q$, express $\log_7 3$ in terms of p and q.

9. **(i)** Plot the graph of $y = \ln x^2$ for $-4 \leqslant x \leqslant 4$ using a scale of 2 cm to represent 1 unit on both the x-axis and the y-axis.

(ii) Plot the graph of $y = e^{\frac{x}{2}}$ on the same graph paper.

(iii) What is the relationship between the two graphs? Why is this so?

10. **Moore's Law.** Dr. Gordan Moore, co-founder of Intel, speculated in 1965 that the number of transistors in an integrated circuit would double every two years. His speculation, now known as Moore's Law, can be expressed as $N = N_0(2^{0.5t})$, where N is the number of transistors, t is the number of years after 1965 and N_0 is a constant.

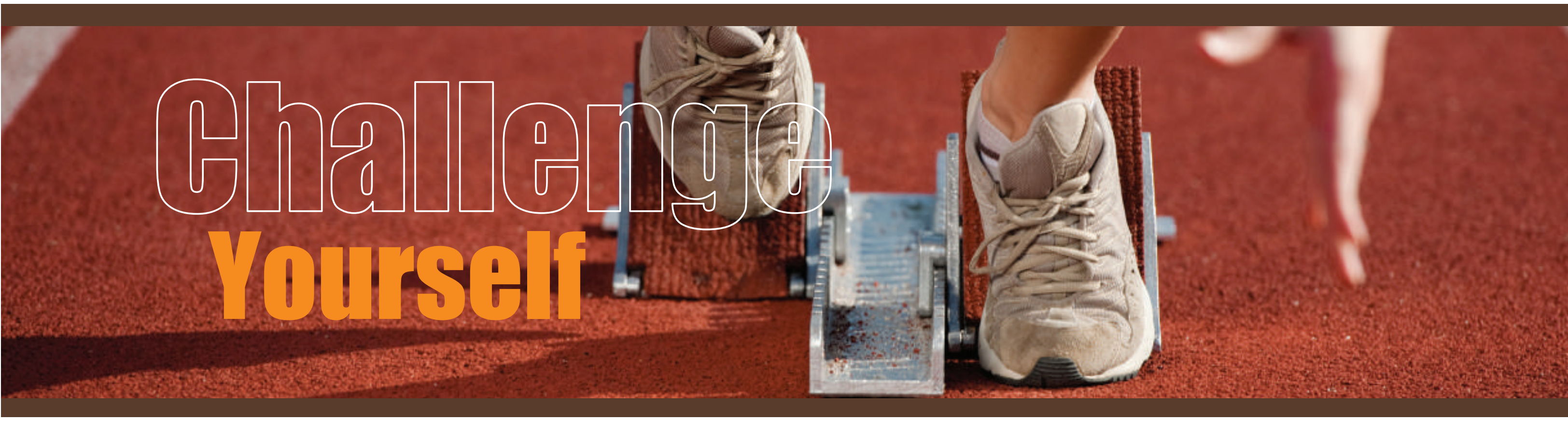
(i) In 1965, a silicon chip contained 240 transistors. Find the value of N_0.

(ii) Find the year in which the number of transistors will exceed one million.

(iii) According to Moore's Law, how many transistors would there be on an integrated circuit in 2013? Express your answer to the nearest billion.

INTERNET RESOURCES

Read more about Moore's Law on the Internet.



Challenge Yourself

1. If $2^{2x} = 5^y = 10^{3z}$, show that $2xy - 3yz - 6xz = 0$.

2. **(i)** Find a pair of different positive integers a and b such that $a^b = b^a$.

(ii) Show that there is only one such pair of different positive integers that satisfies the above equation.

3. **(i)** If m, n and k are rational numbers, and $k > 0$, prove that $m + n\sqrt{k} = p + q\sqrt{k} \Rightarrow m = p$ and $n = q$.

(ii) Hence, or otherwise, express $\sqrt{34 + 24\sqrt{2}}$ in the form $a + b\sqrt{2}$.

4. **(i)** Find the value of

$$\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \ldots + \frac{1}{\sqrt{2024}+\sqrt{2025}}.$$

(ii) Hence, or otherwise, show that

$$88\frac{1}{45} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{2025}} < 89.$$

5. **(a)** If $10^n \leqslant x < 10^{n+1}$, show that $n \leqslant \lg x < n + 1$.

(b) Hence or otherwise, find the total number of digits in 2013^{2013}.

REVISION EXERCISE B1

1. Find, in ascending powers of x, the first three terms in the expansion of $(1 + 2x)^{12}$. Hence, find the coefficient of x^2 in the expansion of $(2 + 3x + 4x^2)(1 + 2x)^{12}$.

2. In the expansion of $(1 - 2x)^4(1 + 3x)^n$, where n is a positive integer, the coefficient of x^2 is 189. Find the value of n. Hence, find the coefficient of x.

3. Write down and simplify the first three terms in the expansion of $(1 + kx)^6$ in ascending powers of x. Given that the coefficient of x^2 in the expansion of $(1 + 6x + 4x^2)(1 + kx)^6$ is 31, find the possible values of k. Hence, find the coefficient of x in the expansion of $(1 + 6x + 4x^2)(1 + kx)^6$.

4. Solve each of the following equations.
(a) $3^{2x+2} - 15 \times 3^x + 1 = 0$ **(b)** $4^{x^2} - 16^{6-2x} = 0$

5. The area of rectangle $ABCD$ is $(46 + 18\sqrt{3})$ cm² and its width is $(4 + 2\sqrt{3})$ cm. Find its length in the form $(a + b\sqrt{3})$ cm, where a and b are integers.

6. Simplify
(a) $\lg 20 + 7\lg\frac{15}{16} + 5\lg\frac{24}{25} + \lg\left(\frac{80}{81}\right)^3$, **(b)** $\frac{\lg x^2 + \lg y^2}{\lg xy}$.

7. Solve the equation $\lg x = 2 + \lg y = 2\lg\frac{2}{3} - \lg\frac{9}{125} - 2\lg\frac{5}{9}$.

8. Solve each of the following equations.
(a) $\lg(3x + 1) - \lg\frac{4}{5} - \lg x = 1 - \lg(3x - 1)$
(b) $2(\log_5 x)^2 + 3 = 7\log_5 x$

9. Given that $\frac{1}{2}\log_3 x + \log_3 y = 2\log_3 z$, express x in terms of y and z.

10. Sketch the graph of $y = \ln x$ for $x > 0$. Insert on your sketch, the additional straight line required to find the number of solutions of the equation $x^3 = e^{1-2x}$.

REVISION EXERCISE B2

1. Expand, in ascending powers of x, the first three terms of each of the following.

(a) $\left(1+\frac{x}{2}\right)^8$ **(b)** $(3-x)^6$

Hence, find the coefficient of x^2 in the expansion of $\left(1+\frac{x}{2}\right)^8(3-x)^6$.

2. Given that $(1 + px + qx^2)(1 - 2x)^6 = 1 - 9x + 19x^2 + \ldots$, calculate the value of p and of q.

3. Expand $(1 + x + 2ax^2)^n$, where n is a positive integer, in ascending powers of x as far as the term in x^2. Given that the coefficients of x and x^2 are 6 and 27 respectively, find the value of a and of n. Hence, find the coefficient of x^3 in the expansion.

4. Solve each of the following equations.

(a) $5^x = 2(5^{\frac{x}{2}}) + 15$ **(b)** $2(9^{x-1}) - 5(3^x) = 27$

5. Without using a calculator, find the value of k such that

$$\left(\frac{1}{\sqrt{6}} - \frac{\sqrt{24}}{3} + \frac{49}{\sqrt{294}}\right) \times \frac{3}{\sqrt{2}} = k\sqrt{3}\,.$$

6. Given that $\log_a 3 = x$ and $\log_a 5 = y$, express each of the following in terms of x and y.

(a) $\log_a 1\frac{2}{3}$ **(b)** $\log_a \frac{25a^3}{9}$

(c) $\log_a \frac{225}{a^4}$ **(d)** $\log_a \frac{27\sqrt{a}}{125}$

7. Solve the simultaneous equations

$$\log_4 x + \log_4 y = 5,$$
$$(\log_4 x)(\log_4 y) = 6.$$

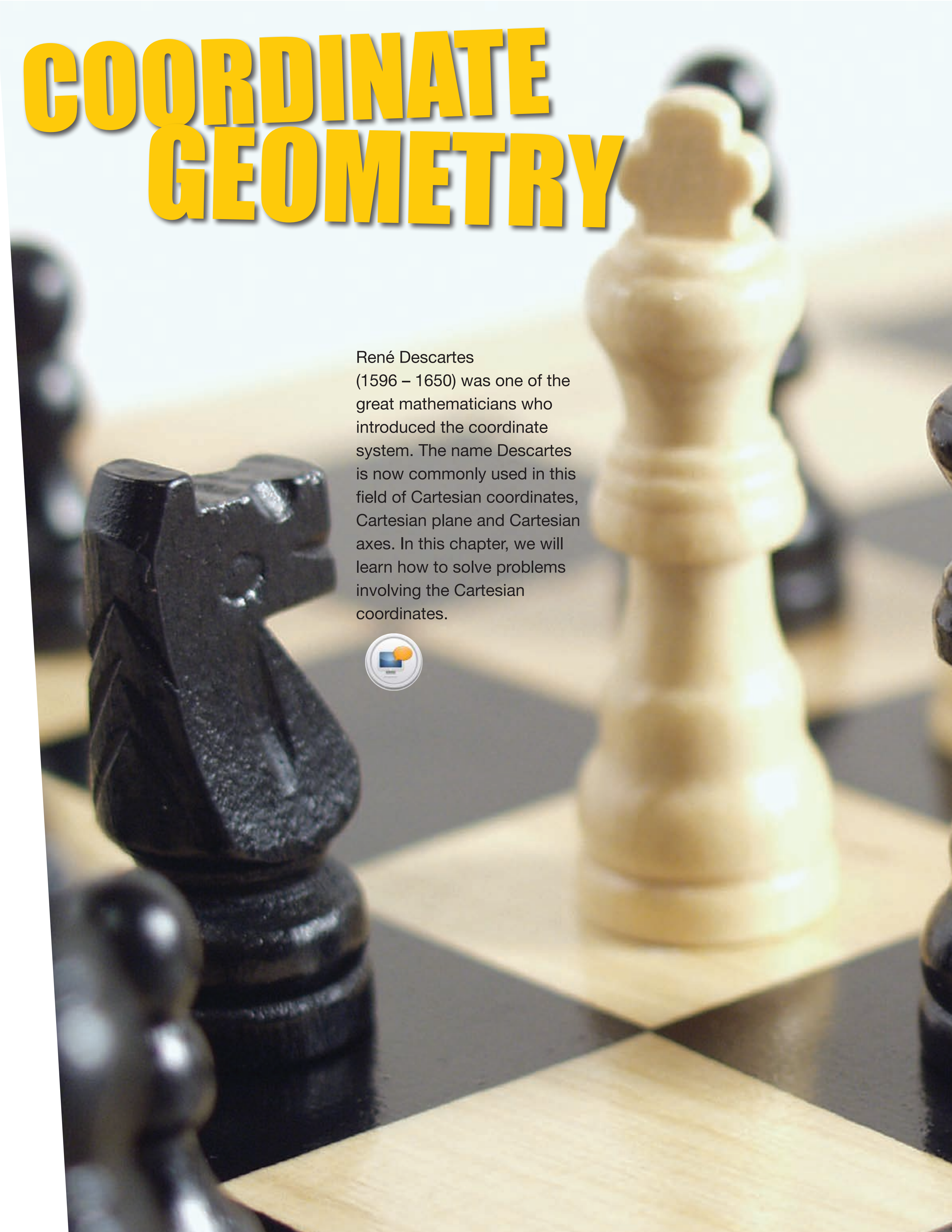
8. Solve each of the following equations.

(a) $\log_3 (4x + 1) = 1 + 2\log_3 2x$ **(b)** $e^x + 15e^{-x} = 8$

9. Given that $2\log_3 a - \log_3 b + 3\log_3 c = 1$, express b in terms of a and c.

10. Draw the graph of $y = e^x$ for $-2 \leqslant x \leqslant 2$. By drawing a suitable straight line on the graph, state the number of solutions of the equation $e^x - x - 2 = 0$.

COORDINATE GEOMETRY



René Descartes (1596 – 1650) was one of the great mathematicians who introduced the coordinate system. The name Descartes is now commonly used in this field of Cartesian coordinates, Cartesian plane and Cartesian axes. In this chapter, we will learn how to solve problems involving the Cartesian coordinates.

CHAPTER 5

Learning Objectives

At the end of this chapter, you should be able to:

- find the midpoint of a line segment,
- find the gradients of parallel lines and the gradients of perpendicular lines,
- find the area of rectilinear figures,
- solve geometry problems.

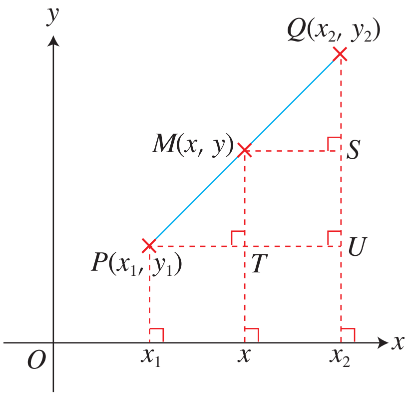
5.1 MIDPOINT OF A LINE SEGMENT

Recap

In O-level mathematics, we have learnt how to calculate the length of a line segment. Given two points $P(x_1, y_1)$ and $Q(x_2, y_2)$, the length of the line segment PQ is given by

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \; .$$

Midpoint Of a Line Segment



Let $M(x, y)$ be the midpoint of PQ. We will now find the coordinates of M in terms of x_1, y_1, x_2 and y_2.

MS and PU are horizontal lines while MT and QU are vertical lines.

Note that the coordinates of T, U and S are (x, y_1), (x_2, y_1) and (x_2, y) respectively.

In ΔPMT and ΔMQS,

$$\angle MPT = \angle QMS \text{ (corr. } \angle\text{s, } PU \,//\, MS),$$

$$\angle MTP = \angle QSM = 90^\circ \text{ and}$$

$$PM = QM.$$

$\therefore \Delta PMT$ and ΔMQS are congruent.

i.e. $\quad PT = MS, \qquad\qquad MT = QS$

$$x - x_1 = x_2 - x, \qquad y - y_1 = y_2 - y$$

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}$$

$\therefore$ The coordinates of the midpoint of $P(x_1, y_1)$ and $Q(x_2, y_2)$ are

$$\boxed{\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).}$$

Worked Example 1

(Midpoint of a Line Segment)

(a) Find the coordinates of the midpoint of the line segment joining

(i) $A(-3, 4)$ and $B(5, 4)$, **(ii)** $B(5, 4)$ and $C(5, 1)$,

(iii) $D(2, 5)$ and $E(-6, -1)$.

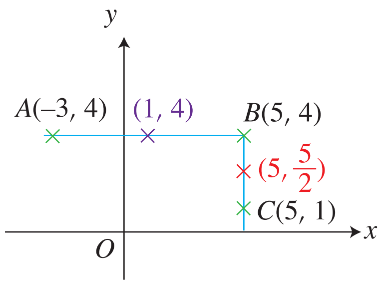
(b) If the coordinates of the midpoint of the line segment joining $P(-5, 3)$ and $Q(x, y)$ are $(3, 4)$, find the value of x and of y.

Solution

(a) **(i)** Midpoint of $AB = \left(\dfrac{-3+5}{2}, 4\right)$

$= (1, 4)$

Note that AB is a horizontal line.



(ii) Midpoint of $BC = \left(5, \dfrac{4+1}{2}\right)$

$= \left(5, \dfrac{5}{2}\right)$

Note that BC is a vertical line.

(iii) Midpoint of $DE = \left(\dfrac{x_1+x_2}{2}, \dfrac{y_1+y_2}{2}\right)$

$= \left(\dfrac{2+(-6)}{2}, \dfrac{5+(-1)}{2}\right)$

$= (-2, 2)$

(b) $(3, 4) = \left(\dfrac{-5+x}{2}, \dfrac{3+y}{2}\right)$

i.e. $3 = \dfrac{-5+x}{2}$, $\quad 4 = \dfrac{3+y}{2}$

$6 = -5 + x$, $\quad 8 = 3 + y$

$x = 11$, $\quad y = 5$

Thinking Time

(i) If two points lie on a horizontal line (see Worked Example 1 **(a)(i)**), what will the coordinates of the midpoint be? Which coordinate will be the same as that of the two points? Can we find the coordinates of the midpoint without using the formula?

(ii) In Worked Example 1 **(a)(ii)**, the two points lie on a vertical line. How do we find the coordinates of the midpoint without using the formula?

Practise Now 1

Similar Questions:
Exercise 5A
Questions 1(a)-(f)

(a) Find the coordinates of the midpoint of the line segment joining

(i) $(2, 3)$ and $(6, 3)$, **(ii)** $(1, 4)$ and $(1, -2)$,

(iii) $(5, 3)$ and $(-1, 7)$.

(b) If $(2, 0)$ is the midpoint of the line segment joining $A(8, -3)$ and $B(x, y)$, find the value of x and of y.

Worked Example 2

(Finding the 4th Vertex of a Parallelogram)

If $A(2, -4)$, $B(7, 1)$ and $C(-1, 5)$ are three vertices of a parallelogram $ABCD$, find the midpoint of AC. Hence, find the coordinates of D.

Solution

$$\text{Midpoint of } AC = \left(\frac{2+(-1)}{2}, \frac{-4+5}{2}\right)$$
$$= \left(\frac{1}{2}, \frac{1}{2}\right)$$

Let the coordinates of D be (h, k).

Midpoint of AC = Midpoint of BD

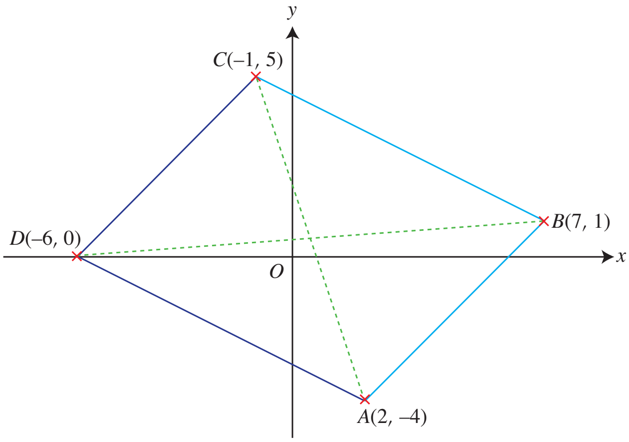
$$\left(\frac{1}{2}, \frac{1}{2}\right) = \left(\frac{7+h}{2}, \frac{1+k}{2}\right)$$

i.e.
$$\frac{1}{2} = \frac{7+h}{2}, \qquad \frac{1}{2} = \frac{1+k}{2}$$
$$1 = 7 + h, \qquad 1 = 1 + k$$
$$h = -6, \qquad k = 0$$

$\therefore D(-6, 0)$

PROBLEM-SOLVING TIP

The diagonals of a parallelogram bisect each other.



Practise Now 2

Similar Questions:
Exercise 5A
Questions 2-5

$ABCD$ is a parallelogram. The coordinates of A, B and C are $(-3, 5)$, $(2, 7)$ and $(4, 6)$ respectively. Find

(i) the midpoint of AC,

(ii) the coordinates of D,

(iii) the lengths of the diagonals AC and BD.

Thinking Time

Given any 3 vertices of a parallelogram, we are able to use the formula for the midpoint to obtain the fourth vertex. Can this rule be applied to a square, a rectangle and a rhombus? What about a trapezium and a kite? You may use a dynamic geometry software to explore this.

Exercise 5A

1 Find the coordinates of the midpoint of the line segment joining each of the following pairs of points.

(a) $(4, 4)$ and $(8, 4)$

(b) $(0, -2)$ and $(0, 6)$

(c) $(1, 1)$ and $(7, 3)$

(d) $(3, -2)$ and $(-2, 7)$

(e) $(2a + b, 3b - a)$ and $(b - a, 2a - b)$

(f) $(ah^2, 2ah)$ and $(ak^2, 4ak)$

2 Three of the vertices of a parallelogram $ABCD$ are $A(-3, 1)$, $B(4, 9)$ and $C(11, -3)$. Find

(i) the midpoint of the diagonal AC,

(ii) the fourth vertex D.

3 Three of the vertices of a parallelogram $PQRS$ are $P(-1, -2)$, $Q(3, 5)$ and $R(9, 1)$. Find the midpoint of PR and use it to find the fourth vertex S.

4 The line $x + 2y = 5$ meets the curve $5x^2 + 4y^2 = 29 - 12x$ at the points A and B. Find the coordinates of the midpoint of AB.

5 The line $5x + y = 17$ intersects the curve $5x^2 + y^2 = 49$ at the points P and Q.

(i) Find the coordinates of the midpoint of PQ.

(ii) Calculate the length of PQ.

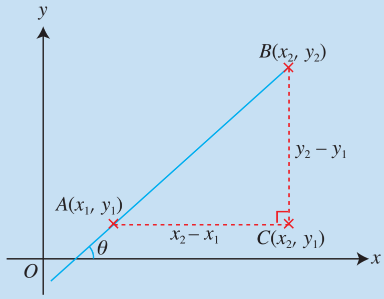
6 In ΔPQR, the midpoints of the sides PQ, QR and PR are $A(-2, 3)$, $B(5, -1)$ and $C(-4, -7)$ respectively. Find the coordinates of P, Q and R.

5.2 PARALLEL AND PERPENDICULAR LINES

In Secondary 1, we have learnt that the gradient of a line is the measure of the ratio of the vertical change (or rise) to the horizontal change (or run),

$$\text{i.e. Gradient} = \frac{\text{vertical change}}{\text{horizontal change}} \quad \text{or} \quad \frac{\text{rise}}{\text{run}}$$

Now, let us explore the relationship between the gradient and $\tan\theta$, where θ is the angle made by the line with the positive direction of the x-axis.



1. In the figure, the coordinates of A and B are (x_1, y_1) and (x_2, y_2) respectively.
State the gradient of the line segment AB in terms of x_1, y_1, x_2 and y_2.

2. Let θ be the angle made by the line segment AB with the positive direction of the x-axis and $0° \leqslant \theta < 90°$.

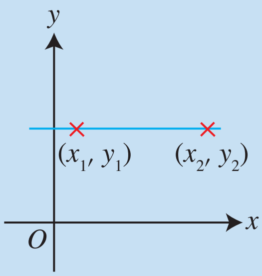
(a) In the right-angled triangle ACB, state the angle whose measure is θ.

(b) What is the value of $\tan \theta$?

(c) What is the relationship between $\tan \theta$ and the gradient of the line?

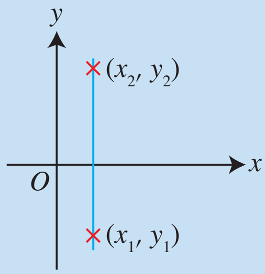
3.

(a)



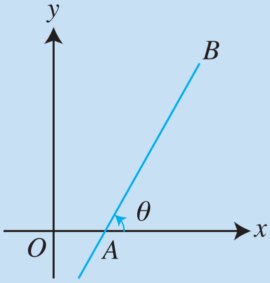
Find the gradient of a horizontal line, showing your working clearly.
Which axis is this line parallel to?

(b)



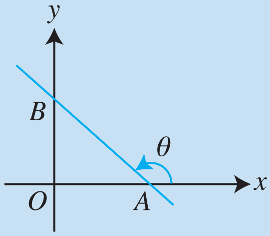
Find the gradient of a vertical line, showing your working clearly.
Which axis is this line parallel to?

(c)



If the line AB makes an acute angle θ with the positive direction of the x-axis, will the gradient of the line be positive or negative? Explain your answer.

(d)



If the line AB makes an obtuse angle θ with the positive direction of the x-axis, will the gradient of the line be positive or negative? Explain your answer.

In summary,

$$\boxed{\text{Gradient of a line} = \tan\theta = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2}, \text{ where } x_2 \neq x_1}$$

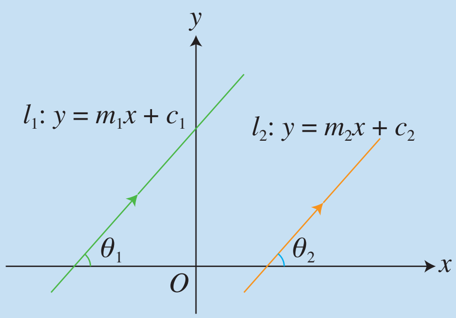
Thinking Time

1. Given three points $A(2, 3)$, $B(3, 5)$ and $C(5, 9)$ which lie on a Cartesian plane, find the gradients of AB, BC and AC. What can you say about the three points?
2. Given that the gradients of AB and BC are equal, what can you say about the points A, B and C?

Work in pairs.
Let us consider the following pair of lines

$$l_1: y = m_1x + c_1$$
$$l_2: y = m_2x + c_2.$$

Case 1:

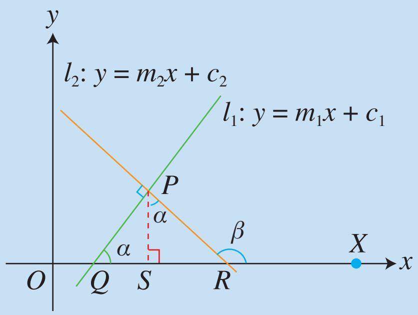


Let θ_1 and θ_2 be the angles that the lines l_1 and l_2 make with the positive direction of the x-axis respectively.

(i) If l_1 and l_2 are parallel, what can we say about θ_1 and θ_2? What can we say about the gradients of these 2 lines?

(ii) If $m_1 = m_2$, what can we say about l_1 and l_2?

Case 2:



Let the lines l_1 and l_2 intersect and form right angles at P. They intersect the x-axis at Q and R respectively. If $\angle PQR = \alpha$, then $\tan \alpha = m_1$; if $\angle PRX = \beta$, then $\tan \beta = m_2$. Let PS be perpendicular to the x-axis, then $\angle SPR = \alpha$.

(i) Consider ΔPQS. Since $m_1 = \tan \alpha$, write down an expression for m_1 in terms of the lengths.

(ii) Consider ΔPRS. Since $m_2 = \tan \beta$, write down an expression for m_2 in terms of the lengths.

Hence, show that $m_2 = -\dfrac{1}{m_1}$.

(iii) If l_1 and l_2 are perpendicular, what can we say about m_1 and m_2?

(iv) If $m_1m_2 = -1$, what can we say about l_1 and l_2?

In summary, for two lines l_2 and l_2,
(a) l_1 is **parallel** to $l_2 \Leftrightarrow$ their gradients are equal,
(b) l_1 is **perpendicular** to $l_2 \Leftrightarrow$ the product of their gradients is -1.

Worked Example 3

(Parallel Lines and Collinear Points)

The coordinates of 4 points are $O(0, 0)$, $A(2, k)$, $B(2k, 9)$ and $C(3k, 2k + 7)$. Find the value(s) of k if

(a) the points O, A and B are collinear,

(b) OA is parallel to BC.

Solution

(a) If O, A and B are collinear, they lie on the same straight line,

i.e. gradient of OA = gradient of OB.

$$\frac{k-0}{2-0} = \frac{9-0}{2k-0}$$
$$\frac{k}{2} = \frac{9}{2k}$$
$$2k^2 = 18$$
$$k^2 = 9$$
$$k = \pm 3$$

INFORMATION

You can also use gradient of OB = gradient of AB, or gradient of OA = gradient of AB to find k.

(b) OA is parallel to BC,

i.e. gradient of OA = gradient of BC.

$$\frac{k-0}{2-0} = \frac{2k+7-9}{3k-2k}$$
$$\frac{k}{2} = \frac{2k-2}{k}$$ (simplify both expressions before cross-multiplying)
$$k^2 = 4k - 4$$
$$k^2 - 4k + 4 = 0$$
$$(k-2)^2 = 0$$
$$k = 2$$

Practise Now 3

Similar Questions:
Exercise 5B
Questions 1, 6, 8(a), 9

The coordinates of 4 points are $A(0, 9)$, $B(k + 1, k + 4)$, $C(2k, k + 3)$ and $D(2k + 2, k + 6)$. Find the value(s) of k if

(a) the points A, B and C are collinear,

(b) AB is parallel to CD.

Worked Example 4

(Perpendicular Lines)

The vertices of ΔABC are $A(0, -5)$, $B(-2, 1)$ and $C(10, k)$. Find the value of k if $\angle ABC = 90°$.

Solution

Since $\angle ABC = 90°$,

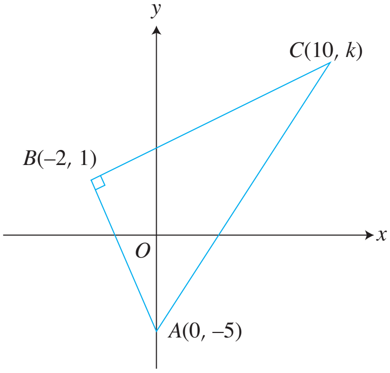
gradient of AB × gradient of $BC = -1$.

$$\left(\frac{1-(-5)}{-2-0}\right) \times \left(\frac{k-1}{10-(-2)}\right) = -1$$

i.e. $$(-3)\left(\frac{k-1}{12}\right) = -1$$

$$k - 1 = 4$$

$$k = 5$$



Class Discussion

Can you use Pythagoras' Theorem to verify your answer to Worked Example 4?

Practise Now 4

Similar Questions:
Exercise 5B
Questions 2-5, 8(b), 10

1. The vertices of ΔABC are $A(0, 1)$, $B(-1, -2)$ and $C(2, k)$. Find the value of k if $\angle ABC = 90°$.

2. The coordinates of 3 points are $A(1, 3)$, $B(5, 1)$ and $C(k, -1)$.

(i) Find the coordinates of the midpoint, M, of AB.

(ii) If MC is perpendicular to AB, find the value of k.

(iii) Calculate the length of MC.

Exercise 5B

1 The coordinates of 4 points are $O(0, 0)$, $A(2, 3k)$, $B(4k, 6)$ and $C(10k, 7)$. Find the value(s) of k if
(a) the points O, A and B are collinear,
(b) OA is parallel to BC.

2 The coordinates of 3 points are $A(1, 1)$, $B(-1, 4)$ and $C(6, k)$. Find the value of k if AB is perpendicular to BC.

3 The vertices of ΔABC are $A(-1, -3)$, $B(2, 3)$ and $C(k + 5, k)$. Find the value of k if AB is perpendicular to BC.

4 Show that $P(-1, 3)$, $Q(6, 8)$ and $R(11, 1)$ are the vertices of an isosceles triangle. Determine whether $\angle PQR$ is a right angle, showing your working clearly.

5 Given that A is the point $(0, 4)$ and B is $(6, 6)$, find
(i) the point C on the x-axis such that $AB = BC$,
(ii) the point D on the y-axis such that $\angle ABD = 90°$.

6 A point P is equidistant from $R(-2, 4)$ and $S(6, -4)$ and its x-coordinate is twice its y-coordinate.
(i) Find the coordinates of P.
(ii) Hence, determine whether P, R and S are collinear, showing your working clearly.

7 The line $x + 3y = 1$ intersects the curve $5y = 20 - 3x - x^2$ at the points P and Q. Calculate the gradient of PQ.

8 The coordinates of 3 points are $A(-1, -6)$, $B(3, -12)$ and $C(k, 6)$. Find the value of k if
(a) A, B and C are collinear,
(b) AB is perpendicular to AC.

9 Determine whether the 4 points $(2, 1)$, $(-1, -5)$, $(1, 5)$ and $(-2, -1)$ are the vertices of a parallelogram, showing your working clearly.

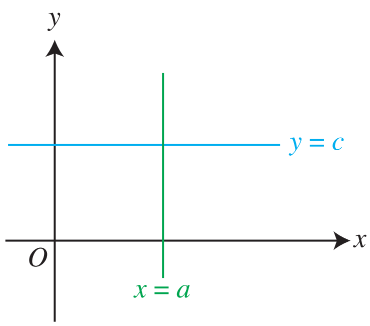
10 Determine whether the 4 points $(5, 8)$, $(7, 5)$, $(3, 5)$ and $(5, 2)$ are the vertices of a rhombus, showing your working clearly.

5.3 MORE PROBLEMS ON EQUATIONS OF STRAIGHT LINES

Recap

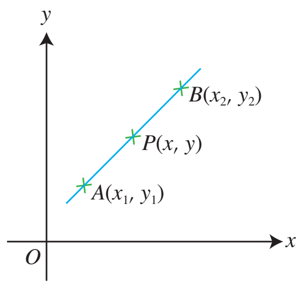
A horizontal line is parallel to the x-axis and its equation is of the form $y = c$.

A vertical line is parallel to the y-axis and its equation is of the form $x = a$.



Equation Of An Oblique Line

We have learnt that in order to find the equation of the straight line joining two given points $A(x_1, y_1)$ and $B(x_2, y_2)$, we consider a point $P(x, y)$ lying on the same straight line.



gradient of AP = gradient of AB

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

i.e. Equation of a straight line passing through two given points is

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

This can also be written as

$$\boxed{y - y_1 = m(x - x_1),}$$

since $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.

Thinking Time

(i) When we want to find the equation of a straight line, what information do we need? Describe the strategy that you would use based on the type of information given.

(ii) Draw a line segment and sketch the perpendicular bisector of the line segment. How can you find the equation of the perpendicular bisector?

Worked Example 5

(Equation of an Oblique Line)

Find the equation of the straight line that passes through the points $A(1, 2)$ and $B(3, 7)$.

Solution

Gradient of $AB = \dfrac{7-2}{3-1} = \dfrac{5}{2}$

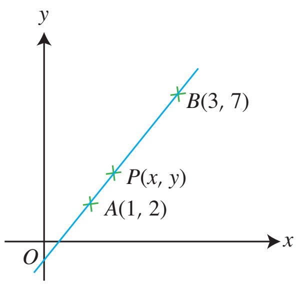
Using $y - y_1 = m(x - x_1)$,

we have $y - 2 = \dfrac{5}{2}(x - 1)$.

$$2y - 4 = 5x - 5$$

$$2y = 5x - 1$$

$\therefore$ Equation of the straight line is $2y = 5x - 1$



INFORMATION

$y = mx + c$ is known as the **gradient-intercept form**; $ax + by + c = 0$, where a, b and c are constants, is known as the **general form**.

Practise Now 5

Similar Questions:
Exercise 5C
Questions 2(a)-(d), 4

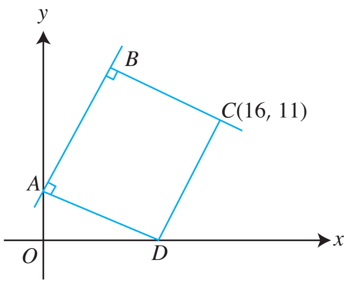
1. Find the equation of the line passing through
   (a) $A(1, -2)$ and $B(4, 7)$,
   (b) $C(3, 8)$ and $D(5, -6)$.
2. Find the equation of the line with gradient 2 and passing through $(1, 4)$.

Worked Example 6

(Oblique Lines)

The figure shows a quadrilateral $ABCD$ in which the points A and D lie on the y-axis and x-axis respectively. C is the point (16, 11) and the equation of AB is $y = 2x + 4$. If AD and BC are perpendicular to AB, find

(i) the coordinates of A and of D,
(ii) the equation of BC,
(iii) the coordinates of B,
(iv) the length of CD.



Solution

(i) At A, $x = 0$, i.e. $y = 2(0) + 4 = 4$.

$\therefore A(0, 4)$

Let D be the point $(k, 0)$.

Gradient of $AB = 2 \Rightarrow$ Gradient of $AD = -\frac{1}{2}$

i.e. $\frac{4-0}{0-k} = -\frac{1}{2}$

$k = 8$

$\therefore D(8, 0)$

ATTENTION

The y-coordinate of a point on the x-axis is 0 and the x-coordinate of a point on the y-axis is 0.

(ii) Gradient of BC = Gradient of $AD = -\frac{1}{2}$

Equation of BC is $y - 11 = -\frac{1}{2}(x - 16)$

$2y + x = 38$

(iii) To find the coordinates of B, we solve $y = 2x + 4$ and $2y + x = 38$ simultaneously.

$$2(2x + 4) + x = 38$$
$$5x = 30$$
$$x = 6$$
$$y = 2(6) + 4 = 16$$

$\therefore B(6, 16)$

(iv) Length of $CD = \sqrt{(16-8)^2 + (11-0)^2}$

$= \sqrt{185}$

$= 13.6$ units

RECALL

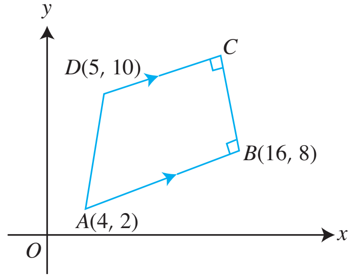
Length of a line segment $= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

Practise Now 6

Similar Questions:
Exercise 5C
Questions 3, 6, 8

The figure shows a trapezium $ABCD$ in which AB is parallel to DC and BC is perpendicular to both AB and DC. The coordinates of A, B and D are (4, 2), (16, 8) and (5, 10) respectively. Find

(i) the equation of DC and of BC,
(ii) the coordinates of C,
(iii) the length of CD.



Worked Example 7

(Finding the 4th vertex of a Parallelogram and the Equation of the Perpendicular Bisector of a Line Segment)

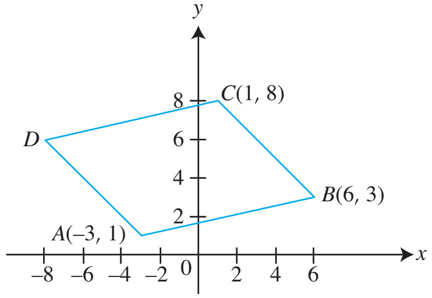
In the figure, the coordinates of three points A, B and C are $A(-3, 1)$, $B(6, 3)$ and $C(1, 8)$.

(i) Find the gradient of BC and of AB.

(ii) If H is the point on the y-axis such that B, C and H lie on the same line, find the coordinates of the point H.

(iii) Find the coordinates of the point D such that $ABCD$ is a parallelogram.

(iv) Find the equation of the perpendicular bisector of BC.



Solution

(i) Gradient of $BC = \dfrac{8-3}{1-6} = -1$

Gradient of $AB = \dfrac{3-1}{6-(-3)} = \dfrac{2}{9}$

(ii) Let H be the point $(0, h)$.

Gradient of BC = Gradient of BH

$$-1 = \frac{3-h}{6-0}$$
$$h = 9$$

$\therefore H(0, 9)$

(iii) Let the coordinates of D be (x_D, y_D). Since $ABCD$ is a parallelogram, the diagonals bisect each other.

i.e. midpoint of BD = midpoint of AC

$$\left(\frac{6+x_D}{2}, \frac{3+y_D}{2}\right) = \left(\frac{-3+1}{2}, \frac{1+8}{2}\right)$$
$$= \left(-1, \frac{9}{2}\right)$$

i.e. $\dfrac{6+x_D}{2} = -1, \quad \dfrac{3+y_D}{2} = \dfrac{9}{2}$

$x_D = -8, \quad y_D = 6$

$\therefore D(-8, 6)$

(iv) Gradient of perpendicular bisector of $BC = 1$ $\quad (m_1m_2 = -1)$

Midpoint of $BC = \left(\dfrac{1+6}{2}, \dfrac{8+3}{2}\right) = \left(\dfrac{7}{2}, \dfrac{11}{2}\right)$

Equation of perpendicular bisector of BC is $y - \dfrac{11}{2} = 1\left(x - \dfrac{7}{2}\right)$

i.e. $y = x + 2$

Practise Now 7

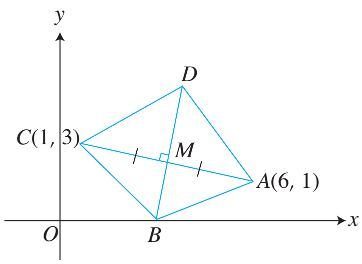
Similar Questions:
Exercise 5C
Questions 7, 9-14

The figure shows a quadrilateral $ABCD$ where A is $(6, 1)$, B lies on the x-axis and C is $(1, 3)$. The diagonal BD bisects AC at right angles at M. Find

(i) the equation of BD,

(ii) the coordinates of B,

(iii) the coordinates of D such that $ABCD$ is a parallelogram.



Basic Level | Intermediate Level | Advanced Level

Exercise 5C

1 Find the equation of the perpendicular bisector of the line segment joining the points $(1, 1)$ and $(2, 4)$.

2 Find the equation of the line passing through the point

(a) $(-2, 5)$ and parallel to the line $3y + 7 = 29$,

(b) $(-1, -6)$ and perpendicular to the line $42 - 7y = 5$,

(c) $(4, 8)$ and parallel to the line $3x + y = 17$,

(d) $(2, -3)$ and perpendicular to the line $y + 2x = 13$.

3 The triangle formed by $A(-2, -3)$, $B(1, 3)$ and $C(10, k)$ is right-angled at A. Find

(i) the value of k,

(ii) the area of ΔABC,

(iii) the length of AM, where M is the midpoint of BC.

4 Find the equation of the straight line that is parallel to $2y - x = 7$ and bisects the line segment joining the points $(3, 1)$ and $(1, -5)$.

5 Find the equation of the line segment joining the points whose x-coordinates on the curve $y = 2x^2 - 3$ are -1 and 1.

6 The coordinates of three points are $A(-1, -3)$, $B(2, 3)$ and $C(6, k)$. If AB is perpendicular to BC, find

(i) the value of k,

(ii) the gradient of AC,

(iii) the acute angle that AC makes with the x-axis.

7 The coordinates of three points are $A(0, 4)$, $B(10, 8)$ and $C(k, 1)$.

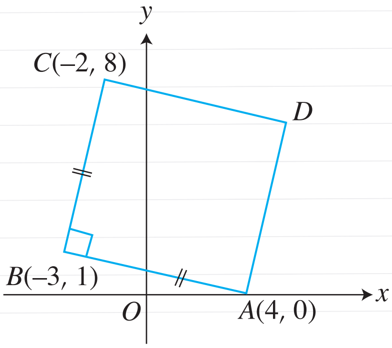
(i) Find the coordinates of the midpoint, M, of AB.

(ii) If MC is the perpendicular bisector of AB, find k.

(iii) Determine if M lies on the perpendicular bisector of AC, explaining your answer clearly.

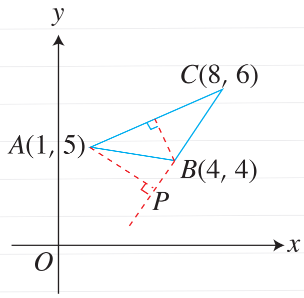
8 Given that the x-intercept of a line is twice its y-intercept and that the line passes through the point of intersection of the lines $3y + x = 3$ and $4y - 3x = 5$, find the equation of this line.

9 The coordinates of three points are $A(4, 0)$, $B(-3, 1)$ and $C(-2, 8)$.



(i) Find the gradient of the line segment joining the points $A(4, 0)$ and $C(-2, 8)$.
(ii) Given that B is the point $(-3, 1)$, explain why $AB = BC$ and that $\angle ABC = 90°$, showing your working clearly.
(iii) If A, B and C are the three vertices of a square, find the coordinates of the fourth vertex, D.
(iv) Hence, or otherwise, calculate the area of the square.

10 The coordinates of ΔABC are $A(1, 5)$, $B(4, 4)$ and $C(8, 6)$. Given that P is the foot of the perpendicular from A to BC, find
(i) the equation of AP,
(ii) the coordinates of P,
(iii) the lengths of AP, BC and AC,
(iv) the area of ΔABC,
(v) the length of the perpendicular from B to AC.



11 The coordinates of three points are $A(-1, -6)$, $B(3, 12)$ and $C(k, 6)$. Find the value(s) of k if
(a) A, B and C are collinear,
(b) AB is perpendicular to AC,
(c) BC is perpendicular to AC,
(d) the intersection of the lines AB and BC have an infinite number of solutions.

12 The coordinates of the vertices of a triangle ABC are $A(1, 2)$, $B(6, 7)$ and $C(7, 2)$. Find the equations of the perpendicular bisectors of
(a) AB,
(b) BC.
Hence, find the coordinates of the point equidistant from A, B and C.

Exercise 5C

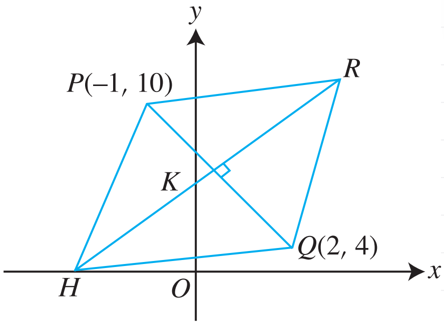
13 The coordinates of the points P and Q are $P(-1, 10)$ and $Q(2, 4)$.

(i) Find the equation of the perpendicular bisector of the line joining P and Q.

(ii) If the perpendicular bisector of PQ cuts the x- and y-axes at points H and K respectively, calculate the length of HK, giving your answer correct to one decimal place.

(iii) Find the coordinates of R such that $PRQH$ is a parallelogram.

(iv) Show that $PRQH$ is a rhombus.

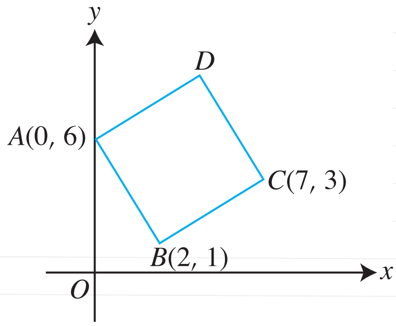


14 Show that the three points A, B and C, whose coordinates are $(0, 6)$, $(2, 1)$ and $(7, 3)$ respectively, are the three possible vertices of a square. Find

(i) the coordinates of the point where the diagonals intersect,

(ii) the gradient of the diagonal BD,

(iii) the coordinates of D, the fourth vertex of the square.

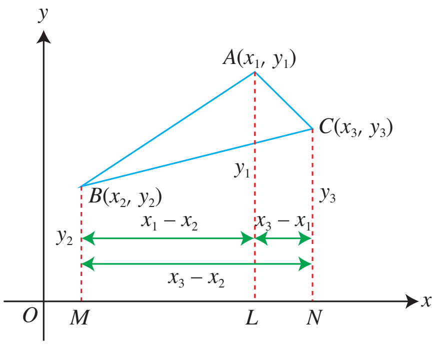


5.4 AREA OF RECTILINEAR FIGURES

If we are given the coordinates of the three vertices of a right-angled triangle ABC, where $\angle ABC = 90°$, we are able to find the length of the base AB and the height BC, before using the formula $\frac{1}{2} \times \text{base} \times \text{height}$ to calculate the area of the triangle. If a triangle is not a right-angled triangle, then can we find the area if the coordinates of the vertices are given?

We shall now learn how to deduce a formula to obtain the area of any triangle when its vertices are given.

The figure shows a triangle ABC such that $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ are in an anticlockwise direction.



Area of ΔABC = Area of trapezium $BMLA$ + Area of trapezium $ALNC$ – Area of trapezium $BMNC$

$$= \frac{1}{2}(y_1 + y_2)(x_1 - x_2) + \frac{1}{2}(y_1 + y_3)(x_3 - x_1) - \frac{1}{2}(y_2 + y_3)(x_3 - x_2)$$

$$= \frac{1}{2}[(x_1y_1 - x_2y_1 + x_1y_2 - x_2y_2) + (x_3y_1 - x_1y_1 + x_3y_3 - x_1y_3) - (x_3y_2 - x_2y_2 + x_3y_3 - x_2y_3)]$$

$$= \frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1 - x_2y_1 - x_3y_2 - x_1y_3)$$

The above complicated formula can be arranged and written as

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix},$$

where the products taken in the direction $\searrow$ are given positive signs,

i.e. $\frac{1}{2}(x_1y_2 + x_2y_3 + x_3y_1)$, and

the products taken in the direction $\nearrow$ are given negative signs,

i.e. $\frac{1}{2}(-x_2y_1 - x_3y_2 - x_1y_3)$.

Note that this rule is only applicable when *the vertices are considered in an anticlockwise direction*. If the vertices are taken in a *clockwise direction*, the final area obtained will be *negative*.

Work in pairs.

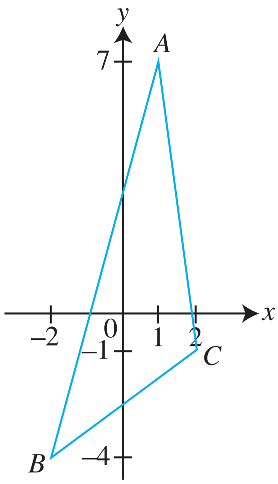
Besides using the formula given above, when the coordinates of the vertices are given, are there other ways of finding the area of a triangle?

Worked Example 8

(Area of a Triangle)

Find the area of the triangle with vertices $A(1, 7)$, $B(-2, -4)$ and $C(2, -1)$.

Solution



PROBLEM-SOLVING TIP

When using this formula to find the area, it does not matter which vertex is selected first, but the coordinates must be arranged in an anticlockwise direction.

Taking the coordinates in an anticlockwise direction,

Area of ΔABC

$$= \frac{1}{2}\begin{vmatrix} 1 & -2 & 2 & 1 \\ 7 & -4 & -1 & 7 \end{vmatrix}$$

$$= \frac{1}{2}\{[(1)(-4) + (-2)(-1) + (2)(7)] - [(7)(-2) + (-4)(2) + (-1)(1)]\}$$

$$= \frac{1}{2}[(-4 + 2 + 14) - (-14 - 8 - 1)]$$

$$= 17\frac{1}{2} \text{ units}^2$$

Practise Now 8

Similar Questions:
Exercise 5D
Questions 1(a)-(b), 5, 6, 8

1. Find the area of the triangle with vertices $A(2, 1)$, $B(-1, -5)$ and $C(5, 8)$.
2. The points $(0, -2)$, $(6, 8)$ and $(-4, -1)$ form a triangle. Find the area of this triangle.

Thinking Time

What can you say about the three points $A(x_1, y_1)$, $B(x_2, y_2)$ and $C(x_3, y_3)$ if the area of $\Delta ABC = \frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix} = 0$?

Class Discussion

Discuss with your classmate how you can use the idea of finding the area of a triangle to prove that the area of a n-sided polygon with vertices $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$, $D(x_4, y_4)$, …, $N(x_n, y_n)$ is

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & \ldots & x_n & x_1 \\ y_1 & y_2 & y_3 & y_4 & \ldots & y_n & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\left[\left(\text{sum of all products} \searrow\right) - \left(\text{sum of all products} \nearrow\right)\right]$$

where A, B, C, D, …, N are taken in an anticlockwise direction.

Search on the Internet to find out what a convex polygon and a concave polygon are.

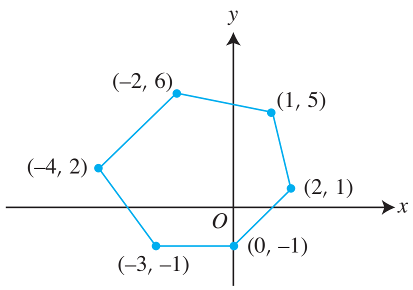
Discuss with your classmates if this formula is applicable to a concave polygon.

Worked Example 9

(Area of a Polygon)

Find the area of the polygon whose vertices are (–2, 6), (1, 5), (2, 1), (0, –1), (–3, –1) and (–4, 2).

Solution



PROBLEM-SOLVING TIP

Make a sketch of the polygon to locate the vertices before considering them in an anticlockwise direction. It does not matter which vertex is taken first.

$$\text{Area of polygon} = \frac{1}{2}\begin{vmatrix} 0 & 2 & 1 & -2 & -4 & -3 & 0 \\ -1 & 1 & 5 & 6 & 2 & -1 & -1 \end{vmatrix}$$

$$= \frac{1}{2}\{[(0)(1) + (2)(5) + (1)(6) + (-2)(2) + (-4)(-1) + (-3)(-1)] - [(-1)(2) + (1)(1) + (5)(-2) + (6)(-4) + (2)(-3) + (-1)(0)]\}$$

$$= \frac{1}{2}[(0 + 10 + 6 - 4 + 4 + 3) - (-2 + 1 - 10 - 24 - 6 + 0)]$$

$= 30$ units^2

Practise Now 9

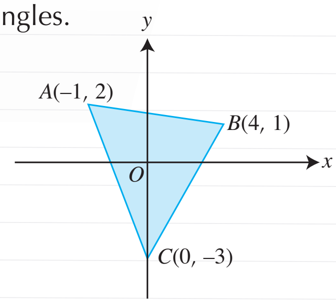
Find the area of the polygon whose vertices are (–2, –4), (–4, 6), (0, 8), (1, 6), (3, 5) and (5, 4).

Similar Questions:
Exercise 5D
Questions 2(a)-(b), 3, 4, 7, 9

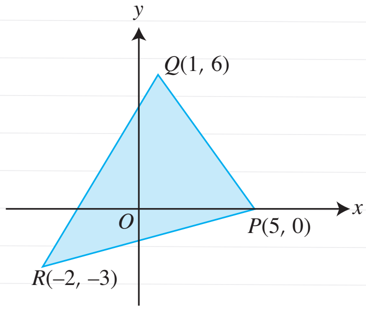
Exercise 5D

1 Find the area of each of the following triangles.

(a)

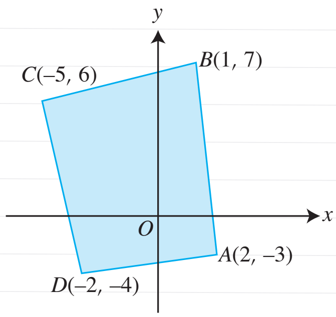


(b)

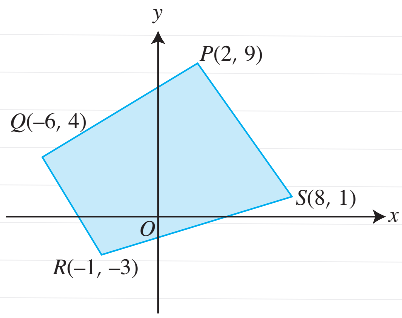


2 Find the area of each of the following quadrilaterals.

(a)



(b)

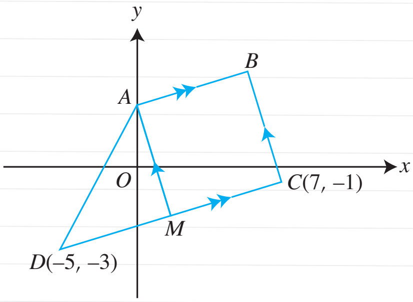


3 A polygon has vertices $A(-2, -5)$, $B(-3, 1)$, $C(0, 9)$, $D(1, 8)$ and $E(3, 5)$.

(i) Find the area of the polygon $ABCDE$.

(ii) Show that the area of ΔACE is 25 units^2.

4 $ABCD$ is a trapezium, where AB is parallel to DC. AM is the perpendicular bisector of DC and AM is parallel to BC. It is given that A lies on the y-axis and the coordinates of C and D are $(7, -1)$ and $(-5, -3)$ respectively.



Find

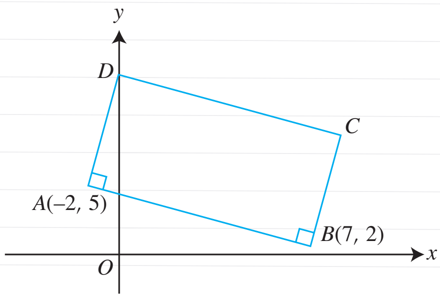
(i) the coordinates of M and A,

(ii) the equation of AB and of BC,

(iii) the coordinates of B,

(iv) the area of the trapezium $ABCD$.

5 The figure shows a rectangle $ABCD$ in which A is $(-2, 5)$ and B is $(7, 2)$. D is a point on the y-axis.



Find

(i) the equation of AB and of AD,

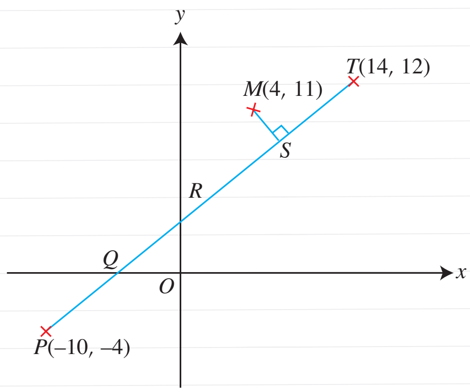
(ii) the coordinates of C and of D.

The line DA produced cuts the x-axis at the point K. Find

(iii) the coordinates of the point K,

(iv) the area of ΔBAD and of ΔBAK.

6 In the figure, $PQRST$ is a straight line cutting the x-axis at Q and the y-axis at R. The coordinates of P, T and M are $(-10, -4)$, $(14, 12)$ and $(4, 11)$ respectively, where MS is the perpendicular from M to PT.

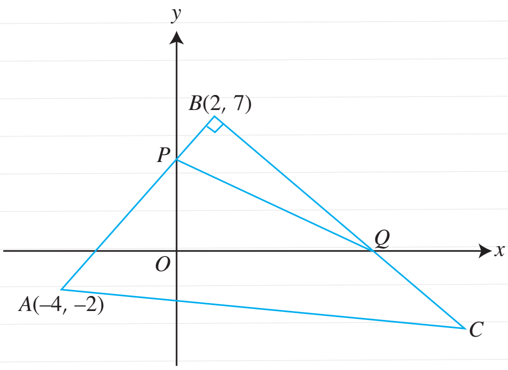


Find
(i) the equation of PQ and of MS,
(ii) the coordinates of Q, R and S,
(iii) the area of ΔPMS.

7 Four points have coordinates $A(3, -2)$, $B(6, 5)$, $C(3, 7)$ and $D(h, 0)$. Find
(i) the area of ΔABC,
(ii) the value of h if the area of ΔACD is equal to the area of ΔABC,
(iii) the coordinates of the point K if $ACBK$ is a parallelogram,
(iv) the area of the parallelogram $ACBK$.

8 The straight line $y = 8 - 2x$ intersects the line $y = k$, the x-axis and the y-axis at the points A, B and C respectively. Given that the area of ΔAOC is 10 units^2, where O is the origin, find
(i) the coordinates of A,
(ii) the length of AB.

9 The figure shows a right-angled triangle where $\angle ABC = 90°$. The coordinates of A and B are $(-4, -2)$ and $(2, 7)$ respectively.



(i) Find the equation of the line BC.
(ii) Find the coordinates of P and of Q, where P lies on the y-axis and Q is the intersection of line BC and the x-axis.
(iii) Given that $BC = 2AB$, find the coordinates of C.
(iv) Find the area of the quadrilateral $APQC$.

SUMMARY

1. If $A(x_1, y_1)$ and $B(x_2, y_2)$ are two distinct points,

(a) Midpoint of $AB = \left(\dfrac{x_1 + x_2}{2}, \dfrac{y_1 + y_2}{2}\right)$

(b) Gradient of $AB = \tan\theta = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{y_1 - y_2}{x_1 - x_2}$, where $x_2 \neq x_1$

2. For two lines l_1 and l_2,

(a) l_1 is parallel to $l_2 \Leftrightarrow$ the gradients are equal,

(b) l_1 is perpendicular to $l_2 \Leftrightarrow$ the products of the gradients is -1,

3. Equation of a straight line

(a) Equation of a line parallel to the x-axis, cutting the y-axis at $(0, c)$ is $y = c$

(b) Equation of a line parallel to the y-axis, cutting the x-axis at $(a, 0)$ is $x = a$

(c) Equation of a line passing through a given point (x_1, y_1) and having gradient m is

$y - y_1 = m(x - x_1)$

4. Area of a rectilinear figure

Note: All vertices are taken in an anticlockwise direction.

(a) Area of a triangle with vertices (x_1, y_1), (x_2, y_2) and (x_3, y_3) is

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_1 \\ y_1 & y_2 & y_3 & y_1 \end{vmatrix}$$

$$= \frac{1}{2}[(x_1y_2 + x_2y_3 + x_3y_1) - (x_2y_1 + x_3y_2 + x_1y_3)]$$

(b) Area of a n-sided convex polygon with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), …, (x_n, y_n) is

$$\frac{1}{2}\begin{vmatrix} x_1 & x_2 & x_3 & x_4 & \ldots & x_n & x_1 \\ y_1 & y_2 & y_3 & y_4 & \ldots & y_n & y_1 \end{vmatrix}$$

$$= \frac{1}{2}\left[\left(\text{sum of all products} \searrow\right) - \left(\text{sum of all products} \nearrow\right)\right]$$

Review Exercise 5

1. The line $y = x + 2$ cuts the curve $2x^2 + y^2 = 5xy + 8$ at A and B. Find the coordinates of the midpoint of AB.

2. Find the equation of the line which passes through the point of intersection of the lines $2x - 3y = 1$ and $4y = 2 - x$ and which is perpendicular to the line $3x + 5y = 13$.

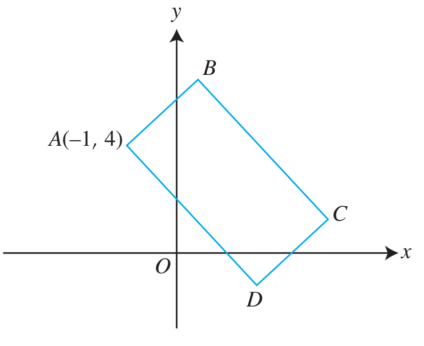
3. Find the equation of the line passing through the point of intersection of the lines $4y = 3x + 6$ and $5y = 12x - 9$ and which is parallel to the line $y = 2x - 13$.

4. The equations of the sides AB, BC and CA of the triangle ABC are $3y = 4x + 13$, $y = -1$ and $y = 1 - 2x$ respectively.
   (i) Find the coordinates of A, B and C.
   (ii) If $ABCD$ is a parallelogram, find the coordinates of D.

5. The straight line $y = 2x + 3$ meets the curve $xy + 20 = 5y$ at the points A and B. Find the coordinates of the midpoint of AB.

6. The straight line $y = 2x - 2$ meets the circle $x^2 + y^2 = 8$ at the points A and B. Find the length of AB, giving your answer correct to one decimal place.

7. The figure shows a rectangle $ABCD$, where A is the point $(-1, 4)$. The equation of BC is $4y + 5x = 52$. Find
   (i) the equation of AD,
   (ii) the equation of AB,
   (iii) the coordinates of B.
   If the length of BC is twice the length of AB, find the coordinates of C and of D. Hence, find the area of $ABCD$.



8. The figure shows a rhombus $ABCD$, where $A(-2, 10)$ and $C(4, 4)$ are opposite corners. The midpoint of AC is M and B lies on the x-axis.

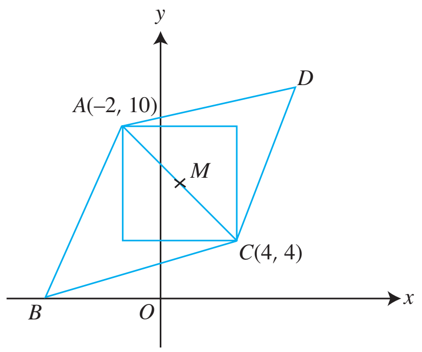
   (a) Find

   (i) the coordinates of M and of B,

   (ii) the equation of BD,

   (iii) the area of $ABCD$.

   (b) Prove that $\angle ABC$ is not a right angle.



9. $ABCD$ is a square, where A is the point $(0, 2)$ and C is the point $(8, 4)$. AC and BD are diagonals of the square and they intersect at M. Find

   (i) the coordinates of M,

   (ii) the equation of BD,

   (iii) the length of AM,

   (iv) the coordinates of the points B and D,

   (v) the area of $ABCD$.

10. The figure shows a parallelogram $ABCD$. The coordinates of A, B and C are $(2, 2)$, $(0, 8)$ and $(8, 10)$ respectively.

   (a) Find

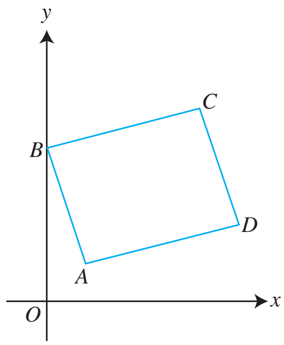
   (i) the coordinates of the point of intersection of the diagonals AC and BD,

   (ii) the coordinates of D,

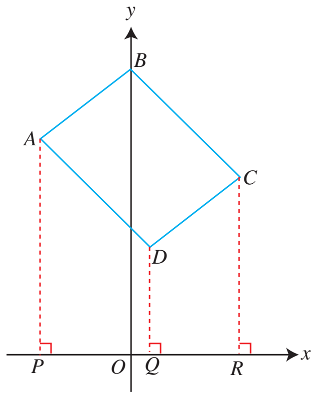
   (iii) the equation of the diagonal BD,

   (iv) the area of the parallelogram $ABCD$.

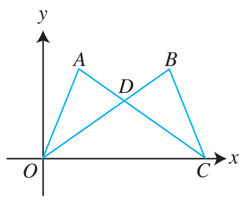
   (b) Explain why the diagonals AC and BD are not perpendicular to each other.



1. The figure shows a parallelogram $ABCD$. Given that $AP = 12$ units, $BO = 16$ units, $CR = 10$ units, $PO = 5$ units and $OQ = 2$ units, find the coordinates of A, B, C and D. Hence, find the area of $ABCD$.

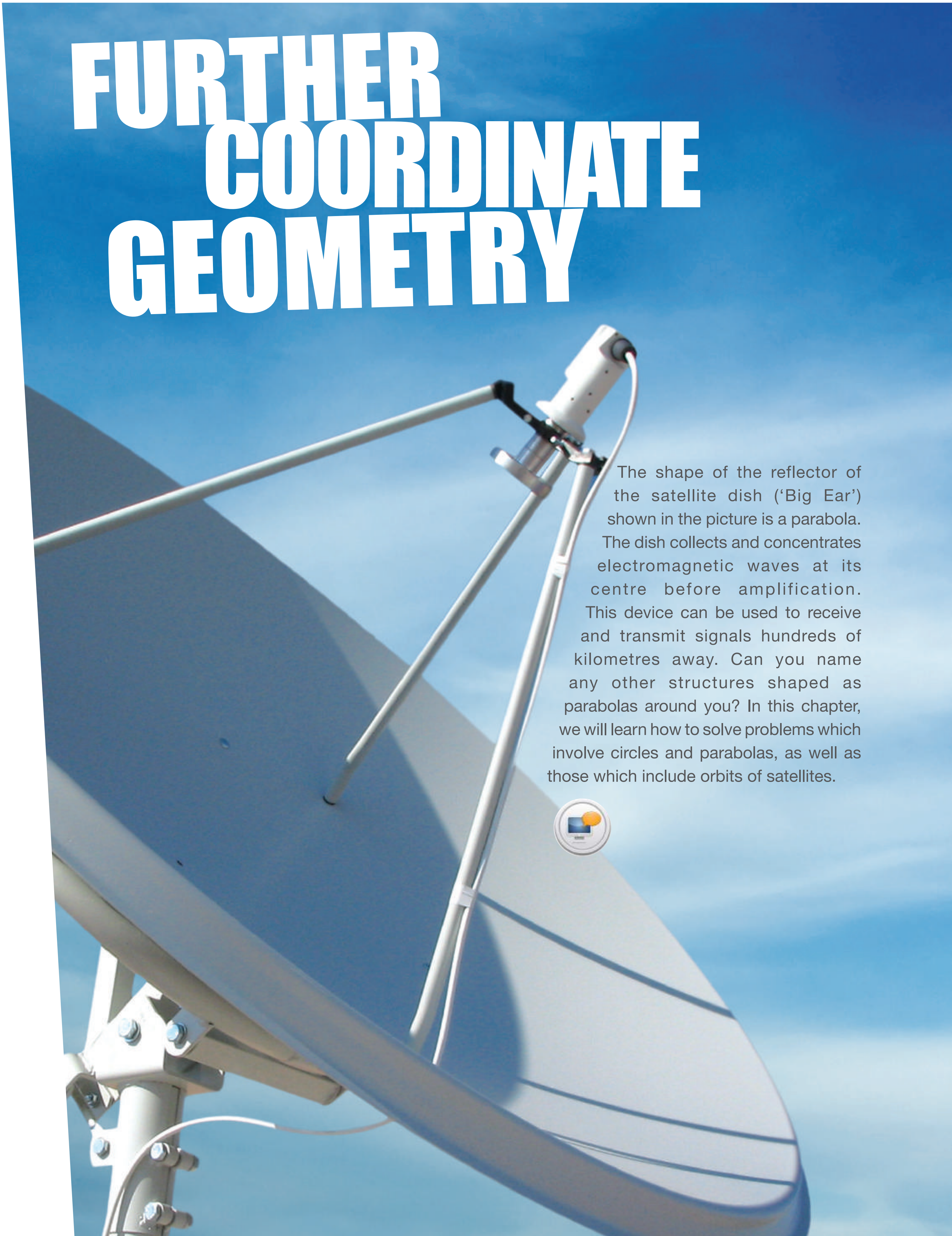


2. In the figure, $OA = BC = 3$ units, $OB = AC = 4$ units and $OC = 5$ units. Find
   (i) the coordinates of A, B, C and D,
   (ii) the area of the polygon $OADBC$.



3. Three vertices of a parallelogram are (0, 0), (−1, 6) and (4, 1). Find all the possible positions of the fourth vertex. Hence, find the coordinates of the point of intersection of the diagonals of the parallelogram in each case.

FURTHER COORDINATE GEOMETRY



The shape of the reflector of the satellite dish ('Big Ear') shown in the picture is a parabola. The dish collects and concentrates electromagnetic waves at its centre before amplification. This device can be used to receive and transmit signals hundreds of kilometres away. Can you name any other structures shaped as parabolas around you? In this chapter, we will learn how to solve problems which involve circles and parabolas, as well as those which include orbits of satellites.

CHAPTER 6

Learning Objectives

At the end of this chapter, you should be able to:

- find the equation of a circle given its centre and radius,
- find the centre and radius of a circle given its equation,
- sketch the graphs of the form $y = ax^n$, where n is a simple rational number,
- sketch the graphs of the form $y^2 = kx$, where k is a real number,
- find the coordinates of the point(s) of intersection of a line and a parabola or circle,
- solve problems involving circles and parabolas.

6.1 EQUATION OF A CIRCLE

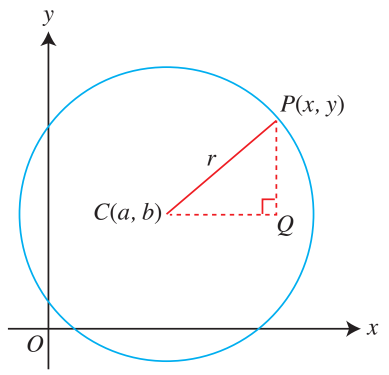
Recap

We have learnt how to draw the graphs of linear functions, $y = mx + c$, and the graphs of quadratic functions $y = ax^2 + bx + c$. In this chapter, we shall learn how to find the equation of a circle, and to sketch the graphs of power functions $y = ax^n$, where n is a simple rational number and parabolas with equations in the form $y^2 = kx$, where k is a real number.

Equation of a Circle in Standard Form

Thinking Time

How can we derive the equation of a circle in standard form?



Let $P(x, y)$ be any point on the circumference of a circle with centre $C(a, b)$ and radius r, where $r > 0$.

Then $\qquad CP = r$

(a) Consider ΔCPQ. Using Pythagoras' Theorem, write down an equation in terms of x, y, a, b and r.

(b) If $a = 0$ and $b = 0$,

(i) state the coordinates of the centre of the circle,

(ii) write down the equation of the circle for this particular case.

In summary, the **standard form** of the equation of a circle with centre $C(a, b)$ and radius r is given by:

$$\boxed{(x - a)^2 + (y - b)^2 = r^2, \text{ where } r > 0.}$$

1. Use a suitable graphing software to plot the circles whose equations are given below. Copy and complete the table by finding the coordinates of the centre of each circle and its radius.

No.	Equation of Circle	Coordinates of Centre of Circle	Radius of Circle
(a)	$x^2 + y^2 = 9$		
(b)	$(x - 1)^2 + (y - 3)^2 = 16$		
(c)	$(x + 2)^2 + (y + 4)^2 = 25$		
(d)	$x^2 + (y - 5)^2 = 1$		
(e)	$\left(x - \frac{1}{4}\right)^2 + (y + 1.5)^2 = 4$		

2. By observing the second and the last columns of the table, how do you find the radius of a circle just by looking at its equation?

3. By observing the second and the third columns of the table, how do you find the coordinates of the centre of a circle just by looking at its equation?

4. The **standard form** of the equation of a circle is $(x - a)^2 + (y - b)^2 = r^2$, where $r > 0$.

(i) Discuss with your classmate the effect of increasing the value of a on the circle. Explain your answer.

(ii) Discuss with your classmate the effect of decreasing the value of b on the circle. Explain your answer.

Worked Example 1

(Finding Coordinates of the Centre and the Radius of a Circle)

Write down the coordinates of the centre and the radius of each of the following circles.

(a) $x^2 + y^2 = 36$ **(b)** $(x - 2)^2 + (y + 3)^2 = 25$ **(c)** $\left(x + \frac{1}{2}\right)^2 + y^2 - 49 = 0$

Solution

(a) Centre (0, 0)

Radius = $\sqrt{36} = 6$

(b) Centre (2, –3)

Radius = $\sqrt{25} = 5$

(c) $\left(x + \frac{1}{2}\right)^2 + y^2 - 49 = 0$

$\Rightarrow \left(x + \frac{1}{2}\right)^2 + y^2 = 49$

Centre $\left(-\frac{1}{2}, 0\right)$

Radius = $\sqrt{49} = 7$

Practise Now 1

Similar Questions:
Exercise 6A
Questions 1(a)-(c)

Write down the coordinates of the centre and the radius of each of the following circles.

(a) $x^2 + y^2 = 81$ **(b)** $(x + 4)^2 + (y - 6)^2 = 100$

(c) $x^2 + \left(y + \frac{1}{3}\right)^2 - 16 = 0$

Thinking Time

A satellite is an object that orbits the earth.

(a) Given that a weather satellite orbits the earth such that its distance from the centre of the earth is always 9000 km, write down an equation that it satisfies, in terms of x and y, where x and y are the longitudinal and latitudinal distances from the centre of the earth respectively in kilometres, as shown on a map.

(b) Your classmate says that the equation of the orbit of another satellite around the earth can be written as $x^2 + y^2 = 3240$. Explain why he is incorrect.

Journal Writing

Search on the Internet for the radius of each of the following structures.

(i) The Singapore Flyer **(ii)** The London Eye

(iii) The Star of Nanchang **(iv)** The Great Beijing Wheel

(v) The Great Berlin Wheel

Assuming that the coordinates of the centre of each structure are (0, 0), write down the equation of each of them.

Worked Example 2

(Finding the Equation of a Circle given the Centre and the Radius)

Find the equation of the circle with centre $C(2, 3)$ and radius 5, leaving your answer in standard form.

Solution

Equation of the circle is $(x - 2)^2 + (y - 3)^2 = 5^2$

i.e. $(x - 2)^2 + (y - 3)^2 = 25$

Practise Now 2

Similar Questions:
Exercise 6A
Questions 2(a)-(e)

Find the equation of each of the following circles.

(a) centre $(-1, 3)$, radius 4

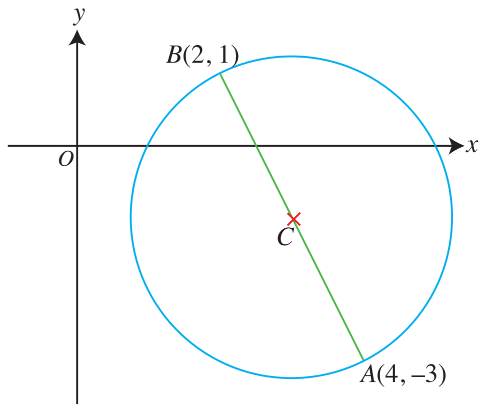
(b) centre $(4, -2)$, radius 7

Worked Example 3

(Finding the Equation of a Circle given the End Points of a Diameter)

Find the equation of a circle with diameter AB, where the coordinates of A and B are $(4, -3)$ and $(2, 1)$ respectively.

Solution



PROBLEM-SOLVING TIP

It is advisable to sketch the circle to better visualise the position of its centre.

The centre of the circle, C, is the midpoint of AB.

$$\text{Coordinates of } C = \left(\frac{4+2}{2}, \frac{-3+1}{2}\right)$$
$$= (3, -1)$$

$$\text{Radius of circle, } AC = \sqrt{(4-3)^2 + (-3+1)^2}$$
$$= \sqrt{5}$$

Equation of the circle is $(x - 3)^2 + (y + 1)^2 = \left(\sqrt{5}\right)^2$

i.e. $(x - 3)^2 + (y + 1)^2 = 5$

RECALL

Midpoint of $A(x_1, y_1)$ and $B(x_2, y_2)$ is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

Practise Now 3

Similar Questions:
Exercise 6A
Questions 3, 4, 6, 7

1. Find the equation of a circle with diameter PQ, where the coordinates of P and Q are $(-2, 1)$ and $(4, 3)$ respectively.
2. Find the equation of a circle which touches the x-axis and whose centre is $C(-3, 5)$.

Equation of a Circle in General Form

If you expand $(x+3)^2 + (y-7)^2 = 36$, you will obtain $x^2 + y^2 + 6x - 14y + 22 = 0$. This is called the general form of the equation of the circle:
$x^2 + y^2 + 2gx + 2fy + c = 0$, where $g = 3$, $f = -7$ and $c = 22$.

Work in pairs.

1. Use a suitable graphing software to plot the circles whose equations are given below. Then complete the table by finding the values of g, f, c and $g^2 + f^2 - c$, the coordinates of the centre of each circle and its radius.

No.	Equation of Circle	g	f	c	$g^2 + f^2 - c$	Coordinates of Centre of Circle	Radius of Circle
(a)	$x^2 + y^2 + 8x + 10y + 16 = 0$						
(b)	$x^2 + y^2 + 3x - 4y - 6 = 0$						
(c)	$x^2 + y^2 - 5x - 9y + \frac{51}{2} = 0$						
(d)	$x^2 + y^2 - 7x - 3.75 = 0$						
(e)	$x^2 + y^2 + \frac{9}{2}y + \frac{7}{2} = 0$						

2. By observing the various columns of the table, write down the coordinates of the centre of a circle in terms of g, f and/or c.

3. By observing the various columns of the table, write down the radius of a circle in terms of g, f and/or c.

4. ***Without*** using any graphing software, write down the radius and the coordinates of the centre of the following circles.

No.	Equation of Circle	g	f	c	Coordinates of Centre of Circle	Radius of Circle, $\sqrt{g^2+f^2-c}$
(a)	$x^2+y^2+12x-2y+1=0$					
(b)	$x^2+y^2-11y=\frac{3}{2}$					
(c)	$4x^2+4y^2-24x+20y+25=0$					

5. Express g, f and c in terms of a, b and/or r.
6. *Without* using any graphing software, write down the radius of the circle whose equation is $x^2+y^2+6x-8y+25=0$. Now use the graphing software to graph the equation. What do you obtain?
7. Without using any graphing software, find the value of g^2+f^2-c for the equation $x^2+y^2+6x-8y+26=0$. Is this a circle? Explain your answer. Now use the graphing software to graph the equation. What do you obtain?
8. From Questions 6 and 7, how does the value of g^2+f^2-c affect whether $x^2+y^2+2gx+2fy+c=0$ is the equation of a circle?

From the discussion, the equation $(x-a)^2+(y-b)^2=r^2$ can be written in the form

$$\boxed{x^2+y^2+2gx+2fy+c=0,} \text{ where } g^2+f^2-c>0,$$

such that $(-g, -f)$ is the centre and $\sqrt{g^2+f^2-c}$ is the radius.

This is the equation of the circle in the **general form**.

Thinking Time

By using the method of completing the square, show that the equation $x^2+y^2+2gx+2fy+c=0$ can be expressed as $(x-a)^2+(y-b)^2=r^2$. Write down the values of a, b and r in terms of g, f and c respectively.

Worked Example 4

(Finding the Centre and the Radius of a Circle)

Find the centre and the radius of the circle $2x^2 + 2y^2 - 3x + 4y = -1$.

Solution

Method 1:

Rewriting the equation, we have

$$2x^2 + 2y^2 - 3x + 4y + 1 = 0$$

i.e. $$x^2 + y^2 - \frac{3}{2}x + 2y + \frac{1}{2} = 0$$

Comparing this with the general form of a circle

$x^2 + y^2 + 2gx + 2fy + c = 0$,

$$2g = -\frac{3}{2} \qquad \text{i.e. } -g = \frac{3}{4},$$

$$2f = 2 \qquad \text{i.e. } -f = -1,$$

$$\text{and } c = \frac{1}{2}.$$

$\therefore$ Centre of circle $= (-g, -f) = \left(\frac{3}{4}, -1\right)$

Radius of circle $= \sqrt{g^2 + f^2 - c}$

$$= \sqrt{\left(-\frac{3}{4}\right)^2 + (1)^2 - \frac{1}{2}}$$

$$= \frac{\sqrt{17}}{4}$$

Method 2:

$$x^2 + y^2 - \frac{3}{2}x + 2y + \frac{1}{2} = 0$$

By completing the square,

$$x^2 - \frac{3}{2}x + \left(-\frac{3}{4}\right)^2 + y^2 + 2y + (1)^2 - \left(-\frac{3}{4}\right)^2 - (1)^2 + \frac{1}{2} = 0$$

$$\left(x - \frac{3}{4}\right)^2 + (y+1)^2 - \frac{9}{16} - 1 + \frac{1}{2} = 0$$

$$\left(x - \frac{3}{4}\right)^2 + (y+1)^2 = \frac{17}{16}$$

Comparing this with the standard form

$(x - a)^2 + (y - b)^2 = r^2$,

centre of circle $= \left(\frac{3}{4}, -1\right)$ and radius $= \sqrt{\frac{17}{16}} = \frac{\sqrt{17}}{4}$.

ATTENTION

When using the general form, always ensure that the right-hand side of the equation is zero.

Class Discussion

Discuss with your classmates which of the two methods in Worked Example 3 you prefer, explaining your reasons clearly.

Practise Now 4

Similar Questions:
Exercise 6A
Questions 1(d)-(f)

Find the centre and the radius of each of the following circles.

(a) $x^2 + y^2 - 6x + 8y + 9 = 0$

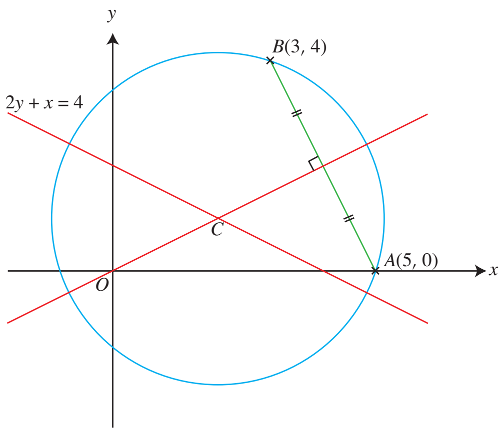
(b) $2x^2 + 2y^2 + 4x - 3y = -2$

Worked Example 5

(Finding the Equation of a Circle given Two Points and Another Condition)

Find the equation of the circle which passes through the points $A(5, 0)$, $B(3, 4)$ and has its centre lying on the line $2y + x = 4$.

Solution



PROBLEM-SOLVING TIP

The centre of the circle lies on the intersection of the line $2y + x = 4$ and the perpendicular bisector of the chord AB.

Midpoint of $AB = \left(\frac{3+5}{2}, \frac{0+4}{2}\right) = (4, 2)$

Gradient of $AB = \frac{4-0}{3-5} = -2$

Gradient of perpendicular bisector of $AB = \frac{1}{2}$

Equation of perpendicular bisector of AB:

$$y - 2 = \frac{1}{2}(x - 4)$$

$$y = \frac{1}{2}x \quad \text{------------------ (1)}$$

The centre also lies on the line

$$2y + x = 4 \quad \text{------------------ (2)}$$

The perpendicular bisector of the chord AB will pass through the centre of the circle.

RECALL

If two lines ℓ_1 and ℓ_2 are perpendicular, then $m_1m_2 = -1$.

Solving (1) and (2),
$x = 2,\ y = 1$
$\therefore$ Centre of circle = (2, 1)

Radius of circle, $r = \sqrt{(2-5)^2 + (1-0)^2}$
$= \sqrt{10}$

$\therefore$ Equation of the circle is

$$(x-2)^2 + (y-1)^2 = \left(\sqrt{10}\right)^2$$

i.e. $(x-2)^2 + (y-1)^2 = 10$ (in the standard form)
$x^2 + y^2 - 4x - 2y - 5 = 0$ (in the general form)

ATTENTION

The centre of a circle lies on the perpendicular bisector of any chord.

INFORMATION

You can give the answer either in the standard form or in the general form since the question does not specify.

Practise Now 5

Similar Questions:
Exercise 6A
Questions 12-14

1. Find the equation of the circle which passes through the points $A(2, 6)$ and $B(-2, -2)$ and has its centre lying on the line $y = x + 1$.

2. The line $y = 2x + 5$ cuts the circle $x^2 + y^2 = 10$ at two points A and B.
(i) Find the coordinates of A and of B.
(ii) Find the equation of the perpendicular bisector of AB and show that it passes through the centre of the circle.
(iii) Given that the perpendicular bisector cuts the circle at P and Q, show that the x-coordinates of P and Q are $k\sqrt{2}$ and $-k\sqrt{2}$ respectively, where k is an integer to be found.

Thinking Time

1. Given a line and a circle, describe possible cases using diagrams.
2. If we have a line and a parabola, how can we find the coordinates of the points of intersection?

Describe the method to find the coordinates of the points of intersection of a line and a curve.

Basic Level | Intermediate Level | Advanced Level

Exercise 6A

1 Find the centre and the radius of each of the following circles.
(a) $x^2 + y^2 = 49$
(b) $2x^2 + 2y^2 = 5$
(c) $(x+2)^2 + (y-3)^2 = 16$
(d) $x^2 + y^2 = 2x + 8$
(e) $x^2 + y^2 - x - 5y + 4 = 0$
(f) $4x^2 + 4y^2 - 6x + 10y = \frac{1}{2}$

2 Find the equation of each of the following circles.
(a) centre $(0, 0)$, radius 3
(b) centre $(-2, 3)$, radius 4
(c) centre $\left(\frac{1}{2}, -\frac{2}{5}\right)$, radius $1\frac{1}{2}$
(d) centre $(4, -1)$, passing through $(-2, 0)$
(e) centre $(3, -4)$, passing through $(9, -4)$

3 A diameter of a circle has its end points at $A(0, -1)$ and $B(2, 3)$. Find the equation of the circle.

4 Find the equation of the circle with the points $(-3, 2)$ and $(3, -2)$ as the end points of its diameter.

5 Find the points of intersection of the line $y = x + 4$ and the circle $(x + 1)^2 + (y - 4)^2 = 25$.

6 A diameter of a circle $x^2 + y^2 - 4x - 2y + 3 = 0$ passes through $(1, 3)$. Find the equation of this diameter.

7 Find the equation of a circle which touches the x-axis and whose centre is $C(5, 4)$.

8 Find the equation of the circle which passes through the point $(4, 2)$ and has a centre at $(2, -2)$.

9 The line $x - y - 3 = 0$ intersects the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ at two points A and B. Find the length of AB.

10 Given that a circle passes through the point $A(4, -7)$ and its centre is at the origin, find
(i) the equation of the circle,
(ii) the length of the chord passing through A, with gradient 2.

11 Given a circle with radius $\frac{5}{2}\sqrt{2}$ and centre $\left(-1\frac{1}{2}, \frac{1}{2}\right)$, find
(i) the equation of the circle,
(ii) the length of the chord $4x - 3y - 5 = 0$ of the circle.

12 Given that a circle which passes through the points $P(3, 5)$ and $Q(-1, 3)$ has radius $\sqrt{10}$, find
(i) the equation of the circle,
(ii) the equation of the line which passes through the centre and the midpoint of PQ.

13 Find the equation of the circle which passes through the points $A(0, 1)$ and $B(3, -2)$ and has its centre lying on the line $y + 3x = 2$.

14 Find the equation of the circle which passes through the points $P(-6, 5)$ and $Q(2, 1)$ and has its centre lying on the line $y - x = 4$.

15 The line $y + x = 0$ cuts the circle $x^2 + y^2 - 8x + 8y + 10 = 0$ at A and B. Show that the x-coordinates of A and B are $a + b\sqrt{11}$ and $a - b\sqrt{11}$ respectively, where a and b are integers to be found.

16 Explain why the line $4y = x - 3$ touches the circle $x^2 + y^2 - 4x - 8y + 3 = 0$.

17 The line $x + 7y = 25$ cuts the circle $x^2 + y^2 = 25$ at two points A and B.
(i) Find the coordinates of A and of B.
(ii) Find the equation of the perpendicular bisector of AB. Does it pass through the centre of the circle? Show your working clearly.
(iii) Find the coordinates of the points where the perpendicular bisector cuts the circle.

18 A straight line with a y-intercept of -3 intersects the circle $x^2 + y^2 - 27x + 41 = 0$ at $P(2, 3)$. Find
(i) the coordinates of the point Q at which the line meets the circle again,
(ii) the equation of the perpendicular bisector of PQ and determine whether it passes through the centre of the circle, showing your working clearly.

6.2 GRAPHS OF $y^2 = kx$

We have learnt that the graph of a quadratic function $y = ax^2 + bx + c$ is in the shape of a **parabola**. In this section, we will explore the graphs of the function $y^2 = kx$, which is also called a **parabola**.

Investigation

Graphs of $y^2 = kx$

Use a suitable graphing software to plot the graph of $y^2 = kx$ for $k = -1$ and $k = 1$. Repeat the above for

(i) $k = -2$ and $k = 2$, **(ii)** $k = -3$ and $k = 3$,
(iii) $k = -4$ and $k = 4$, **(iv)** $k = -5$ and $k = 5$.

1. All the curves pass through a particular point on the coordinate axes. What are the coordinates of this point?

2. Each pair of the curves is symmetrical about one of the axes. Name the axis.

3. State the equation of the line of symmetry of the curve $y^2 = kx$, where k is a constant.

4. Consider the curve $y^2 = kx$. How does the value of k affect the shape of the curve?

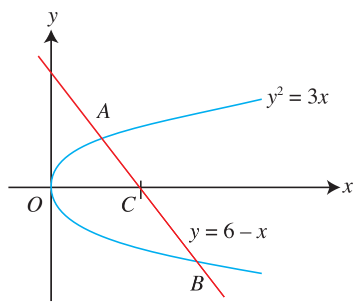
Worked Example 6

(Problem involving a Parabola and a Straight Line)

In the figure, the line $y = 6 - x$ cuts the curve $y^2 = 3x$ at the points A and B, and the x-axis at C. Find

(i) the coordinates of A, B and C,

(ii) the ratio $AC : CB$.



Solution

(i) $y = 6 - x$ -------------- (1)

$y^2 = 3x$ ----------------- (2)

Subst. (1) into (2):

$$\begin{aligned}(6 - x)^2 &= 3x\\ 36 - 12x + x^2 &= 3x\\ x^2 - 15x + 36 &= 0\\ (x - 3)(x - 12) &= 0\\ x = 3 \quad &\text{or} \quad x = 12\\ y = 3 \quad &\phantom{\text{or}} \quad y = -6\end{aligned}$$

$\therefore$ $A(3, 3)$ and $B(12, -6)$

Subst. $y = 0$ into (1): $x = 6$

$\therefore$ $C(6, 0)$

(ii)
$$\begin{aligned}AC &= \sqrt{(3-6)^2 + (3-0)^2}\\ &= \sqrt{18}\\ &= 3\sqrt{2}\\ CB &= \sqrt{(6-12)^2 + (0+6)^2}\\ &= \sqrt{72}\\ &= 6\sqrt{2}\end{aligned}$$

$$\begin{aligned}\therefore AC : CB &= 3\sqrt{2} : 6\sqrt{2}\\ &= 1 : 2\end{aligned}$$

Class Discussion

In Worked Example 6, to find the ratio $AC : CB$, we made use of the formula to find the length of the line segment AC and of CB. Discuss with your classmates how you can make use of similar triangles to obtain the answer. Which method do you prefer and why?

Practise Now 6

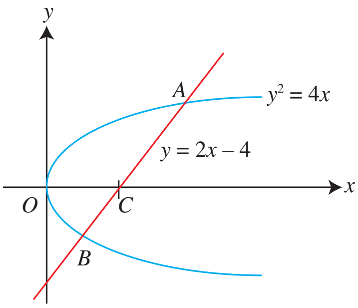
Similar Questions:
Exercise 6B
Questions 3, 4

In the figure, the line $y = 2x - 4$ cuts the curve $y^2 = 4x$ at A and B and the x-axis at C. Find

(i) the coordinates of A, B and C,

(ii) the length of AB,

(iii) the ratio $AC : CB$.



Thinking Time

Sketch the graphs of $y = x^2$ and $y^2 = x$.

1. Both graphs pass through a particular point. What are the coordinates of this point?
2. What can you say about the symmetry of the graphs?
3. For the function $y = x^2$, $y \geqslant 0$ for all values of x. Explain why.
4. In which quadrants does the graph of $y^2 = x$ lie? What can you say about the values of x?
5. How are the two graphs related?

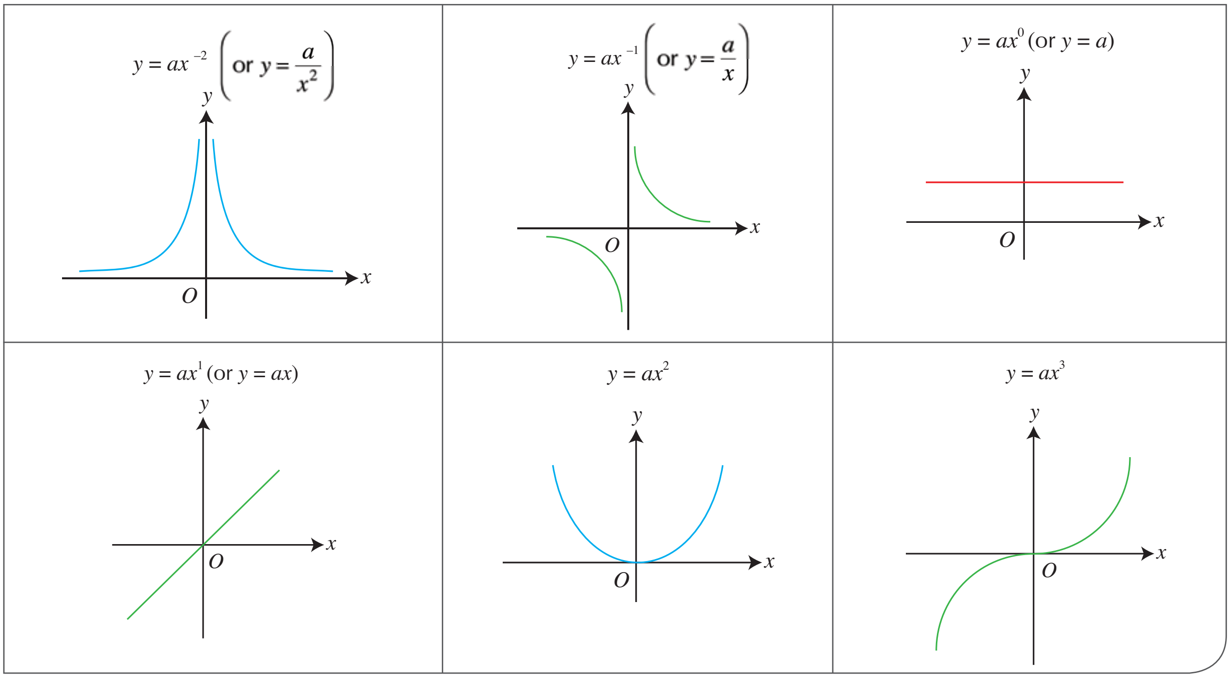
Journal Writing

Search on the Internet for examples of the applications of parabolas in real life and in the sciences. Explain how parabolas are used in each case.

6.3 GRAPHS OF POWER FUNCTIONS

We have learnt the graphs of $y = ax^n$, where $n = -2, -1, 0, 1, 2$ and 3.

The figures below show the shapes of the graphs of $y = ax^n$, where $a > 0$.



Note that when n is even, the curve is symmetrical about the y-axis;
when n is odd, the curve has a rotational symmetry about the origin.

Graphs of $y = ax^n$

We shall now explore the graphs of $y = ax^n$, where n is a simple rational number, e.g. $y = x^{\frac{1}{2}}$ $\left(\text{or } y = \sqrt{x}\right)$.

Investigation

Graphs of $y = x^n$

1. Use a suitable graphing software to plot the graph of $y = x^n$, for each of the following cases and answer the questions below.

   (i) $n = \frac{1}{2}$, $n = \frac{1}{3}$, $n = \frac{1}{5}$, $n = \frac{2}{3}$ **(ii)** $n = \frac{3}{2}$, $n = \frac{5}{3}$, $n = \frac{8}{3}$

   (iii) $n = -\frac{1}{2}$, $n = -\frac{1}{3}$, $n = -\frac{1}{5}$, $n = -\frac{2}{3}$, $n = -\frac{3}{2}$

   (a) What can you say about the index?
   (b) For $x > 0$, what happens to the value of y as x increases?
   (c) For $x > 0$, what happens to the value of y as x approaches 0?
   (d) Does the equation $y = 0$ have any solution? Justify your answer.
2. Repeat the above for $y = -x^n$.
3. For a particular value of n, how are the graphs of $y = x^n$ and $y = -x^n$ related for $x > 0$?
4. Using a graphing software, investigate how the value of a affects the graph of $y = ax^{\frac{1}{3}}$.

Now that we are familiar with the graphs of $y^2 = kx$ and $y = ax^n$, let us consider the case where $k = 1$, $a = 1$ and $n = \frac{1}{2}$, i.e. the graphs of $y^2 = x$ and $y = x^{\frac{1}{2}}$.

Thinking Time

1. Sketch the graphs of $y^2 = x$ and $y = x^{\frac{1}{2}}$. How are they related? Explain your answer.
2. In the investigation on the previous page, describe how the value of a affects the graph of $y = ax^n$ for different values of n, when $x > 0$.

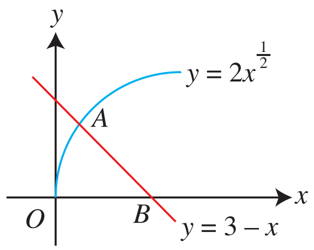
Worked Example 7

(Problem involving a Curve and a Straight Line)

In the figure, the line $y = 3 - x$ intersects the curve $y = 2x^{\frac{1}{2}}$ and the x-axis at the points A and B respectively. Find

(i) the coordinates of A and of B,

(ii) the equation of the perpendicular bisector of AB.



Solution

(i)

$$y = 3 - x \quad \text{---------------} \ (1)$$

$$y = 2x^{\frac{1}{2}} \quad \text{---------------} \ (2)$$

Solving (1) and (2),

$$2x^{\frac{1}{2}} = 3 - x$$

Squaring both sides,

$$\begin{aligned} 4x &= (3 - x)^2 \\ x^2 - 10x + 9 &= 0 \\ (x - 1)(x - 9) &= 0 \\ x = 1 \quad &\text{or} \quad x = 9 \\ y = 2 \quad &\phantom{\text{or}} \quad y = -6 \end{aligned}$$

$\therefore$ Since A lies in the first quadrant, the coordinates of A are $(1, 2)$.

Since B lies on the x-axis, $y = 0$

i.e. $x = 3$

$\therefore B(3, 0)$

(ii) Midpoint of $AB = \left(\frac{1+3}{2}, \frac{2+0}{2}\right) = (2, 1)$

Gradient of $AB = \frac{2-0}{1-3} = -1$

Gradient of perpendicular bisector of $AB = 1$

Equation of perpendicular bisector of AB:

$$\begin{aligned} y - 1 &= 1(x - 2) \\ y &= x - 1 \end{aligned}$$

PROBLEM-SOLVING TIP

We must substitute each pair of values into both sets of equations to eliminate any extraneous solutions.

Practise Now 7

Similar Questions:
Exercise 6B
Questions 5-7

The line $y = x - 2$ intersects the curve $y = x^{\frac{1}{2}}$ and the x-axis at the points A and B respectively. Find

(i) the coordinates of A and of B,

(ii) the equation of the perpendicular bisector of AB.

Basic Level | Intermediate Level | Advanced Level

Exercise 6B

1 Sketch the graph of each of the following equations for the specified range of values of x.

(a) $y = \frac{1}{2}x^{\frac{1}{2}}, x \geqslant 0$

(b) $y = -x^{\frac{2}{3}}, x \geqslant 0$

(c) $y = \frac{2}{3}x^{-\frac{1}{3}}, x > 0$

(d) $y = 4x^{\frac{5}{3}}, x \geqslant 0$

(e) $y = -2x^{\frac{3}{2}}, x \geqslant 0$

(f) $y = -x^{-\frac{3}{2}}, x > 0$

2 Sketch the graph of each of the following equations.

(a) $y^2 = 4x$ **(b)** $y^2 = -2x$

(c) $y^2 = \frac{1}{2}x$ **(d)** $y^2 = -\frac{1}{3}x$

(e) $y^2 + 5x = 0$ **(f)** $2y^2 + 3x = 0$

3 The line $y = 4 - x$ intersects the curve $y^2 = 2x$ at the points A and B. Find

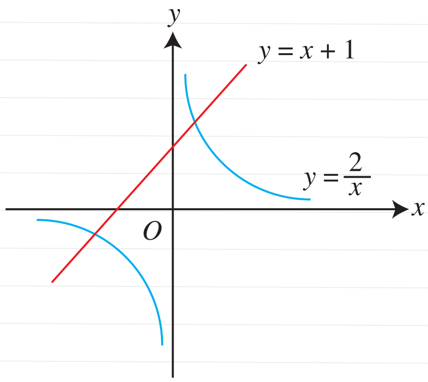
(i) the coordinates of A and of B,

(ii) the length of AB.

4 In the figure, the line $y = x + 1$ intersects the curve $y = \frac{2}{x}$ at the points A and B. Find

(i) the coordinates of A and of B,

(ii) the area of ΔOAB, where O is the origin.



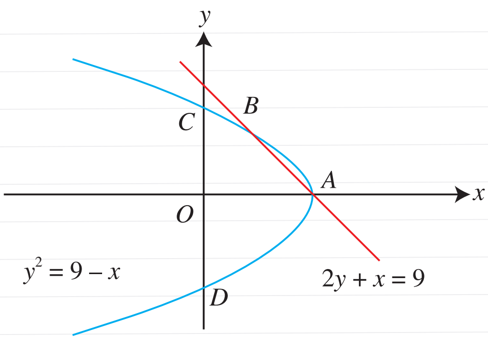
5 The coordinates of the points of intersection of the line $y = x + 2$ and the curve $y = \frac{1}{x}$ are given by $a \pm b\sqrt{c}$. Find the values of a, b and c.

6 The line $2y = x - 3$ intersects the curve $y = x^{\frac{1}{2}}$ and the x-axis at the points A and B respectively. Find

(i) the coordinates of A and of B,

(ii) the equation of the perpendicular bisector of AB.

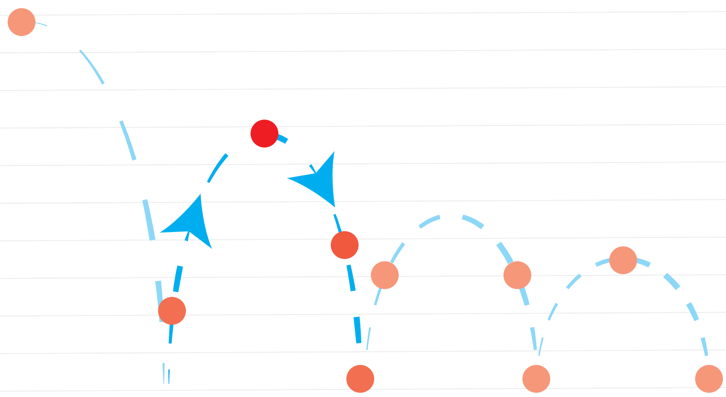
7 In the figure, the line $2y + x = 9$ intersects the curve $y^2 = 9 - x$ at the points A and B.



(i) Find the coordinates of A and of B.

(ii) Given that the curve cuts the y-axis at the points C and D, find the coordinates of C and of D. Hence, find the area of the quadrilateral $ABCD$.

8 The diagram shows the path of a bouncing ball.



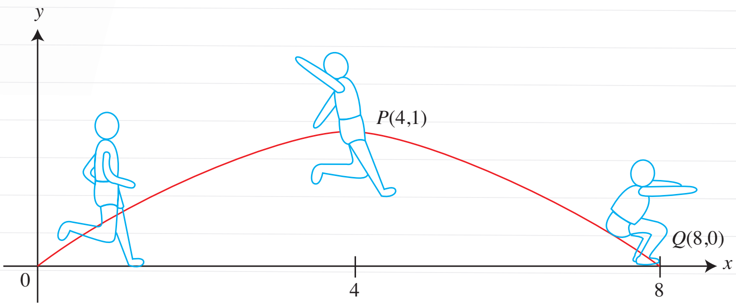
The equation of the parabola of the outlined path of the bouncing ball is $y = kx - x^2$.

(i) Given that the curve passes through the point $P(6, 36)$, find the value of k.

(ii) Hence, sketch the graph of the curve in the positive region of x and y, stating clearly the coordinates of the maximum point and the point where the curve cuts the x-axis.

Exercise 6B

9 The diagram shows the path of the centre of the athlete's body during a long jump.



INFORMATION

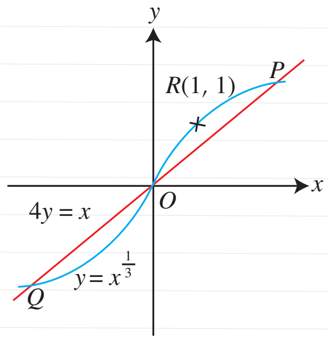
World record for long jump:
Men – Mike Powell (USA), 8.95 m in 1991; Women – Galina Chistoyakova (URS), 7.52 m in 1988

The equation for the parabola is represented by $ay = bx - x^2$. The highest point above the ground reached by the centre of the athlete's body is $P(4, 1)$ and the estimated landing point of the centre of the athlete's body on the ground is $Q(8, 0)$. Find

(i) the value of a and of b,

(ii) the distance between P and Q.

10 The line $4y = x$ intersects the curve $y = x^{\frac{1}{3}}$ at the points P, Q and O, where O is the origin.



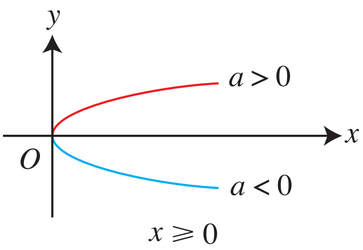
(i) Find the coordinates of P and of Q.

(ii) Show that the length of PQ is $4\sqrt{17}$ units.

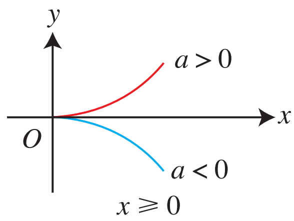
(iii) Given that $R(1, 1)$ lies on the curve, find the area of ΔPQR. Hence, explain clearly why the perpendicular distance from R to the line $4y = x$ is $\dfrac{3\sqrt{17}}{17}$ units.

SUMMARY

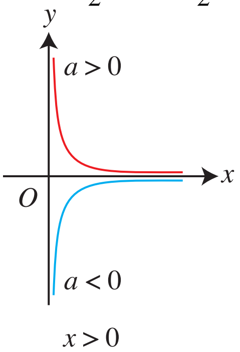
1. The equation of a circle in the standard form is $(x - a)^2 + (y - b)^2 = r^2$, where $r > 0$, (a, b) is the centre and r is the radius.
2. The general form of the equation of a circle is $x^2 + y^2 + 2gx + 2fy + c = 0$, where $g^2 + f^2 - c > 0$, $(-g, -f)$ is the centre and $\sqrt{g^2 + f^2 - c}$ is the radius.
3. Graphs of the form $y = ax^n$, where n is a rational number
   (a) n is a positive rational number less than 1 (e.g. $\frac{1}{2}$ and $\frac{1}{3}$)



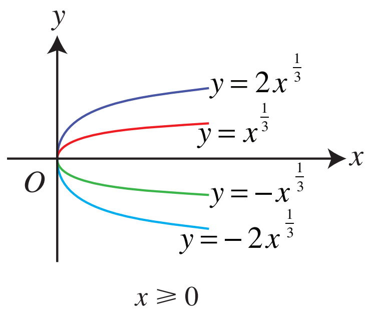
   (b) n is a positive rational number greater than 1 (e.g. $\frac{3}{2}$ and $\frac{5}{3}$)



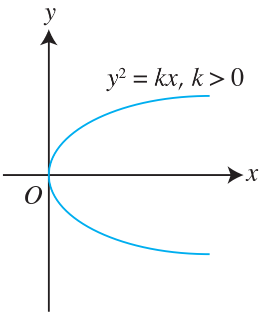
   (c) n is a negative rational number (e.g. $-\frac{1}{2}$ and $-\frac{3}{2}$)



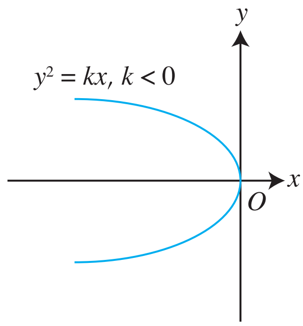
   (d) Effect of varying a



4. Graphs of the form $y^2 = kx$, where k is a real number
   (a) $k > 0$



   (b) $k < 0$



Review Exercise 6

1. Find the equation of the circle, where P is the centre and PQ is the radius.
   (a) $P(2, 7)$, $Q(5, 11)$
   (b) $P(-2, 9)$, $Q(4, 6)$

2. A weather satellite orbits planet P such that the equation of its path can be modelled by the equation
$$x^2 + y^2 - 16x - 12y + 75 = 0,$$
where x and y are the longitudinal and latitudinal distances from the centre of P respectively in kilometres, as shown on a map. State the coordinates of the centre and the radius of the orbit.

3. Find the centre and the radius of each of the following circles.
   (a) $x^2 + y^2 - 4x - 10y + 20 = 0$
   (b) $x^2 + y^2 + 4x - 4y - 17 = 0$
   (c) $3x^2 + 3y^2 = 16$
   (d) $(x + 1)(x - 5) + (y - 2)(y - 4) = 0$

4. Find the equation of the circle with centre $(0, 2)$ and radius 4. Hence, find the coordinates of the points where the circle cuts the x- and y-axes.

5. A diameter of a circle has end points at $A(-5, 0)$ and $B(9, 0)$.
   (i) Find the equation of the circle.
   (ii) If $C(2, k)$ lies on the circle, find the value of k. Hence, determine whether ΔABC is an isosceles right-angled triangle, showing your working clearly.

6. Find the equation of the circle which passes through the points $A(2, 1)$ and $B(3, -2)$ and has its centre lying on the line $y + x = 0$.

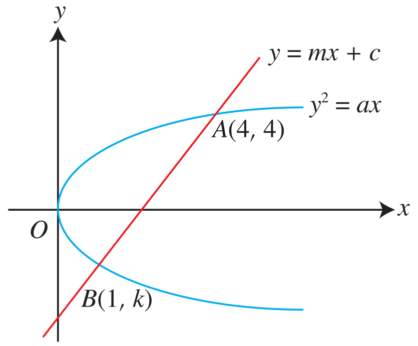
7. The coordinates of P, Q and R are (x, y), $(9, 0)$ and $(1, 0)$ respectively. Given that $PQ = 3PR$,
   (i) find a relationship between x and y,
   (ii) determine whether this relationship represents a circle, showing your working clearly.

8. **(i)** Given two points $P(1, 4)$ and $Q(-1, -2)$, find the equation of the circle with PQ as its diameter.
   (ii) Does the point $R(-3, 2)$ lie on the circle? Hence, determine whether $\angle PRQ$ is $90°$, explaining your working clearly.

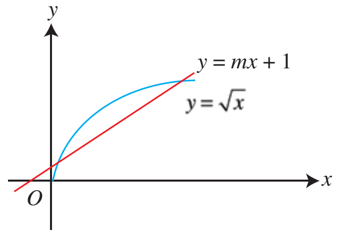
9. The curve $y^2 = -8x$ and the line $y = 2x + 6$ intersect at the points A and B. Show that the x-coordinates of A and B are $a + b\sqrt{7}$ and $a - b\sqrt{7}$ respectively. Find the value of a and of b.

10. The line $y + x = 6$ cuts the circle $x^2 + y^2 - 2x - 6y + 6 = 0$ at two points A and B.
   (i) Find the midpoint of AB.
   (ii) Find the equation of the perpendicular bisector of AB. Hence, show that it passes through the centre of the circle.
   (iii) Given that the perpendicular bisector cuts the circle at P and Q, show that the x-coordinates of P and Q are $a + b\sqrt{2}$ and $a - b\sqrt{2}$ respectively. Find the value of a and of b.

11. In the figure, the line $y = mx + c$ intersects the curve $y^2 = ax$ at $A(4, 4)$ and $B(1, k)$. Find the values of a, m, c and k.



12. In the figure, the line $y = mx + 1$ intersects the curve $y = \sqrt{x}$ at two distinct points. Given that $0 < m < k$, find the value of k.



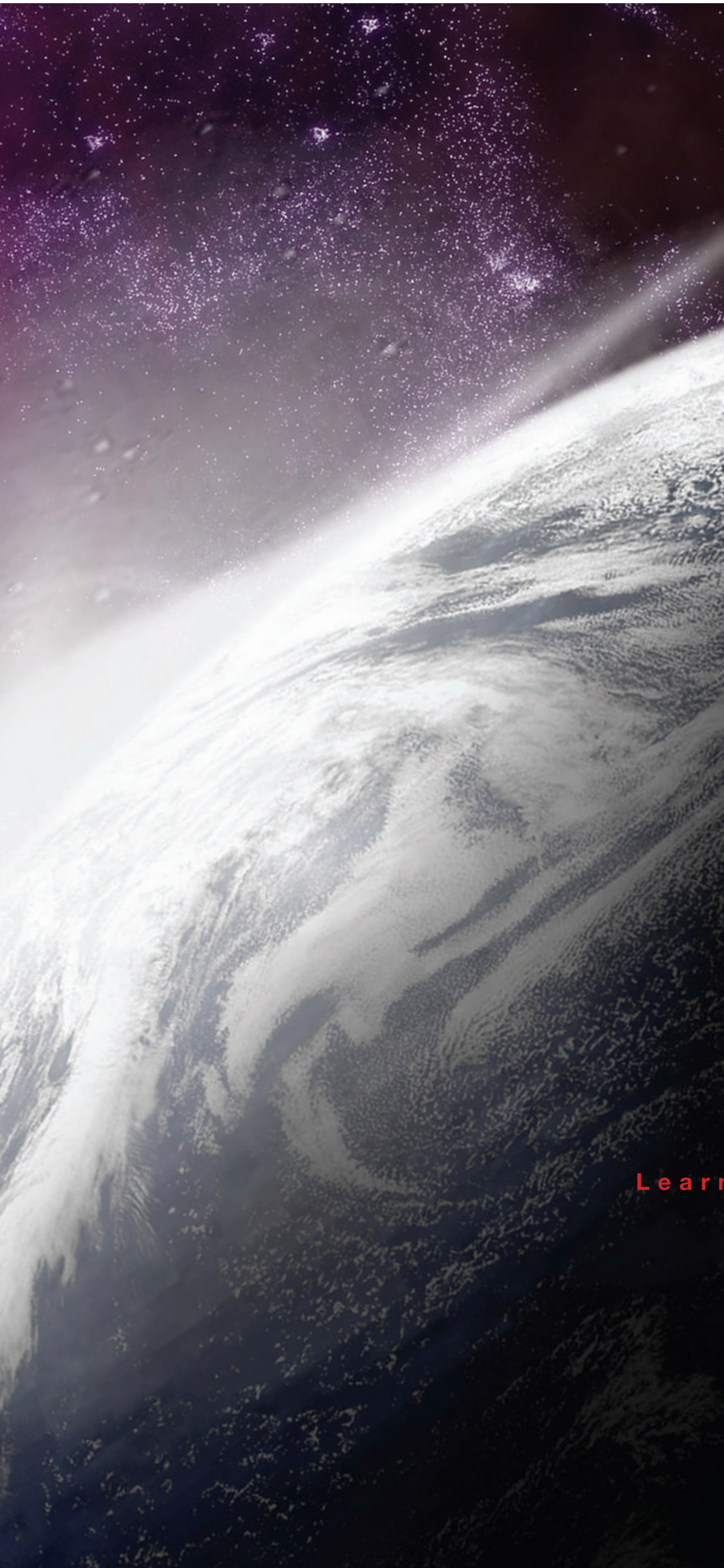
13. The line $2y = 7x + 1$ intersects the curve $y = \dfrac{4}{x^2}$ at the point P. Find the coordinates of P. Explain clearly why P is the only point of intersection.

1. Given that the y-axis is a tangent to the circle with radius 5 units and the circle passes through the point $(1, -4)$, find the possible equations of the circle.

2. Given three circles,
C_1: $(x + 1)^2 + (y - 3)^2 = 1$
C_2: $2x^2 + 2y^2 + 3x + y - 9 = 0$
C_3: $4x^2 + 4y^2 + 7x - 11y = 0$,
show that the centres of C_1 and C_2 are the ends of a diameter of C_3.

3. The circle $x^2 + y^2 - 6x - 9y + 8 = 0$ cuts the x-axis at A and B and the y-axis at C and D. Prove that $OA \times OB = OC \times OD$, where O is the origin.

LINEAR LAW



CHAPTER 7

excluded from the N(A) syllabus

We know that the Earth takes about 365.25 days to revolve around the Sun. This is called the orbital period of the Earth. How long do other planets take to revolve around the Sun? Johannes Kepler (1571-1630) discovered his third law of planetary motion in 1619 based on analyses of numerous astronomical data obtained by observing the movements of the planets around the Sun.

However, the formula for his third law on the orbital period of a planet is not linear: when the data are plotted, the points lie on a curve. How can we draw a curve to best fit the data? Are we able to obtain the equation of a curve? In this chapter, we will learn to transform non-linear relationships into linear form. This is called the **Linear Law**.

Learning Objectives

At the end of this chapter, you should be able to:

- transform a non-linear relationship to a linear form to determine the unknown constants from the straight line graph.

7.1 WHY STUDY LINEAR LAW?

Some phenomena in the sciences or in the real world can be **modelled** using an equation. We can conduct an experiment and collect some data and plot them on a graph. If all the points lie on a straight line, then the relationship between the two variables can be easily derived. This relationship can be used to predict further values which in turn serve to confirm the law experimentally. So how do we find their equations?

In this chapter, we will learn how to determine the equations of non-linear graphs by first converting them to a straight line.

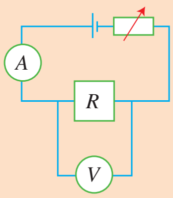
Investigation

Linear Law in the Sciences

Consider the following science experiments, which illustrate two examples of why we study Linear Law.

Part A: To Find a Formula for the Resistance of a Resistor

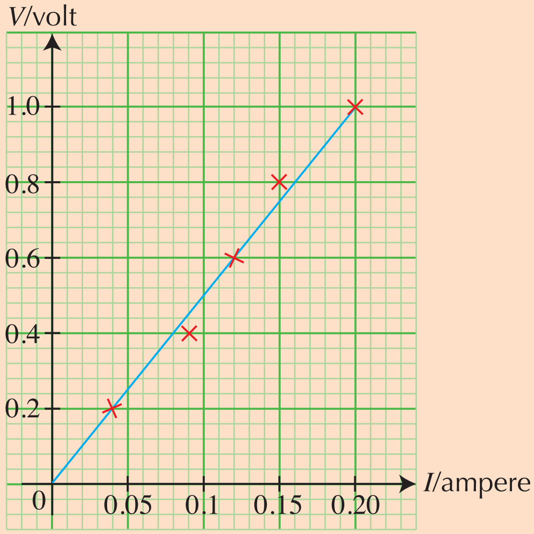
In the first experiment, we want to determine the resistance of a resistor. A circuit is set up as shown in Fig. 7.1(a). In this chapter, we will learn to transform given relationships of experimental data into linear form.



(a)

I/ampere	V/volt
0.04	0.2
0.09	0.4
0.12	0.6
0.15	0.8
0.20	1.0

(b)



(c)

Fig. 7.1

A graph of V against I is then plotted as shown in Fig. 7.1(c).

We observe that the points almost lie in a straight line, so a **straight line** can be drawn as shown above.

1. What is the gradient, m, of the straight line?

2. What is the V-intercept, c, of the straight line?

3. What is the equation of the straight line?

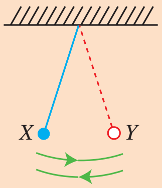
In other words, we have **modelled** the relationship between the potential difference, V volts, across a resistor and the current, I amperes, flowing through it by the equation $V = 5I$ by finding the unknowns m and c of the straight line graph.

The resistance of the resistor is given by $R = \frac{V}{I} = 5$ ohms.

Part B: To Find a Formula for the Period of a Pendulum

In the second experiment, the time taken for a simple pendulum to travel from X to Y and back to X (i.e. the period), T seconds, is measured for different lengths of string, L m, as shown in Fig. 7.2(a). The data collected are shown in Fig. 7.2(b).

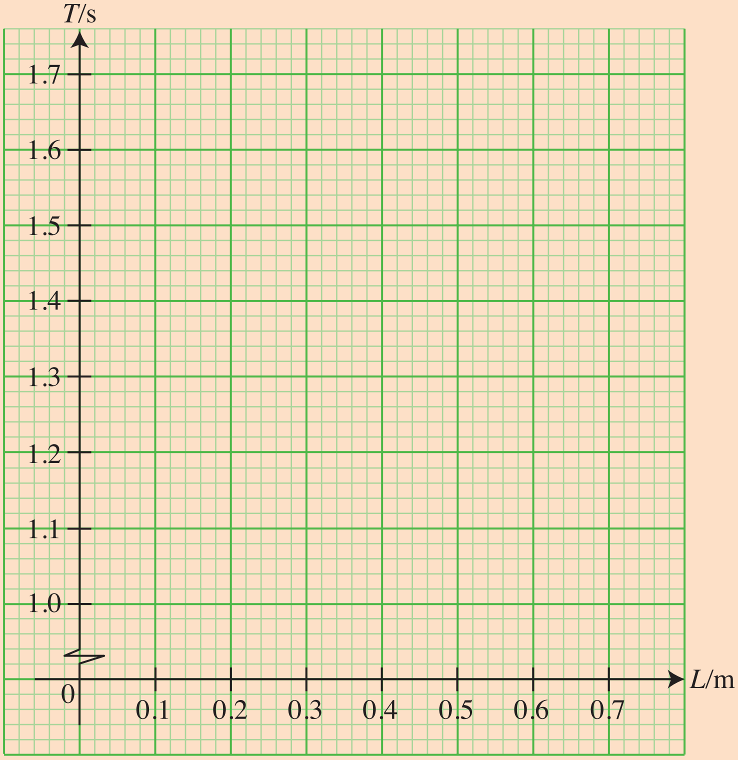
1. Copy the grid in Fig. 7.2(c) and plot the data.
2. Describe the graph obtained.



(a)

L/m	T/s
0.3	1.10
0.4	1.26
0.5	1.41
0.6	1.55
0.7	1.67

(b)



(c)

Fig. 7.2

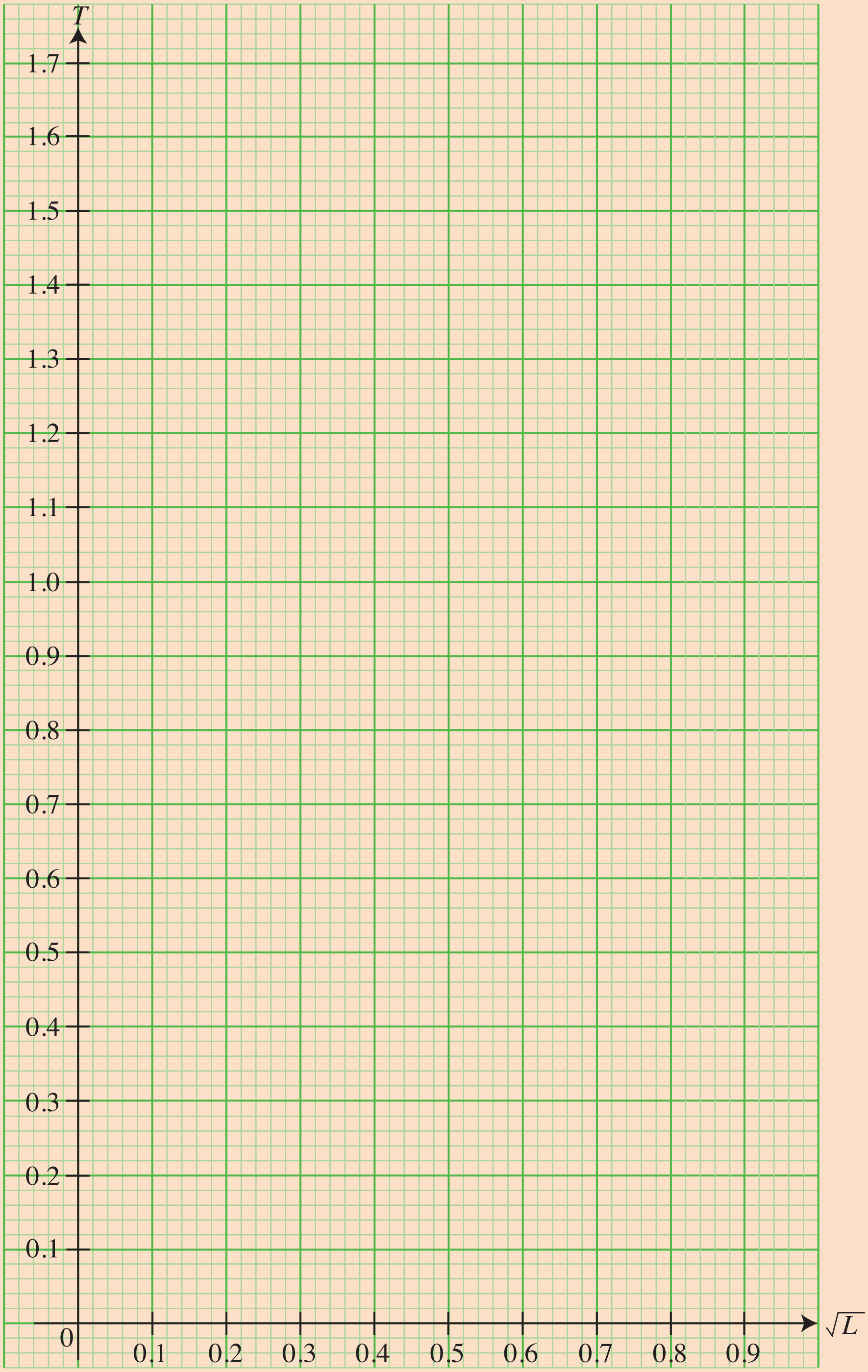
3. How can we find the equation of the graph in Fig. 7.2(c)?
4. If it is given that the equation of the curve is in the form $T = k\sqrt{L}$, how can we find the value of k? If we just choose any point (L, T) on the curve and substitute it into the equation, will the value of k be accurate? If we can find the equation of the curve using only one point, then why do we need so many data points?

   If we can draw a best-fitting curve, then we can use a point on the curve to determine the value of k. However, unlike the line of best fit, it is not easy to determine what constitutes a best-fitting curve. Hence, we need to use another method to find a relationship between T and L.

5. Let $Y = T$ and $X = \sqrt{L}$. Copy and complete the table in Fig. 7.2(d).
6. Copy the grid in Fig. 7.2(e) and plot the data.

$X = \sqrt{L}$	$Y = T$
0.55	1.10
	1.26
	1.41
	1.55
	1.67

(d)



(e)

Fig. 7.2

7. **(i)** Describe the graph obtained.
   (ii) What is the gradient, m, of the straight line?
   (iii) What is the Y-intercept, c, of the straight line?
   (iv) What is the equation of the straight line?
   (v) Since $Y = T$ and $X = \sqrt{L}$, what will the original equation be?

Therefore, the relationship between the period, T, of a simple pendulum and the length of the string, L, can be **modelled** by the equation $T = 2\sqrt{L}$.

Hence, we can obtain a relationship between T and L if we can convert the original equation into the linear form $Y = mX + c$, where the unknowns m and c can be found from the line of best fit. This is called the **Linear Law**. This is a reason why we study Linear Law: so that we can find the non-linear relation.

How can we tell that the equation of the curve is in the form $T = k\sqrt{L}$ and not other forms such as $T = kL^2$ or $T = k\sqrt[3]{L}$? How do we know that we have to plot $Y = T$ against $X = \sqrt{L}$ in order to obtain a straight line graph?

We have to use trial and error to determine what variables to plot in order to obtain a *straight line graph*.

7.2 CONVERTING FROM A NON-LINEAR EQUATION TO A LINEAR FORM

Before we can model more complicated situations using the Linear Law, we need to learn how to choose suitable variables for Y and X so that we can convert the original non-linear equation into the linear form $Y = mX + c$, where m is the gradient of the straight line and c is the Y-intercept.

Worked Example 1

(Convert Non-Linear Equation to Linear Form)

Convert the non-linear equation $y = -ax + \frac{b}{x}$, where a and b are constants, into the linear form $Y = mX + c$.

Solution

Method 1:

Multiply the equation throughout by x:

$$\underbrace{xy}_{Y} = \underset{m}{\underset{\uparrow}{-a}}\;\underbrace{x^2}_{X} + \underset{c}{\underset{\uparrow}{b}}$$

Let $Y = xy$ and $X = x^2$.

Then the equation becomes $Y = mX + c$, where $m = -a$ and $c = b$.

Method 2:

Divide the original equation throughout by x:

$$\frac{y}{x} = -a + \frac{b}{x^2}$$

i.e. $$\underbrace{\frac{y}{x}}_{Y} = \underset{m}{\underset{\uparrow}{b}}\underbrace{\left(\frac{1}{x^2}\right)}_{X} \underset{c}{\underset{\uparrow}{-\,a}} \text{ -------- } (*)$$

Let $Y = \frac{y}{x}$ and $X = \frac{1}{x^2}$.

Then the equation becomes $Y = mX + c$, where $m = b$ and $c = -a$.

PROBLEM-SOLVING TIP

The variables X and Y in $Y = mX + c$ must contain only the original variables x and/or y, but they must *not* contain the original unknown constants a and/or b. Similarly, the constants m and c must contain only the original unknown constants a and/or b, but they must *not* contain the original variables x and/or y.

Method 3:

Divide the equation (*) in Method 2 throughout by b: $\frac{1}{b}\left(\frac{y}{x}\right) = \frac{1}{x^2} - \frac{a}{b}$

Rearranging, we have: $\frac{1}{x^2} = \frac{1}{b}\left(\frac{y}{x}\right) + \frac{a}{b}$

Refer to Worked Example 1.

Discuss with your classmates what variables to choose for X and Y in Method 3, and what the unknowns m and c are.

Observe that there may be more than one method to convert a non-linear equation into the linear form. Which method do you prefer for this example?

Practise Now 1

Similar Questions:
Exercise 7A
Questions 1(a)-(f), 2(g), (h)

Convert each of the following non-linear equations, where a, b, h, k, p and q are constants, into the linear form $Y = mX + c$. State what the variables X and Y and the constants m and c represent. There may be more than one method to do this.

(a) $y = ax^2 + bx$ **(b)** $y = x^3 + ax + b$ **(c)** $y = a\sqrt{x} + \frac{b}{\sqrt{x}}$

(d) $\frac{1}{y} = ax + \frac{b}{x}$ **(e)** $hx + ky = xy$ **(f)** $y = \frac{px}{x+q}$

Worked Example 2

(Convert Exponential Equation into Linear Form)

Convert $y = ae^{bx}$, where a and b are constants, into the linear form.

Solution

$$y = ae^{bx}$$
$$\ln y = \ln ae^{bx} \quad \text{(take ln on both sides)}$$
$$= \ln a + \ln e^{bx}$$
$$= \ln a + bx \ln e$$
$$= bx \ln e + \ln a$$
$$\underbrace{\ln y}_{Y} = \underbrace{b}_{m}\underbrace{x}_{X} + \underbrace{\ln a}_{c} \quad \text{since } \ln e = 1$$

Let $Y = \ln y$ and $X = x$.

Then the equation becomes $Y = mX + c$, where $m = b$ and $c = \ln a$.

PROBLEM-SOLVING TIP

The mathematical constant e has a fixed value, unlike the unknown constants a and/or b. Use ln if base e appears in the equation; use lg if base 10 appears in the equation; otherwise, we can use either ln or lg.

Practise Now 2

Similar Questions:
Exercise 7A
Questions 1(g), (h) 2(a)-(f)

Convert each of the following non-linear equations, where a, b, h, n, p and q are constants, into the linear form $Y = mX + c$. State what the variables X and Y and the constants m and c represent. There may be more than one method to do this.

(a) $y = pe^{qx}$ **(b)** $y = ax^{-n}$ **(c)** $\frac{a}{y} = b^x$

(d) $y^p x^q = 10$ **(e)** $y = x^3 + a^{x-h}$ **(f)** $y^n = hx^2$

7.3 CONVERTING FROM A LINEAR FORM TO A NON-LINEAR EQUATION

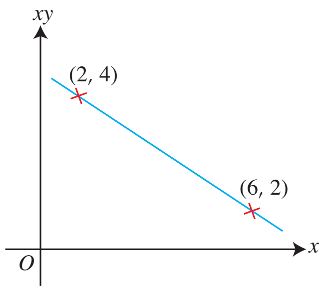
Worked Example 3

(Convert Linear Form to Non-Linear Equation)

The variables x and y are connected by the equation $y = \frac{a}{x} - b$, where a and b are constants. When a graph of xy against x is drawn, the resulting line passes through the points (2, 4) and (6, 2) as shown in the figure. Find

(i) the value of a and of b,

(ii) the value of y when $y = \frac{6}{x}$.



Solution

(i) Let $Y = xy$ and $X = x$. Then the equation of the straight line is $Y = mX + c$.

$$\text{Gradient } m = \frac{Y_2 - Y_1}{X_2 - X_1} = \frac{2-4}{6-2} = -\frac{1}{2}$$

$\therefore$ Equation of the straight line is $Y = -\frac{1}{2}X + c$

Since $(2, 4)$ lies on the straight line,

$$4 = -\frac{1}{2}(2) + c$$
$$4 = -1 + c$$
$$c = 5$$

$\therefore$ Equation of the straight line is

$Y = -\frac{1}{2}X + 5.$

ATTENTION

Since (2, 4) lies on the straight line $Y = mX + c$, then $Y = 4$ when $X = 2$ (and not $y = 4$ when $x = 2$). Alternatively,

$$Y - Y_1 = m(X - X_1)$$
$$Y - 4 = -\frac{1}{2}(X - 2)$$
$$Y - 4 = -\frac{1}{2}X + 1$$
$$Y = -\frac{1}{2}X + 5$$

Method 1:

Since $Y = xy$ and $X = x$, the original

non-linear equation is $xy = -\frac{1}{2}x + 5$ -------- (1)

$$y = -\frac{1}{2} + \frac{5}{x}$$
$$y = \frac{5}{x} - \frac{1}{2}$$

Comparing with the given equation $y = \frac{a}{x} - b$,

then $a = 5$ and $b = \frac{1}{2}$.

Method 2:

The original non-linear equation is $y = \frac{a}{x} - b$.

$xy = a - bx$ (multiply throughout by x)

i.e. $xy = -bx + a$ -------- (2)

Since $Y = xy$ and $X = x$, then $\underbrace{xy}_{Y} = \underbrace{-b}_{m}\underbrace{x}_{X} + \underbrace{a}_{c}$ is in the form $Y = mX + c$, where $m = -b$ and $c = a$.

$\therefore\ b = -m = -\left(-\frac{1}{2}\right) = \frac{1}{2}$

and $a = c = 5$

(ii) When $y = \frac{6}{x}$, $xy = 6$.

Subst. $xy = 6$ into (1) or (2): $6 = -\frac{1}{2}x + 5$

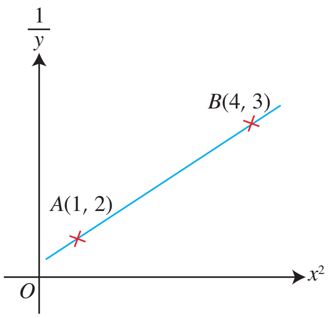
$x = -2$

Subst. $x = -2$ into $y = \frac{6}{x}$: $y = \frac{6}{-2} = -3$

Practise Now 3

Similar Questions:
Exercise 7A
Questions 3-7

1. The variables x and y are connected by the equation $y = \frac{a}{x^2 + b}$, where a and b are constants. When a graph of $\frac{1}{y}$ against x^2 is drawn, the resulting line passes through the points (1, 2) and (4, 3) as shown in the figure. Find



(i) the value of a and of b,

(ii) the value of y when $2x^2y = 1$.

2. The variables x and y are related by the equation $y = kx^n$, where k and n are constants. When a graph of lg y against lg x is plotted, the resulting line passes through the point (4, 1) and cuts the lg y-axis at 3. Find

(i) the value of k and of n,

(ii) the value of y when $y = x$.

Just For Fun

Search on the Internet for 'Presentation Helpdesk Who Wants to be a Millionaire', then search within the website for 'Linear Law (2009)', where you can play *Who Wants to be a Millionaire*, on questions relating to Linear Law.

Exercise 7A

1 Convert each of the following non-linear equations, where a, b, h, k, p and q are constants, into the linear form $Y = mX + c$. State what the variables X and Y and the constants m and c represent. There may be more than one method to do this.

(a) $y = ax^3 - bx$ **(b)** $y = x^2 - bx + a$

(c) $y = \dfrac{h}{x^2} + k$ **(d)** $y = hx^2 + \dfrac{k}{x}$

(e) $\dfrac{1}{y} = \dfrac{p}{\sqrt{x}} + q\sqrt{x}$ **(f)** $ay + xy = bx$

(g) $y = px^n$ **(h)** $y = 10^{ax + b}$

2 Convert each of the following non-linear equations, where a, b, h, k, p and q are constants, into the linear form $Y = mX + c$. State what the variables X and Y and the constants m and c represent. There may be more than one method to do this.

(a) $y = ae^{-bx}$ **(b)** $ay = h^x$

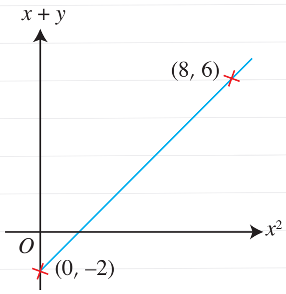
(c) $x^a y^b = \mathrm{e}$ **(d)** $y^n = h^2x^3$

(e) $\dfrac{1}{y} = x^2 + a^{b-x}$ **(f)** $y^x = pq^{x^2}$

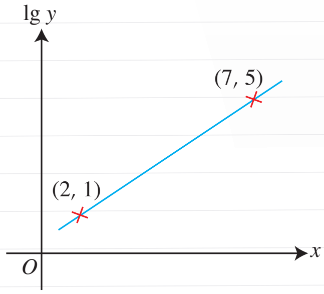
(g) $y = \dfrac{hx^2}{x + k}$ **(h)** $k^2y + 1 = x^3y + hxy$

3 Each of the following shows part of a straight line graph obtained by plotting values of the variables indicated. Express y in terms of x.

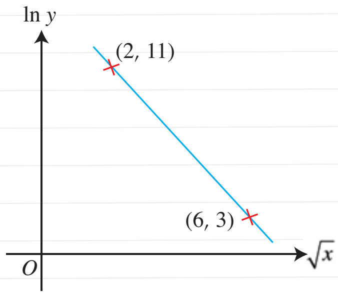
(a)



(b)



(c)



4 The variables x and y are connected by the equation $ay = x + bx^2$. When a graph of $\dfrac{y}{x}$ is plotted against x, the resulting line has a gradient of 5 and an intercept on the $\dfrac{y}{x}$-axis of 0.25. Find the value of a and of b.

5 The variables x and y are related in such a way that when a graph of x against $\dfrac{x^2}{y}$ is drawn, the resulting line has a gradient of 3 and an x-intercept of 4.

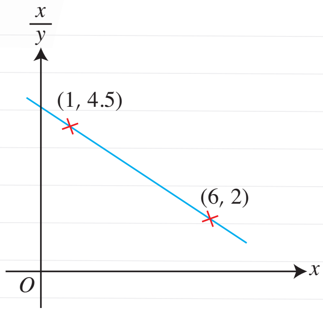
(i) Express y in terms of x.

(ii) Find the value(s) of x when $y = 48$.

6 The variables x and y are connected by the equation $y = ax^b$, where a and b are constants. When a graph of $\lg y$ against $\lg x$ is plotted, the resulting line has a gradient of 1.5 and a $\lg y$-intercept of 1.2. Find the value of a and of b.

Exercise 7A

7 The figure shows part of the straight line drawn to represent the curve $2x = 3ay + bxy$. Find the value of a and of b.



8 A straight line graph is drawn for $\lg y$ against x to represent the curve $y = A^{x+B}$. Given that the line passes through the points (0, 2) and (5, 12), find the value of A and of B. Hence, find the value of x when $\frac{\lg y}{x} = 5$.

9 Variables x and y are related by the equation of the form $y = 10^{a(x+b)}$, where a and b are constants. When $\lg y$ is plotted against x, a straight line which passes through the points (1, –2) and (5, 4) is obtained. Find the value of a and of b. Hence, find the value of $\lg y^{2x}$ when $x = 3$.

7.4 APPLICATIONS OF LINEAR LAW

In Section 7.1, we have discussed the rationale for studying Linear Law: to find non-linear equations in science experiments. In this section, we will apply what we have learnt in Sections 7.2 and 7.3 to convert a non-linear equation into the linear form to determine the unknowns in order to find the original non-linear equation.

Worked Example 4

(Determining Unknown Constants from the Straight Line Graph)

The pairs of values of x and y in the table below are obtained from an experiment. It is believed that they obey the relation $y = ax^2 + bx$, where a and b are constants.

x	2	4	6	8	10	12
y	6.0	18.8	35.0	40.0	93.0	123.7

One of the values of y is believed to be inaccurate. Using a scale of 1 cm to represent 1 unit on both axes, plot the values of $\frac{y}{x}$ against x and draw a straight line to fit the data.

(i) Use your graph to estimate the value of a and of b.

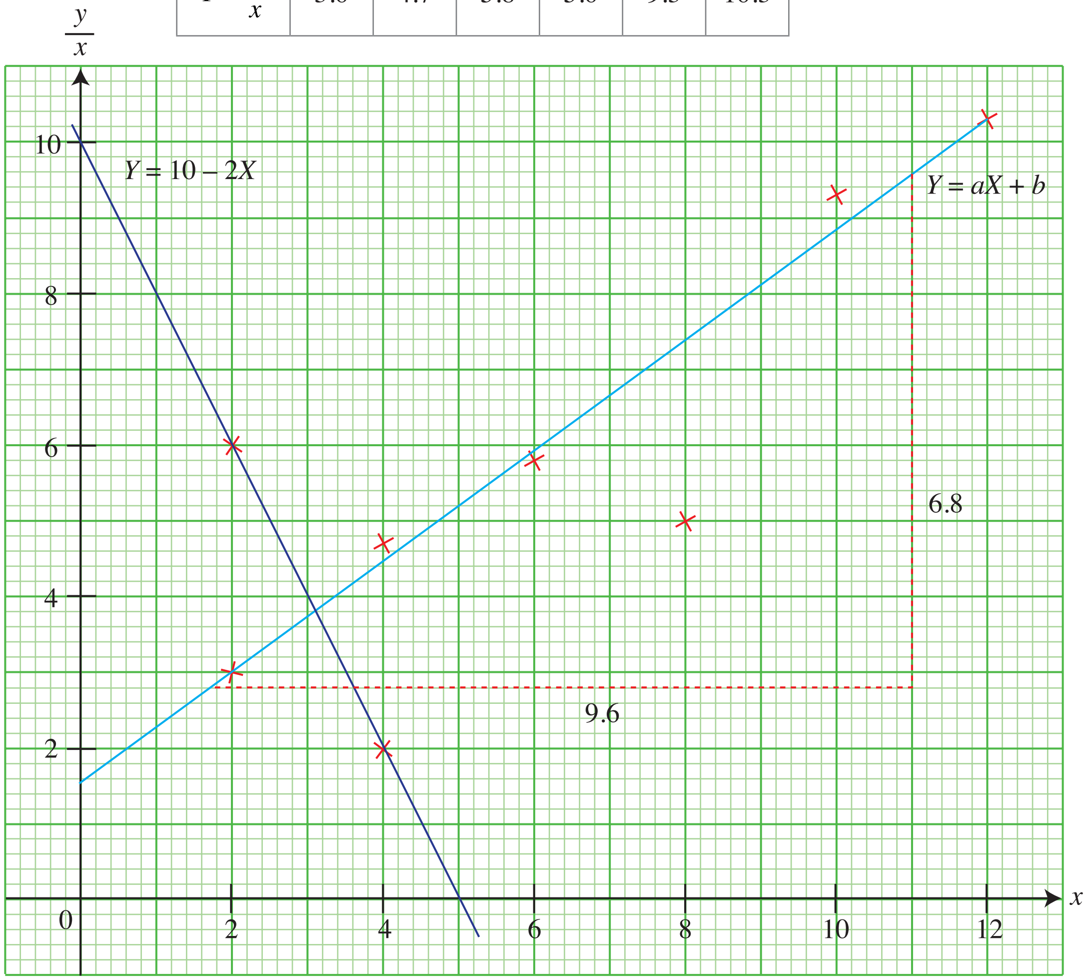
(ii) Determine which value of y is inaccurate and estimate its correct value.

(iii) On the same diagram, draw the line representing the equation $y = 10x - 2x^2$ and hence find the value of x for which $a + \frac{b}{x} = -2 + \frac{10}{x}$.

Solution

Let $Y = \frac{y}{x}$ and $X = x$.

$X = x$	2	4	6	8	10	12
$Y = \frac{y}{x}$	3.0	4.7	5.8	5.0	9.3	10.3



(i) $y = ax^2 + bx$

$\frac{y}{x} = ax + b$ (divide throughout by x)

Since $Y = \frac{y}{x}$ and $X = x$, then $\frac{y}{x} = ax + b$ is in the form $Y = mX + c$, where gradient $m = a$ and Y-intercept $c = b$.

From the graph,

$$\text{gradient } m \approx \frac{6.8}{9.6} \approx 0.71$$

and Y-intercept $c \approx 1.6$.

$\therefore a \approx 0.71$ and $b \approx 1.6$

(ii) Because the point (8, 5.0) lies very far from the straight line in the graph, $y = 40.0$ is inaccurate.

From the graph,

when $X = x = 8$, $Y = \frac{y}{x} \approx 7.4$.

Thus $\frac{y}{8} \approx 7.4$, i.e. $y \approx 7.4 \times 8 \approx 59$.

PROBLEM-SOLVING TIP

For this type of questions, the answers may not be accurate to 3 significant figures, so we just use the approximation sign $\approx$ to *estimate* the values without specifying the degree of accuracy.

INFORMATION

Although $7.4 \times 8 = 59.2$, the answer cannot be so accurate because 7.4 is at most correct to 2 significant figures so we just estimate y to be ≈ 59.

(iii) $y = 10x - 2x^2$

$\frac{y}{x} = 10 - 2x$ (divide throughout by x)

Since $Y = \frac{y}{x}$ and $X = x$, then $Y = 10 - 2X$.

To plot $Y = 10 - 2X$:

X	0	2	4
Y	10	6	2

The graph of $Y = 10 - 2X$ is shown on the same diagram as the graph in **(ii)**.

$y = ax^2 + bx$ -------- (1)

$y = 10x - 2x^2$ -------- (2)

Equating (1) and (2): $ax^2 + bx = 10x - 2x^2$

$a + \frac{b}{x} = \frac{10}{x} - 2$ (divide throughout by x^2)

$a + \frac{b}{x} = -2 + \frac{10}{x}$

Hence, to solve $a + \frac{b}{x} = -2 + \frac{10}{x}$ is the same as solving (1) and (2) simultaneously. This is the same as finding the point of intersection of the two graphs. From the graph, the value of $x \approx 3.1$.

Practise Now 4

Similar Questions:
Exercise 7B
Questions 1, 2, 5, 7

The table below shows experimental values of two variables x and y. It is known that x and y are related by an equation of the form $y = x^2 + ax - b$, where a and b are constants. One of the values of y is believed to be inaccurate.

x	1.5	2	2.5	3	3.5
y	6.0	6.4	8.3	9.1	11

Using a horizontal scale of 1 cm to represent 1 unit, draw a suitable horizontal axis.

Using a vertical scale of 2 cm to represent 1 unit, draw a vertical axis from –3 to 7.

On your axes, draw a straight line graph of $y - x^2$ against x to represent the above data.

(i) Use your graph to estimate the value of a and of b.

(ii) Determine which value of y is inaccurate and estimate its correct value.

(iii) On the same diagram, draw the line representing the equation $\frac{y-1}{x} = x$ and hence find the value of x for which $\frac{a}{b} - \frac{1}{bx} = \frac{1}{x}$.

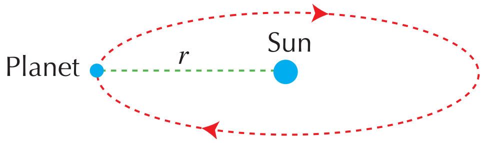
PROBLEM-SOLVING TIP

Since we need to find the Y-intercept, the X-axis must start from 0.

Worked Example 5

(Application of Linear Law)

In the beginning of the chapter, it was mentioned that Kepler discovered his Third Law of Planetary Motion, which states that the period T, in years, of a planet's orbit around the Sun, is given by $T = kr^n$, where r is the furthest distance, in metres, of the planet from the Sun, and k and r are constants to be determined.



The values of r and T for some planets are given in the table below.

Planet	Mercury	Venus	Earth	Mars	Jupiter
r/m	58.3×10^9	108×10^9	150×10^9	227×10^9	778×10^9
T/years	0.24	0.62	1.00	1.88	11.9

(i) Using a scale of 2 cm to represent 1 unit on the ln r-axis and 1 cm to represent 1 unit on the ln T-axis, draw a straight line graph of ln T against ln r to represent the above data.

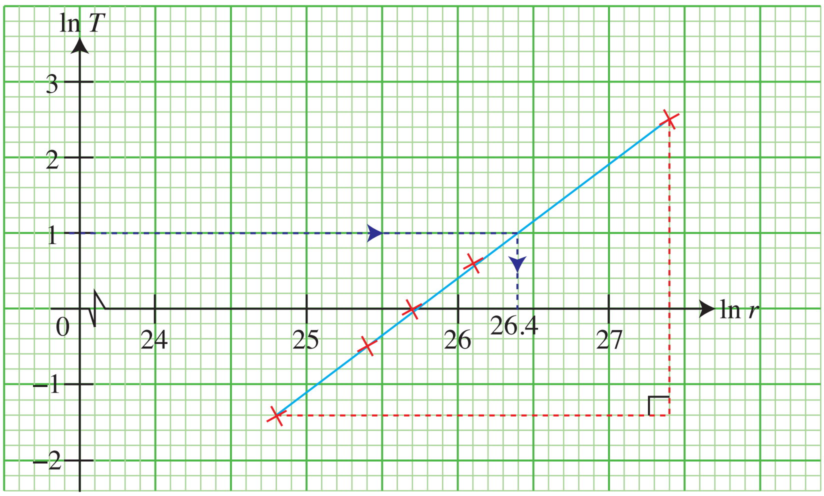
(ii) Use your graph to estimate the value of n and of k, giving your answer correct to 2 significant figures.

(iii) An unknown object was observed to revolve around the Sun with a period of e years. Estimate from your graph its furthest distance from the Sun, leaving your answer in standard form correct to 2 significant figures.

Solution

(i) Let $X = \ln r$ and $Y = \ln T$.

Planet	Mercury	Venus	Earth	Mars	Jupiter
$X = \ln r$	24.8	25.4	25.7	26.1	27.4
$Y = \ln T$	−1.43	−0.48	0	0.63	2.48



(ii) From the graph, gradient of line $= \dfrac{Y_2 - Y_1}{X_2 - X_1}$

$$= \frac{2.50 - (-1.40)}{27.4 - 24.8}$$

$$= \frac{3.90}{2.6}$$

$$= 1.5 \text{ (to 2 s.f.)}$$

Since the Y-intercept cannot be observed directly from the graph, we substitute the coordinates of the point (27.4, 2.50) into the equation of the line.

$$Y = mX + c$$
$$2.50 = 1.5 \times 27.4 + c$$
$$c = -38.6$$

Converting $T = kr^n$ into linear form,

$$\ln T = \ln (kr^n)$$
$$= \ln k + \ln r^n$$
$$\underbrace{\ln T}_{Y} = \underbrace{n}_{m} \underbrace{\ln r}_{X} + \underbrace{\ln k}_{c}$$

$\therefore$ Gradient, $n = 1.5$

and Y-intercept, $c = \ln k = -38.6$

$$k = e^{-38.6} = 1.7 \times 10^{-17} \text{ (to 2 s.f.)}$$

(iii) $T = e$

$Y = \ln T = \ln e = 1$

Draw the line $Y = 1$ on the same graph.

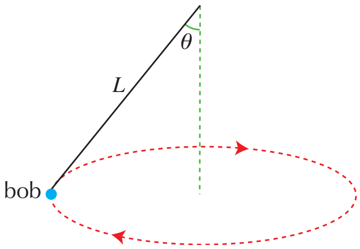
From the graph, $X = \ln r = 26.4$

$$r = e^{26.4} = 2.9 \times 10^{11} \text{ m (to 2 s.f.)}$$

Practise Now 5

Similar Questions:
Exercise 7B
Questions 3, 4, 6, 8-11

The figure shows a pendulum where the bob rotates in a circle in a horizontal plane. The string is at a *fixed* angle θ from the vertical. An experiment is conducted to measure the period of oscillation, T, in seconds, of the pendulum for different lengths L, in metres, of the string. The data collected are shown in the table below.



L/m	0.3	0.5	0.7	0.9	1.1
T/s	1.023	1.279	1.563	1.824	1.959

It is believed that T and L are connected by the equation $T = aL^n$, where a and n are constants.

(i) Draw a suitable straight line graph of lg T against lg L, using a scale of 1 cm to represent 0.05 unit on the lg L-axis and a scale of 1 cm to represent 0.02 unit on the lg T-axis.

(ii) Use the graph to estimate the value of a and of n.

(iii) It is known that $a = 2\pi\sqrt{\dfrac{\cos\theta}{g}}$, where g is the acceleration due to gravity (≈ 9.8 m/s^2) and θ is the angle of the string from the vertical.

(a) State the formula relating the period T of oscillation of a conical pendulum in terms of its length L and the angle θ from the vertical.

(b) Find the angle θ used in this experiment.

Exercise 7B

1 The following table gives experimental values of two variables x and y which are known to be connected by a relation of the form $xy = a + bx$.

x	0.4	0.6	0.8	1.0	1.2
y	22.0	15.3	12.0	10.0	8.7

Using a horizontal scale of 4 cm to represent 1 unit and a vertical scale of 1 cm to represent 1 unit, plot a graph of y against $\frac{1}{x}$ and use it to estimate the value of a and of b.

2 The table below gives experimental values of x and y. It is known that the true values of x and y are connected by a law $y = ax^3 + bx^2$, where a and b are constants.

x	4	5	6	7	8
y	0	–51.3	–142	–293	–511

Using a horizontal scale of 2 cm to represent 1 unit and a vertical scale of 1 cm to represent 1 unit, plot the graph of $\frac{y}{x^2}$ against x. Use your graph to estimate

(i) the value of a and of b,

(ii) the value of x when $\frac{y}{x^2} = 2$.

3 The table below shows two variables x and y which are known to be connected by a law in the form $y = kb^x$.

x	1	2	3	4	5
y	30	75	190	550	1200

Using a horizontal scale of 2 cm to represent 1 unit and a vertical scale of 4 cm to represent 1 unit, plot the graph of lg y against x. Use your graph to

(i) estimate the value of k and of b,

(ii) estimate the value of y when $x = 2.5$,

(iii) determine which value of y is inaccurate and estimate its correct value.

4 The table below shows experimental values of x and y which are connected by an equation in the form $yx^n = c$, where n and c are constants.

x	2	3	4	5	6
y	100	82	72	63	57

Using a suitable scale, plot the graph of ln y against ln x. Use your graph to estimate

(i) the value of n and of c,

(ii) the value of y when $x = 2.51$.

5 The table below shows experimental values of two variables x and y obtained from an experiment.

x	1	2	3	4	5	6
y	5.1	25.2	42.3	60.5	98	137

It is believed that x and y are related by the equation in the form $y = ax + bx^2$, where a and b are constants. Express this equation in a form suitable for plotting a straight line graph. Plot the graph. Hence,

(i) use your graph to estimate the value of a and of b,

(ii) determine which value of y is inaccurate and estimate its correct value.

On the same diagram, draw the line representing the equation $y = 38x - 5x^2$ and hence use your graph to find the value of x for which $(b + 5)x = 38 - a$.

6 The table shows experimental values of two variables, x and y.

x	1.5	3	4.5	6	7	8
y	11.92	5.61	2.63	1.25	0.75	0.46

Using the vertical axis for ln y and the horizontal axis for x, plot ln y against x to obtain a straight line graph. Use your graph to

(i) express y in terms of x,

(ii) estimate the value of y when $\sqrt{x} = 1.8$.

Basic Level | Intermediate Level | Advanced Level

Exercise 7B

7 The table below shows experimental values of two variables x and y.

x	1.0	2.0	3.2	4.5	5.0	7.8
y	0.7	1.3	1.8	2.3	2.4	3.2

Use a suitable scale to plot a graph of $\frac{1}{y}$ against $\frac{1}{x}$ for the given values. Use your graph to

(i) express y in terms of x,

(ii) estimate the value of y when $x = 0.15$.

8 **Resistance to Motion**. A particle moving in a certain liquid with speed x m/s experiences a resistance to motion y N. It is believed that x and y are related by the equation $y = hx^k$, where h and k are constants. The table below shows some experimental values of x and y.

x	5	10	15	20	25	30
y	27	76	140	215	300	395

(i) Express the equation in a form suitable for drawing a straight line graph.

(ii) Draw the graph using suitable scales and use it to estimate the value of h and of k.

(iii) Use your graph to find the approximate value of y when $x = 18$.

9 The table below shows experimental values of x and y which are connected by an equation of the form $y = ax^b$.

x	2	5	10	15	25
y	10	63	250	465	1540

Explain how a straight line graph may be drawn to represent the given equation and draw it for the given data.

(i) Use the graph to estimate the value of a and of b.

(ii) Determine which value of y is inaccurate and estimate its correct value.

10 **Drug Concentration.** In a study on drug therapy, a drug dosage is administered to a volunteer and blood samples are taken at various time intervals to find out how the drug diffuses into the body. The concentration of the drug in the blood, C mg/l, is found to be related to the time, t minutes, by the equation $C = Ae^{-kt}$, where A and k are constants. Some values of t and C have been recorded in the table below.

t	5	25	50	75	90	120
C	92.8	68.7	47.2	32.5	25.9	16.5

(i) Express the equation in a form suitable for drawing a straight line graph.

(ii) Draw the graph using suitable scales and use it to estimate the value of A and of k.

(iii) Estimate the initial drug concentration.

(iv) Use your graph to estimate the drug concentration 1 hour after the drug has been administered.

11 **Cooling Curve.** The temperature, T °C, of a cup of hot coffee, as it cools to room temperature, can be modelled by the equation $T = 25 + ka^{-t}$, where t is the time of cooling in minutes and a and k are constants.

t	2	4	6	8	10	12
T	70	56	47	40	35	32

(i) Express the equation in a form suitable for drawing a straight line graph.

(ii) Draw the graph using suitable scales and use it to estimate the value of a and of k.

(iii) Estimate the initial temperature of the cup of hot coffee.

(iv) Use your graph to estimate the time taken for the temperature to reach 37 °C.

(v) Estimate the room temperature.

SUMMARY

1. To convert a non-linear equation involving x and y, express the equation in the form $Y = mX + c$, where X and Y are expressions in x and/or y.

e.g. if $ax^2 + by^3 = 1$, then $y^3 = -\frac{a}{b}x^2 + \frac{1}{b}$

i.e. $Y = y^3$, $X = x^2$, $m = -\frac{a}{b}$, $c = \frac{1}{b}$

e.g. if $y = e^{\frac{b+x}{a}}$, then $\ln y = \frac{1}{a}x + \frac{b}{a}$

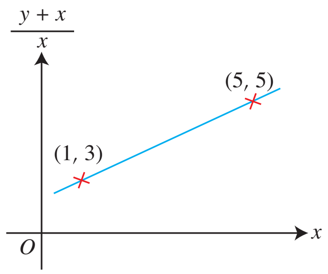
i.e. $Y = \ln y$, $X = x$, $m = \frac{1}{a}$, $c = \frac{b}{a}$

2. The variables X and Y must contain only the original variables x and/or y, but they must not contain the original unknown constants such as a and b.
Similarly, the constants m and c must contain only the original unknown constants such as a and/or b, but they must not contain the original variables x and y.

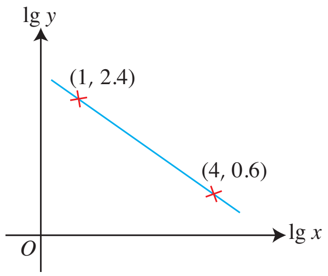
Review Exercise 7

1. The figure shows part of a straight line obtained by plotting $\frac{y+x}{x}$ against x.

(i) Express y in terms of x.
(ii) Find the value of y when $x = 2$.



2. The two variables x and y are connected by the equation $y = kx^n$. The graph of $\lg y$ against $\lg x$ is given. Find the value of k and of n.



3. The variables x and y are related in such a way that when a graph of xy against x^2 is drawn, the resulting line has a gradient of $\frac{1}{3}$ and an xy-intercept of 6. Express y in terms of x.

4. The variables x and y are related in such a way that when $\frac{y}{x^2}$ is plotted against $\sqrt[3]{x}$, a straight line is obtained which passes through (3, 1) and (15, 7). Express y in terms of x.

5. Variables x and y are related by the equation $\frac{x}{2p}+\frac{y^2}{3q}=1$. When a graph of y^2 against x is drawn, the resulting line has a gradient of -1.5 and an intercept of 9 on the y^2-axis. Find the value of p and of q.

6. Variables x and y are connected by an equation of the form $y = Ab^{-x}$, where A and b are constants. When lg y is plotted against x, a straight line is obtained which passes through $\left(2, \frac{1}{2}\right)$ and $(9, -4)$. Find the value of A and of b.

7. The variables x and y are connected by an equation of the form $y = kx^{-n}$, where k and n are constants. When lg y is plotted against lg x, a straight line is obtained which passes through (2, 3) and (8, 7). Find the value of k and of n.

8. The table below gives experimental values of x and y. It is known that x and y are connected by a law $y = ax^3 + bx^2$, where a and b are constants. Plot the graph of $\frac{y}{x^2}$ against x and estimate the value of a and of b.

x	4	5	6	7	8
y	18	75	180	340	580

9. The table below shows some experimental values of two variables x and y.

x	1	2	3	4	5	6
y	15.5	10.0	9.2	9.5	10.3	11.3

It is believed that x and y are related by the equation $y = hx + \frac{k}{x}$, where h and k are constants.

(i) Using a suitable scale, plot the graph of xy against x^2.

(ii) Use your graph to find

(a) the approximate value of h and of k,

(b) the value of x for which $y = \frac{43}{x}$.

10. **Exponential Growth of Bacteria**. The number of bacteria, N, in a petri dish, is observed to increase with time, t hours. Measured values of N and t are given in the following table.

t	2	4	6	8	10
N	61	150	402	915	2250

It is known that N and t are related by the equation $N = N_0e^{kt}$, where N_0 and k are constants.

(i) Plot In N against t for the given data and draw a straight line graph.

(ii) Use your graph to estimate the value of k and of N_0.

(iii) Estimate the time required for the number of bacteria to increase to 1800, giving your answer correct to the nearest minute.

11. The table below shows experimental values of two variables, x and y.

x	1.0	2.0	3.0	4.0	5.0	6.0
y	32.8	16.8	8.62	4.37	2.25	1.15

It is known that x and y are related by an equation of the form $\sqrt[3]{y} = pq^x$, where p and q are constants.

(i) Express $\sqrt[3]{y} = pq^x$ in a form suitable for plotting a straight line graph, stating the variables that must be used for the horizontal and vertical axes.

(ii) Plot the graph using the given data and use your graph to estimate the value of p and of q.

(iii) Use your graph to estimate the value of x when $y = 10$.

12. The table below shows experimental values of two variables x and y. It is known that x and y are related by an equation of the form $y = x^2 + ax - b$, where a and b are constants.

x	1	2	3	4	5
y	5.7	6.4	9.1	13.8	20.5

Using a scale of 2 cm to represent 1 unit on the horizontal and vertical axes, draw a straight line graph of $y - x^2$ against x to represent the above data.

(i) Use your graph to estimate the value of a and of b.

(ii) On the same diagram, draw the line representing the equation $y - 2 = x(x + 1)$ and hence find the value of x for which $a - \frac{2}{x} = 1 + \frac{b}{x}$.

Linear Law can only be applied when there are only two unknown constants in the original non-linear equation. What happens if the non-linear equation has more than two unknown constants, e.g. $y = ax^2 + bx + c$ or $y = ke^{ax + b}$?

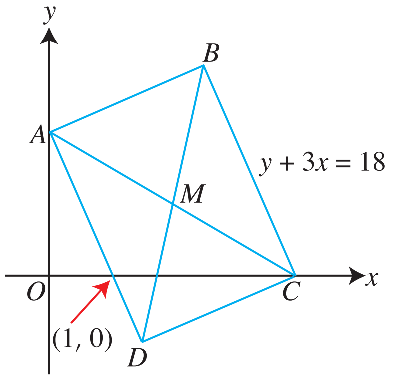
With the advancement in technology, we can actually plot a curve of best fit to fit a given set of data directly. Use a suitable graphing software to plot the data below and then plot the curve of best fit.

x	–1.1	0.3	0.8	1.4	1.8	2.5	3.0
y	–3.9	0.9	1.8	2.2	2.2	1.6	0.6

What is the equation relating x and y?

REVISION EXERCISE C1

1. The figure shows a rectangle $ABCD$ with points A and C on the coordinate axes. The equation of BC is $y + 3x = 18$. AD cuts the x-axis at $(1, 0)$. Given that AC and BD intersect at M, find
(i) the equation of AD,
(ii) the coordinates of C and of M,
(iii) the equation of AB,
(iv) the coordinates of B and of D.

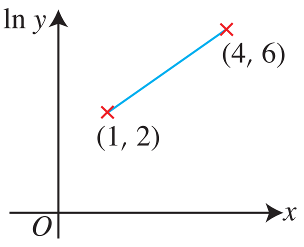


2. Find the area of ΔABC whose coordinates are $A(3, 3)$, $B(-1, 0)$ and $C(5, -3)$. Hence, or otherwise, find the perpendicular distance from C to AB.

3. Find the equation of the circle with centre $(-2, 5)$ and radius 5 units. The line $y = 2x + 4$ cuts the circle at points P and Q. Find the length of the chord PQ.

4. The line $y + 7x = 34$ cuts the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at two points, A and B. Find
(i) the coordinates of A and of B,
(ii) the equation of the perpendicular bisector of AB and show that it passes through the centre of the circle,
(iii) the coordinates of the points where the perpendicular bisector cuts the circle, giving your answers in surd form.

5. The figure shows part of a straight line obtained by plotting $\ln y$ against x. Express y in terms of x.



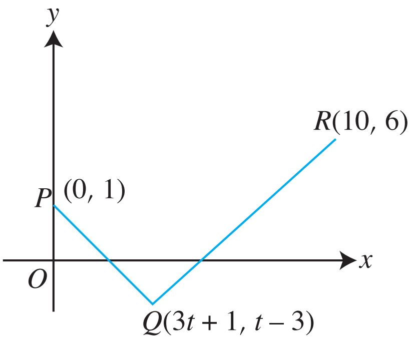
6. The following table gives experimental values of two variables x and y.

x	2	3	4	5	6
y	40	23	18	15	13

It is known that x and y are related by the equation in the form $x^2y = a + bx^2$, where a and b are constants. Using the given data, draw the graph of y against $\frac{1}{x^2}$ and use it to estimate the value of a and of b.

REVISION EXERCISE C2

1. The figure shows the coordinates of three of the vertices of a rectangle $PQRS$. P, Q and R are the points (0, 1), $(3t + 1, t - 3)$ and (10, 6) respectively. Find
(i) the value of t,
(ii) the equation of RS,
(iii) the coordinates of S,
(iv) the equation of the perpendicular bisector of QS,
(v) the area of $PQRS$.

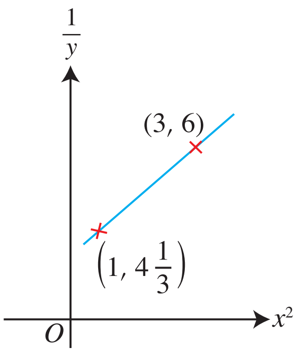


2. A straight line through the point (1, 18) intersects the curve $xy = 30$ at the point (2, 15). Find the coordinates of the point at which the line meets the curve again.

3. Given that a circle with centre (2, 5) touches the x-axis, find the equation of the circle.

4. A diameter of a circle has end points at $P(-4, 3)$ and $Q(4, -3)$.
(i) Find the equation of the circle.
(ii) If $R(k, 4)$ lies on the circle, find the value of k, where $k < 0$. Hence, prove that $\triangle PQR$ is a right-angled triangle.

5. The figure shows part of a straight line obtained by plotting $\frac{1}{y}$ against x^2. Express y in terms of x.



6. The following table gives experimental values of two variables x and y.

x	1	2	3	4
y	28	200	1590	7950

It is known that x and y are related by the equation in the form $y = ab^x$, where a and b are constants. Explain how a straight line graph may be plotted. Hence, plot this straight line and use it to estimate the value of a and of b.

TRIGONOMETRIC FUNCTIONS AND EQUATIONS

CHAPTER 8

When a musical sound wave is changed into a visual image by an oscilloscope, it has a regular pattern which repeats itself in a period of seconds. Have you ever wondered why sound waves can be represented by a trigonometric equation? In this chapter, we will learn about trigonometric functions and equations.

Learning Objectives

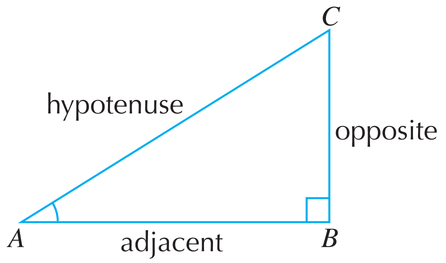
At the end of this chapter, you should be able to:

- evaluate trigonometric ratios of special angles without using a calculator,
- state the amplitude, periodicity and symmetries related to sine and cosine functions,
- sketch the graphs of $y = a \sin (bx) + c$, $y = a \sin\left(\frac{x}{b}\right) + c$, $y = a \cos (bx) + c$, $y = a \cos\left(\frac{x}{b}\right) + c$ and $y = a \tan (bx)$, where a is real, b is a positive integer and c is an integer,
- sketch the graphs of $y = |f(x)|$, where $f(x)$ is trigonometric,
- use $\frac{\sin A}{\cos A} = \tan A$, $\frac{\cos A}{\sin A} = \cot A$, $\frac{1}{\sin A} = \operatorname{cosec} A$, $\frac{1}{\cos A} = \sec A$ and $\frac{1}{\tan A} = \cot A$,
- state the principal values of $\sin^{-1} y$, $\cos^{-1} y$ and $\tan^{-1} y$,
- relate trigonometric functions of any angle to that of its basic (reference) angle,
- solve trigonometric equations and problems involving trigonometric equations.

8.1 TRIGONOMETRIC RATIOS OF SPECIAL ANGLES

Recap

Recall the following about trigonometric ratios.



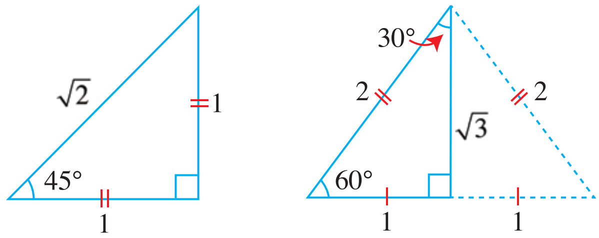
$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}} = \frac{BC}{AC}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}} = \frac{AB}{AC}$$

$$\tan A = \frac{\text{opposite}}{\text{adjacent}} = \frac{BC}{AB}$$

Trigonometric Ratios of Special Angles

The trigonometric ratios of special angles measuring 30°, 45° and 60° (or $\frac{\pi}{6}$, $\frac{\pi}{4}$ and $\frac{\pi}{3}$) can be obtained from the triangles below.



From the above diagrams, the trigonometric ratios of special angles can be calculated.

	$\theta = 30° = \frac{\pi}{6}$	$\theta = 45° = \frac{\pi}{4}$	$\theta = 60° = \frac{\pi}{3}$
$\sin \theta$	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}\left(= \frac{\sqrt{2}}{2}\right)$	$\frac{\sqrt{3}}{2}$
$\cos \theta$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}\left(= \frac{\sqrt{2}}{2}\right)$	$\frac{1}{2}$
$\tan \theta$	$\frac{1}{\sqrt{3}}\left(= \frac{\sqrt{3}}{3}\right)$	1	$\sqrt{3}$

ATTENTION

Some values may involve surds, so rationalise the denominators where necessary.

PROBLEM-SOLVING TIP

Remember the 2 triangles instead of memorising the actual value of each trigonometric ratio in the table.

Worked Example 1

(Trigonometric Ratios of Special Angles)

Without using a calculator, find the exact value of each of the following.

(a) $\dfrac{\sin 30^\circ}{\tan 60^\circ}$ **(b)** $\dfrac{\tan\frac{\pi}{4}\cos\frac{\pi}{3}}{\sin\frac{\pi}{4}+\tan\frac{\pi}{4}}$

Solution

(a)

$$\frac{\sin 30^\circ}{\tan 60^\circ} = \frac{\frac{1}{2}}{\sqrt{3}} = \frac{1}{2\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{6}$$

(b)

$$\frac{\tan\frac{\pi}{4}\cos\frac{\pi}{3}}{\sin\frac{\pi}{4}+\tan\frac{\pi}{4}} = \frac{1\times\frac{1}{2}}{\frac{\sqrt{2}}{2}+1} = \frac{\frac{1}{2}}{\frac{\sqrt{2}+2}{2}} = \frac{1}{\sqrt{2}+2} = \frac{1}{\sqrt{2}+2}\times\frac{\sqrt{2}-2}{\sqrt{2}-2} = \frac{\sqrt{2}-2}{2-4} = \frac{2-\sqrt{2}}{2}$$

RECALL

- $\dfrac{1}{\sqrt{a}} = \dfrac{1}{\sqrt{a}} \times \dfrac{\sqrt{a}}{\sqrt{a}} = \dfrac{\sqrt{a}}{a}$
- $\dfrac{1}{\sqrt{a}+\sqrt{b}} = \dfrac{1}{\sqrt{a}+\sqrt{b}} \times \dfrac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}} = \dfrac{\sqrt{a}-\sqrt{b}}{a-b}$

Practise Now 1

Similar Questions:
Exercise 8A
Questions 1(a)-(d), 7(a)-(d)

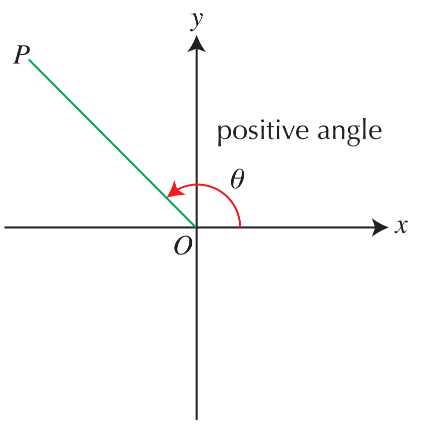
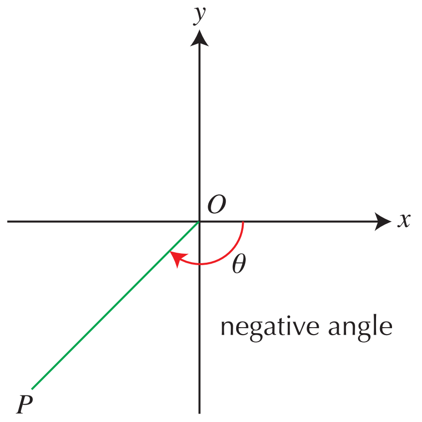
Without using a calculator, find the exact value of each of the following.

(a) $\dfrac{\tan 30^\circ}{\cos 60^\circ}$ **(b)** $\dfrac{\sin 30^\circ \cos 60^\circ}{\cos 45^\circ - \tan 45^\circ}$ **(c)** $\dfrac{\cos\frac{\pi}{4}}{\tan\frac{\pi}{4}+\tan^2\frac{\pi}{6}}$

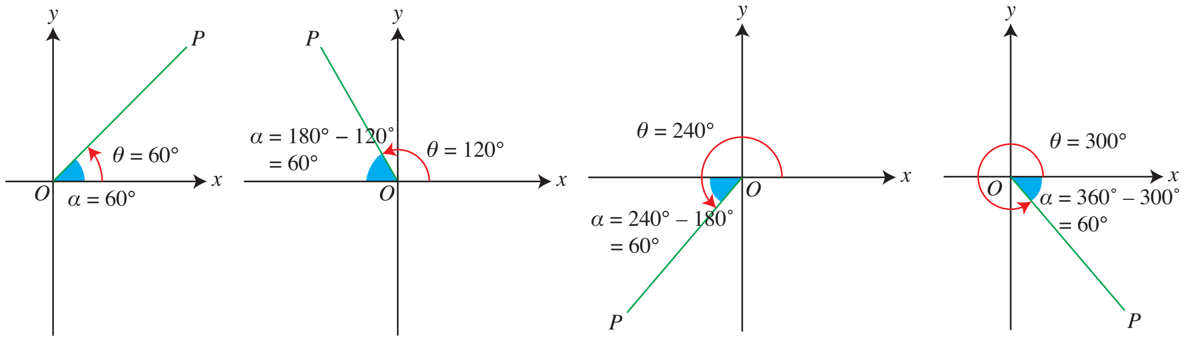
8.2 GENERAL ANGLES

Consider angles in a Cartesian plane. The x- and y-axes divide the plane into four quadrants.

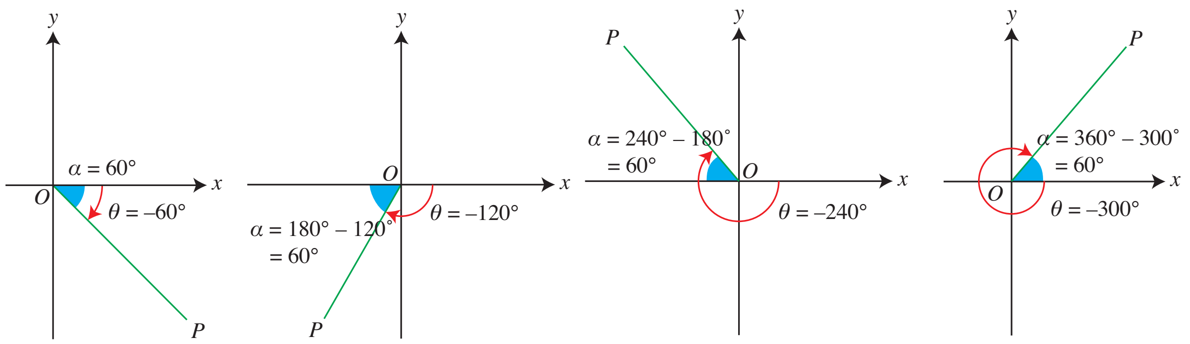
Consider the line OP. O is the origin and P is a point in the plane. The line OP is free to rotate in the xy-plane. The angle θ is measured in an anticlockwise direction from the **positive x-axis** and is said to be in the **quadrant** where OP lies.

OP rotates anticlockwise	*OP* rotates clockwise
	

The figure below shows 4 examples where OP rotates in an *anticlockwise* direction. The values of θ are **positive**, with $\alpha = 60°$.

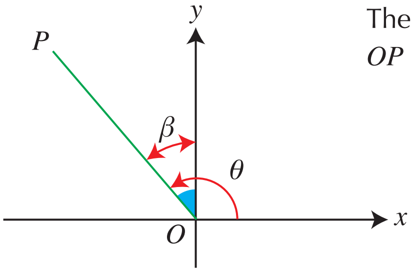


The figure below shows 4 examples where OP rotates in a *clockwise* direction. The values of θ are **negative**, with $\alpha = 60°$.



In all the above examples, α is always a *positive acute angle* between the line OP and the x-axis. The angle α is known as the **basic acute angle** or the **reference angle**.

Serious Misconception



The acute angle, β, between the y-axis and the line OP is ***not*** the basic angle.

Worked Example 2

(Identifying the Quadrant in which OP lies)

Draw a diagram showing the quadrant which the rotating line OP lies in for an angle of

(a) $495°$, **(b)** $-\frac{5\pi}{6}$.

In each case, indicate clearly the direction of rotation and find the basic acute angle, α.

Solution

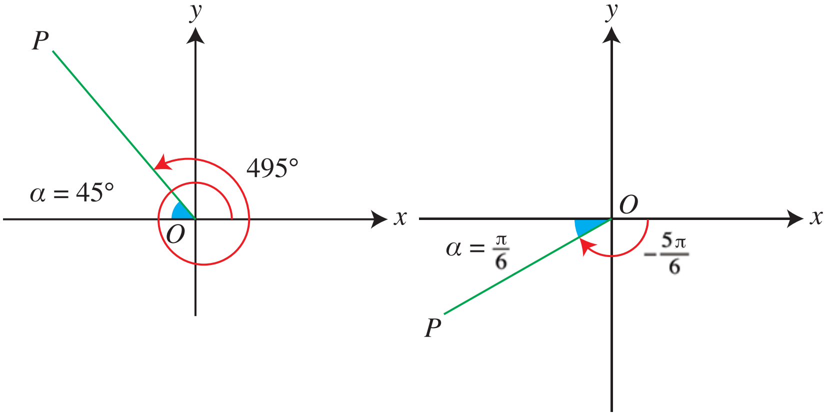
(a) $495° = 360° + 135°$

$= 360° + (180° - 45°)$

$\alpha = 45°$

(b) $-\frac{5\pi}{6} = -\left(\pi - \frac{\pi}{6}\right)$

$\alpha = \frac{\pi}{6}$



Practise Now 2

Similar Questions:
Exercise 8A
Questions 2-4, 8(a)-(h)

Draw separate diagrams to show the quadrant which the rotating line OP lies in for an angle of

(a) $250°$, **(b)** $-380°$, **(c)** $\frac{2\pi}{3}$, **(d)** $-\frac{4\pi}{5}$.

In each case, indicate clearly the direction of rotation and find the basic angle, α.

Worked Example 3

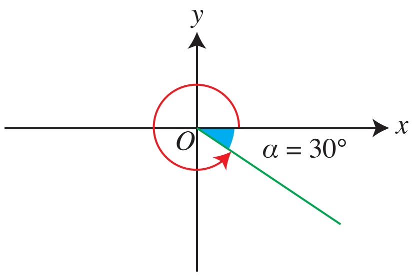
(Finding the Angle when given α and the Quadrant)

Given the basic angle, α, and the quadrant in which θ lies, find the value of θ.

(a) $30°$, 4^{th} quadrant, $0° < \theta < 360°$ **(b)** $\frac{\pi}{4}$, 3^{rd} quadrant, $0 < \theta < 2\pi$

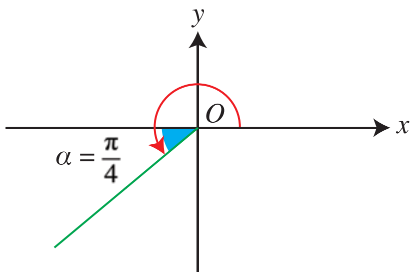
Solution

(a)



$\theta = 360° - 30° = 330°$

(b)



$\theta = \pi + \frac{\pi}{4} = \frac{5\pi}{4}$

Practise Now 3

Similar Questions:
Exercise 8A
Questions 5(a)-(f), 9(a)-(e)

Given the basic angle, α, and the quadrant in which θ lies, find the value of θ.

(a) $35°$, 4^{th} quadrant, $0° < \theta < 360°$ **(b)** $\frac{\pi}{4}$, 4^{th} quadrant, $0 < \theta < 2\pi$

(c) $45°$, 3^{rd} quadrant, $-720° < \theta < -360°$ **(d)** $\frac{\pi}{3}$, 2^{nd} quadrant, $-4\pi < \theta < -2\pi$

8.3 TRIGONOMETRIC RATIOS OF GENERAL ANGLES

Recap

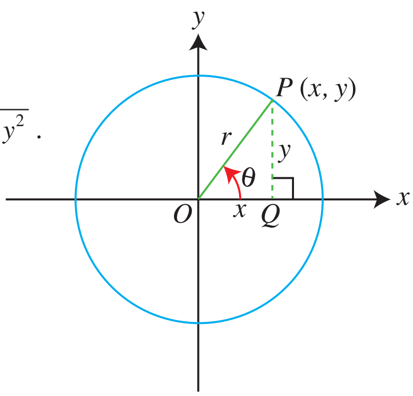
The trigonometric ratios of acute and obtuse angles are as follows.
Given that θ is an acute angle,

$$\sin (180° - \theta) = \sin \theta,$$
$$\cos (180° - \theta) = -\cos \theta,$$
$$\tan (180° - \theta) = -\tan \theta.$$

In general, we define trigonometric ratios of any angle θ in any quadrant as:

$$\sin \theta = \frac{y}{r}, \quad \cos \theta = \frac{x}{r} \quad \text{and} \quad \tan \theta = \frac{y}{x}, \ x \neq 0,$$

where x and y are the coordinates of the point P in the xy-plane and $r = \sqrt{x^2 + y^2}$.

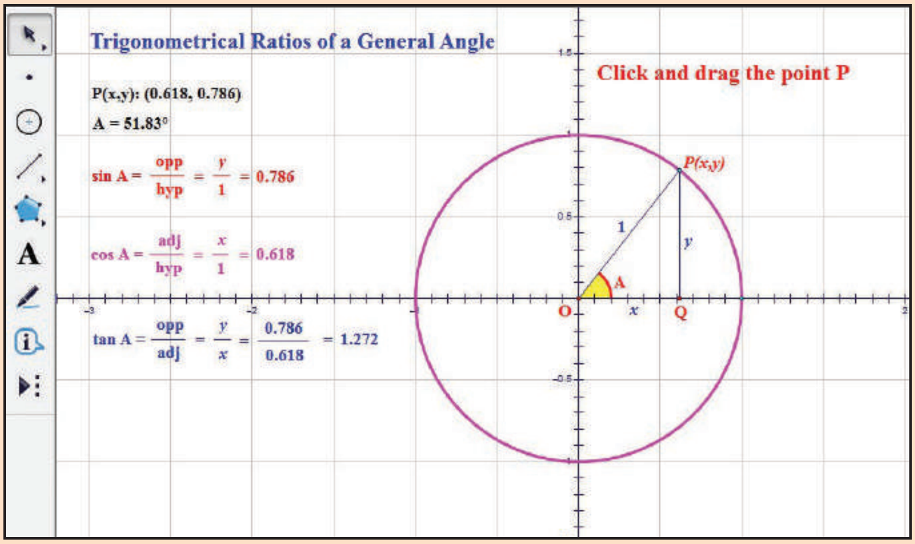


Investigation

Signs of sin A, cos A and tan A

In this investigation, we will explore the signs of sin A, cos A and tan A in the four quadrants.

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template *ASTC*.



$P(x, y)$ is a point on the circumference of a unit circle (i.e. a circle with radius 1 unit). The length of the hypotenuse (hyp) of ΔOPQ is 1 unit. The lengths of the opposite side (opp) and the adjacent side (adj) with respect to $\angle A$ are y units and x units respectively.

1. Write the values of sin A, cos A and tan A in terms of x and y.

$$\sin A = \frac{\text{opp}}{\text{hyp}} = \frac{\square}{\square}; \ \cos A = \frac{\text{adj}}{\text{hyp}} = \frac{\square}{\square}; \ \tan A = \frac{\text{opp}}{\text{adj}} = \frac{\square}{\square}.$$

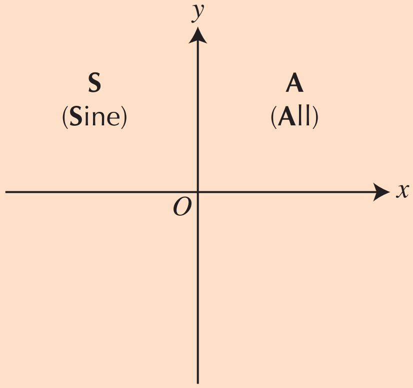
2. Click and drag the point P to investigate how the signs of the trigonometric ratios of $\angle A$ differ when $\angle A$ lies in each of the four **quadrants**. Copy and complete the table below.

	$\angle A$ in 1st quadrant	$\angle A$ in 2nd quadrant	$\angle A$ in 3rd quadrant	$\angle A$ in 4th quadrant
x-coordinate of P	positive	negative		
y-coordinate of P	positive			
$\sin A = \frac{y}{1}$	$\frac{y(+)}{1(+)} =$ positive			
$\cos A = \frac{x}{1}$				
$\tan A = \frac{y}{x}$		$\frac{y(+)}{x(-)} =$ negative		

INFORMATION

You can memorise the diagram on the right as '**A**ll **S**tudents are **T**rendy and **C**lever', or you may want to think of other creative ways to remember this.

3. In each of the four quadrants, which trigonometric ratios are positive? Copy and complete the figure below. The first two quadrants have been done for you: A (All) means all the trigonometric ratios are positive; S (Sine) means only those of sine are positive.



Worked Example 4

(Finding Trigonometric Ratios)

Without using a calculator, find the exact value of each of the following.

(a) $\tan 150°$ **(b)** $\cos\left(-\frac{\pi}{3}\right)$

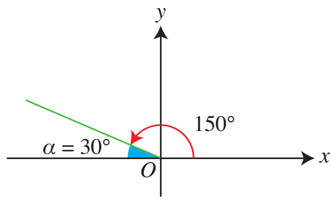
Solution

(a) $150°$ lies in the 2^{nd} quadrant

$\Rightarrow \tan 150°$ is negative.

i.e. $\alpha = 180° - 150° = 30°$

$\therefore \tan 150° = -\tan 30° = -\frac{\sqrt{3}}{3}$

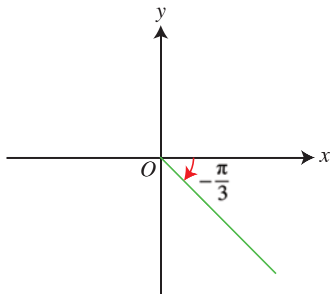


(b) $-\frac{\pi}{3}$ lies in the 4^{th} quadrant

$\Rightarrow \cos\left(-\frac{\pi}{3}\right)$ is positive.

i.e. $\alpha = \frac{\pi}{3}$

$\therefore \cos\left(-\frac{\pi}{3}\right) = \cos\frac{\pi}{3} = \frac{1}{2}$



Discuss with your classmates whether the values of the trigonometric ratios will be the same if the angles rotate by the same amount in the clockwise and anticlockwise directions. For example, do you think

$\sin(-120°) = \sin 120°$?

$\cos(-45°) = \cos 45°$?

$\tan(-300°) = \tan 300°$?

Practise Now 4

Similar Questions:
Exercise 8A
Questions 6(a)-(h)

Without using a calculator, find the exact value of each of the following.

(a) $\tan 210°$ **(b)** $\cos 240°$ **(c)** $\sin\left(-\frac{\pi}{4}\right)$ **(d)** $\tan\frac{5\pi}{3}$

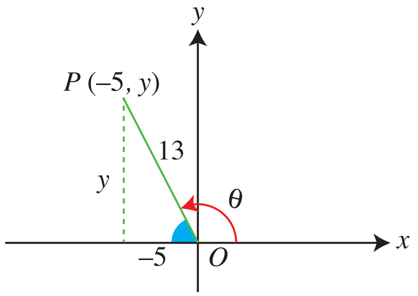
Worked Example 5

(Finding Trigonometric Ratios)

Given that $\cos\theta = -\frac{5}{13}$ and $\tan\theta < 0$, evaluate $\sin\theta$ and $\tan\theta$ without using a calculator.

Solution

Since $\tan\theta < 0$ and $\cos\theta < 0$, θ lies in the 2nd quadrant.



ATTENTION

$\cos\theta < 0 \Rightarrow \theta$ lies in the 2nd or 3rd quadrant
$\tan\theta < 0 \Rightarrow \theta$ lies in the 2nd or 4th quadrant

By Pythagoras' Theorem, $y^2 + (-5)^2 = 13^2$

$$y^2 = 144$$

Since $y > 0$, $y = 12$

$\therefore\ \sin\theta = \frac{12}{13}$ and $\tan\theta = -\frac{12}{5}$

Practise Now 5

Similar Questions:
Exercise 8A
Questions 11-13, 18

1. Given that $\sin\theta = \frac{3}{5}$ and $\tan\theta < 0$, evaluate $\cos\theta$ and $\tan\theta$ without using a calculator.

2. Given that $\tan\theta = -\frac{5}{12}$ such that $\frac{3\pi}{2} < \theta < 2\pi$, evaluate $\sin\theta$ and $\cos\theta$ without using a calculator.

Worked Example 6

(Finding Trigonometric Ratios)

Given that both A and B are angles in the same quadrant, $\sin A = -\frac{3}{5}$ and $\tan B = \frac{2}{5}$, find the value of each of the following without using a calculator.

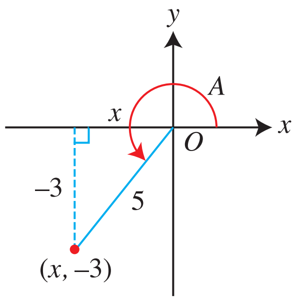
(a) $\cos A$ **(b)** $\tan A$ **(c)** $\sin B$ **(d)** $\cos B$

Solution

A and B lie in the 3rd quadrant.

ATTENTION

$\sin A < 0 \Rightarrow A$ lies in the 3rd or 4th quadrant
$\tan B > 0 \Rightarrow B$ lies in the 1st or 3rd quadrant



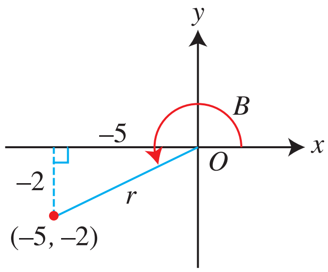
$x^2 + (-3)^2 = 5^2$

$x^2 = 25 - 9 = 16$

Since $x < 0$, $x = -4$.

(a) $\cos A = -\frac{4}{5}$

(b) $\tan A = \frac{-3}{-4} = \frac{3}{4}$



$r^2 = (-2)^2 + (-5)^2 = 29$

Since $r > 0$, $r = \sqrt{29}$.

(c) $\sin B = -\frac{2}{\sqrt{29}} = -\frac{2\sqrt{29}}{29}$

(d) $\cos B = -\frac{5}{\sqrt{29}} = -\frac{5\sqrt{29}}{29}$

Practise Now 6

Similar Questions:
Exercise 8A
Questions 14, 15

Given that both A and B are angles in the same quadrant, $\cos A = -\frac{4}{5}$ and $\tan B = \frac{1}{3}$, find the value of each of the following without using a calculator.

(a) $\sin A$ **(b)** $\tan A$ **(c)** $\sin B$ **(d)** $\cos B$

Worked Example 7

(Finding Trigonometric Ratios)

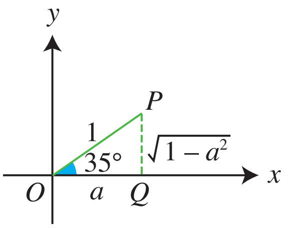
Given that $\cos 35° = a$, express each of the following in terms of a.

(a) $\sin 35°$ **(b)** $\cos 145°$ **(c)** $\tan 215°$ **(d)** $\cos 55°$

Solution

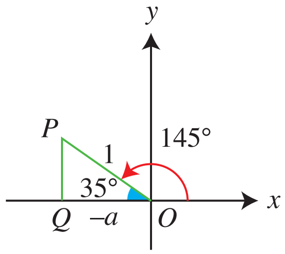
(a) In ΔOPQ, $OP = 1$, $OQ = a$.

$\therefore PQ = \sqrt{1-a^2}$



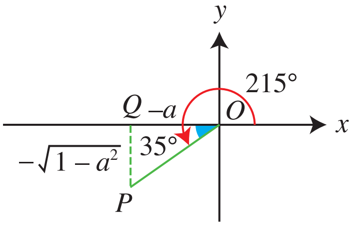
$\therefore \sin 35° = \dfrac{PQ}{OP} = \sqrt{1-a^2}$

(b) 145° lies in the 2nd quadrant.



$\therefore \cos 145° = -\cos 35° = -a$

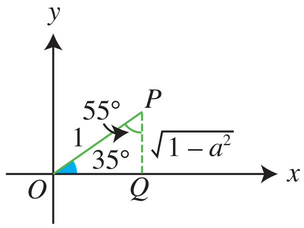
(c) 215° lies in the 3rd quadrant.



$\therefore \tan 215° = \tan 35°$

$= \dfrac{PQ}{OQ} = \dfrac{\sqrt{1-a^2}}{a}$

(d) In ΔOPQ, $\angle OPQ = 90° - 35° = 55°$



$\therefore \cos 55° = \dfrac{PQ}{OP}$

$= \dfrac{\sqrt{1-a^2}}{1} = \sqrt{1-a^2}$

Practise Now 7

Similar Questions:
Exercise 8A
Questions 16, 17, 19

Given that $\sin 50° = a$, express each of the following in terms of a.

(a) $\cos 130°$ **(b)** $\tan 230°$ **(c)** $\sin 310°$

(d) $\cos 310°$ **(e)** $\tan 40°$ **(f)** $\sin 40°$

Negative Angles

Worked Example 8

(Finding the Trigonometric Ratios of Negative Angles)

Given that $0° < \alpha < 90°$, express the trigonometric ratios of $-\alpha$ in terms of the ratios of its basic angle.

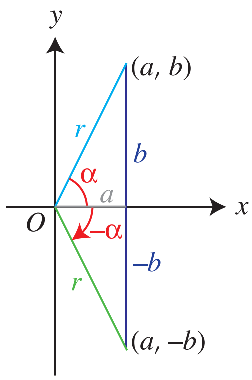
Solution

$-\alpha$ lies in the 4th quadrant, i.e. basic angle = α

$$\sin(-\alpha) = \frac{-b}{r} = -\sin\alpha$$

$$\cos(-\alpha) = \frac{a}{r} = \cos\alpha$$

$$\tan(-\alpha) = \frac{-b}{a} = -\tan\alpha$$



For any angle θ,

$$\boxed{\begin{aligned} \sin(-\theta) &= -\sin\theta \\ \cos(-\theta) &= \cos\theta \\ \tan(-\theta) &= -\tan\theta. \end{aligned}}$$

Practise Now 8

Similar Questions:
Exercise 8A
Questions 10(a)-(d)

In each of the following, express $\sin\theta$, $\cos\theta$ and $\tan\theta$ in terms of the ratios of its basic angle.

(a) $\theta = -80°$ **(b)** $\theta = -100°$ **(c)** $\theta = -255°$ **(d)** $\theta = -300°$

Thinking Time

Let us generalise the conversion between the trigonometric ratios of any angle and its basic angle, α, in different quadrants. Copy and complete the table below.

2nd quadrant	3rd quadrant	4th quadrant	
$\sin(180° - \alpha) = \sin\alpha$	$\sin(180° + \alpha) = -\sin\alpha$	$\sin(360° - \alpha) = -\sin\alpha$	$\sin(-\alpha) = -\sin\alpha$
$\cos(180° - \alpha) =$	$\cos(180° + \alpha) =$	$\cos(360° - \alpha) =$	$\cos(-\alpha) =$
$\tan(180° - \alpha) =$	$\tan(180° + \alpha) =$	$\tan(360° - \alpha) =$	$\tan(-\alpha) =$

Exercise 8A

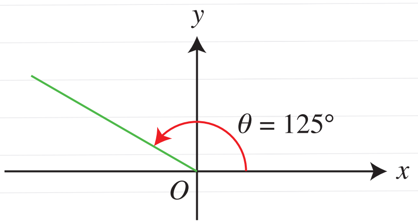
1 Without using a calculator, find the exact value of each of the following.

(a) $\sin 30° + \cos 60°$ **(b)** $\tan 45° - \cos 60°$

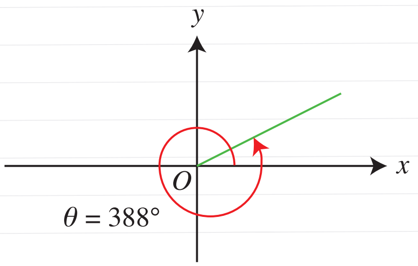
(c) $\tan \frac{\pi}{3} \sin \frac{\pi}{6}$ **(d)** $\sin \frac{\pi}{4} \cos \frac{\pi}{4} - \tan \frac{\pi}{4}$

2 In each of the following figures, find the basic angle of θ.

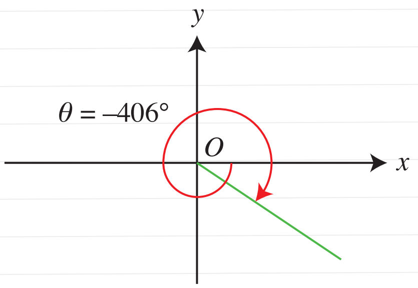
(a)



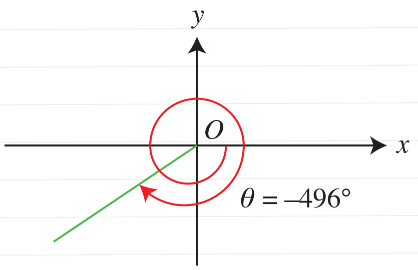
(b)



(c)



(d)



3 Draw each of the following angles in an xy-plane. Indicate clearly the direction of rotation and state the quadrant in which the angle lies.

(a) $260°$ **(b)** $-408°$

(c) $\frac{9\pi}{8}$ **(d)** $-\frac{3\pi}{4}$

4 Express each of the following in terms of the trigonometric ratios of its basic angle.

(a) $\sin 340°$ **(b)** $\cos 125°$

(c) $\sin \frac{5\pi}{4}$ **(d)** $\tan \frac{5\pi}{3}$

5 In each of the following cases, name the quadrant in which the angle θ lies.

(a) $\sin \theta > 0$, $\cos \theta < 0$

(b) $\sin \theta < 0$, $\cos \theta > 0$

(c) $\sin \theta < 0$, $\tan \theta > 0$

(d) $\sin \theta < 0$, $\tan \theta < 0$

(e) $\cos \theta < 0$, $\tan \theta < 0$

(f) $\cos \theta > 0$, $\tan \theta < 0$

6 Without using a calculator, find the exact value of each of the following.

(a) $\sin 150°$ **(b)** $\cos 240°$

(c) $\cos 330°$ **(d)** $\tan 225°$

(e) $\sin(-120°)$ **(f)** $\cos \frac{2\pi}{3}$

(g) $\sin\left(-\frac{\pi}{4}\right)$ **(h)** $\tan\left(-\frac{3\pi}{4}\right)$

7 Without using a calculator, find the exact value of each of the following.

(a) $\dfrac{\sin 45° \cos 45°}{\sin 30° + \cos 60°}$ **(b)** $\dfrac{\sin 60° - \tan 60°}{\tan 45° \cos 30°}$

(c) $\dfrac{\cos^2 \frac{\pi}{6} - \tan^2 \frac{\pi}{4}}{\sin^2 \frac{\pi}{3}}$ **(d)** $\dfrac{\cos \frac{\pi}{6} - \tan \frac{\pi}{4}}{\sin \frac{\pi}{3} + \cos \frac{\pi}{3}}$

Basic Level | Intermediate Level | Advanced Level

Exercise 8A

8 Draw a diagram showing which quadrant the rotating line OP lies in for an angle of
(a) $162°$, **(b)** $235°$,
(c) $-215°$, **(d)** $520°$,
(e) $\frac{7\pi}{3}$, **(f)** $-\frac{15\pi}{4}$,
(g) $\frac{13\pi}{6}$, **(h)** $-3\frac{2}{3}\pi$.
In each case, indicate clearly the direction of rotation and find the basic angle, α.

9 Given that α is the basic angle of θ in each of the following, find θ.
(a) $\alpha = 35°$, $0° < \theta < 180°$
(b) $\alpha = 42°$, $0° < \theta < 90°$ or $180° < \theta < 270°$
(c) $\alpha = 65°$, $0° < \theta < 360°$
(d) $\alpha = \frac{\pi}{4}$, $0 < \theta < \frac{3\pi}{2}$
(e) $\alpha = \frac{\pi}{5}$, $0 < \theta < 2\pi$

10 Express each of the following in terms of the trigonometric ratios of its basic angle.
(a) $\cos(-320°)$ **(b)** $\tan(-64°)$
(c) $\sin\left(-\frac{2\pi}{3}\right)$ **(d)** $\tan\left(-\frac{7\pi}{4}\right)$

11 Given that $\cos\theta = -\frac{4}{5}$ and $180° < \theta < 270°$, evaluate $\sin\theta$ and $\tan\theta$ without using a calculator.

12 Given that $\frac{\pi}{2} < \theta < 2\pi$ and that $\tan\theta = \frac{3}{4}$, calculate the value of $\sin\theta$ and of $\cos\theta$ without using a calculator.

13 Given that $270° < A < 360°$ and that $\sin A = -\frac{5}{13}$, find the value of $\cos A$ and of $\tan A$ without using a calculator.

14 Given that $\tan A = -\frac{5}{12}$ and $\cos B = -\frac{1}{\sqrt{3}}$, where A and B are in the same quadrant, find the value of each of the following without using a calculator.
(a) $\sin A$ **(b)** $\cos A$
(c) $\sin B$ **(d)** $\tan B$

15 Given that $\tan A = -\frac{3}{4}$ and $\tan B = 2$, where both A and B are between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, find the value of each of the following without using a calculator.
(a) $\sin A$ **(b)** $\cos A$
(c) $\sin B$ **(d)** $\cos B$

16 Given that $\sin 50° = p$, express each of the following in terms of p.
(a) $\sin 230°$ **(b)** $\cos 50°$
(c) $\tan(-130°)$ **(d)** $\sin 40°$

17 Given that $\cos 160° = -q$, express each of the following in terms of q.
(a) $\cos 20°$ **(b)** $\sin 160°$
(c) $\tan(-20°)$ **(d)** $\cos 70°$

18 Given that $\cos\theta = \frac{1}{2}$ and $\frac{3\pi}{2} < \theta < 2\pi$, find the value of $\frac{\cos\theta - \sin\theta}{\tan\theta + \sin\theta}$ without using a calculator.

19 It is given that $\cos\theta = k$ and $\sin\theta = k\sqrt{3}$, where k is negative and $0 < \theta < 2\pi$.
(i) In which quadrant does θ lie?
(ii) Find the value of $\tan\theta$.
(iii) Evaluate θ and find the value of k.
(iv) Draw the angle θ in an xy-plane, indicating clearly the direction of rotation.

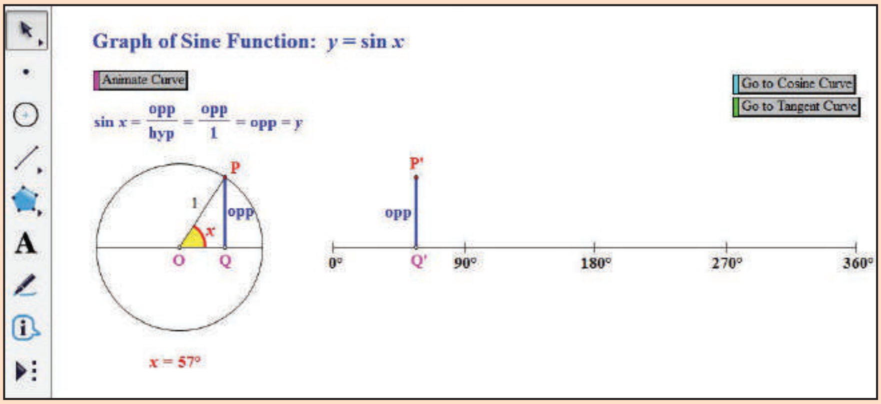
8.4 GRAPHS OF TRIGONOMETRIC FUNCTIONS

We have learnt about trigonometric ratios. What is meant by a trigonometric function? In this section, we will define trigonometric functions by considering line segments related to a unit circle.

Investigation

Trigonometric graphs

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template *Trigo Graphs*.



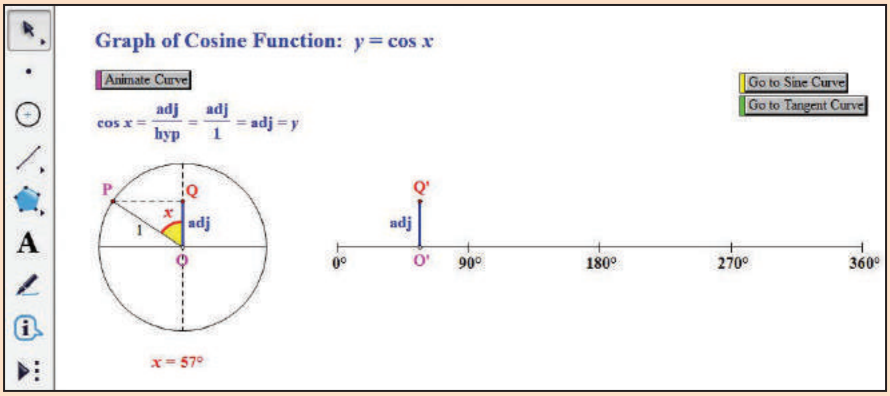
Section A: Graph of Sine Function

$P(x, y)$ is a point on the circumference of a unit circle (i.e. a circle with radius 1 unit). The length of the hypotenuse (hyp) of ΔOPQ is 1 unit. Since the y-coordinate of $P(x, y)$ is y, the length of the opposite side (opp) is y units.

1. Copy and fill in the blank: $\sin x = \frac{\text{opp}}{\text{hyp}} = \frac{\square}{1} = y$, i.e. $y = \sin x$.
2. What do you notice about the lengths of PQ and $P'Q'$?
3. Click on the button 'Animate Curve' and the point P' will trace out the graph of $y = \sin x$. As P moves along the circumference of the unit circle, $\angle x$ will change. Explain how the sine curve $y = \sin x$ is related to the lengths of PQ and $P'Q'$.
4. When $\angle x$ is in the third quadrant, what do you notice about the curve? Can you explain why this is so?
5. Sketch the graph of $y = \sin x$, showing clearly the maximum and minimum points, and the points where the graph cuts the x-axis.

Section B: Graph of Cosine Function

Click on the button 'Go to Cosine Curve':



Notice that $\cos x = \dfrac{\text{adj}}{\text{hyp}} = \dfrac{\square}{1} = y$, i.e. $y = \cos x$.

Since the y-coordinate of P (which is the same as the y-coordinate of Q) is equal to the length of the adjacent side, there is a need to make the adjacent side of the triangle to be in the direction of the y-axis.

That is why the triangle lies in a different quadrant from the one in the previous template.

Since $OQ = OQ'$, then we can trace out the cosine curve using the point Q'.

6. Click on the button 'Animate Curve' and the point P' will trace out the graph of $y = \cos x$. Explain how the cosine curve $y = \cos x$ is related to the lengths of OQ and $O'Q'$.

7. When $\angle x$ is in the second quadrant, what do you notice about the curve? Can you explain why this is so?

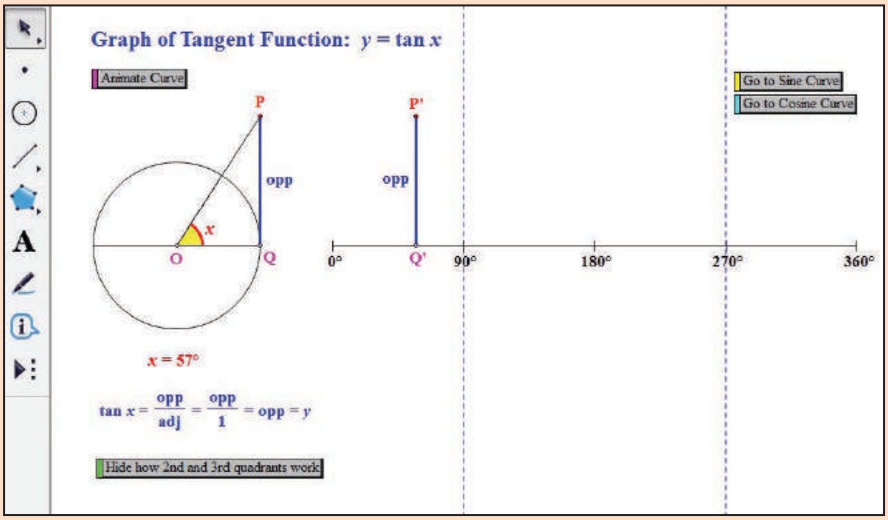
8. Sketch the graph of $y = \cos x$, showing clearly the maximum and minimum points.

INFORMATION

A tangent to a circle is a line that touches the circle at only one point and does not have a fixed length, while PQ is a line segment on the tangent.

Section C: Graph of Tangent Function

Click on the button 'Go to Tangent Curve':



Notice that this template is different from the first two templates in that the adjacent side, OQ, is equal to 1 unit (not the hypotenuse).

9. Fill in the blank: $\tan x = \dfrac{\text{opp}}{\text{adj}} = \dfrac{\square}{1} = y$, i.e. $y = \tan x$.

Notice than $PQ = y = \tan x$, but PQ is a line segment on the tangent to the unit circle at Q. Do you realise that the tangent ratio is related to the tangent of a circle?

10. Click on the button 'Animate Curve' and the point P' will trace out the graph of $y = \tan x$. Explain how the tangent curve $y = \tan x$ is related to the lengths of PQ and $P'Q'$.

11. When $\angle x$ is in the second quadrant, what do you notice about the curve? Can you explain why this is so? Click on the button 'Show how 2nd and 3rd quadrants work' and read the explanations.

INFORMATION

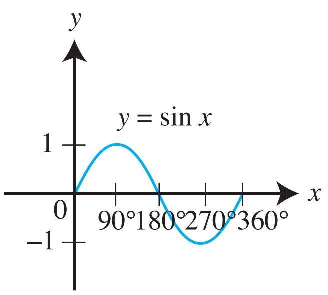
These lines are called the **asymptotes**.

12. Does the curve cut the vertical lines $x = 90°$ and $x = 270°$?

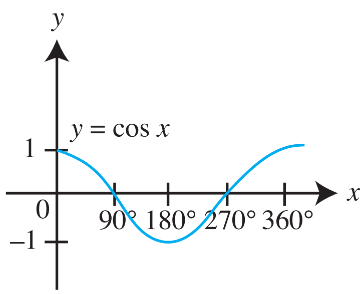
13. Sketch the graph of $y = \tan x$.

From the investigation, we observe the following:

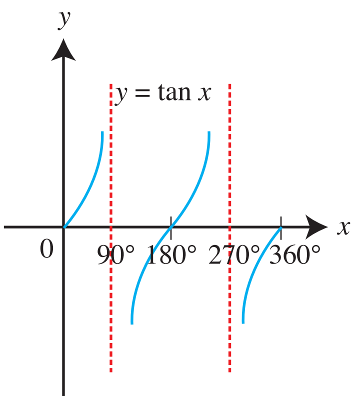
(a) The graph of $y = \sin x$ for $0° \leqslant x \leqslant 360°$ is



(b) The graph of $y = \cos x$ for $0° \leqslant x \leqslant 360°$ is



(c) The graph of $y = \tan x$ for $0° \leqslant x \leqslant 360°$ is



Use the graphs of the trigonometric functions to answer the questions below.

1. Copy and complete the table below.

x	$0°$	$90°$	$180°$	$270°$	$360°$
$y = \sin x$	0			-1	
$y = \cos x$		0			1
$y = \tan x$		undefined		undefined	

2. Copy and complete the table below.

	Maximum value	**Value of x corresponding to the maximum value**	**Minimum value**	**Value of x corresponding to the minimum value**
$y = \sin x$	1			
$y = \cos x$			-1	
$y = \tan x$	does not exist	not applicable		

3. Amplitude and Period

(a) Amplitude

The **amplitude** of the sine or cosine function is *half* the difference between the maximum and minimum values of the function.

(i) What is the amplitude of $y = \sin x$?

(ii) What is the amplitude of $y = \cos x$?

(b) Period

The sine curve repeats itself at every interval of 360°. A function that repeats itself is called a **periodic function**. The length of the interval over which the curve repeats itself is called the **period**.

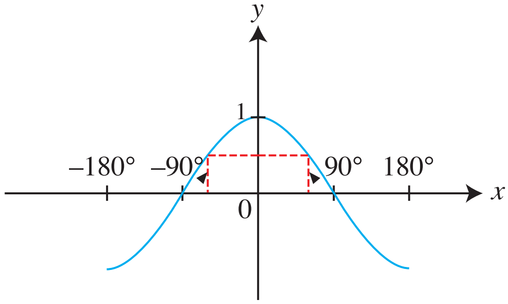
(i) The period of $y = \sin x$ is 360°.

(ii) What is the period of $y = \cos x$?

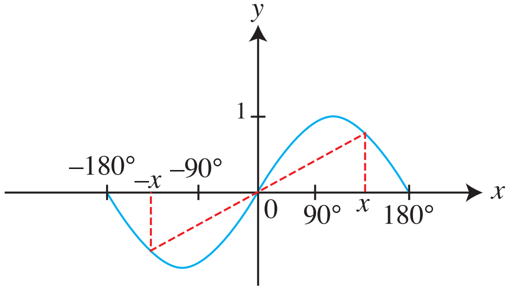
(iii) What is the period of $y = \tan x$?

Thinking Time

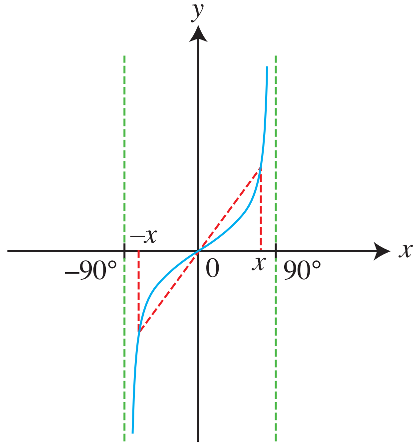
1. The graph of $y = \cos x$ is symmetrical about the y-axis. From this symmetry, how are the values of $\cos(-x)$ related to those of $\cos x$?



2. The graph of $y = \sin x$ has a rotational symmetry about the origin. From this symmetry, how are the values of $\sin(-x)$ related to those of $\sin x$?



3. The graph of $y = \tan x$ has a rotational symmetry about the origin. From this symmetry, how are the values of $\tan(-x)$ related to those of $\tan x$?



Principal Values

For the function $y = \sin x$, $-90° \leqslant x \leqslant 90°$ (or $-\frac{\pi}{2} \leqslant x \leqslant \frac{\pi}{2}$) are chosen to be the **principal values** of x, or $\sin^{-1} y$.

Using a calculator, we get $\sin^{-1}(-1) = -90°$ and $\sin^{-1}(1) = 90°$.

Further examples include $\sin^{-1}\left(\frac{1}{2}\right) = 30°$ and $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -60°$.

Hence, for the function $y = \sin x$, when $-1 \leqslant y \leqslant 1$, the **principal values** of x, or $\sin^{-1} y$, are:

$$\boxed{-90° \leqslant \sin^{-1} y \leqslant 90° \text{ (or } -\frac{\pi}{2} \leqslant \sin^{-1} y \leqslant \frac{\pi}{2}\text{)}}$$

For the function $y = \cos x$, when $-1 \leqslant y \leqslant 1$, the **principal values** of x, or $\cos^{-1} y$, are:

$$\boxed{0° \leqslant \cos^{-1} y \leqslant 180° \text{ (or } 0 \leqslant \cos^{-1} y \leqslant \pi\text{)}}$$

For the function $y = \tan x$, the **principal values** of x, or $\tan^{-1} y$, are:

$$\boxed{-90° < \tan^{-1} y < 90° \text{ (or } -\frac{\pi}{2} < \tan^{-1} y < \frac{\pi}{2}\text{)}}$$

Thinking Time

How can you find the value of x when $y = \sin x$?
What does x represent?
Is the value of x unique? Use the graph of $y = \sin x$ to explain your answer.
What is the principal value of x?

8.5 FURTHER TRIGONOMETRIC GRAPHS

In Section 8.4, we have learnt the graphs of $y = \sin x$, $y = \cos x$ and $y = \tan x$. Now, by making use of a graphing software to explore the functions $y = a \sin bx + c$, $y = a \cos bx + c$ and $y = a \tan bx$, we shall observe how the values of a, b and c affect the graphs.

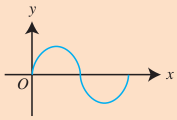
Investigation

Further Trigonometric Graphs

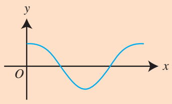
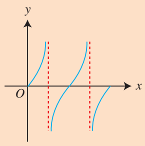
Part A: Amplitude of trigonometric graphs

1. By using a suitable graphing software, set the axes for $0^\circ \leqslant x \leqslant 360^\circ$.

2. Plot the graphs of $y = \sin x$, $y = 2 \sin x$ and $y = 3 \sin x$ on the same screen. Copy and complete the table below.

Graph	Maximum value of the graph	Minimum value of the graph	Sketch of graph	Amplitude	Period
$y = \sin x$	1 when $x = 90^\circ$	-1 when $x = 270^\circ$		1	360°
$y = 2 \sin x$	____ when $x =$ ____	____ when $x =$ ____			
$y = 3 \sin x$	____ when $x =$ ____	____ when $x =$ ____			

3. Repeat steps 1 and 2 for the graphs of $y = a \cos x$ and $y = a \tan x$. Copy and complete the table below.

Graph	Maximum value of the graph	Minimum value of the graph	Sketch of graph	Amplitude	Period
$y = \cos x$	1 when $x = 0^\circ, 360^\circ$	-1 when $x = 180^\circ$		1	360°
$y = 2 \cos x$					
$y = 3 \cos x$					
$y = \tan x$	does not exist	does not exist		N.A.	180°
$y = 2 \tan x$					
$y = 3 \tan x$					

How does the value of a affect the graphs?

Part B: Period of trigonometric graphs

1. Repeat step 1 in Part A to investigate how the values of b affect the graph of $y = \sin bx$. Observe the graphs and copy and complete the table below.

Graph	Maximum value of the graph	Minimum value of the graph	Sketch of graph	Amplitude	Period
$y = \sin x$					
$y = \sin 2x$					
$y = \sin 3x$					
$y = \sin \frac{1}{2}x$					
$y = \sin \frac{1}{3}x$					

2. Explore the graphs of $y = \cos bx$ and $y = \tan bx$, for $b = 1, 2, 3, \frac{1}{2}$ and $\frac{1}{3}$.

How does the value of b affect the graphs?

Part C: Graphs of the form $y = \sin x + c$ and $\cos x + c$

1. Repeat step 1 in Part A to investigate how the values of c affect the graph of $y = \sin x + c$ and $y = \cos x + c$.

2. Plot the graphs of $y = \sin x$, $y = \sin x + 4$ and $y = \sin x - 3$ on the same screen.

3. Plot the graphs of $y = \cos x$, $y = \cos x + 2$ and $y = \cos x - 5$ on the same screen.

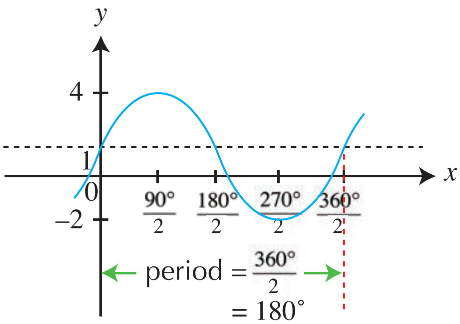
How does the value of c affect the graphs?

Let us consider the following trigonometric graphs.

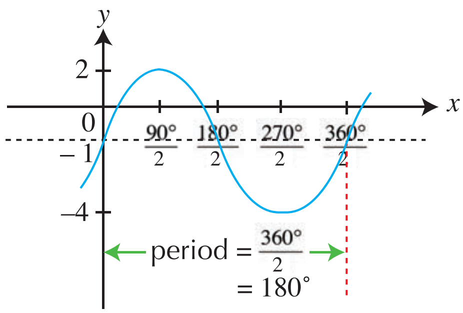
1. For $y = 3 \sin 2x$ and $y = 3 \cos 2x$,

- Maximum value = 3
- Minimum value = –3
- Period = $\frac{360°}{2} = 180°$
- Amplitude = 3

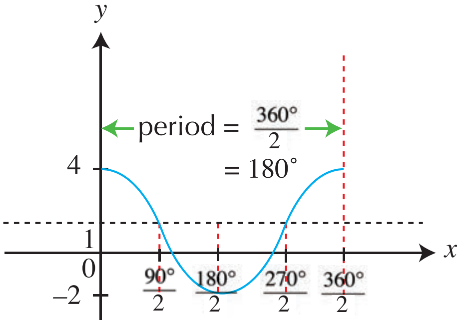
2. For $y = 3 \sin 2x + 1$,



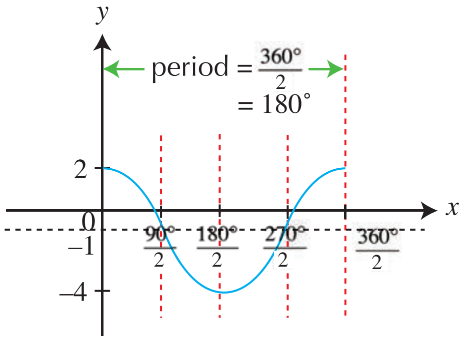
For $y = 3 \sin 2x - 1$,



For $y = 3 \cos 2x + 1$,



For $y = 3 \cos 2x - 1$,



Worked Example 9

(Sketching of $y = a \sin bx$)

The current, y (in amperes), in an alternating current (A.C.) circuit, is given by $y = 2 \sin (6\pi t)$, where t is the time in minutes.

(i) Find the amplitude and the period of this function.

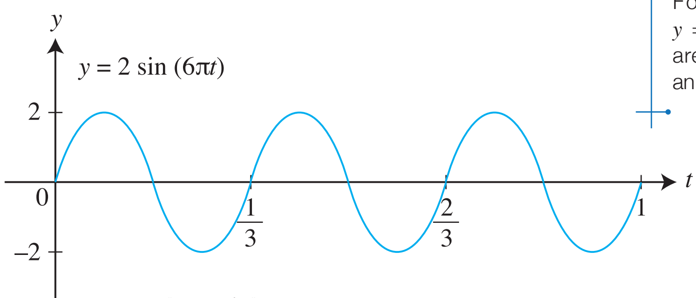
(ii) Sketch the graph of $y = 2 \sin (6\pi t)$ for $0 \leqslant t \leqslant 1$. Hence, state the maximum and minimum values of the current.

Solution

(i) Amplitude = 2 amperes

Period $= \dfrac{2\pi}{6\pi} = \dfrac{1}{3}$ minute

(ii)



Maximum value of the current is 2 amperes,
minimum value of the current is –2 amperes.

PROBLEM-SOLVING TIP

For the graph of $y = 2 \sin 6\pi t$, there are 3 sine curves in an interval of 1 minute.

Practise Now 9

Similar Questions:
Exercise 8B
Questions 2(a), (b), 3(a)-(c)

The current, y (in amperes), in an alternating current (A.C.) circuit, is given by $y = 2 \cos (4\pi t)$, where t is the time in minutes.

(i) Find the amplitude and the period of this function.

(ii) Sketch the graph of $y = 2 \cos (4\pi t)$ for $0 \leqslant t \leqslant 1$. Hence, state the maximum and minimum values of the current.

Worked Example 10

(Sketching of $y = a \cos bx + c$)

The equation of a curve is $y = 2 \cos x + 1$ for $0 \leqslant x \leqslant 2\pi$.

(i) Write down the maximum and minimum values of y.

(ii) State the amplitude, the period and the range of the function.

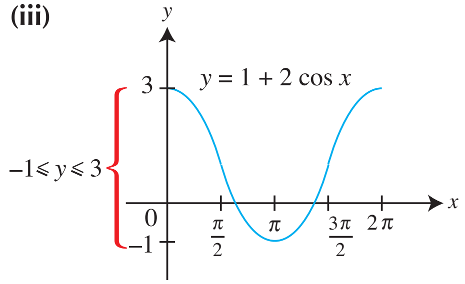
(iii) Hence, sketch the graph of $y = 2 \cos x + 1$ for $0 \leqslant x \leqslant 2\pi$.

Solution

(i) Maximum value of $y = 3$
Minimum value of $y = -1$

(ii) Amplitude = 2
Period = 2π
Range is $-1 \leqslant y \leqslant 3$

(iii)



Practise Now 10

Similar Questions:
Exercise 8B
Questions 2(c)-(e), 3(d)-(e), 4, 7, 8

The equation of a curve $y = 3 \sin x + 2$ for $0 \leqslant x \leqslant 2\pi$.

(i) Write down the maximum and minimum values of y.

(ii) State the amplitude, the period and the range of the function.

(iii) Hence, sketch the graph of $y = 3 \sin x + 2$ for $0 \leqslant x \leqslant 2\pi$.

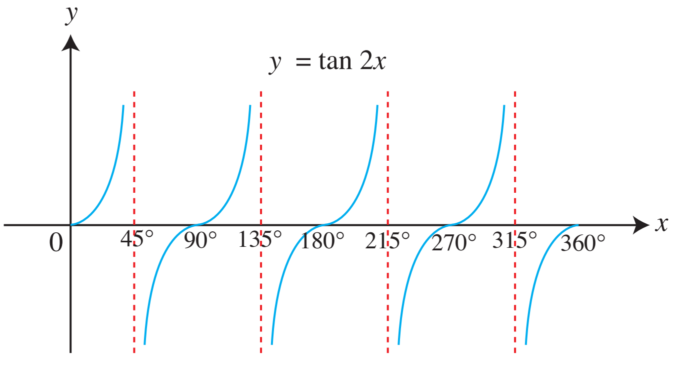
Worked Example 11

(Sketching of $y = \tan bx$)

Sketch the graph of $y = \tan 2x$ for $0° \leqslant x \leqslant 360°$.
State the period of the function.

Solution

Period = $\dfrac{180°}{2} = 90°$



Practise Now 11

Similar Question:
Exercise 8B
Question 4(c)

Sketch the graph of $y = \tan \dfrac{x}{2}$ for $0° \leqslant x \leqslant 360°$.

(i) State the period of the function.

(ii) Explain why the amplitude of a tangent function is not defined.

Worked Example 12

(Maximum and Minimum Value of a Trigonometric Function)

The function $f(x) = a \cos x + b$, where $a > 0$, has a maximum value of 8 and a minimum value of -2. Find the value of a and of b.

Solution

Since $a > 0$, $f(x)$ has a maximum value when $\cos x = 1$ and a minimum value when $\cos x = -1$.

i.e. $a(1) + b = 8$ and $a(-1) + b = -2$

$a + b = 8$ --- (1) $-a + b = -2$ --- (2)

RECALL

$-1 \leqslant \cos x \leqslant 1$

(1) + (2): $2b = 6$

$b = 3$

Subst. $b = 3$ into (1): $a + 3 = 8$

$a = 5$

$\therefore a = 5, b = 3$

Practise Now 12

Similar Questions:
Exercise 8B
Questions 1(a)-(e), 5, 6

The function $f(x) = a \sin x + b$, where $a > 0$, has a maximum value of 6 and a minimum value of -2.

(i) Find the value of a and of b.

(ii) State the amplitude and period of the function.

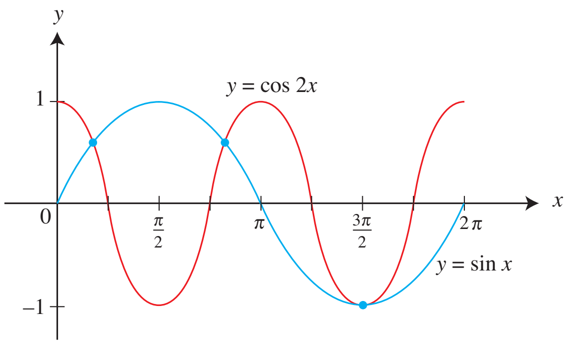
Worked Example 13

(Finding the Number of Solutions of a Trigonometric Equation)

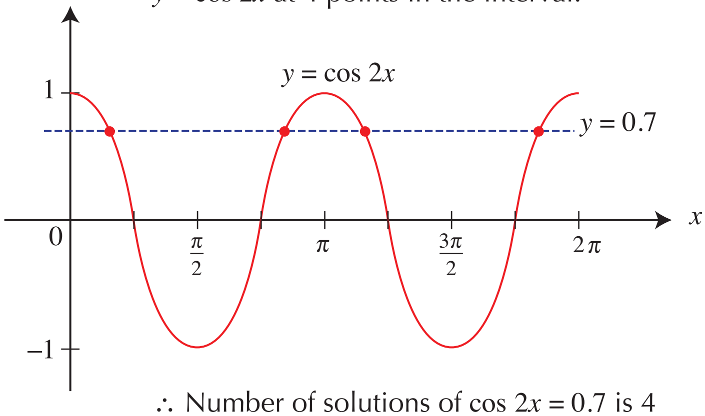
On the same diagram, sketch the curves $y = \sin x$ and $y = \cos 2x$ for the interval $0 \leqslant x \leqslant 2\pi$, labelling each curve clearly. State the number of solutions, in this interval, of each of the following equations.

(a) $\cos 2x = 0.7$ **(b)** $\sin x = \cos 2x$ **(c)** $\sin x - \cos 2x = 1$

Solution

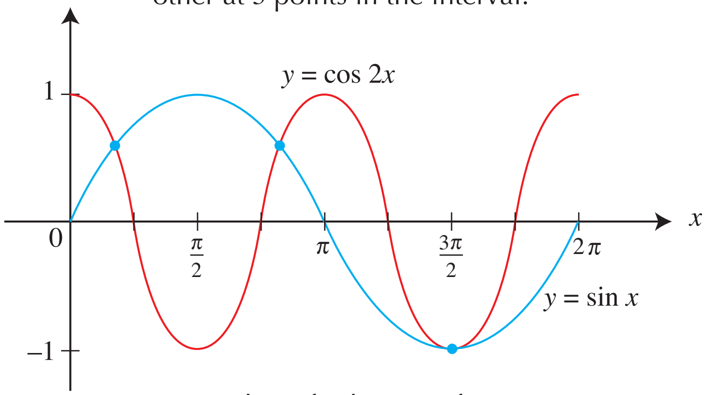


(a) The horizontal line $y = 0.7$ cuts the curve $y = \cos 2x$ at 4 points in the interval.



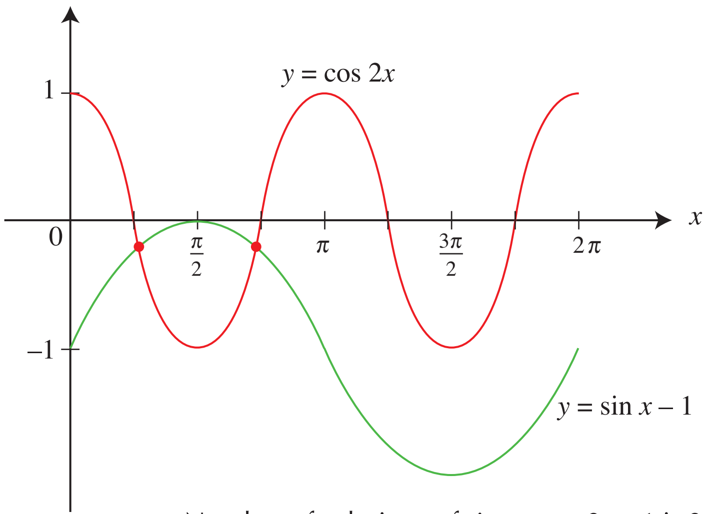
$\therefore$ Number of solutions of $\cos 2x = 0.7$ is 4

(b) The curves $y = \sin x$ and $y = \cos 2x$ intersect each other at 3 points in the interval.



$\therefore$ Number of solutions of $\sin x = \cos 2x$ is 3

(c) Rewrite $\sin x - \cos 2x = 1$ as $\sin x - 1 = \cos 2x$. The curve $y = \sin x - 1$ cuts the curve $y = \cos 2x$ at 2 points in the interval.



$\therefore$ Number of solutions of $\sin x - \cos 2x = 1$ is 2

PROBLEM-SOLVING TIP

In Worked Example 13(c), it is also possible to rewrite $\sin x - \cos 2x = 1$ as $\sin x = \cos 2x + 1$, then count the number of intersections of $y = \sin x$ with $y = \cos 2x + 1$.

Practise Now 13

Similar Questions:
Exercise 8B
Questions 9-11

On the same diagram, sketch the curves $y = \cos x$ and $y = \sin 2x$ for the interval $0 \leqslant x \leqslant 2\pi$, labelling each curve clearly. State the number of solutions, in this interval, of each of the following equations.

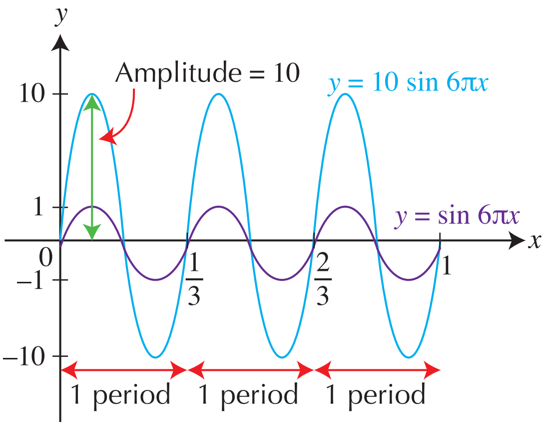
(a) $\sin 2x = 0.6$ **(b)** $\cos x = \sin 2x$ **(c)** $\cos x - \sin 2x = 1$

Applications of Trigonometric Graphs in Sound Waves

The **sine curve** is a mathematical function that models periodic phenomena. The sound wave produced by a tuning fork displayed on an oscilloscope is exactly the same as the sine curve.

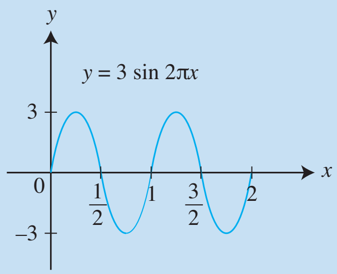
An example is $y = 10 \sin 6\pi x$, where y is the **amplitude**, which represents the loudness of the sound, and x is the time in minutes.

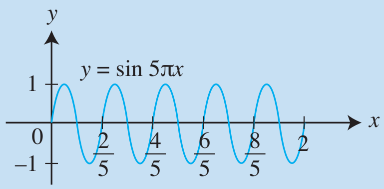
The graph below shows 2 functions $y = \sin 6\pi x$ and $y = 10 \sin 6\pi x$, $0 \leqslant x \leqslant 1$.

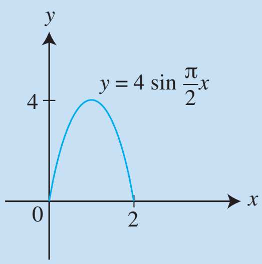


The amplitude of the curve $y = \sin 6\pi x$ is 1 unit and the **period**, i.e. the time taken for one complete cycle, is $\frac{1}{3}$ minute, or 20 seconds. The greater the amplitude of a sound wave, the louder the sound.

The graphs show the sound waves produced by 3 different tuning forks, where y is the amplitude, which represents the loudness of the sound, and x is the time in minutes.







Using the graphs, copy and complete the table below.

Sound waves represented by	Amplitude	Period
$y = 3 \sin 2\pi x$		
$y = \sin 5\pi x$		
$y = 4 \sin \frac{\pi}{2} x$		

Which trigonometric function in the table corresponds to

(a) the loudest sound?

(b) the softest sound?

Worked Example 14

(Problem involving a Roller-coaster)

In a theme park, the car of a roller-coaster reaches y metres above the ground when it is horizontally x metres away from the starting point, where $y = 15 + 7 \sin 2x$ for $0 \leqslant x \leqslant \pi$. Explain clearly how you would obtain

(i) the maximum height of the car above the ground,

(ii) the minimum height of the car above the ground.

Solution

(i) $y = 15 + 7 \sin 2x$

When $\sin 2x = 1$, we obtain the maximum distance of the car above the ground.

Hence, maximum height $= 15 + 7(1) = 22$ m.

(ii) Similarly, when $\sin 2x = -1$, we obtain the minimum distance of the car above the ground.

Hence, minimum height $= 15 + 7(-1) = 8$ m.

Practise Now 14

Similar Questions:
Exercise 8B
Questions 12, 13

In a theme park, the car of a roller-coaster reaches y metres above the ground when it is horizontally x metres away from the starting point, where $y = 15 \sin 2x + 20$ for $0 \leqslant x \leqslant \pi$. Explain clearly how you would obtain

(i) the maximum height of the car above the ground,

(ii) the minimum height of the car above the ground.

Journal Writing

Search on the Internet about how sine/cosine curves relate to

(a) the ferris wheel,

(b) the roller-coaster.

For example, consider the motion of the capsule of the ferris wheel. If we draw a graph to represent its motion, what will the graph look like? Why is this a trigonometric function?

Basic Level | Intermediate Level | Advanced Level

Exercise 8B

1 Find the maximum and minimum values of each of the following.

(a) $3 \cos \theta$ **(b)** $-2 \sin \theta$

(c) $7 \cos (\theta - 45°)$ **(d)** $8 \sin \theta + 3$

(e) $6 - 5 \cos \theta$

2 Sketch each of the following graphs for $0° \leqslant x \leqslant 360°$, stating the amplitude and period.

(a) $y = 4 \sin x$ **(b)** $y = 5 \cos x$

(c) $y = 2 + \sin x$ **(d)** $y = 2 \sin x + 5$

(e) $y = 1 + 3 \cos x$

3 Sketch each of the following graphs for $0 \leqslant x \leqslant 2\pi$, stating the amplitude and the period.

(a) $y = 2 \sin 2x$ **(b)** $y = 3 \cos 2x$

(c) $y = 3 \sin \frac{1}{2}x$ **(d)** $y = 2 \sin 2x - 1$

(e) $y = \cos \frac{1}{3}x - 2$

4 Sketch each of the following graphs on separate diagrams.

(a) $y = 1 + \cos 2x$ for $0° \leqslant x \leqslant 180°$

(b) $y = 4 + 2 \sin 2x$ for $0 \leqslant x \leqslant \pi$

(c) $y = \tan 3x$ for $0° \leqslant x \leqslant 180°$

Exercise 8B

5 The function $f(x) = a \cos x + b$, where $a > 0$, has a maximum value of 7 and a minimum value of 3. Find the value of a and of b.

6 The function $g(x) = p \sin x + q$, where $p > 0$, has a maximum value of 10 and a minimum value of –4. Find the value of p and of q.

7 Sketch each of the following graphs for $0 \leqslant x \leqslant 2\pi$ and state the range of values of y corresponding to $0 \leqslant x \leqslant 2\pi$.

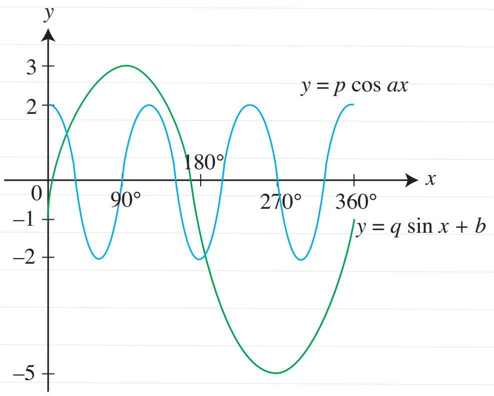
(a) $y = 2 + 3 \cos x$ **(b)** $y = 3 \sin x - 1$
(c) $y = 3 + 2 \cos 2x$ **(d)** $y = 2 \sin x + 3$

8 Sketch the graph of $y = \sin 3x + 2$ for $0 \leqslant x \leqslant \frac{2}{3}\pi$. State the range of values of y corresponding to $0 \leqslant x \leqslant \frac{2}{3}\pi$.

9 On the same diagram, sketch the curves $y = \sin 2x$ and $y = \cos 3x$ for the interval $0° \leqslant x \leqslant 360°$, labelling each curve clearly. State the number of solutions in this interval of the equation $\sin 2x = \cos 3x$.

10 On the same diagram, sketch the graphs of $y = \sin 2x$ and $y = 2 + 3 \sin x$ for $0° \leqslant x \leqslant 360°$. Hence, state the number of solutions in this interval of the equation $\sin 2x - 3 \sin x = 2$.

11



The diagram above shows the graphs of $y = p \cos ax$ and $y = q \sin x + b$ for $0° \leqslant x \leqslant 360°$.

(i) State the values of a, p and q. Explain clearly how these values are obtained and how they relate to the amplitude and the period of the functions.
(ii) State the value of b and explain clearly how its value affects the curve.

12 In a theme park, the car of a roller-coaster is y metres above the ground when it is horizontally x metres away from the starting point, where $y = 18 + 12 \sin 2x$ for $0 \leqslant x \leqslant \pi$. Explain clearly how you would obtain

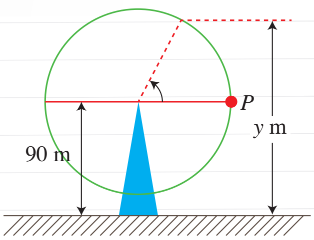
(i) the maximum height of the car above the ground,
(ii) the minimum height of the car above the ground.

13 The sound waves produced by 3 tuning forks A, B and C are represented by $y = \sin 4\pi x$, $y = 5 \sin 3\pi x$ and $y = 7 \sin 4\pi x$ respectively, where y is the amplitude, which represents the loudness of the sound, and x is the time in minutes.

(i) On the same axes, sketch the graphs of the 3 functions for $0 \leqslant x \leqslant 1$.
(ii) Which tuning fork produces
  (a) the loudest sound?
  (b) the softest sound?

14 On the same diagram, sketch the graphs of $y = 2 \sin 2x$ and $y = 2 - \frac{3x}{2\pi}$ for the interval $0 \leqslant x \leqslant 2\pi$. Hence, state the number of solutions of the equation $2(1 - \sin 2x) = \frac{3x}{2\pi}$ in the given interval.

15 The diagram shows a ferris wheel of diameter 160 m and whose centre is 90 m above the ground. The wheel takes 36 minutes to make one complete revolution. A girl is now at point P, 90 m above the ground.



(i) Given the equation $y = a + b \sin 2\pi t$, where y is the height, in metres, of the girl's position above the ground and t is the time in minutes after passing P, find the value of a and of b.
(ii) Hence, sketch the graph of $y = a + b \sin 2\pi t$ for $0 \leqslant t \leqslant 72$.
(iii) Explain clearly how you would obtain
(a) the maximum height of the girl above the ground,
(b) the minimum height of the girl above the ground.

excluded from the N(A) syllabus

8.6 Graphs of $y = |f(x)|$, where $f(x)$ is trigonometric

We have learnt about modulus functions in Sections 2.6 and 2.7. In this section, we shall learn about modulus functions involving trigonometric expressions.

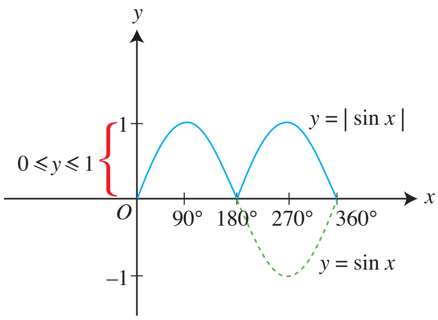
Worked Example 15

(Graph of a Modulus Trigonometric Function)
Sketch the graph of $y = |\sin x|$ for the domain $0° \leqslant x \leqslant 360°$, stating the range of values of y.

Solution

Step 1: Sketch the graph of $y = \sin x$, with the portion showing $\sin x < 0$ dotted.
Step 2: Reflect the dotted portion about the x-axis to obtain the graph of $y = |\sin x|$.

$\therefore$ Range of values of y is $0 \leqslant y \leqslant 1$



Practise Now 15

Similar Questions:
Exercise 8C
Questions 1(a)-(e)

1. Sketch the graph of $y = |\cos x|$ for $0° \leqslant x \leqslant 360°$, stating the range of values of y.
2. Sketch the graph of $y = \left|2\cos\frac{1}{2}x\right|$ for $0 \leqslant x \leqslant 2\pi$, stating the range of values of y.

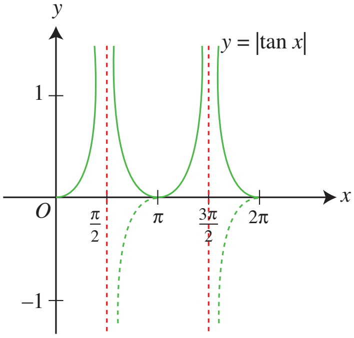
Worked Example 16

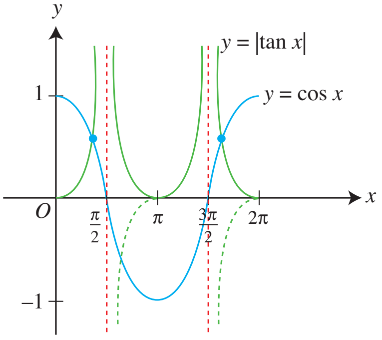
(Finding the Number of Solutions of a Trigonometric Equation)

On the same diagram, sketch the graphs of $y = |\tan x|$ and $y = \cos x$ for $0 \leqslant x \leqslant 2\pi$. Hence, state the number of roots of the equation $|\tan x| = \cos x$ for $0 \leqslant x \leqslant 2\pi$.

Solution

To sketch the graph of $y = |\tan x|$, we reflect the dotted portion of $y = \tan x$ about the x-axis.





From the graph, there are 2 roots.

ATTENTION

The number of roots of the equation refers to the number of points of intersection of the graphs $y = |\tan x|$ and $y = \cos x$.

Practise Now 16

Similar Questions:
Exercise 8C
Questions 2, 3

1. On the same diagram, sketch the graphs of $y = |\sin x|$ and $y = \tan x$ for $0 \leqslant x \leqslant 2\pi$. Hence, state the number of solutions of the equation $\tan x = |\sin x|$ for $0 \leqslant x \leqslant 2\pi$.
2. On the same diagram, sketch the graphs of $y = |\tan x|$ and $y = 1 + \cos x$ for $0° \leqslant x \leqslant 360°$. Hence, state the number of roots of the equation $|\tan x| = 1 + \cos x$ for $0° \leqslant x \leqslant 360°$.

Worked Example 17

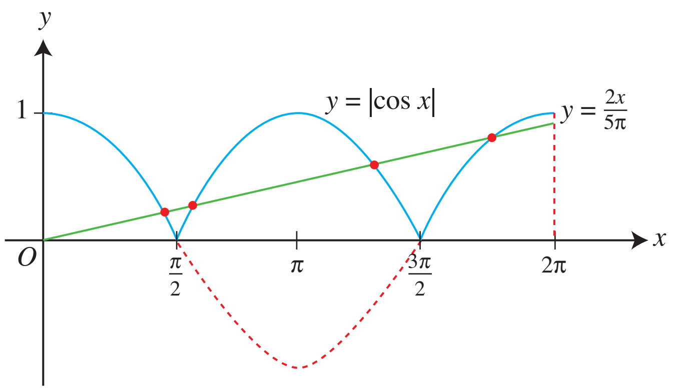
(Finding the Number of Solutions of a Trigonometric Equation)

On the same diagram, sketch the graphs of $y = |\cos x|$ and $y = \frac{2x}{5\pi}$ for $0 \leqslant x \leqslant 2\pi$. Hence, state the number of solutions of the equation $2x = 5\pi|\cos x|$ for $0 \leqslant x \leqslant 2\pi$.

Solution

Step 1: Sketch the graph of $y = |\cos x|$.

Step 2: The graph of $y = \frac{2x}{5\pi}$ is a straight line with gradient $\frac{2}{5\pi}$ and passing through the origin. To obtain another point on the straight line, take $x = 2\pi$, i.e. $y = 0.8$.



From the graph, there are 4 solutions.

Practise Now 17

Similar Questions:
Exercise 8C
Questions 4, 5

1. On the same diagram, sketch the graphs of $y = |\sin x|$ and $y = \frac{x}{2\pi}$ for $0 \leqslant x \leqslant 2\pi$. Hence, state the number of solutions of the equation $2\pi|\sin x| = x$ for $0 \leqslant x \leqslant 2\pi$.

2. On the same diagram, sketch the graphs of $y = |3\cos x|$ and $y = \frac{x}{\pi} - 1$ for $0 \leqslant x \leqslant 2\pi$. Hence, state the number of solutions of the equation $x = 3\pi|\cos x| + \pi$ for $0 \leqslant x \leqslant 2\pi$.

Exercise 8C

excluded from the N(A) syllabus

1 Sketch each of the following graphs for $0° \leqslant x \leqslant 360°$.

(a) $y = |3 \cos x|$ **(b)** $y = |4 \sin x|$

(c) $y = |2 \cos 2x|$ **(d)** $y = |1 + 2 \cos x|$

2 On the same diagram, sketch the graphs of $y = |\tan x|$ and $y = 1 + \sin x$ for $0° \leqslant x \leqslant 360°$. Hence, state the number of solutions of the equation $|\tan x| = 1 + \sin x$ in the interval.

3 On the same diagram, sketch the graphs of $y = \left|\cos \frac{1}{2}x\right|$ and $y = \sin x$ for $0 \leqslant x \leqslant \pi$. Hence, state the number of solutions of the equation $\sin x = \left|\cos \frac{1}{2}x\right|$ in the given interval.

4 On the same diagram, sketch the graphs of $y = |2 \sin x|$ and $y = \frac{6x}{5\pi}$ for $0 \leqslant x \leqslant 2\pi$. Hence, state the number of solutions of the following equations in the given interval.

(a) $|5\pi \sin x| = 3x$ **(b)** $5\pi \sin x = 3x$

5 The function f is defined, for $0 \leqslant x \leqslant 2\pi$, as $f(x) = |2 \sin x|$.

(i) Sketch the graph of f.

(ii) Explain clearly how you would deduce the value of k for which the equation $3 \cos x + k = |2 \sin x|$ has 3 solutions for $0 \leqslant x \leqslant 2\pi$. State the value of k.

8.7 COSECANT, SECANT AND COTANGENT RATIOS

For any angle θ, the trigonometric ratios of cosecant, secant and cotangent are defined as

$$\text{cosecant } \theta = \text{cosec } \theta = \frac{1}{\sin \theta},$$

$$\text{secant } \theta = \sec \theta = \frac{1}{\cos \theta},$$

$$\text{cotangent } \theta = \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}.$$

The values of these ratios are determined from the reciprocals of the sine, cosine and tangent ratios. The signs of these ratios also follow exactly those of the reciprocals, i.e. cosec θ is positive when $\sin \theta$ is positive, i.e. when θ lies in the first and second quadrants.

In which quadrants will $\sec \theta$ be positive? In which quadrants will $\cot \theta$ be positive?

Worked Example 18

(Finding Trigonometric Ratios)

Given that $\sin\theta = \frac{3}{4}$ and θ is acute, find each of the following without using a calculator.

(i) $\sec\theta$ **(ii)** $\cot\theta$

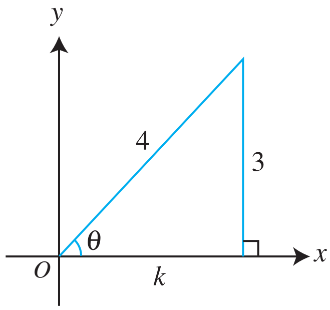
Solution

θ lies in the first quadrant.

$$k^2 + 3^2 = 4^2 \quad \text{(Pythagoras' Theorem)}$$
$$k = \sqrt{7}$$

(i) Since $\cos\theta = \frac{\sqrt{7}}{4}$, $\sec\theta = \frac{4}{\sqrt{7}} = \frac{4\sqrt{7}}{7}$.

(ii) Since $\tan\theta = \frac{3}{\sqrt{7}}$, $\cot\theta = \frac{\sqrt{7}}{3}$.



Practise Now 18

Similar Questions:
Exercise 8D
Questions 1-6

1. Given that $\sin\theta = \frac{2}{3}$ and θ is acute, find each of the following without using a calculator.
(i) $\operatorname{cosec}\theta$ **(ii)** $\sec\theta$ **(iii)** $\cot\theta$

2. Given that $\cos\theta = -\frac{3}{5}$ and θ lies in the 2nd quadrant, find each of the following without using a calculator.
(i) $\sec\theta$ **(ii)** $\operatorname{cosec}\theta$ **(iii)** $\cot\theta$

Basic Level | Intermediate Level | Advanced Level

Exercise 8D

1 Given that $\sin x = \frac{4}{5}$ and that x is acute, find each of the following without using a calculator.
(i) $\sec x$ **(ii)** $\operatorname{cosec} x$ **(iii)** $\cot x$

2 Given that $\tan x = -\frac{1}{2}$ and $90° < x < 270°$, find each of the following without using a calculator.
(i) $\sec x$ **(ii)** $\operatorname{cosec} x$ **(iii)** $\cot x$

3 Given that $\cos x = -\frac{1}{4}$ and $\pi < x < 2\pi$, find each of the following without using a calculator.
(i) $\sec x$ **(ii)** $\operatorname{cosec} x$ **(iii)** $\cot x$

4 Given that $\sec x = 2$ and $0° < x < 180°$, find each of the following without using a calculator.
(i) $\sin x$ **(ii)** $\operatorname{cosec} x$ **(iii)** $\cot x$

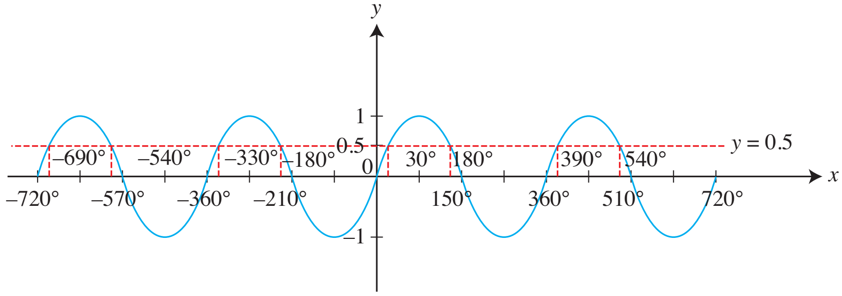
5 Given that $\operatorname{cosec} x = 3$ and that x is acute, find each of the following without using a calculator.
(i) $\cos(90° - x)$ **(ii)** $\tan(90° - x)$
(iii) $\cot x$

6 Given that $\cot x = 2.5$ and that x is acute, find each of the following without using a calculator.
(i) $\sin x$ **(ii)** $\sin\left(\frac{\pi}{2} - x\right)$ **(iii)** $\sec\left(\frac{\pi}{2} - x\right)$

8.8 TRIGONOMETRIC EQUATIONS

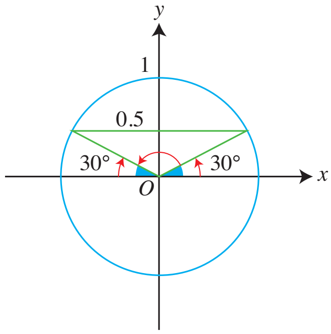
In this section, we will learn how to solve equations involving trigonometric ratios of the forms $\sin x = k$, $\cos x = k$ and $\tan x = k$.

The graph below shows some of the solutions to the equation $\sin x = 0.5$.



From the above, we see that there are many solutions to $\sin x = k$ $(-1 \leqslant k \leqslant 1)$. These are expressed as $x = \sin^{-1} k$. The value of $x = \sin^{-1} k$ in the range $-90° < \sin^{-1} k < 90°$ is known as the **principal value** of $\sin^{-1} k$. Therefore, the principal value of $\sin^{-1} 0.5$ is 30°.

The solutions to the equation $\sin x = 0.5$ can also be obtained by considering the unit circle cutting the line $y = 0.5$. The solutions lie in the 1st and 2nd quadrants.



Basic angle = 30°

For the domain $0° < x < 360°$,
$x = 30°$ and $x = 180° - 30° = 150°$

Similarly, this applies to the equations $\cos x = k$ and $\tan x = k$.

Worked Example 19

(Solving a Trigonometric Equation using ASTC)

Solve the equation $\cos x = -\frac{1}{3}$ such that $0° \leqslant x \leqslant 360°$.

Solution

Let $\cos \alpha = \frac{1}{3}$, where α is the basic angle.

Then $\alpha = \cos^{-1} \frac{1}{3} = 70.53°$.

Since $\cos x = -\frac{1}{3}$ is negative, then x must lie in the 2nd or 3rd quadrant.

$\therefore x = 180° - \alpha$ or $x = 180° + \alpha$

$= 180° - 70.53°$ $\quad$ $= 180° + 70.53°$

$= 109.5°$ (to 1 d.p.) $\quad$ $= 250.5°$ (to 1 d.p.)

PROBLEM-SOLVING TIP

Since the basic angle α is always acute, then $\cos \alpha$ must be positive.

ATTENTION

Always leave your answer in degrees to 1 decimal place; intermediate working must be correct to at least 2 decimal places.

Practise Now 19

Similar Questions:
Exercise 8E
Questions 1-4, 6(c)

1. Find all the angles between 0° and 360° which satisfy the equation $\cos x = -0.81$.
2. Solve the equation $\sin x = 0.43$ for $0° \leqslant x \leqslant 360°$.

Worked Example 20

(Solving a Trigonometric Equation)
Solve the equation $\tan 2A = 1$ for $-\pi < A < \pi$.

Solution

Let $\tan \alpha = 1$, where α is the basic angle.

Then $\alpha = \tan^{-1} 1$

$= \frac{\pi}{4}$ (special angle)

RECALL

Think of *Special Angles* in Section 8.1 whose tangent is equal to 1: the special angle is 45° or $\frac{\pi}{4}$.

Since $\tan 2A = 1$ is positive, then $2A$ must be in the 1st or 3rd quadrant.
Since $-\pi < A < \pi$, then $-2\pi < 2A < 2\pi$.

For $0 < A < 2\pi$,

$$2A = \alpha \text{ or } \pi + \alpha$$
$$= \frac{\pi}{4} \text{ or } \pi + \frac{\pi}{4}$$
$$= \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$
$$A = \frac{\pi}{8} \text{ or } \frac{5\pi}{8}$$

For $-2\pi < 2A < 0$,

$$2A = \alpha - 2\pi \text{ or } (\pi + \alpha) - 2\pi$$
$$= \frac{\pi}{4} - 2\pi \text{ or } \left(\pi + \frac{\pi}{4}\right) - 2\pi$$
$$= -\frac{7\pi}{4} \text{ or } -\frac{3\pi}{4}$$
$$A = -\frac{7\pi}{8} \text{ or } -\frac{3\pi}{8}$$

Practise Now 20

Similar Questions:
Exercise 8E
Questions 6(a), (b), 7(a)

1. Find all the angles between $-\pi$ and π which satisfy the equation $\tan x = -3$.
2. Solve the equation $\tan \frac{1}{2}x = \sqrt{7}$ for $-\pi \leqslant x \leqslant \pi$.

Worked Example 21

(Solving a Trigonometric Equation)
Find all the angles between 0° and 360° which satisfy the equation $\sin (x - 41°) = -0.945$.

Solution

Let $\sin \alpha = 0.945$, where α is the basic angle.
Then $\alpha = \sin^{-1} 0.945 = 70.91°$.

Since $\sin (x - 41°) = -0.945$ is negative, then $x - 41°$ must be in the 3rd or 4th quadrant.

$\therefore\ x - 41° = 180° + \alpha$ or $x - 41° = 360° - \alpha$

$= 180° + 70.91°$ $\qquad$ $= 360° - 70.91°$

$x = 291.9°$ (to 1 d.p.) $\qquad$ $x = 330.1°$ (to 1 d.p.)

ATTENTION

When the question states 'between 0° and 360°', it means that x cannot be equal to 0° or 360°. If x can be equal to 0° or 360°, we write it as 'from 0° to 360°'.

Practise Now 21

Similar Questions:
Exercise 8E
Questions 5(a)-(f), 6(d), 7(b)-(e)

1. Solve each of the following equations.

(a) $\sin(x - 50°) = 0.736$, $0° < x < 360°$

(b) $\tan\left(\frac{1}{2}x + \frac{\pi}{6}\right) = 0.479$, $0 < x < 2\pi$

2. Find all the angles between 0° and 360° inclusive which satisfy the equation $\cos^2\theta = \frac{1}{4}$.

Thinking Time

The x- and y-axes divide the plane into four quadrants. How do you determine the trigonometric ratios of the angles 0°, 90°, 180°, 270° and 360°?

Some trigonometric equations cannot be solved by using the ASTC Method because the angle does not lie in any quadrant, e.g. when the angle is 0°, 90°, 180°, 270° or 360°. Hence, we make use of the graphs of trigonometric functions to solve such equations.

Worked Example 22

(Solving a Trigonometric Equation)

Find all the angles between 0° and 360° inclusive which satisfy the equation $\sin x \cos(x - 120°) = 0$.

ATTENTION

$\sin x \cos(x - 120°)$ means $\sin x \times \cos(x - 120°)$.

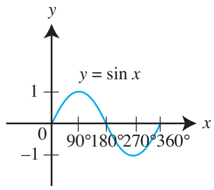
Solution

$\sin x \cos(x - 120°) = 0$

$\sin x = 0$ or $\cos(x - 120°) = 0$

When $\sin x = 0$,

$x = 0°, 180°, 360°$ (from the graph)

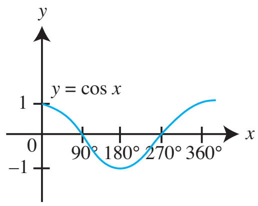


PROBLEM-SOLVING TIP

For $\sin x = 0$, think of the sine curve (you may want to sketch it). What values of x satisfy $\sin x = 0$? For $\cos(x - 120°) = 0$, just think of the curve $y = \cos\theta$ first. What values of θ satisfy $\cos\theta = 0$?

Since $0° \leqslant x \leqslant 360°$,

$-120° \leqslant x - 120° \leqslant 240°$.



When $\cos(x - 120°) = 0$,

$x - 120° = 90°$ or $270°$ (rejected) or $270° - 360°$ (from the graph of $y = \cos x$)

$= 90°$ or $-90°$

$x = 210°$ or $30°$

$\therefore x = 0°, 30°, 180°, 210°$ or $360°$

Practise Now 22

Similar Questions:
Exercise 8E
Questions 8(a), 9(a), (b)

1. Find all the angles between 0° and 360° inclusive which satisfy the equation $\cos x \sin (x + 30°) = 0$.

2. Solve $\tan \frac{1}{2} x \sin 2x = 0$ for $0 \leqslant x \leqslant 2\pi$.

Worked Example 23

(Trigonometric Equation with Common Factors)

Solve $\sin x \cos x = \cos^2 x$ for $0° \leqslant x \leqslant 360°$.

Solution

Since **cos *x* can be 0**, we cannot divide both sides of the equation $\sin x \cos x = \cos^2 x$ by $\cos x$.

So $\sin x \cos x - \cos^2 x = 0$

$\cos x (\sin x - \cos x) = 0$

$\cos x = 0$ or $\sin x - \cos x = 0$

If $\cos x = 0$, then $x = 90°$ or $270°$. (If we divide by $\cos x$, we will miss out these solutions where $\cos x = 0$.)

If $\sin x - \cos x = 0$, then $\sin x = \cos x$.

Since **sin *x* and cos *x* cannot be 0 at the same time**, then $\cos x \neq 0$ and so we can divide both sides of the equation by $\cos x$.

$\therefore \frac{\sin x}{\cos x} = 1$

$\tan x = 1$

Let $\tan \alpha = 1$ where α is the basic angle.

Then $\alpha = \tan^{-1} 1$

$= 45°$ (use Special Angle)

Since $\tan x = 1$ is positive, then x must be in the 1st or 3rd quadrant.

$\therefore$ $x = \alpha$ or $x = 180° + \alpha$

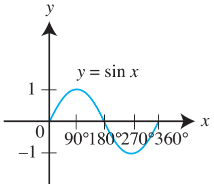
$= 45°$ or $= 180° + 45°$

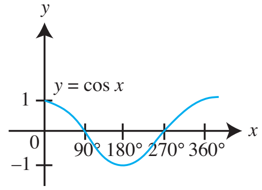
or $= 225°$

$\therefore x = 45°, 90°, 225°$ or $270°$.

ATTENTION

To understand why $\sin x$ and $\cos x$ cannot be 0 at the same time:





From the graphs, when $\sin x = 0$, $\cos x = -1$ or 1; when $\cos x = 0$, $\sin x = -1$ or 1, so $\sin x$ and $\cos x$ cannot be 0 at the same time.

Practise Now 23

Similar Questions:
Exercise 8E
Questions 8(b)-(d), 9(c), (d), 10(a)-(d)

1. Solve $2 \sin x \cos x = \cos x$ for $0° \leqslant x \leqslant 360°$.

2. Find all the angles between 0 and 2π inclusive which satisfy the equation $\tan^2 x - \tan x = 0$.

Exercise 8E

1 Solve each of the following equations for values of x between 0° and 360°.
(a) $\sin x = 0.7245$ **(b)** $\sec x = 2.309$
(c) $\cos x = -0.674$ **(d)** $\tan x = -1.37$

2 Solve each of the following equations for $0° \leqslant x \leqslant 180°$.
(a) $\sin 2x = 0.45$ **(b)** $\cot 2x = 0.699$
(c) $\operatorname{cosec} 2x = -2.342$ **(d)** $\cos 2x = -0.74$

3 Find x in each of the following cases for $0° < x < 360°$.
(a) $\sin x = -0.3782$ and $\cos x$ is positive
(b) $\tan x = -2.361$ and $\sec x$ is negative
(c) $\cos x = -0.4713$ and $\tan x$ is positive
(d) $\sin \frac{1}{2}x = 0.1796$ and $\cot x$ is negative

4 Find all the angles between 0° and 360° which satisfy each of the following equations.
(a) $\sin x = \cos 47°$ **(b)** $\cos x = \tan 38°$
(c) $\cot x = -\sec 183°$ **(d)** $\cos 2x + \cos 24° = 0$
(e) $\tan 3x - 3 \sin 30° = 0$

5 Solve each of the following equations for $0° \leqslant x \leqslant 360°$.
(a) $3 \sin 2x = -1.76$
(b) $5 \cos 3x = -3.45$
(c) $\operatorname{cosec} (2x + 15°) = 1.333$
(d) $\cos (2x - 15°) = -0.145$
(e) $\tan (2x - 78°) = -1.57$
(f) $\sin\left(\frac{1}{2}x - 27°\right) = 0.6$

6 Find all the angles x, where $-\pi < x < \pi$, that satisfy each of the following equations.
(a) $\cos x = 0.75$ **(b)** $\tan x = 2$
(c) $\dfrac{7 \sin x + 6}{1 - \sin x} = 3$ **(d)** $5 \tan\left(x + \dfrac{\pi}{4}\right) = -6$

7 Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.
(a) $\sin x = -0.84$
(b) $\cos (2x - 0.1) = 0.4$
(c) $\tan\left(x - \dfrac{\pi}{4}\right) = -1.48$
(d) $2 \sin (2x + 0.4) = -0.83$

8 Explain why $3 \cos (x + 0.3) = -4$ has no solution.

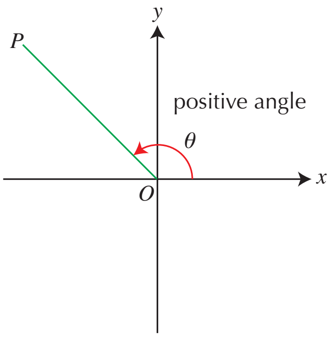
9 Solve each of the following equations for $0° \leqslant x \leqslant 360°$.
(a) $\tan x \sin (x - 25°) = 0$
(b) $\sin^2 x - \sin x \cos x = 0$
(c) $\tan^2 x = \dfrac{9}{4} \tan x$
(d) $\sin x \cos x = 3 \cos x$

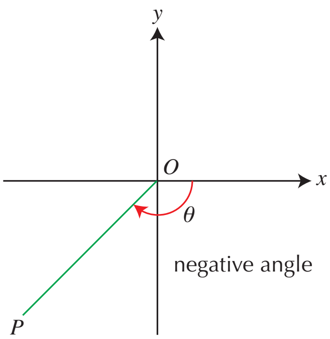
10 Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.
(a) $\cos 2x \sin\left(x - \dfrac{\pi}{4}\right) = 0$
(b) $3 \tan^2 x = 1$
(c) $\sin^2 x = \dfrac{1}{9} \sin x$
(d) $\sin 2x \tan 2x = \sin 2x$

11 Solve each of the following equations for $0° \leqslant x \leqslant 360°$.
(a) $\cos x - 3 \sin x = 0$
(b) $\sec x = 3 \cos x$
(c) $\sin x = \tan x$
(d) $4 \cos x = 2 \cot x$

SUMMARY

1. Angles measured anticlockwise from the positive direction of the x-axis are positive while angles measured clockwise from the positive direction of the x-axis are negative.



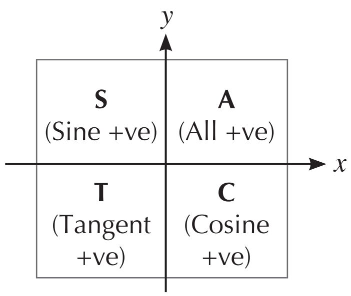


2. In general, we define trigonometric ratios of any angle θ in any quadrant as:

$$\sin\theta = \frac{y}{r},\ \cos\theta = \frac{x}{r} \text{ and } \tan\theta = \frac{y}{x},\ x \neq 0,$$

where x and y are coordinates of the point P in the xy-plane and $r = \sqrt{x^2 + y^2}$.

3. Trigonometric ratios of any angle can be expressed in terms of ratios of the basic angle.



4. For any angle θ,

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta.$$

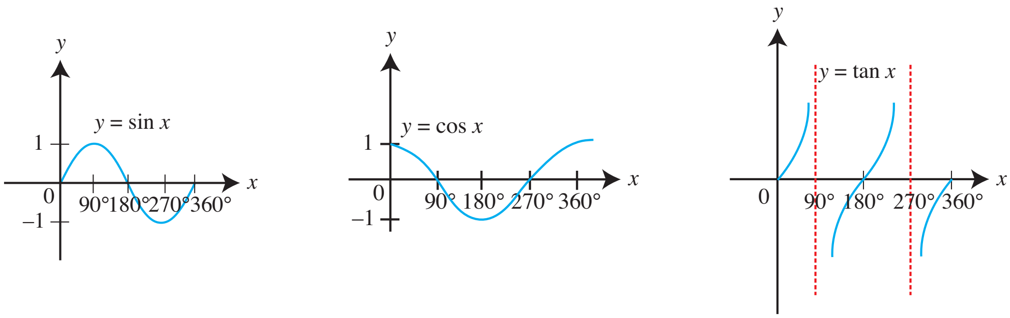
5. For $-1 \leqslant y \leqslant 1$,

Function	Principal value of x
$y = \sin x$	$-90° \leqslant \sin^{-1} y \leqslant 90°$ or $-\frac{\pi}{2} \leqslant \sin^{-1} y \leqslant \frac{\pi}{2}$
$y = \cos x$	$0° \leqslant \cos^{-1} y \leqslant 180°$ or $0 \leqslant \cos^{-1} y \leqslant \pi$

For all real values of y,

Function	Principal value of x
$y = \tan x$	$-90° < \tan^{-1} y < 90°$ or $-\frac{\pi}{2} < \tan^{-1} y < \frac{\pi}{2}$

6. Graphs of basic trigonometric functions:



7. Important identities:

$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$

$\sec\theta = \dfrac{1}{\cos\theta}$

$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$

Review Exercise 8

1. Without using a calculator, find the exact value of each of the following.

(a) $\dfrac{\sin 60^\circ \cos 30^\circ}{\tan 45^\circ}$ **(b)** $\left(\sin\dfrac{\pi}{6} - \cos\dfrac{\pi}{3}\right)\tan\dfrac{\pi}{4}$ **(c)** $\dfrac{\sin 60^\circ - \cos 60^\circ}{\tan 45^\circ + \cos 30^\circ}$

2. Sketch the graphs of each of the following functions for $0 \leqslant x \leqslant \pi$, stating the range, amplitude and period.

(a) $f(x) = 2 + 5\sin 3x$ **(b)** $f(x) = 1 + \cos x$ **(c)** $f(x) = 1 + 2\sin\dfrac{x}{2}$

3. Sketch each of the following curves for $0^\circ \leqslant x \leqslant 360^\circ$.

(a) $y = |\sin 4x|$ **(b)** $y = \left|3\cos\dfrac{x}{2}\right|$ **(c)** $y = |\tan 2x|$

4. On the same diagram, sketch the graphs of $y = |\sin 2x|$ and $y = \dfrac{1}{2}\left(1 + \dfrac{x}{\pi}\right)$ for the domain $0 \leqslant x \leqslant \pi$. Hence, state the number of solutions in this domain of the equation $1 + \dfrac{x}{\pi} = 2|\sin 2x|$.

5. On the same diagram, sketch the graphs of $y = |3\sin 2x|$ and $y = 2 - \dfrac{3x}{2\pi}$ for $0 \leqslant x \leqslant \pi$.
Hence, state the number of solutions in this interval for the equation $|3\pi\sin 2x| = 2\pi - \dfrac{3x}{2}$.

6. Solve each of the following equations for $0^\circ \leqslant x \leqslant 360^\circ$.

(a) $\sin(2x - 15^\circ) = \dfrac{1}{2}$ **(b)** $3\cos^2 x = \sqrt{3}\sin x\cos x$ **(c)** $\sec 2x = -\sqrt{3}$

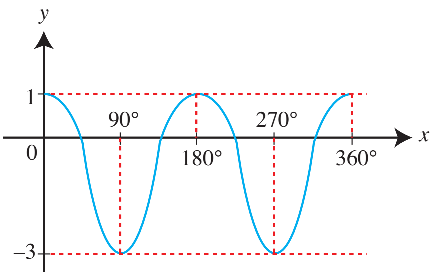
(d) $4\cot x = 5\cos x$ **(e)** $3\tan^2 x + 5\tan x = 0$ **(f)** $\sin x = 3\cos x$

7. The function f is defined as $f(x) = p + q\sin mx$, where p, q and m are positive integers, in the interval $0^\circ \leqslant x \leqslant 360^\circ$.

(a) Find the value of q and of m if the function f has an amplitude of 3 and a period of 120°, explaining your method clearly.

(b) Given that the maximum value of f is 2, sketch the graph of f. Hence, write down the value of p.

8.



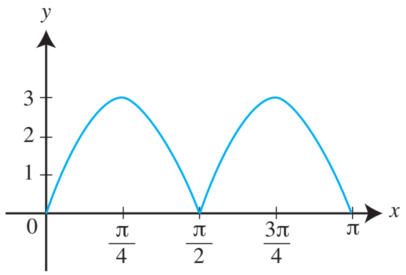
The diagram shows part of the curve of $y = a\cos bx + c$.

(i) Write down the amplitude of y.

(ii) Explain why the period of the function y is 180°.

(iii) Write down the values of a, b and c. Hence, solve the equation $a\cos bx + c = 0$ for $0^\circ \leqslant x \leqslant 360^\circ$.

9. The diagram shows the graph of $y = |a\sin bx|$ or $0 \leqslant x \leqslant \pi$.



(i) Find the possible values of a and explain clearly how you arrive at these values.

(ii) State the value of b.

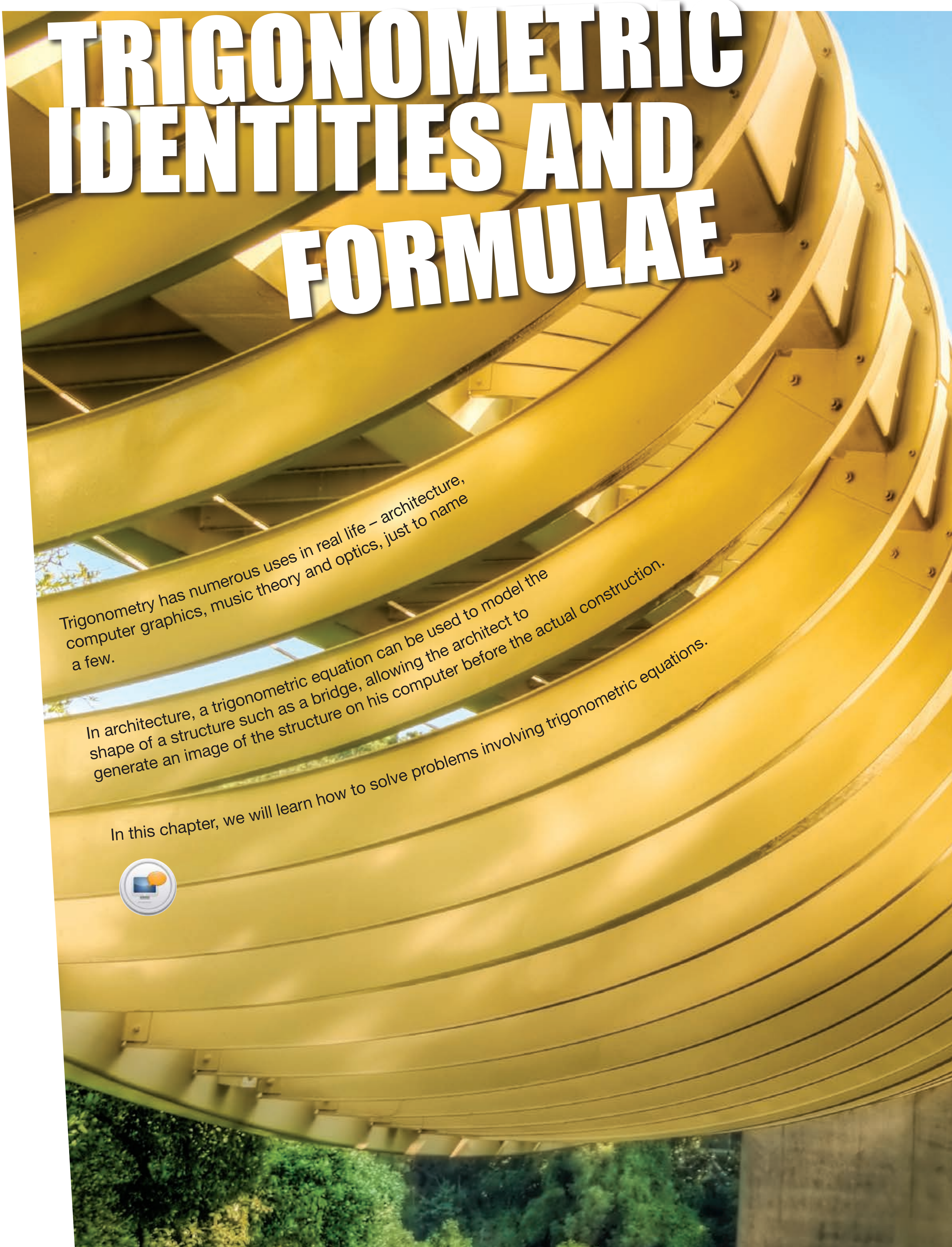
(iii) By inserting a straight line on the graph, find the number of solutions of the equation $|a\sin bx| - \dfrac{2x}{\pi} = 0$ for $0 \leqslant x \leqslant \pi$.

1. Given that $p = \dfrac{1 + \cos x}{\sin x}$ and $\dfrac{1}{p} = \dfrac{1 - \cos x}{\sin x}$, find $\sin x$ and $\cos x$ in terms of p.

2. Sketch each of the following graphs for $0 \leqslant x \leqslant 2\pi$, stating the range for the given domain.

(a) $y = \dfrac{2}{\frac{1}{2} + |2\cos x|}$ **(b)** $y = \dfrac{1}{1 - \frac{1}{2}|\sin x|}$

3. Given that $4\tan x = 3$, where x is acute, find the value of $\dfrac{3\sin x + \sec x}{3\cot x + \operatorname{cosec} x}$.



TRIGONOMETRIC IDENTITIES AND FORMULAE

Trigonometry has numerous uses in real life – architecture, computer graphics, music theory and optics, just to name a few.

In architecture, a trigonometric equation can be used to model the shape of a structure such as a bridge, allowing the architect to generate an image of the structure on his computer before the actual construction.

In this chapter, we will learn how to solve problems involving trigonometric equations.

CHAPTER 9

Learning Objectives

At the end of this chapter, you should be able to:

- prove trigonometric identities,
- apply addition formulae and double angle formulae to simplify trigonometric expressions,
- solve equations of the type $a \sin \theta \pm b \cos \theta = c$ and $a \cos \theta \pm b \sin \theta = c$,
- solve problems involving trigonometric equations.

9.1 TRIGONOMETRIC IDENTITIES

Recap

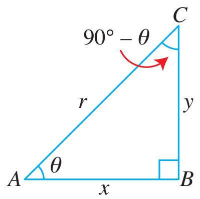
In Chapter 8, we have learnt that for any angle θ, the trigonometric ratios of cosecant, secant and cotangent are defined as

$$\text{cosecant } \theta = \operatorname{cosec} \theta = \frac{1}{\sin \theta},$$
$$\text{secant } \theta = \sec \theta = \frac{1}{\cos \theta},$$
$$\text{cotangent } \theta = \cot \theta = \frac{1}{\tan \theta} = \frac{\cos \theta}{\sin \theta}.$$

In this section, we shall explore the relationship between these trigonometric functions.

Consider the right-angled triangle shown on the right.

$$\sin \theta = \frac{y}{r}, \text{ i.e. } \operatorname{cosec} \theta = \frac{r}{y}$$
$$\cos \theta = \frac{x}{r}, \text{ i.e. } \sec \theta = \frac{r}{x}$$
$$\tan \theta = \frac{y}{x}, \text{ i.e. } \cot \theta = \frac{x}{y}$$



Using Pythagoras' Theorem,

$$x^2 + y^2 = r^2 \text{ ---- } (*)$$

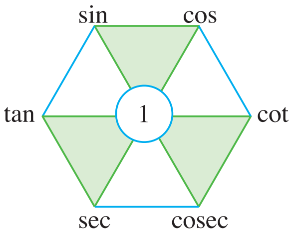


Fig. 9.1

$(*) \div r^2$: $\dfrac{x^2}{r^2} + \dfrac{y^2}{r^2} = 1$, i.e. $\cos^2 \theta + \sin^2 \theta = 1$

$(*) \div y^2$: $\dfrac{x^2}{y^2} + 1 = \dfrac{r^2}{y^2}$, i.e. $\cot^2 \theta + 1 = \operatorname{cosec}^2 \theta$

$(*) \div x^2$: $1 + \dfrac{y^2}{x^2} = \dfrac{r^2}{x^2}$, i.e. $1 + \tan^2 \theta = \sec^2 \theta$

To help you remember these identities, a regular hexagon is drawn as shown in Fig. 9.1.

From Fig. 9.1, any of the following rules may be applied to obtain the various trigonometric identities:

1. Any trigonometric ratio is equal to the product of its two immediate neighbours.

i.e. $\sin \theta = \tan \theta \cos \theta$, $\cot \theta = \cos \theta \operatorname{cosec} \theta$, $\sec \theta = \tan \theta \operatorname{cosec} \theta$

$\cos \theta = \sin \theta \cot \theta$, $\tan \theta = \sin \theta \sec \theta$, $\operatorname{cosec} \theta = \sec \theta \cot \theta$

2. The trigonometric ratios at opposite ends of the same diagonal are reciprocals of each other.

i.e. $\sin\theta = \dfrac{1}{\operatorname{cosec}\theta}$, $\tan\theta = \dfrac{1}{\cot\theta}$, $\cos\theta = \dfrac{1}{\sec\theta}$

3. Consider the three shaded parts. In each shaded part, the sum of the squares of ratios/values at the two *adjacent* vertices (one of which may be 1) is equal to the square of the trigonometric ratio/value at the third vertex.

i.e. $\sin^2\theta + \cos^2\theta = 1$, $\tan^2\theta + 1 = \sec^2\theta$, $1 + \cot^2\theta = \operatorname{cosec}^2\theta$

We shall make use of these identities to solve trigonometric equations.

Worked Example 1

(Solving Trigonometric Equations)

Find all the angles between 0 and 2π which satisfy the equations

(a) $2\cos^2 x - 3\cos x + 1 = 0$, **(b)** $6\tan^2 x = 14 + 7\sec x$.

ATTENTION

Note that the angles required in this question are in radians.

Solution

(a)
$$2\cos^2 x - 3\cos x + 1 = 0$$
$$(2\cos x - 1)(\cos x - 1) = 0$$
$$\cos x = \frac{1}{2} \quad \text{or} \quad \cos x = 1$$
$$\alpha = \frac{\pi}{3} \qquad x = 0, 2\pi \text{ (not applicable)}$$
$$x = \frac{\pi}{3},\ 2\pi - \frac{\pi}{3}$$
$$= \frac{\pi}{3},\ \frac{5\pi}{3}$$
$$\therefore\ x = \frac{\pi}{3},\ \frac{5\pi}{3}$$

(b)
$$6\tan^2 x = 14 + 7\sec x$$
$$6(\sec^2 x - 1) = 14 + 7\sec x \quad (1 + \tan^2 x = \sec^2 x)$$
$$6\sec^2 x - 7\sec x - 20 = 0$$
$$(2\sec x - 5)(3\sec x + 4) = 0$$
$$\sec x = \frac{5}{2} \quad \text{or} \quad \sec x = -\frac{4}{3}$$
$$\cos x = \frac{2}{5} \qquad \cos x = -\frac{3}{4}$$
$$\alpha = 1.159 \qquad \alpha = 0.7227$$
$$x = 1.16,\ 2\pi - 1.1593 \qquad x = \pi - 0.72273,\ \pi + 0.72273$$
$$= 1.16,\ 5.12 \text{ (to 3 s.f.)} \qquad = 2.42 \text{ or } 3.86 \text{ (to 3 s.f.)}$$
$$\therefore\ x = 1.16, 2.42, 3.86, 5.12$$

RECALL

$\cos x > 0$ in the 1st and 4th quadrants.
$\cos x < 0$ in the 2nd and 3rd quadrants.

Practise Now 1

Similar Questions:
Exercise 9A
Questions 2(a)-(d), 4(a)-(d)

1. Find all the angles between 0 and 2π which satisfy the equation $9\sec^2 x = 24\sec x - 16$.

2. Find all the angles between 0 and 2π which satisfy the equation $3\sin^2 z - 5\cos z = 5$.

Worked Example 2

(Using Identities to Solve Equations)

Solve the equation $2\cos\theta + 2\sin\theta = \dfrac{1}{2\cos\theta - 2\sin\theta}$ for $0° \leqslant \theta \leqslant 360°$.

Solution

$$2\cos\theta + 2\sin\theta = \frac{1}{2\cos\theta - 2\sin\theta}$$

$$4\cos^2\theta - 4\sin^2\theta = 1$$

$$4\cos^2\theta - 4(1 - \cos^2\theta) = 1 \quad (\sin^2\theta = 1 - \cos^2\theta)$$

$$4\cos^2\theta - 4 + 4\cos^2\theta = 1$$

$$8\cos^2\theta = 5$$

$$\cos^2\theta = \frac{5}{8}$$

$$\cos\theta = \pm\sqrt{\frac{5}{8}}$$

$$\alpha = 37.76° \quad \text{(to 2 d.p.)}$$

$$\theta = 37.76°,\ 180° - 37.76°,\ 180° + 37.76°,\ 360° - 37.76°$$

$\therefore\ \theta = 37.8°, 142.2°, 217.8°, 322.2°$ (to 1 d.p.)

RECALL

$(a + b)(a - b) = a^2 - b^2$

Practise Now 2

Similar Questions:
Exercise 9A
Questions 1(a)-(d), 3(a)-(d), 5

1. Solve the equation $4\sin x + 4\cos x = \dfrac{1}{\cos x - \sin x}$ for $0° \leqslant x \leqslant 360°$.

2. Solve the equation $4\cos^2\theta = 9 - 2\sec^2\theta$ for $0 \leqslant \theta \leqslant 2\pi$.

Basic Level | Intermediate Level | Advanced Level

Exercise 9A

1 Solve each of the following equations for $0° \leqslant x \leqslant 360°$.

(a) $2\sec^2 x - \tan x - 3 = 0$

(b) $2\cos^2 x - 5\sin x = 5$

(c) $2\operatorname{cosec}^2 x + \cot x - 8 = 0$

(d) $2\sin^2 x - \cos x + 1 = 0$

2 Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.

(a) $2\cot^2 x + 5 = 7\operatorname{cosec} x$

(b) $\tan^2 x = 7 - 2\sec x$

(c) $2\cos x = 3\sin^2 x$

(d) $2\sec^2 x - 5\tan x = 0$

3 Solve each of the following equations for $0° \leqslant x \leqslant 360°$.

(a) $2\cos x - \sin x = \dfrac{3}{2\cos x + \sin x}$

(b) $5\tan x + 6 = \dfrac{2}{\cos^2 x}$

(c) $3\cos x + 3 = \dfrac{5}{\operatorname{cosec}^2 x}$

(d) $\dfrac{3}{\cos^2 x} = 7 + \dfrac{4}{\cot x}$

4 Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.

(a) $\cos^2 x - \sin^2 x = -1$

(b) $\sin x + \cos x \cot x = 2$

(c) $4 + \sin x \tan x = 4 \cos x$

(d) $\dfrac{8}{1 - \cos^2 x} - 14 \operatorname{cosec} x + 5 = 0$

5 Express the equation $\cot x = \dfrac{13 + \sin x}{5 \cos x}$ in the form $a \sin^2 x + b \sin x + c = 0$, where a, b and c are constants. Hence, solve the equation $\cot x = \dfrac{13 + \sin x}{5 \cos x}$ for $0° \leqslant x \leqslant 360°$.

9.2 PROVING OF IDENTITIES

A trigonometric identity is an equation that holds true for all values of the angles involved in the expression.
For instance, $1 + \tan^2 A = \sec^2 A$ is true for all values of A.

Strategy 1: Think of what identities you can use

- It is sometimes useful to express every expression in terms of $\sin x$ and $\cos x$, e.g. $\tan x$, $\operatorname{cosec} x$, $\sec x$ and $\cot x$ can all be expressed as $\dfrac{\sin x}{\cos x}$, $\dfrac{1}{\sin x}$, $\dfrac{1}{\cos x}$ and $\dfrac{\cos x}{\sin x}$ respectively.
- Look out for expressions such as $\sin^2 x + \cos^2 x$, $\sec^2 x - \tan^2 x$ and $\operatorname{cosec}^2 x - \cot^2 x$ because they are all equal to 1.

Worked Example 3

(Use of Trigonometric Identities)

Prove the identity $(\cot A + \tan A) \cos A = \operatorname{cosec} A$.

Solution

$$\begin{aligned}
\text{LHS} &= (\cot A + \tan A) \cos A \\
&= \left(\frac{\cos A}{\sin A} + \frac{\sin A}{\cos A}\right) \cos A \\
&= \left(\frac{\cos^2 A + \sin^2 A}{\sin A \cos A}\right) \cos A \\
&= \left(\frac{1}{\sin A \cos A}\right) \cos A \qquad (\cos^2 A + \sin^2 A = 1) \\
&= \frac{1}{\sin A} \\
&= \operatorname{cosec} A = \text{RHS (proven)}
\end{aligned}$$

PROBLEM-SOLVING TIP

Express $\cot A$ and $\tan A$ in terms of $\cos A$ and $\sin A$.

Practise Now 3

Similar Questions:
Exercise 9B
Questions 1(a)-(g), 2(a)-(e)

Prove each of the following identities.

(a) $\cos^2 A + \cot^2 A \cos^2 A = \cot^2 A$

(b) $\dfrac{1+\sin A-\sin^2 A}{\cos A} = \cos A + \tan A$

Strategy 2: Start from the more complicated expression

It is usually easier to simplify a more complicated expression than to try to make a simple expression more complicated.

Worked Example 4

(Simplifying a More Complicated Expression)

Prove the identity $2\tan x = \dfrac{\cos x}{\operatorname{cosec} x-1} + \dfrac{\cos x}{\operatorname{cosec} x+1}$.

Solution

$$\begin{aligned}
\text{RHS} &= \frac{\cos x}{\operatorname{cosec} x-1} + \frac{\cos x}{\operatorname{cosec} x+1}\\
&= \frac{\cos x(\operatorname{cosec} x+1)+\cos x(\operatorname{cosec} x-1)}{(\operatorname{cosec} x-1)(\operatorname{cosec} x+1)}\\
&= \frac{\cos x\operatorname{cosec} x+\cos x+\cos x\operatorname{cosec} x-\cos x}{\operatorname{cosec}^2 x-1}\\
&= \frac{2\cos x\operatorname{cosec} x}{\cot^2 x} && (\operatorname{cosec}^2 x - 1 = \cot^2 x)\\
&= \frac{2\cos x\left(\frac{1}{\sin x}\right)}{\frac{\cos^2 x}{\sin^2 x}} && (\text{express as } \sin x \text{ and } \cos x)\\
&= \frac{2\sin x}{\cos x}\\
&= 2\tan x = \text{LHS (proven)}
\end{aligned}$$

Practise Now 4

Similar Questions:
Exercise 9B
Questions 2(f), 3(a)-(e)

Prove each of the following identities.

(a) $(1+\tan\theta)^2 + (1-\tan\theta)^2 = 2\sec^2\theta$

(b) $\dfrac{1+\sec A}{\tan A+\sin A} = \operatorname{cosec} A$

Worked Example 5

(Simplifying a More Complicated Expression)

Prove the identity $\dfrac{1+\cos x}{\sin x \cos x} = \tan x + \cot x + \operatorname{cosec} x$.

Solution

Method 1:

$$\begin{aligned}
\text{LHS} &= \frac{1+\cos x}{\sin x \cos x} \\
&= \frac{1}{\sin x \cos x} + \frac{\cos x}{\sin x \cos x} \\
&= \frac{1}{\sin x \cos x} + \operatorname{cosec} x \\
&= \frac{\sin^2 x + \cos^2 x}{\sin x \cos x} + \operatorname{cosec} x \qquad \text{(replace 1 with } \sin^2 x + \cos^2 x\text{)} \\
&= \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x} + \operatorname{cosec} x \\
&= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \operatorname{cosec} x \\
&= \tan x + \cot x + \operatorname{cosec} x = \text{RHS (proven)}
\end{aligned}$$

Method 2:

$$\begin{aligned}
\text{RHS} &= \tan x + \cot x + \operatorname{cosec} x \\
&= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x} + \frac{1}{\sin x} \\
&= \frac{\sin^2 x + \cos^2 x + \cos x}{\sin x \cos x} \\
&= \frac{1+\cos x}{\sin x \cos x} = \text{LHS (proven)}
\end{aligned}$$

Which method do you prefer?

Practise Now 5

Similar Questions:
Exercise 9B
Questions 3(f)-(s), 4(a)-(c), 5

Prove each of the following identities.

(a) $\dfrac{1+\sin x}{1-\sin x} = (\sec x + \tan x)^2$

(b) $\operatorname{cosec} x\,(\operatorname{cosec} x - \sin x) + \dfrac{\sin x - \cos x}{\sin x} = \operatorname{cosec}^2 x - \cot x$

Basic Level | Intermediate Level | Advanced Level

Exercise 9B

1 Prove each of the following identities.

(a) $\cos x = \sin x \cot x$

(b) $\tan x = \sin x \sec x$

(c) $\cot x = \cos x \operatorname{cosec} x$

(d) $\operatorname{cosec} x - \sin x = \cos x \cot x$

(e) $\dfrac{\cos^2 x}{1-\sin x} = 1 + \sin x$

(f) $1 - \dfrac{\sin^2 x}{1+\cos x} = \cos x$

(g) $1 + 2\sec^2 x = 2\tan^2 x + 3$

(g) $\dfrac{1}{\tan x + \cot x} = \sin x \cos x$

(h) $\dfrac{\sin x}{1+\cos x} + \dfrac{1+\cos x}{\sin x} = 2\operatorname{cosec} x$

(i) $\tan^2 x + \cot^2 x + 2 = \operatorname{cosec}^2 x \sec^2 x$

(j) $\cos^2 x - \sin^2 x = \cos^4 x - \sin^4 x$

(k) $(\cos x + \cot x)\sec x = 1 + \operatorname{cosec} x$

(l) $(1+\cot x)^2 + (1-\cot x)^2 = \dfrac{2}{\sin^2 x}$

(m) $\tan^2 x - \sin^2 x = \tan^2 x \sin^2 x$

(n) $\cot x(\sec^2 x - 1) = \tan x$

(o) $(1-\cos x)(1+\sec x) = \sin x \tan x$

(p) $(1-\sin x)\left(1+\dfrac{1}{\sin x}\right) = \cos x \cot x$

(q) $\cos^2 x + \cot^2 x \cos^2 x = \cot^2 x$

(r) $\dfrac{1-\cot^2 x}{1+\cot^2 x} = \sin^2 x - \cos^2 x$

(s) $\sec^4 x - \tan^4 x = \dfrac{1+\sin^2 x}{\cos^2 x}$

2 Prove each of the following identities.

(a) $\dfrac{\sin^2 x}{1-\sin^2 x} = \sec^2 x - 1$

(b) $\dfrac{1+\cot x}{1+\tan x} = \dfrac{1}{\tan x}$

(c) $1 + \dfrac{\cos^2 x}{\sin x - 1} = -\sin x$

(d) $\dfrac{1+\sin x}{1+\operatorname{cosec} x} = \sin x$

(e) $\dfrac{1+\sec x}{1+\cos x} = \sec x$

(f) $\dfrac{1-\sec^2 x}{(1+\cos x)(1-\cos x)} = -\sec^2 x$

3 Prove each of the following identities.

(a) $\dfrac{\sin^2 x(1+\cot^2 x)}{\cos^2 x} = \tan^2 x + 1$

(b) $\dfrac{1-\cos^2 x}{1-\sin^2 x} + \tan x \cot x = \sec^2 x$

(c) $\dfrac{\cos x}{1-\sin^2 x - \cos^2 x + \sin x} = \cot x$

(d) $\dfrac{1-\sin^2 x}{1-\cos^2 x} + \tan x \cot x = \dfrac{1}{\sin^2 x}$

(e) $\dfrac{1}{\sin x + 1} - \dfrac{1}{\sin x - 1} = \dfrac{2}{\cos^2 x}$

(f) $\dfrac{1-2\cos^2 x}{\sin x \cos x} = \tan x - \cot x$

4 Prove each of the following identities.

(a) $\dfrac{\cos A - \cos B}{\sin A + \sin B} = \dfrac{\sin B - \sin A}{\cos A + \cos B}$

(b) $(\cos^2 \theta - 2)^2 - 3\sin^2 \theta = \cos^4 \theta + \sin^2 \theta$

(c) $\dfrac{1}{1-\dfrac{1}{1-\dfrac{1}{1-\operatorname{cosec}^2 x}}} = \operatorname{cosec}^2 x$

5 If $p = \dfrac{1+\cos x}{\sin x}$, prove that $\dfrac{1}{p} = \dfrac{1-\cos x}{\sin x}$.

Hence, find $\sin x$ and $\cos x$ in terms of p.

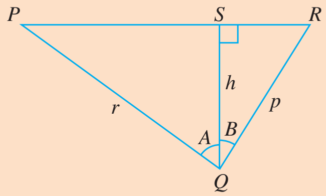
9.3 ADDITION FORMULAE

Given a compound angle $(A + B)$, are we able to find a formula for the expansion of, say, $\sin(A + B)$ in terms of $\sin A$, $\sin B$, $\cos A$ and $\cos B$? Let us attempt the investigation below to find out.

Investigation

Expansion of $\sin(A + B)$

In this investigation, we will derive a formula for the expansion of $\sin(A + B)$.



In the figure, QS is perpendicular to PR, $\angle A$ and $\angle B$ are acute angles, $PQ = r$, $QS = h$ and $QR = p$.

Consider ΔRQS and ΔPQS. Express the ratios $\cos B$ and $\cos A$ in terms of h, r and p.

By writing the areas of ΔPQR, ΔPQS and ΔRQS in terms of $(A + B)$, A and B respectively, copy and complete the table below.

Area of ΔPQR	Area of ΔPQS	Area of ΔRQS
$\frac{1}{2}rp\sin(A + B)$		

Write an expression to show how the 3 areas are related.
Hence, derive the expression for $\sin(A + B)$.

Thinking Time

By replacing B with $(-B)$ in the above formula, show that

$$\sin(A - B) = \sin A\cos B - \cos A\sin B.$$

Let us use the relationship $\cos\theta = \sin(90° - \theta)$ to find the formula for the expansion of $\cos(A + B)$.

$$\begin{aligned}\cos(A + B) &= \sin[90° - (A + B)]\\ &= \sin[(90° - A) - B]\\ &= \sin(90° - A)\cos B - \cos(90° - A)\sin B\\ &= \cos A\cos B - \sin A\sin B\end{aligned}$$

If we replace B with $-B$ in the above formula, what expression do we obtain for the expansion of $\cos(A - B)$?

By writing $\tan(A + B)$ as $\dfrac{\sin(A + B)}{\cos(A + B)}$ and $\tan(A - B)$ as $\dfrac{\sin(A - B)}{\cos(A - B)}$, obtain similar expressions for the expansion of $\tan(A + B)$ and $\tan(A - B)$.

In general, the **Addition Formulae** are

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$
$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$
$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}.$$

Thinking Time

Do you think $\sin(A+B)$ is equal to $\sin A + \sin B$?

1. Choose $A = 60°$ and $B = 20°$. What is $A + B$?
2. Find the values of the following using a calculator, leaving your answer correct to 3 significant figures.

$\sin(A+B)$	$\sin A + \sin B$	$\sin(A-B)$	$\sin A - \sin B$

3. From the table above, is $\sin(A+B) = \sin A + \sin B$?
   Is $\sin(A-B) = \sin A - \sin B$?
4. Choose a different set of values of A and B to check if:
   (i) $\sin(A+B) = \sin A + \sin B$?
   (ii) $\sin(A-B) = \sin A - \sin B$?
5. Use a calculator to investigate whether:
   (i) $\cos(A+B) = \cos A + \cos B$?
   (ii) $\cos(A-B) = \cos A - \cos B$?
   (iii) $\tan(A+B) = \tan A + \tan B$?
   (iv) $\tan(A-B) = \tan A - \tan B$?

Serious Misconceptions

$\sin(A+B) \neq \sin A + \sin B$ $\quad$ $\cos(A+B) \neq \cos A + \cos B$ $\quad$ $\tan(A+B) \neq \tan A + \tan B$

$\sin(A-B) \neq \sin A - \sin B$ $\quad$ $\cos(A-B) \neq \cos A - \cos B$ $\quad$ $\tan(A-B) \neq \tan A - \tan B$

Worked Example 6

(Use of the Addition Formulae to Find Trigonometric Ratios)
Without using a calculator, find the value of each of the following, giving each answer in surd form.

(a) $\cos 75°$
(b) $\sin 37°\cos 23° + \cos 37°\sin 23°$
(c) $\dfrac{1 + \tan 15°}{1 - \tan 15°}$

Solution

(a) $\cos 75° = \cos(45° + 30°)$

$$= \cos 45° \cos 30° - \sin 45° \sin 30°$$

$$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}} \times \frac{1}{2}$$

$$= \frac{\sqrt{3}-1}{2\sqrt{2}} \qquad \text{(rationalise the denominator)}$$

$$= \frac{\sqrt{6}-\sqrt{2}}{4}$$

PROBLEM-SOLVING TIP

We choose $75° = 45° + 30°$ because $45°$ and $30°$ are special angles. Then we apply the expansion of $\cos(A + B)$.

(b) $\sin 37° \cos 23° + \cos 37° \sin 23°$

$$= \sin(37° + 23°) \qquad (\sin(A+B) = \sin A \cos B + \cos A \sin B)$$

$$= \sin 60°$$

$$= \frac{\sqrt{3}}{2}$$

(c) $\dfrac{1 + \tan 15°}{1 - \tan 15°} = \dfrac{\tan 45° + \tan 15°}{1 - \tan 45° \tan 15°} \qquad (\tan 45° = 1)$

$$= \tan(45° + 15°) \qquad \left(\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}\right)$$

$$= \tan 60°$$

$$= \sqrt{3}$$

Practise Now 6

Similar Questions:
Exercise 9C
Questions 1(a)-(f), 2(a)-(h)

Without using a calculator, express each of the following in surd form.

(a) $\cos 15°$ **(b)** $\sin 187° \cos 52° - \cos 187° \sin 52°$ **(c)** $\dfrac{1 - \tan 15°}{1 + \tan 15°}$

Worked Example 7

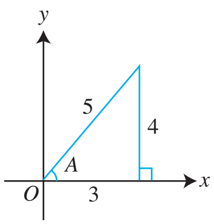
(Use of the Addition Formulae)

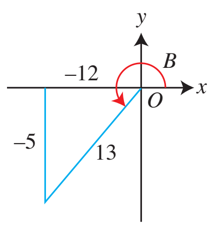
Given that $\cos A = \frac{3}{5}$ and $\sin B = -\frac{5}{13}$, where A is acute and $90° < B < 270°$, find, without using a calculator, the value of each of the following.

(i) $\sin(A + B)$ **(ii)** $\cos(A - B)$ **(iii)** $\tan(A + B)$

Solution

A is acute, i.e. A lies in the first quadrant.
Since $\sin B < 0$ and $90° < B < 270°$, B lies in the third quadrant.





PROBLEM-SOLVING TIP

Draw right-angled triangles in the appropriate quadrants to find the values of the other trigonometric ratios.

From the diagrams, we have

$$\sin A = \frac{4}{5},\quad \tan A = \frac{4}{3},\quad \cos B = -\frac{12}{13},\quad \tan B = \frac{5}{12}.$$

(i) $\sin (A + B) = \sin A \cos B + \cos A \sin B$

$$= \left(\frac{4}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{3}{5}\right)\left(-\frac{5}{13}\right)$$

$$= -\frac{63}{65}$$

ATTENTION

$\sin (A + B) \neq \sin A + \sin B$
$\cos (A - B) \neq \cos A - \cos B$
$\tan (A + B) \neq \tan A + \tan B$

(ii) $\cos (A - B) = \cos A \cos B + \sin A \sin B$

$$= \left(\frac{3}{5}\right)\left(-\frac{12}{13}\right) + \left(\frac{4}{5}\right)\left(-\frac{5}{13}\right)$$

$$= -\frac{56}{65}$$

(iii) $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

$$= \frac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{4}{3} \times \frac{5}{12}}$$

$$= 3\frac{15}{16}$$

Practise Now 7

Similar Questions:
Exercise 9C
Questions 3-7

1. Given that $\sin A = \frac{3}{5}$ and $\tan B = \frac{5}{12}$, where both A and B lie between 90° and 270°, without using a calculator, find the value of each of the following.
   (i) $\sin (A - B)$ **(ii)** $\cos (A + B)$ **(iii)** $\tan (A - B)$
2. Given that $\sin A = \frac{3}{5}$ and $\cos B = -\frac{4}{5}$, where both A and B are obtuse, without using a calculater, find the value of
   (i) $\tan (A + B)$, **(ii)** $\cos (A + B)$, **(iii)** $\sin (A - B)$.

Worked Example 8

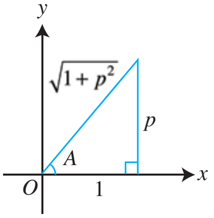
(Algebraic Fractions in Addition Formulae)

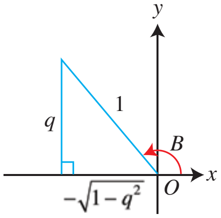
Given that $\tan A = p$ and $\sin B = q$, where A is acute and B is obtuse, find each of the following in terms of p and q.

(i) $\sin (A + B)$ **(ii)** $\cos (B - A)$ **(iii)** $\tan (A - B)$

Solution

A and B must lie in the 1st and 2nd quadrants respectively.





(i) $\sin(A+B) = \sin A \cos B + \cos A \sin B$

$$= \left(\frac{p}{\sqrt{1+p^2}}\right)\left(-\sqrt{1-q^2}\right) + \left(\frac{1}{\sqrt{1+p^2}}\right)(q)$$

$$= \frac{q - p\sqrt{1-q^2}}{\sqrt{1+p^2}}$$

(ii) $\cos(B-A) = \cos B \cos A + \sin B \sin A$

$$= \left(-\sqrt{1-q^2}\right)\left(\frac{1}{\sqrt{1+p^2}}\right) + q\left(\frac{p}{\sqrt{1+p^2}}\right)$$

$$= \frac{pq - \sqrt{1-q^2}}{\sqrt{1+p^2}}$$

(iii) $\tan(A-B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$$= \frac{p - \dfrac{-q}{\sqrt{1-q^2}}}{1 + p\left(\dfrac{-q}{\sqrt{1-q^2}}\right)}$$

$$= \frac{\dfrac{p\sqrt{1-q^2}+q}{\sqrt{1-q^2}}}{\dfrac{\sqrt{1-q^2}-pq}{\sqrt{1-q^2}}}$$

$$= \frac{p\sqrt{1-q^2}+q}{\sqrt{1-q^2}} \div \frac{\sqrt{1-q^2}-pq}{\sqrt{1-q^2}}$$

$$= \frac{p\sqrt{1-q^2}+q}{\sqrt{1-q^2}} \times \frac{\sqrt{1-q^2}}{\sqrt{1-q^2}-pq}$$

$$= \frac{p\sqrt{1-q^2}+q}{\sqrt{1-q^2}-pq}$$

Practise Now 8

Similar Questions:
Exercise 9C
Questions 10-12

Given that $\tan A = p$ and $\tan B = q$, where A and B are acute angles, find each of the following in terms of p and q.

(i) $\sin(A+B)$ **(ii)** $\cos(A-B)$ **(iii)** $\tan(A+B)$

Worked Example 9

(Use of the Addition Formulae)

It is given that A and B are acute angles such that $\cos(A-B)=\frac{16}{25}$ and $\sin A\sin B=\frac{6}{25}$. Without using a calculator, find the value of each of the following.

(i) $\cos A\cos B$ **(ii)** $\cos(A+B)$ **(iii)** $2\tan A\tan B$

Solution

(i)
$$\begin{aligned}\cos(A-B)&=\cos A\cos B+\sin A\sin B\\ \frac{16}{25}&=\cos A\cos B+\frac{6}{25}\\ \cos A\cos B&=\frac{16}{25}-\frac{6}{25}\\ &=\frac{10}{25}\\ &=\frac{2}{5}\end{aligned}$$

(ii)
$$\begin{aligned}\cos(A+B)&=\cos A\cos B-\sin A\sin B\\ &=\frac{2}{5}-\frac{6}{25}\\ &=\frac{10}{25}-\frac{6}{25}\\ &=\frac{4}{25}\end{aligned}$$

(iii)
$$\begin{aligned}2\tan A\tan B&=2\left(\frac{\sin A}{\cos A}\right)\left(\frac{\sin B}{\cos B}\right)\\ &=2\left(\cfrac{\frac{{}^{3}\cancel{6}}{\cancel{25}\,{}^{5}}}{\frac{{}^{1}\cancel{2}}{\cancel{5}\,{}^{1}}}\right)\\ &=\frac{6}{5}\\ &=1\frac{1}{5}\end{aligned}$$

Practise Now 9

Similar Questions:
Exercise 9C
Questions 8, 9, 13, 14

1. It is given that A and B are acute angles such that $\sin(A+B)=\frac{7}{8}$ and $\cos A\sin B=\frac{1}{4}$. Without using a calculator, find the value of each of the following.

(i) $\sin A\cos B$ **(ii)** $\sin(A-B)$ **(iii)** $\dfrac{2\tan A}{3\tan B}$

2. Given that $\dfrac{\cos(A+B)}{\cos(A-B)}=\dfrac{3}{4}$, prove that $\cos A\cos B=7\sin A\sin B$.

Hence, find a relationship between $\tan A$ and $\tan B$.

Exercise 9C

Do not use a calculator for this exercise.

1 Express each of the following as a single trigonometric ratio.

(a) sin 37° cos 25 ° + cos 37° sin 25°
(b) cos 25° cos 15° – sin 25° sin 15°
(c) sin 126° cos 23° – cos 126° sin 23°
(d) cos 75° cos 24° + sin 75° sin 24°
(e) $\dfrac{\tan 27° + \tan 13°}{1 - \tan 27° \tan 13°}$
(f) $\dfrac{\tan 37° - \tan 12°}{1 + \tan 37° \tan 12°}$

2 Use the addition formulae to express each of the following in surd form.

(a) sin 15° **(b)** sin 105°
(c) sin 75° **(d)** cos 105°
(e) cos 345° **(f)** tan 75°
(g) tan 255° **(h)** tan 285°

3 Given that $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{12}{13}$ and that A and B are acute, find the value of

(i) $\sin (A + B)$, **(ii)** $\cos (A + B)$,
(iii) $\tan (A - B)$.

4 Given that $\sin A = \dfrac{5}{13}$ and $\sin B = \dfrac{3}{5}$, where A is obtuse and B is acute, find the value of

(i) $\sin (A + B)$, **(ii)** $\cos (A - B)$,
(iii) $\tan (A + B)$.

5 Given that $\tan A = -\dfrac{3}{4}$ and $\cos B = -\dfrac{5}{13}$ such that both A and B lie between 180° and 360°, find the value of

(i) $\sin (A + B)$, **(ii)** $\cos (A - B)$,
(iii) $\tan (A + B)$.

6 Given that $\cos A = \dfrac{8}{17}$ and $\sin B = -\dfrac{3}{5}$ and that A and B lie in the same quadrant, find the value of

(i) $\sin (A - B)$, **(ii)** $\cos (A + B)$,
(iii) $\tan (A - B)$.

7 Given that $\sin A = -\dfrac{15}{17}$ and $\cos B = -\dfrac{4}{5}$ and that A and B lie in the same quadrant, find the value of

(i) $\sin (A + B)$, **(ii)** $\cos (A + B)$,
(iii) $\tan (A - B)$.

8 Show that $\cos (\theta - 40°) = \sin (\theta + 50°)$.
Hence, or otherwise, solve the equation $\cos (\theta - 40°) = 4 \sin (\theta + 50°)$, giving all solutions between 0° and 360°.

9 Given that A and B are acute angles, $\sin (A - B) = \dfrac{16}{65}$ and $\sin A \cos B = \dfrac{36}{65}$, find the value of

(i) $\cos A \sin B$, **(ii)** $\sin (A + B)$,
(iii) $3 \tan A \cot B$.

10 $\angle A$ is an acute angle such that $\tan A = a$. Find the value of each of the following in terms of a.

(i) $\tan (45° + A)$ **(ii)** $\sin (60° + A)$
(iii) $\cos (A - 30°)$

11 Given that $\tan A = a$, $\sin B = b$ and that A and B are acute, find the value of each of the following in terms of a and/or b.

(i) $\sin (A + B)$ **(ii)** $\cos (A - B)$
(iii) $\tan (A - B)$

Exercise 9C

12 Given $\sin A = x$, $\cos B = y$ and that $90° < A < 180°$ and $270° < B < 360°$, find the value of each of the following in terms of x and/or y.

(i) $\sin (A + B)$ **(ii)** $\cos (A - B)$ **(iii)** $\tan 2A$ **(iv)** $\sin 2B$ **(v)** $\cos 2B$

13 Given that $\cos (A+B) = \frac{3}{5}$ and $\cos A \cos B = \frac{7}{10}$, where both A and B are acute angles, find the value of

(i) $\tan A \tan B$, **(ii)** $\tan (A - B)$, **(iii)** $\sin \left(\frac{A+B}{2}\right)$.

14 Show that $\frac{\sin (A+B) - \sin (A-B)}{\cos (A+B) + \cos (A-B)} = \tan B$. Given that $\tan (A+B) = -\frac{3}{4}$ and $\tan (A-B) = \frac{5}{12}$, where $180° \leqslant A + B \leqslant 360°$ and $0° \leqslant A - B \leqslant 180°$, write down the values of $\sin (A + B)$, $\cos (A + B)$, $\sin (A - B)$ and $\cos (A - B)$. Hence, calculate the value of $\tan B$.

9.4 DOUBLE ANGLE FORMULAE

From the Addition Formulae, more identities can be derived.

Consider $\sin (A + B) = \sin A \cos B + \cos A \sin B$

If $B = A$, $\sin (A + A) = \sin A \cos A + \cos A \sin A$

$$\sin 2A = 2 \sin A \cos A$$

Consider $\cos (A + B) = \cos A \cos B - \sin A \sin B$

If $B = A$, $\cos (A + A) = \cos A \cos A - \sin A \sin A$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$\cos^2 A = 1 - \sin^2 A$: $\cos 2A = 1 - 2 \sin^2 A$

$\sin^2 A = 1 - \cos^2 A$: $\cos 2A = 2 \cos^2 A - 1$

Consider $\tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$

If $B = A$, $\tan (A + A) = \frac{\tan A + \tan A}{1 - \tan A \tan A}$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

Summarising, we have

$$\begin{aligned} \sin 2A &= 2 \sin A \cos A \\ \cos 2A &= \cos^2 A - \sin^2 A \\ &= 1 - 2 \sin^2 A \\ &= 2 \cos^2 A - 1 \\ \tan 2A &= \frac{2 \tan A}{1 - \tan^2 A}. \end{aligned}$$

Thinking Time

By replacing A with $2A$, $3A$, $4A$ and $\frac{A}{2}$, explain clearly how each of the following can be derived.

(a) $\sin 4A = 2 \sin 2A \cos 2A$ **(b)** $\cos 6A = \cos^2 3A - \sin^2 3A$

(c) $\tan 8A = \dfrac{2 \tan 4A}{1 - \tan^2 4A}$ **(d)** $\sin A = 2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$

(e) $\cos A = 1 - 2 \sin^2 \dfrac{A}{2}$ **(f)** $\tan A = \dfrac{2 \tan \frac{A}{2}}{1 - \tan^2 \frac{A}{2}}$

Worked Example 10

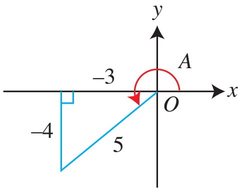
(Double Angle Formulae)

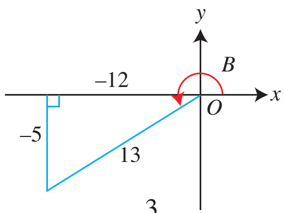
Given that $\sin A = -\frac{4}{5}$, $\cos B = -\frac{12}{13}$ and that A and B are in the same quadrant, find each of the following without the use of a calculator.

(i) $\sin 2A$ **(ii)** $\tan 2B$ **(iii)** $\cos \frac{B}{2}$

Solution

Since $\sin A < 0$ and $\cos B < 0$, both A and B lie in the third quadrant.





(i) From the diagram, we have $\cos A = -\frac{3}{5}$.

$$\sin 2A = 2 \sin A \cos A = 2\left(-\frac{4}{5}\right)\left(-\frac{3}{5}\right) = \frac{24}{25}$$

(ii) From the diagram, we have $\tan B = \frac{5}{12}$.

$$\tan 2B = \frac{2 \tan B}{1 - \tan^2 B} = \frac{2\left(\frac{5}{12}\right)}{1 - \left(\frac{5}{12}\right)^2} = \frac{120}{119}$$

(iii) Using $\cos B = 2 \cos^2 \frac{B}{2} - 1$,

$$-\frac{12}{13} = 2 \cos^2 \frac{B}{2} - 1$$

$$2 \cos^2 \frac{B}{2} = \frac{1}{13}$$

$$\cos^2 \frac{B}{2} = \frac{1}{26}$$

$$\cos \frac{B}{2} = \pm \frac{1}{\sqrt{26}}$$

Since $180° < B < 270°$, we have $90° < \frac{B}{2} < 135°$,

i.e. $\frac{B}{2}$ lies in the second quadrant.

$$\therefore \cos \frac{B}{2} = -\frac{1}{\sqrt{26}} = -\frac{\sqrt{26}}{26}$$

ATTENTION

$\sin 2A \neq 2 \sin A$

$\cos \frac{B}{2} \neq \frac{1}{2} \cos B$

$\tan 2B \neq 2 \tan B$

PROBLEM-SOLVING TIP

Consider the quadrant which B lies in before determining the quadrant in which $\frac{B}{2}$ lies.

Practise Now 10

Similar Questions:
Exercise 9D
Questions 1(a)-(e), 5, 6, 7

1. Given that $\sin A = \frac{3}{5}$, $\tan B = -\frac{5}{12}$ and that A and B are in the same quadrant, find the value of each of the following without using a calculator.

(i) $\cos 2A$ **(ii)** $\sin 2B$ **(iii)** $\tan \frac{B}{2}$

2. Given that $\tan A = \frac{8}{15}$, $\sin B = -\frac{4}{5}$ and that A and B each lie between 90° and 270°, find the value of each of the following without using a calculator.

(i) $\sin (A + B)$ **(ii)** $\cos \frac{B}{2}$

Worked Example 11

(Use of the Double Angle Formulae)

Given that $\cos 4x = \frac{7}{18}$ and that $270° \leqslant 4x \leqslant 360°$, find, without using a calculator, the value of

(i) $\cos 2x$, **(ii)** $\sin 2x$, **(iii)** $\cos x$,
(iv) $\sin x$, **(v)** $\sin 3x$.

Solution

(i) Since $270° \leqslant 4x \leqslant 360°$, we have

$$135° \leqslant 2x \leqslant 180°,$$

i.e. $2x$ lies in the second quadrant.

$$\cos 4x = 2\cos^2 2x - 1$$

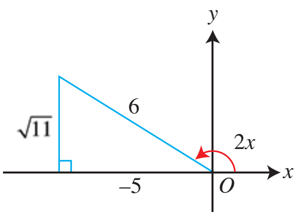
$$\frac{7}{18} = 2\cos^2 2x - 1$$

$$2\cos^2 2x = \frac{25}{18}$$

$$\cos^2 2x = \frac{25}{36}$$

$$\cos 2x = -\frac{5}{6}$$ (2x lies in the 2nd quadrant, i.e. $\cos 2x < 0$)

(ii)



$$\sin 2x = \frac{\sqrt{11}}{6}$$

PROBLEM-SOLVING TIP

Draw a right-angled triangle for angle $2x$ based on the ratio obtained in **(i)**.

(iii) Since $270° \leqslant 4x \leqslant 360°$, we have

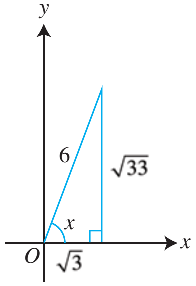
$67.5° \leqslant x \leqslant 90°$, i.e. x lies in the first quadrant.

$$\begin{aligned}
\cos 2x &= 2\cos^2 x - 1 \\
-\frac{5}{6} &= 2\cos^2 x - 1 \\
2\cos^2 x &= \frac{1}{6} \\
\cos^2 x &= \frac{1}{12} \\
\cos x &= \frac{1}{\sqrt{12}} \quad \text{(x lies in the 1st quadrant, i.e. } \cos x > 0) \\
&= \frac{\sqrt{12}}{12} \\
&= \frac{2\sqrt{3}}{12} \\
&= \frac{\sqrt{3}}{6}
\end{aligned}$$

PROBLEM-SOLVING TIP

Draw a right-angled triangle for angle x based on the ratio obtained in **(iii)**.

(iv)



$$\sin x = \frac{\sqrt{33}}{6}$$

INFORMATION

You can also write $\sin 3x$ as $\sin (x + 2x)$.

(v)
$$\begin{aligned}
\sin 3x &= \sin (2x + x) \\
&= \sin 2x \cos x + \cos 2x \sin x \\
&= \left(\frac{\sqrt{11}}{6}\right)\left(\frac{\sqrt{3}}{6}\right) + \left(-\frac{5}{6}\right)\left(\frac{\sqrt{33}}{6}\right) \\
&= -\frac{\sqrt{33}}{9}
\end{aligned}$$

Practise Now 11

Similar Questions:
Exercise 9D
Questions 2-4, 8, 9

1. Given that $\sin 4x = -\frac{24}{25}$ and that $270° \leqslant 4x \leqslant 360°$, find, without using a calculator, the value of

(i) $\cos 2x$, **(ii)** $\cos x$, **(iii)** $\cos 3x$.

2. Given that $\tan 2A = \frac{5}{12}$ and that A is acute, find the value of each of the following without using a calculator.

(i) $\tan A$ **(ii)** $\sin 4A$

Worked Example 12

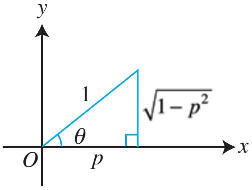
(Algebraic Fractions in Double Angle Formulae)

If $\cos \theta = p$, where θ is an acute angle, express each of the following in terms of p.

(i) $\sin 4\theta$ **(ii)** $\sin \frac{1}{2}\theta$

Solution

(i)
$$\begin{aligned}\sin 4\theta &= 2 \sin 2\theta \cos 2\theta \\ &= 2(2 \sin \theta \cos \theta)(2 \cos^2 \theta - 1) \\ &= 4\left(\sqrt{1-p^2}\right)(p)(2p^2 - 1) \\ &= 4p\sqrt{1-p^2}(2p^2 - 1)\end{aligned}$$



(ii)
$$\begin{aligned}\cos \theta &= 1 - 2\sin^2 \tfrac{1}{2}\theta \\ p &= 1 - 2\sin^2 \tfrac{1}{2}\theta \\ 2\sin^2 \tfrac{1}{2}\theta &= 1 - p \\ \sin^2 \tfrac{1}{2}\theta &= \frac{1-p}{2} \\ \therefore \sin \tfrac{1}{2}\theta &= \sqrt{\frac{1-p}{2}}\end{aligned}$$

($\sin \frac{1}{2}\theta > 0$ because θ is acute)

Practise Now 12

Similar Questions:
Exercise 9D
Questions 12-14

If $\sin \theta = a$, where θ is an acute angle, express each of the following in terms of a.

(i) $\sin 2\theta$ **(ii)** $\cos 4\theta$ **(iii)** $\cos^2 \frac{1}{2}\theta$

Worked Example 13

(Equations involving the Double Angle Formulae)

Solve each of the following equations for the given range of x.

(a) $3 \sin 2x + 5 \sin x = 0$, $0° \leqslant x \leqslant 360°$

(b) $2 \cos 2x - 3 \cos x - 1 = 0$, $0 \leqslant x \leqslant 2\pi$

Solution

(a)
$$\begin{aligned}3 \sin 2x + 5 \sin x &= 0 \\ 3(2 \sin x \cos x) + 5 \sin x &= 0 \\ \sin x (6 \cos x + 5) &= 0\end{aligned}$$

$\sin x = 0$ or $\cos x = -\frac{5}{6}$

$x = 0°, 180°, 360°$

$\alpha = 33.56°$ (to 2 d.p.)

$x = 180° - 33.56°, 180 + 33.56°$

$= 146.4°, 213.6°$ (to 1 d.p.)

$\therefore x = 0°, 146.4°, 180°, 213.6°, 360°$

(b)
$$\begin{aligned} 2\cos 2x - 3\cos x - 1 &= 0 \\ 2(2\cos^2 x - 1) - 3\cos x - 1 &= 0 \\ 4\cos^2 x - 3\cos x - 3 &= 0 \\ \cos x &= \frac{-(-3) \pm \sqrt{(-3)^2 - 4(4)(-3)}}{2(4)} \\ &= 1.319 \text{ or } -0.5687 \quad \text{(to 4 s.f.)} \\ &\text{(no solution)} \end{aligned}$$

When $\cos x = -0.5687$,

$$\begin{aligned} \alpha &= 0.9659 \quad \text{(to 4 s.f.)} \\ x &= \pi - 0.9659,\ \pi + 0.9659 \\ &= 2.18,\ 4.11 \quad \text{(to 3 s.f.)} \end{aligned}$$

PROBLEM-SOLVING TIP

We replace $\cos 2x$ with $2\cos^2 x - 1$ so that we are able to obtain a quadratic equation in $\cos x$.

Practise Now 13

Similar Questions:
Exercise 9D
Questions 10(a)-(d), 11(a)-(d)

1. Solve each of the following equations for $0° \leqslant x \leqslant 360°$.
(a) $2\sin 2x - 3\cos x = 0$ **(b)** $2\cos 2x + 3\sin x - 2 = 0$

2. Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.
(a) $3\cos 2x - 5\cos x = 0$ **(b)** $\tan 2x = 3\tan x$

Basic Level | Intermediate Level | Advanced Level

Exercise 9D

Do not use a calculator for questions 1 to 9 and 14.

1 Express each of the following as a single trigonometric ratio.
(a) $2\sin 25° \cos 25°$ **(b)** $1 - 2\sin^2 27°$
(c) $\dfrac{2\tan 35°}{1 - \tan^2 35°}$ **(d)** $\cos^2 25° - \sin^2 25°$
(e) $2\cos^2 13° - 1$

2 Given that $\cos 2A = \frac{119}{169}$ and that A is acute, find the value of
(i) $\tan 2A$, **(ii)** $\tan A$,
(iii) $\cos A$, **(iv)** $\sin A$.

3 Given that $\tan A = \frac{1}{2}$, find the value of $\tan 2A$ and of $\tan 3A$.

4 Given that $\cos 2A = \frac{2}{3}$ and that A is acute, find the value of
(i) $\sin 2A$, **(ii)** $\cos A$,
(iii) $\tan A$, **(iv)** $\tan \frac{1}{2}A$.
Leave your answers in surd form where necessary.

5 Given that $\sin A = \frac{3}{5}$ and that A is obtuse, find the value of
(i) $\cos 2A$, **(ii)** $\tan 2A$,
(iii) $\sin \frac{1}{2}A$, **(iv)** $\cos 4A$.

6 Given that $\sin A = \frac{15}{17}$, $\cos B = -\frac{3}{5}$ and that A and B are in the same quadrant, find the value of
(i) $\sin 2A$, **(ii)** $\cos \frac{1}{2}A$,
(iii) $\cos 2B$.

Exercise 9D

7 Given that $\sin X = -\frac{5}{13}$, $\cos Y = -\frac{4}{5}$ and that X and Y are in the same quadrant, find the value of

(i) $\sin\frac{X}{2}$, **(ii)** $\cos 2Y$,

(iii) $\cos 2X$.

8 Given that $\cos A = -\frac{4}{5}$, where A is obtuse, find the value of

(i) $\sin 2A$, **(ii)** $\sin 4A$,

(iii) $\tan\frac{1}{2}A$.

9 Given that $\tan x = -2$ and that $\frac{\pi}{2} < x < \frac{3\pi}{2}$, find the value of

(i) $\cos 2x$, **(ii)** $\tan 2x$,

(iii) $\sin 4x$.

10 Solve each of the following equations for $0° \leqslant x \leqslant 360°$.

(a) $4 \sin 2x - \sin x = 0$

(b) $5 \sin 2x + 3 \cos x = 0$

(c) $4 \cos 2x - 3 \sin x + 1 = 0$

(d) $5 \tan 2x + 7 \tan x = 0$

11 Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.

(a) $3 \sin 2x - 2 \cos x = 0$

(b) $7 \sin 2x + 3 \sin x = 0$

(c) $5 \cos 2x + 11 \sin x - 8 = 0$

(d) $3 \tan 2x - 8 \tan x = 0$

12 Given that A is acute and that $\cos A = x$, find expressions for each of the following.

(i) $\tan^2 A$ **(ii)** $\sin 2A$

(iii) $\cos 4A$ **(iv)** $\sin\frac{1}{2}A$

13 State the limits between which A must lie, given that A is acute and

(a) $\tan 2A$ is negative,

(b) $\cos 3A$ is negative,

(c) both $\tan 2A$ and $\cos 3A$ are negative.

14 Given that $\tan\theta = \frac{3}{4}$, where $180° < \theta < 270°$, without using a calculator, find the value of

(i) $\cos 2\theta$, **(ii)** $\sin\frac{\theta}{2}$,

(iii) $\tan(270° + 2\theta)$.

9.5 FURTHER PROVING OF IDENTITIES

In the table below, the various identities learnt in this chapter have been grouped together to help you remember them. The general guidelines used for proving identities in the earlier part of this chapter are also applicable here.

Addition Formulae	Double Angle Formulae
$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$ $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$ $\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B}$	$\sin 2A = 2 \sin A \cos A$ $\cos 2A = \cos^2 A - \sin^2 A$ $= 1 - 2\sin^2 A$ $= 2\cos^2 A - 1$ $\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$

Worked Example 14

(Use of $\tan A = \frac{\sin A}{\cos A}$ and $\cot A = \frac{\cos A}{\sin A}$)

Prove the identity $\tan A + \cot A = \frac{2}{\sin 2A}$.

Solution

$$\begin{aligned}\text{LHS} &= \tan A + \cot A\\ &= \frac{\sin A}{\cos A} + \frac{\cos A}{\sin A}\\ &= \frac{\sin^2 A + \cos^2 A}{\sin A \cos A}\\ &= \frac{1}{\frac{1}{2}\sin 2A} \qquad (\sin^2 A + \cos^2 A = 1,\ \sin 2A = 2\sin A\cos A)\\ &= \frac{2}{\sin 2A} = \text{RHS (proven)}\end{aligned}$$

Practise Now 14

Similar Questions:
Exercise 9E
Questions 1(a)-(f)

1. Prove the identity $1 - \tan^2 A = \cos 2A \sec^2 A$.

2. Prove the identity $\dfrac{\sin\left(A+\frac{\pi}{4}\right)}{\cos\left(A+\frac{\pi}{4}\right)} + \dfrac{\cos\left(A+\frac{\pi}{4}\right)}{\sin\left(A+\frac{\pi}{4}\right)} = 2\sec 2A$

Worked Example 15

(Use of the Double Angle Formulae for sin 2*A* and cos 2*A*)

Prove that $\operatorname{cosec} A = \frac{\cos 2A}{\sin A} + \frac{\sin 2A}{\cos A}$.

Solution

Method 1:
$$\begin{aligned}\text{RHS} &= \frac{\cos 2A}{\sin A} + \frac{\sin 2A}{\cos A}\\ &= \frac{1 - 2\sin^2 A}{\sin A} + \frac{2\sin A\cos A}{\cos A}\\ &= \frac{1}{\sin A} - 2\sin A + 2\sin A\\ &= \operatorname{cosec} A = \text{LHS (proven)}\end{aligned}$$

Method 2:
$$\begin{aligned}\text{RHS} &= \frac{\cos 2A}{\sin A} + \frac{\sin 2A}{\cos A}\\ &= \frac{\cos 2A\cos A + \sin 2A\sin A}{\sin A\cos A}\\ &= \frac{\cos(2A - A)}{\sin A\cos A} \qquad (\cos(2A - A) = \cos 2A\cos A + \sin 2A\sin A)\\ &= \frac{\cos A}{\sin A\cos A}\\ &= \frac{1}{\sin A}\\ &= \operatorname{cosec} A = \text{LHS (proven)}\end{aligned}$$

Which method do you prefer?

Practise Now 15

Similar Questions:
Exercise 9E
Questions 2(a)-(h)

1. Prove the identity cosec $2A$ + cot $2A$ = cot A.
2. Prove the identity $\dfrac{2\sin(A-B)}{\cos(A-B)-\cos(A+B)}=\cot B-\cot A$

Worked Example 16

(Proving of Identities)
Prove that cot A + tan $2A$ = cot A sec $2A$.

Solution

$$\begin{aligned}\text{LHS} &= \cot A+\tan 2A\\ &= \frac{\cos A}{\sin A}+\frac{\sin 2A}{\cos 2A}\\ &= \frac{\cos 2A\cos A+\sin 2A\sin A}{\sin A\cos 2A}\\ &= \frac{\cos(2A-A)}{\sin A\cos 2A}\\ &= \left(\frac{\cos A}{\sin A}\right)\left(\frac{1}{\cos 2A}\right)\end{aligned}$$

= cot A sec $2A$ = RHS (proven)

Practise Now 16

Similar Questions:
Exercise 9E
Questions 2(i)-(m)

1. Prove the identity $\dfrac{\sin 4A}{1+\cos 4A}=\tan 2A$.
2. Prove the identity $\dfrac{1-\cos 2A+\sin A}{\sin 2A+\cos A}=\tan A$

Basic Level | Intermediate Level | Advanced Level

Exercise 9E

1 Prove each of the following identities.

(a) $\dfrac{\sin 2x}{1-\cos 2x}=\cot x$

(b) $\cos^4 x-\sin^4 x=\cos 2x$

(c) $\tan x+\cot x=2\operatorname{cosec} 2x$

(d) $\left(\sin\frac{1}{2}x+\cos\frac{1}{2}x\right)^2=1+\sin x$

(e) $\dfrac{1}{\cos 2x}=\dfrac{\sec^2 x}{2-\sec^2 x}$

(f) $\operatorname{cosec} 2x-\cot 2x=\tan x$

(g) $(\tan x-\operatorname{cosec} x)^2-(\cot x-\sec x)^2$
$=2(\operatorname{cosec} x-\sec x)$

(h) $\cos(60°+A)+\sin(30°+A)=\cos A$

(i) $\sin 3A=3\sin A-4\sin^3 A$

(j) $\operatorname{cosec} 2A-\tan A=\cot 2A$

(k) $(\tan A+\cot A)\sin 2A=2$

2 Prove each of the following identities.

(a) $\dfrac{\cos x}{1+\cos 2x}+\dfrac{\sin x}{1-\cos 2x}=\dfrac{1}{2}(\sec x+\operatorname{cosec} x)$

(b) $2\cos^2\left(\dfrac{\pi}{4}-A\right)=1+\sin 2A$

(c) $\dfrac{1-\tan^2 x}{1+\tan^2 x}=\cos 2x$

(d) $\dfrac{1-\sin x}{\cos x}=\dfrac{2}{\cos x}-\dfrac{\cos x}{1-\sin x}$

(e) $\dfrac{\cos A-\sin A}{\cos A+\sin A}=\sec 2A-\tan 2A$

(f) $\dfrac{\sin 2A}{1+\cos 2A}=\tan A$

(g) $\dfrac{\cos A-\sin A}{\cos A+\sin A}+\dfrac{\cos A+\sin A}{\cos A-\sin A}=2\sec 2A$

(h) $\dfrac{1+\tan A}{1-\tan A}+\dfrac{1-\tan A}{1+\tan A}=2\sec 2A$

(i) $\dfrac{2}{\cot A\tan 2A}=1-\tan^2 A$

(j) $\tan\left(\dfrac{\pi}{4}+\dfrac{A}{2}\right)=\tan A+\sec A$

(k) $\dfrac{\tan\left(\dfrac{\pi}{4}+A\right)+\tan\left(\dfrac{\pi}{4}-A\right)}{\tan\left(\dfrac{\pi}{4}+A\right)-\tan\left(\dfrac{\pi}{4}-A\right)}=\operatorname{cosec} 2A$

(l) $\dfrac{2\sin x-\sin 2x}{2\sin x+\sin 2x}=\tan^2\dfrac{x}{2}$

(m) $\dfrac{\sin A+\cos A}{\sin A-\cos A}=\dfrac{1+\sin 2A}{1-2\cos^2 A}$

3 Prove the identity $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}$

4 Prove the identity $\dfrac{\cos^3 A-\sin^3 A}{\cos A-\sin A}=\dfrac{2+\sin 2A}{2}$.

9.6 R-FORMULAE

Earlier in this chapter, we have learnt the Addition Formulae and the Double Angle Formulae. In this section, we will learn how to solve equations of the form $a\sin\theta+b\cos\theta=c$ by using the R-formulae which is derived below.

Let $a\sin\theta+b\cos\theta=R\sin(\theta+\alpha)$, where R is a positive constant and α is an acute angle.

$$\begin{aligned}R\sin(\theta+\alpha)&=R(\sin\theta\cos\alpha+\cos\theta\sin\alpha)\\&=R\sin\theta\cos\alpha+R\cos\theta\sin\alpha\end{aligned}$$

Equating the coefficients of $\sin\theta$ and $\cos\theta$ in the identity, we have

$$a=R\cos\alpha \text{ -------------- (1)}$$
$$b=R\sin\alpha. \text{ -------------- (2)}$$

(2) ÷ (1): $\dfrac{b}{a}=\dfrac{\sin\alpha}{\cos\alpha}$, i.e. $\tan\alpha=\dfrac{b}{a}$

Squaring (1) and (2) and adding them, we have

$$\begin{aligned}a^2+b^2&=R^2\cos^2\alpha+R^2\sin^2\alpha\\&=R^2(\cos^2\alpha+\sin^2\alpha)\\&=R^2.\end{aligned}$$

Taking the positive square root, we have

$$R=\sqrt{a^2+b^2}\,.$$

$\therefore$ $a\sin\theta+b\cos\theta$ can be written as $R\sin(\theta+\alpha)$, where $R=\sqrt{a^2+b^2}$ and $\tan\alpha=\dfrac{b}{a}$.

Now, we have learnt how to express $a\sin\theta+b\cos\theta$ in the form $R\sin(\theta+\alpha)$. Let us now explore how to express $a\sin\theta-b\cos\theta$, $a\cos\theta+b\sin\theta$ and $a\cos\theta-b\sin\theta$ in similar forms.

Investigation

$a \cos \theta \pm b \sin \theta$

Using a similar approach, show that

$a \sin \theta - b \cos \theta = R \sin (\theta - \alpha)$, where $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \frac{b}{a}$.

Find similar expressions for $a \cos \theta + b \sin \theta$ and $a \cos \theta - b \sin \theta$.

Summarising, we have

$$a \sin \theta \pm b \cos \theta = R \sin (\theta \pm \alpha)$$
$$a \cos \theta \pm b \sin \theta = R \cos (\theta \mp \alpha),$$

where $R = \sqrt{a^2 + b^2}$, $\tan \alpha = \frac{b}{a}$ and α is acute.

Worked Example 17

(Expressing $a \sin \theta + b \cos \theta$ in the form $R \sin (\theta + \alpha)$)

Express $4 \sin \theta + 3 \cos \theta$ in the form $R \sin (\theta + \alpha)$, where R and α are constants such that α is acute. Hence, solve the equation $4 \sin \theta + 3 \cos \theta = 4.8$ for values of θ between 0° and 360°.

Solution

Let $4 \sin \theta + 3 \cos \theta = R \sin (\theta + \alpha) = R \sin \theta \cos \alpha + R \cos \theta \sin \alpha$.

Equating coefficients of $\sin \theta$ and $\cos \theta$,

$$4 = R \cos \alpha \text{ ----- (1)}$$
$$3 = R \sin \alpha \text{ ----- (2)}$$

(2) ÷ (1): $\tan \alpha = \frac{3}{4}$

$$\alpha = 36.87° \text{ (to 2 d.p.)}$$

Squaring and adding (1) and (2),

$$4^2 + 3^2 = R^2$$
$$R = 5$$

$$\therefore 4 \sin \theta + 3 \cos \theta = 5 \sin (\theta + 36.87°)$$
$$4 \sin \theta + 3 \cos \theta = 4.8$$
$$5 \sin (\theta + 36.87°) = 4.8$$
$$\sin (\theta + 36.87°) = 0.96$$
$$\theta + 36.87° = 73.74°, 180° - 73.74°$$
$$= 73.74°, 106.26°$$
$$\theta = 36.9°, 69.4° \text{ (to 1 d.p.)}$$

ATTENTION

We usually work with 2 decimal places for angles in degrees in the intermediate steps for greater accuracy in the answer.

Practise Now 17

Similar Questions:
Exercise 9F
Questions 5(a)-(d), 6(a)-(d), 7(a)-(d)

1. Express $4 \sin \theta + 9 \cos \theta$ in the form $R \sin (\theta + \alpha)$, where R and α are constants such that α is acute. Hence, solve the equation $4 \sin \theta + 9 \cos \theta = 1.6$, where $0° \leqslant \theta \leqslant 360°$.
2. Express $5 \cos \theta - 7 \sin \theta$ in the form $R \cos (\theta + \alpha)$, where R and α are constants. Hence, solve the equation $5 \cos \theta - 7 \sin \theta = 1.8$ in the range $0 \leqslant \theta \leqslant 2\pi$.

Maximum And Minimum Values using the R-formulae

Given the expression $a \sin \theta + b \cos \theta$, how do we determine its maximum value and the corresponding value of θ?

$$a \sin \theta + b \cos \theta = \sqrt{a^2 + b^2} \sin (\theta + \alpha)$$

Recalling from Chapter 8, $-1 \leqslant \sin (\theta + \alpha) \leqslant 1$.

Since $\sqrt{a^2 + b^2} > 0$, we have

$$-\sqrt{a^2 + b^2} \leqslant \sqrt{a^2 + b^2} \sin (\theta + \alpha) \leqslant \sqrt{a^2 + b^2}.$$

$\therefore$ The maximum value of $a \sin \theta + b \cos \theta$ is $\sqrt{a^2 + b^2}$ and it occurs when $\sin (\theta + \alpha) = 1$.

Similarly, the minimum value of $a \sin \theta + b \cos \theta$ is $-\sqrt{a^2 + b^2}$ and it occurs when $\sin (\theta + \alpha) = -1$.

Worked Example 18

(Maximum and Minimum Values)

Express $8 \sin \theta - 5 \cos \theta$ in the form $R \sin (\theta - \alpha)$, where R and α are constants such that α is acute. Hence, find the maximum and minimum values of the expression $8 \sin \theta - 5 \cos \theta$ such that $0° < \theta < 360°$ and the values of θ for which these values occur.

Solution

Let $8 \sin \theta - 5 \cos \theta = R \sin (\theta - \alpha) = R \sin \theta \cos \alpha - R \cos \theta \sin \alpha$.

Equating coefficients of $\sin \theta$ and $\cos \theta$,

$$8 = R \cos \alpha \text{ ----- (1)}$$

$$5 = R \sin \alpha \text{ ----- (2)}$$

(2) ÷ (1): $\tan \alpha = \dfrac{5}{8}$

$\alpha = 32.01°$ (to 2 d.p.)

Squaring and adding (1) and (2),

$$8^2 + 5^2 = R^2$$

$$R = \sqrt{89}$$

$\therefore 8 \sin \theta - 5 \cos \theta = \sqrt{89} \sin (\theta - 32.01°)$

Maximum value of $8 \sin \theta - 5 \cos \theta = \sqrt{89}$

when $\sin (\theta - 32.01°) = 1$

$\theta - 32.01° = 90°$

$\theta = 122.0°$ (to 1 d.p.)

Minimum value of $8 \sin \theta - 5 \cos \theta = -\sqrt{89}$

when $\sin (\theta - 32.01°) = -1$

$\theta - 32.01° = 270°$

$\theta = 302.0°$ (to 1 d.p.)

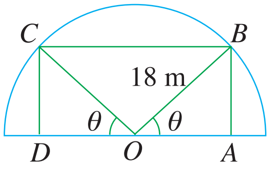
Practise Now 18

Similar Questions:
Exercise 9F
Questions 1-4, 8-10

Express $8 \sin \theta - 7 \cos \theta$ in the form $R \sin (\theta - \alpha)$, where R and α are constants such that α is acute. Hence, find the maximum and minimum values of the expression $8 \sin \theta - 7 \cos \theta$ such that $0° < \theta < 360°$ and the values of θ for which these values occur.

Worked Example 19

(Cross Section of a Tunnel)



The figure shows the sketch of the cross-section of a tunnel in the shape of a semicircle, centre O, and $ABCD$ is a rectangle. The radius of the tunnel is 18 m, $\angle AOB = \angle COD = \theta$ and the perimeter of the rectangle $ABCD$ is P m.

(i) Show that $P = 36 \sin \theta + 72 \cos \theta$.
(ii) Express P in the form $R \sin (\theta + \alpha)$, where R is positive and α is acute.
(iii) State the maximum value of P and the corresponding value of θ.

Solution

(i) In ΔAOB, $\sin \theta = \frac{AB}{18}$, i.e. $AB = 18 \sin \theta$,

$\cos \theta = \frac{OA}{18}$, i.e. $OA = 18 \cos \theta$.

Since $AB = CD$ and $AD = BC = 2OA$,

$$P = 2(18 \sin \theta) + 2(2 \times 18 \cos \theta)$$
$$= 36 \sin \theta + 72 \cos \theta$$

(ii) $R = \sqrt{36^2 + 72^2} = 36\sqrt{5}$ and $\tan \alpha = \frac{72}{36} = 2$, i.e. $\alpha = 63.43°$ (to 2 d.p.)

$\therefore$ $36 \sin \theta + 72 \cos \theta = 36\sqrt{5} \sin (\theta + 63.43°)$

(iii) Maximum value of $36 \sin \theta + 72 \cos \theta = 36\sqrt{5}$

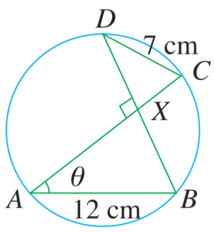
when $\sin (\theta + 63.43°) = 1$

$$\theta + 63.43° = 90°$$
$$\theta = 26.6° \text{ (to 1 d.p.)}$$

Practise Now 19

Similar Questions:
Exercise 9F
Questions 11-15

The figure shows the logo of a brand that a designer is working on. The chords AC and BD of the circle intersect at right angles at the point X. It is given that $\angle BAC = \theta$, where θ varies, $AB = 12$ cm and $CD = 7$ cm.



(i) Show that $AC = 12 \cos \theta + 7 \sin \theta$.
(ii) Express AC in the form $R \cos (\theta - \alpha)$, where R is positive and α is acute.
(iii) State the maximum length of AC and the corresponding value of θ.

Exercise 9F

1 Express each of the following in the form $R\sin(\theta + \alpha)$. State the maximum and minimum values and the corresponding value of θ in each case.
(a) $2\sin\theta + 5\cos\theta$ **(b)** $\sqrt{3}\sin\theta + \cos\theta$

2 Express each of the following in the form $R\sin(\theta - \alpha)$. State the maximum and minimum values and the corresponding value of θ in each case.
(a) $\sqrt{7}\sin\theta - \cos\theta$ **(b)** $\sqrt{5}\sin\theta - \sqrt{3}\cos\theta$

3 Express the following in the form $R\cos(\theta + \alpha)$. State the maximum and minimum values and the corresponding value of θ in each case.
(a) $3\cos\theta - 2\sin\theta$ **(b)** $2\cos\theta - \sqrt{3}\sin\theta$

4 The displacement, x cm, of a particle t seconds after passing a fixed point O, is given by $x = 2\cos t + 5\sin t$. By expressing $2\cos t + 5\sin t$ in the form $R\cos(t - \alpha)$, where R is a positive constant and α is an acute angle, find the maximum and minimum displacements of the particle and the corresponding value of t in each case.

5 Find all the angles between 0° and 360° which satisfy each of the following equations.
(a) $2\cos\theta - 7\sin\theta = 1$
(b) $\sqrt{2}\cos\theta - \sqrt{3}\sin\theta = \sqrt{5}$
(c) $3\cos\theta + 4\sin\theta = 2$
(d) $\sin\theta - \sqrt{2}\cos\theta = 0.8$

6 Find all the angles between 0 and 2π which satisfy each of the following equations.
(a) $3\cos x + 2\sin x = 1.2$
(b) $2\sin x - \cos x = 0.8$
(c) $4\cos x - 3\sin x = 1.9$
(d) $5\sin x + 2\cos x = 1.4$

7 Solve each of the following equations for $0^\circ \leqslant x \leqslant 360^\circ$.
(a) $5\cos 2x + 4\sin 2x = 2.3$
(b) $3\cos 2x - 2\sin 2x = \sqrt{2}$
(c) $4\sin 3x + 3\cos 3x = \sqrt{3}$
(d) $7\sin 3x - 6\cos 3x = 3.8$

8 Wave motion may be modelled by a sine or cosine curve. The equations of two waves can be modelled by $y = 12\cos x$ and $y = -5\sin x$, where x and y are the horizontal and vertical displacements respectively. Given that these two waves are superimposed such that the resultant vertical displacement is given by $y = 12\cos x - 5\sin x$,
(i) express $12\cos x - 5\sin x$ in the form $R\cos(x + \alpha)$, where R is a positive constant and α is an acute angle,
(ii) find the maximum and minimum vertical displacements of the wave and the corresponding values of x between 0° and 360°.

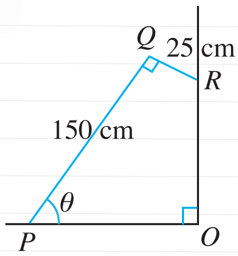
9 Express $5\sin x - 12\cos x$ in the form $R\sin(x - \alpha)$, where R is a positive constant and α is an acute angle. Hence, find
(i) all the solutions, in the range $0^\circ \leqslant x \leqslant 360^\circ$, of the equation $5\sin x - 12\cos x = 7$,
(ii) the maximum and minimum values of $(5\sin x - 12\cos x)^2$.

10 Express $3\sin x + 4\cos x$ in the form $R\sin(x + \alpha)$, where R is a positive constant and α is an acute angle. Hence, find the maximum and minimum values of each of the following.
(a) $3\sin x + 4\cos x + 1$
(b) $(3\sin x + 4\cos x)^2 + 1$
(c) $(3\sin x + 4\cos x)^3 + 1$
(d) $(3\sin x + 4\cos x)^4 + 1$
Explain why it is not possible for $(3\sin x + 4\cos x)^{2n} + 1$, where n is a positive integer, to have a negative value.

Basic Level | Intermediate Level | Advanced Level

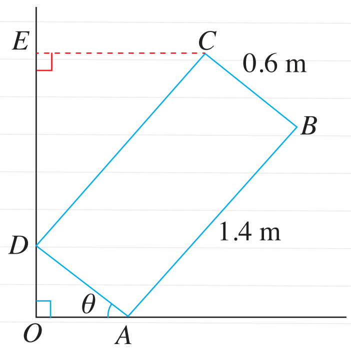
Exercise 9F

11 The figure shows a gardening tool PQR leaning against a vertical wall OR. It is given that $QR = 25$ cm, $PQ = 150$ cm, $\angle PQR = \angle POR = 90°$ and $\angle OPQ = \theta$.



(i) Given that OP is the distance between the foot of the wall and the tip of the handle of the gardening tool, show that $OP = (25 \sin \theta + 150 \cos \theta)$ cm.
(ii) Express OP in the form $R \sin (\theta + \alpha)$, where R is a positive constant and α is an acute angle.
(iii) What is the greatest distance between the foot of the wall and the tip of the handle of the tool? Write down the value of θ when it occurs.
(iv) Find the value of θ for which $OP = 90$ cm.

12 The figure shows the plan view of a rectangular table $ABCD$ which is positioned at a corner of a room. Given that $AB = 1.4$ m, $BC = 0.6$ m, $\angle OAD = \theta$ and $\angle AOD = \angle CED = 90°$, where θ varies.

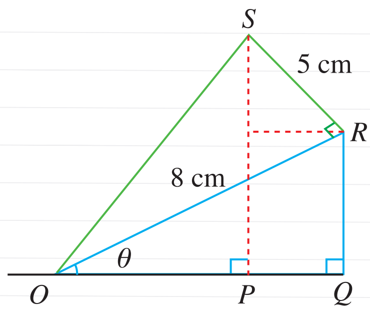


(i) Show that $OE = 1.4 \cos \theta + 0.6 \sin \theta$.
(ii) Express OE in the form $R \cos (\theta - \alpha)$, where R is positive and α is acute.
(iii) Find the value of θ when OE is a maximum.
(iv) Find the value of θ when $OE = 1.1$ m.

13 The current, I amperes, of a circuit, at time t seconds, is given by $I = 3 \cos t + 2 \sin t$. By expressing I in the form $R \cos (t - \alpha)$, where R is positive and $0 < \alpha < \frac{\pi}{2}$, find the maximum current flowing through the circuit and the time at which it first occurs.

14 Using the identity $\cos 2\theta = 2 \cos^2\theta - 1$ and $a \cos \theta + b \sin \theta = \sqrt{a^2 + b^2} \cos (\theta - \alpha)$, where α is a constant, find the maximum and minimum values of $11 \cos^2 \theta + 3 \sin \theta + 6 \sin \theta \cos \theta + 5$.
Hence, or otherwise, solve the equation $11 \cos^2 \theta + 3 \sin^2 \theta + 6 \sin \theta \cos \theta + 5 = 15$ for $0° \leqslant x \leqslant 360°$.

15 The figure shows two triangles, ΔORS and ΔOQR. It is given that $OR = 8$ cm, $RS = 5$ cm, $\angle OPS = \angle OQR = \angle ORS = 90°$ and $\angle QOR = \theta$, where θ varies.



(i) Show that $OP = 8 \cos \theta - 5 \sin \theta$.
(ii) Given that $OP = R \cos (\theta + \alpha)$, find the positive value of R and the acute angle α.
(iii) Find the value of θ when $OP = 7$ cm.
(iv) Express SP in the form $R \sin (\theta + \alpha)$.
(v) Show that the area of triangle OPS, A cm^2, is given by $A = \frac{89}{4} \sin (2\theta + 2\alpha)$.
(vi) Find the value of θ for which A is a maximum.

SUMMARY

1. **Trigonometric Identities**
   - $\sin^2 A + \cos^2 A = 1$
   - $\tan^2 A + 1 = \sec^2 A$
   - $1 + \cot^2 A = \text{cosec}^2 A$

2. **Addition Formulae**
   - $\sin (A + B) = \sin A \cos B + \cos A \sin B$
   - $\sin (A - B) = \sin A \cos B - \cos A \sin B$
   - $\cos (A + B) = \cos A \cos B - \sin A \sin B$
   - $\cos (A - B) = \cos A \cos B + \sin A \sin B$
   - $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$
   - $\tan (A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

3. **Double Angle Formulae**
   - $\sin 2A = 2 \sin A \cos A$
   - $\cos 2A = \cos^2 A - \sin^2 A$
     $= 1 - 2 \sin^2 A$
     $= 2 \cos^2 A - 1$
   - $\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

4. ***R*-Formula**
   - $a \sin \theta \pm b \cos \theta = R \sin (\theta \pm \alpha)$
   - $a \cos \theta \pm b \sin \theta = R \cos (\theta \mp \alpha)$

   where $R = \sqrt{a^2 + b^2}$ and $\tan \alpha = \dfrac{b}{a}$

Review Exercise 9

1. Given that $90° < A < 180°$ and that $\cos^2 A = \dfrac{8}{9}$, find, without using a calculator, the value of
   (i) $\sin A$, **(ii)** $\tan A$, **(iii)** $\cot 2A$, **(iv)** $\sin 4A$.

2. Given that $\cos A = \dfrac{4}{5}$ and $\cos B = -\dfrac{12}{13}$ and that A and B are between $0°$ and $180°$, find, without using a calculator, the value of
   (i) $\sin (A + B)$, **(ii)** $\sin \dfrac{1}{2} A$, **(iii)** $\text{cosec } 2A$, **(iv)** $\sec 4A$.

3. Given that A is acute and that $\cos A = \frac{p^2 - 1}{p^2 + 1}$, where $p > 1$, express each of the following in terms of p.

(i) $\sin A$ **(ii)** $\cos \frac{1}{2} A$ **(iii)** $\tan \frac{1}{2} A$ **(iv)** $\sin 2A$

4. Solve each of the following equations for $0° \leqslant x \leqslant 360°$.

(a) $\operatorname{cosec} 2x = 2$

(b) $\cot (2x - 15°) = 1$

(c) $2 \cos (2x - 30°) = -\sqrt{3}$

(d) $\operatorname{cosec} 2x = \sec 50°$

(e) $3 \sin x \cos x = 1$

(f) $4 \cot x - 4 \operatorname{cosec} x + 5 \sin x = 0$

(g) $3 \cos x \cot x = 5$

(h) $\cot x = 6 - 5 \tan x$

(i) $\tan \frac{1}{2} x = 1.2$

(j) $3 \cos 2x + 8 \sin x + 5 = 0$

(k) $4 \cos x - 5 \sin x = 1.3$

(l) $9 \sin x - 5 \cos x = 2.4$

(m) $12 \sin 2x + 5 \cos 2x = 2$

(n) $7 \cos 3x - 6 \sin 3x = 4$

5. Prove each of the following identities.

(a) $\frac{\sin (A + B)}{\sin (A - B)} = \frac{\tan A + \tan B}{\tan A - \tan B}$

(b) $\cot (A + B) = \frac{\cot A \cot B - 1}{\cot A + \cot B}$

(c) $\frac{(1 + \tan A)^2 - 2 \tan^2 A}{1 + \tan^2 A} = \sin 2A + \cos 2A$

(d) $\frac{1 + \cos A}{\sin A} + \frac{\sin A}{1 + \cos A} = \frac{2}{\sin A}$

(e) $\frac{\cot A \cos A}{\cot A + \cos A} = \frac{1 - \sin A}{\cos A}$

(f) $\frac{1 + \sin A}{1 - \sin A} = (\tan A + \sec A)^2$

(g) $\operatorname{cosec} x - \sin x = \cos x \cot x$

(h) $2 \operatorname{cosec}^2 2x + 2 \cot 2x \operatorname{cosec} 2x = \operatorname{cosec}^2 x$

(i) $1 + (1 - \cos^2 x)(1 - \cot^2 x) = 2 \sin^2 x$

(j) $\frac{1}{1 + \cos x} + \frac{1}{1 - \cos x} = 2 \operatorname{cosec}^2 x$

(k) $\frac{\cos x}{1 - \tan x} + \frac{\sin x}{1 - \cot x} = \cos x + \sin x$

(l) $\tan x + \cot x = \sec x \operatorname{cosec} x$

(m) $\frac{1 + \cos 2A + \sin 2A}{1 - \cos 2A + \sin 2A} = \cot A$

6. Given that $\cos A = -\frac{3}{5}$, where $0° < A < 180°$ and $\tan B = \frac{5}{12}$, where $180° < B < 360°$, find the value of each of the following without using a calculator.

(i) $\tan (A + B)$ **(ii)** $\sec (-2A)$ **(iii)** $\sin 2A$

7. Given that $\tan 2A = \frac{5}{12}$ and that A is acute, find the value of each of the following without using a calculator.

(i) $\tan A$ **(ii)** $\sin 4A$ **(iii)** $\cos\left(\frac{\pi}{4} - 2A\right)$

(iv) $\sec\left(2A + \frac{\pi}{3}\right)$ **(v)** $\tan 3A$

8. Given that $\sin A = a$ and A is acute, express each of the following in terms of a.

(i) $\sin 2A$ **(ii)** $\cot (A + 45°)$ **(iii)** $\sec (30° + A)$

9. Given that $\tan A = \frac{3}{4}$ and $\tan (A + B) = 2$, where both A and B are acute angles, find the value of each of the following without using a calculator.

(i) $\tan B$ **(ii)** $\sin (A - B)$ **(iii)** $\operatorname{cosec} 2A$

(iv) $\sin^2 2A$ **(v)** $\cos^2 2B$

10. Given that $\sin^2 x = \frac{8}{9}$ and that $90° < x < 180°$, find the value of each of the following without using a calculator.

(i) $\tan x$ **(ii)** $\cos 2x$ **(iii)** $\cos \frac{x}{2}$

11. Given that $\tan A = 3 \tan B$, show that $\tan (A - B) = \frac{\sin 2B}{3 - 2\cos^2 B}$.

12. Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.

(a) $4 \sin 2x - 3 \sin x = 0$ **(b)** $2 \sin 2x = 3 \cos x$

(c) $3 \cos 2x - \cos x = 2$ **(d)** $4 \sin x - 3 \cos x + 1 = 0$

(e) $2 \cos \left(x + \frac{\pi}{12}\right) + 3 \sin \left(x + \frac{\pi}{12}\right) = 1.8$

13. Solve each of the following equations for $0° \leqslant \theta \leqslant 180°$.

(a) $\sin 4\theta \cos 2\theta = \cos 4\theta \sin 2\theta$ **(b)** $\cos 3\theta \cos \theta = \sin 3\theta \sin \theta$

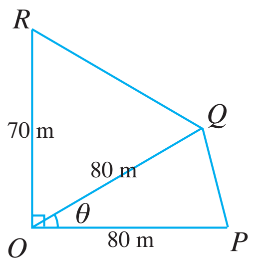
14. Express $12 \sin x - 5 \cos x$ in the form $R \sin (x - \alpha)$, where R is positive and α is acute.

(i) Find the values of x between 0° and 360° for which $12 \sin x - 5 \cos x = 7.5$.

(ii) Find the maximum and minimum values of $12 \sin x - 5 \cos x - 9$ and the corresponding angles between 0° and 360°.

(iii) Sketch the graph of $y = 12 \sin x - 5 \cos x - 9$ for $0° \leqslant x \leqslant 360°$.

15. The figure shows two triangular plots of land OPQ and OQR. It is given that $OP = OQ = 80$ m, $OR = 70$ m, $\angle POR$ is a right angle and $\angle POQ = \theta$ radians.



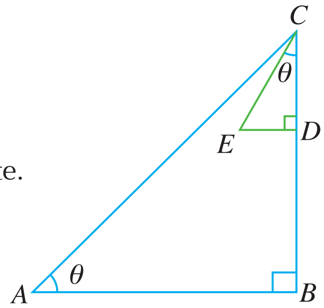
(i) Show that the sum of the areas of the two plots of land, A m², is given by $A = 3200 \sin \theta + 2800 \cos \theta$.

(ii) Express A in the form $R \cos (\theta - \alpha)$, where R is positive and α is acute.

(iii) Hence, find the value of θ for which A will be a maximum.

(iv) Find the value of θ when $A = 4000$ m².

16. In the figure, $\angle ABC = \angle CDE = 90^\circ$, $CE = 7$ cm, $AC = 24$ cm and $\angle BAC = \angle DCE = \theta$, where θ is an acute angle that varies.



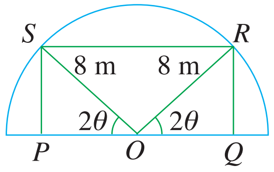
(i) If $k = AB + BD$, show that $k = 24 \sin \theta + 17 \cos \theta$.

(ii) Express k in the form $R \sin (\theta + \alpha)$, where R is positive and α is acute.

(iii) State the maximum value of k and the corresponding value of θ.

(iv) Find the value of θ for which $k = 23$ cm.

17. An exhibition hall is located within a dome. The cross section of the dome is a semicircle, centre O and radius 8 m, with a rectangle $PQRS$ inscribed inside as shown. Given that $\angle POS = \angle QOR = 2\theta$ radians, show that the perimeter of the rectangle $PQRS$, k m, is given by $k = 16 \sin 2\theta + 32 \cos 2\theta$.



(i) Express k in the form $R \sin (2\theta + \alpha)$, where R is positive and α is acute.

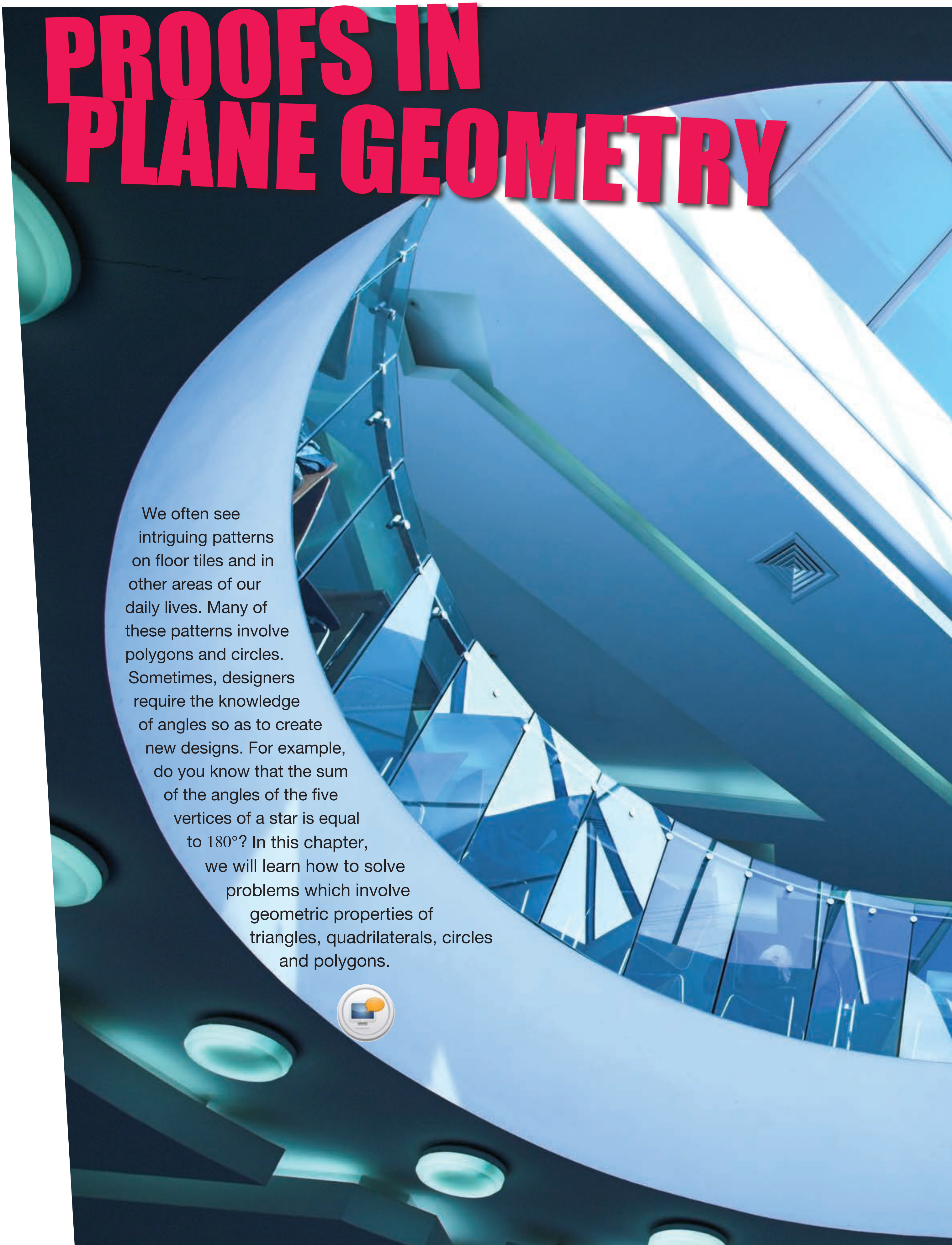
(ii) Find the value of θ for which $k = 25$.

(iii) Find the maximum value of k and the value of θ at which the maximum value occurs.

Challenge Yourself

1. Given that $270^\circ < 4x < 360^\circ$ and that $\cos 4x = \frac{17}{32}$, find the values of $\cos x$, $\cos 2x$ and $\cos 3x$.

2. Prove the identity $\sin^4 A + \cos^4 A = \frac{3}{4} + \frac{1}{4}\cos 4A$.



PROOFS IN PLANE GEOMETRY

We often see intriguing patterns on floor tiles and in other areas of our daily lives. Many of these patterns involve polygons and circles. Sometimes, designers require the knowledge of angles so as to create new designs. For example, do you know that the sum of the angles of the five vertices of a star is equal to $180°$? In this chapter, we will learn how to solve problems which involve geometric properties of triangles, quadrilaterals, circles and polygons.

CHAPTER 10

excluded from the N(A) syllabus

Learning Objectives

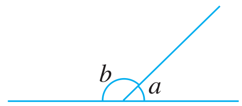
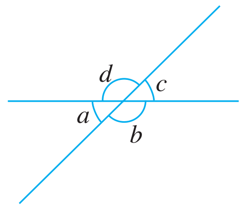
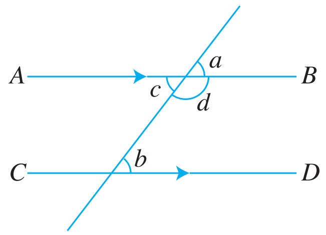
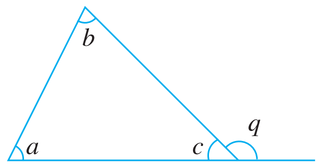
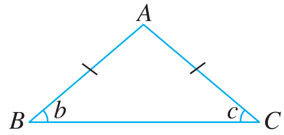
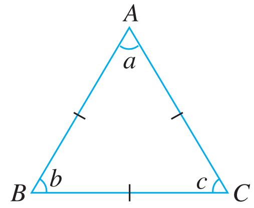
At the end of this chapter, students should be able to solve problems using the following:

- properties of parallel lines cut by a transversal, perpendicular and angle bisectors, triangles, special quadrilaterals and circles,
- congruent and similar triangles,
- Midpoint Theorem,
- Tangent-Chord Theorem (Alternate Segment Theorem).

10.1 BASIC PROOFS IN PLANE GEOMETRY

Recap

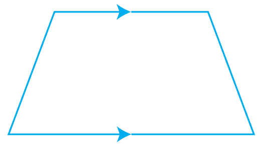
Some of the geometric properties that we have learnt are listed below and we may need to use them in this chapter.

No.	Geometric Property	
1	**Adjacent angles on a straight line are supplementary** (adj. ∠s on a str. line) • $\angle a + \angle b = 180°$	
2	**Vertically opposite angles are equal** (vert. opp. ∠s) • $\angle a = \angle c$ • $\angle b = \angle d$	
3	**Angles formed by parallel lines:** • Corresponding angles are equal, i.e. $\angle a = \angle b$ (corr. ∠s, AB // CD) • Alternate angles are equal, i.e. $\angle b = \angle c$ (alt. ∠s, AB // CD) • Interior angles are supplementary, i.e. $\angle b + \angle d = 180°$ (int. ∠s, AB // CD)	
4	**Angle properties of triangles:** • $\angle a + \angle b + \angle c = 180°$ (∠ sum of Δ) • $\angle a + \angle b = \angle q$ (ext. ∠ = sum of int. opp. ∠s)	
	• Isosceles Δ: $\angle b = \angle c$ (base ∠s of isos. Δ), $AB = AC$	
	• Equilateral Δ: $\angle a = \angle b = \angle c = 60°$ (∠s of equilateral Δ), $AB = AC = BC$	

5 **Properties of quadrilaterals:**

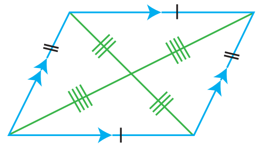
(a) Trapezium

- At least one pair of opposite sides are parallel



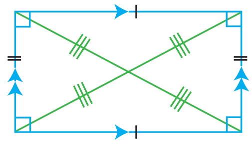
(b) Parallelogram

- Two pairs of opposite sides are parallel
- Opposite sides are equal in length
- Opposite angles are equal
- Diagonals bisect each other



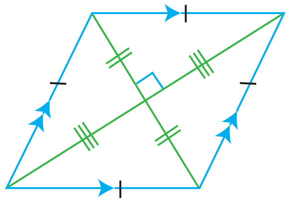
(c) Rectangle

- Has all the properties of a parallelogram
- All angles are right angles
- Diagonals are equal in length



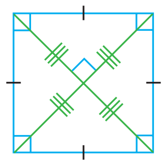
(d) Rhombus

- Two pairs of opposite sides are parallel
- All sides are equal in length
- Opposite angles are equal
- Diagonals bisect each other at right angles
- Diagonals bisect the interior angles



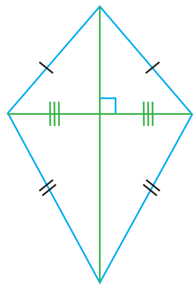
(e) Square

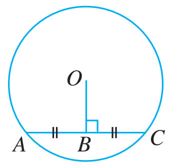
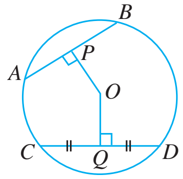
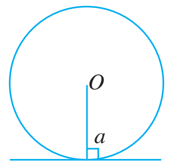
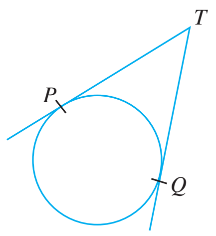
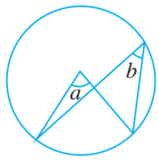
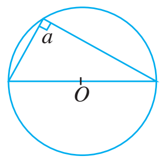
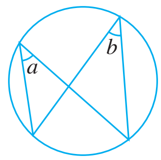
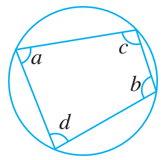
- Has all the properties of a rhombus
- All angles are right angles
- Diagonals are equal in length



(f) Kite

- Two pairs of adjacent sides are equal
- Diagonals are perpendicular
- The longer diagonal bisects the other diagonal



6	**Symmetry properties of circles:** • Perpendicular bisector of chord passes through the centre i.e. $OB \perp AC \Leftrightarrow AB = BC$ ($\perp$ bisector of a chord passes through centre) • Equal chords are equidistant from the centre $AB = CD \Leftrightarrow OP = OQ$	 
	• Tangent is perpendicular to the radius at the point of contact, i.e. $\angle a = 90°$ (tan. $\perp$ rad.) • Tangents from an external point are equal in length, i.e. $TP = TQ$	 
7	**Angle properties of circles:** • Angle at the centre is equal to twice the angle at the circumference ($\angle$ at centre = 2 $\angle$ at $\odot^{ce}$), i.e. $\angle a = 2\angle b$ • Angle in a semicircle is a right angle (rt. $\angle$ in a semicircle), i.e. $\angle a = 90°$ • Angles in the same segment are equal ($\angle$s in the same segment), i.e. $\angle a = \angle b$ • Angles in opposite segments are supplementary ($\angle$s in opp. segments are supp.), i.e. $\angle a + \angle b = 180°$ $\angle c + \angle d = 180°$	   

Thinking Time

By considering the properties of triangles, quadrilaterals and circles, discuss whether each of the statements below is true and give reasons to support your answer.

(a) All equilateral triangles are isosceles triangles.
(b) All parallelograms are rectangles.
(c) All circles are similar figures.

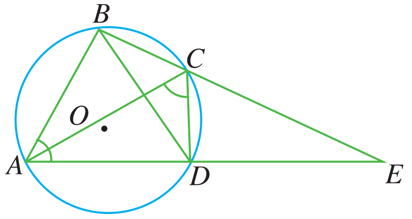
Worked Example 1

(Proving using Angle Properties of Circles)

In the figure, O is the centre of the circle. BCE and ADE are straight lines. Given that $\angle BAD = \angle ACD$, prove that

(i) CD bisects $\angle ACE$,

(ii) $BD = AD$.



Solution

(i) $\angle BCD = 180° - \angle BAD$ ($\angle$s in opp. segments are supp.)

$\therefore \angle DCE = 180° - \angle BCD$ (adj. $\angle$s on a str. line)

$= 180° - (180° - \angle BAD)$

$= \angle BAD$

$= \angle ACD$ (given)

$\therefore CD$ bisects $\angle ACE$. (proven)

(ii) $\angle ABD = \angle ACD$ ($\angle$s in the same segment)

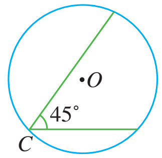
$\angle ACD = \angle BAD$ (given)

Since $\angle ABD = \angle BAD$, ΔABD is isosceles.

$\therefore AD = BD$ (sides of isos. Δ) (proven)

Just For Fun

The figure shows the plan of a circular hall of a jewellery exhibition. C is a hidden video camera which scans an angle of 45°. How many more of such video cameras must be installed on the walls of the hall so that they will cover the entire hall? Indicate the position where each video camera must be mounted.



How many video cameras would required if each one can scan an area of

(i) 35°?

(ii) 60°?

(iii) 90°?

(iv) 100°?

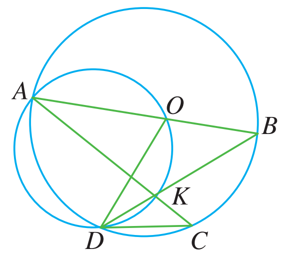
Practise Now 1

Similar Questions:
Exercise 10A
Questions 1-8

1. In the figure, O is the centre of the circle $ABCD$. The smaller circle $AOKD$ intersects the larger circle at A and D. Given that AOB, AKC and BKD are straight lines, prove that

(i) $DK = CK$,

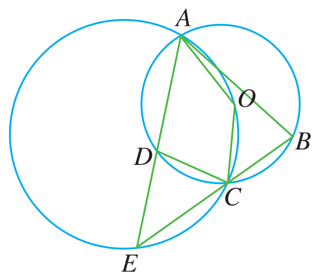
(ii) BKD bisects $\angle ODC$.



2. In the figure, O is the centre of the circle $ABCD$ and it intersects the circle $AOCE$ at A and C. Given that ECB and ADE are straight lines, prove that

(i) $DE = CE$,

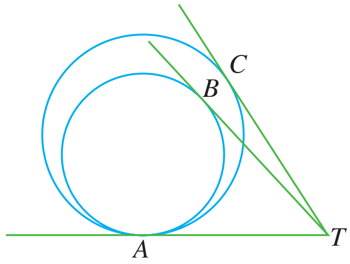
(ii) $\angle CDE = \angle BAD$.



Worked Example 2

(Proving using Symmetry Properties of Circles)

In the figure, TA is the common tangent to the two circles which touch internally at A. TA and TB are tangents to the smaller circle and TA and TC are tangents to the larger circle. Prove that a circle with centre at T can be drawn to pass through the points A, B and C.



Solution

Consider the smaller circle:

$TA = TB$ (tangents from an external point)

Consider the larger circle:

$TA = TC$ (tangents from an external point)

i.e. $TA = TB = TC$

$\therefore$ A circle with centre T can be drawn to pass through the points A, B and C.

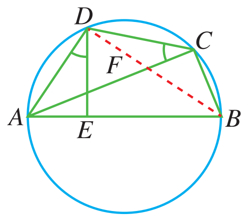
Practise Now 2

Similar Questions:
Exercise 10A
Questions 9-14

In the figure, AB is the diameter of the circle and $\angle ADE = \angle DCA$.

(i) By considering $\angle DAB$, prove that $\angle DEB = 90°$.

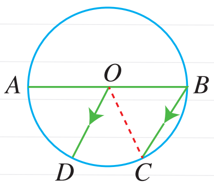
(ii) By making use of the property that angles in opposite segments are supplementary, prove that a circle can be drawn to pass through B, C, F and E.



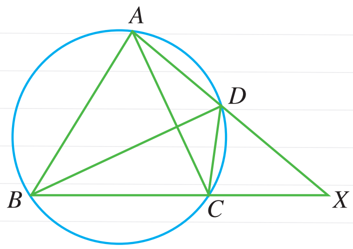
Basic Level | Intermediate Level | Advanced Level

Exercise 10A

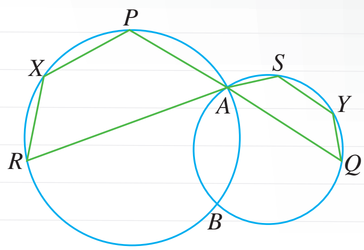
1 In the figure, $OD \parallel BC$ and AOB is the diameter of the circle. Prove that $\angle AOD = \angle COD$.



2 In the figure, A, B, C and D are points on the circle. BC and AD are produced to meet at X. Given that CD bisects $\angle BDX$, prove that ΔABC is isosceles.



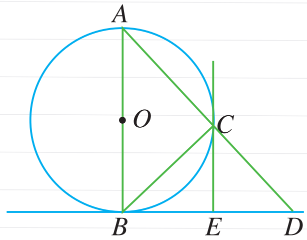
3 In the figure, PAQ and RAS are straight lines. Prove that $\angle PXR = \angle QYS$.



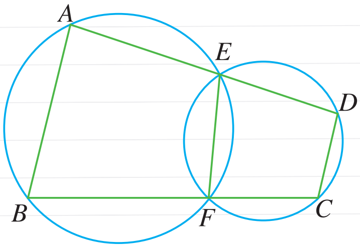
4 In the figure, BD and CE are tangents to the circle, of which AB is a diameter and ACD is a straight line. Prove that

(i) $\angle ABC = \angle ECD$,

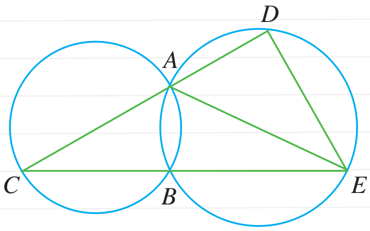
(ii) $BE = ED$.



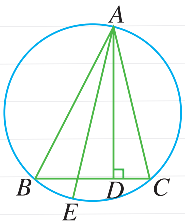
5 In the figure, two circles intersect at E and F. AED and BFC are straight lines. Prove that $BA \parallel CD$.



6 In the figure, the smaller circle ABC intersects the larger circle $ABED$ at A and B. Given that CAD and CBE are straight lines and that AC is the diameter of the smaller circle ABC, prove that AE is the diameter of the larger circle.



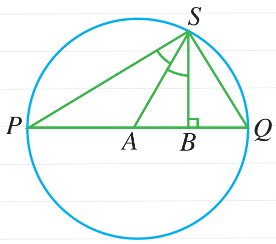
7 In the figure, $\angle ADC = 90°$ and AE is the diameter of the circle. By joining B and E, prove that $\angle CAD = \angle BAE$.



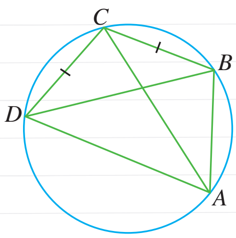
8 In the figure, PQ is a diameter of the circle. S is a point on the circumference. A and B are points on PQ such that $\angle SBQ = 90°$ and AS bisects $\angle PSB$. Prove that

(i) $\angle SPQ = \angle BSQ$,

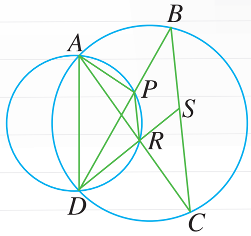
(ii) $\angle ASQ = \angle SAQ$.



9 In the figure, A, B, C and D are points on the circle. Given that $CD = CB$, prove that AC bisects $\angle BAD$.



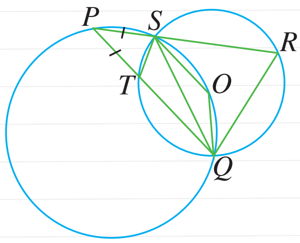
10 In the figure, $ABCD$ and $APRD$ are two circles intersecting at A and D. ARC and BPD are straight lines. DR produced meets BC at S. Prove that PR is parallel to BC.



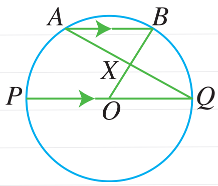
11 In the figure, O is the centre of the smaller circle $QRST$ and that PSR and PTQ are straight lines.

(i) Prove that $PQ = PR$.

(ii) Given that $PS = PT$, show that TS is parallel to QR.



12 In the figure, O is the centre of the circle, PQ is a diameter and AB is a chord which is parallel to PQ. AQ and OB intersect at X. Prove that $\angle BXQ = 3\angle PQA$.

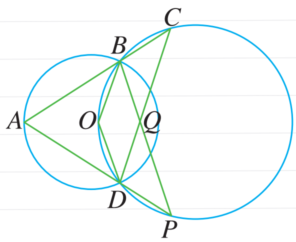


13 In the figure, O is the centre of the smaller circle passing through the points A, B and D. Given that ABC and ADP are straight lines and that O, D, P, C and B are points on the larger circle, prove that

(i) $\angle ACD + 2\angle BAD = 180°$,

(ii) ΔACD is an isosceles triangle,

(iii) $\angle PQC + 3\angle BAD = 360°$.

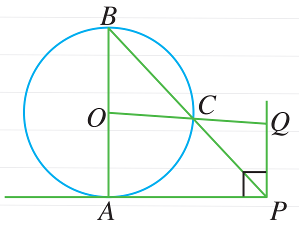


Exercise 10A

14 In the figure, O is the centre of the circle and AB is a diameter. The tangent at A meets BC produced at P. The perpendicular through P meets OC produced at Q. Prove that

(i) AB is parallel to PQ,

(ii) ΔPQC is an isosceles triangle.



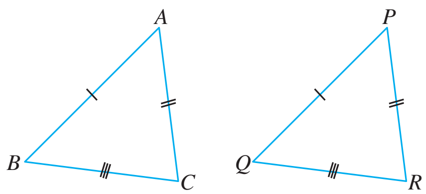
10.2 PROOFS USING CONGRUENCE AND SIMILARITY TESTS

Congruence Tests for Triangles

We have learnt the following tests of congruence for two triangles, which can be used to prove that ΔABC is congruent to ΔPQR (denoted by $\Delta ABC \equiv \Delta PQR$).

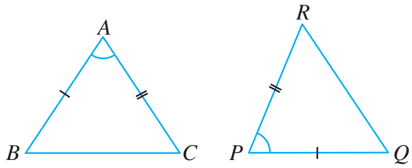
(i) SSS Congruence Test

$AB = PQ$, $AC = PR$ and $BC = QR$



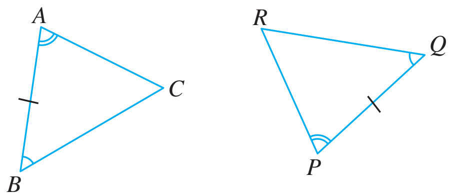
(ii) SAS Congruence Test

$AB = PQ$, $AC = PR$ and $\angle A = \angle P$



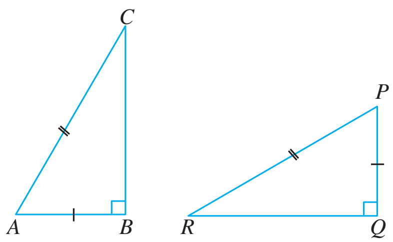
(iii) ASA Congruence Test

$\angle A = \angle P$, $\angle B = \angle Q$ and $AB = PQ$



(iv) RHS Congruence Test (only for right-angled triangles)

$\angle B = \angle Q = 90^\circ$, $AC = PR$ and $AB = PQ$



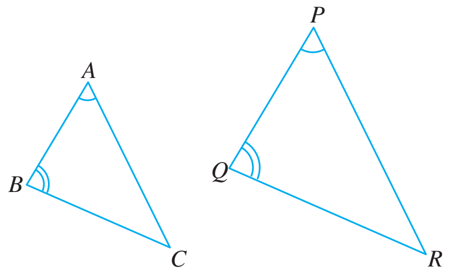
Note: When congruent triangles are named, the letters must be written in the correct order so that it is easier to identify corresponding vertices.

Similarity Tests for Triangles

ΔABC is similar to ΔPQR if the corresponding angles are equal or the lengths of the corresponding sides are proportional.

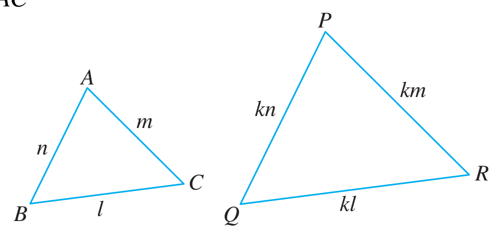
(i) AA Similarity Test

$\angle A = \angle P$ and $\angle B = \angle Q$ (It follows that $\angle C$ must be equal to $\angle R$.)



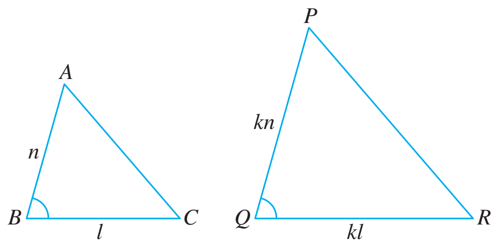
(ii) SSS Similarity Test

$\frac{PQ}{AB} = \frac{QR}{BC} = \frac{PR}{AC}$, i.e. the ratios of their corresponding sides are equal.



(iii) SAS Similarity Test

$\frac{PQ}{AB} = \frac{QR}{BC}$, and $\angle B = \angle Q$, i.e. the ratios of two of the corresponding sides are equal and the included angles are equal.



Proofs Involving Congruence And Similarity Tests

Let us prove this statement:

The diagonals of a parallelogram bisect each other.

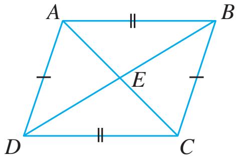


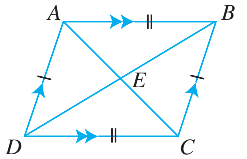
Fig. 10.1

If we label the parallelogram *ABCD* and the point of intersection of the two diagonals *AC* and *BD* as *E* (see Fig. 10.1), 'the diagonals bisect each other' means that $AE = CE$ and $BE = DE$.

When proving the above statement, we cannot start by saying that ΔABE and ΔCDE are congruent (SSS Congruence Test) because this makes use of $AE = CE$ and $BE = ED$ which is what we are supposed to prove.

Thus we need to prove $AE = CE$ and $BE = DE$, so we cannot make use of this information to do the proof. How do we prove that ΔABE and ΔCDE are congruent? We can show the pictorial working as follows:

Given Conditions

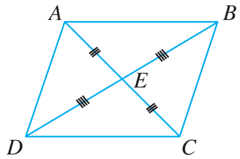


To prove

$AE = CE$

$BE = DE$

Conclusions



Then we think of what properties we can use, e.g. what triangles should we consider? Can we prove that ΔABE and ΔCDE are congruent? We can show the pictorial working as follows:

Pictorial Working

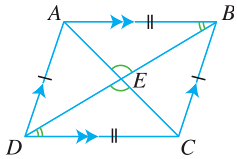


Fig. 10.2

Thinking Time

To prove that the diagonals of a parallelogram bisect each other, refer to the pictorial working in Fig. 10.2. Copy and complete the proof below:

In ΔABE and ΔCDE,

$\angle AEB = \angle$________ (vert. opp. ∠s)

$AB = DC$ (opp. sides of //gram)

$\angle ABE = \angle$________ (alt. ∠s, $AB // DC$)

$\therefore\ \Delta ABE \equiv$ ________

$\therefore$ ΔABE and ΔCDE are congruent. (________)

Since the two triangles are congruent,

$AE = CE$ and $BE =$ ________.

Hence, the two diagonals bisect each other. (proven)

PROBLEM-SOLVING TIP

To write the vertices in the correct order, we can link the corresponding vertices like this:

$A \leftrightarrow C$

$B \leftrightarrow D$

$E \leftrightarrow E$

Class Discussion

Work in pairs to prove the following:

If the diagonals of a quadrilateral bisect each other, then the quadrilateral is a parallelogram.

You can follow the example on the previous page and draw a pictorial representation of the given conditions, the conclusions and the pictorial working. You will discover that this proof uses a different congruence test from the above proof. Why is this so?

Thinking Time

Justify whether each of the statements below is true.

(a) Base angles of an isosceles triangle are equal.

(b) Diagonals of a rhombus are equal in length.

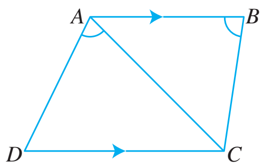
Worked Example 3

(Proving using Similar Triangles)

In the figure, $ABCD$ is a trapezium in which AB is parallel to DC and $\angle ABC = \angle CAD$. Prove that

(i) ΔABC is similar to ΔCAD,

(ii) $AC^2 = AB \times CD$.



Solution

(i) $\angle BAC = \angle ACD$ (alt. $\angle$s, $AB \parallel DC$)

$\angle ABC = \angle CAD$ (given)

$\therefore$ ΔABC is similar to ΔCAD. (AA Similarity Test)

(ii) From **(i)**,

$\frac{AB}{CA} = \frac{AC}{CD}$, i.e. $AB \times CD = AC \times CA$

$\therefore$ $AC^2 = AB \times CD$

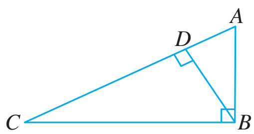
Practise Now 3

Similar Questions:
Exercise 10B
Questions 1, 2, 3, 5-9, 11

1. In the figure, ABC is a right-angled triangle with BD perpendicular to AC.

(i) Prove that ΔABC is similar to ΔBDC.

(ii) Hence, show that $BD \times BC = AB \times DC$.

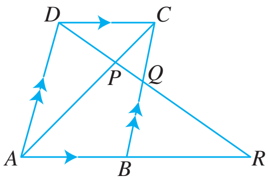


2. In the figure, $ABCD$ is a parallelogram. APC, BQC, ABR and $DPQR$ are straight lines. It is also given that $3DP = PR$.

(i) Prove that ΔPCD and ΔPAR are similar.

(ii) Name a triangle that is similar to ΔBQR.

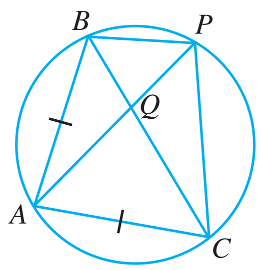
(iii) Name the triangle that is similar to ΔDAP.



Worked Example 4

(Proving using Similar Triangles)

In the figure, $ABPC$ is a circle where $AB = AC$, AQP and BQC are straight lines. By considering a pair of similar triangles, prove that $AB^2 = AP \times AQ$.



Solution

In ΔABP and ΔAQB,

$\angle BAP = \angle QAB$ (common $\angle$)

$\angle ABC = \angle ACB$ (base $\angle$s of isos. ΔABC)

$\angle APB = \angle ACB$ ($\angle$s in same segment)

$\therefore \angle APB = \angle ACB = \angle ABQ$

$\therefore$ ΔABP is similar to ΔAQB. (AA Similarity Test)

$\Rightarrow \dfrac{AB}{AQ} = \dfrac{AP}{AB}$

$\therefore AB^2 = AP \times AQ$

Practise Now 4

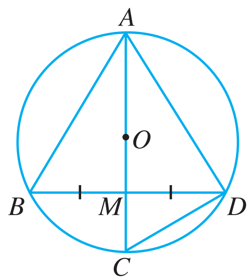
Similar Questions:
Exercise 10B
Questions 4, 10

In the figure, O is the centre of the circle with AOC as its diameter passing through M, the midpoint of BD. Prove that

(i) ΔABM and ΔADM are congruent,

(ii) ΔACD is similar to ΔABM,

(iii) $AB \times CD = AC \times BM$.



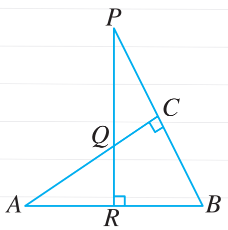
Basic Level | Intermediate Level | Advanced Level

Exercise 10B

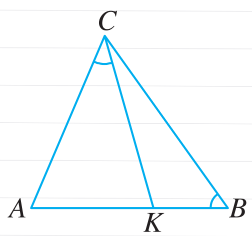
1 In the figure, $\angle ACB = \angle BRP = 90°$, ARB, BCP, AQC and PQR are straight lines. Prove that

(i) ΔABC and ΔPBR are similar,

(ii) ΔAQR and ΔPQC are similar.



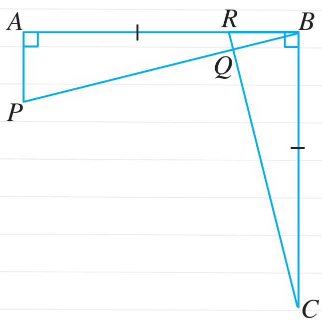
2 In the figure, AKB is a straight line and $\angle ACK = \angle ABC$. Prove that ΔACK and ΔABC are similar.



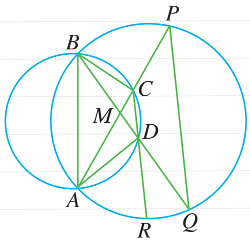
3 In the figure, $\angle PAB = \angle ABC = 90°$, $AB = BC$, $BR = \frac{1}{3}AB$, $AP = \frac{1}{3}BC$, CQR and PQB are straight lines. Prove that

(i) ΔABP and ΔBCR are congruent,

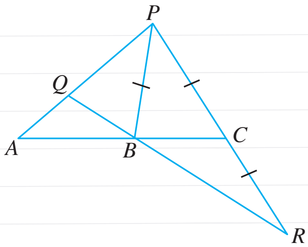
(ii) $\angle BQC = 90°$.



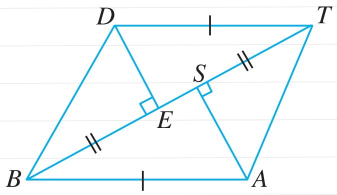
4 The circles $ABCD$ and $ABPQR$ intersect at A and B. Given that ACP, CDR and $BMDQ$ are straight lines, prove that
- **(i)** $\angle BDC = \angle BQP$,
- **(ii)** ΔMPQ and ΔMBA are similar.



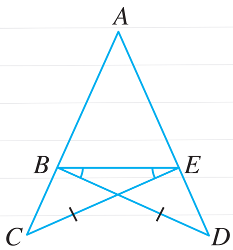
5 In the figure, ABC, PCR and QBR are straight lines. Given that $PB = PC = CR$ and B is the midpoint of AC, prove that
- **(i)** ΔABP is congruent to ΔBCR,
- **(ii)** ΔAQB is an isosceles triangle.



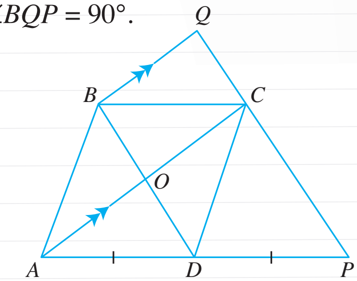
6 In the figure, $BEST$ is a straight line. Given that $BA = DT$, $BE = ST$, DE is perpendicular to BT and AS is perpendicular to BT. Using congruent triangles, prove that DT // BA. Hence, show that $ABDT$ is a parallelogram.



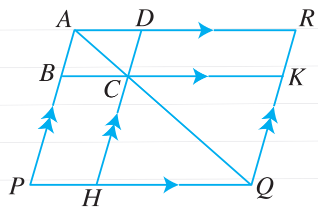
7 In the figure, ABC and AED are straight lines. Given that $BD = EC$ and $\angle EBD = \angle BEC$, prove that
- **(i)** ΔEDB and ΔBCE are congruent,
- **(ii)** $AB = AE$.



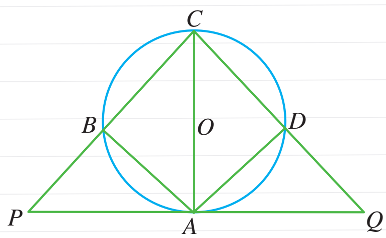
8 In the figure, $ABCD$ is a rhombus, AC is parallel to BQ, $AD = DP$ and ADP and PCQ are straight lines. Prove that
- **(i)** ΔABD and ΔDCP are congruent,
- **(ii)** $\angle BQP = 90°$.



9 In the figure, $ABCD$, $APQR$, $DCKR$, $BPHC$ and $CHQK$ are parallelograms.
- **(i)** State the triangle which is congruent to ΔABC.
- **(ii)** State the triangle which is congruent to ΔCHQ.
- **(iii)** Prove that the area of $BPHC$ is equal to the area of $DCKR$.
- **(iv)** Prove that $AC \times HC = DC \times QC$.



10 In the figure, O is the centre of the circle, AC is a diameter and PAQ is a tangent to the circle at A.
- **(i)** Show that ΔCAP is similar to ΔCBA.
- **(ii)** Prove that $AC^2 = CP \times CB$.
- **(iii)** State a triangle that is similar to ΔCAQ. Hence, or otherwise, prove that $CP \times CB = CD \times CQ$.

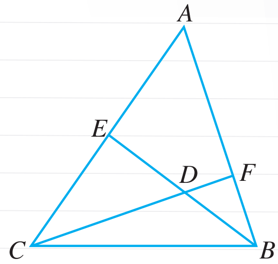


Exercise 10B

11 In the figure, BE and CF are the altitudes of ΔABC. BE and CF intersect at D. Prove that

(i) $BD \times DE = CD \times DF$,

(ii) $AE \times AC = AF \times AB$.

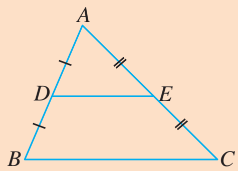


10.3 MIDPOINT THEOREM

Investigation

Midpoint Theorem

If D and E are the midpoints of AB and AC of a given triangle ABC, what can we say about DE and BC? Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template **Midpoint Theorem**.



Given conditions: In ΔABC, the midpoints of AB and AC are D and E respectively.

1. Click and drag to move the points A, B and C to obtain different ΔABC. What two relationships do you observe between the lines DE and BC?

2. Copy and complete the following.

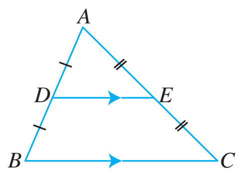
Conclusions: **(a)** The line DE is ____________ to the line BC.

(b) The length of the line DE is ___________ the length of the line BC.

This is called the **Midpoint Theorem**.

From the investigation, we observe that:

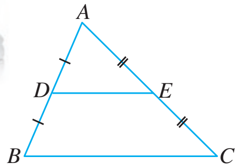
In ΔABC, if the midpoints of AB and AC are D and E respectively, then $DE \mathbin{/\mkern-6mu/} BC$ and $DE = \frac{1}{2}BC$.



Proof of Midpoint Theorem

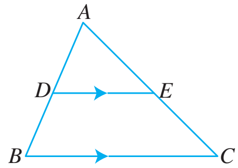
How can we prove the Midpoint Theorem?
For better visualisation, we can draw the given conditions and the conclusions as shown:

Given Conditions



To prove →

Conclusions



Copy and complete the partial proof of the Midpoint Theorem below:

In ΔADE and ΔABC, $\qquad \dfrac{AD}{AB} = \dfrac{AE}{AC} = \dfrac{1}{2}$ (________)

$\angle DAE = \angle$________ (common $\angle$)

ΔADE and ΔABC are similar. (________)

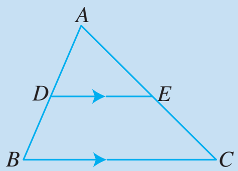
Since the two triangles are similar, $\angle ADE = \angle$________.

$\therefore DE \,//$ ________ ($\angle ADE$ and $\angle$_____ are _____ $\angle$s.)

Since the two triangles are similar, $\dfrac{DE}{BC} =$ _____ $= \dfrac{1}{2}$.

$\therefore DE = \dfrac{1}{2}$______ (proven)

Discuss with your classmate how to prove the following. Then write down the proof.



In ΔABC, if D and E are two points on AB and AC respectively such that $DE \,//\, BC$ and $DE = \dfrac{1}{2}BC$, then D and E are the **midpoints** of AB and AC respectively.

INTERNET RESOURCES

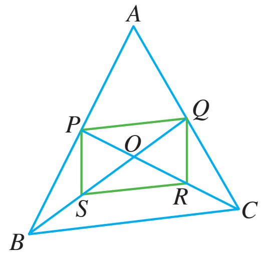
Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template **Midpoint Theorem** to investigate if the result in this activity is true.

The proof of the Midpoint Theorem uses the SAS Similarity Test but the proof of the result in the previous activity uses a *different* test. Can you explain why this is so?

Worked Example 5

(Proving using the Midpoint Theorem)

In ΔABC, BQ and CP intersect at O. P, Q, R and S are the midpoints of AB, AC, OC and OB respectively. Prove that $PQRS$ is a parallelogram. Hence, show that $OS = \frac{1}{3}BQ$.



Solution

In ΔABC and ΔAPQ, P and Q are the midpoints of AB and AC respectively.

i.e. $PQ = \frac{1}{2}BC$ and PQ // BC (Midpoint Theorem)

In ΔOSR and ΔOBC, S and R are the midpoints of OB and OC respectively.

i.e. $SR = \frac{1}{2}BC$ and SR // BC (Midpoint Theorem)

Since $PQ = SR$ and PQ // SR,

$\therefore$ $PQRS$ is a parallelogram.

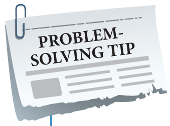
$OQ = OS$ (diagonals of a parallelogram bisect each other)

$OB = 2OS$ (S is the midpoint of OB.)

Then $BQ = OB + OQ$

$= 2OS + OS$

$\therefore$ $OS = \frac{1}{3}BQ$

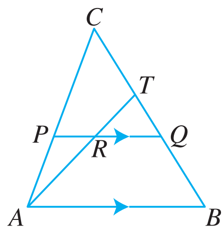


To prove that $PQRS$ is a parallelogram, we can either prove that PQ // SR and $PQ = SR$ or PS // QR and $PS = QR$.

Practise Now 5

Similar Questions:
Exercise 10C
Questions 1-7

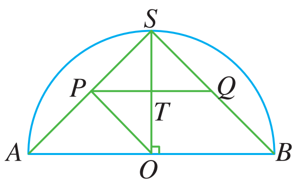
1. In the figure, AB is parallel to PRQ, T and P are the midpoints of CQ and CA respectively and ART is a straight line. Prove that $TR : RA = 1 : 2$.



2. In ΔABS, P and Q are the midpoints of AS and BS. O is the centre of the semicircle ABS and $\angle SOB = 90°$. Prove that

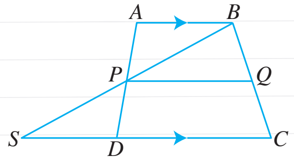
(i) $\angle SPT = \angle OBS$,

(ii) $TQ : AB = 1 : 4$.



Exercise 10C

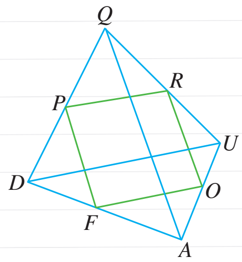
1 In the trapezium $ABCD$, AB // DC and P and Q are the midpoints of sides AD and BC respectively. The extensions of BP and CD intersect at S. Prove that



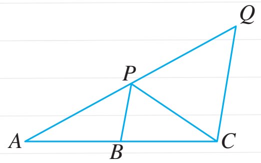
(i) P is the midpoint of BS, hence show that PQ // DC,

(ii) $PQ = \frac{1}{2}(AB + DC)$.

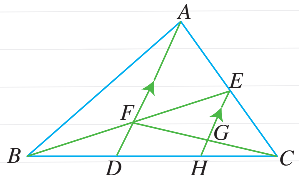
2 Quadrilateral $QUAD$ has an inscribed quadrilateral $PROF$, where P, R, O and F are the midpoints of QD, QU, UA and AD respectively. Prove that quadrilateral $PROF$ is a parallelogram.



3 In the figure, P and B are the midpoints of AQ and AC respectively. Given that PB bisects $\angle APC$, prove that $\angle PCQ$ is an isosceles triangle.



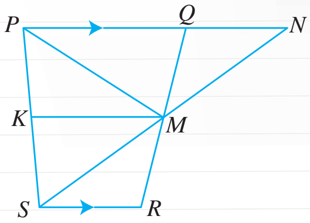
4 In ΔABC, E is the midpoint of AC. AD and BE intersect at F, which is the midpoint of BE. FC and EH intersect at G.



(i) Given that AD // EH, prove that $EH = 2FD$.

(ii) Hence, show that $AF = 3FD$.

5 In the figure, PQ is parallel to SR, M and K are the midpoints of QR and PS respectively and SM produced meets PQ produced at N. Prove that

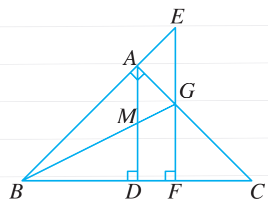


(i) ΔSRM is congruent to ΔNQM,

(ii) area of ΔPSM = area of ΔPMN,

(iii) area of $PQRS = 2 \times$ area of ΔPMN.

6 In ΔABC, $\angle BAC = 90°$, AD is perpendicular to BC and M is the midpoint of AD. BMG and EGF are straight lines where EGF is perpendicular to BC.



(i) Explain why

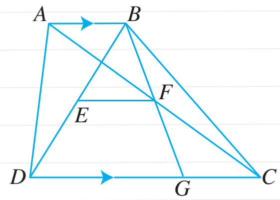
(a) ΔBGF is similar to ΔBMD,

(b) ΔBEG is similar to ΔBAM.

(ii) Hence, show that $EG = GF$.

(iii) Prove that $GF^2 = AG \times GC$.

7 In trapezium $ABCD$, AB // DC and E and F are the midpoints of the diagonals BD and AC respectively. BFG and DGC are straight lines. Prove that

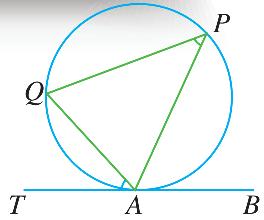


(i) ΔABF and ΔCGF are congruent,

(ii) $EF = \frac{1}{2}(CD - AB)$.

10.4 TANGENT-CHORD THEOREM (ALTERNATE SEGMENT THEOREM)

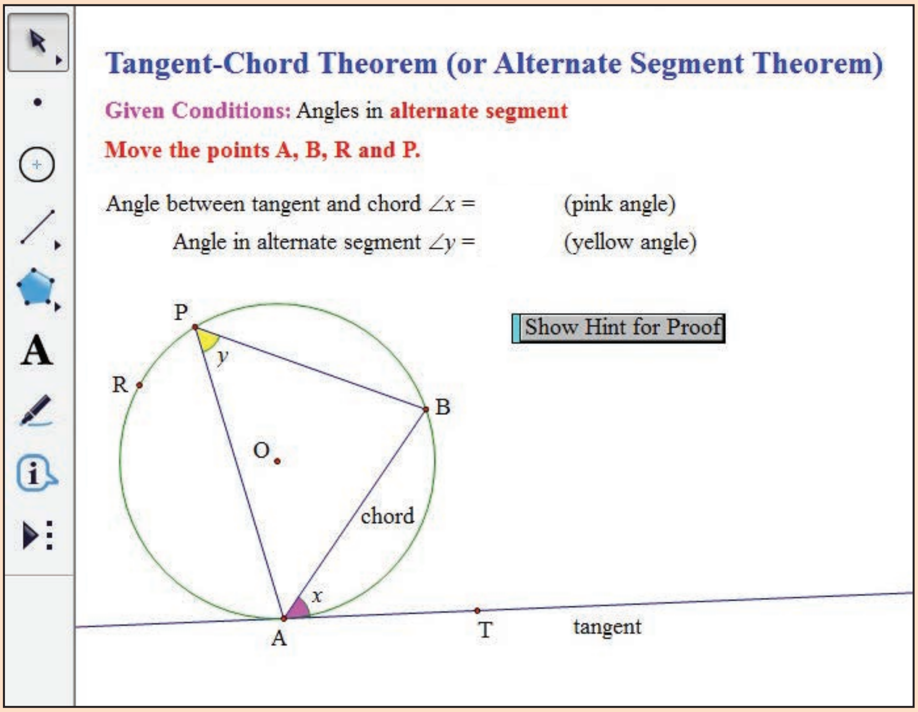
In the figure, TA is the tangent to the circle at A and AQ is a chord at the point of contact. $\angle TAQ$ is the angle between the tangent TA and the chord AQ at A. $\angle QPA$ is the angle subtended by chord AQ in the alternate segment. We will now learn a theorem that states how they are related.



Investigation

Tangent-Chord Theorem

Go to http://www.shinglee.com.sg/StudentResources/ and open the geometry software template Tangent-Chord Theorem as shown below:



Given conditions: Angles in Alternate Segment

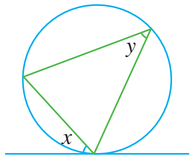
In the above circle, there are two coloured angles. The pink angle, $\angle x$, is the angle between the **tangent** and the **chord** at the point of contact A. The yellow angle, $\angle y$, is an angle subtended by the chord in the **alternate segment** with reference to $\angle x$. Notice that the two angles are on different sides of the chord.

1. Click and drag to move the points A, B, R and P to change the size of the two coloured angles and the radius of the circle. State what you observe about the relationship between $\angle x$ and $\angle y$.

2. Can you prove that your observation in Question 1 is true for all angles x and y? You can click the button [Show Hint for Proof] in the template.

This is called the **Tangent-Chord Theorem** (or **Alternate Segment Theorem**).

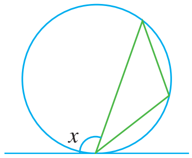
From the above investigation, the **Tangent-Chord Theorem** (or **Alternate Segment Theorem**) states that:

> The angle between a tangent and a chord at the point of contact is equal to the angle subtended by the chord in the alternate segment.
>
> 
>
>
> In the diagram, $\angle x = \angle y$.
>
> Abbreviation: $\angle$s in alt. seg.

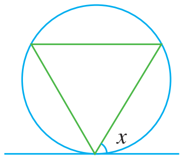
Thinking Time

Identify the angle that is equal to $\angle x$ in each of the following circles and label it as $\angle y$.

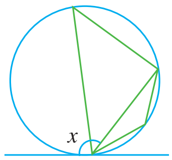
(a)



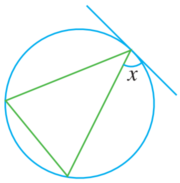
(b)



(c)



(d)



Proof of Tangent-Chord Theorem

Class Discussion

Work in pairs.

Consider the circle in Fig. 10.3, where A, B and C lie on the circumference of the circle and PQ is a tangent to the circle at A.

To prove that $\angle BAQ = \angle ACB$, in Fig. 10.4, draw a diameter AOS and join SB.

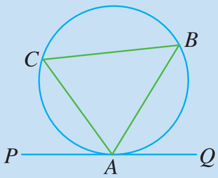


Fig 10.3

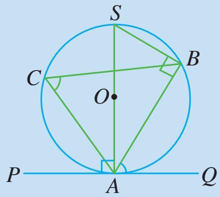


Fig 10.4

Copy and complete the following.

$\angle ABS =$ ___ (rt. $\angle$ in semicircle)

$\angle SAQ =$ ___ (tan. $\perp$ rad.)

$\angle ASB = \angle ACB$ (_______________)

$\angle ASB + \angle SAB = 90°$ (_______________)

$\angle SAB + \angle BAQ = 90°$ (_______________)

$\therefore \angle ASB = \angle BAQ$

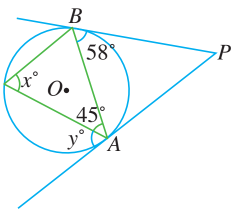
$\angle BAQ = \angle ACB$ (since $\angle ASB =$ _____)

Before we proceed to attempt questions on the proofs involving the Tangent-Chord Theorem, let us first solve some questions that make use of this theorem.

Worked Example 6

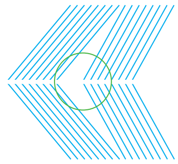
(Tangent-Chord Theorem)

Given that PA and PB are tangents to a circle with centre O, $\angle ABP = 58°$ and $\angle BAC = 45°$, find the value of x and of y.



Just For Fun

Do you see a perfect circle in the figure below?



Solution

$x° = 58°$ (∠s in alt. segment)

$\angle PAB = 58°$ (base ∠s of isosceles ΔPAB)

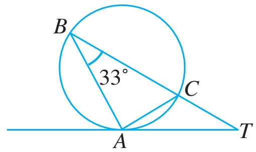
$\therefore\ y° = 180° - 58° - 45°$ (adj. ∠s on a str. line)

$= 77°$

$\therefore\ x = 58,\ y = 77$

Practise Now 6

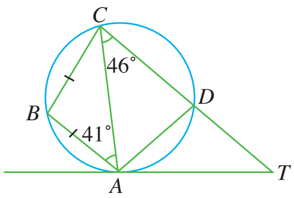
1. BC is a diameter of the circle and TA is the tangent to the circle at A. Given that $\angle ABC = 33°$, find $\angle ATC$.



2. TA is the tangent to the circle at A, $AB = BC$, $\angle BAC = 41°$ and $\angle ACT = 46°$. Find

(i) $\angle CAD$,

(ii) $\angle ATC$.



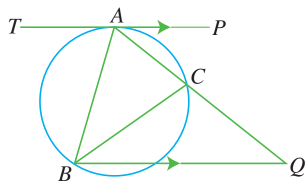
Worked Example 7

(Proving using Tangent-Chord Theorem)

In the figure, AP is a tangent to the circle at A and AP is parallel to BQ. Prove that

(i) ΔABC is similar to ΔAQB,

(ii) $AB^2 = AQ \times AC$.



Solution

(i) $\angle PAQ = \angle AQB$ (alt. ∠s, AP // BQ)

$\angle PAQ = \angle ABC$ (∠s in alt. segment)

i.e. $\angle AQB = \angle ABC$

$\angle BAC = \angle BAQ$ (common ∠)

$\therefore\ \Delta ABC$ is similar to ΔAQB. (AA Similarity Test)

(ii) From **(i)**,

$$\frac{AB}{AQ} = \frac{AC}{AB}$$

i.e. $$AB^2 = AQ \times AC$$

INFORMATION

We can also say that $\angle BAC = \angle BAQ$ (common ∠).

Practise Now 7

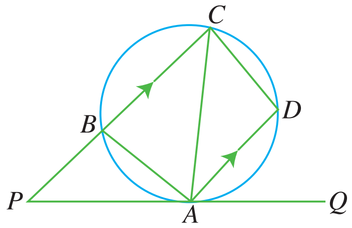
Similar Questions:
Exercise 9C
Questions 1-10

In the figure, PAQ is a tangent to the circle and PBC is parallel to AD. Prove that

(i) $\angle PAB = \angle CAD$,

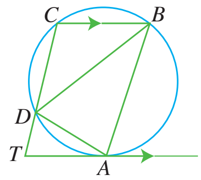
(ii) ΔCAD is similar to ΔPAB,

(iii) $PA^2 = PB \times PC$.



Worked Example 8

(Proving using Tangent-Chord Theorem)



In the figure, TA is a tangent to the circle at A and is parallel to CB. Given that TDC is a straight line, prove that

(i) $\angle ATD = \angle BAD$,

(ii) $BA \times TD = AT \times AD$.

Solution

(i) $\angle ATD + \angle TCB = 180°$ (int. $\angle$s, CB // TA)

$\angle BAD + \angle TCB = 180°$ ($\angle$s in opp. segments are supp.)

$\therefore\ \angle ATD = \angle BAD$

(ii) From **(i)**, $\angle ATD = \angle BAD$ and $\angle TAD = \angle ABD$ ($\angle$s in alt. segments)

ΔBAD is similar to ΔATD. (AA Similarity Test)

$$\frac{BA}{AT} = \frac{AD}{TD}$$

$\therefore\ BA \times TD = AT \times AD$

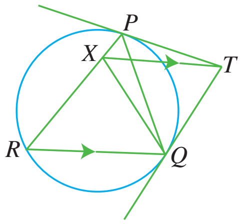
Practise Now 8

Similar Questions:
Exercise 10D
Questions 11-15

In the figure, TP and TQ are tangents to the circle at P and Q respectively. The point X on RP is such that XT is parallel to RQ.

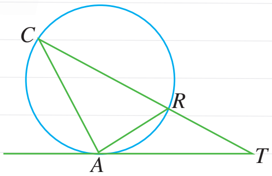
(i) Prove that $\angle PXT = \angle PQT$.

(ii) Given that a circle can be drawn to pass through T, P, X and Q, prove that $\angle PXT = \angle QXT$.



Exercise 10D

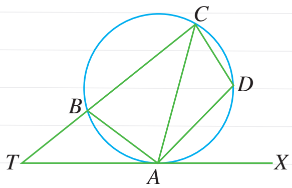
1 In the figure, the tangent to the circle at A meets CR produced at T. Prove that ΔCAT is similar to ΔART. Hence, show that $AC \times RT = AT \times AR$.



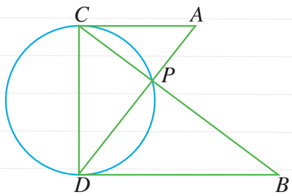
2 In the figure, TAX is a tangent to the circle at A, TBC is a straight line and $\angle TAB = \angle CAD$. Prove that

(i) BC is parallel to AD,

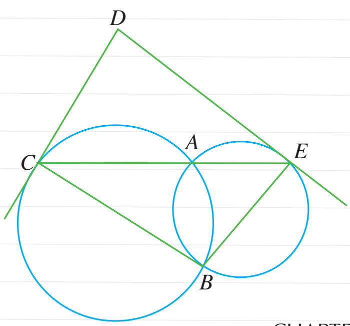
(ii) ΔTAC is similar to ΔCDA.



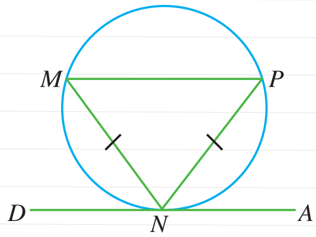
3 In the figure, CD is a diameter of the circle. AC and BD are tangents to the circle at C and D respectively. BC and AD intersect at a point P which lies on the circumference of the circle. Prove that CA is parallel to DB. Hence, show that $CD^2 = AC \times BD$.



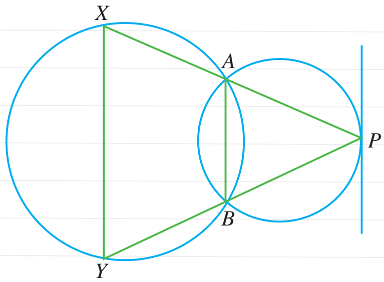
4 Two circles intersect at A and B. CD and DE are tangents to the circles at C and E respectively and CAE is a straight line. Prove that a circle can be drawn to pass through C, D, E and B.



5 DNA is a tangent to the circle at N. Given that $MN = PN$, prove that MP // DA.



6 A and B are the points of intersection of the two circles. PAX and PBY are straight lines. Prove that the tangent to the smaller circle at P is parallel to XY.



7 In the figure, TA and TB are tangents to the circle, centre O, from T.

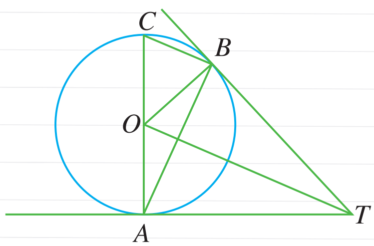
(i) Given that AOC is a diameter of the circle, state a reason why $\angle AOB = 2\angle ACB$.

Hence, prove that

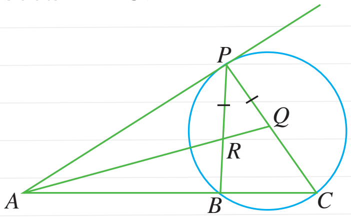
(ii) CB is parallel to OT,

(iii) a circle with a diameter OT can be drawn to pass through A and B,

(iv) ΔOBC is similar to ΔTAB.



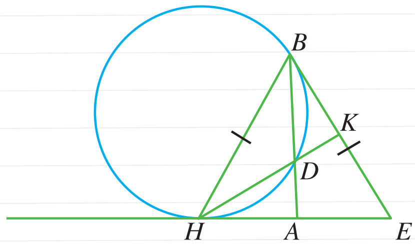
8 In the figure, AP is the tangent to the circle PBC at P. Given that ARQ and ABC are straight lines and $PQ = PR$, prove that ARQ bisects $\angle PAC$.



9 In the figure, HAE is the tangent to the circle at H, $BH = BE$ and KH is the angle bisector of $\angle BHE$ and it cuts the circle at D. Given that BD produced meets HE at A, prove that

(i) $HD = BD$,

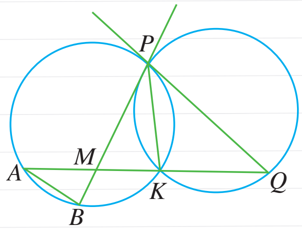
(ii) a circle can be drawn to pass through A, D, K and E.



10 In the figure, BP is a tangent to the circle PQK and PQ is a tangent to the circle $PKBA$. Given that $AMKQ$ is a straight line, prove that

(i) AB is parallel to PQ,

(ii) $MP \times AM = BM \times MQ$.

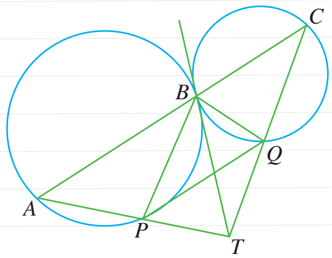


11 In the figure, TB is a common tangent to the two circles which touch each other externally at B. Given that ABC, APT and TQC are straight lines, prove that

(i) a circle can be drawn to pass through T, P, B and Q,

(ii) ΔTPQ is similar to ΔTCA,

(iii) $\angle BAP + \angle PQC = 180°$.

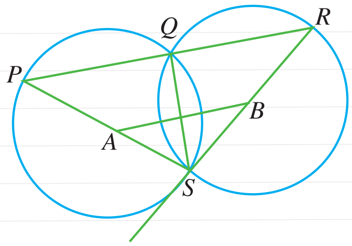


12 In the figure, A and B are the centres of the circles that intersect at Q and S. PAS and RBS are the diameters of the circles and RBS is the tangent to the circle PQS. Prove that

(i) PQR is a straight line,

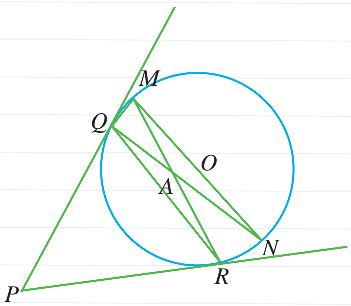
(ii) $PR = 2AB$,

(iii) $QS^2 = PQ \times QR$.

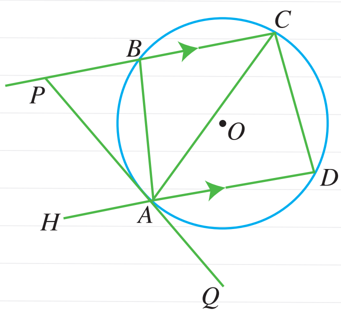


Exercise 10D

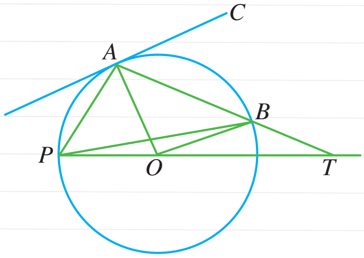
13 PQ and PR are tangents to the circle and MN is a diameter. Prove that
(i) $\angle MAN = 90° + \angle PQR$,
(ii) $\angle QPR + 2\angle MAN = 360°$.



14 In the figure, O is the centre of the circle. PAQ is the tangent to the circle at A, PBC is parallel to HAD.
(i) Name an angle equal to $\angle ACD$, explaining your answer clearly.
(ii) Prove that $\angle APB = \angle ACD$.
(iii) Prove that ΔACD is similar to ΔAPB.

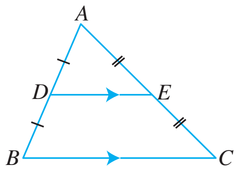


15 CA is a tangent to the circle, centre O, at A. ABT and POT are straight lines. Given that BT is equal to the radius of the circle, prove that
(i) $\angle ABP = 3\angle OBP$,
(ii) $\angle POA = 3\angle BOT$,
(iii) $\angle OBT = 2(\angle BTO + \angle CAB)$



SUMMARY

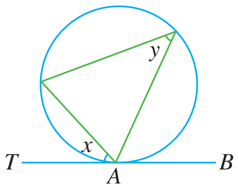
1. Midpoint Theorem



If D and E are the midpoints of AB and AC respectively, then

- $DE \, // \, BC$
- $DE = \frac{1}{2} BC$

2. Tangent-Chord Theorem (or **Alternate Segment Theorem**)

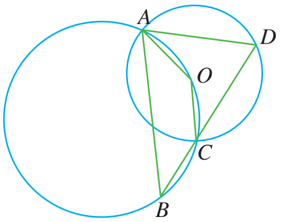


If TAB is a tangent to the circle at A, then

- $\angle x = \angle y$

Review Exercise 10

1. In the figure, O is the centre of the smaller circle ADC and A, O, C and B lie on the larger circle. Given that BCD is a straight line, prove that



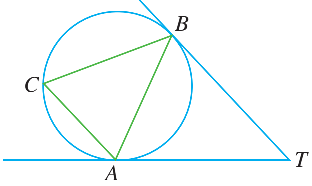
(i) $\angle ABD + 2\angle ADB = 180°$,

(ii) $AB = BD$.

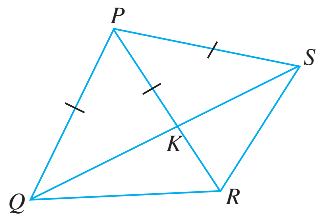
2. In the figure, TA and TB are tangents to the circle and AB is the angle bisector of $\angle TAC$. Prove that

(i) ΔTAB is similar to ΔBAC,

(ii) $AB = BC$.



3. In the figure, $PQ = PR = PS$.



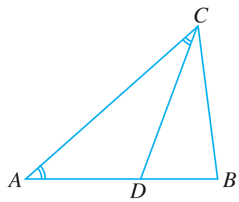
(i) Explain why $\angle PKS = \angle PRS + \angle QSR$.

(ii) Prove that $\angle QSR = \frac{1}{2}\angle QPR$.

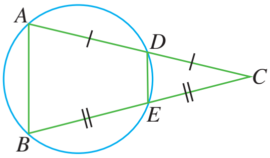
4. In ΔABC, CD is the bisector of $\angle ACB$ and $\angle CAD = \angle ACD$. Prove that

(i) $BC^2 = AB \times BD$,

(ii) $AC \times DB = BC \times CD$.



5. In the circle, chords AD and BE are produced to meet at C such that $AD = DC$ and $BE = EC$. Prove that $AC = BC$.

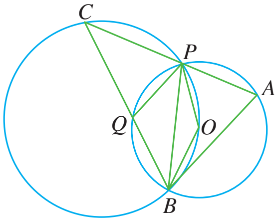


6. In the figure, O is the centre of the smaller circle $APQB$. APC and CQB are straight lines.

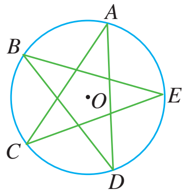
(i) Prove that $\angle CAB = \angle CBA$.

(ii) Explain why $\angle CPQ = \angle CQP$. Hence, show that ΔCPQ is isosceles.

(iii) Hence, or otherwise, show that PQ is parallel to AB.



7. AC, AD, BD, BE and CE are chords in the circle, centre O. Prove that the sum of the acute angles $\angle A$, $\angle B$, $\angle C$, $\angle D$ and $\angle E$ is 180°.

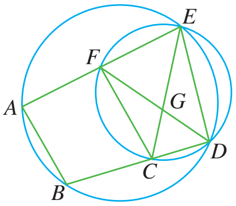


8. Two circles intersect at the points D and E. The points A and B lie on the larger circle while C and F lie on the smaller circle. BCD and AFE are straight lines. Prove that

(i) AB // FC,

(ii) $GE : GD = EF : CD$,

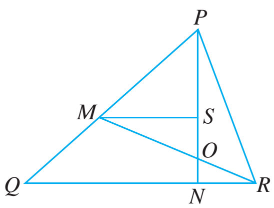
(iii) $CG = CD$, if $EF = FG$.



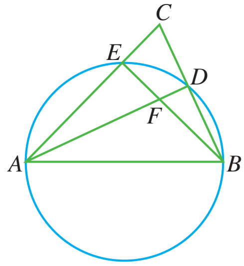
9. In ΔPQR, M is the midpoint of PQ and S is the midpoint on PN. PN and MR intersect at O. Given that $OR : OM = 1 : 2$, show that

(i) ΔNOR is similar to ΔSOM,

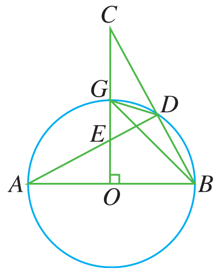
(ii) $OP = 5NO$.



10. AB is a diameter of the circle. Chords AD and BE intersect at F. Given that $AF = BC$, prove that ΔAFE is congruent to ΔBCE. Hence, or otherwise, prove that ΔEAB is an isosceles triangle.



11. AB is a diameter of the circle, centre O. C is a point on OG produced and CB intersects the circle at D. It is also given that OG is perpendicular to AB and OG intersects the chord AD at E. Using similar triangles, prove that $AE \times ED = OE \times EC$.

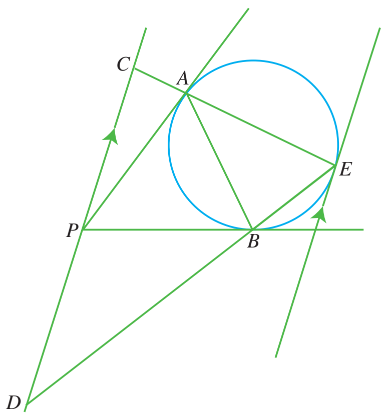


12. In the figure, PA and PB are tangents to the circle at A and B respectively. Line DPC is parallel to the tangent at E. CAE and DBE are straight lines.
Prove that

(i) $PA = PB$,

(ii) $PA = PC$,

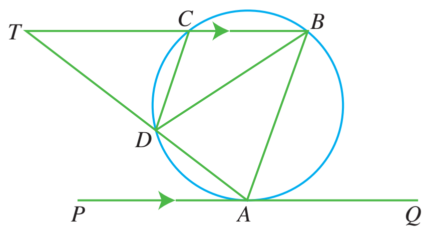
(iii) $PB = PD$.



13. In the figure, PAQ is the tangent to the circle at A. The line TCB is parallel to PAQ and TDA is a straight line. Prove that

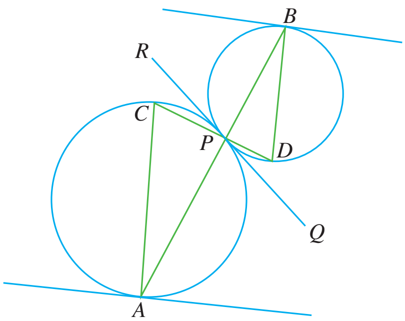
(i) $\angle ADB = \angle TDC$,

(ii) ΔTCD is similar to ΔBAD.

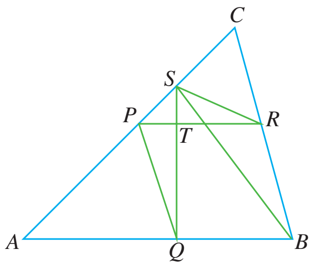


Challenge Yourself

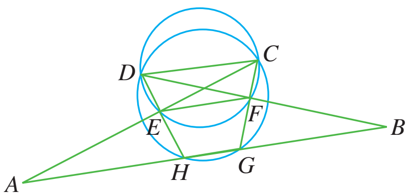
1. Two circles touch each other externally at P such that RPQ is a common tangent. Given that APB and CPD are straight lines, explain why
   (i) the diameters AC and BD are parallel,
   (ii) the line APB is perpendicular to the line CPD,
   (iii) the tangents to the circles at A and B are parallel.



2. In ΔABC, P, Q and R are the midpoints of AC, AB and BC respectively. Lines PR and QS intersect at T. Given that $BS \perp AC$, show that $\angle RPQ = \angle RSQ$.



3. Two circles intersect at C and D. $AHGB$, DEH and CFG are straight lines. Show that ΔAEH and ΔDFC are similar.



REVISION EXERCISE D1

1. Given that x and y are angles in the same quadrant such that $\sin x = \frac{12}{13}$ and $\tan y = -\frac{3}{4}$, find the value of

   (i) $\frac{1}{\sin y} + \frac{1}{\cot x}$, (ii) $\frac{\sin y \cos x}{\cos y + \sin x \sin y}$.

2. Solve each of the following equations for $0° \leqslant x \leqslant 360°$.

   (a) $\sin (2x + 40°) = \cos 70°$ (b) $\cot x \cos x = 1 + \sin x$

   (c) $3 \sin x + 2 \cos x = 3$ (d) $\sin x \cos x = \sqrt{3} \sin^2 x$

3. Prove each of the following identities.

   (a) $\frac{\sin \theta}{1 + \cos \theta} = \operatorname{cosec} \theta - \cot \theta$ (b) $\frac{\cos (A - B) - \cos (A + B)}{\sin (A - B) + \sin (A + B)} = \tan B$

4. Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.

   (a) $2 \sin \frac{x}{2} = 0.25$ (b) $\sin^2 x = \frac{1}{4} \tan x$

5. Given that $\frac{\cos^2 x}{1 + 2 \sin^2 x} = \frac{1}{3}$, where $90° < x < 180°$, find the value of $\frac{2 \sin x}{1 + \cos x}$.

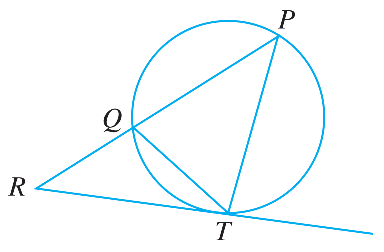
6. Sketch, on the same diagram, the graphs of $y = 1 - 2 \sin x$ and $y = 2 \cos 2x$, for $0 \leqslant x \leqslant 2\pi$. Hence, state the number of solutions in this interval of the equation $\sin x + \cos 2x = \frac{1}{2}$.

7. Express $3 \sin \theta + 6 \cos \theta$ in the form $R \sin (\theta + \alpha)$, where R is a positive constant and α is acute. Hence, or otherwise,

   (i) solve the equation $3 \sin \theta + 6 \cos \theta = 1$ for $0° \leqslant \theta \leqslant 360°$,

   (ii) find the maximum and minimum values of $3 \sin \theta + 6 \cos \theta$ and the corresponding values of θ.

8. In the figure, the tangent to the circle at T meets PQ produced at R. By using similar triangles, prove that $\frac{TP}{TQ} = \frac{TR}{QR}$.



REVISION EXERCISE D2

1. Given that $\sin A = \frac{3}{5}$, $\sin B = \frac{1}{\sqrt{2}}$ and that both A and B are acute, find, without using a calculator, the value of

(i) $\sin (A + B)$, **(ii)** $\tan (A - B)$, **(iii)** $\tan 2A$.

2. Solve each of the following equations for $0° \leqslant x \leqslant 360°$.

(a) $3 \cos (x - 60°) = 1$

(b) $2 \sec^2 x - \tan x = 5$

(c) $\sin x \cos x = \frac{1}{4}$

(d) $6 \sin x \cos x + 2 \sin x - 3 \cos x = 1$

3. Prove each of the following identities.

(a) $\frac{1 - \cos 2\theta}{1 + \cos 2\theta} = \tan^2 \theta$

(b) $\frac{\sec^2 x + 2 \tan x}{1 + 2 \sin x \cos x} = \sec^2 x$

4. Solve each of the following equations for $0 \leqslant x \leqslant 2\pi$.

(a) $4 \sin x + 3 \cos x = 1$

(b) $\cos 2x + 3 \cos x + 2 = 0$

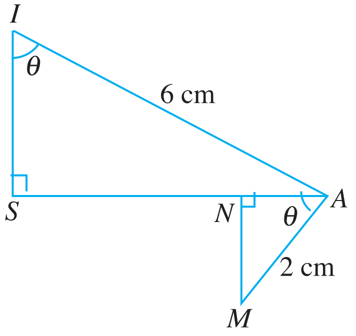
5. By expressing $\sin 3x$ as $\sin (2x + x)$, show that $\sin 3x = 3 \sin x - 4 \sin^3 x$. Hence, solve the equation $\sin 3x = 2 \sin x$ for $0° \leqslant x \leqslant 360°$.

6. Sketch, on the same diagram, the curves $y = \cos 2x$ and $y = |2 \sin x|$ for the interval $0° \leqslant x \leqslant 360°$, labelling each curve. State the number of solutions in this interval of the equation $|2 \sin x| = \cos 2x$.

7. In the figure, $AI = 6$ cm, $AM = 2$ cm and $\angle SIA = \angle MAN = \theta$ radian. IS and MN are each perpendicular to SA. Given that $P = IS + SN + NM$, show that $P = 8 \sin \theta + 4 \cos \theta$.

(i) Express P in the form $R \sin (\theta + \alpha)$, where $R > 0$ and α is acute. Find the two possible values of θ for which $P = 8.5$.

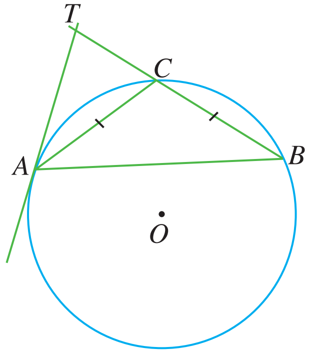
(ii) Find the maximum value of P and the value of θ at which the maximum value occurs.



8. In the figure, O is the centre of the circle and AB is a chord. C is the point on the circumference such that $AC = BC$. BC produced meets the tangent to the circle at A at the point T. Prove that

(i) AC bisects the angle TAB,

(ii) $\angle ACT = 2\angle TAC$.



DIFFERENTIATION & ITS APPLICATIONS

CHAPTER 11

The basic principles of calculus (which consists of differentiation and integration) was independently laid by Isaac Newton (1643-1727) and Gottfried von Leibniz (1646-1716) in the 17^{th} century. Calculus has many applications in the sciences and in real-world contexts which we will learn later in this chapter and in the next few chapters. In this chapter, we will learn how to apply the rules of differentiation as well as to solve problems involving rates of change.

Learning Objectives

At the end of this chapter, you should be able to:

- apply the rules of differentiation to differentiate algebraic expressions,
- find the equations of the tangent and the normal to a curve at a particular point,
- solve problems involving rates of change.

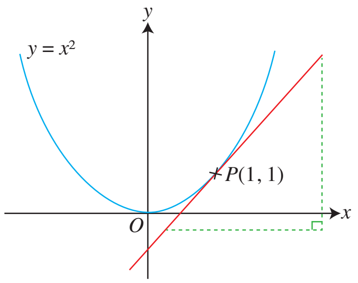
11.1 GRADIENT FUNCTIONS

Recap

We have learnt how to find the gradient of a straight line by using the following formulae:

$$\text{Gradient of a straight line} = \frac{\text{vertical change}}{\text{horizontal change}} \text{ or } \frac{\text{rise}}{\text{run}} \text{ or } \frac{y_2 - y_1}{x_2 - x_1}.$$

We have also learnt how to find the gradient of a curve at a point by drawing a **tangent** to the curve at that point before finding the gradient of the tangent. For example, to find the gradient of the curve $y = x^2$ at the point $P(1, 1)$, we have to plot the curve on a sheet of graph paper, then draw the tangent to the curve at the point $P(1, 1)$. Finally, we find the gradient of the tangent using the above formulae for the gradient of a straight line.

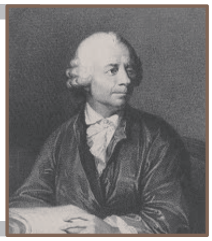


What happens if we want to find the gradient of the curve $y = x^2$ at another point $Q(2, 4)$?
Is this method of finding the gradient of a curve at any point by drawing a tangent each time very tedious?

Moreover, the answer obtained is only an *estimate* due to the inaccuracy of plotting the points on a curve and drawing a tangent.

Therefore, there is a need to learn a more accurate method of finding the gradient of a curve at *any* point. This method is called **differentiation**.

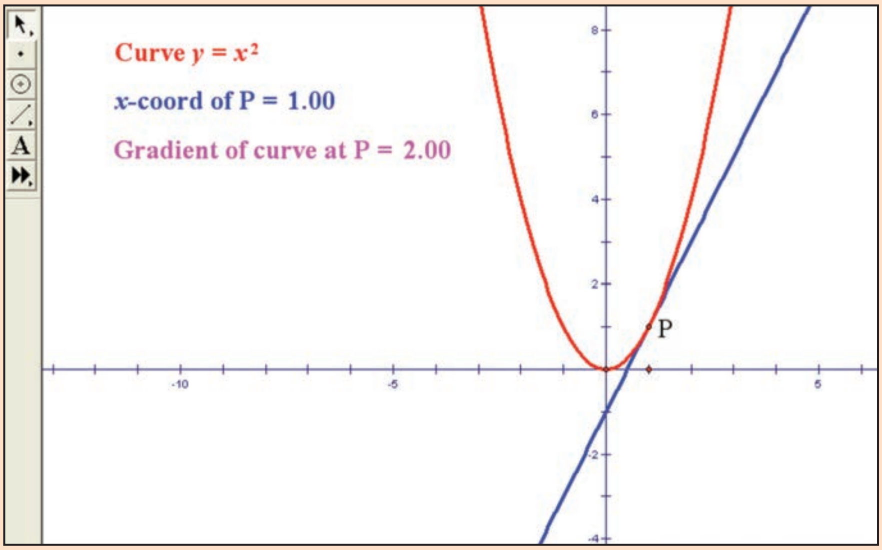
Leonhard Euler (1707-1783) was a great Swiss mathematician and physicist famous for many contributions such as his work on infinitesimal calculus, which involves finding slopes of curves and other problems. Euler made important contributions to calculus and trigonometry in their modern form. He also showed the importance of the number e in the study of various branches of mathematics.



Investigation

Gradient of a Curve

Go to http://www.shinglee.com.sg/StudentResources and open the template *Gradient of Curve* as shown below:



1. Click on the point P and drag to change the point of contact between the curve and the tangent. Copy and complete the table below to find the gradient of the curve $y = x^2$ at the point P.

x-coordinate of P	–4	–3	–2	–1	0	1	2	3	4
Gradient of curve at P	–8	–6			0				

2. Find a general formula, in terms of x, for the gradient of the curve $y = x^2$ at any point $P(x, y)$.

3. Using the formula in question 2, find the gradient of the curve $y = x^2$ at (5, 25) and at (–6, 36).

Derivatives and Gradient Functions

The process of finding the gradient of a curve at any point in the investigation above is called **differentiation**. The formula is called the **derivative** of y with respect to x and is denoted by $\frac{\mathrm{d}y}{\mathrm{d}x}$ (read as 'dee y dee x').

The formula is also called the **gradient function** $f'(x)$ (read as 'f prime x') of the curve $y = f(x)$ because it can be used to find the gradient of the curve at any point.

There are three different notations to describe the above:

(1) If $y = x^2$, then $\frac{\mathrm{d}y}{\mathrm{d}x} = 2x$.

(2) If $f(x) = x^2$, then $f'(x) = 2x$.

(3) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2) = 2x$, i.e. if we *differentiate* x^2 with respect to x, the result is $2x$.

ATTENTION

$\frac{\mathrm{d}y}{\mathrm{d}x}$ is *not* dividing dy by dx.

ATTENTION

The *gradient* of a curve at a particular point is a numerical value while the *gradient function* of a curve is a function.

Investigation

Derivative of x^n

From the previous investigation, we have discovered that $\frac{d}{dx}\left(x^2\right) = 2x$.

Using the same method as the previous investigation for $y = x^3$, $y = x^4$, $y = x^5$ and $y = x^6$, the following results are obtained:

$$\frac{d}{dx}\left(x^3\right) = 3x^2; \quad \frac{d}{dx}\left(x^4\right) = 4x^3; \quad \frac{d}{dx}\left(x^5\right) = 5x^4; \quad \frac{d}{dx}\left(x^6\right) = 6x^5.$$

1. What do you think $\frac{d}{dx}\left(x^7\right)$ is equal to?

2. What do you think $\frac{d}{dx}\left(x^8\right)$ is equal to?

3. In general, if n is a positive integer, what is $\frac{d}{dx}\left(x^n\right)$ equal to?

4. Using the formula in Question 3, what is $\frac{d}{dx}(x)$ equal to? Does your answer make sense?
Hint: Consider $y = x$.

5. Does this rule apply if n is
(a) a negative integer?
(b) a real number?

Thinking Time

Refer to the investigation above.

(a) What does the derivative represent?

(b) We observe that the tangent to the curve at $x = 0$ is horizontal. What is the value of the derivative when the tangent is horizontal?

(c) Consider the values of the derivative from $x = 0$ to $x = 4$. Notice that as the value of the derivative increases, the gradient of the tangent also increases. What can we say about the derivative if the tangent is vertical?

Differentiation of Power Functions

In general,

$$\boxed{\frac{d}{dx}\left(x^n\right) = nx^{n-1}, \text{ where } n \text{ is a real number.}}$$

ATTENTION

How to remember this formula: 'bring down the power n and subtract one from the current power n to get the new power'.

Worked Example 1

(Differentiation of Power Functions)

Find the derivative of each of the following.

(a) $\dfrac{1}{x^2}$ **(b)** $y=\sqrt{x}$ **(c)** $f(x) = 1$

Solution

(a) $\dfrac{d}{dx}\left(\dfrac{1}{x^2}\right)=\dfrac{d}{dx}\left(x^{-2}\right)$

$= -2x^{-2-1}$

$= -2x^{-3}$

$= -\dfrac{2}{x^3}$

(b) $y=\sqrt{x}$

$= x^{\frac{1}{2}}$

$\dfrac{dy}{dx}=\dfrac{1}{2}x^{\frac{1}{2}-1}$

$=\dfrac{1}{2}x^{-\frac{1}{2}}$

$=\dfrac{1}{2\sqrt{x}}$

(c) $f(x) = 1$

$= x^0$

$f'(x) = 0x^{0-1}$

$= 0$

Practise Now 1

Similar Questions:
Exercise 11A
Questions 1(a), 2(a)

Find the derivative of each of the following.

(a) $\dfrac{1}{x}$ **(b)** $f(x)=\sqrt[3]{x}$

11.2 FIVE RULES OF DIFFERENTIATION

Rule 1:

Consider the function $kf(x)$, where k is a constant and $f(x)$ is a function. We call k a **scalar multiple**.

This rule states that $\boxed{\dfrac{d}{dx}\left[kf(x)\right]=k\dfrac{d}{dx}\left[f(x)\right].}$

ATTENTION

How to remember this rule: 'if k is a scalar multiple, leave k alone when you differentiate'.

Rule 2:

If $f(x)$ and $g(x)$ are functions, then

$$\boxed{\frac{d}{dx}\left[f(x)\pm g(x)\right]=\frac{d}{dx}\left[f(x)\right]\pm\frac{d}{dx}\left[g(x)\right].}$$

Rule 2 is also known as the **Addition/Subtraction Rule**.

ATTENTION

How to remember this rule: 'for the sum or difference of functions, just differentiate each function separately'.

Worked Example 2

(Differentiation of Power Functions)

Differentiate each of the following with respect to x.

(a) $-\frac{3}{8x^4} - 6x + 7$ **(b)** $3x^4 + 5x^3 - \frac{4}{\sqrt{x}}$

Solution

(a) $\frac{d}{dx}\left(-\frac{3}{8x^4} - 6x + 7\right) = \frac{d}{dx}\left(-\frac{3}{8}x^{-4} - 6x^1 + 7x^0\right)$ (Addition/Subtraction Rule)

$= -\frac{3}{8}\frac{d}{dx}(x^{-4}) - 6\frac{d}{dx}(x^1) + 7\frac{d}{dx}(x^0)$ (differentiate each term separately)

$= -\frac{3}{8}(-4x^{-5}) - 6\frac{d}{dx}(1x^0) + 7(0x^{-1})$

$= \frac{3}{2x^5} - 6$

(b) Let $f(x) = 3x^4 + 5x^3 - \frac{4}{\sqrt{x}} = 3x^4 + 5x^3 - 4x^{-\frac{1}{2}}$.

$f'(x) = 3(4x^3) + 5(3x^2) - 4\left(-\frac{1}{2}x^{-\frac{3}{2}}\right)$

$= 12x^3 + 15x^2 + 2x^{-\frac{3}{2}}$

$= 12x^3 + 15x^2 + \frac{2}{\sqrt{x^3}}$

Practise Now 2

Similar Questions:
Exercise 11A
Questions 1(b)-(l), 2(b)-(d)

1. Differentiate each of the following with respect to x.

   (a) $\frac{4}{9x^3} + ax$, where a is a constant **(b)** $\frac{3}{x} + \sqrt[3]{x} + 2013$

2. Differentiate $\frac{8}{\sqrt{t}} + \frac{3t}{\sqrt{t}}$ with respect to t.

Worked Example 3

(Finding the Gradient of a Curve)

Find the gradient of the curve $y = (2x - 1)(x + 2)$ at the point (2, 12).

Solution

$y = (2x - 1)(x + 2)$

$= 2x^2 + 3x - 2$

$\frac{dy}{dx} = 4x + 3 - 0$ (differentiate each term separately)

When $x = 2$, $\frac{dy}{dx} = 4(2) + 3$

$= 11$

$\therefore$ Gradient of curve at (2, 12) = 11

ATTENTION

You can differentiate each *term* separately (Addition and Subtraction Rule), but you cannot differentiate each *factor* separately, i.e. $\frac{d}{dx}[(2x-1)(x+2)] \neq (2)(1) = 2$.

Practise Now 3

Similar Questions:
Exercise 11A
Questions 3(a)-(e), 4, 7, 8

1. Find the gradient of the curve $y = (3x - 2)(x + 4)$ at the point (3, 49).
2. Find the gradient of the curve $y = \frac{3x - 9}{2x^2}$ at the point where the curve crosses the x-axis.

Worked Example 4

(Application of Differentiation)
Given that the gradient of the tangent to the curve $y = ax^3 + bx^2 + 3$, at the point (1, 4) is 7, calculate the value of the constants a and b.

Solution

$y = ax^3 + bx^2 + 3$

$\frac{dy}{dx} = 3ax^2 + 2bx$

Since the gradient of the tangent at (1, 4) is 7,

$7 = 3a(1)^2 + 2b(1)$

$3a + 2b = 7$ ---------------------(1)

Since (1, 4) is a point on the curve,

$4 = a(1)^3 + b(1)^2 + 3$

$a + b = 1$ ---------------------(2)

(2) × 2: $2a + 2b = 2$ ---------------------(3)

(1) – (3): $\therefore a = 5$

Substitute $a = 5$ into (2):

$5 + b = 1$

$\therefore b = -4$

Practise Now 4

Similar Questions:
Exercise 11A
Questions 9-11

1. The gradient of the curve $y = ax + \frac{b}{x^2}$ at the point (1, 11) is 2. Calculate the value of a and of b.
2. The curve $y = ax^2 + bx$ has gradients 4 and 14 at $x = 1$ and $x = 2$ respectively. Calculate the value of a and of b.

Basic Level | Intermediate Level | Advanced Level

Exercise 11A

1 Differentiate each of the following with respect to x.

(a) x^3

(b) $10x^5 - 4x^3 + 7x$

(c) $5x^4 + \frac{1}{x} - \frac{1}{x^2}$

(d) $\frac{7}{2}x^4 + \frac{5}{3x^4} + x$

(e) $3x^4 + \frac{1}{3x^4} + \frac{3}{2x^2}$

(f) $(2x + 3)^2$

(g) $(x^2 - 1)^2$

(h) $(x^3 + 2)(x^2 - 1)$

(i) $\left(x^2 + \frac{2}{x}\right)^2$

(j) $\frac{x^2 + 5}{x}$

(k) $\frac{3x^2 + x}{\sqrt{x}}$

(l) $5x^3 + \frac{3}{x} - \frac{1}{\sqrt{x}}$

2 Find the value of $\frac{dy}{dx}$ at the given value of x.

(a) $y = \frac{1}{x^3}$, $x = 1$

(b) $y = 4x^2 - 5\sqrt{x} - \frac{1}{x}$, $x = 4$

(c) $y = (2x + 1)(3x - 4)$, $x = 3$

(d) $y = \frac{x^2 + x + 4}{\sqrt{x}}$, $x = 9$

Exercise 11A

3 Calculate the gradient of the curve at the point where it crosses the given line.

(a) $y = 3x^2 - 4x + 3$, $x = 2$

(b) $y = 5x + \dfrac{1}{x^2}$, $x = 3$

(c) $y = 4x\left(x - \sqrt{x}\right)$, $x = 1$

(d) $y = \dfrac{x-4}{x}$, $y = 3$

(e) $y = \dfrac{2x-3}{x}$, $y = 5$

4 Find the coordinates of the point on the curve $y = x^2 - 5x + 7$ at which the gradient is 3.

5 Differentiate each of the following, where a, b and c are constants, with respect to x.

(a) $5ax^2 + 3bx^3 + 5$ **(b)** $2a^2x + 5a^3b + c^2x^3$

(c) $5a^2 + \dfrac{b}{x^2} + \dfrac{c^2}{b}$ **(d)** $\dfrac{ax^2 + bx}{\sqrt{x}}$

6 Differentiate each of the following with respect to t.

(a) $\dfrac{3t^2 - 4t^3}{t^3}$ **(b)** $\dfrac{3\sqrt{t} - 4}{\sqrt{t}}$

(c) $\sqrt{t} + \dfrac{1}{\sqrt{t}}$ **(d)** $\left(\sqrt{t} + \dfrac{2}{\sqrt{t}}\right)^2$

(e) $\dfrac{7}{2t^4} + \dfrac{2}{\sqrt{t}} + \dfrac{2t}{\sqrt[3]{t}}$ **(f)** $4t + \dfrac{2}{3\sqrt{t}} + 7$

7 Find the gradient of the curve $y = \dfrac{5x-4}{x^2}$ at the point where the curve crosses the x-axis.

8 Find the gradient of the curve $y = 3x^2 - 5x - 1$ at the point where the curve crosses the y-axis.

9 The gradient of the curve $y = ax^2 + bx$, where a and b are constants, at the points $x = 1$ and $x = 3$ are -2 and 10 respectively. Find the value of a and of b.

10 The gradient of the curve $y = \dfrac{a}{x^2} + \dfrac{b}{x}$ at the point $(-1, 5)$ is 4. Find the values of the constants a and b.

11 The gradient of the curve $y = ax - \dfrac{b}{x}$ at the point $\left(\dfrac{1}{2}, -\dfrac{13}{2}\right)$ is 3. Find the values of the constants a and b.

12 Given that $pV = 3600$, find the value of $\dfrac{dp}{dV}$ when $p = 40$.

13 Given that $y = x^3 - 6x^2 + 9x + 5$, find the range of values of x for which $\dfrac{dy}{dx} > 0$.

14 Given that $v = 3t^2 + 5t - 7$, find $\dfrac{dv}{dt}$ and the range of values of t for which $\dfrac{dv}{dt}$ is negative.

Rule 3:

Consider $y = (3x^2 + 1)^{10}$. How do we find its derivative?
As it is very tedious to expand $(3x^2 + 1)^{10}$ before differentiating each term separately, there is a need to learn a more efficient method that allows us to find the derivative without expanding.

Let $u = 3x^2 + 1$. Then u is a function of x.
Now $y = (3x^2 + 1)^{10}$ becomes $y = u^{10}$, i.e. y is a function of u.
Therefore, y is a **composite function** of u and x.

To find the derivative of a composite function,

$$\boxed{\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}.}$$

This is also known as the **Chain Rule**.

Worked Example 5

(Application of Chain Rule)
Find the derivative of $y = (3x^2 + 1)^{10}$.

Solution

Let $u = 3x^2 + 1$. Then $\frac{du}{dx} = 6x$.

Now $y = (3x^2 + 1)^{10}$ becomes $y = u^{10}$, so $\frac{dy}{du} = 10u^9$.

$$\begin{aligned} \therefore \frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= 10u^9 \times 6x \\ &= 10(3x^2 + 1)^9 \times 6x \text{ ------------(*)} \\ &= 60x(3x^2 + 1)^9 \end{aligned}$$

In fact, from (*), we observe that we should *differentiate from the 'outside' to the 'inside'*.

$y = (\underbrace{3x^2 + 1}_{})^{10}$ differentiate 'outside' (power) first

then differentiate 'inside' $(3x^2 + 1)$

$$\begin{aligned} \frac{dy}{dx} &= 10(3x^2 + 1)^9 \times 6x \\ &= 60x(3x^2 + 1)^9 \end{aligned}$$

Practise Now 5

Similar Questions:
Exercise 11B
Questions 1-11

1. Find the derivative of each of the following.
   (a) $y = (4x^3 - 1)^7$ **(b)** $8\sqrt{x^2 + 3x - 4}$ **(c)** $f(x) = \frac{6}{(2 - 5x^4)^3}$

2. Find the gradient of the curve $y = (3x - 8)^6$ at the point where $x = 3$.

Basic Level | Intermediate Level | Advanced Level

Exercise 11B

1 Differentiate each of the following with respect to x.

(a) $(2x+5)^7$ **(b)** $(3-4x)^8$

(c) $\left(\frac{1}{2}x-4\right)^5$ **(d)** $\frac{(5x-3)^6}{8}$

(e) $3(x+4)^5$ **(f)** $\frac{2}{3}\left(\frac{x}{6}-1\right)^4$

2 Differentiate each of the following with respect to x.

(a) $\frac{1}{3x+2}$ **(b)** $\frac{1}{(x^2+2)^3}$

(c) $\frac{5}{(3-4x)^3}$ **(d)** $\frac{6}{2-5x}$

(e) $\frac{12}{2+3x^2}$ **(f)** $\frac{3}{4(5-3x^2)}$

3 Differentiate each of the following with respect to x.

(a) $\sqrt{2x+3}$ **(b)** $\sqrt{4+3x}$

(c) $\sqrt{5x^2+6}$ **(d)** $\sqrt{3x^3-4}$

(e) $\sqrt{2x^2-4x+5}$ **(f)** $\sqrt[3]{2x^2-5}$

(g) $\frac{2}{\sqrt{x-3}}$ **(h)** $\sqrt{18-5x^2}$

4 Find the gradient of the curve $y=(3x^2-5x+3)^3$ at the point where $x=1$.

5 If $y=2t+5+\frac{4}{5t-9}$, find the value of $\frac{dy}{dt}$ where $t=2$.

6 Find the gradient of the curve $y=2+\frac{12}{(3x-4)^2}$ at the point (2, 5).

7 Find the gradient of the curve $y=\frac{24}{(3x-5)^2}$ at the point (2, 24).

8 Differentiate each of the following with respect to x.

(a) $\left(x^2+\frac{3}{x}\right)^5$ **(b)** $\frac{2}{3\sqrt{2x^2-5}}$

(c) $\left(x+\frac{2}{x}\right)^{\frac{1}{3}}$ **(d)** $\frac{6}{\sqrt{2x^2+7x}}$

9 Calculate the coordinates of the point on the curve $y=\sqrt{x^2-4x+8}$ at which $\frac{dy}{dx}=0$.

10 The curve $y=\frac{a}{2+bx}$ passes through the point (1, 1) and the gradient at that point is $\frac{3}{5}$. Calculate the values of the constants a and b.

11 Differentiate each of the following, where a, b and c are constants, with respect to x.

(a) $(ax^2+bx^3+c)^5$ **(b)** $\sqrt{ax^2+bx+c}$

(c) $\frac{1}{\sqrt{a^2-2x}}$ **(d)** $\frac{2}{\sqrt{x^2-3a^2+2b}}$

(e) $\left(ax+\frac{b}{x}\right)^5$ **(f)** $\left(a^2x^2+\frac{b^3}{x^2}\right)^4$

Rule 4:

Consider $y = (3x + 2)\sqrt{4x-1}$. How do we find its derivative?
We cannot differentiate each factor separately.
Neither can we expand $(3x + 2)\sqrt{4x-1}$ before differentiating each term separately. As a result, there is a need to learn another method.

If u and v are functions of x, then

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}\,.}$$

This is known as the **Product Rule**.

ATTENTION

$$\frac{\mathrm{d}}{\mathrm{d}x}(uv) \neq \frac{\mathrm{d}u}{\mathrm{d}x} \times \frac{\mathrm{d}v}{\mathrm{d}x}$$

ATTENTION

How to remember this rule: 'first function × derivative of second function + second function × derivative of first function'.

Worked Example 6

(Application of Product Rule)
Find the derivative of $y = (3x + 2)\sqrt{4x-1}$.

Solution

$$y = (3x+2)\sqrt{4x-1}$$

$$= (3x+2)(4x-1)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underbrace{(3x+2)}_{\text{first}}\ \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left[(4x-1)^{\frac{1}{2}}\right]}_{\text{differentiate second}} + \underbrace{(4x-1)^{\frac{1}{2}}}_{\text{second}}\ \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}(3x+2)}_{\text{differentiate first}}$$

$$= (3x+2) \times \underbrace{\frac{1}{2}\left[(4x-1)^{-\frac{1}{2}}(4)\right]}_{\text{apply Chain Rule}} + (4x-1)^{\frac{1}{2}}(3)$$

$$= \frac{2(3x+2)}{(4x-1)^{\frac{1}{2}}} + 3(4x-1)^{\frac{1}{2}}$$

$$= \frac{2(3x+2)+3(4x-1)}{(4x-1)^{\frac{1}{2}}} \quad \text{(combine into a single fraction)}$$

$$= \frac{18x+1}{\sqrt{4x-1}}$$

ATTENTION

$$(4x-1)^{\frac{1}{2}} = \sqrt{4x-1}$$

Practise Now 6

Similar Questions:
Exercise 11C
Questions 1(a)-(e), 6(a)-(e)

1. Find the derivative of each of the following.
(a) $y = (4x - 3)(2x + 7)^6$ **(b)** $\mathrm{f}(v) = (3 - v^2)\sqrt{5v+4}$

2. Differentiate each of the following with respect to x, where a and b are constants.
(a) $\dfrac{3x}{a - bx^2}$ **(b)** $(a^2 - x^2)(5ax - 3x^3)^4$

Worked Example 7

(Gradient involving Product Rule)

Calculate the gradient of the curve $y = x\sqrt{x+1}$ at the point where $x = 3$. Find the x-coordinate of the point where $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

Solution

$$y = x(x+1)^{\frac{1}{2}}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \underbrace{x}_{\text{first}} \times \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}\left[(x+1)^{\frac{1}{2}}\right]}_{\text{differentiate second}} + \underbrace{(x+1)^{\frac{1}{2}}}_{\text{second}} \times \underbrace{\frac{\mathrm{d}}{\mathrm{d}x}(x)}_{\text{differentiate first}}$$

$$= x \times \frac{1}{2}(x+1)^{-\frac{1}{2}}(1) + (x+1)^{\frac{1}{2}}(1)$$

$$= \frac{x}{2(x+1)^{\frac{1}{2}}} + (x+1)^{\frac{1}{2}}$$

$$= \frac{x}{2\sqrt{x+1}} + \sqrt{x+1}$$

$$= \frac{x + 2(x+1)}{2\sqrt{x+1}}$$

$$= \frac{3x+2}{2\sqrt{x+1}}$$

When $x = 3$, $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(3)+2}{2\sqrt{3+1}}$

$$= \frac{11}{4}$$

$$= 2\frac{3}{4}$$

When $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$, $\frac{3x+2}{2\sqrt{x+1}} = 0$

$$\therefore 3x + 2 = 0$$

$$x = -\frac{2}{3}$$

ATTENTION

A fraction is equal to 0 when its numerator is equal to 0.

Practise Now 7

Similar Questions:
Exercise 11C
Questions 2-5, 7-9

1. Find the x-coordinates of the points on the curve $y = (3x-2)^4(2x+5)^7$ where the gradients of the tangents to the curve at these points are equal to 0.

2. Find the x-coordinates of the points on the curve $y = (2x-3)^3(x+5)^5$ where the tangents are parallel to the x-axis.

Exercise 11C

1 Differentiate each of the following with respect to x.

(a) $(3x + 2)(2 - x^2)$ **(b)** $(x - 3)\sqrt{x + 4}$

(c) $(2x + 3)(x - 3)^4$ **(d)** $(x + 1)^3(x + 3)^5$

(e) $(x + 5)^3(x - 4)^6$

2 Calculate the gradients of the curve $y = (x - 1)^3(x + 3)^2$ at the points where $x = 0$ and $x = 2$.

3 Given that $y = (2x + 3)^5(x + 2)^8$, find the value of $\frac{dy}{dx}$ when $x = -1$.

4 Given that $y = (x + 3)^4(x - 5)^7$, find $\frac{dy}{dx}$ and the values of x for which $\frac{dy}{dx} = 0$.

5 Given that $y = (2x - 3)^3(x + 5)^5$, find the values of x for which $\frac{dy}{dx} = 0$.

6 Differentiate each of the following with respect to x.

(a) $3x\sqrt{4 - 7x^3}$

(b) $(x^2 - 3x)(x + 5)^6$

(c) $(x^3 + x^2)(x - 2)^7$

(d) $(3x - 1)\sqrt{2x^2 + 3}$

(e) $2x\sqrt{(3x - 1)^5}$

7 The equation of a curve is $y = (x - 5)\sqrt{2x - 1}$. Find $\frac{dy}{dx}$ and the x-coordinate of the curve where the gradient is zero.

8 Given that $y = (3x + 2)(2 - x)^{-1}$, find $\frac{dy}{dx}$ and the values of x for which $\frac{dy}{dx} = 8$.

9 Given that $y = (x + 4)\sqrt{9 - x}$, find $\frac{dy}{dx}$ and the value of x for which $\frac{dy}{dx} = 0$.

10 Differentiate each of the following, where a and b are constants, with respect to x.

(a) $(a^2 + x^2)\sqrt{a^2 - x^2}$

(b) $(3ax - b)^3(2ax - b^3)^2$

(c) $(2a^2x^3 + 3x)(2bx - x^3)^2$

(d) $\sqrt[3]{a^2 + x}\,(2ax + 3b)$

(e) $(x + 3ab)\sqrt{a^2 + x^2}$

(f) $(x + a)^3(x^2 - b)^4$

Rule 5:

Consider $y = \frac{4x^2 - 1}{(5x + 2)^3}$. How do we find its derivative?

One way is to rewrite it as $y = (4x^2 - 1)(5x + 2)^{-3}$ before applying the Product Rule. Another way is to apply the **Quotient Rule**, which states that:

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2},$$

where u and v are functions of x.

ATTENTION

$$\frac{d}{dx}\left(\frac{u}{v}\right) \neq \frac{\frac{du}{dx}}{\frac{dv}{dx}}$$

ATTENTION

How to remember this rule: 'denominator × derivative of numerator – numerator × derivative of denominator; divided by the square of the denominator'

Worked Example 8

(Application of Quotient Rule)

Find the derivative of $f(x) = \frac{2x+5}{3x-4}$.

Solution

$$f(x) = \frac{2x+5}{3x-4}$$

$$f'(x) = \frac{\underbrace{(3x-4)}^{\text{denominator}} \times \underbrace{\frac{d}{dx}(2x+5)}^{\text{differentiate numerator}} - \underbrace{(2x+5)}^{\text{numerator}} \times \underbrace{\frac{d}{dx}(3x-4)}^{\text{differentiate denominator}}}{\underbrace{(3x-4)^2}_{\text{square of the denominator}}}$$

$$= \frac{(3x-4)(2)-(2x+5)(3)}{(3x-4)^2}$$

$$= \frac{6x-8-6x-15}{(3x-4)^2}$$

$$= -\frac{23}{(3x-4)^2}$$

Practise Now 8

Similar Questions:
Exercise 11D
Questions 1(a)-(j), 5(a)-(d)

1. Find the derivative of each of the following.

(a) $f(x) = \frac{2x}{4x-3}$ **(b)** $y = \frac{x}{3x-7}$

2. Differentiate each of the following with respect to x.

(a) $\frac{2x^3-1}{(3x+1)^2}$ **(b)** $\frac{2x-7}{\sqrt[3]{2x-5}}$ **(c)** $\sqrt{\frac{1+4x}{3x^2-7}}$

3. Given that $y = \frac{\sqrt{x^2+25}}{x+5}$, calculate the value of $\frac{dy}{dx}$ when $x = 12$.

Worked Example 9

(Application of Quotient Rule)

Given that $y = \dfrac{4x-3}{x^2+1}$, find $\dfrac{dy}{dx}$ and determine the range of values of x for which both y and $\dfrac{dy}{dx}$ are positive.

Solution

$$\frac{dy}{dx} = \frac{(x^2+1)\times\frac{d}{dx}(4x-3)-(4x-3)\times\frac{d}{dx}(x^2+1)}{(x^2+1)^2}$$

$$= \frac{(x^2+1)\times(4)-(4x-3)\times(2x)}{(x^2+1)^2}$$

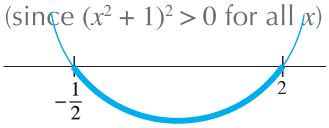
$$= \frac{4x^2+4-8x^2+6x}{(x^2+1)^2}$$

$$= \frac{4+6x-4x^2}{(x^2+1)^2}$$

For $y > 0$, we need $4x - 3 > 0$. (since $x^2 + 1 > 0$ for all x)

i.e. $x > \frac{3}{4}$ -------------------- (1)

For $\frac{dy}{dx} > 0$, we need $4 + 6x - 4x^2 > 0$. (since $(x^2 + 1)^2 > 0$ for all x)

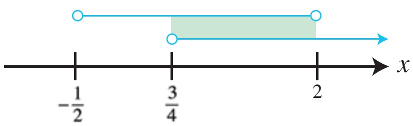


i.e. $2x^2 - 3x - 2 < 0$

$(2x + 1)(x - 2) < 0$

$\Rightarrow -\frac{1}{2} < x < 2$ --------- (2)

Combining (1) and (2),



we get $\frac{3}{4} < x < 2$.

Practise Now 9

Similar Questions:
Exercise 11D
Questions 2-4, 6-9

1. Given that $y = \dfrac{2x}{x^2+1}$, find $\dfrac{dy}{dx}$ and determine the range of values of x for which both y and $\dfrac{dy}{dx}$ are negative.

2. Given that $y = \dfrac{x-2}{x^2+5}$, find $\dfrac{dy}{dx}$ and determine the range of values of x for which both y and $\dfrac{dy}{dx}$ are positive.

Basic Level | Intermediate Level | Advanced Level

Exercise 11D

1 Differentiate each of the following with respect to x.

(a) $\frac{3x+2}{1-4x}$ (b) $\frac{2x-1}{3-x}$

(c) $\frac{3x}{2x-3}$ (d) $\frac{4x+7}{1-2x^2}$

(e) $\frac{2x+8}{3x^2-2}$ (f) $\frac{7x-2}{2x+3}$

(g) $\frac{x^2}{2x-1}$ (h) $\frac{x+1}{x^2+1}$

(i) $\frac{3}{2-x}$ (j) $\frac{x}{x^2+3}$

2 Calculate the gradient of the curve $y = \frac{3x^2-8}{5-2x}$ at the point (2, 4).

3 Given the curve $y = \frac{3x^2}{x+1}$, find the values of x for which $\frac{dy}{dx} = 0$.

4 Calculate the coordinates of the points on the curve $y = \frac{3x^2}{4x-1}$ for which the tangent is parallel to the x-axis.

5 Differentiate each of the following with respect to x.

(a) $\frac{\sqrt{x}}{3+x}$ (b) $\frac{2x}{\sqrt{x-3}}$

(c) $\frac{2x-7}{\sqrt{x+1}}$ (d) $\frac{\sqrt{x}}{x^2+4}$

6 Find the gradient of the tangent to the curve $y = \frac{2x-1}{x^2+3}$ at the point where the curve cuts the x-axis.

7 Given that $y = \frac{x-2}{x^2+5}$, find $\frac{dy}{dx}$ and determine the range of values of x for which both y and $\frac{dy}{dx}$ are positive.

8 Find the values of x for which the gradient of the curve $y = \frac{x-7}{x^2+3}$ is zero, giving your answers correct to 2 decimal places.

9 The line $2x + 9y = 3$ meets the curve $xy + y + 2 = 0$ at the points P and Q. Calculate the gradient of the curve at P and at Q.

10 Differentiate each of the following, where a, b and c are constants, with respect to x.

(a) $\frac{a^2+x^2}{a^2-x^2}$ (b) $\frac{2ax^2+bx}{b^2-3x^3}$

(c) $\frac{ax^2+bx+c}{ax^2-bx-c}$ (d) $\frac{ax^2-bx}{cx^2+bx}$

11 Differentiate each of the following, where a, b and c are constants, with respect to x.

(a) $\sqrt{\frac{a+x}{2a-x}}$ (b) $\frac{ax-b}{\sqrt{bx+c}}$

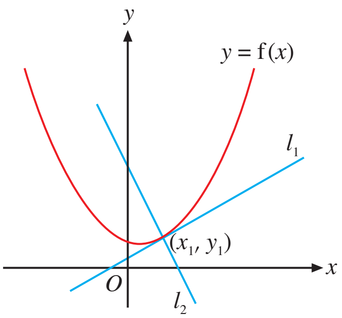
(c) $\frac{a-x}{\sqrt{2ax-x^2}}$ (d) $\sqrt{\frac{ax}{a^2+x^2}}$

11.3 EQUATIONS OF TANGENT AND NORMAL TO A CURVE

Recap

In the previous sections, we have learnt the techniques of differentiating algebraic expressions in the form $y = ax^n$, $y = [f(x)]^n$, $y = uv$ and $y = \frac{u}{v}$, where u and v are functions of x.

We shall now learn how to find the equations of tangents and normals to a curve.



The figure shows a curve $y = f(x)$, where l_1 is the tangent to the curve at the point (x_1, y_1). The gradient of the tangent to the curve at any point x is given by $\frac{dy}{dx}$. If the value of $\frac{dy}{dx}$ at (x_1, y_1) is m, the equation of the tangent is given by

$$\boxed{y - y_1 = m(x - x_1).}$$

ATTENTION

$m \neq \frac{dy}{dx}$, but m is the gradient of the line at a given point.

The line l_2 is perpendicular to the tangent and is called the **normal** to the curve at (x_1, y_1). If $m \neq 0$, the gradient of l_2 is given by $-\frac{1}{m}$ and its equation is given by

$$\boxed{y - y_1 = -\frac{1}{m}(x - x_1).}$$

Worked Example 10

(Equations of a Tangent and a Normal)

Find the equation of the tangent and the normal to the curve $y = x^3 - 2x^2 + 3$ at the point where $x = 2$.

Solution

$\frac{dy}{dx} = 3x^2 - 4x$

When $x = 2$, $\frac{dy}{dx} = 3(2)^2 - 4(2) = 4$

and $y = 2^3 - 2(2)^2 + 3 = 3$

i.e. the gradient of the tangent at (2, 3) is 4.

$\therefore$ Equation of tangent: $y - 3 = 4(x - 2)$

i.e. $y = 4x - 5$

The gradient of the normal at (2, 3) is $-\frac{1}{4}$ and its equation is

$$y - 3 = -\frac{1}{4}(x - 2)$$

i.e. $4y + x = 14$

Practise Now 10

Similar Questions:
Exercise 11E
Questions 1-5

1. Find the equation of the tangent and the normal to the curve $y = 5x^2 - 7x - 1$ at the point where $x = 1$.

2. Find the equation of the tangent and the normal to the curve $y = 2x^3 - 9x^2 + 12x + 7$ at the point where $x = 1\frac{1}{2}$.

3. Find the equation of the tangent and the normal to the curve $y = \frac{x-3}{2x-1}$ when $y = -\frac{1}{3}$.

Worked Example 11

(Problem involving Tangents and Normals)

A point $P(4, 7)$ lies on the curve $y = x^2 - 6x + 15$.

(i) Find the gradient of the curve at P and the equation of the normal at this point.

(ii) Find the coordinates of the point where this normal cuts the curve again.

The tangent at another point Q is perpendicular to the tangent at P.

(iii) Find the x-coordinate of Q.

Solution

(i) $\frac{dy}{dx} = 2x - 6$

At (4, 7), $\frac{dy}{dx} = 2(4) - 6 = 2$

$\therefore$ The gradient of the curve at $P(4, 7)$ is 2 and the gradient of the normal is $-\frac{1}{2}$.

$\therefore$ Equation of the normal at (4, 7): $y - 7 = -\frac{1}{2}(x - 4)$

$$y = -\frac{1}{2}x + 9$$

(ii) When the normal cuts the curve again,

$$x^2 - 6x + 15 = -\frac{1}{2}x + 9$$
$$2x^2 - 12x + 30 = -x + 18$$
$$2x^2 - 11x + 12 = 0$$
$$(2x - 3)(x - 4) = 0$$
$$\therefore \quad x = \frac{3}{2} \quad \text{or} \quad x = 4$$

When $x = \frac{3}{2}$, $y = -\frac{1}{2}\left(\frac{3}{2}\right) + 9$

$$= -\frac{3}{4} + 9$$
$$= \frac{33}{4}.$$

$\therefore$ The normal cuts the curve again at $\left(\frac{3}{2}, \frac{33}{4}\right)$.

(iii) The tangent at Q is perpendicular to the tangent at P,

i.e. $2x - 6 = -\frac{1}{2}$

$$2x = \frac{11}{2}$$
$$x = \frac{11}{4}$$

$\therefore$ The x-coordinate of Q is $\frac{11}{4}$.

Practise Now 11

Similar Questions:
Exercise 11E
Questions 6-13

1. The normal to the curve $y = 2x^2 - 7x$ at the point P is parallel to the line $2y + 2x = 7$. Find
   (i) the coordinates of P,
   (ii) the equation of the tangent to the curve at P,
   (iii) the coordinates of the point on the curve where the normal cuts the curve again.

2. Find the equation of the tangent to the curve $y = \frac{4}{x^3}$ at the point where $x = 2$. This tangent meets the x-axis at A and the y-axis at B. Find the coordinates of A and B and hence find the coordinates of the midpoint of AB. Determine whether the midpoint of AB lies on the curve $y = \frac{4}{x^3}$.

Thinking Time

If $\frac{dy}{dx} = 0$ at (x_1, y_1), what is the equation of the tangent and the normal to the curve at (x_1, y_1)?

What can you say about the tangent and the normal at this point?

Basic Level | Intermediate Level | Advanced Level

Exercise 11E

1 Find the equation of the tangent to each of the following curves at the given value of x.

(a) $y = 2x^2 - 3x + 1,\quad x = 2$

(b) $y = 2x^3 - 4,\quad x = 1$

(c) $y = x^2 - 2x + 3,\quad x = -2$

(d) $y = \dfrac{x-1}{x+1},\quad x = 1$

(e) $y = (x^2 + 3)^2,\quad x = -1$

2 Find the equation of the normal to each of the following curves at the given value of x.

(a) $y = x^2 + 2,\quad x = 1$

(b) $y = 2x^3 + 4x + 3,\quad x = 1$

(c) $y = \dfrac{x}{x+1},\quad x = 2$

(d) $y = 2 + \dfrac{1}{x},\quad x = 1$

(e) $y = x^3 - x^2,\quad x = 2$

3 The equation of a curve is $y = 2x + \dfrac{1}{x}$. Find $\dfrac{dy}{dx}$ and the equation of the normal to the curve at $x = 2$.

4 The normal to the curve $y = 7x - \dfrac{6}{x}$ at $x = 3$ intersects the y-axis at the point P. Calculate the coordinates of P.

5 Find the equation of the normal to the curve $y = \dfrac{x-2}{2x+1}$ at the point where the curve cuts the x-axis.

6 Find the coordinates of the points on the curve $y = 4x + \dfrac{1}{x}$ where the tangents are parallel to the line $y + 5 = 0$.

7 The tangent to the curve $y = 2x^2 - 7x + 3$ at a certain point is parallel to the line $y = x + 2$. Find the equation of this tangent and the coordinates of the point where it cuts the y-axis.

8 The curve $y = ax^3 + bx^2 + c$ passes through the points $(-1, 0)$ and $(0, 5)$. If the tangent to the curve at the point $x = 1$ is parallel to the x-axis, find the values of a, b and c.

9 Find the equation of the tangent to the curve $y = x^2 - 4x + 2$ at the point where $x = 3$. The tangent at another point A on the curve is perpendicular to the tangent at $x = 3$. Calculate the x-coordinate of A.

10 Find the equation of the tangent to the curve $y = (x - 2)^3$ at the point $(3, 1)$. Calculate the coordinates of the point where this tangent meets the curve again.

11 Find the equation of the tangent to the curve $y = x^2 - x - 1$ at the point $A(2, 1)$. Calculate the coordinates of the point on the curve at which the normal is parallel to the tangent at A. On the same diagram, sketch the curve, the tangent and the normal.

12 The normal to the curve $y = 2x^2 + kx + 1$ at the point $(-1, 2)$ is parallel to the line $3y - x = 9$. Find the value of k. Calculate the coordinates of the point where this normal meets the curve again.

13 The equation of a curve is

$$y = 2x^3 - 21x^2 + 78x - 98.$$

Find

(i) the gradient of the tangent at the point $(3, 1)$,

(ii) the x-coordinate of the point at which the tangent of the curve is parallel to the tangent at $(3, 1)$.

11.4 RATES OF CHANGE

Have you ever used a conical cup provided by health clubs, offices, hospitals or other establishments to fill drinking water from a big container?

Discuss the following with your classmates.

1. Is the rate of flow of water from the big container constant throughout?

2. Is the rate of increase in the depth of water in the cup constant throughout?

Consider the 2 cups shown below. Each cup has a height of 12 cm and a volume of 240 cm^3. It takes 16 seconds to fill up each cup. Copy and sketch the graphs of depth against time. What do you observe about the two graphs? What can you say about the gradients?

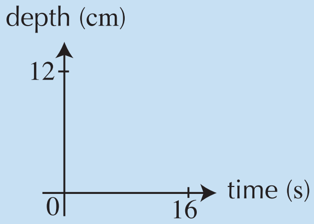


Fig. 11(a)

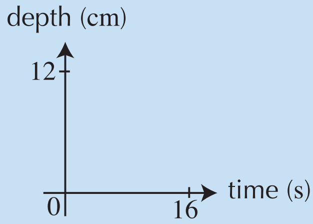


Fig. 11(b)

(a) Discuss with your classmates the shape of the graphs in Fig. 11(a) and Fig. 11(b).

Since it takes 16 seconds to fill up a conical cup of volume 240 cm^3, we say that the **average rate of change** in the volume is 15 cm^3 per second. The height of the conical cup is 12 cm, so the depth of water in the cup increases from 0 cm to 12 cm in 16 seconds.

(b) What is the average rate of change in the depth of the water in the cup? Is the depth of water in the conical cup rising at a constant rate when it is being filled? Can we say that the rate of increase in the depth of water in the conical cup changes as water is dispensed?

Assuming that the flow of water from the big container is constant, the water is said to be dispensed at a **constant rate** of 15 cm^3 per second. The **instantaneous rate of change** of the depth of water at a particular time, t, is the gradient of the curve, which can be found by drawing a tangent to the curve at t or by using differentiation to find the gradient of the tangent at t.

Worked Example 12

(Rate of Change)

Given that $V = \frac{1}{3}\pi t^3 + 2\pi t^2 + 3t$, find the rate of change of V with respect to t when $t = 2$, leaving your answer in terms of π.

Solution

$$\frac{dV}{dt} = \frac{1}{3}\pi(3t^2) + 2\pi(2t) + 3$$
$$= \pi t^2 + 4\pi t + 3$$

When $t = 2$,
$$\frac{dV}{dt} = \pi(2)^2 + 4\pi(2) + 3$$
$$= 12\pi + 3$$

Practise Now 12

Similar Questions:
Exercise 11F
Questions 1, 2, 10, 17, 19

1. Given that $A = 2t^2 - 4t + 5$, find the rate of change of A with respect to t when $t = 2$.
2. Given that $V = \frac{4}{3}t^3 - \frac{3}{4}t^2 + 2t + 3$, find the rate of change of V with respect to t when $t = 3$.

Connected Rates of Change

If two variables x and y both vary with another variable, say t, the rates of change with respect to t, $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are related by $\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$ (Chain Rule).

For example, if the area, A, and the radius, r, of a circle both vary with time t, then the rate of change of A with respect to t, i.e. $\frac{dA}{dt}$, and the rate of change of r with respect to t, i.e. $\frac{dr}{dt}$, are related by $\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$ (Chain Rule).

We shall use the above to calculate the rate of change of one variable as compared to another.

Worked Example 13

(Application of Rates of Change)

The radius of a circle increases at a rate of 0.2 cm/s. Calculate the rate of increase in the area when the radius is 5 cm.

Solution

Let r cm and A cm² be the radius and the area of the circle respectively. It is given that the rate of change of the radius is 0.2 cm/s.

i.e $\frac{dr}{dt} = 0.2$

$$A = \pi r^2$$
$$\frac{dA}{dr} = 2\pi r$$
$$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$$
$$= 2\pi r \times 0.2 = 0.4\pi r$$

When $r = 5$, $\frac{dA}{dt} = 0.4 \times \pi \times 5 = 2\pi$

$\therefore$ The rate of increase in the area is 2π cm²/s.

ATTENTION

$\frac{dA}{dt}$ indicates the rate of change of the area at a given instant.

Practise Now 13

Similar Questions:
Exercise 11F
Questions 3-5, 7

The radius of a sphere increases at a rate of 0.1 cm/s. Calculate the rate of increase of
(i) the surface area,
(ii) the volume,
when the radius is 12 cm, giving each of your answers in terms of π.

Worked Example 14

(Application of Rate of Change)
A hemispherical bowl of radius 8 cm is completely filled with water. The water is then transferred at a steady rate into an empty inverted right circular cone of base radius 8 cm and height 16 cm, with its axis vertical. Given that all the water is transferred in $5\frac{1}{3}$ seconds and that V cm³ represents the amount of water in the cone at any time t, find

(i) $\frac{dV}{dt}$ in terms of π,
(ii) the rate of change of
  (a) the height of water level,
  (b) the horizontal surface area of the cone,
  when the depth of the water is 6 cm.

Solution

(i) Volume of water in the hemispherical bowl $= \frac{2}{3}\pi(8^3) = 341\frac{1}{3}\pi$ cm³

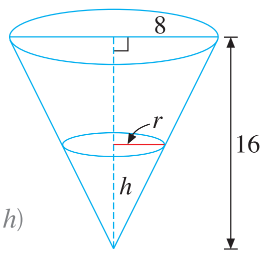
Since the rate of change of V is constant, $\frac{dV}{dt} = \frac{341\frac{1}{3}\pi}{5\frac{1}{3}}$

$= 64\pi$ cm³/s

(ii) For the right circular cone, let the radius be r cm and the height be h cm.

Using similar triangles, we have $\frac{r}{h} = \frac{8}{16}$

i.e. $r = \frac{1}{2}h$



(a) Volume of water in the cone,

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{\pi}{12}h^3 \text{ (replace } r \text{ with } \tfrac{1}{2}h)$$

$$\frac{dV}{dh} = \frac{\pi}{12} \times 3h^2 = \frac{\pi h^2}{4}$$

$$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt} \quad \text{i.e. } 64\pi = \frac{\pi h^2}{4} \times \frac{dh}{dt}$$

$$\therefore \quad \frac{dh}{dt} = \frac{64\pi \times 4}{\pi h^2}$$

When $h = 6$, $\frac{dh}{dt} = \frac{64 \times 4}{6^2} = 7\frac{1}{9}$ cm/s

(b) Area of the horizontal surface, $A = \pi r^2 = \pi\left(\frac{h}{2}\right)^2 = \frac{\pi h^2}{4}$

$$\frac{dA}{dh} = \frac{\pi}{2}h$$

When $h = 6$, $\frac{dA}{dh} = \frac{\pi}{2}(6) = 3\pi$

$$\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt} = 3\pi \times 7\frac{1}{9} = 21\frac{1}{3}\pi \text{ cm}^2\text{/s}$$

Practise Now 14

Similar Questions:
Exercise 11F
Questions 6, 8, 9, 11-16, 18

If a child blows air into a plastic spherical globe at a constant rate of 50 cm^3 per second, find the rate of change of
(i) the radius,
(ii) the surface area of the globe,
when its radius is 20 cm.

Basic Level | Intermediate Level | Advanced Level

Exercise 11F

1 Given that $y = \frac{t}{\sqrt{2t+1}}$, find the rate of change of y with respect to t when $t = 2$.

2 Given that $y = 2t^3 - 6t^2 + 4t - 7$, find the rate of change of y with respect to t and the range of values of t for which $\frac{dy}{dt} \geqslant 4$.

3 The radius of a circular blob of ink is increasing at a constant rate of $\frac{1}{5}$ cm/s. Find the rate at which the area is increasing when the radius is 10 cm.

4 A cube is expanding in such a way that its sides are changing at a rate of 2 cm/s. Find the rate of change of the total surface area of the cube when its volume is 125 cm^3.

5 The area of a circle is increasing at a uniform rate of 8 cm^2/s. Calculate the rate of increase of the radius when the circumference of the circle is 96 cm.

6 A spherical ball is being inflated at a rate of 20 cm^3/s. Find the rate of increase of its radius when its surface area is 100π cm^2.

7 The radius of a spherical balloon increases at a rate of 0.05 cm/s.
(i) Find the rate of increase of the surface area when the volume is 972π cm^3.
(ii) Will the rate of increase of the surface area be the same when the radius is 10 cm and when it is 12 cm? Explain your answer clearly.

8 A container, initially empty, is being filled with water. The depth of the container is h cm and its volume, V cm^3, is given by $V = \frac{1}{3}h^2(h+4)$. Given that the depth of the water increases at a rate of 2 cm/s, find the rate of increase of the volume when $h = 5$.

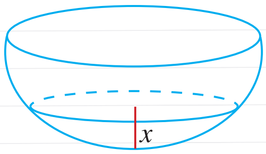
9 A gas in a container changes its volume according to the equation $PV = 3600$, where P is the number of units of pressure and V is the number of units of the volume. Given that P is increasing at a rate of 20 units per second at the instant when $P = 40$, calculate the rate of change in the volume at this instant.

10 Variables x and y are connected by the equation $2xy = 195$. If x is increasing at a rate of $\frac{3}{2}$ units per second, find the rate of change of y at the instant when $x = 15$ units.

11 The base of a closed rectangular box is a square of sides x cm and the height of the box 8 cm. Given that the sides of the square base increase at a constant rate of 0.05 cm/s, find the rate of increase in the volume and in the total surface area, when the total surface area of the box is 210 cm^2.

12 The height, h cm, of a cone remains constant while the radius of the base is increasing at a rate of 5 cm/s. What is the rate of change of the volume of the cone when the base radius is 60 cm?

13



Water is poured into a hemispherical bowl of radius 6 cm at a rate of 20 cm^3/s. After t seconds, the volume of water in the bowl, V cm^3, is given by $V = 6\pi x^2 - \frac{1}{3}\pi x^3$, where x cm is the height of water in the bowl.

(i) Calculate the rate of change of x when $x = 4$ cm, giving your answer in terms of π.

(ii) Will the rate of change of x increase when $x = 5$? Explain your answer clearly.

14 When the height of liquid in a tub is x m, the volume of liquid is V m^3, where $V = 0.05[(3x + 2)^3 - 8]$.

(i) Find an expression for $\frac{dV}{dx}$.

The liquid enters the tub at a constant rate of 0.081 m^{3}s^{-1}.

(ii) Find the rate at which the height of the liquid is increasing when $V = 0.95$.

15 Liquid is poured into a container at a rate of k m^3/s. The volume of liquid in the container is V m^3, where $V = \frac{1}{3}\pi h^2(3k - h)$ and h is the depth of the liquid in the container. Find, in terms of k, the rate at which the depth changes, when $h = \frac{2k}{5}$ m.

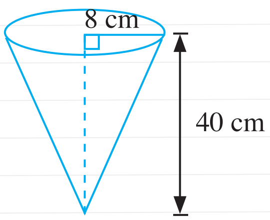
16 The volume of water in a vessel is $\frac{\pi x^2(3a - x)}{4}$ cm^3 when the depth of water in the vessel is x cm. If water is poured into the vessel at a rate of a^3 cm^3/s, calculate the rate at which the level of the water rises when the depth is $\frac{3a}{4}$ cm, giving your answer in terms of a.

17 A viscous liquid is poured onto a flat surface. It forms a circular patch which grows at a steady rate of 8 cm^2/s. Find, in terms of π,

(i) the radius of the patch 25 seconds after pouring has commenced,

(ii) the rate of increase of the radius at this instant.

18



A vessel is in the shape of a right circular cone. The radius of the top is 8 cm and the height is 40 cm. Water is poured into the vessel at a rate of 20 cm^3/s. Calculate the rate at which the water level is rising when

(i) the water level is 12 cm from the vertex,

(ii) the vessel is one-quarter full.

19 An important formula used in the study of light in Physics is $\frac{1}{f} = \frac{1}{u} + \frac{1}{v}$, where f is the focal length of the lens, u is the object distance and v is the image distance from the axis of the lens. For a particular lens, we have $\frac{1}{25} = \frac{1}{u} + \frac{1}{v}$. Given that u is increasing at a rate of 1.2 cm/s, calculate the rate of change of v when $u = 35$.

SUMMARY

1. For a curve $y = \mathrm{f}(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ represents the gradient of the tangent to the curve at a point x. $\frac{\mathrm{d}y}{\mathrm{d}x}$ measures the rate of change of y with respect to x.

2. Five Rules of Differentiation

   - **Rule 1:**

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left[k\mathrm{f}(x)\right] = k\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{f}(x)\right]$$

   - **Rule 2:**

   $$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x) \pm \mathrm{g}(x)] = \frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)] \pm \frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{g}(x)]$$ (Addition/Subtraction Rule)

   - **Rule 3:**

   $$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$ (Chain Rule)

   - **Rule 4:**

   $$\frac{\mathrm{d}}{\mathrm{d}x}(uv) = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$ (Product Rule)

   - **Rule 5:**

   $$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{u}{v}\right) = \frac{v\frac{\mathrm{d}u}{\mathrm{d}x} - u\frac{\mathrm{d}v}{\mathrm{d}x}}{v^2}$$ (Quotient Rule)

3. If $y = \mathrm{f}(x)$, the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ at (x_1, y_1) gives the gradient, m, of the tangent and $-\frac{1}{m}$ gives the gradient of the normal. The equation of the tangent at (x_1, y_1) is $y - y_1 = m(x - x_1)$ and the equation of the normal at (x_1, y_1) is $y - y_1 = -\frac{1}{m}(x - x_1)$.

4. If the variables x and y both vary with another variable, say t, then the rates of change of x and y with respect to t are related by $\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}x} \times \frac{\mathrm{d}x}{\mathrm{d}t}$.

Review Exercise 11

1. Calculate the gradients of the curve at the point(s) where it intersects the given line.

   (a) $y = \dfrac{2 - x}{x^2}$, x-axis (b) $y = \dfrac{16x^3 - 1}{x^2}$, $x = 1$ (c) $y = \sqrt{x} + \dfrac{1}{\sqrt{x}}$, $x = 4$

   (d) $y = x^2 - 4x$, $y = -4$ (e) $y = 3 - 5x - 2x^2$, y-axis (f) $y = 2x^2 - 3x + 1$, $y = 21$

2. Differentiate each of the following with respect to x.

   (a) $(5x - 4)^5$ (b) $\sqrt{2x^3 + 5}$ (c) $(5x^2 + 3)^{\frac{3}{2}}$

   (d) $(2 + 9x^2)^{\frac{1}{3}}$ (e) $\dfrac{3}{(3x - 5)^3}$ (f) $\dfrac{3x - 7}{\sqrt{2x + 9}}$

3. Differentiate each of the following with respect to x, where a, b and c are constants.

   (a) $(ax^2 + bx)^6$ (b) $(2ax + b)^5(5x^2 - ab)^6$ (c) $\dfrac{2ax^2 + bx}{bx^3 - cx}$ (d) $\dfrac{ax + b}{cx^2 - a^2}$

4. Find the coordinates of the points on the curve $y = \dfrac{x^2 - 1}{x}$ at which the gradient of the curve is 5.

5. The gradient of the curve $y = ax^3 + bx$ at the point (3, 1) is 3. Find the values of the constants a and b.

6. The equation of the tangent to the curve $y = kx + \dfrac{6}{x}$ at the point (–2, –19) is $px + qy = c$. Find the values of the integers k, p, q and c, where $c > 0$.

7. Given that $A = 4r^3 - 3r^2 - 18r + 5$, find $\dfrac{\mathrm{d}A}{\mathrm{d}r}$ and hence, the range of values of r for which $\dfrac{\mathrm{d}A}{\mathrm{d}r} < 0$.

8. The straight line $2x + 3y = 7$ meets the curve $xy + 10 = 0$ at the points P and Q. Calculate the gradient of the curve at P and at Q.

9. The equation of a curve is $y = \dfrac{x^2 + 6}{x - 3}$. Find the value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ at the point $x = 4$. Hence, find the equation of the normal to the curve at $x = 4$.

10. The gradient of the curve $y = ax^2 - \dfrac{b}{x^2}$ at the point (1, 8) is 4. Find the values of the constants a and b. With these values, find the equation of the tangent to the curve at the point $x = 3$.

11. The pressure and volume of a gas are related by the formula $PV^{1.5} = 500$. If the pressure increases at a rate of 2 units/second, find the rate of change in the volume when

(a) $V = 20$, **(b)** $P = 30$.

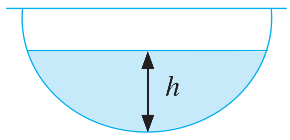
Give each of your answers correct to 1 decimal place.

12. The values of x and y are related by the equation $x^2y = 24$, where $x > 0$. If x increases at a rate of 0.05 units/second, find the rate of change of y when

(a) $x = 2$, **(b)** $y = 12$.

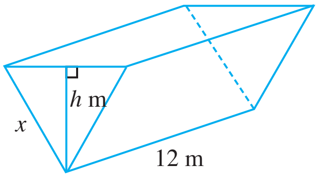
13. The variables x and y are related by the formula $\frac{1}{y} = \frac{1}{10} - \frac{1}{x}$. If x increases at a rate of 2 cm/s, calculate the rate of change of y when $x = 15$ cm.

14. The volume of water in a hemispherical bowl, V cm^3, is given by the formula $V = \frac{1}{3}\pi h^2(24 - h)$, where h is the depth of the water in cm. If water is poured into the bowl at a rate of 8 cm^3/s, find the rate of change of h when $h = 4$ cm.

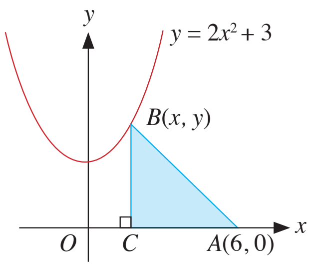


15. The equation of a curve is $y = 5x - 2\sqrt{x}$. The tangent to the curve at $x = 4$ cuts the y-axis at point P. Find the coordinates of P.

16. A water trough of length 12 m has a vertical cross-section in the shape of an equilateral triangle of sides x m and height h m. Express V, the volume of water that the trough can hold in terms of h. When the height of water in the trough is 1.8 m, its depth is decreasing at a rate of 0.2 m/s. Find the rate of change in the volume of water in the trough at this instant.



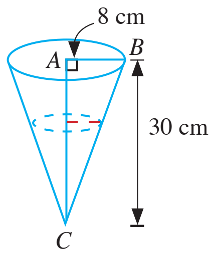
17. The figure shows part of the curve $y = 2x^2 + 3$. The point $B(x, y)$ is a variable point that moves along the curve for $0 < x < 6$. C is a point on the x-axis such that BC is parallel to the y-axis and $A(6, 0)$ lies on the x-axis. Express the area of the triangle ABC, T cm^2, in terms of x, and find an expression for $\frac{dT}{dx}$. Given that when $x = 2$, T is increasing at the rate of 0.8 unit^2/s, find the corresponding rate of change of x at this instant.



18. The figure shows a right circular cone with base radius 8 cm and height 30 cm. Initially, it is filled with water. Water leaks through a small hole at the vertex C at a rate of 5 cm^3/s.

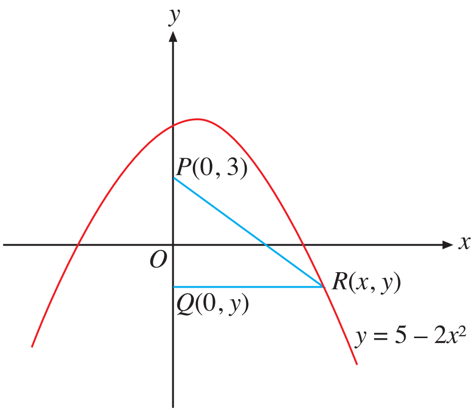
(i) Express V, the volume of water in the cone, in terms of height, h.

(ii) Find the rate of change of h when $h = 12$ cm.

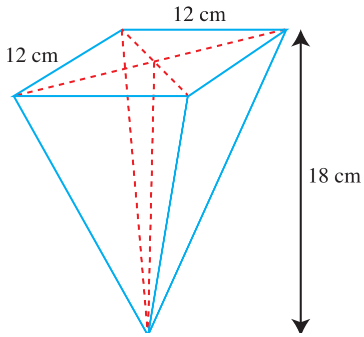


Challenge Yourself

1. The figure shows the curve $y = 5 - 2x^2$, the fixed point $P(0, 3)$ and two variable points, Q and R, where QR is always parallel to the x-axis.
   (a) Express A, the area of ΔPQR, in terms of x, for $x > 1$.
   (b) Given that Q moves away from P at a rate given by $\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{1}{3}$ unit/s, find the rate of change of A when $x = 4$ units.



2. An inverted square pyramid of sides 12 cm and height 18 cm is completely filled with water. Water leaks away at the vertex at a rate of 4 cm^3/s. Calculate the rate of change of
   (i) the height of the water level,
   (ii) the horizontal surface area of the water,
   when the depth of the water is 8 cm.



FURTHER APPLICATIONS OF DIFFERENTIATION

In the supermarket, we see many cylindrical cans that are used to store food. How do manufacturers determine the dimensions of these cans so as to minimise the cost of manufacturing? In this chapter, we will learn how to solve questions which involve finding the minimum amount of metal required to manufacture a can.

CHAPTER 12

Learning Objectives

At the end of this chapter, you should be able to:

- calculate higher derivatives of functions,
- distinguish between increasing and decreasing functions,
- find the nature of a stationary point on a curve and determine whether it is a maximum or a minimum point or a point of inflexion,
- solve practical problems involving maximum and minimum values.

12.1 HIGHER DERIVATIVES

When a function of the form $y = \mathrm{f}(x)$ is differentiated with respect to x, the derivative is another function of x.

If we have $y = x^3 + 2x^2 + 3x + 4$,

then $\dfrac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 4x + 3$.

The function $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ is also known as the first derivative of y with respect to x.

Differentiating $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ with respect to x, we have $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)$, which can be written as $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$. This is called the **second derivative** of y with respect to x. If the function $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is further differentiated with respect to x, the **third derivative** $\dfrac{\mathrm{d}^3y}{\mathrm{d}x^3}$ is obtained.

i.e.

$$y = x^3 + 2x^2 + 3x + 4 \quad \text{or} \quad \mathrm{f}(x) = x^3 + 2x^2 + 3x + 4$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 3x^2 + 4x + 3 \quad \text{or} \quad \mathrm{f}'(x) = 3x^2 + 4x + 3$$

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 4 \quad \text{or} \quad \mathrm{f}''(x) = 6x + 4$$

$$\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = 6 \quad \text{or} \quad \mathrm{f}'''(x) = 6$$

Worked Example 1

(Finding $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$)

It is given that $y = \dfrac{2x^2}{x-3}$, where $x \neq 3$.

(i) Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

(ii) Find the range of values of x for which both $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ and $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ are negative.

(iii) Is $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$? Show working to support your answer.

Solution

(i) $$y = \frac{2x^2}{x-3}$$

$$\frac{dy}{dx} = \frac{(x-3)\times 4x - 2x^2 \times 1}{(x-3)^2} \quad \text{(Quotient Rule)}$$
$$= \frac{4x^2 - 12x - 2x^2}{(x-3)^2}$$
$$= \frac{2x^2 - 12x}{(x-3)^2}$$

$$\frac{d^2y}{dx^2} = \frac{(x-3)^2 \times (4x-12) - (2x^2 - 12x)\times 2(x-3)}{(x-3)^4} \quad \text{(Quotient Rule)}$$
$$= \frac{(x-3)\left[(x-3)(4x-12) - 2\left(2x^2-12x\right)\right]}{(x-3)^4}$$
$$= \frac{4x^2 - 12x - 12x + 36 - 4x^2 + 24x}{(x-3)^3}$$
$$= \frac{36}{(x-3)^3}$$

RECALL

Applying the quotient rule to $y = \frac{u}{v}$, $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$.

(ii) For $\frac{dy}{dx} < 0$,

$$\frac{2x^2 - 12x}{(x-3)^2} < 0$$

Since $(x-3)^2 > 0$ for all values of x and $x \neq 3$,

$$2x^2 - 12x < 0$$
$$2x(x-6) < 0$$

i.e. $0 < x < 6$ ------------ (1)

For $\frac{d^2y}{dx^2} < 0$,

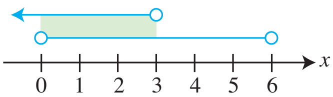
$$\frac{36}{(x-3)^3} < 0$$

Since $36 > 0$,

$$(x-3)^3 < 0$$
$$x - 3 < 0$$

i.e. $x < 3$ ------------ (2)

Combining (1) and (2), we have



$\therefore\ 0 < x < 3$

PROBLEM-SOLVING TIP

Draw a number line to better visualise the region of x which is required.

(iii) Since $\left(\frac{dy}{dx}\right)^2 = \left[\frac{2x^2 - 12x}{(x-3)^2}\right]^2 = \frac{4x^4 - 48x^3 + 144x^2}{(x-3)^4}$

and $\frac{d^2y}{dx^2} = \frac{36}{(x-3)^3}$,

$$\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2.$$

Serious Misconception

Do not confuse $\frac{d^2y}{dx^2}$ for $\left(\frac{dy}{dx}\right)^2$. We have just seen from Worked Example 1 that $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$.

Practise Now 1

Similar Questions:
Exercise 12A
Questions 1-5

1. Given that $y = x^2(5x - 3)$, find
   (i) $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$,
   (ii) the range of values of x for which both $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ are positive.
   (iii) Show that $\frac{d^2y}{dx^2} \neq \left(\frac{dy}{dx}\right)^2$.
2. If $v = t^3 + 2t^2 + 3t + 1$, find the values of t for which $\frac{dv}{dt} = \frac{d^2v}{dt^2}$.

Basic Level | Intermediate Level | Advanced Level

Exercise 12A

1 Find the first and second derivatives of each of the following functions with respect to x.
(a) $y = 3x^4$
(b) $y = 5x - 7$
(c) $y = 3x^2 + 5x - 1$
(d) $y = \frac{1}{x}$
(e) $y = (3x + 2)^{10}$
(f) $y = \sqrt{x - 4}$

2 Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for each of the following functions.
(a) $y = \frac{x^3 + x}{x^2}$
(b) $y = \frac{x - 1}{x^2}$
(c) $y = \frac{3x^3 - x}{\sqrt{x}}$
(d) $y = \frac{x}{x - 1}$
(e) $y = \frac{2x + 5}{x - 1}$
(f) $y = \frac{x^2}{x + 1}$

3 If $y = \frac{3x + 4}{4x - 1}$, find
(i) the coordinates of the points on the curve at which $\frac{dy}{dx} = -\frac{19}{4}$,
(ii) the value of $\frac{d^2y}{dx^2}$ at the point $\left(\frac{1}{2}, \frac{11}{2}\right)$.

4 Find expressions for $f'(x)$ and $f''(x)$ for each of the following, where a, b, c and d are constants.
(a) $y = ax^3 + bx^2 + 7c$
(b) $y = (ax^2 + bx + c)^2$
(c) $y = x\sqrt{x} + a$
(d) $y = \left(ax^2 + bx\right)\sqrt{x}$
(e) $y = \frac{ax^3 + bx}{cx^2}$
(f) $y = \frac{ax + b}{cx - d}$

5 If $y = x^2 + 2x + 3$, show that $\left(\frac{dy}{dx}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^3 = 4y$.

6 If $xy - 3 = 2x^2$, prove that $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = y$.

12.2 INCREASING AND DECREASING FUNCTIONS

Thinking Time

Part 1: Increasing Functions

Consider each of the 3 cases in Fig. 12.1.

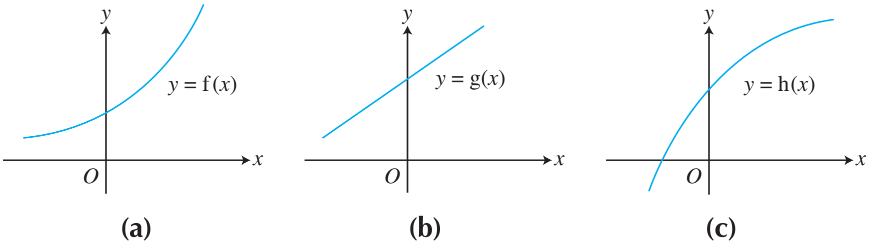


Fig. 12.1

(i) What happens to the value of y as x increases?

(ii) What is the difference between the gradient of the graph in Fig. 12.1(b) and those in Fig. 12.1(a) and Fig. 12.1(c)?

(iii) What can you say about the sign of the gradient at any point?

(iv) What conclusions can you draw about the sign of $\frac{dy}{dx}$ for an increasing function?

Part 2: Decreasing Functions

(i) Sketch possible graphs of decreasing functions.

(ii) What happens to the value of y as x increases?

(iii) What can you say about the sign of the gradient at any point?

(iv) What conclusions can you draw about the sign of $\frac{dy}{dx}$ for a decreasing function?

Consider the two quadratic graphs below.

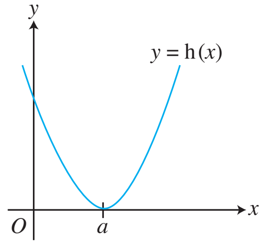


Fig. 12.3

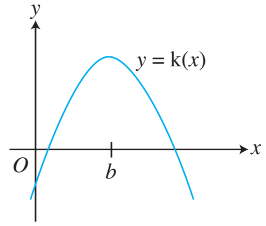


Fig. 12.4

For the function $y = \mathrm{h}(x)$, observe that $\mathrm{h}(x)$ is an increasing function for $x > a$, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ for $x > a$. $\mathrm{h}(x)$ is a decreasing function for $x < a$, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ for $x < a$.

Similarly, for the function $y = \mathrm{k}(x)$, observe that $\mathrm{k}(x)$ is an increasing function for $x < b$, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ for $x < b$. $\mathrm{k}(x)$ is a decreasing function for $x > b$, i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ for $x > b$.

Worked Example 2

(Increasing/Decreasing Function)

It is given that $y = \frac{3x^2}{2x-1}$, where $x > \frac{1}{2}$.

(i) Find $\frac{\mathrm{d}y}{\mathrm{d}x}$.

(ii) Find the range of values of x for which $y = \frac{3x^2}{2x-1}$ is an increasing function.

(iii) State the range of values of x for which $y = \frac{3x^2}{2x-1}$ is a decreasing function.

Solution

(i) $y = \frac{3x^2}{2x-1}$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{(2x-1)\times 6x - 3x^2 \times 2}{(2x-1)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{12x^2 - 6x - 6x^2}{(2x-1)^2}$$

$$= \frac{6x^2 - 6x}{(2x-1)^2}$$

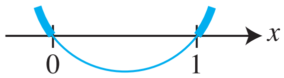
(ii) For y to be an increasing function, $\frac{dy}{dx} > 0$.

$$\frac{6x^2 - 6x}{(2x-1)^2} > 0$$

$6x^2 - 6x > 0$ (since denominator $(2x - 1)^2 > 0$)

$6x(x - 1) > 0$

$x < 0$ or $x > 1$



Given that $x > \frac{1}{2}$,

$\therefore\ x > 1$

(iii) For y to be a decreasing function, $0 < x < 1$, but $x > \frac{1}{2}$.

$\therefore\ \frac{1}{2} < x < 1$

Practise Now 2

Similar Questions:
Exercise 12B
Questions 1(a)-(f), 2(a)-(f), 3

Given that $y = \frac{3x}{x^2+1}$, find $\frac{dy}{dx}$ and find the range of values of x for which $y = \frac{3x}{x^2+1}$ is a decreasing function.

Worked Example 3

(Application of an Increasing Function)

The approximate drug concentration, $C(t)$, in a person's bloodstream t hours after ingesting 5 units of a drug, can be modelled by the function $C(t) = 1.04t - \frac{3t^3}{200}$. Find the time interval when the drug concentration in the bloodstream is increasing.

Solution

$$C(t) = 1.04t - \frac{3t^3}{200}$$

$$C'(t) = 1.04 - \frac{9t^2}{200}$$

For $C(t)$ to be an increasing function, $C'(t) > 0$.

$$1.04 - \frac{9t^2}{200} > 0$$

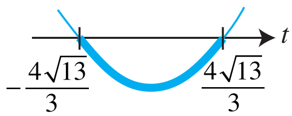
$$208 - 9t^2 > 0$$

$$9t^2 - 208 < 0$$

$$(3t + 4\sqrt{13})(3t - 4\sqrt{13}) < 0$$

$$-\frac{4\sqrt{13}}{3} < t < \frac{4\sqrt{13}}{3}$$

$$-4.81 < t < 4.81$$



Since $t \geqslant 0$,

$\therefore\ 0 \leqslant t < 4.81$

Practise Now 3

Similar Questions:
Exercise 12B
Questions 4, 5

The percentage of a drug being absorbed into the bloodstream x hours after the drug is taken is given by $f(x) = \frac{5x}{4x^2+16}$. Find the time interval during which the rate of absorption of the drug is increasing.

Basic Level | Intermediate Level | Advanced Level

Exercise 12B

1 Find the set of values of x for which each of the following is an increasing function.

(a) $f(x) = x^2 + 10x + 5$

(b) $f(x) = 3x^2 - 12x + 7$

(c) $f(x) = 8 - 6x - 3x^2$

(d) $f(x) = 2x^3 - 3x^2 - 72x + 5$

(e) $f(x) = 4x^3 - 9x^2 - 30x + 7$

(f) $f(x) = x^3 - 12x^2 + 45x + 6$

2 Find the set of values of x for which each of the following is a decreasing function.

(a) $f(x) = 5x^2 - 10x + 3$

(b) $f(x) = 5 - 7x - 2x^2$

(c) $f(x) = 2x^3 - 3x^2 - 12x + 5$

(d) $f(x) = 4x^3 - 15x^2 - 72x + 5$

(e) $f(x) = 4x^3 - 3x^2 - 18x + 1$

(f) $f(x) = x^3 + 3x^2 - 24x + 11$

3 A function is defined by $y = \frac{x^2}{2x-3}$, where $x > 1\frac{1}{2}$. Find the range of values of x for which $y = \frac{x^2}{2x-3}$ is an increasing function.

4 The profit function, $P(x)$, of a manufacturing company producing x articles, can be modelled by the function $P(x) = 2x^3 - 21x^2 + 60x - 5$. Find the range of values of x for which the profit is decreasing.

5 A manufacturing company finds that the profit, $P(x)$, in thousand dollars, is related to the number of workers, x, in hundreds, employed on the factory floor. This relationship can be modelled by the function

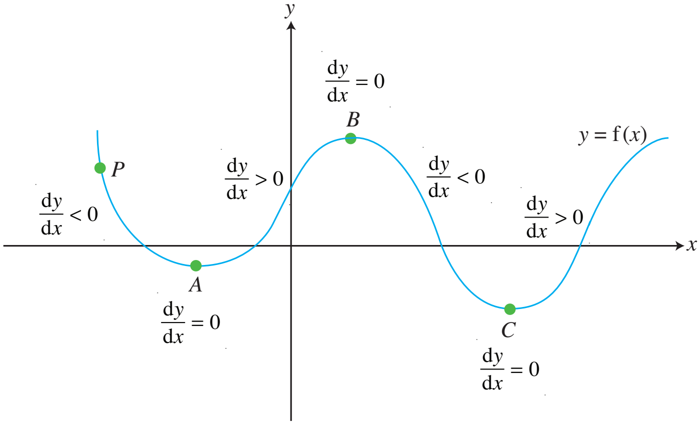
$$P(x) = 2x^3 - 27x^2 + 108x - 5.$$

For what values of x will the profit function be increasing? Briefly explain the significance of your answer.

12.3 STATIONARY POINTS

Maximum and Minimum Points

Consider the graph of $y = f(x)$ shown below.



Notice that along PA, y decreases as x increases, i.e. the gradient of the curve, $\frac{dy}{dx}$, is negative. This part of the function is a decreasing function.

At A, $\frac{dy}{dx}$ is equal to 0. The point A where $\frac{dy}{dx} = 0$ is called a **stationary point** or a turning point. The stationary point A is a **minimum point** as the value of y is less than the other values of y in the neighbourhood.

Along AB, y increases as x increases, i.e. the value of $\frac{dy}{dx}$ becomes positive. This part of the function is an increasing function. Notice that as the curve passes through the point A, the value of $\frac{dy}{dx}$ changes its sign from negative to positive.

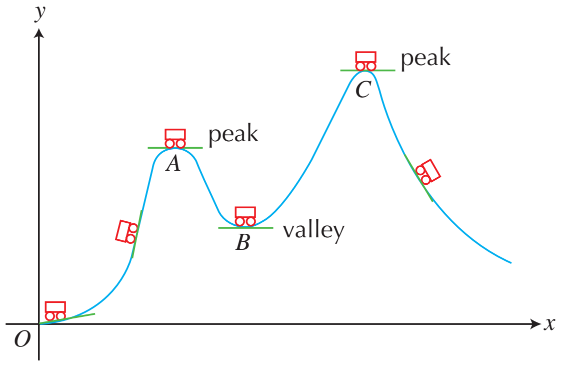
At B, $\frac{dy}{dx}$ is equal to 0. The point B where $\frac{dy}{dx} = 0$ is called a stationary point or a turning point. The stationary point B is a maximum point as the value of y is greater than the other values of y in the neighbourhood.

Along BC, y decreases as x increases, i.e. the value of $\frac{dy}{dx}$ becomes negative. This part of the function is a decreasing function. Notice that as the curve passes through the point B, the value of $\frac{dy}{dx}$ changes its sign from positive to negative.

What is the value of $\frac{dy}{dx}$ at C? State whether it is a maximum or a minimum point.

Analogy for Maximum and Minimum Points

We can think of the roller-coaster car as a graph, where the floor of the roller-coaster car acts as the tangent line at each point on the track.



Using this analogy, it is clear that the gradient of the tangent is positive when the car is travelling uphill from left to right, and when the car is moving downhill from left to right, the gradient of the tangent is negative.

When the car reaches the peak and the valley, the gradient of the tangent is zero.

The peaks A and C are called maximum points.

The valley B is called a minimum point.

Stationary Points of Inflexion

Consider the graphs below.

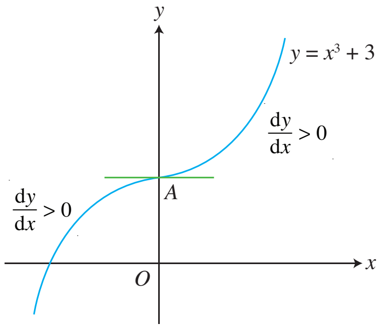


Fig. 12.5

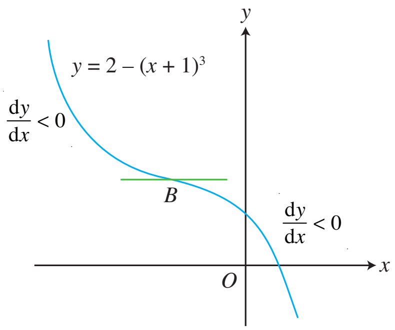


Fig. 12.6

In Fig. 12.5,

$y = x^3 + 3$

$\frac{dy}{dx} = 3x^2$

At $A(0, 3)$,

$\frac{dy}{dx} = 0$

Since $\frac{dy}{dx} = 3x^2 \geqslant 0$ for all values of x, then $\frac{dy}{dx}$ remains positive for both $x < 0$ and $x > 0$, i.e. the sign does not change.

In Fig. 12.6,

$y = 2 - (x + 1)^3$

$\frac{dy}{dx} = -3(x + 1)^2$

At $B(-1, 2)$,

$\frac{dy}{dx} = 0$

Since $\frac{dy}{dx} = -3(x + 1)^2 \leqslant 0$ for all values of x, then $\frac{dy}{dx}$ remains negative for both $x < -1$ and $x > -1$, i.e. the sign does not change.

However, points A and B are neither maximum nor minimum points. They are called **stationary points of inflexion**. Notice that for stationary points of inflexion,

(i) $\frac{dy}{dx} = 0$,

(ii) $\frac{dy}{dx}$ changes from positive to zero, then to positive again; or $\frac{dy}{dx}$ changes from negative to zero, then to negative again.

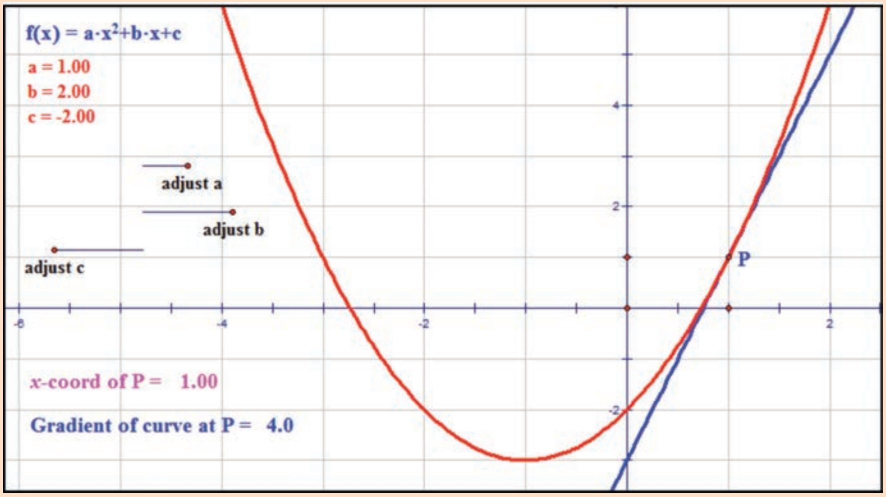
The manner in which the gradient of a curve changes as it passes through a stationary point will help us determine the nature of the stationary point.

In general, stationary points occur when $\frac{dy}{dx} = 0$ and there are 3 types of stationary points – maximum point, minimum point and stationary point of inflexion.

Investigation

Stationary Points using the First Derivative Test

Go to http://www.shinglee.com.sg/StudentResources/ and open the computer algebra software template *Stationary Points*.



The template shows the graph of $y = ax^2 + bx + c$ (red curve) and the tangent (blue line) to the curve at the point of contact $P(x_1, y_1)$.

Part A: Minimum Point

1. Drag and move the point P to change the value of its x-coordinate of P to those in the table below. Record the gradient of the curve at P. Copy and complete the table.

x-coordinate of P	−2	−1	0	1	2	3	4
Gradient of curve at P							

2. What are the coordinates of the point where the gradient $\frac{dy}{dx}$ is 0?

   Is this stationary point a maximum or a minimum point?

3. As x increases from –2 to 4 through the stationary point, what do you notice about the sign of the gradient of the curve?

Part B: Maximum Point

4. Drag the sliders 'adjust a', 'adjust b' and 'adjust c' to change values of a, b and c in the template to obtain the graph of $y = 2 - x - x^2$. Change the value of the x-coordinate of P to those in the table below and record the gradient of the curve at P. Copy and complete the table.

x-coordinate of P	–3	–2	–1	–0.5	0	1	2
Gradient of curve at P							

5. What are the coordinates of the stationary point? Is the stationary point a maximum or a minimum point?

6. As x increases from –3 to 2 through the stationary point, what do you notice about the sign of the gradient $\frac{dy}{dx}$ of the curve?

Part C: Stationary Point of Inflexion

Double-click on $f(x) = ax^2 + bx + c$ in the template and then change the equation of the curve from $y = ax^2 + bx + c$ to $y = ax^3 + bx + c$ by changing the power of x^2 from 2 to 3. Then click OK. Change the equation of the curve to $y = x^3$ by changing the values of a, b and c to –1, 0 and 4 respectively.

7. Change the value of the x-coordinate of P to those in the table below and record the gradient of the curve at P. Copy and complete the table.

x-coordinate of P	–3	–2	–1	0	1	2	3
Gradient of curve at P							

8. What are the coordinates of the stationary point? Is the stationary point a maximum or a minimum point?

9. As x increases from –3 to 3 through the stationary point of inflexion, what do you notice about the sign of the gradient $\frac{dy}{dx}$ of the curve?

INFORMATION

There are other types of points of inflexion that are *not* stationary. In this chapter, we will only deal with stationary points of inflexion. In some countries, 'inflexion' is spelt as 'inflection'.

Part D: Conclusion

10. What do we notice about the value of $\frac{dy}{dx}$ at a stationary point?

11. As x increases through a minimum point, how does the value of $\frac{dy}{dx}$ change?

12. As x increases through a maximum point, how does the value of $\frac{dy}{dx}$ change?

13. As x increases through a stationary point of inflexion, how does the value of $\frac{dy}{dx}$ change?

From the investigation, we conclude that

(i) A **stationary point** is a point on a curve where the gradient $\frac{dy}{dx}$ is equal to 0.

(ii) A minimum point is a stationary point on a curve where the gradient $\frac{dy}{dx}$ changes sign from negative to positive as x increases through the point.

(iii) A maximum point is a stationary point on a curve where the gradient $\frac{dy}{dx}$ changes sign from positive to negative as x increases through the point.

(iv) A **stationary point of inflexion** is a stationary point on a curve where the gradient $\frac{dy}{dx}$ does not change sign as x increases through the point.

We will now learn how to use the first derivative test to determine the nature of a stationary point.

Worked Example 4

(Maximum and Minimum Points)

Find the coordinates of the stationary points on the curve $y = x^3 - 3x + 2$ and determine the nature of these points. Hence, sketch the graph of $y = x^3 - 3x + 2$.

Solution

$$y = x^3 - 3x + 2$$

$$\frac{dy}{dx} = 3x^2 - 3$$

For stationary points, $\frac{dy}{dx} = 0$.

$$3x^2 - 3 = 0$$
$$3x^2 = 3$$
$$x^2 = 1$$
$$x = 1 \quad \text{or} \quad x = -1$$
$$y = 0 \qquad\quad y = 4$$

The stationary points are (1, 0) and (–1, 4).

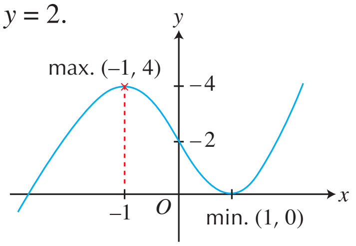
x	–1.1	–1	–0.9
$\frac{dy}{dx}$	$3(-1.1)^2 - 3$ = positive	0	$3(-0.9)^2 - 3$ = negative
Sketch of tangent			
Outline of graph			

(–1, 4) is a maximum point.

x	0.9	1	1.1
$\frac{dy}{dx}$	$3(0.9)^2 - 3$ = negative	0	$3(1.1)^2 - 3$ = positive
Sketch of tangent			
Outline of graph			

(1, 0) is a minimum point.

When $x = 0$, $y = 2$.



PROBLEM-SOLVING TIP

Substitute $x = 0$ into the equation of the curve to find the y-intercept.

Practise Now 4

Similar Questions:
Exercise 12C
Questions 1(a), (b), (d), (e), (f), 2(a), (b), (d)

Find the coordinates of the stationary points of the curve $y = x^3 - 3x^2 - 9x + 5$ and determine the nature of the stationary points. Sketch the graph of $y = x^3 - 3x^2 - 9x + 5$.

Worked Example 5

(Stationary Point of Inflexion)

Find the stationary point of the curve $y = x^3 - 1$ and determine its nature. Hence, sketch the curve $y = x^3 - 1$.

Solution

$$y = x^3 - 1$$

$$\frac{dy}{dx} = 3x^2$$

For a stationary point, $\frac{dy}{dx} = 0$.

$$3x^2 = 0$$

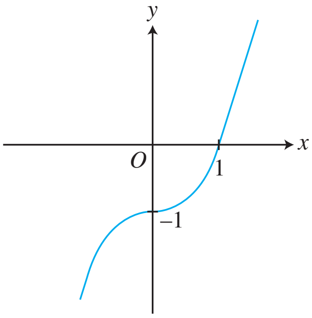
$$x = 0$$

$$y = -1$$

The stationary point is $(0, -1)$.

x	-0.1	0	0.1
$\frac{dy}{dx}$	$3(-0.1)^2$ = positive	0	$3(0.1)^2$ = positive
Sketch of tangent			
Outline of graph			

There is a stationary point of inflexion at $(0, -1)$.



PROBLEM-SOLVING TIP

We are able to obtain the x-intercept of this curve easily by substituting in $y = 0$.

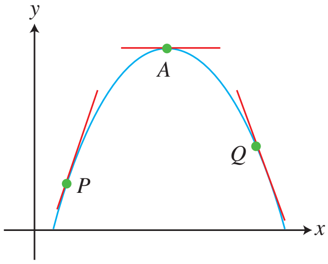
Practise Now 5

Similar Questions:
Exercise 12C
Question 1(c), 2(c)

Find the stationary points of the curve $y = x^4 - x^3$ and determine the nature of these stationary points. Hence, sketch the curve $y = x^4 - x^3$.

Second Derivative Test

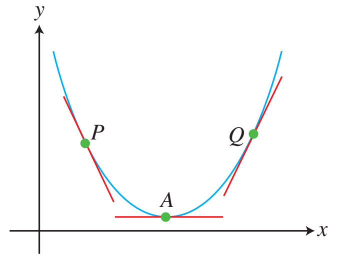
The **second derivative test** is another method of determining the nature of a stationary point. The gradient of the tangent to the curve (i.e. $\frac{dy}{dx}$) at P is positive. Then the gradient decreases to zero at the maximum point A, before decreasing further to a negative value at Q.



Since the gradient, $\frac{dy}{dx}$, decreases across a maximum point as x increases (from left to right), then the rate of change of $\frac{dy}{dx}$ (i.e. $\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$) is negative, i.e. for $\frac{d^2y}{dx^2} < 0$, $\frac{dy}{dx}$ is a decreasing function.

Therefore, a point is a maximum when $\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} < 0$.

The gradient of the tangent to the curve (i.e. $\frac{dy}{dx}$) at P is negative. Then the gradient increases to zero at the minimum point A, before increasing further to a positive value at Q. Since the gradient, $\frac{dy}{dx}$, increases across a minimum point as x increases (from left to right), then the rate of change of (i.e. $\frac{d^2y}{dx^2}$) is positive, i.e. for $\frac{d^2y}{dx^2} > 0$, $\frac{dy}{dx}$ is an increasing function.



Therefore, $\boxed{\text{a point is a minimum when } \frac{dy}{dx} = 0 \text{ and } \frac{d^2y}{dx^2} > 0.}$

What happens when $\frac{d^2y}{dx^2} = 0$? The second derivative test fails and we have to use the first derivative test to determine the nature of the stationary point, as shown in Worked Example 7.

Worked Example 6

(Second Derivative Test)

Find the coordinates of the stationary points on the curve $y = 2x^3 + 3x^2 - 12x + 7$ and determine the nature of these stationary points. Hence, sketch the curve $y = 2x^3 + 3x^2 - 12x + 7$.

Solution

$y = 2x^3 + 3x^2 - 12x + 7$

$\frac{dy}{dx} = 6x^2 + 6x - 12$

For stationary points, $\frac{dy}{dx} = 0$.

$$6x^2 + 6x - 12 = 0$$
$$x^2 + x - 2 = 0$$
$$(x + 2)(x - 1) = 0$$
$$x = -2 \quad \text{or} \quad x = 1$$
$$y = 27 \qquad y = 0$$

The stationary points are (–2, 27) and (1, 0).

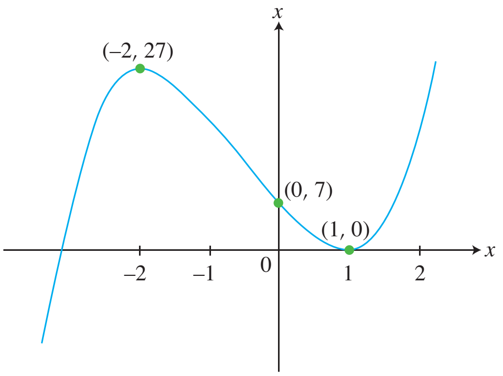
$\frac{d^2y}{dx^2} = 12x + 6$

When $x = -2$, $\frac{d^2y}{dx^2} = -18 < 0$.

When $x = 1$, $\frac{d^2y}{dx^2} = 18 > 0$.

$\therefore$ (–2, 27) is a maximum point and (1, 0) is a minimum point.

When $x = 0$, $y = 7$.



Practise Now 6

Similar Questions:
Exercise 12C
Questions 3(a), (b), (d), 4-15

Find the coordinates of the stationary points on the curve $y = 2x^3 - 3x^2 - 12x + 6$ and determine the nature of these stationary points. Sketch the curve $y = 2x^3 - 3x^2 - 12x + 6$.

Worked Example 7

($\frac{d^2y}{dx^2}$ and Point of Inflexion)

Find the coordinates of the stationary points of the curve $y = x^4 - 4x^3 + 2$ and determine the nature of the stationary points. What can you say about the stationary point when $\frac{d^2y}{dx^2} = 0$ at that value of x?

Solution

$$y = x^4 - 4x^3 + 2$$

$$\frac{dy}{dx} = 4x^3 - 12x^2$$

For stationary points, $\frac{dy}{dx} = 0$.

$$4x^3 - 12x^2 = 0$$

$$4x^2(x - 3) = 0$$

$$x = 0 \quad \text{or} \quad x = 3$$

$$y = 2 \qquad\quad y = -25$$

The stationary points are $(0, 2)$ and $(3, -25)$.

$$\frac{d^2y}{dx^2} = 12x^2 - 24x$$

When $x = 0$, $\frac{d^2y}{dx^2} = 0$.

x	-0.1	0	0.1
$\frac{dy}{dx}$	$4(-0.1)^3 - 12(-0.1)^2$ = negative	0	$4(0.1)^3 - 12(0.1)^2$ = negative
Sketch of tangent			
Outline of graph			

$\therefore$ $(0, 2)$ is a stationary point of inflexion.

When $x = 3$, $\frac{d^2y}{dx^2} = 36 > 0$.

$\therefore$ $(3, -25)$ is a minimum point.

ATTENTION

When $\frac{d^2y}{dx^2} = 0$, we use the First Derivative Test to determine the nature of the stationary point.

Practise Now 7

Similar Questions:
Exercise 12C
Questions 3(c), 12, 16

Find the coordinates of the stationary points of the curve $y = 2x(x - 3)^3 + 1$ and determine the nature of the stationary points.

Class Discussion

Work in pairs.

Worked Example 7 illustrates the case where $\frac{d^2y}{dx^2} = 0$, and the first derivative test has to be used to determine the nature of the stationary point.

(i) By using a suitable graphing software, draw the graph of $y = x^4 - 4x^3 + 2$. Write down the coordinates of the stationary points and the nature of the stationary points.

(ii) Without using a graphing software, find the coordinates of the stationary points of the curve

(a) $y = 3x^3 + 18x^2 + 27x - 5$, **(b)** $y = x^4 - 2x^3$, **(c)** $y = 2x(x - 3)^3 + 1$,

and determine the nature of the stationary points.

(iii) Now, use a graphing software to check your answers in part **(ii)**.

From the discussion, we can conclude that in the case where $\frac{d^2y}{dx^2} = 0$, the second derivative test fails to tell us the nature of the stationary point. Hence, we need to make use of the first derivative test to determine the nature of the stationary point.

Worked Example 8

(Minimum Gradient of a Curve)

Find the minimum gradient of the curve $y = 2x^3 - 9x^2 + 5x + 3$ and the value of x when the minimum gradient occurs.

Solution

$y = 2x^3 - 9x^2 + 5x + 3$

$\frac{dy}{dx} = 6x^2 - 18x + 5$, which is the gradient function

Let $z = 6x^2 - 18x + 5$.

To find the minimum gradient, we need to find the value of x when z has a stationary value, i.e. to find the value of x when $\frac{dz}{dx} = 0$.

$$\frac{dz}{dx} = 12x - 18$$

and $\frac{d^2z}{dx^2} = 12$

For stationary values of z, $\frac{dz}{dx} = 0$.

$$12x - 18 = 0$$
$$x = 1.5$$

when $x = 1.5$, $z = 6(1.5)^2 - 18(1.5) + 5 = -8.5$

when $x = 1.5$, $\frac{d^2z}{dx^2} = 12 > 0$.

$\therefore$ The minimum gradient of the curve is -8.5 and it occurs when $x = 1.5$.

Practise Now 8

Find the minimum gradient of the curve $y = 5x^3 - 24x^2 + 7x - 5$ and the value of x when the minimum gradient occurs.

Similar Question:
Exercise 12C
Question 14

Exercise 12C

1 Find the coordinates of the stationary points of each of the following curves and determine the nature of the stationary points.

(a) $y = x^2 + 3x + 1$
(b) $y = 2x - x^2 - 3$
(c) $y = x^3 - 3x^2 + 3x - 7$
(d) $y = 3x^3 - 3x^2 - 8x + 5$
(e) $y = x^2(3 - x)$
(f) $y = 4x(x^2 - 12)$

2 Using the first derivative test, find the coordinates of the stationary points of each of the following curves and determine the nature of the stationary points.

(a) $y = 27x^3 + \frac{1}{x} + 1$ **(b)** $y = \frac{(x-3)^2}{x}$
(c) $y = 4 - x^3$
(d) $y = x^3 + \frac{1}{2}x^2 - 2x + 3$

3 Find the coordinates of the stationary points of each of the following curves and determine the nature of the stationary points.

(a) $y = 2x^3 - 9x^2 + 12x + 1$
(b) $y = x^3 + 3x^2 - 9x + 7$
(c) $y = 3x^4 - 4x^3 + 5$
(d) $y = 2x^2 + \frac{500}{x}$

4 Find the coordinates of the stationary points of each of the following curves and determine the nature of the stationary points.

(a) $y = 2x^4 - 8x^3 - 25x^2 + 150x$
(b) $y = (x + 2)(x - 1)^2$
(c) $y = x^2 + \frac{128}{x}$
(d) $y = x + \frac{1}{x-1}$

5 The curve $y = 2x^3 - 15x^2 + 36x + 7$ has two turning points P and Q. Find the coordinates of P and Q and determine the nature of the turning points. Sketch the curve $y = 2x^3 - 15x^2 + 36x + 7$.

6 The equation of a curve is $y = \frac{3x+2}{2x-1}$, where $x \neq \frac{1}{2}$.

(i) Find $\frac{dy}{dx}$.
(ii) Explain whether a stationary point exists on the curve, stating the reasons clearly.

7 Given that $y = \frac{x^2 - 2x + 1}{x - 3}$, where $x \neq 3$, find

(i) an expression for $\frac{dy}{dx}$,
(ii) the coordinates of the stationary points and the nature of these stationary points.

8 The curve $y = \frac{2x^2}{x^2+1}$ has one stationary point.

(i) Show that $\frac{dy}{dx} = \frac{4x}{(x^2+1)^2}$.
(ii) Find the coordinates of the turning point and determine the nature of this turning point.

9 Given that $y = \frac{3}{\sqrt{2x-1}}$, where $x > \frac{1}{2}$,

(i) find $\frac{dy}{dx}$,
(ii) explain why the curve $y = \frac{3}{\sqrt{2x-1}}$ does not have a stationary point.

Basic Level | Intermediate Level | Advanced Level

Exercise 12C

10 Given that $y = \frac{2x-3}{3x-5}$, where $x \neq \frac{5}{3}$, find $\frac{dy}{dx}$. Hence, show that the curve $y = \frac{2x-3}{3x-5}$ has no turning points.

11 The curve $y = ax^2 + \frac{b}{x}$, where a and b are constants, has a stationary point at (2, 12).
(i) Find the value of a and of b.
(ii) Determine whether (2, 12) is a maximum or a minimum point.

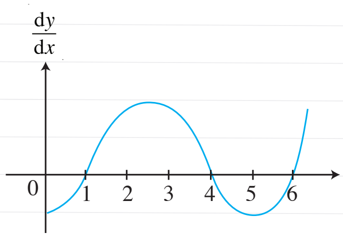
12 The curve $y = ax^4 + bx^3 + 5$, where a and b are constants, has a minimum point at (–1, 4) and (2, 85) is a point on the curve.
(i) Find the value of a and of b.
(ii) Find the coordinates of the other stationary point on the curve and determine the nature of this stationary point.

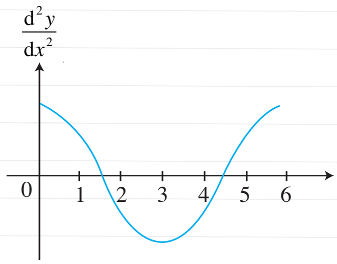
13 The curve $y = ax + \frac{b}{x^2}$, where a and b are constants, has a stationary point at $\left(\frac{1}{2}, 12\right)$.
(i) Find the value of a and of b.
(ii) Determine the nature of the stationary point of the curve.
(iii) Find the range of values of x for which the curve is an increasing function.

14 The curve $y = x^3 + px^2 + qx + 1$ has a stationary point at (2, 21).
(i) Find the value of p and of q.
(ii) Find the coordinates of the other stationary point.
(iii) Determine the nature of these two stationary points.
(iv) Find the minimum gradient of the curve and the value of x when the minimum gradient occurs.

15 The curve $y = px^3 + 3x^2 + 36x + q$ has a stationary point at (2, 51).
(i) Find the value of p and of q.
(ii) Find the coordinates of the other stationary point.
(iii) Determine the nature of these two turning points.
(iv) Find the range of values of x for which y is an increasing function.

16





The graphs of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ of a function $y = f(x)$ are shown in the diagram.

The function $y = f(x)$ passes through the points (1, –2), (4, 5) and (6, 2). Without finding the equation of the function $y = f(x)$,
(i) determine the coordinates of the maximum and minimum points of the graph of y,
(ii) sketch the graph of the function $y = f(x)$.

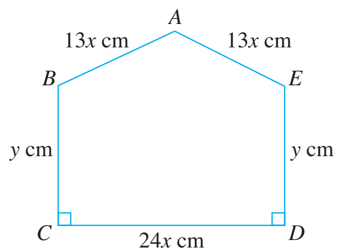
12.4 PROBLEMS ON MAXIMUM AND MINIMUM VALUES

We shall now look at some problems involving maximum and minimum values, such as finding the minimum amount of material to make a container and the maximum area that can be enclosed within a certain shape.

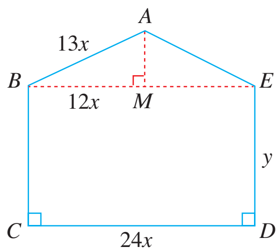
Worked Example 9

(Maximum Area enclosed by a Wire)

A piece of wire of length 240 cm is bent into the shape as shown in the diagram. Express y in terms of x and show that the area, A cm^2, enclosed by the wire is given by $A = 2880x - 540x^2$. Find the value of x and of y for which A is a maximum. Hence, find the maximum area.



Solution



Perimeter of shape is

$13x + 13x + y + y + 24x = 240$

$$y = 120 - 25x$$

In ΔABM, $(13x)^2 = (12x)^2 + AM^2$ (Pythagoras' Theorem)

$$AM^2 = 169x^2 - 144x^2$$
$$= 25x^2$$
$$AM = 5x$$

Area enclosed = Area of ΔABE + Area of rectangle $BCDE$

i.e. $A = \frac{1}{2}(24x)(5x) + 24xy$

$$= 60x^2 + 24x(120 - 25x)$$
$$= 2880x - 540x^2$$
$$\frac{dA}{dx} = 2880 - 1080x$$

For A to be a maximum or a minimum, $\frac{dA}{dx} = 0$.

$$2880 - 1080x = 0$$
$$2880 = 1080x$$
$$x = 2\frac{2}{3}$$
$$y = 120 - 25\left(2\frac{2}{3}\right) = 53\frac{1}{3}$$

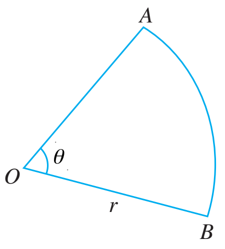
$\frac{d^2A}{dx^2} = -1080 < 0 \Rightarrow A$ is a maximum.

$A = 2880\left(2\frac{2}{3}\right) - 540\left(2\frac{2}{3}\right)^2 = 3840$ cm^2

Practise Now 9

Similar Questions:
Exercise 12D
Questions 3-5, 7, 12

A sector of a circle OAB with radius r cm contains an angle of θ radians between the bounding radii. Given that the perimeter of the sector is 7 cm, express θ in terms of r and show that the area of the sector is $\frac{1}{2}r(7-2r)\text{ cm}^2$. Hence, or otherwise, find the maximum area of this sector as r varies.

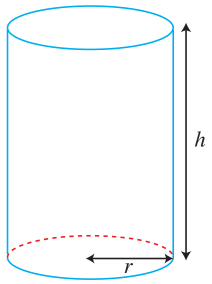


Worked Example 10

(Minimum Surface Area)

Find the least amount of material needed to make an open right cylindrical vessel with a fixed volume of 400π cm³.

Solution



Let the radius and the height of the cylinder be r and h respectively.

Since $V = \pi r^2 h$,

then $\pi r^2 h = 400\pi$.

i.e. $h = \dfrac{400}{r^2}$

Amount of material, $A = \pi r^2 + 2\pi r h$

$$= \pi r^2 + 2\pi r\left(\frac{400}{r^2}\right)$$

$$= \pi r^2 + \frac{800\pi}{r} \text{ ------------ (1)}$$

$$\frac{dA}{dr} = 2\pi r - \frac{800\pi}{r^2}$$

For the amount of material to be a maximum or minimum, $\dfrac{dA}{dr} = 0$.

$$2\pi r - \frac{800\pi}{r^2} = 0$$

$$2\pi r = \frac{800\pi}{r^2}$$

$$r^3 = 400$$

$$r = \sqrt[3]{400}$$

Subst. $r = \sqrt[3]{400}$ into (1):

$$A = \pi r^2 + \frac{800\pi}{r}$$

$$= \pi\left(\sqrt[3]{400}\right)^2 + \frac{800\pi}{\sqrt[3]{400}}$$

$$= 512 \quad \text{(to 3 s.f.)}$$

$$\frac{d^2A}{dr^2} = 2\pi + \frac{1600\pi}{r^3}$$

When $r = \sqrt[3]{400}$,

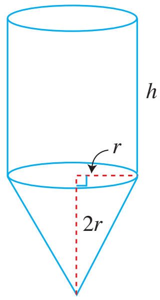
$$\frac{d^2A}{dr^2} = 2\pi + \frac{1600\pi}{r^3} = 18.8 \text{ (to 3 s.f.)} > 0 \Rightarrow A \text{ is a minimum.}$$

$\therefore$ Minimum amount of material required is 512 cm²

Practise Now 10

Similar Questions:
Exercise 12D
Questions 6, 8, 10

The diagram shows an open container which consists of a right circular cylinder of height h cm and radius r cm fixed to a hollow cone of radius r cm and height $2r$ cm. Given that the volume of the container is 60 cm^3,



(i) show that $h = \dfrac{60}{\pi r^2} - \dfrac{2}{3}r$,

(ii) show that the total surface area of the container,

$$A = \frac{120}{r} + \left(\sqrt{5} - \frac{4}{3}\right)\pi r^2,$$

(iii) find the value of r for which A has a stationary value, giving your answer correct to one decimal place. Determine whether the stationary value of A is a maximum or a minimum. Hence, find the stationary value of A.

Thinking Time

A real-life application of optimisation using calculus is to minimise the total surface area of a cylindrical can, given a fixed volume so as to reduce the cost of metal used.

Let A and V be the total surface area and volume of the cylinder with radius r and height h.

(i) Express h in terms of r.

(ii) Show that $A = 2\pi r^2 + \dfrac{2V}{r}$.

(iii) Since the volume is fixed, taking V to be a constant, find $\dfrac{\mathrm{d}A}{\mathrm{d}r}$.

(iv) Hence, show that A is a minimum when $h = 2r$.

(v) For many cylindrical canned food, $h \neq 2r$. Why do you think this is so? What other considerations would most manufacturers make when designing the dimensions of these containers?

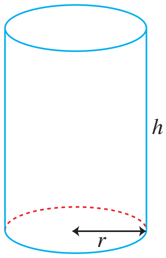
Discuss with your classmates how differentiation can be applied to solve problems such as in business and the sciences. In each case, what does the derivative represent?

Worked Example 11

(Minimum Cost involving Area of Cylinder)

A manufacturer makes closed right circular cylinders each with a volume of 32π cm³. The material for the top and bottom of the cylinder costs 2 cents per cm² while that for the sides of the container costs 1 cent per cm². Find the dimensions of the container that the manufacturer must use so that the cost will be a minimum, assuming there is no wastage when constructing the tin.

Solution



Let the radius and height of the cylinder be r cm and h cm respectively.

$\pi r^2 h = 32\pi$

$h = \dfrac{32}{r^2}$ ------- (1)

Cost function, $C = (2\pi r^2)(2) + (2\pi rh)(1)$ ------- (2)

Subst. (1) into (2): $C = 4\pi r^2 + 2\pi r\left(\dfrac{32}{r^2}\right)$

$$= 4\pi r^2 + 64\pi r^{-1}$$

$$\frac{dC}{dr} = 8\pi r - 64\pi r^{-2}$$

$$= 8\pi r - \frac{64\pi}{r^2}$$

For C to be a minimum, $\dfrac{dC}{dr} = 0$.

$$8\pi r - \frac{64\pi}{r^2} = 0$$

$$8\pi r = \frac{64\pi}{r^2}$$

$$r^3 = 8$$

$$r = 2$$

$$h = \frac{32}{2^2} = 8$$

$$\frac{d^2C}{dr^2} = 8\pi + \frac{128\pi}{r^3}$$

When $r = 2$,

$$\frac{d^2C}{dr^2} = 8\pi + \frac{128\pi}{r^3} = 24\pi > 0$$

C is a minimum when $r = 2$.

$\therefore$ The manufacturer must produce cylinders with radius 2 cm and height 8 cm to achieve minimum cost.

Practise Now 11

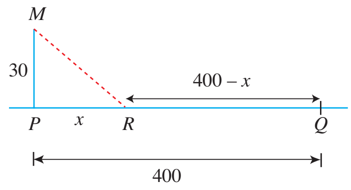
Similar Question:
Exercise 12D
Question 11

A closed box with a square base is to have a volume of 250 cubic metres. The material for the top and bottom of the box costs \$3 per square metre and the material for the sides of the box costs \$1.50 per square metre. What are the dimensions of the box if the cost of the material is to be a minimum?

Worked Example 12

(Minimum Time taken)

A man rows 30 m out to sea from a point P on a straight coast. He reaches a point M such that MP is perpendicular to the coast. He then wishes to get as quickly as possible to a point Q on the coast 400 m from P. If he can row at 40 m/min and cycle at 50 m/min, how far from P should he land?



Solution

Let the man land at the point R, x m from P, i.e. he has to row a distance of $\sqrt{30^2 + x^2}$ m and cycle a distance of $(400 - x)$ m.

$$\text{Total time taken, } T = \frac{\sqrt{30^2 + x^2}}{40} + \frac{400 - x}{50}$$

$$= \frac{1}{40}\left(30^2 + x^2\right)^{\frac{1}{2}} + 8 - \frac{x}{50}$$

$$\frac{dT}{dx} = \frac{1}{40} \times \frac{1}{2}\left(900 + x^2\right)^{-\frac{1}{2}}(2x) - \frac{1}{50}$$

$$= \frac{x}{40\sqrt{900 + x^2}} - \frac{1}{50}$$

For T to be a stationary value, $\frac{dT}{dx} = 0$.

$$\frac{x}{40\sqrt{900 + x^2}} = \frac{1}{50}$$

$$50x = 40\sqrt{900 + x^2}$$

$$5x = 4\sqrt{900 + x^2}$$

$$25x^2 = 16(900 + x^2)$$

$$9x^2 = 14\ 400$$

$$x^2 = 1600$$

$$x = 40$$

x	39.9	40	40.1
$\frac{dT}{dx}$	$\frac{39.9}{40\sqrt{900 + 39.9^2}} - \frac{1}{50}$ = negative	0	$\frac{40.1}{40\sqrt{900 + 40.1^2}} - \frac{1}{50}$ = positive

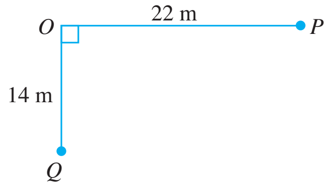
Since $\frac{dT}{dx}$ changes from negative to positive as x passes through 40, T is a minimum when $x = 40$, i.e. the man should land 40 m from P in order to reach Q as quickly as possible.

Note: Since finding $\frac{d^2T}{dx^2}$ is quite long and tedious, the first derivative test is preferred.

Practise Now 12

Similar Question:
Exercise 12D
Question 13

P is a point 22 m due east of a fixed point O and Q is a point 14 m due south of O. A particle A starts at P and moves towards O at a speed of 4 m/s while B starts at Q at the same time as A and moves towards O at a speed of 3 m/s. Find an expression for the distance between A and B t seconds from the start. Hence, find the value of t when the distance between A and B is a minimum and find this minimum distance.



Basic Level

Intermediate Level

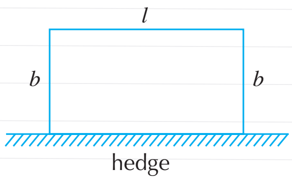
Advanced Level

Exercise 12D

1. The sum of the two numbers x and y is 82. Find the maximum value of its product P, where $P = xy$.

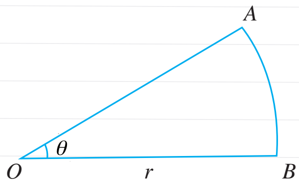
2. Find the minimum value of the sum of a positive number and its reciprocal.

3. A rectangular field is surrounded by a fence on three of its sides and a straight hedge on the fourth side. If the length of the fence is 320 m, find the maximum area of the field enclosed.

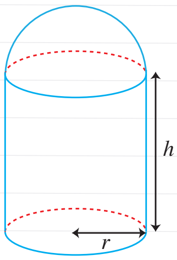


4. A rectangular box has a square base of side x cm. If the sum of one side of the square and the height is 15 cm, express the volume of the box in terms of x. Use this expression to determine the maximum volume of the box.

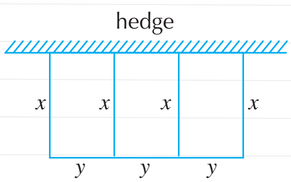
5. A wire of length 80 cm is bent to form a sector OAB of a circle. If $OA = r$ cm and $\angle AOB = \theta$ radian, express θ in terms of r and show that the area of the sector, A cm^2, is given by $A = \frac{1}{2}r(80 - 2r)$. Hence, find the maximum area of the sector as r varies.



6. A solid is formed by joining a hemisphere of radius r cm to a cylinder of radius r cm and height h cm. If the total surface area of the solid is 180π cm^2, calculate the value of r and of h such that the volume of the solid is a maximum.



7. A farmer has 800 m of fencing and wishes to make an enclosure of three equal rectangular sheep pens, using a hedge as one of its sides as shown in the figure. Find the value of x and of y that will make the total area enclosed a maximum.



8. An open tank with a square base is to be made from a thin sheet of metal. Find the length of the square base and the height of the tank so that the least amount of metal is used to make a tank with a capacity of 8 m^3.

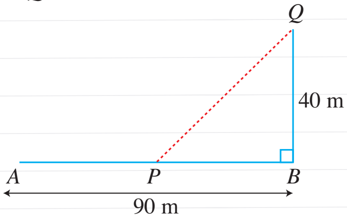
9. There are 40 spherical marbles each of radius x cm and 60 spherical marbles each of radius y cm. If x and y vary such that $x + y = 15$, find the value of x and of y that will make the sum of the volumes a minimum.

10 A matchbox consists of an outer cover, open at both breadth ends, into which slides a rectangular inner box without a top. The length, y cm, of the box is 1.5 times its breadth, x cm. If the volume of the matchbox is to be 30 cm^3 and that the thickness of the material is to be taken as negligible, show that the area, A cm^2, of material used is given by $A = 4.5x^2 + \frac{160}{x}$. If the material used is to be a minimum, find the length and the height of the box.

11 A closed right cylindrical can of radius r cm and height h cm is to be constructed to hold 600 cm^3 of liquid. Express h in terms of r. Find the value of r and of h such that the total surface area of metal used to make the can is a minimum.

12 A piece of wire 24 cm long is cut into two pieces; one is bent to form a square of side x cm and the other to form a circle of radius r cm. Express r in terms of x. Find the value of x for which the sum of the areas of the square and the circle is a minimum.

13 A man is at a point A along a straight road APB. He is to travel along AP, then along PQ, to reach a point Q which is out in a field. Given that $AB = 90$ m and $BQ = 40$ m, if he can run from A to P at a speed of 8 m/s and at 6 m/s from P to Q, find the distance AP so that he will be able to reach Q in the shortest time.



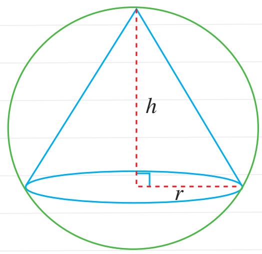
14 The volume of a right circular cone is given by the formula $V = \frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height of the cone. If $r + h = a$ $(a > 0)$, find, in terms of a, an expression for r and for h so that the volume of the cone is a maximum.

15 A right circular cone of base radius r cm and height h cm fits exactly into a sphere of internal radius 12 cm.

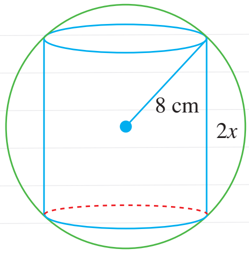
(i) Express r in terms of h.

(ii) If the volume of the cone is V cm^3, show that $V = 8\pi h^2 - \frac{1}{3}\pi h^3$.

(iii) Find the value of r for which V has a stationary value and determine whether this value of V is a maximum or a minimum. Hence, find this volume.



16 A cylinder is placed inside a sphere of radius 8 cm. If the height of the cylinder is $2x$ cm, assuming that the curved edges of the cylinder touch the surface of the sphere, show that the volume of the cylinder is $2\pi x(64 - x^2)$ cm^3. Find the value of x for which there is a maximum volume. Hence, find the maximum volume of the cylinder.



SUMMARY

1. **Increasing/Decreasing Functions**

   - $y = \mathrm{f}(x)$ is an increasing function for a given interval of x if $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$ in that interval.
   - $y = \mathrm{f}(x)$ is an decreasing function for a given interval of x if $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$ in that interval.

2. Stationary points of a function $y = \mathrm{f}(x)$ occur when $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$.

3. **First Derivative Test**

   - Point of inflexion: the sign of $\frac{\mathrm{d}y}{\mathrm{d}x}$ does not change before and after the stationary point
   - Maximum point: the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from positive to negative from the left to the right of the stationary point
   - Minimum point: the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ changes from negative to positive from the left to the right of the stationary point

4. **Second Derivative Test**

   - If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} < 0$, the stationary point is a maximum point.
   - If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} > 0$, the stationary point is a minimum point.
   - If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, the nature of the stationary point can be determined by making use of the First Derivative Test.

Review Exercise 12

1. Find the coordinates of the stationary points of each of the following functions and determine their nature.
   (a) $y = 4x^3 + 15x^2 - 18x + 2$ **(b)** $y = 2x^3 + x^2 - 8x + 8$ **(c)** $y = 2x^4 - x + 3$
   (d) $y = x^3 - 3x^2 + 7$ **(e)** $y = 3 - x^2 - \dfrac{16}{x^2}$ **(f)** $y = 3x^4 - 4x^3 + 5$

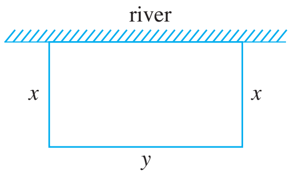
2. Given that $y = \dfrac{(2x-5)^2}{x^2+3}$, find the values of x for which y has a stationary value.

3. Find the coordinates of the stationary points of the curve $y = 2x(x-1)^3$ and determine the nature of these stationary points.

4. Calculate the minimum gradient of the curve $y = 2x^3 - 9x^2 + 5$.

5. Given that $x + y = 5$, calculate the maximum value of $2x^2 + xy - 3y^2$.

6. A farmer wishes to enclose a rectangular pen using a river as one of its sides. If the farmer has 600 m of fencing material, find the dimensions of the pen that will enclose the greatest possible area.



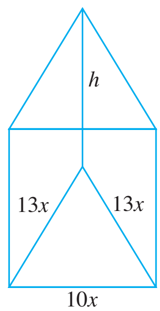
7. A right circular cylindrical vessel enclosed at both ends has a total surface area of 800 cm^2. Calculate the value of the radius that gives the vessel its maximum volume, giving your answer correct to 2 decimal places.

8. Given that the volume of a right solid cylinder of radius r cm is 250π cm^3, find the value of r for which the total surface area of the solid is a minimum.

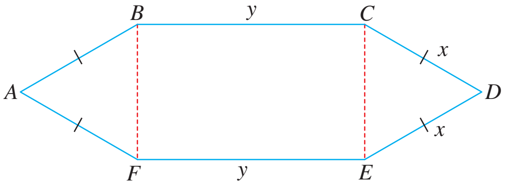
9. A wire of length 80 cm is cut into two pieces, one piece of length $4x$ cm which is used to form a square and the other piece is used to form a circle of radius r cm. Express r in terms of x. Hence, find the total area A cm^2 of the square and the circle in terms of x. Find the value of x for which A is stationary and determine whether it is a maximum or minimum.

10. The petrol consumption of a car in litres per kilometre is advertised to be $C(x) = \dfrac{1}{2000}\left(\dfrac{5000 + x^2}{x}\right)$, where x is the speed of the car in km/h. Find the optimal speed of the car when the petrol consumption will be a minimum, giving your answer correct to 2 decimal places.

11. An open box with a triangular base of sides $13x$ cm, $13x$ cm and $10x$ cm, has a volume of 3840 cm^3. If the height of the box is h cm, express h in terms of x.
    (i) If the total internal surface area of the box is A cm^2, show that $A = \dfrac{2304}{x} + 60x^2$.
    (ii) Find the value of x for which A is stationary and determine whether this value of A is a maximum or a minimum.



12. A wire of length 120 cm is bent to form the perimeter of a frame $ABCDEF$, where ABF and CDE are equilateral triangles of side x cm and $BC = EF = y$ cm.

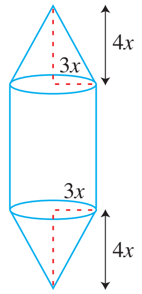


(i) Express y in terms of x.

(ii) Show that the total area of $ABCDEF$, Z cm^2, is given by $Z = \left(\frac{\sqrt{3}}{2} - 2\right)x^2 + 60x$.

(iii) Find the value of x for which Z will be a maximum, leaving your answer in the form $a + b\sqrt{3}$, where a and b are constants.

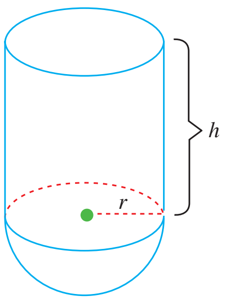
13. A man intends to build a tank of capacity 2160π cm^3 using metal sheets of negligible thickness. The tank is to consist of a right circular cylinder of height h cm and radius $3x$ cm, plus two identical right circular cones fixed at the two ends of the cylinder as shown in the diagram. Each cone has a base radius of $3x$ cm and a height of $4x$ cm.



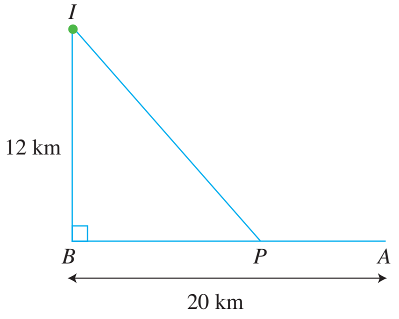
(i) Express the total surface area, A cm^2, of the metal sheet needed to construct the tank in terms of x.

(ii) Find the value of x for which the total surface area will be a minimum.

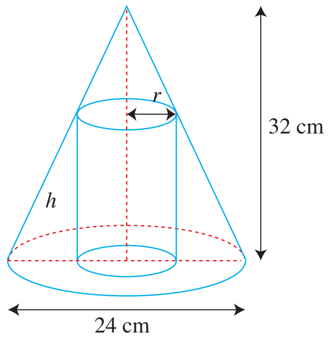
14. The figure shows an open container made up of two parts, a right circular cylinder of radius r cm and height h cm and a hemisphere of radius r cm. The container is made of some thin metal sheets. The cost of the cylindrical surface is \$1.50 per cm^2 and that of the hemispherical surface is \$3.20 per cm^2. If the volume of the container is to be 800 cm^3, find the value of h and of r such that the total cost of constructing the container is a minimum.



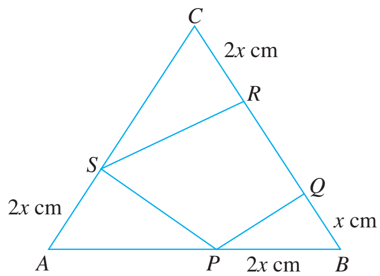
15. A power generating company intends to supply electricity to an island located some 12 km from shore and 20 km from the power plant, A, as shown in the diagram. The costs of laying 1 km of cable on land and undersea are \$500 000 and \$750 000 respectively. The company intends to lay land cables from A to a point P and undersea cables from P to the island, I. Find the length of land cables and undersea cables such that the total cost of laying the cables is a minimum.



16. A solid cylinder of radius r cm and height h cm is to be carved out of a piece of solid wood in the form of a solid cone of base diameter 24 cm and height 32 cm. Calculate the value of r for which the volume of the cylinder carved out will be a maximum and find this maximum volume.



17. The figure shows an equilateral triangle ABC of sides 18 cm. $PQRS$ is a quadrilateral where $PB = CR = AS = 2x$ cm and $BQ = x$ cm. Express the area of $PQRS$ in terms of x. Find the value of x for which the area of $PQRS$ is a minimum and find the minimum area of $PQRS$.



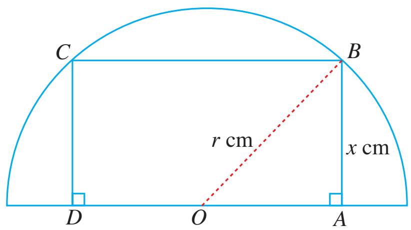
18. The distance between an airport terminal, A, and a factory, F, is x km. The travelling cost of sending goods from F to A is directly proportional to x, and the rental cost of the factory is inversely proportional to x. When $x = 2$, the travelling cost and rental costs are \$1600 and \$90 000 respectively.

(i) Show that the total travelling cost and rental cost of the factory is \$$C$, where $C = 800x + \dfrac{180\,000}{x}$.

(ii) Find the value of x for which C is a minimum. Hence, find the minimum cost.

19. At 8 a.m., ship A is 100 km due north of ship B. Ship A is sailing due south at 20 km/h and ship B is sailing due east at 10 km/h. At what time will A be closest to B?

A rectangle $ABCD$ is inscribed in a semicircle with a fixed radius r cm. Two vertices, B and C, of the rectangle, lie on the arc of the semicircle.



(i) If $AB = x$ cm, show that the perimeter, P cm, of the rectangle is $2x + 4\sqrt{r^2 - x^2}$.

(ii) Prove that as x varies, the maximum value of P occurs when $AB : BC = 1 : k$, where k is to be determined.

DIFFERENTIATION OF TRIGONOMETRIC, LOGARITHMIC & EXPONENTIAL FUNCTIONS AND THEIR APPLICATIONS

Each time a key on a piano is pressed, the soundboard of the piano generates sound. The sound pressure of a musical note can be represented by a logarithmic equation. In this chapter, we will learn how to find the derivatives of trigonometric, logarithmic and exponential functions and to apply them to solve problems.

CHAPTER 13

excluded from the N(A) syllabus

Learning Objectives

At the end of this chapter, you should be able to:

- differentiate trigonometric functions,
- differentiate logarithmic functions and exponential functions,
- solve problems involving the differentiation of trigonometric, logarithmic and exponential functions.

13.1 DERIVATIVES OF TRIGONOMETRIC FUNCTIONS

In Chapter 11, we have learnt how to find the derivatives of algebraic expressions. In this section, we shall focus on obtaining the derivatives of trigonometric functions.

Investigation

Derivatives of $\sin x$ and $\cos x$

1. Use a suitable graphing software to plot the graph of $y = \sin x$, where x is in radians. From the menu, select the option to display the graph of the derivative of $y = \sin x$.
   (i) What is the equation of the graph that you obtain?
   (ii) What can you say about the derivative of $\sin x$?

2. On a new page, plot the graph of $y = \cos x$, where x is in radians. From the menu, select the option to display the graph of the derivative of $y = \cos x$.
   (i) What is the equation of the graph that you obtain?
   (ii) What can you say about the derivative of $\cos x$?

From the investigation, we can conclude that

$$\frac{d}{dx}(\sin x) = \cos x, \text{ where } x \text{ is in radians,}$$
$$\frac{d}{dx}(\cos x) = -\sin x, \text{ where } x \text{ is in radians.}$$

We shall now use the above results to find the derivative of $\tan x$ and other trigonometric functions.

Derivative of $\tan x$

Let $y = \tan x = \dfrac{\sin x}{\cos x}$.

Applying the Quotient Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{(\cos x)^2} \\ &= \frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} \\ &= \sec^2 x\end{aligned}$$

i.e. $\boxed{\frac{d}{dx}(\tan x) = \sec^2 x,}$ where x is in radians

Worked Example 1

(Derivatives of Trigonometric Functions)

Differentiate each of the following with respect to x.

(a) $4 \sin x$ **(b)** $2x^2 \cos x$

(c) $\dfrac{\tan x}{2x+3}$ **(d)** $(3x + 2 \sin x)^4$

PROBLEM-SOLVING TIP

$\dfrac{d}{dx}[kf(x)] = k\dfrac{d}{dx}f(x)$

Solution

(a) $\dfrac{d}{dx}(4 \sin x) = 4 \cos x$

(b) Let $y = 2x^2 \cos x$.

$$\frac{dy}{dx} = 2x^2\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(2x^2) \quad \text{(Product Rule)}$$

$$= 2x^2(-\sin x) + (\cos x)(4x)$$

$$= 4x \cos x - 2x^2 \sin x$$

(c) Let $y = \dfrac{\tan x}{2x+3}$.

$$\frac{dy}{dx} = \frac{(2x+3)\frac{d}{dx}(\tan x) - \tan x\frac{d}{dx}(2x+3)}{(2x+3)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{(2x+3)\sec^2 x - 2\tan x}{(2x+3)^2}$$

(d) Let $y = (3x + 2 \sin x)^4$.

$$\frac{dy}{dx} = 4(3x + 2\sin x)^3 \times \frac{d}{dx}(3x + 2\sin x) \quad \text{(Chain Rule)}$$

$$= 4(3x + 2 \sin x)^3(3 + 2 \cos x)$$

Practise Now 1

Similar Questions:
Exercise 13A
Question 1(a), (b)

1. Differentiate each of the following with respect to x.

 (a) $3 \tan x - 2 \sin x$ **(b)** $3x^5 \sin x$

 (c) $\dfrac{2\cos x}{5x+1}$ **(d)** $(3x + 5 \cos x)^6$

2. Given that $f(x) = \dfrac{x}{\tan x}$, show that $f'(x)$ can be written in the form $a \cot x + bx \operatorname{cosec}^2 x$, where a and b are constants to be determined.

Derivatives of sin($ax + b$), cos($ax + b$) and tan($ax + b$)

Let $y = \sin(ax + b)$ and $u = ax + b$, where a and b are constants and x is in radians.

i.e. $y = \sin u$ $\qquad \dfrac{du}{dx} = a$

$\dfrac{dy}{du} = \cos u$

Applying the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx} \\ &= \cos u \times a \\ &= a\cos(ax + b)\end{aligned}$$

i.e. $\boxed{\dfrac{d}{dx}[\sin(ax+b)] = a\cos(ax+b)}$, where x is in radians

Relating to the differentiation of algebraic expressions which we have learnt in Chapter 11, we have

$$y = \sin(ax+b)$$

differentiate trigonometric function first

then differentiate angle

$$\frac{dy}{dx} = \cos(ax+b) \times a$$

Similarly, we can show that, if x is in radians,

$$\boxed{\begin{aligned}&\frac{d}{dx}[\cos(ax+b)] = -a\sin(ax+b), \\ &\frac{d}{dx}[\tan(ax+b)] = a\sec^2(ax+b).\end{aligned}}$$

INFORMATION

In particular,

$\dfrac{d}{dx}(\sin ax) = a\cos ax$

$\dfrac{d}{dx}(\cos ax) = -a\sin ax$

$\dfrac{d}{dx}(\tan ax) = a\sec^2 ax$

Note: In calculus, unless otherwise stated, all angles are measured in **radians**.

Worked Example 2

(Derivatives of $\sin(ax + b)$, $\cos(ax + b)$ and $\tan(ax + b)$)

Differentiate each of the following with respect to x.

(a) $2\sin\left(3x + \dfrac{\pi}{2}\right)$ **(b)** $3x^2 \tan 5x$ **(c)** $\dfrac{\sin 4x}{\cos\left(\dfrac{\pi}{2} - 2x\right)}$

Solution

(a) $\dfrac{d}{dx}\left[2\sin\left(3x+\dfrac{\pi}{2}\right)\right] = 2\cos\left(3x+\dfrac{\pi}{2}\right) \times 3$

differentiate trigonometric function first

then differentiate angle

$$= 6\cos\left(3x+\frac{\pi}{2}\right)$$

(b) $\dfrac{d}{dx}\left(3x^2 \tan 5x\right) = 3x^2 \dfrac{d}{dx}(\tan 5x) + \tan 5x \dfrac{d}{dx}(3x^2)$ (Product Rule)

$$\begin{aligned} &= 3x^2 (5 \sec^2 5x) + (\tan 5x)(6x) \\ &= 15x^2 \sec^2 5x + 6x \tan 5x \\ &= 3x(5x \sec^2 5x + 2 \tan 5x) \end{aligned}$$

(c) Let $y = \dfrac{\sin 4x}{\cos\left(\dfrac{\pi}{2} - 2x\right)}$.

$$\begin{aligned} \frac{dy}{dx} &= \frac{\cos\left(\dfrac{\pi}{2} - 2x\right)\dfrac{d}{dx}(\sin 4x) - \sin 4x \dfrac{d}{dx}\left[\cos\left(\dfrac{\pi}{2} - 2x\right)\right]}{\cos^2\left(\dfrac{\pi}{2} - 2x\right)} \\ &= \frac{\cos\left(\dfrac{\pi}{2} - 2x\right) \times (4 \cos 4x) - \sin 4x\left[-\sin\left(\dfrac{\pi}{2} - 2x\right) \times (-2)\right]}{\cos^2\left(\dfrac{\pi}{2} - 2x\right)} \\ &= \frac{4 \cos 4x \cos\left(\dfrac{\pi}{2} - 2x\right) - 2 \sin 4x \sin\left(\dfrac{\pi}{2} - 2x\right)}{\cos^2\left(\dfrac{\pi}{2} - 2x\right)} \\ &= \frac{4 \cos 4x \sin 2x - 2 \sin 4x \cos 2x}{\sin^2 2x} \end{aligned}$$

Thinking Time

In Worked Example 2(c), the answer can also be written as $-4 \sin 2x$. Can you explain why this is so?

Hint: Try to recall trigonometric formulae that we have learnt in Chapters 8 and 9 to simplify $\dfrac{\sin 4x}{\cos\left(\dfrac{\pi}{2} - 2x\right)}$ *before finding the derivative.*

Practise Now 2

Similar Questions:
Exercise 13A
Questions 1(c)-(h), 3(a)-(f)

1. Differentiate each of the following with respect to x.

(a) $5 \cos\left(2x + \dfrac{\pi}{4}\right)$ **(b)** $4x \tan\left(3x - \dfrac{\pi}{3}\right)$ **(c)** $\dfrac{2 \sin 5x}{\cos\left(\dfrac{\pi}{2} - 4x\right)}$

2. Given that $f(x) = x \sec \pi x$, find an expression for $f'(x)$, giving your answer in terms of $\sec \pi x$ and $\tan \pi x$ only.

Worked Example 3

(Derivatives of $\sin^n f(x)$, $\cos^n f(x)$ and $\tan^n f(x)$ using the Chain Rule)

Differentiate each of the following with respect to x.

(a) $3\sin^4 x + 2\cos^2 3x$ **(b)** $4\tan^3\left(\frac{x}{2}+\frac{\pi}{3}\right)$

Solution

(a)
$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}(3\sin^4 x + 2\cos^2 3x)\\
&= 3\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)^4 + 2\frac{\mathrm{d}}{\mathrm{d}x}(\cos 3x)^2\\
&= 3\left[4(\sin x)^3\frac{\mathrm{d}}{\mathrm{d}x}(\sin x)\right] + 2\left[2(\cos 3x)\frac{\mathrm{d}}{\mathrm{d}x}(\cos 3x)\right] \quad \text{(Chain Rule)}\\
&= 12\sin^3 x\,(\cos x) + 4\cos 3x\,(-3\sin 3x)\\
&= 12\sin^3 x\cos x - 12\cos 3x\sin 3x
\end{aligned}$$

(b)
$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}\left[4\tan^3\left(\frac{x}{2}+\frac{\pi}{3}\right)\right]\\
&= 4\frac{\mathrm{d}}{\mathrm{d}x}\left[\tan\left(\frac{x}{2}+\frac{\pi}{3}\right)\right]^3\\
&= 4\times 3\left[\tan\left(\frac{x}{2}+\frac{\pi}{3}\right)\right]^2 \times \sec^2\left(\frac{x}{2}+\frac{\pi}{3}\right)\times\frac{1}{2} \quad \text{(Chain Rule)}\\
&= 6\tan^2\left(\frac{x}{2}+\frac{\pi}{3}\right)\sec^2\left(\frac{x}{2}+\frac{\pi}{3}\right)
\end{aligned}$$

Practise Now 3

Similar Questions:
Exercise 13A
Questions 2(a)-(c), 4(d)

Differentiate each of the following with respect to x.

(a) $2\sin^5 x - 5\tan^2\left(2x+\frac{\pi}{2}\right)$ **(b)** $3\cos^3\left(\frac{\pi}{2}-4x\right)$

Worked Example 4

(Value of a Derivative of a Trigonometric Function)

Given that $y = \dfrac{\cos x}{5 + \sin x}$, find the value of $\dfrac{dy}{dx}$ when $x = 0.5$ radian, giving your answer correct to 3 decimal places.

Solution

$$y = \frac{\cos x}{5 + \sin x}$$

$$\frac{dy}{dx} = \frac{(5 + \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(5 + \sin x)}{(5 + \sin x)^2}$$

$$= \frac{(5 + \sin x)(-\sin x) - \cos x(\cos x)}{(5 + \sin x)^2}$$

$$= \frac{-5 \sin x - \sin^2 x - \cos^2 x}{(5 + \sin x)^2}$$

$$= \frac{-5 \sin x - \left(\sin^2 x + \cos^2 x\right)}{(5 + \sin x)^2}$$

$$= \frac{-5 \sin x - 1}{(5 + \sin x)^2}$$

When $x = 0.5$,

$$\frac{dy}{dx} = \frac{-5 \sin 0.5 - 1}{(5 + \sin 0.5)^2}$$

$$= -0.113 \text{ (to 3 d.p.)}$$

RECALL

$\sin^2 x + \cos^2 x = 1$

PROBLEM-SOLVING TIP

Make use of trigonometric identities to simplify expressions where possible.

Practise Now 4

Similar Questions:
Exercise 13A
Questions 7, 8, 9

1. Given that $y = \dfrac{\sin 2x}{3 + \cos 2x}$, find the value of $\dfrac{dy}{dx}$ when $x = 0.8$ radian, giving your answer correct to 3 decimal places.

2. Given that $y = 3 \sin 2x \cos^3 2x$, find the value of $\dfrac{dy}{dx}$ when $x = \dfrac{\pi}{6}$.

Class Discussion

By writing $\sec x$ as $\dfrac{1}{\cos x}$, show that $\dfrac{d}{dx}(\sec x) = \sec x \tan x$.

Discuss with your classmates if you are able to obtain similar expressions for $\dfrac{d}{dx}(\operatorname{cosec} x)$ and $\dfrac{d}{dx}(\cot x)$.

Basic Level | Intermediate Level | Advanced Level

Exercise 13A

1 Differentiate each of the following with respect to x.

(a) $5 \sin x + 3 \tan x$ **(b)** $3x^2 + 4 \tan x$

(c) $3 \cos 2x + \tan 4x$ **(d)** $4 \sin 7x - x$

(e) $\cos 7x + \sin 3x$ **(f)** $\tan\left(7x + \dfrac{\pi}{4}\right)$

(g) $\sin\left(\dfrac{x}{\sqrt{2}} - \dfrac{\pi}{3}\right)$ **(h)** $\cos\left(\dfrac{\pi}{3} - \dfrac{3}{2}x\right)$

2 Find the derivative of each of the following.

(a) $\sin^3 x + 4 \cos x$ **(b)** $\sin 2x - 3 \cos^2 x$

(c) $\sin^5 (2x - 3\pi)$ **(d)** $(2 \sin x - \cos x)^3$

(e) $\tan x^3$ **(f)** $\sec 4x$

3 Differentiate each of the following with respect to x.

(a) $2x^2 + 3x \sin x$

(b) $4x^3 \cos 2x$

(c) $2x^3 \cos x + 5 \tan x$

(d) $3x \cos 5x - 5x \tan 3x$

(e) $\operatorname{cosec} 2x$

(f) $\cot 3x$

4 Find the derivative of each of the following.

(a) $(5x \cos x + 6x^2)^4$

(b) $\sin x \cos 2x$

(c) $\cos 3x \tan 4x$

(d) $3x^2 \tan^2 (x + \pi)$

(e) $\dfrac{3 \sin x}{2x + 1}$

(f) $\dfrac{2 + x}{\sin 2x}$

(g) $\dfrac{\sin x - \cos x}{\sin x + \cos x}$

5 Find $\frac{dy}{dx}$ for each of the following.

(a) $y = \sin \frac{4}{x}$

(b) $y = \cos \frac{3}{x}$

(c) $y = \tan \frac{5}{x}$

(d) $y = \sin^2 3x^2 - \cos \frac{1}{x}$

(e) $y = \sec^2 3x$

(f) $y = \frac{3}{\sin (3x + \pi)}$

(g) $y = \frac{4}{\sec (3x - \pi)}$

(h) $y = \frac{2}{\cos^3 \left(4x + \frac{\pi}{2}\right)}$

6 Differentiate each of the following with respect to x, where a, b, m, p and q are constants.

(a) $p \cos qx + q \sin px$

(b) $a \sin x + bx \cos x$

(c) $x \sin \frac{a}{x}$

(d) $\tan^3 (px + q)$

(e) $\frac{\sin px}{\cos qx}$

(f) $\frac{\sin mx}{\tan px}$

7 Given that $y = 3x \sin 2x$, find the value of $\frac{dy}{dx}$ when $x = 1.2$.

8 Given that $f(x) = \frac{3 + \sin x}{2 \tan x}$, find the value of $f'(x)$ when $x = 0.6$.

9 Given that $y = \sqrt{\sin x - 4 \tan x}$, find the value of $\frac{dy}{dx}$ when $x = 2$.

10 Find the value of h and of k such that $\frac{d}{dx}\left(\frac{2 \sin x}{3 + 4 \cos x}\right) = \frac{h + k \cos x}{(3 + 4 \cos x)^2}$.

11 Given that $y = \sin ax + \cos bx$, where a and b are constants, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.
If $a - b = 2$ and $\frac{d^2y}{dx^2} = 16 - a^2 y$ when $x = 0$, find the value of $a + b$.

12 Given that $\frac{d}{dx}\left(\sqrt[3]{6 + 2 \tan x}\right) = \frac{p \sec^2 x}{q\sqrt[3]{(6 + 2 \tan x)^2}}$, find the value of p and of q.

13.2 DERIVATIVES OF LOGARITHMIC FUNCTIONS

In Section 13.1, we have learnt the derivatives of trigonometric functions. The logarithmic function has been covered in Chapter 4. What is the derivative of the logarithmic function?

Derivative of ln x

Investigation

Derivative of ln x

Use a suitable graphing software to plot the graph of $y = \ln x$. From the menu, select the option to display the graph of the derivative of $y = \ln x$.

(i) What is the equation of the graph that you obtain?

(ii) What can you say about the derivative of ln x?

From the investigation, we can conclude that

$$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x}.}$$

Derivative of Logarithmic Functions

Consider $y = \ln \mathrm{f}(x)$, where $\mathrm{f}(x)$ is a function of x and $\mathrm{f}(x) > 0$ for any value of x.

Let $y = \ln u$ and $u = \mathrm{f}(x)$.

i.e. $\frac{\mathrm{d}y}{\mathrm{d}u} = \frac{1}{u}$ $\qquad$ $\frac{\mathrm{d}u}{\mathrm{d}x} = \mathrm{f}'(x)$

Applying the Chain Rule,

$$\begin{aligned}\frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x} \\ &= \frac{1}{u} \times \mathrm{f}'(x) \\ &= \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\end{aligned}$$

i.e. $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[\ln \mathrm{f}(x)] = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}}$$

In particular, $$\boxed{\frac{\mathrm{d}}{\mathrm{d}x}[\ln (ax+b)] = \frac{a}{ax+b}.}$$

If $y = \log_a x$, then $y = \frac{\ln x}{\ln a}$,

i.e. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\ln a} \times \frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x \ln a}$.

Worked Example 5

(Derivative of ln f(x))

Differentiate each of the following with respect to x.

(a) $\ln (5x^2 + 7x)$ **(b)** $\ln \cos 3x$ **(c)** $\ln \sqrt[3]{x^2 + 5}$

(d) $\log_5 x^3$ **(e)** $\ln \sqrt{\frac{2+x}{2-x}}$

Solution

(a) $\dfrac{d}{dx}\left[\ln(5x^2+7x)\right] = \dfrac{10x+7}{5x^2+7x}$ ← differentiate 'inside' ← copy 'inside'

(b) $$\frac{d}{dx}\left[\ln\cos 3x\right] = \frac{\frac{d}{dx}(\cos 3x)}{\cos 3x}$$
$$= \frac{-\sin 3x \times 3}{\cos 3x}$$
$$= -3\tan 3x$$

(c) $$\frac{d}{dx}\left[\ln\sqrt[3]{x^2+5}\right] = \frac{d}{dx}\left[\ln\left(x^2+5\right)^{\frac{1}{3}}\right]$$
$$= \frac{d}{dx}\left[\frac{1}{3}\ln(x^2+5)\right]$$
$$= \frac{1}{3}\times\frac{d}{dx}[\ln(x^2+5)]$$
$$= \frac{1}{3}\times\frac{\frac{d}{dx}(x^2+5)}{x^2+5}$$
$$= \frac{1}{3}\times\frac{2x}{x^2+5}$$
$$= \frac{2x}{3\left(x^2+5\right)}$$

PROBLEM-SOLVING TIP

Apply relevant logarithmic laws f(x) before using the formula to differentiate the logarithmic function.

RECALL

$$\log_a b = \frac{\log_c b}{\log_c a}$$

(d) $$\frac{d}{dx}\left(\log_5 x^3\right) = 3\times\frac{d}{dx}\left(\log_5 x\right)$$
$$= 3\times\frac{d}{dx}\left(\frac{\ln x}{\ln 5}\right)$$
$$= \frac{3}{\ln 5}\times\frac{d}{dx}(\ln x)$$
$$= \frac{3}{\ln 5}\times\frac{1}{x}$$
$$= \frac{3}{x\ln 5}$$

(e) $$\frac{d}{dx}\left[\ln\sqrt{\frac{2+x}{2-x}}\right] = \frac{1}{2}\times\frac{d}{dx}\ln\left(\frac{2+x}{2-x}\right)$$
$$= \frac{1}{2}\times\frac{d}{dx}[\ln(2+x)-\ln(2-x)]$$
$$= \frac{1}{2}\times\left[\frac{1}{2+x}-\frac{-1}{2-x}\right]$$
$$= \frac{1}{2}\times\left[\frac{1}{2+x}+\frac{1}{2-x}\right]$$
$$= \frac{1}{2}\times\left[\frac{(2-x)+(2+x)}{(2+x)(2-x)}\right]$$
$$= \frac{2}{(2+x)(2-x)}$$

RECALL

$$\log_a\frac{x}{y} = \log_a x - \log_a y$$

In Worked Example 5(e), if we let $f(x) = \ln\sqrt{\dfrac{2+x}{2-x}}$, we will have an alternative method to find the derivative. Try it out and discuss with your classmates if you prefer this method. Explain your choice.

Practise Now 5

Similar Questions:
Exercise 13B
Questions 1(a)-(h), 2(a)-(d), 7(a), (b), (e)-(h)

1. Differentiate each of the following with respect to x.

(a) $\ln(5x+2)$ **(b)** $\ln \sin 4x$ **(c)** $\ln\sqrt[3]{x^3+2x}$

(d) $\log_7(x^5+4)$ **(e)** $\ln\dfrac{x}{\sqrt{x^2+2}}$

2. Given that $e^y = 5x^3 + 2x$, find an expression, in terms of x, for $\dfrac{dy}{dx}$.

Worked Example 6

(Value of a Derivative of a Logarithmic Function)

For each of the following functions, find the value of $\dfrac{dy}{dx}$ when $x = 1$.

(a) $y = 3x^2 \ln x$ **(b)** $y = \dfrac{\ln(3x-1)}{x^2}$

Solution

(a) $y = 3x^2 \ln x$

$$\frac{dy}{dx} = 3x^2 \times \frac{d}{dx}(\ln x) + \ln x \times \frac{d}{dx}(3x^2) \quad \text{(Product Rule)}$$

$$= 3x^2 \times \frac{1}{x} + \ln x \times 6x$$

$$= 3x + 6x \ln x$$

When $x = 1$,

$$\frac{dy}{dx} = 3(1) + 6(1)\ln 1$$

$$= 3$$

(b) $y = \dfrac{\ln(3x-1)}{x^2}$

$$\frac{dy}{dx} = \frac{x^2 \times \frac{d}{dx}[\ln(3x-1)] - \ln(3x-1) \times \frac{d}{dx}\left(x^2\right)}{\left(x^2\right)^2} \quad \text{(Quotient Rule)}$$

$$= \frac{x^2 \times \frac{3}{3x-1} - \ln(3x-1) \times 2x}{x^4}$$

$$= \frac{3x^2 - 2x(3x-1)\ln(3x-1)}{x^4(3x-1)}$$

When $x = 1$,

$$\frac{dy}{dx} = \frac{3(1)^2 - 2(1)[3(1)-1]\ln[3(1)-1]}{(1)^4[3(1)-1]} = 0.114 \text{ (to 3 s.f.)}$$

Practise Now 6

Similar Questions:
Exercise 13B
Questions 3, 6

1. Given that $y = x \ln(\tan 2x)$, find the value of $\frac{dy}{dx}$ when $x = 0.5$.
2. Given that $f(x) = \frac{\ln 2x}{1+x}$, where $x > 0$, find the value of $f'(1.2)$.

Basic Level | Intermediate Level | Advanced Level

Exercise 13B

1 Differentiate each of the following with respect to x.

(a) $\ln(5 - x)$
(b) $\ln(3 - x^3)$
(c) $\ln(x^2 - 2x)$
(d) $\ln(5x - 1)^2$
(e) $\ln(2x + 5)^3$
(f) $\ln\sqrt{1+x}$
(g) $\ln\frac{2}{x}$
(h) $\ln\frac{4}{5-2x}$

2 Differentiate each of the following with respect to x.

(a) $\ln\cos x$
(b) $\ln\tan x$
(c) $\ln\sqrt{\frac{1-3x}{2x+1}}$
(d) $\ln\sqrt[3]{\frac{x^2}{5x+4}}$

3 Given that $f(x) = \ln 5x$, where $x > 0$, find the value of $f'(1)$.

4 Differentiate each of the following with respect to x.

(a) $3x^2 \ln 5x$
(b) $(1 + x^2)^3 \ln x$
(c) $\frac{\ln x}{x}$
(d) $\ln\left(\sqrt{x} - 1\right)^2$

5 Find $\frac{dy}{dx}$ for each of the following.

(a) $y = x^2 \ln x$
(b) $y = x^3 \ln x^2$
(c) $y = (2x^3 + 1)\ln x$
(d) $y = \frac{\ln x}{5x}$

6 Given that $y = x \ln 2x$, find the value of $\frac{dy}{dx}$ and of $\frac{d^2y}{dx^2}$ at the point where $x = \frac{1}{4}$.

7 Find $\frac{dy}{dx}$ for each of the following.

(a) $y = \lg 2x$
(b) $y = \log_5(2x + 3)$
(c) $y = \log_a \sin x$
(d) $y = \log_a(x^3 + 1)$
(e) $e^y = x^2 + 2$
(f) $e^y = 2x^3 + 7x$
(g) $e^y = \cos 2x$
(h) $e^y = \sec x$

8 Given that $y = \frac{\sin 2x}{\ln x^2}$, show that

$$\frac{dy}{dx} = \frac{2x \ln x^{\cos 2x} - \sin 2x}{2x(\ln x)^2}.$$

9 Given that $x = \frac{1}{3}e^{y(2x+5)}$, find the value of $\frac{dy}{dx}$ at the point where the curve intersects the line $x - e^2 = 0$.

13.3 DERIVATIVES OF EXPONENTIAL FUNCTIONS

The exponential function has been covered in Chapter 4. How is the derivative of the exponential function obtained?

Derivative of e^x

Investigation

Use a suitable graphing software to plot the graph of $y = e^x$. From the menu, select the option to display the graph of the derivative of $y = e^x$.

(i) What is the equation of the graph that you obtain?

(ii) What can you say about the derivative of e^x?

Derivative of e^x

From the investigation, we can conclude that

$$\boxed{\frac{d}{dx}(e^x) = e^x.}$$

Derivative of Exponential Functions

Let us use the above result to find the derivative of $e^{f(x)}$, where $f(x)$ is a function of x.

Consider the function $y = e^{f(x)}$.

Let $y = e^u$ and $u = f(x)$.

i.e. $\frac{dy}{du} = e^u$ $\qquad$ $\frac{du}{dx} = f'(x)$

Applying the Chain Rule,

$$\begin{aligned}\frac{dy}{dx} &= \frac{dy}{du} \times \frac{du}{dx}\\ &= e^u \times f'(x)\\ &= f'(x) \times e^{f(x)}\end{aligned}$$

i.e. $$\boxed{\frac{d}{dx}\left[e^{f(x)}\right] = f'(x) \times e^{f(x)}}$$

In particular,

$$\boxed{\begin{aligned}\frac{d}{dx}\left(e^{ax}\right) &= ae^{ax}\\ \frac{d}{dx}\left(e^{ax+b}\right) &= ae^{ax+b}\end{aligned}}$$, where a and b are constants.

Worked Example 7

(Derivative of $e^{f(x)}$)

Differentiate each of the following with respect to x.

(a) e^{4-3x} **(b)** $3x^2 e^{2x}$ **(c)** $e^{2x+1} \sin 2x$ **(d)** $\dfrac{e^{3-4x}}{x}$

Solution

(a) $$\frac{d}{dx}\left(e^{4-3x}\right) = \underbrace{e^{4-3x}}_{\text{copy entire term}} \frac{d}{dx}\underbrace{(4-3x)}_{\text{differentiate 'inside'}}$$

$$= -3e^{4-3x}$$

(b) $$\frac{d}{dx}\left(3x^2e^{2x}\right) = 3x^2 \times \frac{d}{dx}\left(e^{2x}\right) + e^{2x} \times \frac{d}{dx}\left(3x^2\right)$$ (Product Rule)

$$= 3x^2 \times (2e^{2x}) + e^{2x} \times (6x)$$

$$= 6x(x+1)e^{2x}$$

(c) $$\frac{d}{dx}\left(e^{2x+1} \sin 2x\right) = e^{2x+1}\frac{d}{dx}(\sin 2x) + \sin 2x \frac{d}{dx}\left(e^{2x+1}\right)$$ (Product Rule)

$$= e^{2x+1}(2\cos 2x) + \sin 2x(2e^{2x+1})$$

$$= 2e^{2x+1}(\cos 2x + \sin 2x)$$

(d) $$\frac{d}{dx}\left(\frac{e^{3-4x}}{x}\right) = \frac{x\frac{d}{dx}\left(e^{3-4x}\right) - e^{3-4x}\frac{d}{dx}(x)}{x^2}$$ (Quotient Rule)

$$= \frac{x\left(-4e^{3-4x}\right) - e^{3-4x}(1)}{x^2}$$

$$= -\frac{(4x+1)e^{3-4x}}{x^2}$$

Practise Now 7

Similar Questions:
Exercise 13C
Questions 1-5

Differentiate each of the following with respect to x.

(a) $3e^{5x}$ **(b)** $2e^{5-2x}$ **(c)** e^{3x^2-5x}

(d) $e^{5+2x} \cos 2x$ **(e)** $\dfrac{e^{2+5x}}{3x}$

Worked Example 8

(Value of a Derivative of an Exponential Function)

Given that $y = e^{x^2+3x}$, find the value of $\dfrac{dy}{dx}$ at the point where $x = 0$.

Solution

$$\frac{d}{dx}\left(e^{x^2+3x}\right) = e^{x^2+3x}\frac{d}{dx}\left(x^2+3x\right)$$

$$= (2x+3)e^{x^2+3x}$$

When $x = 0$,

$$\frac{dy}{dx} = 3e^0 = 3$$

Practise Now 8

Similar Questions:
Exercise 13C
Questions 6-8

Given that $y = (e^x + e^{2x})^4$, find the value of $\frac{dy}{dx}$ at the point where $x = 0$.

Basic Level | Intermediate Level | Advanced Level

Exercise 13C

1 Differentiate each of the following with respect to x.

(a) $5e^x$ **(b)** e^{3x+1}

(c) e^{-4x} **(d)** e^{x^2+7}

2 Find $\frac{dy}{dx}$ for each of the following.

(a) $y = e^{\sin x}$ **(b)** $y = e^{\cos 2x}$

(c) $y = e^{4\tan x}$ **(d)** $y = 3e^{\sin x - \cos x}$

3 Find $f'(x)$ for each of the following, where p, q and r are constants.

(a) $f(x) = e^{px^2+qx}$ **(b)** $f(x) = e^{r^2+x}$

4 Find $\frac{dy}{dx}$ for each of the following.

(a) $y = e^{\sqrt{x}}$ **(b)** $y = e^{\frac{5}{x}}$

(c) $y = \frac{1}{\sqrt{e^x}}$ **(d)** $y = e^x + \frac{1}{e^x}$

5 Differentiate each of the following with respect to x.

(a) $x^2 e^x$ **(b)** $e^x \cos x$

(c) $e^x \sin 2x$ **(d)** $e^x (\cos x - \sin x)$

(e) $\frac{1}{2}x^2 e^{\sin x}$ **(f)** $\sqrt{x}e^{-x}$

(g) $x^2 e^{-x^2}$ **(h)** $\frac{e^x}{\sqrt{x}}$

6 Given that $y = e^{5x} - e^{\cos x}$, find the value of $\frac{dy}{dx}$ at the point where $x = 0$.

7 Given that $f(x) = \frac{\sin 2x}{e^{3x+1}}$, find an expression for $f'(x)$.

8 Given that $f(x) = \frac{x \cos x}{e^x}$, find an expression for $f'(x)$.

9 Without considering the binomial expansion of $\left(e^x - \frac{1}{e^x}\right)^3$, show that

$$\frac{d}{dx}\left(e^x - \frac{1}{e^x}\right)^3 = 3(e^{2x} - e^{-2x})(e^x - e^{-x}).$$

13.4 FURTHER APPLICATIONS OF DIFFERENTIATION

Let us now take a look at a few examples that involve the differentiation of trigonometric, logarithmic and exponential functions.

Worked Example 9

(Equations of Tangent and Normal)

Find the equation of the tangent and the normal to the curve $y = 2\sin x - 1$ at the point where $x = \frac{\pi}{6}$.

Solution

$$y = 2\sin x - 1$$

$$\frac{dy}{dx} = 2\cos x$$

When $x = \frac{\pi}{6}$,

$$y = 2\sin\frac{\pi}{6} - 1 = 0 \text{ and}$$

$$\frac{dy}{dx} = 2\cos\frac{\pi}{6} = \sqrt{3}.$$

$\therefore$ Equation of tangent is $y - 0 = \sqrt{3}\left(x - \frac{\pi}{6}\right)$

i.e. $y = x\sqrt{3} - \frac{\pi\sqrt{3}}{6}$

Gradient of normal at $x = \frac{\pi}{6}$ is $-\frac{1}{\sqrt{3}}$

$\therefore$ Equation of normal is $y - 0 = -\frac{1}{\sqrt{3}}\left(x - \frac{\pi}{6}\right)$

$$y = -\frac{x}{\sqrt{3}} + \frac{\pi}{6\sqrt{3}}$$

i.e. $y = -\frac{\sqrt{3}}{3}x + \frac{\pi\sqrt{3}}{18}$

RECALL

If two lines are perpendicular to each other, $m_1 m_2 = -1$.

Practise Now 9

Similar Questions:
Exercise 13D
Questions 1, 3-5, 8, 21

1. Find the equation of the tangent and the normal to the curve $y = 4x + \cos 2x$ at the point where $x = \frac{\pi}{12}$.
2. The equation of a curve is $y = \ln(2x^2 + 5x)$, where $x > 0$. Find the x-coordinate of the point on the curve at which the normal to the curve is perpendicular to the line $6y = 2x + 25$.
3. The curve $y = e^{\sqrt{x}-1}$ meets the line $y = 1$ at the point A. Find the coordinates of A and the equation of the normal to the curve at A.

Worked Example 10

(Connected Rates of Change)

Given that $y=\frac{\ln 4x}{2x^3}$, find the rate of change of y at the instant when $y=0$, given that x is changing at a rate of 1.2 units per second.

Solution

$$y=\frac{\ln 4x}{2x^3}$$

$$\frac{dy}{dx}=\frac{2x^3\frac{d}{dx}(\ln 4x)-\ln 4x\frac{d}{dx}\left(2x^3\right)}{\left(2x^3\right)^2} \quad \text{(Quotient Rule)}$$

$$=\frac{2x^3\frac{4}{4x}-\ln 4x\left(6x^2\right)}{\left(2x^3\right)^2}$$

$$=\frac{2x^2-6x^2\ln 4x}{4x^6}=\frac{1-3\ln 4x}{2x^4}$$

When $y=0$,

$\ln 4x=0$

$4x=e^0=1 \qquad \therefore\ x=\frac{1}{4}$

Applying the Chain Rule,

$$\frac{dy}{dt}=\frac{dy}{dx}\times\frac{dx}{dt}$$

$$=\left[\frac{1-3\ln 4x}{2x^4}\right]\times 1.2$$

When $x=\frac{1}{4}$,

$$\frac{dy}{dt}=\frac{1-3\ln 4\left(\frac{1}{4}\right)}{2\left(\frac{1}{4}\right)^4}\times 1.2=153.6\text{ units/s}$$

$\therefore$ y is changing at a rate of 153.6 units/s.

Practise Now 10

Similar Question:
Exercise 13D
Questions 6, 11, 12, 16, 18, 23

1. Given that $y=\ln\left(\frac{3x-4}{2x+5}\right)$, find
   (i) $\frac{dy}{dx}$,
   (ii) the rate of change of x when $x=2$, given that y is changing at a rate of 2.5 units per second.
2. Given that $y=1+2\sin^2 x$ for $0\leqslant x\leqslant\pi$ and that x is increasing at a rate of 0.2 radian per second, find the rate of change of y with respect to time when $x=\frac{\pi}{3}$.
3. Given that $y=3xe^{-2x}+5e^{-3x}$, find $\frac{dy}{dx}$. Given that both x and y vary with time and that x is increasing at a rate of 3 units/s at the instant when $x=\frac{1}{2}$, find the rate of change of y at this instant.

Worked Example 11

(Stationary Point)

Find the value of x between 0 and π for which the curve $y = e^x \cos x$ has a stationary point.

Solution

$y = e^x \cos x$

$$\frac{dy}{dx} = e^x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}\left(e^x\right)$$

$$= e^x(-\sin x) + \cos x\,(e^x)$$

$$= e^x(\cos x - \sin x)$$

For stationary points, $\frac{dy}{dx} = 0$.

i.e. $e^x(\cos x - \sin x) = 0$

$e^x = 0$ or $\cos x - \sin x = 0$

(no solution) $\cos x = \sin x$

$\tan x = 1$

$x = \frac{\pi}{4}$

$\therefore x = \frac{\pi}{4}$

Practise Now 11

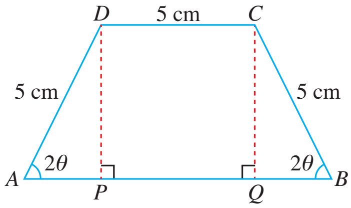
Similar Questions:
Exercise 13D
Questions 2, 7, 10, 13, 15

1. Given that $y = 3x \ln x$, find $\frac{dy}{dx}$. Hence, determine the coordinates of the stationary point of the curve $y = 3x \ln x$.

2. Given that $y = e^x \sin 2x$, find the value of x where $0 \leqslant x \leqslant \frac{\pi}{2}$ for which y has a stationary value. Determine the nature of this stationary value.

Worked Example 12

(Maxima/Minima Problem)



The figure shows a trapezium $ABCD$ where $AD = CD = BC = 5$ cm and $\angle DAB = \angle CBA = 2\theta$ radians, where θ is acute. Show that the area of the trapezium, y cm^2, is given by $100 \sin\theta\cos^3\theta$. Hence, or otherwise, find the maximum area of the trapezium as θ varies.

Solution

In ΔADP,

$$\sin 2\theta = \frac{DP}{5} \quad \text{and} \quad \cos 2\theta = \frac{AP}{5}$$

$$DP = 5\sin 2\theta \qquad AP = 5\cos 2\theta$$

By symmetry,

$$CQ = 5\sin 2\theta \quad \text{and} \quad BQ = 5\cos 2\theta$$

Area of trapezium $ABCD$ is $y = \frac{1}{2}(AB + DC) \times DP$

$$= \frac{1}{2}(10\cos 2\theta + 5 + 5) \times 5\sin 2\theta$$

$$= 5(\cos 2\theta + 1) \times 5\sin 2\theta$$

$$= 25\sin 2\theta(\cos 2\theta + 1)$$

$$= 25(2\sin\theta\cos\theta)(2\cos^2\theta - 1 + 1)$$

$$= 50\sin\theta\cos\theta(2\cos^2\theta)$$

$$= 100\sin\theta\cos^3\theta$$

Applying the Product Rule,

$$\frac{dy}{d\theta} = 100\left[\sin\theta \times \frac{d}{d\theta}\left(\cos^3\theta\right) + \cos^3\theta \times \frac{d}{d\theta}(\sin\theta)\right]$$

$$= 100[\sin\theta \times (3\cos^2\theta)(-\sin\theta) + \cos^3\theta \times (\cos\theta)]$$

$$= 100\cos^2\theta(\cos^2\theta - 3\sin^2\theta)$$

For the area to be a maximum, $\frac{dy}{d\theta} = 0$.

$$100\cos^2\theta(\cos^2\theta - 3\sin^2\theta) = 0$$

$100\cos^2\theta = 0$ or $\cos^2\theta - 3\sin^2\theta = 0$

$\cos^2\theta = 0$ or $3\sin^2\theta = \cos^2\theta$

$\theta = \frac{\pi}{2}$ $\qquad$ $\tan^2\theta = \frac{1}{3}$

$$\tan\theta = \pm\frac{1}{\sqrt{3}}$$

$$\theta = \frac{\pi}{6}, \frac{5\pi}{6}$$

Since θ is acute, $\theta = \frac{\pi}{6}$.

$$\frac{dy}{d\theta} = 100\cos^4\theta - 300\cos^2\theta\sin^2\theta$$

$$= 100\cos^4\theta - 75\sin^2 2\theta$$

$$\frac{d^2y}{d\theta^2} = -400\cos^3\theta\sin\theta - 300\sin 2\theta\cos 2\theta$$

When $\theta = \frac{\pi}{6}$, $\frac{d^2y}{d\theta^2} = -400 \times \frac{3\sqrt{3}}{8} \times \frac{1}{2} - 300 \times \frac{\sqrt{3}}{2} \times \frac{1}{2}$

$$= -75\sqrt{3} - 75\sqrt{3} = -150\sqrt{3} < 0$$

Area is a maximum when $\theta = \frac{\pi}{6}$.

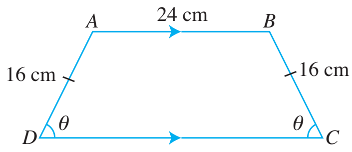
$\therefore$ Maximum area of the trapezium is $y = 100\sin\frac{\pi}{6}\cos^3\frac{\pi}{6}$

$$= 100\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)^3$$

$$= \frac{75\sqrt{3}}{4}\text{ cm}^2$$

Practise Now 12

Similar Question:
Exercise 13D
Question 22



In the figure, $ABCD$ is a trapezium such that $\angle ADC = \angle BCD = \theta$ radian, AB is parallel to DC, $AB = 24$ cm and $AD = BC = 16$ cm. Show that the area of the trapezium, S cm^2, is given by $S = 128\sin\theta\,(3 + 2\cos\theta)$. Hence, find the value of θ for which S has a stationary value. Determine whether this value of θ makes S a maximum or a minimum.

Basic Level | Intermediate Level | Advanced Level

Exercise 13D

1. Find the equation of the tangent and that of the normal to the curve $y = 5\sin 2x$ at the point where $x = \frac{\pi}{6}$.

2. Given that $y = 3\cos x - 5\sin x$, write down an expression for $\frac{dy}{dx}$. Hence, determine the value of x, where $0 \leqslant x \leqslant \pi$, for which y has a stationary point. Determine whether this is a maximum or a minimum point.

3. Find the gradient of the curve $y = 2x^2\ln x$ at the point where $x = 1$. Hence, find the equation of the tangent to the curve at $x = 1$.

4. Find the equation of the tangent to the curve $y = e^{2x}$ at the point $(0, 1)$.

5. Find the coordinates of the point on the curve $y = \ln(9 - 2x)$ at which the normal to the curve is parallel to the line $2y - x + 7 = 0$.

6. Given that $y = 2\sin x - 3\cos 2x$ and that x is increasing at a constant rate of 0.2 unit per second, find the rate of change of y when $x = \frac{\pi}{6}$.

7. Find the coordinates of the stationary point of the curve

   (a) $y = x^2\ln x$, **(b)** $y = x\ln x^2$.

Exercise 13D

8 The equation of a curve is $y = 2\tan\left(x + \frac{\pi}{4}\right)$. Find

(i) the gradient of the tangent to the curve at the point where the curve intersects the y-axis,

(ii) the equation of the normal to the curve at $x = 0$.

9 If $y = x + 2\cos x$, find the maximum value of y for which $0 < x < \pi$.

10 Find the stationary values of $y = \sin x \cos^3 x$ for $0 < x < \pi$.

11 The variables x and y are connected by the equation $y = 2\cos^2\left(x - \frac{\pi}{6}\right)$, where $0 < x < \frac{\pi}{2}$. Given that x is increasing at 0.3 radian per second, find the rate of change of y with respect to time when $x = \frac{\pi}{3}$.

12 The variables x and y are connected by the equation $y = 2\tan\left(3x + \frac{\pi}{8}\right)$. Given that x is decreasing at the rate of 0.25 radian per second, find the rate of change of y with respect to time when $x = \frac{\pi}{24}$.

13 Given that $y = 2\sin^3 x - 3\sin x$, find an expression for $\frac{dy}{dx}$ in terms of $\cos x$ and find the coordinates of the turning points for $0 \leqslant x \leqslant \pi$. Determine the nature of these turning points.

14 Show that the curve $y = \ln\left(\frac{5 - 7x}{8 + x}\right)$ has no stationary point for all real values of x.

15 Given that $y = e^{-x}(\sin x - \cos x)$, find $\frac{dy}{dx}$ and determine, for $0 < x < \pi$, the value of x for which y is stationary. Hence, determine the nature of the stationary point.

16 Due to urbanisation, the population, y thousand units, of a certain colony of insects in a forest is given by $yt^3 = \ln 2t$, where t represents the time in months. Find the rate of change of the population when $t = 3$ and when $t = 9$.

17 The equation of a curve is $y = \ln(3x + 1)$.

(i) Find an expression for $\frac{dy}{dx}$.

(ii) Find the value of $\frac{dy}{dx}$ at the point where $y = \ln 7$.

(iii) Show that as x increases, the gradient of the curve decreases. What value does $\frac{dy}{dx}$ approach as x becomes very large?

18 The sound pressure, P, of a given musical note can be represented by $P = 8 \ln \frac{E}{10^{-12}}$, where E is the sound power in Watts. Given that $E = \sin 0.5t$, find the rate of change of P at $t = \frac{\pi}{2}$ seconds.

19 If $y = 2xe^{x^2 - 3x}$, find $\frac{dy}{dx}$ and the range of values of x for which y is a decreasing function.

20 The mass, m grams, of a radioactive substance present at time t days after the first day it was observed, is given by the formula $m = 36e^{-0.03t}$. Find

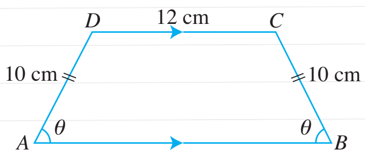
(i) the mass of the substance after 8 days,

(ii) the number of days after which the mass decreases to 15 g,

(iii) the rate at which the mass is decreasing when $t = 20$.

21 Find the gradient of the normal to the curve $y = 2x + 3 \sec^2 2x$ at the point where $x = \frac{\pi}{6}$, giving your answer in the form $a + b\sqrt{3}$, where both a and b are rational numbers.

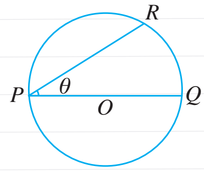
22



The figure shows a trapezium $ABCD$ where $\angle DAB = \angle ABC = \theta$ radian, AB is parallel to DC, $DC = 12$ cm and $AD = BC = 10$ cm. Show that the area of the trapezium, S cm², is given by $S = 20 \sin \theta\,(6 + 5 \cos \theta)$.

Hence, find the value of θ for which S has a stationary value. Determine whether this value of θ makes S a maximum or a minimum.

23 The figure shows a circle centre O and radius 10 cm.



(i) Given that $\angle RPQ = \theta$ radian, find the area, A cm², enclosed by the chord PR, the diameter PQ and the arc QR, giving your answer in terms of θ.

(ii) Given that θ is increasing at a rate of 0.1 radian per second, find the rate of change of A when $\theta = \frac{\pi}{6}$ radian.

24 Show that the least value of $x^2 + 2x \sin \beta + 1$ is $\cos^2 \beta$, where β is a constant.

25 The volume, V cm³, of a cone, is given by the formula $V = kr^3 \cot \theta$, where k is a constant, r is the radius of the cone and θ is a semi-vertical angle. Given that r is fixed and θ is increasing at a rate of 0.2 radian per second, calculate the rate of change of volume with respect to time when $\theta = \frac{\pi}{6}$, leaving your answer in terms of k and r.

SUMMARY

Trigonometric Functions

The following formulae are applicable when x is measured in radians and a, b and n are constants.

1. Derivatives of sin x, cos x and tan x

$\frac{d}{dx}(\sin x) = \cos x$ $\quad$ $\frac{d}{dx}(\cos x) = -\sin x$ $\quad$ $\frac{d}{dx}(\tan x) = \sec^2 x$

2. Derivatives of sin $(ax + b)$, cos $(ax + b)$ and tan $(ax + b)$

$$\frac{d}{dx}\sin(ax+b) = a\cos(ax+b)$$

$$\frac{d}{dx}\cos(ax+b) = -a\sin(ax+b)$$

$$\frac{d}{dx}\tan(ax+b) = a\sec^2(ax+b)$$

Logarithmic Functions

3. $\frac{d}{dx}(\ln x) = \frac{1}{x}$ $\quad$ $\frac{d}{dx}[\ln(ax+b)] = \frac{a}{ax+b}$ $\quad$ $\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$

Exponential Functions

4. $\frac{d}{dx}\left(e^x\right) = e^x$ $\quad$ $\frac{d}{dx}\left(e^{ax+b}\right) = ae^{ax+b}$ $\quad$ $\frac{d}{dx}\left[e^{f(x)}\right] = f'(x) \times e^{f(x)}$

Review Exercise 13

1. Differentiate each of the following with respect to x.
   (a) $2\pi x + 2\cos \pi x$
   (b) $(\sin x + \cos x)^2$
   (c) $\sin^2 (3x - 1)$
   (d) $\cos^4 x$
   (e) $\tan^2 \frac{1}{3}x$
   (f) $\ln (5x - 2)$
   (g) $\ln (2x^2 + 3)$
   (h) $\ln \sin 4x$
   (i) $3e^{2x} + e^{3+x}$
   (j) $\dfrac{3}{5e^{4x-3}} + \sqrt{e^{7x-3}}$

2. Differentiate each of the following with respect to x.
   (a) $\sin^2 x \cos 3x$
   (b) $\sin^5 x \cos^4 x$
   (c) $\dfrac{\sin x}{x^2}$
   (d) $\sin x \ln 5x$
   (e) $3x^2 \ln (2x - 5)$
   (f) $e^x \cos 3x$
   (g) $x^3e^{2x} + 2xe^{4x}$
   (h) $e^{4x} \tan x$
   (i) $3x^2 \ln (2x^2 + 3)$
   (j) $e^{3x} \sin^4 2x$

3. Find the equation of the tangent and the normal to the curve $y = \dfrac{\sin x}{1 + \cos x}$ at the point where $x = \dfrac{\pi}{3}$.

4. Given that $y = \ln (\tan 2x)$, show that $\dfrac{dy}{dx} = 4 \operatorname{cosec} 4x$.

5. (a) Find the coordinates of the point on the curve $y = \ln (7 - 3x)$ at which the tangent to the curve is parallel to the line $y + 3x = 24 - \pi$.
   (b) The equation of a curve is $y = \ln (x^2 + 2x)$, where $x > 0$. Find the coordinates of the point on the curve at which the normal to the curve is parallel to $12y + 5x + 3 = \pi$.

6. Given that $y = \sin 3x + \cos^2 x$ and that x is changing at a rate of 0.2 unit per second, find the rate of change of y with respect to time when $x = \dfrac{\pi}{6}$.

7. Find the maximum and minimum values of the curve $y = \sin x (1 + \cos x)$ for $0 \leqslant x \leqslant 2\pi$.

8. A curve has the equation $y = \dfrac{\cos x}{\sin x - 4}$. Find
   (i) an expression for $\dfrac{dy}{dx}$,
   (ii) the values of x between 0 and 2π for which y is stationary.

9. Given that $y = \ln\left(\dfrac{x^2+5}{x-1}\right)$, find the coordinates of the stationary point and determine its nature.

10. Show that the curve $y = \ln\left(\dfrac{4x+7}{2x-1}\right)$ has no stationary point for all real values of x.

11. The concentration, y units, of a certain drug in the bloodstream, at time t minutes is given by $y = e^{2t} + 2e^{-2t}$. Find the minimum concentration of the drug in the bloodstream and state the value of t when this occurs.

12. Given that $y = e^x \sin 2x$, find $\dfrac{dy}{dx}$ and determine, for $0 \leqslant x \leqslant \pi$, the values of x for which y is stationary.

13. Find the coordinates of the point where the curve $y = \ln(x^2 - 8)$ crosses the positive x-axis. Hence, find the equation of the normal to the curve at this point.

14. Given the curve $y = \dfrac{\sin x + \cos x}{e^{2x}}$ for $0 \leqslant x \leqslant 2\pi$, find $\dfrac{dy}{dx}$. Hence, find the values of x for which the tangent to the curve is parallel to the x-axis.

15. Given that $y = \dfrac{e^{1-2x}}{4-3x}$, find the range of values of x for which $y = \dfrac{e^{1-2x}}{4-3x}$ is a decreasing function.

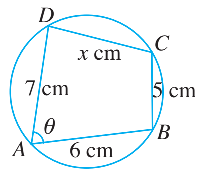
16. Given that $e^y = 5x^2 - 4$, find $\dfrac{dy}{dx}$. Hence, find the rate of change of y at $x = 2$, given that x is increasing at the rate of 1.5 units per second.

17. The amount in an employee's long-term retirement account, \$$R$, increases according to the formula $R = 50\,000 \ln(2t + 3)$, where t is the number of years after he starts his retirement account.

(i) How much will there be in the retirement account after 5 years?

(ii) What is the rate of increase of the amount after 4 years?

18. The figure shows a quadrilateral $ABCD$ in which AB = 6 cm, AD = 7 cm, BC = 5 cm, $CD = x$ cm and $\angle BAD = \theta$ radian, where θ varies.

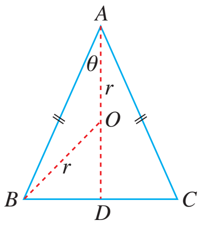


(i) Find the area of the triangle BAD and that of triangle BCD in terms of θ and/or x.

(ii) State the value of θ for which the area of the triangle BAD will be a maximum. Hence, find the maximum area.

(iii) Using this value of θ, find the value of x.

19. The figure shows an isosceles triangle ABC inscribed in a circle of radius r cm and centre at O. Given that $\angle BAO = \theta$ radian and $\angle ADC = \dfrac{\pi}{2}$ radians, show that the area, S cm^2, of ΔABC is given by $S = r^2 \sin 2\theta\,(1 + \cos 2\theta)$. Calculate the value of θ for which S has a stationary value and determine whether this value of θ makes S a maximum or a minimum.



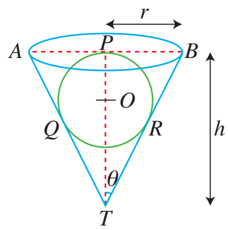
20. The concentration of alcohol in a person's blood, $C\%$, when he consumes alcohol, can be represented by $C = 0.2te^{-0.8t}$, where t is the number of hours after consuming the alcohol.

(i) What is the concentration of alcohol in his blood 2 hours after consuming the alcohol?

(ii) What is the rate of change of alcohol concentration in his blood 1.5 hours after consuming the alcohol?

(iii) What is the maximum level of alcohol concentration in his blood after consuming the alcohol?

21. A sphere with centre O and radius 3 cm fits exactly on the sides of an inverted right circular cone of height h cm, base radius r cm and θ is the angle between TP and TB. If the volume of the cone is V cm^3, express V in terms of h. Hence, find the value of h for which V is stationary and determine its nature.



Challenge Yourself

1. Given that $e^{-y} = \sin x \cos x$, show that $\frac{dy}{dx} = -2 \cot 2x$.

2. Given that $y = \cos [\ln (1 + x)]$, prove that $(1+x)^2 \frac{d^2y}{dx^2} + (1+x)\frac{dy}{dx} + y = 0$.

3. Find the values of the constants a, b and c for which $\frac{d}{dx}\left(\frac{\sin 2x}{1+\cos x}\right) = \frac{a + b\cos^2 x + c\cos^3 x}{(1+\cos x)^2}$.

REVISION EXERCISE E1

1. Differentiate each of the following with respect to x.

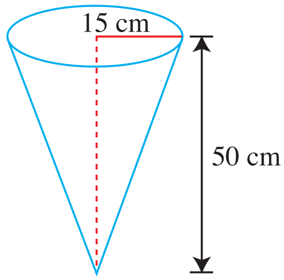
(a) $\dfrac{3x^2 + 2x}{\sqrt{x}}$ **(b)** $\sqrt{x^3 + 2}$

(c) $\sin x\,(1 + \cos x)$ **(d)** $\dfrac{\tan x}{3x + 1}$

(e) $5e^{4x} + e^{-\frac{1}{2}x}$ **(f)** $\ln\left(\dfrac{\sin x + \cos x}{\sin x - \cos x}\right)$

2. Find the point where the tangent to the curve $y = 4x^3 - 3x$ at the point $\left(-\dfrac{1}{2}, 1\right)$ meets the curve again.

3. A hollow right circular cone has a base radius of 15 cm and a height of 50 cm. It contains water which leaks through a small hole in the vertex at a rate of 60 cm³/s. Find the rate at which the water level is falling when the height of the water is 25 cm from the vertex.



4. Find the coordinates of the stationary points of the curve $y = \sin^2 x \cos 2x$ in the range $0 < x < \dfrac{2\pi}{3}$ and determine the nature of these stationary points.

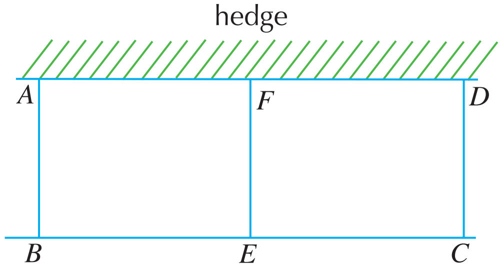
5. Given that $y = Ae^{2x} + Be^{-x} + C$, $\dfrac{dy}{dx} = 6e^{2x} + e^{-x}$ and that (0, 4) lies on the curve $y = Ae^{2x} + Be^{-x} + C$, find the values of the constants A, B and C.

6. If the function $y = a \sin x + \cos x$ satisfies the equation $\dfrac{d^2y}{dx^2} + y = b$, where a and b are constants, find the value of a and of b, given that $\dfrac{dy}{dx} = 2$ when $x = 0$.

7. **(a)** Find the minimum value of the function $y = x^4 - 2x^3 - 2x^2 + 10$.

(b) If $y = (x + b)(x + c)$, where b and c are constants, find the minimum value of y in terms of b and c.

8. Two paddocks, one for sheep and the other for cattle, are to be formed alongside a thick hedge, as shown in the figure. If a farmer has only 240 m of fencing, find the dimensions of the paddocks so that the sum of the area is a maximum.



REVISION EXERCISE E2

1. Differentiate each of the following with respect to x.

(a) $\dfrac{x+2}{x-3}$ **(b)** $\left(2x^2 - \dfrac{3}{x}\right)^5$

(c) $(x^2 + x)\tan x$ **(d)** $\dfrac{x^2}{\sin 2x}$

(e) $e^{4x}\sin(2x+3)$ **(f)** $\ln \dfrac{e^{2x}}{5x}$

2. Sketch the curve $y = \sin x$ for $0 \leqslant x \leqslant \pi$. Find the equation of the normal to the curve at the point $P\left(\dfrac{\pi}{6}, \dfrac{1}{2}\right)$.

3. At any instant, the volume V and the pressure P of a gas are connected by the equation $PV = c$, where c is a constant. Given $V = 30$ cm^3 and $P = 50$ Ns/cm^2 and that the pressure is increasing at a rate of 2.5 Ns/cm^2, find the instantaneous rate of change of the volume.

4. Find the coordinates of the stationary points of the curve $y = 9x + \dfrac{1}{x}$. Determine the nature of each of these stationary points.

5. The curve $y = ax^2 + bx + c$ passes through the origin O and has a minimum value at the point $(1, -2)$. Find the values of a, b and c.

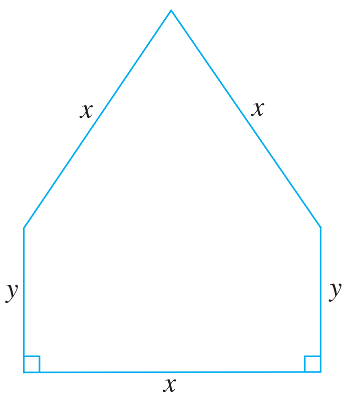
6. Given that $y = \dfrac{2}{3}e^{3x}$, show that y is an increasing function for all real values of x.

7. Given $y = e^x \cos 2x$, find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. Hence, find

(i) the gradient of the tangent to the curve $y = e^x \cos 2x$ when $x = \dfrac{\pi}{2}$,

(ii) the value of $\dfrac{d^2y}{dx^2}$ when $x = \dfrac{\pi}{2}$.

8. A piece of wire of length 18 cm is bent to form the shape as shown in the figure. Express y in terms of x. Hence, show that the area enclosed, A cm^2, is given by $A = \left(\dfrac{\sqrt{3}}{4} - \dfrac{3}{2}\right)x^2 + 9x$. Determine the value of x for which A will be a maximum. Hence, find the maximum area.



INTEGRATION

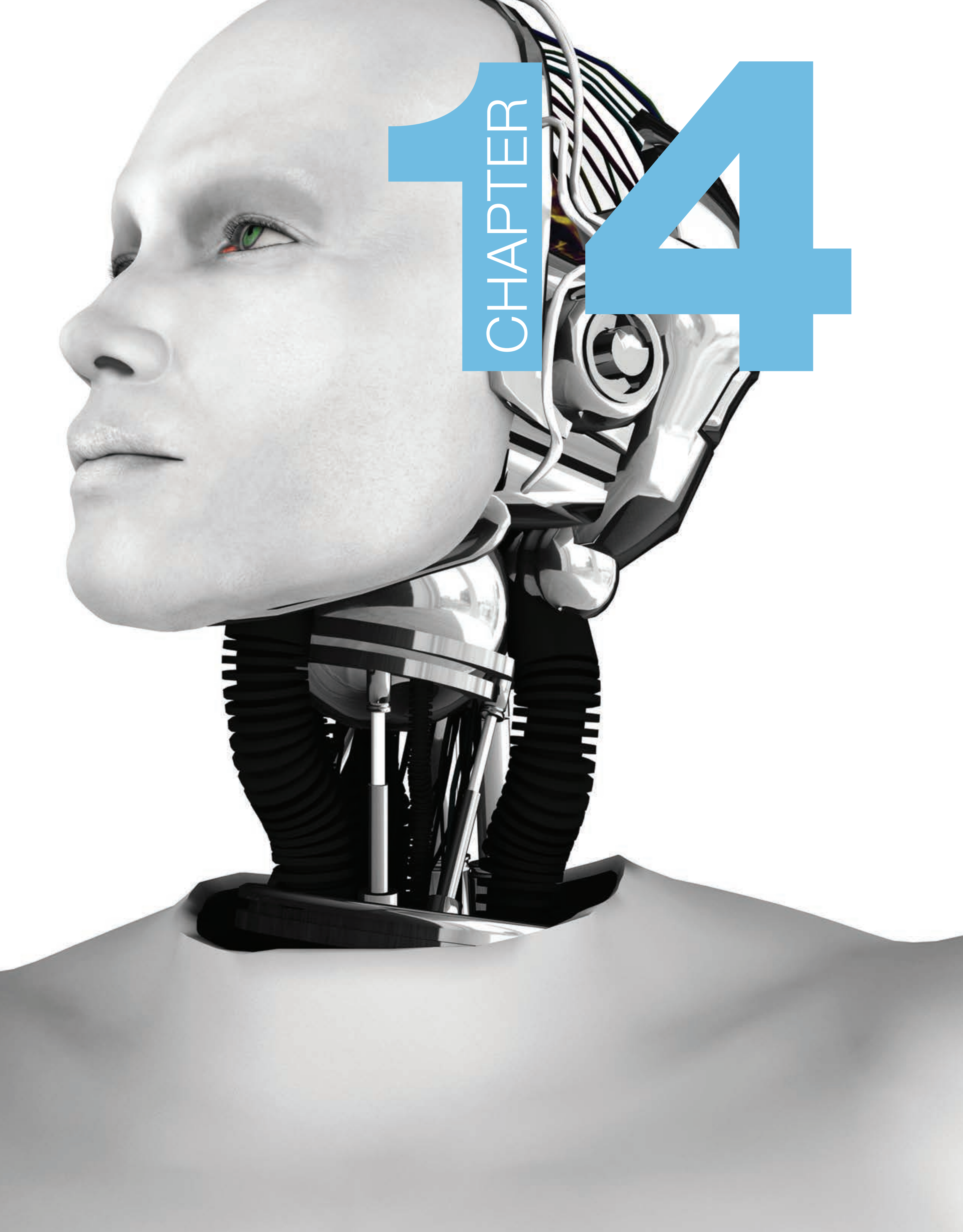
Integration is the reverse process of differentiation. Integration has many uses. It can be used in economics to derive cost functions or in the medical field to estimate growth rates of bacteria. In this chapter, we will learn how to integrate expressions and to apply integration to solve problems.

Learning Objectives

At the end of this chapter, you should be able to:

- use the rule that $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{F}(x)] = \mathrm{f}(x) \Rightarrow \int \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(x) + c$,
- integrate simple algebraic expressions and terms involving linear function,
- integrate trigonometric functions,
- integrate logarithmic and exponential functions,
- apply integration to solve problems.

CHAPTER 14



14.1 INTEGRATION AS THE REVERSE OF DIFFERENTIATION

In Chapter 11, we have learnt that: if $y = x^2$, then $\frac{dy}{dx} = 2x$.

What happens if we do the reverse, i.e. given $\frac{dy}{dx} = 2x$, how do we find y?

Clearly, if $\frac{dy}{dx} = 2x$, then $y = x^2$ is a possible answer, but is it the only answer?

Investigation

Integration as the Reverse of Differentiation

1. Differentiate each of the following.

(i) $y = x^2$ **(ii)** $y = x^2 + 3$ **(iii)** $y = x^2 - 7$

(iv) $y = x^2 + 100$ **(v)** $y = x^2 + c$, where c is a constant

2. What are the derivatives of the above equal to?

3. Therefore, if $\frac{dy}{dx} = 2x$, what are the possible expressions for y?

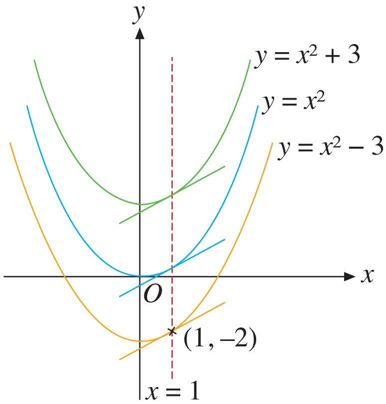
These expressions obtained are called indefinite integrals.

Differentiation is the process of obtaining the derivative $\frac{dy}{dx}$ from y.

The *reverse* process of obtaining y from $\frac{dy}{dx}$ is called **integration**.

Integration is the reverse of differentiation.

The integration of $2x$ gives a family of curves $y = x^2 + c$, having the same gradient function $\frac{dy}{dx} = 2x$. The cases $c = -3, 0, 3$ are shown in the graph. The gradient of each of the curves at the point where $x = 1$ is 2.



When more information is given, the value of c can be found and we get the equation of a particular curve. For example, given that (1, –2) lies on the curve, then $-2 = 1^2 + c$, i.e. $c = -3$. Therefore, the equation of the curve is $y = x^2 - 3$.

From the investigation on the previous page, we observe that there are infinitely many solutions for y if its derivative is given. Symbolically, we write:

If $\frac{dy}{dx} = 2x$, then $y = \int 2x \, dx = x^2 + c$, where c is an **arbitrary constant**.

$\int 2x \, dx$ is called the **indefinite integral** of $2x$ with respect to x ('indefinite' because it has infinitely many solutions).

Thinking Time

In general, what is an indefinite integral? How is this related to its derivative? Can you describe it?

Integration of Power Functions

In Chapter 11, we have learnt the following:

$$\frac{d}{dx}(3x^4) = 12x^3$$

Hence,

$$\int 12x^3 \, dx = \frac{12x^4}{4} + c, \text{ i.e. } \int x^3 \, dx = \frac{x^4}{4} + c .$$

In general,

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + c, \text{ where } n \neq -1 \text{ and } c \text{ is a constant.}$$

ATTENTION

How to remember this formula: 'index n plus 1 to get the new index, then divide by the new index'.

Worked Example 1

(Integration of Power Functions)

Find the integral of each of the following.

(a) x^2 **(b)** $\frac{dy}{dx} = 1$

Solution

(a)
$$\int x^2 \, dx = \frac{x^{2+1}}{2+1} + c$$
$$= \frac{x^3}{3} + c$$

(b)
$$\frac{dy}{dx} = 1$$
$$\therefore y = \int 1 \, dx$$
$$= \int x^0 \, dx$$
$$= \frac{x^{0+1}}{0+1} + c$$
$$= x + c$$

Practise Now 1

Similar Question:
Exercise 14A
Question 1(a)

Find the integral of each of the following.

(a) x^3 **(b)** $\frac{dy}{dx} = \sqrt{x}$

14.2 TWO RULES OF INTEGRATION

In Chapter 11, we have learnt 5 Rules of Differentiation, but only the first two rules can be reversed for integration.

Rule 1 of Integration

This rule states that

$$\boxed{\int kf(x)\,dx = k\int f(x)\,dx,}\text{ where } k \text{ is a scalar multiple.}$$

ATTENTION

How to remember this rule: 'if k is a scalar multiple, leave k alone when you integrate'.

Worked Example 2

(Application of Rule 1 of Integration)

Integrate each of the following with respect to x.

(a) $-\dfrac{3}{8x^4}$ **(b)** $\dfrac{dy}{dx} = k$

Solution

(a)
$$\begin{aligned}\int\left(-\frac{3}{8x^4}\right)dx &= \int\left(-\frac{3}{8}x^{-4}\right)dx\\ &= -\frac{3}{8}\int x^{-4}dx\\ &= -\frac{3}{8}\left(\frac{x^{-3}}{-3}\right)+c\\ &= \frac{1}{8x^3}+c\end{aligned}$$

(b)
$$\frac{dy}{dx} = k$$
$$\begin{aligned}\therefore y &= \int k\,dx\\ &= \int kx^0\,dx\\ &= k\int x^0\,dx\\ &= k\left(\frac{x^1}{1}\right)+c\\ &= kx + c\end{aligned}$$

ATTENTION

If k is scalar **multiple**, leave k alone when you integrate (Rule 1 of Integration), but if k is a constant **term**, you must integrate k to give $kx + c$ (see Worked Example 2b).

Practise Now 2

Similar Questions:
Exercise 14A
Questions 1(b)-(d)

1. Integrate each of the following with respect to x.

(a) $\dfrac{4}{9x^3}$ **(b)** $\dfrac{dy}{dx} = 3$ **(c)** $f'(x) = ax$, where a is a constant

2. Integrate $\dfrac{6}{\sqrt[3]{t}}$ with respect to t.

Rule 2 of Integration

If $\mathrm{f}(x)$ and $\mathrm{g}(x)$ are functions, then

$$\int [\mathrm{f}(x) \pm \mathrm{g}(x)]\,\mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x \pm \int \mathrm{g}(x)\,\mathrm{d}x.$$

This is also known as the **Addition/Subtraction Rule**.

ATTENTION

How to remember this rule: 'for the sum or difference of functions, just integrate each function separately'.

Worked Example 3

(Application of Rule 2 of Integration)

The gradient of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = (3x - 1)(x + 2)$. Given that the curve passes through the point (2, 3), find the equation of the curve.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = (3x - 1)(x + 2)$$
$$= 3x^2 + 5x - 2$$
$$\therefore y = \int (3x^2 + 5x - 2)\,\mathrm{d}x$$
$$= x^3 + \frac{5}{2}x^2 - 2x + c$$

Since the curve passes through (2, 3), then $3 = 2^3 + \frac{5}{2}(2^2) - 2(2) + c$.

Solving, $c = -11$.

$\therefore$ The equation of the curve is $y = x^3 + \frac{5}{2}x^2 - 2x - 11$.

Thinking Time

In Worked Example 3, $\int (3x^2 + 5x - 2)\,\mathrm{d}x = x^3 + \frac{5}{2}x^2 - 2x + c$.

If we integrate each term separately, there should be a constant for each integration, so we should have:

$$\int (3x^2 + 5x - 2)\,\mathrm{d}x = (x^3 + c) + \left(\frac{5}{2}x^2 + c\right) - 2x + c.$$

Discuss which of the two is the correct answer and explain your reasons clearly.

Practise Now 3

Similar Questions:
Exercise 14A
Questions 3-10

1. Find **(a)** $\int 3x(x - 7)\,\mathrm{d}x$, **(b)** $\int \frac{5u^3 + 2\sqrt{u}}{u}\,\mathrm{d}u$, **(c)** $\int \left(\frac{2x^2 + 3}{4x}\right)^2 \mathrm{d}x$.

2. The gradient of a curve at a point (x, y) on the curve is given by $2 - 3x + 4x^2$. Given that the curve passes through the point (1, 1), find the equation of the curve.

Worked Example 4

(Application of Integration)

A publisher of textbooks found that the rate of change in the cost, \$$y$, of producing x thousand books is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2000}{\sqrt{x}}$ and that the overhead cost is \$80 000. Find the cost function, y, and hence find the cost of producing 12 000 books, giving your answer correct to the nearest dollar.

Solution

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2000}{\sqrt{x}}$$

$$\therefore\ y = \int \frac{2000}{\sqrt{x}}\,\mathrm{d}x$$

$$= \int 2000x^{-\frac{1}{2}}\,\mathrm{d}x$$

$$= \frac{2000x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

$$= 4000\sqrt{x} + c$$

When $x = 0$, $y = 80\ 000$,

i.e. $80\ 000 = 4000\sqrt{0} + c$

$c = 80\ 000$

$\therefore\ y = 4000\sqrt{x} + 80\ 000$

When $x = 12\ 000$,

$y = 4000\sqrt{12\ 000} + 80\ 000$

$= 518\ 178$

$\therefore$ The cost of producing 12 000 books is \$518 178.

Practise Now 4

Similar Question:
Exercise 14A
Question 11

The rate of change of the revenue, \$$R$, from the sales of x units of goods in a company is given by $\frac{\mathrm{d}R}{\mathrm{d}x} = 4\sqrt[3]{x^2} + x$. Find the revenue function, R. Hence, find the revenue of the company if 1000 units of goods are sold.

Hint: The company earns no revenue if no goods are sold.

Basic Level | Intermediate Level | Advanced Level

Exercise 14A

1 Integrate each of the following with respect to x.

(a) x^5 **(b)** $\frac{1}{3}\sqrt{x}$

(c) $\frac{2}{3x^3}$ **(d)** $\frac{4}{5\sqrt[3]{x}}$

2 Find each of the following indefinite integrals.

(a) $\int (3x^3 - 4\sqrt{x} + 3)\,dx$

(b) $\int (6x^2 - \frac{4}{x^2})\,dx$

(c) $\int \left(5 - \frac{1}{\sqrt{x}} + \frac{1}{x^3}\right) dx$

(d) $\int \frac{x^4 + 5x}{2x^3}\,dx$

3 Given that $\frac{dy}{dt} = 2t^2 + 3t + 4$ and $y = 17$ when $t = 1$, find the value of y when $t = 2$.

4 Find each of the following indefinite integrals.

(a) $\int \frac{3x}{2\sqrt[5]{x^2}}\,dx$

(b) $\int \frac{(3x-1)^2}{5x^4}\,dx$

(c) $\int \frac{3x^7 + x^2}{2\sqrt[3]{x}}\,dx$

(d) $\int (x - 3\sqrt{x})^2\,dx$

(e) $\int (1 + \sqrt[4]{x})(1 - \sqrt[4]{x})\,dx$

(f) $\int \left(\sqrt[3]{x} + \frac{2}{\sqrt[3]{x}}\right)^2 dx$

5 Given that the gradient of a curve is $2x^2 + 7x$ and that the curve passes through the origin, determine the equation of the curve.

6 The rate of change of A with respect to r is given by $\frac{dA}{dr} = 4r + 7$. If $A = 12$ when $r = 1$, find A in terms of r.

7 The marginal revenue function, $\frac{ds}{dt}$, of a company can be approximated by the equation $\frac{ds}{dt} = 3t^2 - 7$, where s is the revenue function in millions of dollars and t is the time in years after its startup. Given that the startup cost was \$6 million, find an expression for s.

8 A curve is such that $\frac{dy}{dx} = k\sqrt[3]{x}$, where k is a constant and that it passes through the points (1, 4) and (8, 16). Find the equation of the curve.

9 The gradient of a curve at the point (x, y) on the curve is given by $\frac{x^2 - 4}{x^2}$. Given that the curve passes through the point (2, 7), find the equation of the curve.

10 A curve with $\frac{dy}{dx} = kx + 3$, where k is a constant, passes through the point $P(3, 19)$. Given that the gradient of the normal to the curve at the point P is $-\frac{1}{15}$, find

(i) the value of k,

(ii) the equation of the curve,

(iii) the coordinates of the turning point on the curve.

11 The rate of change of the profit, \$$P$, from the sales of x items in a company is given by $\frac{dP}{dx} = 3x^2 - 8x + 5$ and the overhead of the company is \$2500 (i.e. profit = –\$2500 when no sales are made). Find the profit function, \$$P$, and calculate the profit when 20 items are sold.

14.3 INTEGRATION OF A FUNCTION INVOLVING A LINEAR FACTOR

In Chapter 11, we have learnt the following:

$$\frac{d}{dx}\left[\frac{1}{11\times 3}(3x+2)^{11}\right] = (3x+2)^{10}$$

Hence,

$$\int (3x+2)^{10}\,dx = \frac{(3x+2)^{11}}{11\times 3}+c\,.$$

In general,

$$\frac{d}{dx}\left[\frac{1}{(n+1)(a)}(ax+b)^{n+1}\right] = (ax+b)^n$$

$$\therefore \int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{(n+1)(a)}+c\,,\ n\neq -1 \text{ and } a\neq 0$$

Worked Example 5

(Integration of $(ax + b)^n$)

Integrate each of the following with respect to x.

(a) $(2x + 7)^6$ **(b)** $\dfrac{3}{\sqrt{5x+4}}$

Solution

(a) $$\int (2x+7)^6\,dx = \frac{(2x+7)^{6+1}}{(6+1)(2)}+c$$
$$= \frac{1}{14}(2x+7)^7+c$$

(b) $$\frac{3}{\sqrt{5x+4}}\,dx = 3\int (5x+4)^{-\frac{1}{2}}\,dx$$
$$= 3\left[\frac{(5x+4)^{-\frac{1}{2}+1}}{\left(-\frac{1}{2}+1\right)(5)}\right]+c$$
$$= \frac{6}{5}\sqrt{5x+4}+c$$

PROBLEM-SOLVING TIP

We can only use the above formula for $\int [f(x)]^n\,dx$ when $f(x)$ is a linear factor. We cannot use the formula for expressions such as $\int (ax^2+bx+c)^n\,dx$ or $\int (ax^3+b)^n\,dx$.

Practise Now 5

Similar Questions:
Exercise 14B
Questions 1-6

Integrate each of the following with respect to x.

(a) $(5x + 4)^{11}$ **(b)** $\dfrac{2}{\sqrt{3x+5}}$ **(c)** $\left(\dfrac{2}{4x-7}\right)^4$

Exercise 14B

1 Integrate each of the following with respect to x.

(a) $(3x-7)^4$ **(b)** $\sqrt{5x+1}$

(c) $2(5x-6)^4$ **(d)** $\sqrt{(2x+7)^3}$

(e) $\dfrac{5}{\sqrt{2-7x}}$ **(f)** $\dfrac{3}{\sqrt{(5-4x)^3}}$

(g) $\left(\dfrac{2}{3x-1}\right)^4$ **(h)** $\dfrac{3}{4(5-2x)^3}$

2 Find the equation of the curve which passes through the point (0, 4) and for which $\dfrac{dy}{dx}=(2x+1)^3$.

3 Given that the gradient of a curve is $\sqrt{7-2x}$ and that the curve passes through the point $\left(\dfrac{3}{2}, 4\right)$, determine the equation of the curve.

4 The rate of expenditure, \$$m$, on the maintenance of a machine, is given by $\dfrac{dm}{dt}=\sqrt{3t+6}$, where t is the number of years the machine is in use. Given that the expenditure is \$7000 after 1 year, find m in terms of t.

5 Find

(a) $\displaystyle\int\sqrt{\frac{9}{6x-5}}\,dx$,

(b) $\displaystyle\int\frac{5(3x+4)^3+(3x+4)}{\sqrt{3x+4}}\,dx$.

Hence, find the range of values for each integral to be valid, showing your working clearly.

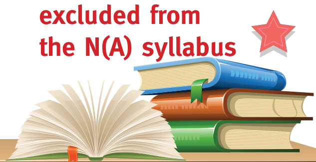
6 Integrate each of the following, where p, m and q are constants, with respect to x.

(a) $(px+q)^5$ **(b)** $(px+q)^m$

(c) $\sqrt[m]{px+q}$ **(d)** $(p^m x-q^2)^m$

14.4 INTEGRATION OF TRIGONOMETRIC FUNCTIONS

excluded from the N(A) syllabus



Recall that $\dfrac{d}{dx}(\sin x)=\cos x$, $\dfrac{d}{dx}(\cos x)=-\sin x$ and $\dfrac{d}{dx}(\tan x)=\sec^2 x$.

Since integration is the reverse of differentiation, we have

$$\int\cos x\,dx = \sin x + c$$
$$\int\sin x\,dx = -\cos x + c$$
$$\int\sec^2 x\,dx = \tan x + c$$

In each of the above cases, x is measured in **radians** and c denotes an arbitrary constant. As in differentiation, integration of trigonometric functions is performed only when the angles involved are measured in radians.

In general, if a and b are constants,

Since $\frac{d}{dx}[\sin(ax+b)] = a\cos(ax+b)$,
then $\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b)+c$

Since $\frac{d}{dx}[\cos(ax+b)] = -a\sin(ax+b)$,
then $\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b)+c$

Since $\frac{d}{dx}[\tan(ax+b)] = a\sec^2(ax+b)$,
then $\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b)+c$

where c is an arbitrary constant.

Worked Example 6

(Integration of $\sin ax$, $\cos ax$ and $\sec^2 ax$)

Integrate each of the following with respect to x.

(a) $2\sin 3x + \frac{1}{2}\cos 2x$ **(b)** $\frac{3}{4\cos^2 5x}$ **(c)** $\tan^2 x$

Solution

(a) $$\int\left(2\sin 3x + \frac{1}{2}\cos 2x\right)dx = 2\times\left(-\frac{1}{3}\cos 3x\right)+\frac{1}{2}\times\left(\frac{1}{2}\sin 2x\right)+c$$
$$= -\frac{2}{3}\cos 3x + \frac{1}{4}\sin 2x + c$$

(b) $$\int \frac{3}{4\cos^2 5x}\,dx = \int \frac{3}{4}\sec^2 5x\,dx$$
$$= \frac{3}{4}\left(\frac{\tan 5x}{5}\right)+c$$
$$= \frac{3}{20}\tan 5x + c$$

(c) $$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx$$
$$= \tan x - x + c$$

ATTENTION

There is **no such rule** as $\int \tan^2 x\,dx = \frac{\tan^3 x}{3}+c$.

Practise Now 6

Similar Questions:
Exercise 14C
Questions 1(a)-(f), 2-4

Integrate each of the following with respect to x.

(a) $5\sin 4x + \frac{3}{4}\cos\frac{1}{2}x$ **(b)** $\tan^2 3x$ **(c)** $3x + \frac{4}{7\cos^2 6x}$

Worked Example 7

(Integration of sin $(ax + b)$, cos $(ax + b)$ and sec² $(ax + b)$)

Find each of the following integrals.

(a) $\int [x^2 + \sec^2 (3 - 2x)]\, dx$ **(b)** $\int \sin\left(\frac{3}{4}x + 2\right) dx$

(c) $\int \frac{2}{5}\cos\left(\frac{\pi}{4} - 2x\right) dx$

Solution

(a) $\int [x^2 + \sec^2 (3 - 2x)]\, dx = \frac{x^3}{3} + \frac{\tan(3-2x)}{-2} + c$

$= \frac{x^3}{3} - \frac{1}{2}\tan(3-2x) + c$

(b) $\int \sin\left(\frac{3}{4}x + 2\right) dx = \frac{-\cos\left(\frac{3}{4}x+2\right)}{\frac{3}{4}} + c$

$= -\frac{4}{3}\cos\left(\frac{3}{4}x + 2\right) + c$

(c) $\int \frac{2}{5}\cos\left(\frac{\pi}{4} - 2x\right) dx = \frac{2}{5}\left[\frac{\sin\left(\frac{\pi}{4} - 2x\right)}{-2}\right] + c$

$= -\frac{1}{5}\sin\left(\frac{\pi}{4} - 2x\right) + c$

ATTENTION

π is a constant.

Practise Now 7

Similar Questions:
Exercise 14C
Questions 5-11

1. Find each of the following integrals.

(a) $\int \left[\sqrt{x} + \cos(5x + 3)\right] dx$ **(b)** $\int [2 + \sin(3x + \pi)]\, dx$

(c) $\int \tan^2\left(2x + \frac{\pi}{2}\right) dx$

2. The gradient of a curve is given by $\frac{dy}{dx} = \frac{1}{2}x - 2\sin 2x$. Given that the curve passes through the origin, find the equation of the curve.

Exercise 14C excluded from the N(A) syllabus

1 Integrate each of the following with respect to x.

(a) $2\cos 3x - \sin 5x$ **(b)** $x + \cos\frac{x}{2}$

(c) $2\sec^2\frac{1}{2}x$ **(d)** $\cos(3x-2)$

(e) $7\sin(5x-3)$ **(f)** $1+\tan^2 x$

2 If $\frac{dy}{dx} = \sin x + \cos x$, find an expression for y if $y = 2$ when $x = \frac{\pi}{4}$.

3 If $\frac{dy}{dx} = 1 - \sin 2x$, find an expression for y if $y = \frac{3}{2}$ when $x = 0$.

4 If $\frac{dy}{dx} = 1 + \frac{1}{2}\cos 2x$, find an expression for y if $y = 3$ when $x = \frac{\pi}{4}$.

5 Find each of the following indefinite integrals.

(a) $\int\left(\frac{1}{\sqrt{x}} + \sin\frac{3x}{2}\right)dx$

(b) $\int\left(\frac{1}{x^2} - \sec^2 3x\right)dx$

(c) $\int\left(\cos 3x - \frac{1}{\cos^2 4x}\right)dx$

(d) $\int\frac{3+2\cos^2 x}{4\cos^2 x}\,dx$

(e) $\int\pi\sin\left(\frac{\pi x}{4} - \frac{\pi}{4}\right)dx$

(f) $\int\pi\cos\left(\pi x - \frac{\pi}{2}\right)dx$

6 Integrate each of the following, where p and q are constants, with respect to x.

(a) $2\sec^2\frac{px}{q}$ **(b)** $\cos(px-q)$

(c) $\tan^2(px+q)$ **(d)** $p\tan^2 x - qx + p$

7 The growth rate of a specimen of bacteria can be approximately modelled by the equation $\frac{dy}{dx} = \tan^2 x$, where x is the time in months after the bacteria has been cultivated and y is the number of units of bacteria. Given that when $x = \frac{\pi}{4}$, $y = \frac{\pi}{2}$, find an expression for y in terms of x.

8 Given that the gradient of the tangent to a curve is $\frac{dy}{dx} = \sec^2\frac{x}{2}$ and that $y = 5\sqrt{3}$ when $x = \frac{\pi}{3}$, find the equation of the curve in terms of x and a constant c.

9 Find an expression for f(x) such that

$$f'(x) = 2\cos^2\left(5x - \frac{\pi}{4}\right) + \sin^2 x - x.$$

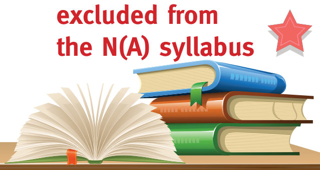
10 Given that $\frac{dx}{dt} = 2\cos\left(3t + \frac{\pi}{2}\right) + \sin\frac{t}{2}$ and that $x = \frac{2}{3}$ when $t = 0$, find the value of x when $t = \pi$.

11 Show that $\frac{d}{dx}\left(\frac{x}{\cos x}\right) = \frac{\cos x + x\sin x}{\cos^2 x}$.

Hence, or otherwise, find the equation of the curve whose gradient function is $\frac{\cos x + x\sin x}{\cos^2 x}$ and passes through the point (0, 2).

14.5 INTEGRATION OF FUNCTIONS OF THE FORMS $\frac{1}{x}$ AND $\frac{1}{ax+b}$

excluded from the N(A) syllabus



In Section 14.1, we have learnt that $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c$, where $n \neq -1$.

In the case of $n = -1$, how do we find the integral of $\frac{1}{x}$?

Recall that, for $x > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}(\ln x) = \frac{1}{x} \text{ and}$$

for $ax + b > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}x}[\ln(ax+b)] = \frac{a}{ax+b}.$$

Since integration is the reverse of differentiation, we have

$$\int \frac{1}{x} \, \mathrm{d}x = \ln x + c, \; x > 0, \text{ and}$$

$$\int \frac{1}{ax+b} \, \mathrm{d}x = \frac{1}{a} \ln(ax+b) + c, \; ax + b > 0.$$

Thinking Time

What happens when $\int x^{-1} \, \mathrm{d}x$ is evaluated using the formula $\int x^n \, \mathrm{d}x = \frac{x^{n+1}}{n+1} + c$?

How can we find the integral of x^{-1}?

Worked Example 8

(Integration of $\dfrac{1}{ax+b}$)

Integrate each of the following with respect to x.

(a) $\dfrac{1}{2x}$ **(b)** $\dfrac{1}{2x+3}$ **(c)** $\dfrac{5}{3-4x}$

Solution

(a)
$$\int \frac{1}{2x}\,\mathrm{d}x = \frac{1}{2}\int \frac{1}{x}\,\mathrm{d}x$$
$$= \frac{1}{2}\ln x + c$$

PROBLEM-SOLVING TIP

We can only manipulate the integral by isolating or extracting a constant from it.

(b)
$$\int \frac{1}{2x+3}\,\mathrm{d}x = \frac{1}{2}\ln(2x+3) + c$$

(c)
$$\int \frac{5}{3-4x}\,\mathrm{d}x = 5\int \frac{1}{3-4x}\,\mathrm{d}x$$
$$= 5\left(-\frac{1}{4}\right)\ln(3-4x) + c$$
$$= -\frac{5}{4}\ln(3-4x) + c$$

Practise Now 8

Similar Questions:
Exercise 14D
Questions 1(a)-(d), 4(a)-(f), 5(a), (b)

Integrate each of the following with respect to x.

(a) $\dfrac{1}{4x}$ **(b)** $\dfrac{1}{5x+3}$ **(c)** $\dfrac{3}{7x+1}$ **(d)** $\dfrac{5}{6-9x}$

The invention of infinitesimal calculus, consisting of differential and integral calculus, is primarily attributed to the independent works of **Gottfried von Leibniz** (1646 – 1716) and Isaac Newton (1642 – 1727). Leibniz was a German mathematician and philosopher who, according to his notebooks, first used integral calculus to find the area under the graph of $y = \mathrm{f}(x)$. He also introduced notations such as the d and $\int$ used for differentials and integrals respectively, which are still widely used today. Search on the Internet to find out more about Leibniz and his contributions to calculus.

Thinking Time

The students were asked to find $\int \frac{3}{5x}\,dx$.

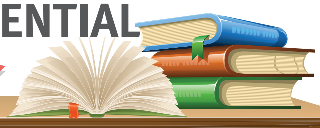
By using the formula $\int \frac{a}{ax+b}\,dx = \ln(ax+b)+c$,

Ali gave his answer as: $\int \frac{3}{5x}\,dx = \frac{3}{5}\int \frac{1}{x}\,dx = \frac{3}{5}\ln x + c$ and

Kumar gave his answer as: $\int \frac{3}{5x}\,dx = \frac{3}{5}\int \frac{2}{2x}\,dx = \frac{3}{5}\ln 2x + c$.

Why are there two different answers? Explain your reasoning.

14.6 INTEGRATION OF EXPONENTIAL FUNCTIONS excluded from the N(A) syllabus



In Chapter 13, we have learnt that

$$\frac{d}{dx}(e^x) = e^x,$$

$$\frac{d}{dx}(e^{ax}) = ae^{ax} \text{ and}$$

$$\frac{d}{dx}(e^{ax+b}) = ae^{ax+b}.$$

Since integration is the reverse of differentiation, we have

$$\boxed{\begin{aligned} &\int e^x\,dx = e^x + c \\ &\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c \end{aligned}}$$

Note the difference.

index is a constant

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

base is a variable

but

index is a variable

$$\int e^x\,dx = e^x + c$$

base is a constant

Worked Example 9

(Integration of e^{ax+b})

Find each of the following indefinite integrals.

(a) $\int e^{3x}\,dx$ **(b)** $\int e^{2-5x}\,dx$

Solution

(a) $\int e^{3x}\,dx = \frac{1}{3}e^{3x} + c$

(b) $\int e^{2-5x}\,dx = \frac{1}{-5}e^{2-5x} + c = -\frac{1}{5}e^{2-5x} + c$

Practise Now 9

Similar Questions:
Exercise 14D
Questions 2(a)-(d), 3(a)-(d), 5(c), (d)

1. Find each of the following indefinite integrals.

(a) $\int e^{2x+\pi}\,dx$ **(b)** $\int e^{3-4x}\,dx$ **(c)** $\int \frac{e^{3x}+1}{e^{3x}}\,dx$

2. Find a formula for y in terms of x if $\frac{dy}{dx} = \frac{x^2-1}{\sqrt{x}} - \sqrt{e^{-\frac{4}{3}x}}$ and $y = \frac{1}{2}$ when $x = 0$. Calculate y when $x = 1$.

Basic Level | Intermediate Level | Advanced Level

Exercise 14D

excluded from the N(A) syllabus

1 Integrate each of the following with respect to x.

(a) $2 + \frac{3}{x}$ **(b)** $\left(2 + \frac{1}{x}\right)^2$

(c) $\frac{1}{3x-7}$ **(d)** $\frac{1}{5-x}$

2 Integrate each of the following with respect to x.

(a) $3e^x$ **(b)** $4e^{\frac{x}{2}}$

(c) $\frac{1}{5}e^{-x}$ **(d)** e^{2x+7}

3 Find each of the following indefinite integrals.

(a) $\int 5e^{2x+\pi}\,dx$ **(b)** $\int (e^{2x}-1)^2\,dx$

(c) $\int \frac{2+e^{3x}}{e^{x+1}}\,dx$ **(d)** $\int \frac{2e^x + 9x^2e^{3x}}{18x^2e^x}\,dx$

4 Find each of the following indefinite integrals.

(a) $\int \left(3 + \frac{5}{x}\right)dx$ **(b)** $\int \frac{x^3+2x}{x^2}\,dx$

(c) $\int \left(2 + \frac{3}{x}\right)^2 dx$ **(d)** $\int \frac{\pi + 5x}{2x^2}\,dx$

(e) $\int \left(3 - \frac{4}{x}\right)^2 dx$ **(f)** $\int \frac{3}{4-5x}\,dx$

5 Integrate each of the following with respect to x.

(a) $\left(2x + \frac{1}{x^2}\right)^2$ **(b)** $\frac{2x-4}{x^2-4x+4}$

(c) $\frac{3e^{3x}+e^x}{2e^{2x}}$ **(d)** $\frac{\left(e^{2x}+3e^x\right)^2}{e^{3x}}$

14.7 FURTHER EXAMPLES OF INTEGRATION

Some of the examples in this section make use of the concept that if

$$\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{F}(x)] = \mathrm{f}(x), \text{ then } \int \mathrm{f}(x)\,\mathrm{d}x = \mathrm{F}(x) + c$$

to solve problems on integration.

Sometimes, we may be able to integrate a complicated expression by

(i) simplifying it using partial fractions, or

(ii) manipulating it using trigonometric identities.

Worked Example 10

(Integration involving Trigonometric Formulae)

Prove that $(\cos\theta - 2\sin\theta)^2 = 2\frac{1}{2} - 2\sin 2\theta - 1\frac{1}{2}\cos 2\theta$.

Hence, find $\int (2\sin x - \cos x)^2\,\mathrm{d}x$.

Solution

$$\begin{aligned}
\text{LHS} &= (\cos\theta - 2\sin\theta)^2 \\
&= \cos^2\theta - 4\sin\theta\cos\theta + 4\sin^2\theta \\
&= \left(\frac{1+\cos 2\theta}{2}\right) - 2(2\sin\theta\cos\theta) + 4\left(\frac{1-\cos 2\theta}{2}\right) \\
&= \frac{1}{2} + \frac{1}{2}\cos 2\theta - 2\sin 2\theta + 2 - 2\cos 2\theta \\
&= 2\frac{1}{2} - 2\sin 2\theta - 1\frac{1}{2}\cos 2\theta = \text{RHS}
\end{aligned}$$

RECALL

- $\cos 2\theta = 2\cos^2\theta - 1$
- $\cos 2\theta = 1 - 2\sin^2\theta$
- $\sin 2\theta = 2\sin\theta\cos\theta$

$$\begin{aligned}
\int (2\sin x - \cos x)^2\,\mathrm{d}x &= \int (\cos x - 2\sin x)^2\,\mathrm{d}x \\
&= \int \left(2\frac{1}{2} - 2\sin 2x - 1\frac{1}{2}\cos 2x\right)\mathrm{d}x \\
&= 2\frac{1}{2}x - 2\left[\frac{-\cos 2x}{2}\right] - 1\frac{1}{2}\left[\frac{\sin 2x}{2}\right] + c \\
&= \frac{5}{2}x + \cos 2x - \frac{3}{4}\sin 2x + c
\end{aligned}$$

Practise Now 10

Prove that $\frac{\sin\theta}{\operatorname{cosec}\theta - \cot\theta} = 1 + \cos\theta$. Hence, find $\int \frac{2\sin\theta}{\operatorname{cosec}\theta - \cot\theta}\,\mathrm{d}\theta$.

Similar Questions:
Exercise 14E
Questions 8-11, 15, 16

Worked Example 11

(Integration as the Reverse of Differentiation)

Given that $y = \ln\left(x + \sqrt{x^2 - 1}\right)$, find $\frac{dy}{dx}$. Hence, find $\int \frac{5}{4\sqrt{x^2 - 1}}\,dx$.

Solution

$$y = \ln\left(x + (x^2 - 1)^{\frac{1}{2}}\right)$$

$$\frac{dy}{dx} = \frac{1 + \frac{1}{2}(x^2 - 1)^{-\frac{1}{2}}(2x)}{x + \sqrt{x^2 - 1}}$$

$$= \frac{1 + \frac{x}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}}$$

$$= \frac{\frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}}$$

$$= \frac{1}{\sqrt{x^2 - 1}}$$

$$\int \frac{5}{4\sqrt{x^2 - 1}}\,dx = \frac{5}{4}\int \frac{1}{\sqrt{x^2 - 1}}\,dx$$

$$= \frac{5}{4}\ln\left(x + \sqrt{x^2 - 1}\right) + c$$

Practise Now 11

Similar Question:
Exercise 14E
Question 5

Given that $y = \ln\left[\frac{7}{\sqrt{6x + 1}}\right]$, find $\frac{dy}{dx}$. Hence, find $\int \frac{9}{7(6x + 1)}\,dx$.

Worked Example 12

(Integration as the Reverse of Differentiation)

Given that $y = x^2\sqrt{3x - 2}$, find $\frac{dy}{dx}$. Hence, find $\int \frac{15x^2 - 8x + 7}{2\sqrt{3x - 2}}\,dx$.

Solution

$$\frac{dy}{dx} = x^2\frac{d}{dx}(3x - 2)^{\frac{1}{2}} + (3x - 2)^{\frac{1}{2}}\frac{d}{dx}(x^2)$$

$$= x^2\left[\frac{1}{2}(3x - 2)^{-\frac{1}{2}}(3)\right] + \sqrt{3x - 2}\,(2x) = \frac{3x^2}{2\sqrt{3x - 2}} + 2x\sqrt{3x - 2}$$

$$= \frac{3x^2 + 4x(3x - 2)}{2\sqrt{3x - 2}} = \frac{3x^2 + 12x^2 - 8x}{2\sqrt{3x - 2}} = \frac{15x^2 - 8x}{2\sqrt{3x - 2}}$$

$$\int \frac{15x^2 - 8x + 7}{2\sqrt{3x-2}}\,dx = \int \frac{15x^2 - 8x}{2\sqrt{3x-2}}\,dx + \int \frac{7}{2\sqrt{3x-2}}\,dx$$

$$= \int \frac{15x^2 - 8x}{2\sqrt{3x-2}}\,dx + \frac{7}{2}\int (3x-2)^{-\frac{1}{2}}\,dx$$

$$= x^2\sqrt{3x-2} + \frac{\frac{7}{2}(3x-2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)(3)} + c$$

$$= x^2\sqrt{3x-2} + \frac{7}{3}\sqrt{3x-2} + c$$

Practise Now 12

Similar Questions:
Exercise 14E
Questions 1-4

1. Given that $y = x\sqrt{4+3x^2}$, show that $\frac{dy}{dx} = \frac{4+6x^2}{\sqrt{4+3x^2}}$.
   Hence, integrate $\frac{2+3x^2}{\sqrt{4+3x^2}}$ with respect to x.

2. Show that $\frac{d}{dx}\left(\frac{x}{\sqrt{1-x^2}}\right) = \frac{1}{(1-x^2)\sqrt{1-x^2}}$. Hence, find $\int \frac{2}{(1-x^2)\sqrt{1-x^2}}\,dx$.

Worked Example 13

(Integration by using Partial Fractions)

Express $\frac{x^2 - 5x + 21}{(x+1)(x-2)^2}$ in partial fractions. Hence, find $\int \frac{x^2 - 5x + 21}{(x+1)(x-2)^2}\,dx$.

Solution

Let $\frac{x^2 - 5x + 21}{(x+1)(x-2)^2} = \frac{A}{x+1} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

$$= \frac{A(x-2)^2 + B(x-2)(x+1) + C(x+1)}{(x+1)(x-2)^2}$$

$\therefore\ x^2 - 5x + 21 = A(x-2)^2 + B(x-2)(x+1) + C(x+1)$

Let $x = 2$: $4 - 10 + 21 = C(3)$ $\quad\therefore C = 5$

Let $x = -1$: $1 + 5 + 21 = A(-1-2)^2$ $\quad\therefore A = 3$

Let $x = 0$: $21 = 3(-2)^2 + B(-2)(1) + 5(1)$ $\quad\therefore B = -2$

$$\therefore \frac{x^2 - 5x + 21}{(x+1)(x-2)^2} = \frac{3}{x+1} - \frac{2}{x-2} + \frac{5}{(x-2)^2}$$

$$\therefore \int \frac{x^2 - 5x + 21}{(x+1)(x-2)^2}\,dx = \int \frac{3}{x+1}\,dx - \int \frac{2}{x-2}\,dx + \int \frac{5}{(x-2)^2}\,dx$$

$$= 3\int \frac{1}{x+1}\,dx - 2\int \frac{1}{x-2}\,dx + \int 5(x-2)^{-2}\,dx$$

$$= 3\ln(x+1) - 2\ln(x-2) + \frac{5(x-2)^{-1}}{(-1)(1)} + c$$

$$= 3\ln(x+1) - 2\ln(x-2) - \frac{5}{x-2} + c$$

Practise Now 13

Similar Questions:
Exercise 14E
Questions 12-14

1. Express $\frac{7x+3}{(x-1)(2x+3)}$ in partial fractions. Hence, find $\int \frac{7x+3}{(x-1)(2x+3)}\,dx$.

2. Express $\frac{2}{3x-2}+\frac{3}{(3x-2)^2}$ as a single fraction. Hence, integrate $\frac{6x-1}{2(3x-2)^2}$ with respect to x.

Worked Example 14

(Integration as the Reverse of Differentiation)

Given that $y = 3e^{2x}(\sin 2x + \cos 2x)$, find $\frac{dy}{dx}$. Hence, find $\int 5e^{2x}\cos 2x\,dx$.

Solution

$$y = 3e^{2x}(\sin 2x + \cos 2x)$$

$$\frac{dy}{dx} = 3e^{2x}\frac{d}{dx}(\sin 2x + \cos 2x) + (\sin 2x + \cos 2x)\frac{d}{dx}(3e^{2x})$$

$$= 3e^{2x}(2\cos 2x - 2\sin 2x) + (\sin 2x + \cos 2x)(6e^{2x})$$

$$= 6e^{2x}\cos 2x - 6e^{2x}\sin 2x + 6e^{2x}\sin 2x + 6e^{2x}\cos 2x$$

$$= 12e^{2x}\cos 2x$$

$$\Rightarrow \int 12e^{2x}\cos 2x\,dx = 3e^{2x}(\sin 2x + \cos 2x) + c$$

$$\Rightarrow \frac{5}{12}\int 12e^{2x}\cos 2x\,dx = \frac{5}{12}\left[3e^{2x}(\sin 2x + \cos 2x)\right] + \frac{5}{12}c$$

$$\Rightarrow \int 5e^{2x}\cos 2x\,dx = \frac{5}{4}\left[e^{2x}(\sin 2x + \cos 2x)\right] + c', \text{ where } c' = \frac{5}{12}c$$

$$= \frac{5}{4}e^{2x}(\sin 2x + \cos 2x) + c'$$

Practise Now 14

Similar Questions:
Exercise 14E
Questions 6, 7

1. Given that $y = e^x(\cos x - \sin x)$, find $\frac{dy}{dx}$. Hence, find $\int 5e^x \sin x\,dx$.

2. **(i)** Find $\frac{d}{dt}(e^{-t^2})$.

   (ii) If the rate of growth of the profit (in million of dollars) from a new invention is approximated by $P'(t) = 3te^{-t^2}$ where t is the time measured in years, find the profit function, $P(t)$, for the new invention, given that the profit in the second year that the new invention is in operation is \$20 000.

Basic Level | Intermediate Level | Advanced Level

Exercise 14E

1 Given that $y = 2x\sqrt{3x+1}$, show that $\frac{dy}{dx} = \frac{9x+2}{\sqrt{3x+1}}$. Hence, find $\int \frac{9x+2}{2\sqrt{3x+1}}\,dx$.

2 Given that $y = \frac{2x}{\sqrt{3x+4}}$, show that $\frac{dy}{dx} = \frac{3x+8}{\sqrt{(3x+4)^3}}$. Hence, find $\int \frac{6x+16}{\sqrt{(3x+4)^3}}\,dx$.

3 Show that $\frac{d}{dx}\left(5x\sqrt{3x^2+7}\right) = \frac{5(6x^2+7)}{\sqrt{3x^2+7}}$.

Hence, find $\int \frac{6x^2+7}{4\sqrt{3x^2+7}}\,dx$.

4 Given that $y = \frac{4x}{\sqrt{3x-2x^2}}$, find $\frac{dy}{dx}$.

Hence, find $\int \frac{7x}{\sqrt{(3x-2x^2)^3}}\,dx$.

5 ★ Differentiate $5x^2 \ln x$ with respect to x.

Hence, find $\int 4x \ln x\,dx$.

6 ★ Given that $y = e^{2x}(5x-4)$, find $\frac{dy}{dx}$.

Hence, find $\int 3x\,e^{2x}\,dx$.

7 ★ Given that $y = e^{3x}(\sin 3x + \cos 3x)$, show that $\frac{dy}{dx} = 6e^{3x}\cos 3x$.

Hence, find $\int e^{3x}\cos 3x\,dx$.

8 ★ Prove that $7\cos^2 x + 5\sin^2 x = 6 + \cos 2x$.

Hence, find $\int (7\cos^2 x + 5\sin^2 x)\,dx$.

9 ★ Given that $y = \frac{2\sin x}{1+\cos x}$, show that $\frac{dy}{dx} = \frac{2}{1+\cos x}$. Hence, find $\int \frac{3}{7(1+\cos x)}\,dx$.

10 ★ Prove that $\cos^4 x - \sin^4 x = \cos 2x$.

Hence, find $\int \frac{\cos^4 x - \sin^4 x}{3}\,dx$.

11 ★ Given that $y = 3\sin x\cos x$, find $\frac{dy}{dx}$ and express it in terms of $\cos^2 x$.

Hence, find $\int 7\cos^2 x\,dx$.

12 ★ Express $\frac{x-19}{(x+2)(2x-3)}$ in partial fractions.

Hence, find $\int \frac{2x-38}{3(x+2)(2x-3)}\,dx$.

13 ★ Express $\frac{27-7x}{(2x+3)(x-1)^2}$ in partial fractions.

Hence, find $\int \frac{27-7x}{5(2x+3)(x-1)^2}\,dx$.

14 ★ **(i)** Find $\frac{d}{dx}\left[\ln(x^2+3)\right]$.

(ii) Express $\frac{18-15x}{(2x+5)(x^2+3)}$ in partial fractions. Hence, find

$$\int \frac{6-5x}{(2x+5)(x^2+3)}\,dx.$$

15 ★ Prove that $\sin 3x = 3\sin x - 4\sin^3 x$.

Hence, find $\int 3\sin^3 x\,dx$.

16 ★ Given that $y = 2\tan^3 x$, find $\frac{dy}{dx}$. Hence, find $\int 5\sec^4 x\,dx$.

SUMMARY

1. If $\dfrac{d}{dx}[F(x)] = f(x)$, then $\int f(x)\,dx = F(x) + c$, where c is a constant.

2. Two Rules of Integration

   (a) **Rule 1:** $\int kf(x)dx = k\int f(x)dx$

   (b) **Rule 2:** $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$ (Addition/Subtraction Rule)

3. **Integration formulae**

   (a) A function involving a linear factor: $\int (ax+b)^n dx = \dfrac{(ax+b)^{n+1}}{(n+1)(a)} + c,\ n \neq -1$ and $a \neq 0$

   (b) Trigonometric functions:

   - $\int \cos(ax+b)dx = \dfrac{1}{a}\sin(ax+b) + c$
   - $\int \sin(ax+b)dx = -\dfrac{1}{a}\cos(ax+b) + c$
   - $\int \sec^2(ax+b)dx = \dfrac{1}{a}\tan(ax+b) + c$

   (c) Functions of the form $\dfrac{1}{x}$ and $\dfrac{1}{ax+b}$:

   - $\int \dfrac{1}{x}dx = \ln x + c,\ x > 0$
   - $\int \dfrac{1}{ax+b}dx = \dfrac{1}{a}\ln(ax+b) + c,\ ax+b > 0$

   (d) Exponential functions:

   - $\int e^x\,dx = e^x + c$
   - $\int e^{ax+b}\,dx = \dfrac{1}{a}e^{ax+b} + c$

Review Exercise 14

1. Find each of the following indefinite integrals.

   (a) $\int \left(t + \dfrac{1}{t^2}\right)dt$ (b) $\int \dfrac{t^2+2}{t^4}dt$ (c) $\int (2x+9)^5\,dx$ (d) $\int \sqrt{(5x-7)^3}\,dx$

   (e) $\int \dfrac{7}{\sqrt{2-3x}}dx$ (f) $\int 7\sin 2x\,dx$ (g) $\int 2\cos(7x+5)\,dx$

2. Find each of the following indefinite integrals.

   (a) $\int e^{3x+1}\,dx$ (b) $\int 5e^{2x+3}\,dx$ (c) $\int e^x(3e^{2x} - 4)\,dx$ (d) $\int \dfrac{e^x + 3e^{2x}}{4e^x}dx$

   (e) $\int \dfrac{3+x}{2x^2}dx$ (f) $\int \dfrac{(5-x)^2}{3x}dx$ (g) $\int \dfrac{2}{3x-4}dx$ (h) $\int \dfrac{2+3e^{4x}}{e^{x+3}}dx$

3. A curve is such that $\frac{dy}{dx} = ax - 3$, where a is a constant. Given that the gradient of the normal to the curve is $-\frac{1}{5}$ at the point (2, −1), find
(i) the value of a,
(ii) the equation of the curve.

4. Given that $\frac{dy}{dx} = 2x - 4x^3$ and that $y = 3$ when $x = 1$, find y in terms of x.

5. Given that the gradient of a curve is $2x(x + 3) + 4$ and that the curve passes through (0, 5), determine the equation of the curve.

6. Find the equation of the curve which passes through the point (1, 5) and for which the gradient function is $2 - \frac{1}{x^2}$.

7. The rate of the amount, x units, of a chemical compound absorbed by an organism is given by $\frac{dx}{dt} = 0.2e^{-0.1t}$, where t is the time in hours. If there is no absorption when $t = 0$, find the number of units of the chemical compound absorbed in 8 hours.

8. The curve for which $\frac{dy}{dx} = e^{3x}$ passes through the point (0, −1). Find the equation of the curve.

9. Given that $y = \frac{2x - 3}{\sqrt{(4x + 3)}}$, show that $\frac{dy}{dx} = \frac{4(x + 3)}{\sqrt{(4x + 3)^3}}$. Hence, find $\int \frac{x + 3}{\sqrt{(4x + 3)^3}}\, dx$.

10. Given that $y = \ln \sin x$, find $\frac{dy}{dx}$. Hence, find $\int 3 \cot x \, dx$.

11. Given that $y = 3x^2 \ln 4x$, find $\frac{dy}{dx}$. Hence, find $\int x \ln 4x \, dx$.

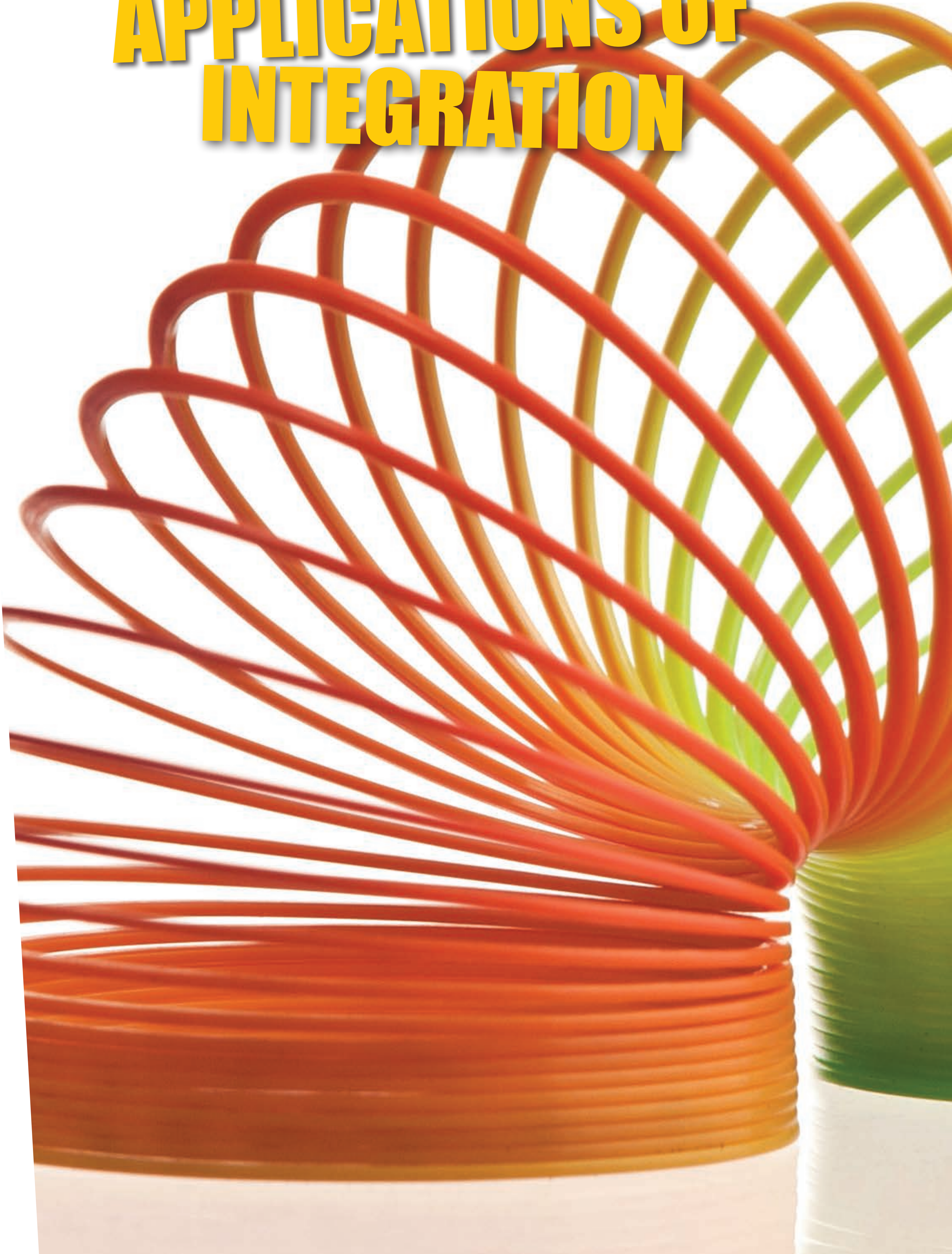
12. (i) Find $\frac{d}{dx}\left[\ln (3x^2 + 5)\right]$.
(ii) Express $\frac{17x^2 - 21x + 5}{(2x - 3)(3x^2 + 5)}$ in partial fractions.
Hence, find $\int \frac{17x^2 - 21x + 5}{(2x - 3)(3x^2 + 5)}\, dx$.

1. (i) Find $\frac{d}{dx}\left[\ln (8x^3 - 27)\right]$.
(ii) Without using partial fractions, find $\int \frac{27x^2}{(2x - 3)(4x^2 + 6x + 9)}\, dx$.

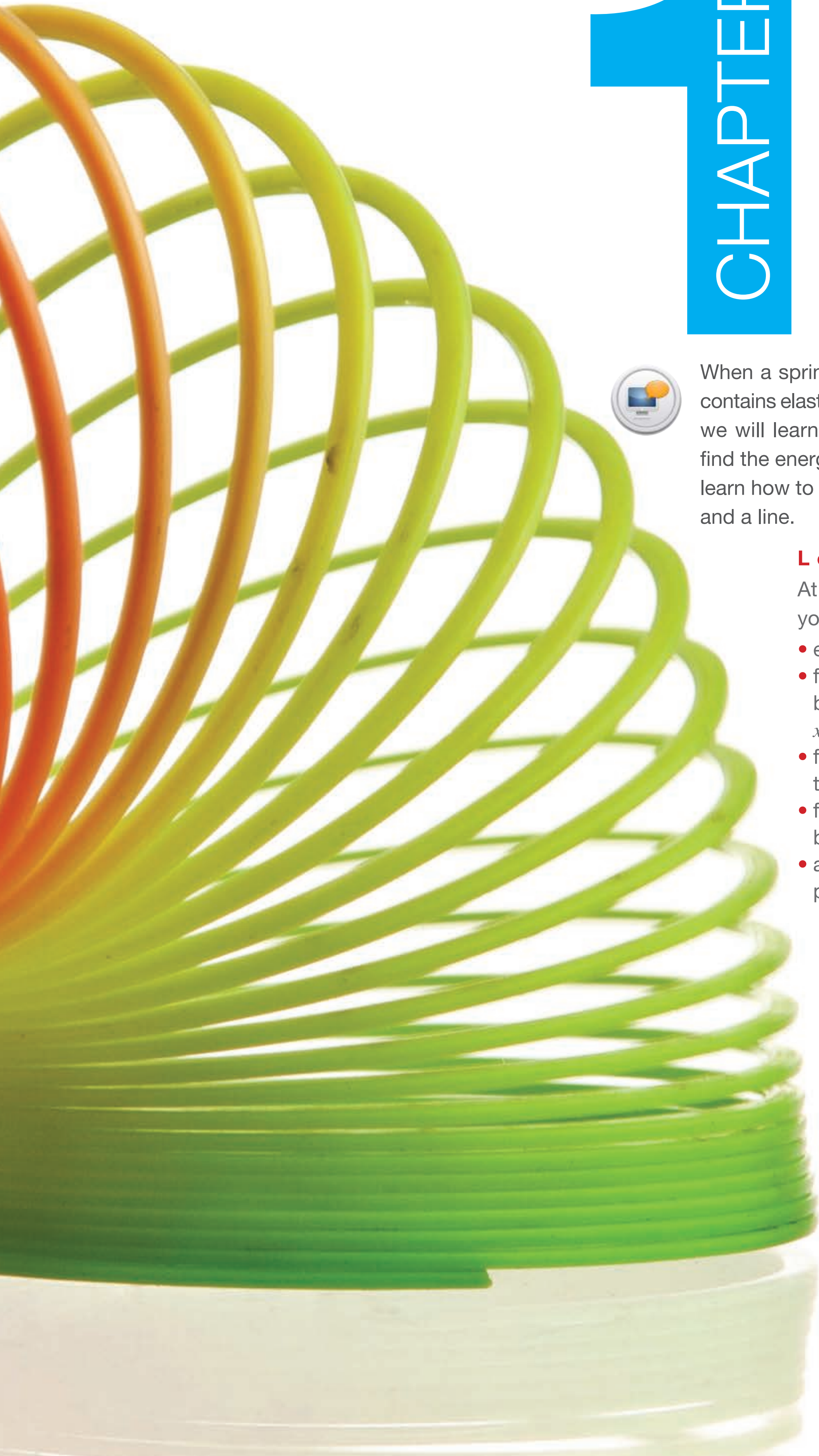
2. Find
$$\int \frac{8}{1 + 8x + 24x^2 + 32x^3 + 16x^4}\, dx.$$

3. Given that $z = x\frac{dy}{dx}$, express $\frac{dz}{dx}$ in terms of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. Hence show that the equation $x\frac{d^2y}{dx^2} + \frac{dy}{dx} = 3x^2$ can be reduced to $x\frac{dy}{dx} = x^3 + c$. If y has a stationary point at (1, 2), find an expression for y.

APPLICATIONS OF INTEGRATION



CHAPTER 15



When a spring is compressed or stretched, it contains elastic potential energy. In this chapter, we will learn that integration can be used to find the energy stored in a spring. We will also learn how to find the area bounded by a curve and a line.

Learning Objectives

At the end of this chapter, you should be able to:

- evaluate definite integrals,
- find the area under a curve between the x-axis and the lines $x = a$ and $x = b$,
- find the area of a region below the x-axis,
- find the area of a region bounded by a curve and line(s),
- apply definite integrals to solve problems.

15.1 DEFINITE INTEGRALS

In Chapter 14, we studied the techniques to integrate algebraic, trigonometric and exponential functions. We shall apply these techniques to find the area under a curve and to solve other problems.

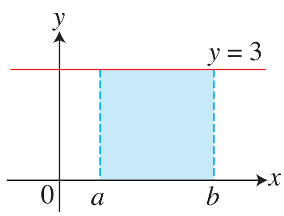
Let us first understand what a definite integral represents by considering a simple example.

The figure shows the graph of $y = \mathrm{f}(x) = 3$.

$$\int \mathrm{f}(x)\,\mathrm{d}x = \int 3\,\mathrm{d}x$$

$$= 3x + c$$

$$= \mathrm{F}(x) + c, \text{ where } \mathrm{F}(x) = 3x$$



The shaded area bounded by $y = 3$, the line $x = a$ and $x = b$ and the x-axis is

$$A = 3(b - a)$$

$$= 3b - 3a$$

$$= \mathrm{F}(b) - \mathrm{F}(a)$$

We can write $\mathrm{F}(b) - \mathrm{F}(a)$ simply as $\left[\mathrm{F}(x)\right]_a^b$.

In this example, $\left[\mathrm{F}(x)\right]_a^b = \left[3x\right]_a^b$.

Since $\int \mathrm{f}(x)\,\mathrm{d}x = 3x + c$,

Then $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \left[3x + c\right]_a^b$

$$= \left[\mathrm{F}(x) + c\right]_a^b$$

$$= [\mathrm{F}(b) + c] - [\mathrm{F}(a) + c]$$

$$= \mathrm{F}(b) - \mathrm{F}(a).$$

Notice that c cancels out in the process.

$\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ is called a **definite integral**.

A definite integral may be taken to be the area under a curve.

Hence, the evaluation of the definite integral can be written as

$$\boxed{\int_a^b \mathrm{f}(x)\,\mathrm{d}x = \left[\mathrm{F}(x)\right]_a^b = \mathrm{F}(b) - \mathrm{F}(a).}$$

Thinking Time

Explain what $\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ represents.

Write down the differences between an indefinite integral and the definite integral.

Worked Example 1

(Integration of ax^n)

Evaluate

(a) $\displaystyle\int_0^2 x^3\,\mathrm{d}x$, **(b)** $\displaystyle\int_{-2}^0 5x^4\,\mathrm{d}x$, **(c)** $\displaystyle\int_2^3 \frac{6}{x^2}\,\mathrm{d}x$.

Solution

(a)

$$\int_0^2 x^3\,\mathrm{d}x = \left[\frac{x^4}{4}\right]_0^2$$

$$= \left(\frac{2^4}{4}\right) - \left(\frac{0^4}{4}\right) = 4$$

(b)

$$\int_{-2}^0 5x^4\,\mathrm{d}x = \left[\frac{5x^5}{5}\right]_{-2}^0$$

$$= [x^5]_{-2}^0$$

$$= (0^5) - (-2)^5$$

$$= 32$$

(c)

$$\int_2^3 \frac{6}{x^2}\,\mathrm{d}x = \int_2^3 6x^{-2}\,\mathrm{d}x$$

$$= \left[\frac{6x^{-1}}{-1}\right]_2^3$$

$$= \left[\frac{-6}{x}\right]_2^3 = \left(\frac{-6}{3}\right) - \left(\frac{-6}{2}\right) = 1$$

Thinking Time

(1) Let $\mathrm{f}(x) = \sin x$. Calculate $\displaystyle\int_0^\pi \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^\pi 3\,\mathrm{f}(x)\,\mathrm{d}x$.

(2) Let $\mathrm{f}(x) = \mathrm{e}^{3x+5}$. Calculate $\displaystyle\int_0^1 \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_0^1 \frac{1}{3}\,\mathrm{f}(x)\,\mathrm{d}x$.

From the results in Worked Example 1 and in **(1)** and **(2)** above, can you find the relationship between $\displaystyle\int_a^b \mathrm{f}(x)\,\mathrm{d}x$ and $\displaystyle\int_a^b k\,\mathrm{f}(x)\,\mathrm{d}x$?

Practise Now 1

Similar Questions:
Exercise 15A
Questions 2(a)-(k)

Evaluate each of the following.

(a) $\displaystyle\int_1^3 2x^2\,\mathrm{d}x$ **(b)** $\displaystyle\int_1^4 \frac{5}{\sqrt{x}}\,\mathrm{d}x$ **(c)** $\displaystyle\int_1^3 \frac{x^3+4x}{x^5}\,\mathrm{d}x$

Worked Example 2

(Integration of $(ax + b)^n$)

Evaluate

(a) $\int_0^2 (3x+2)^2 \, dx$, **(b)** $\int_0^5 \sqrt{x+4} \, dx$.

Solution

(a) $$\int_0^2 (3x+2)^2 \, dx = \left[\frac{(3x+2)^3}{(3)(3)} \right]_0^2$$

$$= \frac{[3(2)+2]^3}{9} - \frac{2^3}{9}$$

$$= 56$$

RECALL

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + c$$

(b) $$\int_0^5 \sqrt{x+4} \, dx = \int_0^5 (x+4)^{\frac{1}{2}} \, dx$$

$$= \left[\frac{(x+4)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)(1)} \right]_0^5$$

$$= \left[\frac{2}{3}(x+4)^{\frac{3}{2}} \right]_0^5$$

$$= \frac{2}{3}(9)^{\frac{3}{2}} - \frac{2}{3}(4)^{\frac{3}{2}}$$

$$= 12\frac{2}{3}$$

PROBLEM-SOLVING TIP

You can also take out the constant $\frac{2}{3}$ before substituting in the limits, i.e. $\frac{2}{3}\left[(x+4)^{\frac{3}{2}}\right]_0^5$.

Practise Now 2

Similar Questions:
Exercise 15A
Questions 3(a)-(f)

Evaluate

(a) $\int_1^3 (2x-1)^3 \, dx$, **(b)** $\int_0^9 \sqrt{2x+9} \, dx$.

Worked Example 3

(Integration of Trigonometric Functions)

Evaluate

(a) $\int_0^{\frac{\pi}{3}} (2\cos x + 1)\,\mathrm{d}x$,

(b) $\int_0^{\frac{\pi}{6}} (2\sin 3x + 3\tan^2 2x)\,\mathrm{d}x$.

Solution

(a) $\int_0^{\frac{\pi}{3}} (2\cos x + 1)\,\mathrm{d}x = [2\sin x + x]_0^{\frac{\pi}{3}}$

$$= \left(2\sin\frac{\pi}{3} + \frac{\pi}{3}\right) - (2\sin 0 + 0)$$

$$= 2\left(\frac{\sqrt{3}}{2}\right) + \frac{\pi}{3}$$

$$= \sqrt{3} + \frac{\pi}{3}$$

(b) $\int_0^{\frac{\pi}{6}} (2\sin 3x + 3\tan^2 2x)\,\mathrm{d}x$

$$= \int_0^{\frac{\pi}{6}} [2\sin 3x + 3(\sec^2 2x - 1)]\,\mathrm{d}x$$

$$= \left[2\left(\frac{-\cos 3x}{3}\right) + 3\left(\frac{\tan 2x}{2}\right) - 3x\right]_0^{\frac{\pi}{6}}$$

$$= \left[-\frac{2}{3}\cos\frac{\pi}{2} + \frac{3}{2}\tan\frac{\pi}{3} - 3\left(\frac{\pi}{6}\right)\right] - \left[-\frac{2}{3}\cos 0 + \frac{3}{2}\tan 0 - 0\right]$$

$$= \left(\frac{3}{2}\sqrt{3} - \frac{\pi}{2}\right) - \left(-\frac{2}{3}\right)$$

$$= \frac{3\sqrt{3}}{2} + \frac{2}{3} - \frac{\pi}{2}$$

RECALL

$\tan^2 x + 1 = \sec^2 x$

ATTENTION

Always remember to substitute both the limits into the expression after doing the integration.

Practise Now 3

Similar Questions:
Exercise 15A
Questions 4(a)-(f)

Evaluate

(a) $\int_0^{\frac{\pi}{4}} 3\sin 2x\,\mathrm{d}x$, **(b)** $\int_0^{\frac{\pi}{4}} (\cos 3x + 2\tan^2 x)\,\mathrm{d}x$.

Worked Example 4

(Integration of $\frac{1}{ax+b}$)

Evaluate each of the following integrals.

(a) $\int_1^4 \frac{2x+3}{x^2}\,dx$ (b) $\int_{-1}^0 \frac{3}{4-2x}\,dx$

Solution

(a) $$\int_1^4 \frac{2x+3}{x^2}\,dx = \int_1^4 \left(\frac{2}{x}+3x^{-2}\right)dx = \left[2\ln x+\frac{3x^{-1}}{-1}\right]_1^4$$

$$= \left[2\ln x-\frac{3}{x}\right]_1^4 = \left(2\ln 4-\frac{3}{4}\right)-(2\ln 1-3)$$

$$= 2\ln 4+2\frac{1}{4}$$

(b) $$\int_{-1}^0 \frac{3}{4-2x}\,dx = 3\int_{-1}^0 \frac{1}{4-2x}\,dx = 3\left(-\frac{1}{2}\right)[\ln(4-2x)]_{-1}^0$$

$$= \left[-\frac{3}{2}\ln(4-2x)\right]_{-1}^0 = \left(-\frac{3}{2}\ln 4\right)-\left\{-\frac{3}{2}\ln[4-2(-1)]\right\}$$

$$= -\frac{3}{2}\ln 4+\frac{3}{2}\ln 6 = \frac{3}{2}(\ln 6-\ln 4)$$

$$= \frac{3}{2}\ln\frac{6}{4} = \frac{3}{2}\ln\frac{3}{2}$$

ATTENTION

We can only take out a constant factor from an integral.

Practise Now 4

Similar Questions:
Exercise 15A
Questions 7(a)-(f)

Evaluate each of the following integrals.

(a) $\int_0^3 \frac{3}{2x+1}\,dx$ (b) $\int_1^3 \frac{(2x+3)^2}{3x}\,dx$

Worked Example 5

(Integration of e^x and e^{ax+b})

Evaluate each of the following, giving your answer correct to 2 decimal places.

(a) $\int_1^2 e^{1+x}\,dx$ (b) $\int_0^2 (2e^{2x}+3)\,dx$

Solution

(a) $$\int_1^2 e^{1+x}\,dx = \left[e^{1+x}\right]_1^2 = e^3-e^2 = 12.70 \text{ (to 2 d.p.)}$$

(b) $$\int_0^2 (2e^{2x}+3)\,dx = \left[e^{2x}+3x\right]_0^2 = (e^4+6)-(e^0+0) = 59.60 \text{ (to 2 d.p.)}$$

Practise Now 5

Similar Questions:
Exercise 15A
Questions 5(a)-(f), 6(a)-(f)

1. Evaluate each of the following, giving your answer correct to 2 decimal places.

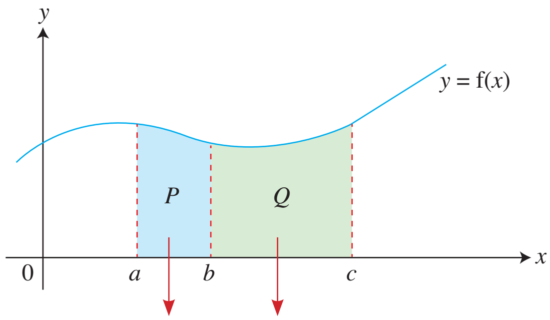
(a) $\int_0^3 e^{3-x}\,dx$ (b) $\int_0^1 (3e^{3x}+1)\,dx$

2. Given that $\int_0^{\pi} (2+3\cos\frac{1}{2}x)\,dx = \int_{-1}^{k} e^{2x+1}$, find the value of k, giving your answer correct to 3 decimal places.

Thinking Time

Using $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$, deduce the value of $\int_a^a f(x)\,dx$.

An important result of definite integration can be explained by using the concept of area.



$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx$$

Area of P + Area of Q = Total area

Worked Example 6

(Use of Definite Integration Results)

Given that $\int_1^6 f(x)\,dx = 9$, evaluate

(a) $\int_1^6 3\,f(x)\,dx$, **(b)** $\int_1^6 [f(x)+2]\,dx$.

Solution

(a) $\int_1^6 3\,f(x)\,dx = 3\int_1^6 f(x)\,dx = 3(9) = 27$

(b) $\int_1^6 [f(x)+2]\,dx$

$= \int_1^6 f(x)\,dx + \int_1^6 2\,dx$

$= 9 + [2x]_1^6$

$= 9 + [2(6) - 2(1)] = 19$

Practise Now 6

Similar Questions:
Exercise 15A
Questions 1(a), (b)

Given that $\int_1^4 f(x)\,dx = 5$, evaluate

(a) $\int_4^1 2\,f(x)\,dx$, **(b)** $\int_1^3 f(x)\,dx + \int_3^4 [f(x)+3]\,dx$.

Basic Level | Intermediate Level | Advanced Level

Exercise 15A

1 Given that $\int_1^4 f(x)\,dx = 10$, find

(a) $\int_1^4 5\,f(x)\,dx$, (b) $\int_1^4 [f(x) - 4]\,dx$.

2 Evaluate each of the following integrals.

(a) $\int_0^9 2x^{\frac{3}{2}}\,dx$ (b) $\int_1^8 5\sqrt[3]{x^4}\,dx$

(c) $\int_2^3 \frac{1}{x^2}\,dx$ (d) $\int_1^{16} \frac{2}{\sqrt[4]{x}}\,dx$

(e) $\int_{-1}^1 (x^2 + 3x)\,dx$ (f) $\int_1^2 \left(x + \frac{1}{x^2}\right)dx$

(g) $\int_1^4 \sqrt{x}(2 - x)\,dx$ (h) $\int_0^3 (3x + 2)(x - 5)\,dx$

(i) $\int_0^1 x^2(2\sqrt{x} - 3)\,dx$ (j) $\int_1^2 \left(3 - 2x^2 + \frac{4}{x^3}\right)dx$

(k) $\int_1^2 \frac{(x + 2)(x - 2)}{x^2}\,dx$

3 Evaluate each of the following integrals.

(a) $\int_0^2 (2x - 3)^3\,dx$ (b) $\int_4^7 \sqrt{x - 3}\,dx$

(c) $\int_1^{13} \sqrt{x + 3}\,dx$ (d) $\int_{-4}^0 \frac{4}{\sqrt{1 - 2x}}\,dx$

(e) $\int_1^3 \frac{4}{(3x + 2)^2}\,dx$ (f) $\int_1^5 (2x - 1)^{-2}\,dx$

4 Evaluate each of the following integrals.

(a) $\int_0^{\frac{\pi}{2}} (2x + \cos x)\,dx$ (b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\sin 2x - \cos x)\,dx$

(c) $\int_0^{\frac{\pi}{2}} 2\cos\frac{x}{2}\,dx$ (d) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2 x\,dx$

(e) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{\cos^2 x}\,dx$ (f) $\int_0^{\frac{\pi}{12}} \sec^2 3x\,dx$

5 ★ Evaluate each of the following integrals.

(a) $\int_0^1 e^{\frac{x}{3}}\,dx$ (b) $\int_0^{\frac{1}{3}} e^{6x}\,dx$

(c) $\int_0^1 e^{-x}\,dx$ (d) $\int_0^1 e^{3x}\,dx$

(e) $\int_0^1 20e^{-4t}\,dt$ (f) $\int_0^2 4e^{-2t}\,dt$

6 ★ Evaluate each of the following, giving your answer correct to 2 decimal places.

(a) $\int_0^1 \left(\frac{1}{e^x} + e^{3x}\right)^2 dx$ (b) $\int_0^{\frac{\pi}{2}} (e^{-\frac{x}{2}} + \sin 2x)\,dx$

(c) $\int_0^1 \frac{4}{e^{1-2x}}\,dx$ (d) $\int_0^1 (e^{3x} - e^x)\,dx$

(e) $\int_0^2 \frac{e^{3x} - 4}{2e^x}\,dx$ (f) $\int_0^3 \left(e^{4-x}\right)dx$

7 ★ Evaluate each of the following, giving your answer correct to 2 decimal places.

(a) $\int_1^5 \frac{5}{x}\,dx$ (b) $\int_0^1 \frac{1}{3x + 1}\,dx$

(c) $\int_3^6 \frac{3 + 2x}{4x^2}\,dx$ (d) $\int_3^5 \frac{3}{x - 2}\,dx$

(e) $\int_1^5 \frac{(2 + x)^2}{3x}\,dx$ (f) $\int_1^4 \left(4 - \frac{3}{x}\right)^2 dx$

8 ★ Hooke's Law states that the force, F, required to compress the spring, is directly proportional to the length, x, by which the spring is compressed. Given $F = kx$ and that the work done, W, required to compress the spring is $\int_{x_1}^{x_2} F\,dx$, where x_1 is the compressed length of the spring, find an expression for W in terms of k, x_1 and x_2.

9 ★ Find $\frac{d}{dx}(\ln \cos x)$. Hence, find the value of $\int_0^{\frac{\pi}{4}} \tan x\,dx$, giving your answer in its exact form.

15.2 FURTHER EXAMPLES ON DEFINITE INTEGRALS

Worked Example 7

(Definite Integration involving Partial Fractions)

Express $\frac{x+17}{(x-3)(x+2)}$ in the form $\frac{A}{x-3}+\frac{B}{x+2}$. Hence, evaluate

$\int_4^6 \frac{x+17}{(x-3)(x+2)}\,\mathrm{d}x$, giving your answer correct to 2 decimal places.

Solution

$$\frac{x+17}{(x-3)(x+2)} = \frac{A}{x-3}+\frac{B}{x+2}$$

$$= \frac{A(x+2)+B(x-3)}{(x-3)(x+2)}$$

Multiplying throughout by $(x-3)(x+2)$,

$$x + 17 = A(x + 2) + B(x - 3)$$

Let $x = -2$: $\quad -2 + 17 = B(-2 - 3)$

$$B = -3$$

Let $x = 3$: $\quad 3 + 17 = A(3 + 2)$

$$A = 4$$

$$\therefore \frac{x+17}{(x-3)(x+2)} = \frac{4}{x-3}-\frac{3}{x+2}$$

$$\int_4^6 \frac{x+17}{(x-3)(x+2)}\,\mathrm{d}x = \int_4^6 \left(\frac{4}{x-3}-\frac{3}{x+2}\right)\mathrm{d}x$$

$$= [4\ln(x-3) - 3\ln(x+2)]_4^6$$

$$= [4\ln(6-3) - 3\ln(6+2)] - [4\ln(4-3) - 3\ln(4+2)]$$

$$= 4\ln 3 - 3\ln 8 - 4\ln 1 + 3\ln 6$$

$$= 3.53 \text{ (to 2 d.p.)}$$

Practise Now 7

Similar Question:
Exercise 15B
Question 4

Express $\frac{13x+12}{(x+1)(2x+3)}$ in the form $\frac{A}{x+1}+\frac{B}{2x+3}$. Hence, evaluate

$\int_0^3 \frac{13x+12}{(x+1)(2x+3)}\,\mathrm{d}x$, giving your answer correct to 2 decimal places.

We shall use the rule that if $\frac{\mathrm{d}}{\mathrm{d}x}[\mathrm{f}(x)] = \mathrm{g}(x)$, then $\int_a^b \mathrm{g}(x)\,\mathrm{d}x = [\mathrm{f}(x)]_a^b$ to integrate expressions that do not fall into the standard categories.

Worked Example 8

(Definite Integration of an Algebraic Expression)

Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\sqrt{1+x^3}}\right) = \frac{2-x^3}{2\sqrt{(1+x^3)^3}}$. Hence, evaluate $\int_0^2 \frac{4-2x^3}{\sqrt{(1+x^3)^3}}\,\mathrm{d}x$.

Solution

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\sqrt{1+x^3}}\right) = \frac{(1+x^3)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}x}(x) - x\frac{\mathrm{d}}{\mathrm{d}x}(1+x^3)^{\frac{1}{2}}}{1+x^3} \qquad \text{(Quotient Rule)}$$

$$= \frac{(1+x^3)^{\frac{1}{2}}(1) - x\left[\frac{1}{2}(1+x^3)^{-\frac{1}{2}}(3x^2)\right]}{1+x^3}$$

$$= \left(\sqrt{1+x^3} - \frac{3x^3}{2\sqrt{1+x^3}}\right)\frac{1}{(1+x^3)}$$

$$= \frac{2(1+x^3) - 3x^3}{2\sqrt{1+x^3}(1+x^3)} = \frac{2-x^3}{2\sqrt{(1+x^3)^3}}$$

Since $\int_0^2 \frac{2-x^3}{2\sqrt{(1+x^3)^3}}\,\mathrm{d}x = \left[\frac{x}{\sqrt{1+x^3}}\right]_0^2$,

$$\therefore \quad 4\int_0^2 \frac{2-x^3}{2\sqrt{(1+x^3)^3}}\,\mathrm{d}x = \left[\frac{4x}{\sqrt{1+x^3}}\right]_0^2.$$

i.e.
$$\int_0^2 \frac{4-2x^3}{\sqrt{(1+x^3)^3}}\,\mathrm{d}x = \left[\frac{4(2)}{\sqrt{1+2^3}}\right] - \left[\frac{0}{\sqrt{1}}\right]$$

$$= \frac{8}{3}$$

$$= 2\frac{2}{3}$$

Practise Now 8

Similar Questions:
Exercise 15B
Questions 1, 2, 5

1. Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left[(2x+5)\sqrt{4x-5}\right] = \frac{12x}{\sqrt{4x-5}}$.

Hence, evaluate $\int_2^4 \frac{6x}{\sqrt{4x-5}}\,\mathrm{d}x$, giving your answer correct to 2 decimal places.

2. Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x}{1+3x}\right) = \frac{2}{(1+3x)^2}$. Hence, evaluate $\int_1^4 \left(\frac{4}{1+3x}\right)^2 \mathrm{d}x$.

Worked Example 9

(Definite Integration of a Trigonometric Function)

If $y=\dfrac{\sin x}{1+\cos x}$, show that $\dfrac{dy}{dx}=\dfrac{1}{1+\cos x}$. Hence, evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{6}{1+\cos x}\,dx$.

Solution

$$y=\frac{\sin x}{1+\cos x}$$

$$\frac{dy}{dx}=\frac{(1+\cos x)\frac{d}{dx}(\sin x)-\sin x\frac{d}{dx}(1+\cos x)}{(1+\cos x)^2}$$

$$=\frac{(1+\cos x)\cos x-\sin x(-\sin x)}{(1+\cos x)^2}$$

$$=\frac{\cos x+\cos^2 x+\sin^2 x}{(1+\cos x)^2}$$

$$=\frac{1}{1+\cos x}$$

RECALL

$\sin^2\theta+\cos^2\theta=1$

$$\therefore \int_0^{\frac{\pi}{4}}\frac{1}{1+\cos x}\,dx=\left[\frac{\sin x}{1+\cos x}\right]_0^{\frac{\pi}{4}}$$

$$\int_0^{\frac{\pi}{4}}\frac{6}{1+\cos x}\,dx=\left[\frac{6\sin x}{1+\cos x}\right]_0^{\frac{\pi}{4}}$$

$$=\left(\frac{6\sin\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}\right)-\left(\frac{6\sin 0}{1+\cos 0}\right)$$

$$=\frac{\frac{6\sqrt{2}}{2}}{1+\frac{\sqrt{2}}{2}}-0$$

$$=\frac{6\sqrt{2}}{2+\sqrt{2}}\times\frac{2-\sqrt{2}}{2-\sqrt{2}} \quad \text{(rationalise the denominator)}$$

$$=6\sqrt{2}-6$$

Practise Now 9

Similar Questions:
Exercise 15B
Questions 3, 6-9

1. Differentiate $3x\cos x$ with respect to x. Hence, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} 2x\sin x\,dx$.

2. Use the derivatives of $\sin x$ and $\cos x$ to show that $\dfrac{d}{dx}(\tan x)=\sec^2 x$ and that $\dfrac{d}{dx}(\sec x)=\sec^2 x\sin x$. Hence, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}}\frac{1+\sin x}{\cos^2 x}\,dx$.

Worked Example 10

(Definite Integration of an Exponential Function)
Differentiate xe^{2x} with respect to x. Hence, evaluate $\int_0^1 2x\,e^{2x}\,dx$.

Solution

Let $y = xe^{2x}$.

$$\frac{dy}{dx} = x\frac{d}{dx}(e^{2x}) + e^{2x}\frac{d}{dx}(x)$$
$$= 2x\,e^{2x} + e^{2x}$$
$$\int_0^1 (2x\,e^{2x} + e^{2x})\,dx = \left[xe^{2x}\right]_0^1$$
$$\int_0^1 2x\,e^{2x}\,dx + \int_0^1 e^{2x}\,dx = \left[xe^{2x}\right]_0^1 \quad \left(\int [f(x)+g(x)]\,dx = \int f(x)\,dx + \int g(x)\,dx\right)$$
$$\int_0^1 2x\,e^{2x}\,dx + \left[\frac{1}{2}e^{2x}\right]_0^1 = \left[xe^{2x}\right]_0^1$$
$$\therefore \int_0^1 2x\,e^{2x}\,dx = \left[xe^{2x}\right]_0^1 - \frac{1}{2}\left[e^{2x}\right]_0^1$$
$$= \left[(1)(e^{2(1)}) - (0)(e^{2(0)})\right] - \frac{1}{2}\left[e^{2(1)} - e^{2(0)}\right]$$
$$= e^2 - 0 - \frac{1}{2}e^2 + \frac{1}{2}$$
$$= \frac{1}{2}(e^2 + 1)$$

Practise Now 10

Similar Questions:
Exercise 15B
Questions 10, 11, 13

1. Differentiate $(2x - 1)e^{2x}$ with respect to x. Hence, evaluate $\int_0^2 4x\,e^{2x}\,dx$, giving your answer correct to 2 decimal places.
2. Differentiate e^{2x^2} with respect to x. Hence, evaluate $\int_0^{1.5} xe^{2x^2}\,dx$.

Worked Example 11

(Definite Integration of $\ln f(x)$)
Differentiate $3x^2 \ln x$ with respect to x. Hence, find the value of $\int_1^3 5x \ln x\,dx$, giving your answer correct to 2 decimal places.

Solution

$$\frac{d}{dx}(3x^2 \ln x) = 3x^2\frac{d}{dx}(\ln x) + \ln x\frac{d}{dx}(3x^2)$$
$$= 3x^2\left(\frac{1}{x}\right) + \ln x\,(6x) = 3x + 6x \ln x$$

Since $\int_1^3 (3x + 6x \ln x)\,dx = \left[3x^2 \ln x\right]_1^3$,

multiplying both sides by $\frac{5}{6}$ to obtain $\int_1^3 5x \ln x\,dx$ on the LHS,

$$\frac{5}{6}\int_1^3 (3x + 6x \ln x)\,dx = \frac{5}{6}\left[3x^2 \ln x\right]_1^3,$$
$$\therefore \int_1^3 5x \ln x\,dx = \left[\frac{5}{2}x^2 \ln x\right]_1^3 - \int_1^3 \frac{5}{2}x\,dx = \left[\frac{5}{2}x^2 \ln x\right]_1^3 - \left[\frac{5}{2}\left(\frac{x^2}{2}\right)\right]_1^3$$
$$= \frac{5}{2}\left[(3)^2 \ln 3 - (1)^2 \ln 1\right] - \frac{5}{2}\left[\frac{3^2}{2} - \frac{1^2}{2}\right]$$
$$= 14.72 \text{ (to 2 d.p.)}$$

Practise Now 11

Similar Question:
Exercise 15B
Question 12

1. Differentiate $x \ln x$ with respect to x. Hence, or otherwise, evaluate $\int_1^4 \ln x \, dx$.

2. Differentiate $5x^3 \ln x$ with respect to x. Hence, evaluate $\int_1^3 7x^2 \ln x \, dx$, giving your answer correct to 2 decimal places.

Basic Level | Intermediate Level | Advanced Level

Exercise 15B

1 If $y = (1+3x)^{\frac{3}{2}}$, show that $\frac{dy}{dx} = \frac{9}{2}(1+3x)^{\frac{1}{2}}$.

Hence, evaluate $\int_0^1 (1+3x)^{\frac{1}{2}} \, dx$.

2 Given that $y = \sqrt{2x^3+9}$, show that $\frac{dy}{dx} = \frac{3x^2}{\sqrt{2x^3+9}}$. Hence, evaluate

$\int_0^2 \frac{x^2}{\sqrt{2x^3+9}} \, dx$.

3 Prove that $(\sin 2x + \cos 2x)^2 = 1 + \sin 4x$.

Hence, evaluate $\int_0^{\frac{\pi}{4}} (\sin 2x + \cos 2x)^2 \, dx$.

4 **(i)** Express $\frac{3x-5}{(x-1)(x-3)}$ in the form $\frac{A}{x-1} + \frac{B}{x-3}$.

(ii) Hence, evaluate $\int_4^6 \frac{3x-5}{(x-1)(x-3)} \, dx$.

(iii) Explain why it is not possible to find the value of $\int_1^6 \frac{3x-5}{(x-1)(x-3)} \, dx$.

5 Show that $\frac{d}{dx}\left(x\sqrt{x^2+4}\right) = \frac{2x^2+4}{\sqrt{x^2+4}}$.

Hence, evaluate $\int_0^4 \frac{x^2+2}{2\sqrt{x^2+4}} \, dx$.

6 **(i)** Use the derivatives of $\sin x$ and $\cos x$ to show that $\frac{d}{dx}(\cot x) = -\operatorname{cosec}^2 x$.

(ii) Show that $\frac{1+\cos 2x}{1-\cos 2x} = \operatorname{cosec}^2 x - 1$.

(iii) Use the results obtained in **(i)** and **(ii)** to evaluate $\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{1+\cos 2x}{1-\cos 2x} \, dx$.

7 The annual rate of revenue of a company can be modelled by the function $x \sin x$. Then the total revenue for n years is given by $\int_0^n x \sin x \, dx$.

(i) Find the derivative of $x \cos x$ with respect to x.

(ii) Hence, evaluate $\int_0^{\frac{n}{2}} x \sin x \, dx$.

(iii) Explain what is meant by your answer in **(ii)**.

8 Given that $\cos 3x = 4\cos^3 x - 3\cos x$, find $\int_0^{\frac{\pi}{3}} 4(3+\cos^3 x) \, dx$.

9 **(i)** Prove that $\tan^4 \theta = \tan^2 \theta \sec^2 \theta - \sec^2 \theta + 1$.

(ii) Show that $\frac{d}{d\theta}\left(\frac{1}{3}\tan^3 \theta\right) = \tan^2 \theta \sec^2 \theta$.

(iii) Hence, evaluate $\int_0^{\frac{\pi}{3}} \tan^4 \theta \, d\theta$.

Exercise 15B

10 Differentiate $e^{2\sqrt{x}}$ with respect to x.
Hence, evaluate $\int_0^4 \frac{e^{2\sqrt{x}}}{\sqrt{x}}\,dx$.

11 Given that $y = 2xe^{3x}$, find $\frac{dy}{dx}$. Hence, evaluate $\int_0^1 xe^{3x}\,dx$.

12 Differentiate $x^2 \ln x - 3x$ with respect to x.
Hence, evaluate $\int_1^4 x \ln x\,dx$.

13 If $\int e^{-2x}\,f(x)\,dx = e^{-2x}\sin 4x + c$, find f($x$).

15.3 AREA UNDER A CURVE

In the previous section, we have observed that $\int_a^b 3\,dx = 3(b - a)$ is the area bounded by the horizontal line $y = 3$, the x-axis and the lines $x = a$ and $x = b$. What happens if it is a curve instead of a horizontal line, i.e. how do you find the area under a curve?

Investigation

Area under a Curve

1. Evaluate the following definite integrals $\int_a^b f(x)\,dx$. Then use a graphing software to plot each of the corresponding functions $y = f(x)$ and find the area bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$. Copy and complete the table.

No.	$y = f(x)$	Evaluate $\int_a^b f(x)\,dx$	Area bounded by graph of $y = f(x)$, x-axis, $x = a$ and $x = b$ (from graphing software)	Is the graph of $y = f(x)$ between $x = a$ and $x = b$ completely above or below the x-axis?
(i)	$y = x^2 - 2x - 3$	$\int_3^4 (x^2 - 2x - 3)\,dx =$		above x-axis
(ii)	$y = x^2 - 2x - 3$	$\int_2^3 (x^2 - 2x - 3)\,dx =$		
(iii)	$y = 2x + 3$	$\int_{-1}^2 (2x + 3)\,dx =$		
(iv)	$y = 2x + 3$	$\int_{-4}^{-2} (2x + 3)\,dx =$		
(v)	$y = \sin x$	$\int_0^{\pi} \sin x\,dx =$		
(vi)	$y = \sin x$	$\int_{\pi}^{2\pi} \sin x\,dx =$		

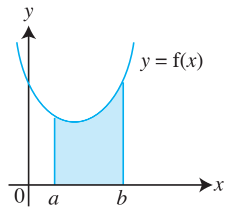
2. What can you conclude about the relationship between the **definite integral** $\int_a^b f(x)\,dx$ and the **area** bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$?
   (a) If $y = f(x) \geqslant 0$ for $a \leqslant x \leqslant b$ (i.e. the graph of $y = f(x)$ between $x = a$ and $x = b$ lies completely *above* the x-axis), how can we find the area bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$?
   (b) If $y = f(x) \leqslant 0$ for $a \leqslant x \leqslant b$ (i.e. the graph of $y = f(x)$ between $x = a$ and $x = b$ lies completely *below* the x-axis), how can we find the area bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$?
3. Your classmate says that the area bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$. Is he correct? Explain your answer.
4. How do you find the area bounded by the graph of $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$, if the graph of $y = f(x)$ intersects the x-axis between $x = a$ and $x = b$?
5. What is the difference between the indefinite integral $\int f(x)\,dx$ and the definite integral $\int_a^b f(x)\,dx$?

From the investigation, we can conclude that the area of the region bounded by the curve $y = f(x)$, the lines $x = a$ and $x = b$, and the x-axis is given by the definite integral

$$\int_a^b f(x)\,dx, \text{ where } f(x) \geqslant 0 \text{ and } a \leqslant x \leqslant b.$$

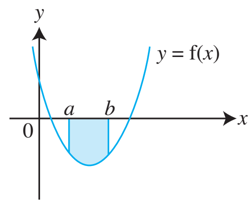
In general, the area enclosed by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ ($a < b$) is

$\int_a^b f(x)\,dx$ when $f(x) \geqslant 0$ (i.e. above the x-axis)



and

$\left|\int_a^b f(x)\,dx\right|$ when $f(x) < 0$ (i.e. below the x-axis)

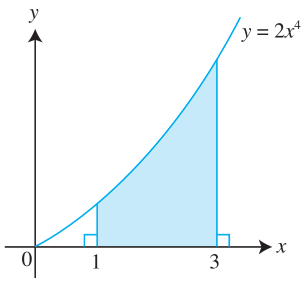


Worked Example 12

(Area under a Curve)

The figure shows part of the curve $y = 2x^4$.

Find the area of the shaded region.



Solution

The area of the shaded region is

$$\int_1^3 2x^4 \, dx = \left[\frac{2}{5}x^5\right]_1^3$$

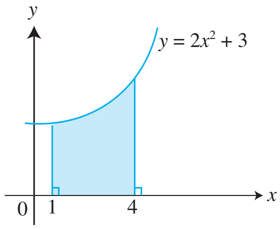
$$= \frac{2}{5}(3^5 - 1^5)$$

$$= \frac{484}{5} \text{ or } 96\frac{4}{5} \text{ units}^2 .$$

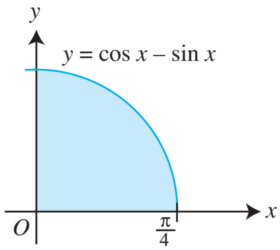
Practise Now 12

Similar Questions:
Exercise 15C
Questions 1(a)-(c)

1. Find the area of the shaded region.



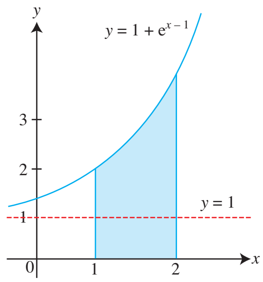
2. The figure shows part of the curve $y = \cos x - \sin x$. Calculate the area of the shaded region, leaving your answer in surd form.



Worked Example 13

(Area under a Curve)

The figure shows part of the curve $y = 1 + e^{x-1}$.
Find the area of the shaded region.



Solution

Area enclosed $= \displaystyle\int_1^2 y \, dx$

$$= \int_1^2 (1 + e^{x-1}) \, dx$$

$$= \left[x + e^{x-1}\right]_1^2$$

$$= (2 + e^1) - (1 + e^0)$$

$$= e$$

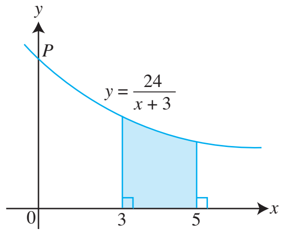
$$= 2.72 \text{ units}^2 \text{ (to 3 s.f.)}$$

Practise Now 13

Similar Questions:
Exercise 15C
Questions 4-11

1. Sketch the curve $y = 2 + e^{2x}$ and find the area enclosed by the curve, the coordinate axes and the line $x = 2$.

2. The figure shows part of the curve $y = \frac{24}{x+3}$. Find the area enclosed by the curve, the x-axis and the lines $x = 3$ and $x = 5$, giving your answer correct to 3 decimal places.

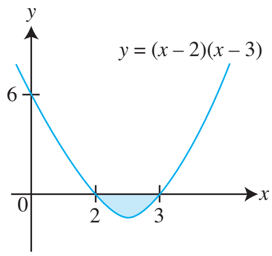


Area below the x-axis

What happens if we need to find the area of the region below the x-axis? If we simply use $\int_a^b f(x)\,dx$, the answer obtained will be negative. Therefore, for a region that lies below the x-axis, the area is given as $\left|\int_a^b f(x)\,dx\right|$. Worked Example 14 illustrates this.

Worked Example 14

(Area below the x-axis)
The figure shows part of the curve $y = (x - 2)(x - 3)$. Find the area of the shaded region.



Solution

$$\int_2^3 (x^2 - 5x + 6)\,dx$$

$$= \left[\frac{x^3}{3} - \frac{5x^2}{2} + 6x\right]_2^3$$

$$= \left[\frac{3^3}{3} - \frac{5}{2}(3)^2 + 6(3)\right] - \left[\frac{2^3}{3} - \frac{5}{2}(2)^2 + 6(2)\right]$$

$$= -\frac{1}{6}$$

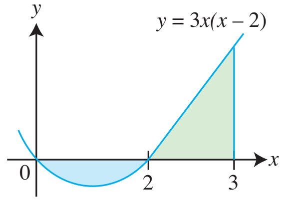
Notice that the shaded region is below the x-axis and the calculated value is negative.

Area of the shaded region $= \left|-\frac{1}{6}\right| = \frac{1}{6}$ units^2

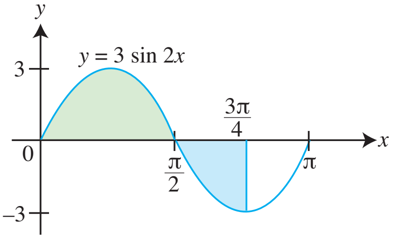
Practise Now 14

Similar Questions:
Exercise 15C
Questions 2(a)-(c)

1. The figure shows part of the curve $y = 3x(x - 2)$. Find the total area enclosed by the curve $y = 3x(x - 2)$, the x-axis and the line $x = 3$.



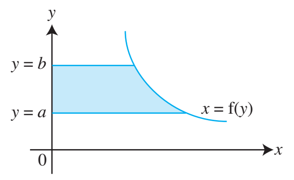
2. Find the total area of the shaded regions enclosed by the curve $y = 3 \sin 2x$, the x-axis and the line $x = \frac{3\pi}{4}$.



Area enclosed by a curve and the y-axis

The area enclosed by the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$ $(a < b)$ is given by

$$\int_a^b x\,dy \text{ when } x = f(y) \geqslant 0$$

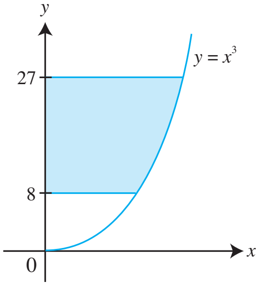


Worked Example 15

(Area enclosed by the curve and the y-axis)

The figure shows part of the sketch of the curve $y = x^3$. Calculate the area enclosed by the curve, the y-axis, the lines $y = 8$ and $y = 27$.

Solution



$y = x^3$

$\therefore x = y^{\frac{1}{3}}$

$$\text{Area enclosed} = \int_8^{27} x\,dy$$

$$= \int_8^{27} y^{\frac{1}{3}}\,dy$$

$$= \left[\frac{y^{\frac{4}{3}}}{\frac{4}{3}}\right]_8^{27}$$

$$= \left[\frac{3}{4}(\sqrt[3]{y})^4\right]_8^{27}$$

$$= \frac{3}{4}(\sqrt[3]{27})^4 - \frac{3}{4}(\sqrt[3]{8})^4$$

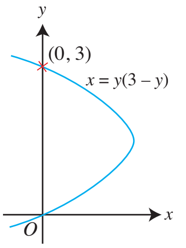
$$= 60\frac{3}{4} - 12$$

$$= 48\frac{3}{4} \text{ units}^2$$

Practise Now 15

Similar Questions:
Exercise 15C
Question 3

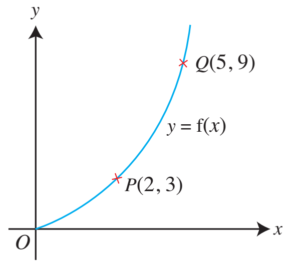
The figure shows the sketch of the curve $x = y(3 - y)$. Find the area enclosed by the curve and the y-axis.



Worked Example 16

(Applications of Definite Integrals)

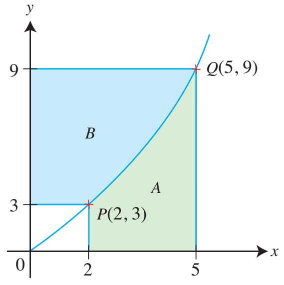
The figure shows part of the graph of $y = \mathrm{f}(x)$. Given that the points $O(0, 0)$, $P(2, 3)$ and $Q(5, 9)$ lie on the curve, find the value of $\int_2^5 y\,\mathrm{d}x + \int_3^9 x\,\mathrm{d}y$.



Solution

From the figure, we see that $\int_2^5 y\,\mathrm{d}x$ gives the area A and $\int_3^9 x\,\mathrm{d}y$ gives the area B.

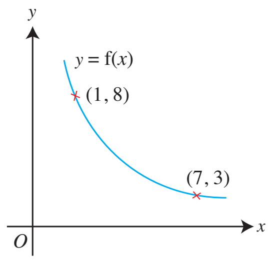
$$\therefore \int_2^5 y\,\mathrm{d}x + \int_3^9 x\,\mathrm{d}y = (5 \times 9) - (2 \times 3)$$
$$= 39$$



Practise Now 16

Similar Question:
Exercise 15C
Question 12

The figure shows part of the curve $y = \mathrm{f}(x)$. Given that $(1, 8)$ and $(7, 3)$ lie on the curve and that $\int_1^7 y\,\mathrm{d}x = 29$, find the value of $\int_3^8 x\,\mathrm{d}y$.



Thinking Time

By considering the previous worked examples, describe the different methods of finding

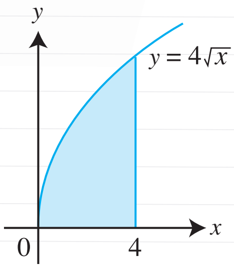
(a) the area bounded by a curve, the x-axis and the lines $x = a$ and $x = b$,

(b) the area bounded by a curve, the y-axis and the lines $y = a$ and $y = b$,

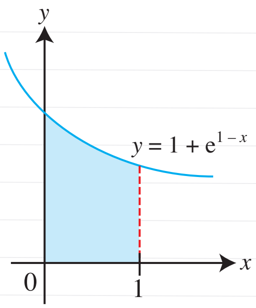
Exercise 15C

1 For each of the following figures, find the area of the shaded region.

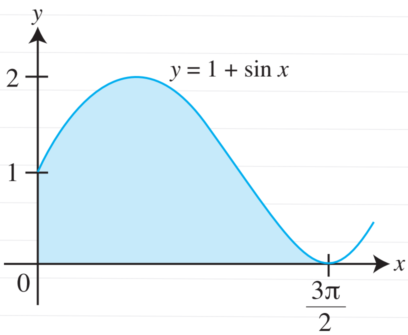
(a)



(b)

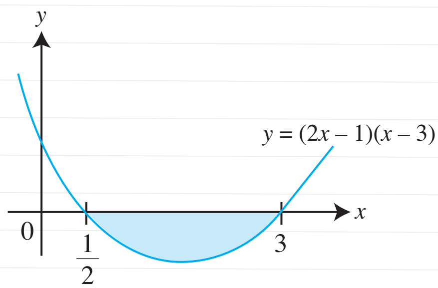


(c)

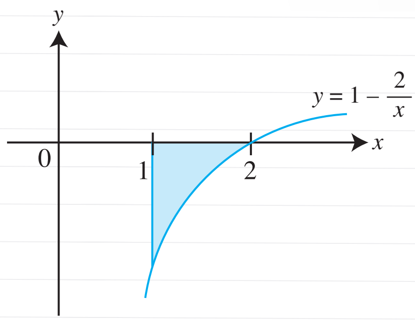


2 For each of the following figures, find the area of the shaded region, giving your answer correct to 3 significant figures where necessary.

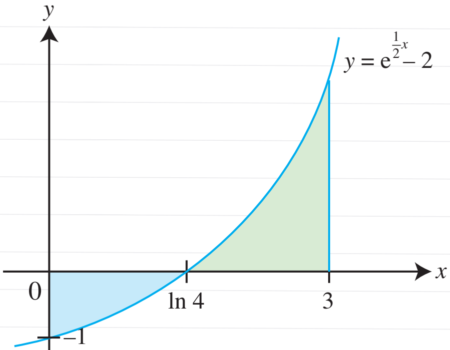
(a)



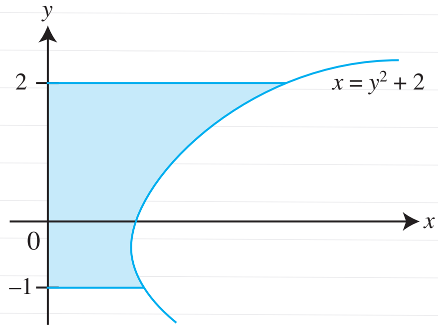
(b)



(c)



3 Find the area of the shaded region in the figure.



4 Sketch the curve $y = 4x - x^2$. Find the area of the region bounded by the curve and the x-axis.

5 Find the area of the region bounded by $y = 16 - x^2$, the line $x = -2$, the line $x = 3$ and the x-axis.

6 The surface of the cross section of a hump can be approximately modelled by the equation $y = \sin 2x$, where x and y are the horizontal and vertical distances in metres, measured from the bottom-left corner of its cross section respectively.

(i) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \sin 2x \, dx$.

(ii) Given that the horizontal length of the hump is $\frac{\pi}{4}$ m, deduce the cross-sectional area of the hump.

7 Evaluate $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}} 4\cos 3x \, dx$. On your sketch of $y = 4\cos 3x$, shade the region whose area is given by this integral.

8 Find the area of the region bounded by $y = x(3 - x)$ and the x-axis.

(a) Show that the line $x = \frac{3}{2}$ divides the region into two parts with equal areas.

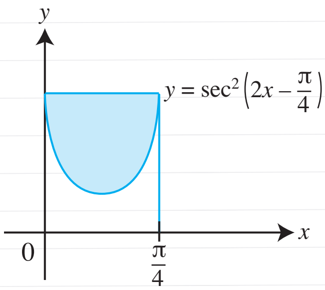
(b) Show that the line $x = 1$ divides the region into two parts whose areas are in the ratio of 7 : 20.

9 Sketch the curve $y = \frac{8}{x}$ for $x > 0$ and find the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 3$.

10 **(i)** Differentiate $\tan\left(2x - \frac{\pi}{4}\right)$ with respect to x.

The figure shows part of the curve $y = \sec^2\left(2x - \frac{\pi}{4}\right)$.

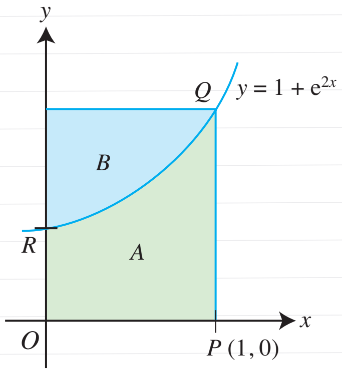
(ii) Calculate the shaded area.



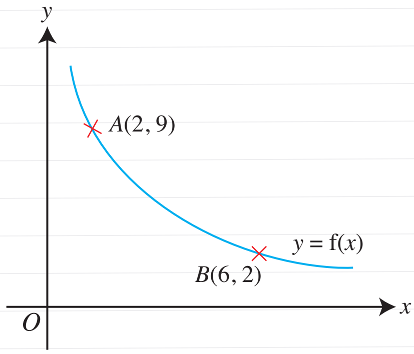
11 The figure shows part of the curve $y = 1 + e^{2x}$ and P is $(1, 0)$. Calculate

(i) the coordinates of Q and of R,

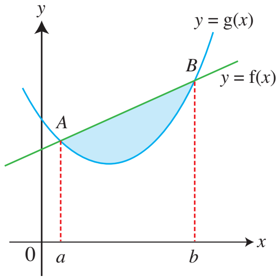
(ii) the area of the shaded region A and hence calculate the area of the shaded region B.



12 The figure shows part of the curve $y = f(x)$. Given that the points $A(2, 9)$ and $B(6, 2)$ lie on the curve and that $\displaystyle\int_2^6 y \, dx = 11$, find the numerical value of $\displaystyle\int_2^9 x \, dy$.



Area bounded by a line and a curve



The figure shows the region bounded by the line $y = \mathrm{f}(x)$ and the curve $y = \mathrm{g}(x)$. Suppose that the line and the curve intersect at the points A and B, where the x-coordinates of A and B are a and b respectively.

Then the shaded area enclosed between $y = \mathrm{f}(x)$ and $y = \mathrm{g}(x)$
= (area under $y = \mathrm{f}(x)$ between $x = a$, $x = b$ and the x-axis)
– (area under $y = \mathrm{g}(x)$ between $x = a$, $x = b$ and the x-axis)

$$= \int_a^b \mathrm{f}(x)\,\mathrm{d}x - \int_a^b \mathrm{g}(x)\,\mathrm{d}x$$

$$= \int_a^b [\mathrm{f}(x) - \mathrm{g}(x)]\,\mathrm{d}x$$

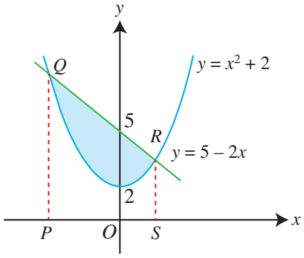
Worked Example 17

(Area bounded by a Line and a Curve)

Find the area enclosed by the curve $y = x^2 + 2$ and the line $y = 5 - 2x$.

Solution

In the sketch, the line $y = 5 - 2x$ intersects the curve $y = x^2 + 2$ at Q and R.



So $\qquad 5 - 2x = x^2 + 2$

$$x^2 + 2x - 3 = 0$$
$$(x + 3)(x - 1) = 0$$
$$x = -3 \text{ or } x = 1$$

The line and curve intersect at $x = -3$ and $x = 1$.

Area enclosed is $\displaystyle\int_{-3}^{1} [(5 - 2x) - (x^2 + 2)]\,\mathrm{d}x$

$$= \int_{-3}^{1} (3 - 2x - x^2)\,\mathrm{d}x$$

$$= \left[3x - x^2 - \frac{x^3}{3}\right]_{-3}^{1}$$

$$= \left[3(1) - (1)^2 - \frac{1^3}{3}\right] - \left[3(-3) - (-3)^2 - \frac{(-3)^3}{3}\right]$$

$$= 10\frac{2}{3} \text{ units}^2$$

Class Discussion

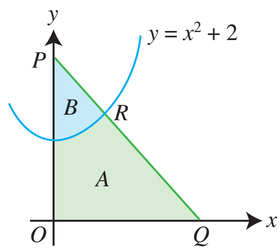
In Worked Example 17, we used the formula $\int_a^b [f(x) - g(x)]\,dx$ to find the area enclosed by the curve and the line. Discuss with your classmate to find an alternative method to solve the question. By using this method, check if your answer tallies with the one given above.

Practise Now 17

Similar Questions:
Exercise 15D
Questions 1-8

1. Find the area enclosed by the curve $y = 4x - x^2$ and the line $y = 2x$.

2. The figure shows part of the curve $y = x^2 + 2$ and the line $y + 2x = 17$. Find



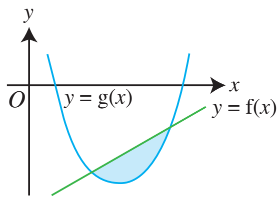
(i) the coordinates of P, Q and R,

(ii) the area of the shaded region A,

(iii) the ratio of $\dfrac{\text{area of shaded region } A}{\text{area of shaded region } B}$.

Thinking Time

The figure shows the area enclosed by the curve $y = g(x)$ and the line $y = f(x)$. Notice that the area enclosed is under the x-axis. Will the area enclosed be negative?



Explain clearly how we can use calculus to find the area enclosed.

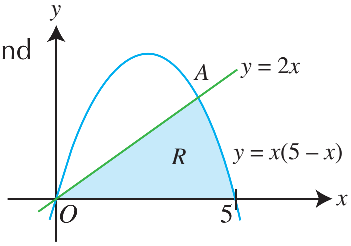
Worked Example 18

(Area bounded by a Line and a Curve)

The figure shows part of the curve $y = x(5 - x)$ and the line $y = 2x$ intersecting at the point A. Find

(i) the coordinates of A,

(ii) the area of the shaded region R.



Solution

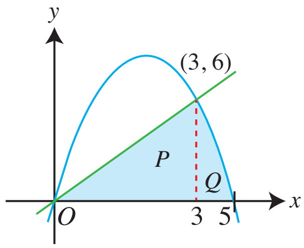
(i) At A,

$$x(5 - x) = 2x$$
$$5x - x^2 = 2x$$
$$x^2 - 3x = 0$$
$$x(x - 3) = 0$$
$$x = 0 \text{ (rejected) or } x = 3$$

When $x = 3$,

$$y = 2(3) = 6$$

$\therefore A(3, 6)$



(ii) Area of shaded region = area of P + area of Q

$$= \frac{1}{2}(3)(6) + \int_3^5 (5x - x^2)\,dx$$

$$= 9 + \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_3^5$$

$$= 9 + \left[\left(\frac{5(5)^2}{2} - \frac{5^3}{3}\right) - \left(\frac{5(3)^2}{2} - \frac{3^3}{3}\right)\right]$$

$$= 16\frac{1}{3} \text{ units}^2$$

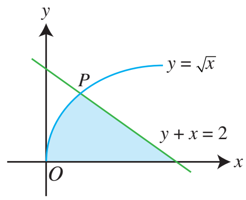
Practise Now 18

Similar Question:
Exercise 15D
Question 10

The figure shows part of the curve $y = \sqrt{x}$ and the line $y + x = 2$ intersecting at the point P. Find

(i) the coordinates of P,

(ii) the area of the shaded region.



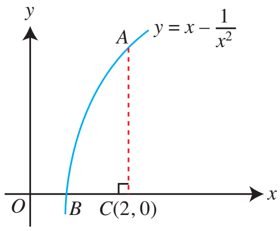
Worked Example 19

(Applications of Definite Integrals)

The figure shows part of the curve $y = x - \dfrac{1}{x^2}$.

Given that C is the point $(2, 0)$ and the curve cuts the x-axis at B, find the coordinates of A and of B.

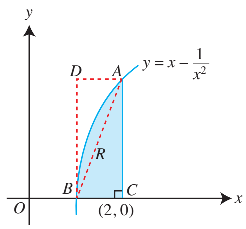
Explain briefly why $\dfrac{7}{8} < \displaystyle\int_1^2 \left(x - \frac{1}{x^2}\right) dx < \frac{7}{4}$.



Solution

When $x = 2$, $y = 2 - \frac{1}{4} = \frac{7}{4}$

$\therefore A\left(2, \frac{7}{4}\right)$



When $y = 0$, $x - \frac{1}{x^2} = 0$

$$x^3 - 1 = 0$$
$$x = 1$$

$\therefore B(1, 0)$

The value of $\int_1^2 \left(x - \frac{1}{x^2}\right) \mathrm{d}x$ gives the area of the region R enclosed by the curve, the line $x = 2$ and the x-axis.

$\therefore$ Area of $\Delta ABC < \int_1^2 \left(x - \frac{1}{x^2}\right) \mathrm{d}x <$ Area of rectangle $BCAD$

$$\frac{1}{2} \times 1 \times \frac{7}{4} < \int_1^2 \left(x - \frac{1}{x^2}\right) \mathrm{d}x < 1 \times \frac{7}{4}$$

$$\frac{7}{8} < \int_1^2 \left(x - \frac{1}{x^2}\right) \mathrm{d}x < \frac{7}{4}$$

Practise Now 19

Similar Question:
Exercise 15D
Question 10

Sketch the graph of $y = \frac{15}{x}$ for $3 \leqslant x \leqslant 5$.

Hence, explain why $6 < \int_3^5 \frac{15}{x} \mathrm{d}x < 10$.

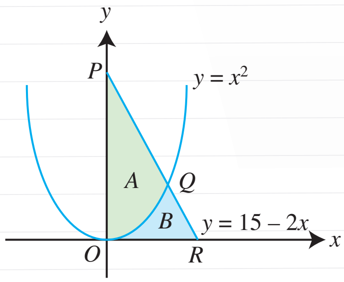
Basic Level | Intermediate Level | Advanced Level

Exercise 15D

1 Sketch the curve $y = x^2 + 1$ and the line $y = 5$. Calculate the area of the region enclosed by the curve and the line.

2 Find the coordinates of the points of intersection of the line $y = x - 1$ and the curve $y = 1 - x^2$. Hence, calculate the area of the region bounded by the curve $y = 1 - x^2$ and the line $y = x - 1$.

3



The figure shows part of the curve $y = x^2$ and part of the line $y = 15 - 2x$. Find

(i) the coordinates of P, Q and R,

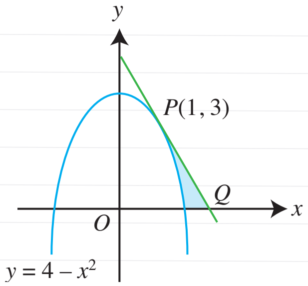
(ii) the ratio of the area of the shaded region A to the area of the shaded region B.

Basic Level | Intermediate Level | Advanced Level

Exercise 15D

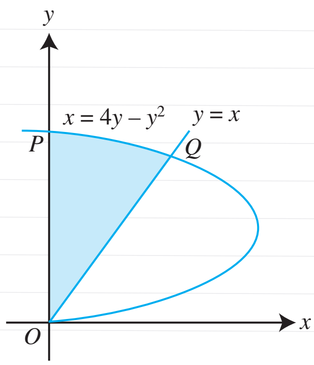
4 Sketch the curves $y = \sin x$ and $y = \cos x$ for $0 \leqslant x \leqslant \frac{\pi}{2}$. Calculate the area of the region enclosed by
(a) the curves and the x-axis,
(b) the curves and the y-axis.

5



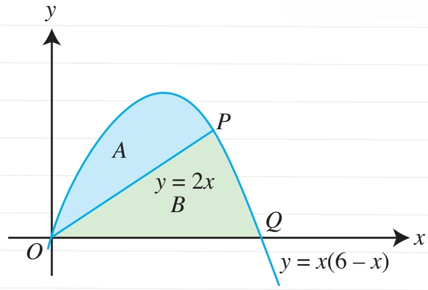
(i) The tangent at $P(1, 3)$ to the curve $y = 4 - x^2$ cuts the x-axis at the point Q. Prove that the x-coordinate of Q is $\frac{5}{2}$.
(ii) Calculate the area of the shaded region.

6



The figure shows part of the curve $x = 4y - y^2$ and part of the line $y = x$. Find
(i) the coordinates of P and of Q,
(ii) the area of the shaded region.

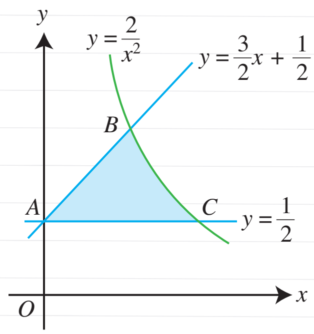
7



The figure shows part of the curve $y = x(6 - x)$ and part of the line $y = 2x$. Find
(i) the coordinates of P and of Q,
(ii) the ratio of the area of A to the area of B.

8 Find the area enclosed by the curve $y = 1 + e^x$, the x- and y-axes and the line $x = 3$.

9



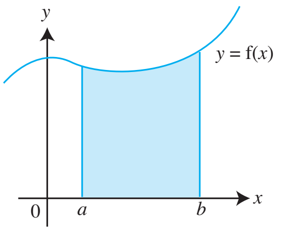
The figure shows part of the curve $y = \frac{2}{x^2}$. The straight line $y = \frac{3}{2}x + \frac{1}{2}$ cuts the y-axis at $A\left(0, \frac{1}{2}\right)$ and the curve at $B(1, 2)$. The line $y = \frac{1}{2}$ cuts the curve at $C\left(2, \frac{1}{2}\right)$. Calculate the area of the shaded region ABC.

10 Sketch the graph of $y = 6 - \frac{18}{x}$ in the interval $2 \leqslant x \leqslant 9$. Hence explain briefly why $12 < \int_3^9 \left(6 - \frac{18}{x}\right) dx < 24$.

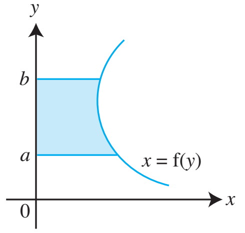
SUMMARY

1. If $\int f(x)\,dx = F(x) + c$, then $\int_a^b f(x)\,dx = [F(x)]_a^b = F(b) - F(a)$.

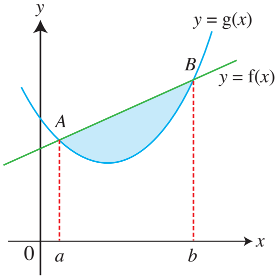
2. The area bounded by the curve $y = f(x)$, the x-axis and the lines $x = a$ and $x = b$ is given by $\int_a^b f(x)\,dx$ if $f(x) \geqslant 0$.



3. The area bounded by the curve $x = f(y)$, the y-axis and the lines $y = a$ and $y = b$ is given by $\int_a^b f(y)\,dy$ if $f(y) \geqslant 0$.



4. The area bounded by the line $y = f(x)$ and the curve $y = g(x)$ from $x = a$ to $x = b$ is given by $\int_a^b [f(x) - g(x)]\,dx$.



Review Exercise 15

1. Evaluate each of the following, giving your answer correct to 3 significant figures where necessary.

(a) $\displaystyle\int_0^1 \left(\frac{3}{2x+1}\right)^2 \mathrm{d}x$ (b) $\displaystyle\int_0^{\frac{\pi}{2}} \sin\left(2x+\frac{\pi}{3}\right)\mathrm{d}x$

(c) $\displaystyle\int_1^7 \left(3-\frac{2}{x}\right)^2 \mathrm{d}x$ (d) $\displaystyle\int_2^5 \frac{(2+x^2)^2}{3x}\mathrm{d}x$

(e) $\displaystyle\int_{-1}^1 \frac{4}{e^{2x}}\mathrm{d}x$ (f) $\displaystyle\int_1^2 e^{-\frac{3}{4}x}\mathrm{d}x$

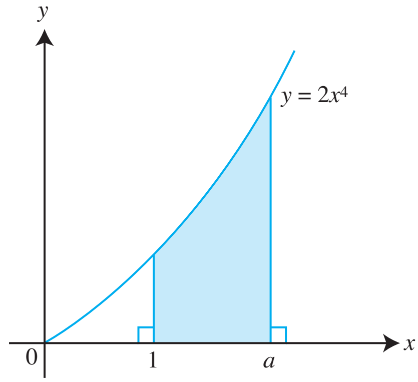
2. Find the possible values of p if

$$\int_3^p (5-x)^5\ \mathrm{d}x = \frac{21}{2}.$$

3. Find the values of k between 0 and 2π if

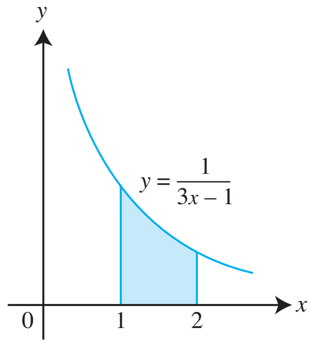
$$\int_0^k (\sin 2x - \sin x)\ \mathrm{d}x = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin\frac{1}{2}x\ \mathrm{d}x.$$

4. The shaded region in the figure is bounded by the curve $y = 2x^4$, the x-axis and the lines $x = 1$ and $x = a$. Given that the area of the region is 12.4 units^2, calculate the value of a.



5. Sketch the graph of $y=\dfrac{10}{x}$ for $1 \leqslant x \leqslant 4$ and hence explain briefly why $7.5 < \displaystyle\int_1^4 \frac{10}{x}\ \mathrm{d}x < 18.5$.

6. A sketch of the curve $y=\dfrac{1}{3x-1}$ is as shown. Find the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$. Give your answer correct to 3 decimal places.



7. Sketch the curve $y = 2x^2 + 3$ and the line $y = 5$. Calculate the area of the region enclosed by the curve and the line.

8. (i) Differentiate $x \ln (2 + x)$ with respect to x.

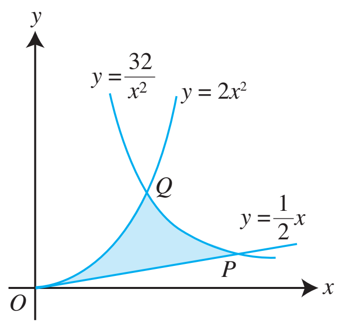
(ii) Express $\dfrac{x}{2+x}$ in the form $a+\dfrac{b}{2+x}$, where a and b are constants.

(iii) Calculate the area enclosed by the curve $y = \ln (2 + x)$, the x-axis and the line $x = 5$. Give your answer correct to 3 decimal places.

9. The diagram shows part of the curve $y = 2x^2$, part of the curve $y=\dfrac{32}{x^2}$ and part of the line $y=\dfrac{1}{2}x$. Find

(i) the coordinates of P and of Q,

(ii) the area of the shaded region.

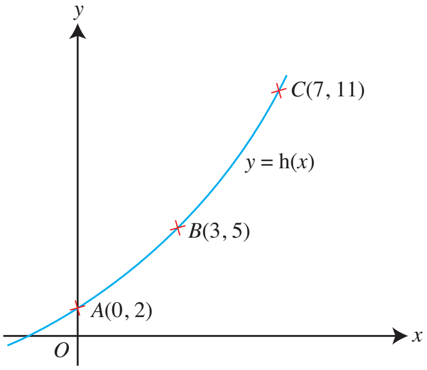


10. Express $\frac{2x+1}{x+3}$ in the form $p+\frac{q}{x+3}$. Hence, evaluate $\int_0^1 \frac{2x+1}{x+3}$, giving your answer correct to 3 significant figures.

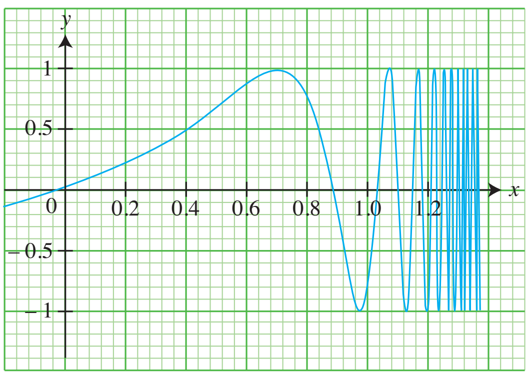
11. Given that $y=\frac{1}{2}x\sin 2x$, show that $\frac{dy}{dx}=\frac{1}{2}\sin 2x + x\cos 2x$. Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} x\cos 2x \, dx$.

12. Express $\frac{1}{x^3}+\frac{3}{(1-2x)^2}$ as a single fraction.
Hence, find the value of
$\int_1^2 \frac{3t^3+4t^2-4t+1}{4t^5-4t^4+t^3} dt$.

13. The figure shows part of the curve $y = h(x)$. Given that the points $A(0, 2)$, $B(3, 5)$ and $C(7, 11)$ lie on the curve, find the numerical value of $\int_3^7 y\,dx + \int_5^{11} x\,dy$.



1. The figure shows the graph of $y=\frac{\sin x}{\cos^3 x}$. Find the area enclosed by the curve, the x-axis and the line $x=\frac{\pi}{6}$.



2. Find the area enclosed by the curve $y=(6x-3)\sqrt{x^2-x+3}$, the x-axis and the line $x = 3$.

KINEMATICS

CHAPTER 16

excluded from the N(A) syllabus

Kinematics is the study of the motion of objects. Kinematics has many applications in fields such as movie production, 3D-animation and computer game programming. Machines that make use of gears and the projectile motion of balls in sports also make use of kinematics.

In this chapter, we will learn how to apply kinematics to solve simple problems which involve the motion of a particle in a straight line.

Learning Objectives

At the end of this chapter, you should be able to:

- solve problems involving displacement, velocity and acceleration of a particle moving in a straight line.

16.1 APPLICATION OF DIFFERENTIATION IN KINEMATICS

Recap

In Chapter 11, we have learnt that for the function $y = \mathrm{f}(x)$, $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ measures the rate of change of y with respect to x.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ indicates that y increases with respect to x, i.e. **y is an increasing function**.

$\dfrac{\mathrm{d}y}{\mathrm{d}x} < 0$ indicates that y decreases with respect to x, i.e. **y is a decreasing function**.

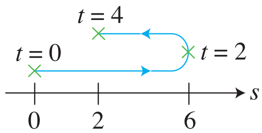
We also have learnt about motion with constant acceleration as well as distance-time and speed-time graphs.

This chapter covers the motion of particles that changes with time, i.e. the displacement, velocity and acceleration as functions of time, and how to solve such problems using differentiation and integration.

Displacement

Consider a toy train moving along a straight track. Assume that it moves such that s m denotes its distance from a fixed point O, t seconds after passing O.

When $t = 0$, $s = 0$;
when $t = 2$, $s = 6$;
when $t = 4$, $s = 2$.



We say that at time $t = 2$, the displacement of the toy train is 6 m. At $t = 2$, the particle stops and reverses its direction of motion. At $t = 4$, its displacement is 2 m.

The **displacement** of a particle from a fixed point is the distance of the particle from the fixed point.

However, the total distance moved by the toy train from $t = 0$ to $t = 4$ is equal to 6 m + (6 – 2) m = 6 m + 4 m = 10 m.

Velocity

Suppose a particle moves such that its displacement, s, from a fixed point O, is related to the time t. The rate of change of displacement with respect to time is then given by $\dfrac{\mathrm{d}s}{\mathrm{d}t}$, which is the instantaneous **velocity**, v, of the particle at a given instant.

i.e. If s is a function of t,

$$\boxed{v = \frac{\mathrm{d}s}{\mathrm{d}t}\,.}$$

Class Discussion

Work in pairs.

Consider the motion of a particle which moves in a straight line such that its displacement, s m, from a fixed point O, t seconds after leaving O, is given by $s = 50 + 40t - 5t^2$.

(i) Find an expression, in terms of t, for the velocity of the particle.

(ii) Find the values of s and of v when $t = 0$, 4 and 8.

(iii) By illustrating the motion of the particle on a diagram, describe its motion from $t = 0$ to $t = 8$. You should include the motion of the particle when $t = 4$.

(iv) When the velocity is positive, what can you say about the direction of motion of the particle?

In conclusion, when $\frac{ds}{dt}$ is positive, the object is moving away from a fixed point and when $\frac{ds}{dt}$ is negative, it is moving towards the fixed point.

When a particle is **instantaneously at rest**, $v = 0$.

Acceleration

If the velocity, v, is a function of t, then the rate of change of velocity with respect to time is given by $\frac{dv}{dt}$, which is called the **acceleration**.
i.e. If v is a function of t,

$$a = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Consider the particle mentioned above.

$$\text{Since } v = \frac{ds}{dt} = 40 - 10t,$$
$$a = \frac{dv}{dt} = -10.$$

The particle has a constant acceleration of -10 m s^{-2}.

Thinking Time

Describe the motion of the particle if it has a constant acceleration of

(a) -10 m s^{-2},

(b) 10 m s^{-2}.

Positive acceleration indicates that velocity increases with respect to time.

Negative acceleration indicates that velocity decreases with respect to time.

Worked Example 1

(Finding the Velocity and the Acceleration when given the Displacement)

A particle moves in a straight line so that, t seconds after passing a fixed point O, its displacement, s metres, is given by $s = t^3 - 6t^2 + 9t + 56$. Find

(i) its velocity when $t = 4$,
(ii) its acceleration when $t = 4$,
(iii) the values of t when it is instantaneously at rest,
(iv) the total distance travelled in the first 2.5 seconds of its motion,
(v) the distance of the particle from O when $t = 2.5$.

Solution

(i) $s = t^3 - 6t^2 + 9t + 56$

$v = \frac{\mathrm{d}s}{\mathrm{d}t} = 3t^2 - 12t + 9$

When $t = 4$, $v = 3(4)^2 - 12(4) + 9 = 9$.

$\therefore$ Its velocity when $t = 4$ is 9 m s^{-1}.

(ii) $v = 3t^2 - 12t + 9$

$a = \frac{\mathrm{d}v}{\mathrm{d}t} = 6t - 12$

When $t = 4$, $a = 6(4) - 12 = 12$.

$\therefore$ Its acceleration when $t = 4$ is 12 m s^{-2}.

(iii) When the particle is instantaneously at rest, its velocity is zero.

i.e. $3t^2 - 12t + 9 = 0$

$t^2 - 4t + 3 = 0$

$(t - 1)(t - 3) = 0$

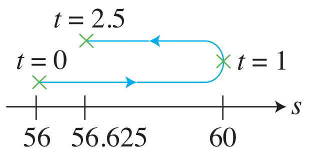
$t = 1$ or $t = 3$

$\therefore$ It is instantaneously at rest when $t = 1$ and when $t = 3$.

ATTENTION

When a particle is instantaneously at rest, $v = \frac{\mathrm{d}s}{\mathrm{d}t} = 0$.

(iv)



When $t = 0$, $s = 56$.

When $t = 1$, $s = (1)^3 - 6(1)^2 + 9(1) + 56 = 60$.

When $t = 2.5$, $s = (2.5)^3 - 6(2.5)^2 + 9(2.5) + 56$
$= 56.625$.

Distance travelled from $t = 1$ to $t = 2.5$ is $60 - 56.625 = 3.375$

$\therefore$ Total distance travelled in the first 2.5 seconds is $4 + 3.375 = 7.375$ m

PROBLEM-SOLVING TIP

Always draw a diagram to illustrate the motion of the particle.

(v) When $t = 2.5$, $s = 56.625$.

$\therefore$ The distance of the particle from O when $t = 2.5$ is 56.625 m.

ATTENTION

The **total distance travelled** is different from the distance of the particle from O. The latter is the magnitude of the displacement of the particle from O.

Thinking Time

In Worked Example 1, in order to calculate the total distance travelled in the first 2.5 seconds, we substitute $t = 0$, 1 and 2.5 to find the displacement at each of these times.

(i) Explain why it is necessary to find the displacement when $t = 1$.

(ii) Explain why it is not necessary to find the displacement when $t = 2$.

Practise Now 1

Similar Questions:
Exercise 16A
Questions 1, 2, 6, 7, 12

A particle moves in a straight line so that, t seconds after passing a fixed point O, its displacement, s metres, is given by $s = 4t^3 - 15t^2 + 18t$. Find

(i) its velocity when $t = 3$,

(ii) its acceleration when $t = 3$,

(iii) the values of t when it is instantaneously at rest,

(iv) the total distance travelled in the first 2 seconds of its motion,

(v) the distance of the particle from O when $t = 2$.

Worked Example 2

(Problem involving Trigonometric Expressions)

A particle moves in a straight line so that its displacement, s metres, from a fixed point A, is given by $s = 6t + 2\cos 3t$, where t is the time in seconds after passing A. Find

(i) the initial position of the particle,

(ii) an expression for the velocity and the acceleration of the particle in terms of t,

(iii) the time at which the particle first comes to a rest and its distance from A at this instant.

Solution

(i) $s = 6t + 2\cos 3t$

When $t = 0$, $s = 6(0) + 2\cos 0 = 2$.

$\therefore$ The particle is initially 2 m away from A.

(ii)
$$s = 6t + 2\cos 3t$$
$$v = \frac{ds}{dt} = 6 - 6\sin 3t$$
$$a = \frac{dv}{dt} = -18\cos 3t$$

RECALL

$\frac{d}{dx}[\sin(ax)] = a\cos ax$

$\frac{d}{dx}[\cos(ax)] = -a\sin ax$

(iii) When $v = 0$,
$$6 - 6\sin 3t = 0$$
$$6\sin 3t = 6$$
$$\sin 3t = 1$$
$$3t = \frac{\pi}{2}$$
$$t = \frac{\pi}{6}$$

$\therefore$ The particle first comes to a rest when $t = \frac{\pi}{6}$.

When $t = \frac{\pi}{6}$,
$$s = 6\left(\frac{\pi}{6}\right) + 2\cos 3\left(\frac{\pi}{6}\right)$$
$$= \pi$$

$\therefore$ The distance from A at this instant is π m.

Practise Now 2

Similar Questions:
Exercise 16A
Questions 3, 5, 13

A particle moves in a straight line so that its distance, s cm, from a fixed point O, is given by $s = 3t + 4 \sin 2t$, where t is the time in seconds after leaving O. Find

(i) its distance from O at $t = 1.1$,
(ii) the velocity of the particle when $t = \frac{\pi}{6}$,
(iii) the acceleration of the particle when it first comes to rest.

Worked Example 3

(Problem involving Vertical Motion)

A stone thrown into the air rises a distance of s metres in t seconds, where $s = 20t - 5t^2$.

(i) Find the velocity of the stone when $t = 1.5$ and when $t = 3.5$.
(ii) Explain what is meant by a negative velocity.
(iii) Find the maximum height reached by the stone.
(iv) Find the values of t when the particle is 15 m above the ground. Explain why there are two answers.

Solution

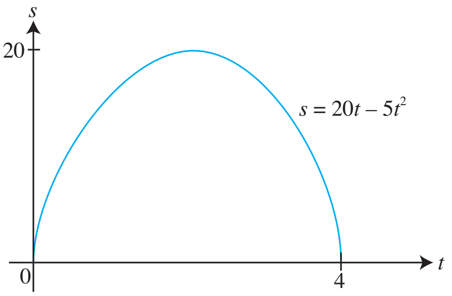
(i)
$$s = 20t - 5t^2$$
$$v = \frac{ds}{dt} = 20 - 10t$$

When $t = 1.5$,
$v = 20 - 10(1.5) = 5$

When $t = 3.5$,
$v = 20 - 10(3.5) = -15$

$\therefore$ The velocities are 5 m s^{-1} and -15 m s^{-1} respectively.

(ii) Negative velocity means that the stone is falling down and is moving towards the ground.



(iii) At maximum height,
$$v = \frac{ds}{dt} = 0$$
$$20 - 10t = 0$$
$$t = 2$$

When $t = 2$,
$s = 20(2) - 5(2)^2 = 20$

$\therefore$ The maximum height reached by the stone is 20 m.

(iv) When $s = 15$,
$$20t - 5t^2 = 15$$
$$t^2 - 4t + 3 = 0$$
$$(t - 1)(t - 3) = 0$$
$$t = 1 \text{ or } t = 3$$

$\therefore$ The particle is 15 m above the ground when $t = 1$ and $t = 3$.

The two answers mean that 1 second after the stone is thrown up, it is 15 m from the ground; 3 seconds later, on its way down, it is 15 m from the ground.

Practise Now 3

Similar Question:
Exercise 16A
Question 4

An object is thrown into the air from the top of a tall building and its height above the ground, s metres, after t seconds, is given by $s = 40 + 30t - 5t^2$.

(i) Write down the height of the building.

(ii) Find the velocity of the object when $t = 2$ and when $t = 4$. Explain the significance of the values of the velocity at these two times.

(iii) Find the values of t when the object is at a height of 60 m above the ground. Explain the significance of these two values of t in relation to the motion of the object.

Worked Example 4

(Problem involving Exponential Expressions)

A computer animation shows a cartoon rabbit moving in a straight line so that, at time t seconds after motion has begun, its displacement, s m, from a fixed point O, is given by $s = 6\left(\frac{2}{e^t} - \frac{3}{e^{2t}}\right)$.

(i) Find the time when the rabbit is instantaneously at rest.

(ii) Given that the rabbit pauses its motion once to pick up a carrot, find the total distance it travels during the first 2 seconds of its motion.

Solution

(i) $s = 6\left(\frac{2}{e^t} - \frac{3}{e^{2t}}\right) = 6(2e^{-t} - 3e^{-2t})$

$$v = \frac{ds}{dt} = 6(-2e^{-t} + 6e^{-2t}) = 12\left(\frac{3}{e^{2t}} - \frac{1}{e^t}\right)$$

When $v = 0$,

$$12\left(\frac{3}{e^{2t}} - \frac{1}{e^t}\right) = 0$$

$$\frac{3}{e^{2t}} - \frac{1}{e^t} = 0$$

$$3e^t = e^{2t}$$

$$3e^t - e^{2t} = 0$$

$$e^t(3 - e^t) = 0$$

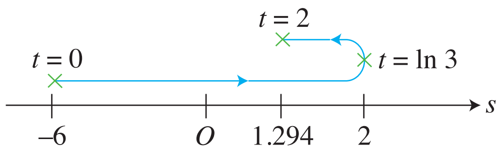
$e^t = 0$ (no solution) or $e^t = 3$

$t = \ln 3 = 1.10$ (to 3 s.f.)

(ii) When $t = 0$, $s = -6$.

When $t = \ln 3$, $s = 6\left(\frac{2}{e^{\ln 3}} - \frac{3}{e^{2\ln 3}}\right) = 2$.

When $t = 2$, $s = 6\left(\frac{2}{e^2} - \frac{3}{e^4}\right) = 1.294$.



∴ Total distance travelled in the first 2 seconds is
$(6 + 2) + (2 - 1.294) = 8.71$ m (to 3 s.f.)

ATTENTION

We substitute $t = \ln 3$ to find the value of s when the particle reverses its direction of motion.

Practise Now 4

Similar Questions:
Exercise 16A Questions 8-11

In a children's computer game, the motion of a firetruck P is such that its displacement, s metres from a fixed point O, can be modelled by $s = 200(e^{-2t} - e^{-3t})$, where t is the time in seconds after passing O. Find

(i) the time when P is at instantaneous rest,

(ii) the total distance travelled by P during the first 3 seconds of its motion.

Basic Level | Intermediate Level | Advanced Level

Exercise 16A

1 A remote-controlled car travels in a straight line so that its distance, s cm, from a fixed point O, is given by $s = 77t + 13t^2 - t^3$, where t is the time in seconds after passing O. Find

(i) the time when the car is instantaneously at rest before it reverses its direction of motion,

(ii) the distance travelled in the first 4 seconds,

(iii) the distance travelled in the fourth second,

(iv) the acceleration of the car when $t = 4$.

2 A vehicle moves in a straight line so that its displacement, s metres, from a fixed point O, is given by $s = 2t^3 - 3t^2 - 12t + 6$, where t is the time in seconds after motion has begun. Find

(i) an expression for the velocity and the acceleration at time t,

(ii) the value of t when the vehicle is instantaneously at rest,

(iii) the minimum velocity attained by the vehicle.

3 A particle moves in a straight line so that its displacement, s metres, from a fixed point O, is given by $s = 2t + \cos 3t$, where t is the time in seconds after passing O. Find

(i) the time when the particle first comes to a stop,

(ii) the velocity and acceleration of the particle when $t = \frac{\pi}{12}$.

4 A stone is thrown downwards from the top of a tall building such that its height, h metres, above the ground, t seconds later, is given by $h = 100 - 5t - 5t^2$. Find

(i) the initial speed of the stone,

(ii) the time when the stone hits the ground and its speed at that instant,

(iii) the height of the building.

5 A particle moves in a straight line such that its displacement, s metres from a fixed point O at time t seconds, is given by $s = k \sin^2 t$, where k is a positive constant and $0 \leqslant t \leqslant \pi$.

(i) Find an expression for the velocity and the acceleration of the particle.

(ii) Find the values of t when the velocity of the particle is

(a) zero, **(b)** a maximum.

6 A body moves in a straight line so that its distance, s cm, from a fixed point O, is given by $s = 24t^2 - t^3$, where t is the time in seconds after passing O. Find

(i) the time when the body returns to O and the velocity at this instant,

(ii) the greatest distance of the body from O.

7 A cyclist travels in a straight line so that his displacement, S metres, from a fixed point O, is given by $S = 4 + 15t - t^3$, where t is the time in seconds after passing a point X on the line.

(i) Find the distance OX.

(ii) Find the velocity of the cyclist at $t = 2$.

(iii) Calculate the acceleration of the cyclist when he is instantaneously at rest.

(iv) Calculate the distance travelled by the cyclist during the third second of his motion.

8 A particle moves in a straight line so that, t seconds after leaving a fixed point O, its velocity, v m s^{-1}, is given by $v = e^{2t} - 4e^t + 12$. Find

(i) the time at which the particle has a velocity of 17 m s^{-1},

(ii) the initial acceleration of the particle.

9 A particle P moves along the x-axis such that, at time t seconds after passing the origin O, its displacement, s metres from O, is given by $s = 200(e^{-2t} - e^{-3t})$. Find

(i) the time when s is a maximum,

(ii) the acceleration of P at this instant.

10 A computer animation shows a cartoon caravan moving in a straight line so that its displacement, s metres, from a fixed point O, is given by $s = e^{-2t} \sin 2t$, where t is the time in seconds after it passes O. Given that the caravan frequently pauses its motion to refuel its engine, find the time when it first comes to an instantaneous rest.

11 A particle moves in a straight line such that its displacement, s metres, from a starting point O, is given by $s = \ln (2t + 3)$.

(i) Find the initial position of the particle.

(ii) Find the velocity of the particle when $t = 5$.

(iii) Show that the particle is decelerating for all values of t.

Exercise 16A

12 A toy train moves in a straight line so that its displacement, s metres, from a fixed point, is given by $s = 2t^3 - 4t^2 + 2t - 1$, where t is the time measured from the start of the motion. Find

(i) the values of t when the toy train is at rest,

(ii) the time interval(s) during which the velocity is positive,

(iii) the time interval(s) during which the velocity is negative.

13 A particle moves in a straight line so that its displacement, s metres, from a fixed point O, is given by $s = 2\sin t + \cos t$, where t is the time in seconds after passing O. Find expressions for its velocity, v m s^{-1}, and acceleration a m s^{-2}, at time t. Hence, or otherwise, show that $v^2 + s^2 = 5$ and $s = -a$.

16.2 APPLICATION OF INTEGRATION IN KINEMATICS

In Section 16.1, we have learnt that when a particle moves in a straight line such that its displacement from a fixed point is given as s, then the velocity is given as $v = \frac{ds}{dt}$ and the acceleration $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Conversely,

$$\boxed{s = \int v \, dt \text{ and } v = \int a \, dt.}$$

In this section, we will apply both differentiation and integration to solve problems involving the motion of a particle.

Worked Example 5

(Finding the Displacement when given the Velocity)

An object P moves in a straight line so that its velocity, v m s^{-1}, is given by $v = 2t^2 - 7t + 3$, where t is the time in seconds after leaving a point A which is 5 m away from O. Find

(i) an expression, in terms of t, for the displacement of P from O,

(ii) the values of t when P is instantaneously at rest,

(iii) the distance of P from O at $t = 2$,

(iv) the total distance travelled by P in the first 5 seconds of its motion.

Solution

(i) $v = 2t^2 - 7t + 3$

$$s = \int v \, dt$$
$$= \int (2t^2 - 7t + 3) \, dt \quad \text{(integrate each term separately)}$$
$$= \frac{2}{3}t^3 - \frac{7}{2}t^2 + 3t + c$$

When $t = 0$, $s = 5$ $\therefore$ $c = 5$

$$\therefore s = \frac{2}{3}t^3 - \frac{7}{2}t^2 + 3t + 5$$

(ii) When P is instantaneously at rest, $v = 0$.

i.e. $2t^2 - 7t + 3 = 0$

$$(2t - 1)(t - 3) = 0$$
$$t = \frac{1}{2} \text{ or } t = 3$$

$\therefore$ P is instantaneously at rest when $t = \frac{1}{2}$ and $t = 3$.

(iii) $s = \frac{2}{3}t^3 - \frac{7}{2}t^2 + 3t + 5$

When $t = 2$,

$$s = \frac{2}{3}(2)^3 - \frac{7}{2}(2)^2 + 3(2) + 5 = 2\frac{1}{3}$$

$\therefore$ P is $2\frac{1}{3}$ m away from O.

(iv) When $t = 0$, $s = 5$.

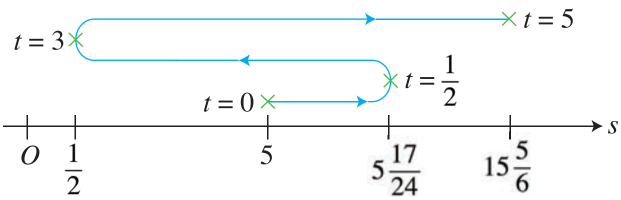
When $t = \frac{1}{2}$, $s = \frac{2}{3}\left(\frac{1}{2}\right)^3 - \frac{7}{2}\left(\frac{1}{2}\right)^2 + 3\left(\frac{1}{2}\right) + 5 = 5\frac{17}{24}$.

When $t = 3$, $s = \frac{2}{3}(3)^3 - \frac{7}{2}(3)^2 + 3(3) + 5 = \frac{1}{2}$.

When $t = 5$, $s = \frac{2}{3}(5)^3 - \frac{7}{2}(5)^2 + 3(5) + 5 = 15\frac{5}{6}$.

PROBLEM-SOLVING TIP

Always draw a diagram to illustrate the motion of the object.



$\therefore$ Total distance travelled in the first 5 seconds

$$= \left(5\frac{17}{24} - 5\right) + \left(5\frac{17}{24} - \frac{1}{2}\right) + \left(15\frac{5}{6} - \frac{1}{2}\right)$$
$$= 21\frac{1}{4} \text{ m}$$

Practise Now 5

Similar Questions:
Exercise 16B
Questions 3-5

A particle P moves in a straight line so that its velocity, v m s^{-1}, is given by $v = 2 - 2t + 4t^2$ m s^{-1}, where t is the time in seconds after passing O. Find

(i) an expression, in terms of t, for the displacement of P from O,

(ii) the distance of P from O at $t = 3$,

(iii) the total distance travelled by P in the first 6 seconds of its motion.

Worked Example 6

(Problem involving Trigonometric Expressions)

A particle starts from a point O and moves in a straight line with a velocity, v m s^{-1}, given by $v = t + \sin 2t$, where t is the time in seconds after leaving O. Calculate the displacement of the particle when $t = \frac{\pi}{2}$ and its acceleration at this instant.

Solution

$v = t + \sin 2t$

$s = \int v \, dt = \int (t + \sin 2t) \, dt$

$\quad = \frac{t^2}{2} - \frac{1}{2}\cos 2t + c$

When $t = 0$, $s = 0$ $\therefore$ $c = 0 - 0 + \frac{1}{2}\cos 0 = \frac{1}{2}$.

Hence, $\quad s = \frac{t^2}{2} - \frac{1}{2}\cos 2t + \frac{1}{2}$

When $t = \frac{\pi}{2}$, $s = \frac{\left(\frac{\pi}{2}\right)^2}{2} - \frac{1}{2}\cos 2\left(\frac{\pi}{2}\right) + \frac{1}{2} = \frac{\pi^2}{8} + 1$

$\therefore$ Displacement of the particle when $t = \frac{\pi}{2}$ is $\left(\frac{\pi^2}{8} + 1\right)$ m

$a = \frac{dv}{dt}$

$\quad = 1 + 2\cos 2t$

When $t = \frac{\pi}{2}$, $a = 1 + 2\cos 2\left(\frac{\pi}{2}\right) = -1$

$\therefore$ Acceleration of the particle when $t = \frac{\pi}{2}$ is -1 m s^{-2}

RECALL

$\int \sin (Ax + B) \, dx$

$= -\frac{1}{A}\cos (Ax + B) + c$

Thinking Time

In Worked Example 6, the acceleration of the particle when $t = \frac{\pi}{2}$ is -1 m s^{-2}.

(i) What is the difference in the motion of the particle if it has an acceleration of 1 m s^{-2} instead?

(ii) What happens to the instantaneous velocity of the particle changes when it has an acceleration of -1 m s^{-2}.

Practise Now 6

Similar Questions:
Exercise 16B
Questions 6-8

A particle starts from a point O and moves in a straight line with a velocity, v m s^{-1}, given by $v = 2 + 12\sin\frac{1}{2}t$, where t is the time in seconds after leaving O. Find

(i) the displacement of the particle when $t = \frac{\pi}{3}$ and its acceleration at this instant,

(ii) the time at which the particle first attains a speed of 8 m s^{-1},

(iii) the acceleration of the particle when $t = 1$.

Worked Example 7

(Problem involving Exponential Expressions)

A particle travels in a straight line so that its velocity, v m s^{-1}, is given by $v = 6e^{-\frac{1}{3}t}$, where t is the time in seconds after leaving a fixed point O. Calculate, correct to 2 decimal places,

(i) the acceleration of the particle when $t = 3$,

(ii) the distance travelled in the third second of motion after leaving O.

Solution

(i) $v = 6e^{-\frac{1}{3}t}$

$$a = \frac{dv}{dt} = 6\left(-\frac{1}{3}\right)e^{-\frac{1}{3}t}$$

$$= -2e^{-\frac{1}{3}t}$$

When $t = 3$,

$$a = -2e^{-\frac{1}{3}(3)}$$

$$= -2e^{-1}$$

$$= -0.74 \text{ (to 2 d.p.)}$$

∴ Acceleration of the particle when $t = 3$ is -0.74 m s^{-2}

(ii) Distance travelled in the third second

$$= \int_2^3 v \, dt$$

$$= \int_2^3 6e^{-\frac{1}{3}t} \, dt$$

$$= \left[\frac{6e^{-\frac{1}{3}t}}{-\frac{1}{3}}\right]_2^3$$

$$= \left[-18e^{-\frac{1}{3}t}\right]_2^3$$

$$= \left(-18e^{-1}\right) - \left(-18e^{-\frac{2}{3}}\right)$$

$$= 2.62 \text{ m (to 2 d.p.)}$$

ATTENTION

The distance travelled in the third second refers to that from $t = 2$ to $t = 3$, whereas the distance moved in the first 3 seconds refers to the total distance from $t = 0$ to $t = 3$.

PROBLEM-SOLVING TIP

We can use $\int_a^b v \, dt$ to find the total distance travelled between $t = a$ and $t = b$ only if the particle travels in the same direction during this time interval.

Practise Now 7

Similar Questions:
Exercise 16B
Questions 9, 10, 12

A particle travels in a straight line so that its velocity, v m s^{-1}, is given by $v = 5(2 - 5e^{-t})$, where t is the time in seconds after leaving a fixed point O. Calculate, correct to 2 decimal places,

(i) the acceleration of the particle when $t = 2$,

(ii) the total distance travelled by the particle in the first 3 seconds.

Worked Example 8

(Finding the Displacement and Velocity when given the Acceleration)

A particle moving in a straight line passes a fixed point O with a velocity of 16 m s^{-1}. The acceleration of the particle, a m s^{-2}, is given by $a = 2t - 10$, where t is the time after passing O. Find

(i) the values of t when the particle is instantaneously at rest,

(ii) the displacement of the particle when $t = 5$,

(iii) the total distance travelled by the particle in the first 5 seconds of its motion,

(iv) the distance travelled by the particle in the 5th second.

Solution

(i) $a = 2t - 10$

$v = \int a \, dt = \int (2t - 10) \, dt$

$= t^2 - 10t + c$

When $t = 0$, $v = 16$ $\therefore c = 16$.

$$v = t^2 - 10t + 16$$

When the particle is instantaneously at rest, $v = 0$.

i.e. $t^2 - 10t + 16 = 0$

$(t - 2)(t - 8) = 0$

$t = 2$ or $t = 8$

$\therefore$ The particle is instantaneously at rest when $t = 2$ and $t = 8$.

(ii) $v = t^2 - 10t + 16$

$s = \int v \, dt = \int (t^2 - 10t + 16) \, dt$

$= \frac{1}{3}t^3 - 5t^2 + 16t + c_1$

When $t = 0$, $s = 0$ $\therefore c_1 = 0$.

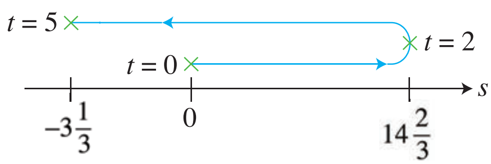
$s = \frac{1}{3}t^3 - 5t^2 + 16t$

$\therefore$ Displacement of the particle when $t = 5$ is $\frac{1}{3}(5)^3 - 5(5)^2 + 16(5) = -3\frac{1}{3}$ m

(iii) When $t = 0$, $s = 0$.

When $t = 2$, $s = \frac{1}{3}(2)^3 - 5(2)^2 + 16(2) = 14\frac{2}{3}$.

When $t = 5$, $s = -3\frac{1}{3}$.

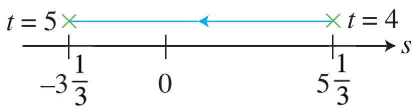


$\therefore$ Total distance travelled in the first 5 seconds

$= 14\frac{2}{3} + \left(14\frac{2}{3} + 3\frac{1}{3}\right) = 32\frac{2}{3}$ m

(iv) When $t = 4$, $s = \frac{1}{3}(4)^3 - 5(4)^2 + 16(4) = 5\frac{1}{3}$.

When $t = 5$, $s = -3\frac{1}{3}$.



$\therefore$ Total distance travelled in the 5th second

$= 5\frac{1}{3} + 3\frac{1}{3} = 8\frac{2}{3}$ m

ATTENTION

The distance travelled in the 5th second refers to the distance travelled between $t = 4$ and $t = 5$.

INFORMATION

The distance travelled in the 5th second can also be found from $\int_4^5 v \, dt$.

Practise Now 8

Similar Questions:
Exercise 16B
Questions 2, 11

A particle moves in a straight line so that, t seconds after passing a fixed point O, its acceleration, a cm s^{-2}, is given by $a = 4t - 16$. Given that it passes O with a velocity of 30 cm s^{-1},

(i) find an expression for the velocity of the particle in terms of t,
(ii) show that the particle comes to instantaneous rest when $t = 3$,
(iii) find the displacement of the particle when $t = 3$,
(iv) find the total distance travelled by the particle in the first 3 seconds of its motion,
(v) find the distance travelled by the particle in the 3rd second.

Basic Level | Intermediate Level | Advanced Level

Exercise 16B

1 A particle, moving in a straight line, passes through a fixed point O. Its velocity, v m s^{-1}, t seconds after passing O, is given by $v = t^2 + 2t + 1$.
(i) Find an expression for the displacement of the particle from O.
(ii) Calculate the distance of the particle from O when
(a) $t = 2$, **(b)** $t = 5$.

2 A toy truck travelling in a straight line passes a fixed point O with a velocity of 4 cm s^{-1}. Its acceleration, a cm s^{-2}, is given by $a = 2t + 1$, where t is the time in seconds after passing O. Find
(i) its velocity when $t = 3$, **(ii)** its distance from O when $t = 3$.

3 A particle travelling in a straight line passes a fixed point O. Its velocity, v cm s^{-1}, t seconds after passing O, is given by $v = 10 + 26t - 3t^2$. Find
(i) the maximum value of v, **(ii)** the distance of the particle from O when $t = 2$.

Exercise 16B

4 A particle travels in a straight line so that its velocity v m s^{-1}, is given by $v = 8 + \frac{1}{2}t^2$, where t is the time in seconds after leaving O. Find

(i) the velocity of the particle when the acceleration is 1 m s^{-2},

(ii) the distance travelled during the 4$^{\text{th}}$ second.

5 A body starts from a fixed point O and moves in a straight line with a velocity, v m s^{-1}, given by $v = 4t - t^2$, where t is the time in seconds after leaving O. Find

(i) the distance travelled by the body before it comes to rest,

(ii) the time taken before the body returns to O.

6 A particle travels in a straight line so that, t seconds after passing through a fixed point O, its velocity, v m s^{-1}, is given by $v = 2t^2 + (1 - 10k)t + 18k - 1$, where k is a constant.

(i) Find an expression for the acceleration of the particle in terms of k and t.

(ii) Given that the acceleration of the particle at $t = 5$ is 1 m s^{-2}, find the value of k.

(iii) Using the value of k found in **(ii)**, find the range of values of t for which the particle is moving in the positive direction.

(iv) State the values of t when the particle comes to an instantaneous rest.

(v) Find an expression for the displacement of the particle.

7 A particle moves along a straight line with a velocity, v m s^{-1}, given by $v = 4 - 8 \sin 2t$, where t is the time in seconds after passing a fixed point O.

(i) Determine the range of values of v.

(ii) Find the distance of the particle from O at the instant when the particle first comes to an instantaneous rest.

8 In a factory, a machine part P, which is connected to a food cutter, moves in a straight line from a fixed point O so that, t seconds after passing O, its velocity, v m s^{-1}, is given by $v = 2\cos\frac{t}{2} + 1$, where $0 \leqslant t \leqslant 2\pi$. Find

(i) the value of t when P first comes to an instantaneous rest, giving your answer in terms of π,

(ii) the total distance travelled by P from $t = 0$ to $t = 2\pi$.

9 A particle moves in a straight line from a fixed point O so that its velocity, v m s^{-1}, is given by $v = 2e^{3t} + 5e^{-3t}$, where t is the time in seconds after leaving O. Calculate

(i) the acceleration of the particle when $t = 1$,

(ii) the total distance travelled by the particle in the first 2 seconds of its motion.

10 A particle moves in a straight line so that, t seconds after leaving a fixed point A, its velocity, v m s^{-1}, is given by $v = 4e^t - 5e^{1-2t}$. Find

(i) the value of t when the particle is instantaneously at rest,

(ii) the distance travelled by the particle in the 2$^{\text{nd}}$ second of its motion,

(iii) the average speed of the particle for the first 3 seconds of its motion.

11 A particle travelling in a straight line passes a fixed point O with a velocity of 8 m s^{-1} and an acceleration of –2 m s^{-2}. It moves in such a manner that, t seconds after passing O, its acceleration a m s^{-2}, is given by $a = h + kt$. During the first second after passing O, it moves through a distance of 9 m. Find the value of h and of k.

12 A particle moves in a straight line such that, t seconds after leaving a fixed point O, its velocity, v m s^{-1}, is given by $v = 8 - e^{-2t}$. Sketch the graph of $v = 8 - e^{-2t}$.

(i) Write down the initial velocity of the particle.

(ii) If t becomes very large, what value will v approach? Explain your answer clearly.

(iii) Find the acceleration of the particle when $t = 3$, giving your answer in cm s^{-2} correct to 3 decimal places.

(iv) Find the distance travelled by the particle in the first 4 seconds of its journey, giving your answer correct to 2 decimal places.

SUMMARY

1. Relationship between displacement (s), velocity (v) and acceleration (a)

$$\begin{tikzcd}
\text{displacement } (s) \arrow[r, bend left, "v = \frac{ds}{dt}"] & \text{velocity } (v) \arrow[l, bend left, "s = \int v\,dt"] \arrow[r, bend left, "a = \frac{dv}{dt} = \frac{d^2s}{dt^2}"] & \text{acceleration } (a) \arrow[l, bend left, "v = \int a\,dt"]
\end{tikzcd}$$

2. When a particle is instantaneously at rest, $v = 0$.

Review Exercise 16

1. A particle starts from a point A and moves in a straight line so that its displacement, s metres, from a fixed point O, t seconds after leaving A, is given by $s = 5 + t(t-5)^2$.
   (i) Obtain expressions for the velocity and acceleration of the particle in terms of t.
   (ii) Find the distance OA.
   (iii) Given that the particle is next at A when $t = T$, find the value of T.
   (iv) Find the values of t when the particle is instantaneously at rest.
   (v) Find the total distance travelled by the particle in the first T seconds on its motion.

2. An object moves in a straight line so that its displacement, s metres, from a fixed point O at time t seconds, is given by $s = bt + ct^3$, where b and c are constants. Given that when $t = 4$, the velocity is zero and the acceleration is 12 m s^{-2}, find the value of b and of c. Hence, find the velocity and acceleration when $t = 1$.

3. A particle moves in a straight line so that its displacement, s metres, from a fixed point O, after t seconds, is given by
$$s = \frac{2}{3}t^3 - \frac{19}{2}t^2 + 35t - 5.$$
   The particle reverses its direction of motion at time t_1 and t_2, where $t_2 > t_1$. Find
   (i) the value of t_1 and of t_2,
   (ii) the distance travelled by the particle between $t = 0$ and $t = t_1$,
   (iii) the total distance travelled by the particle from $t = 0$ to $t = t_2$,
   (iv) the acceleration of the particle when $t = t_2$.

4. A stone is thrown vertically upwards so that its height, s metres, above the ground, after t seconds, is given by $s = 40t - 5t^2$. Find
   (i) its initial velocity,
   (ii) its height above the ground when it is instantaneously at rest,
   (iii) its velocity when it is 15 m above the ground, giving your answer correct to the nearest m s^{-1}.

5. A particle passes a fixed point O and moves in a straight line with a velocity of $4e^{1-2t} \text{ m s}^{-1}$, where t is the time in seconds after passing O. Find
   (i) the acceleration of the particle when $t = 1$,
   (ii) the distance travelled by the particle in the first 3 seconds of its motion.

6. A particle moves in a straight line such that, t seconds after passing a fixed point O, its velocity, $v \text{ m s}^{-1}$, is given by $v = 2e^{\frac{t}{8}+1}$. Calculate
   (i) the velocity of the particle when the acceleration is $\frac{1}{4}e^2 \text{ m s}^{-2}$,
   (ii) the displacement of the particle from O when $t = 4$.

7. The velocity, $v \text{ km h}^{-1}$, of a train, which moves along a straight track from station P to station Q at which it next stops, is given by $v = 1 - \sin\frac{\pi t}{2}$, where t is the time in hours after leaving P. Calculate
   (i) the time taken for the train to travel from P to Q,
   (ii) the acceleration of the train when $t = \frac{1}{2}$.

8. A particle P moves along a straight line so that after t seconds, its displacement, s metres, from a fixed point O, is given by $s = te^{-\frac{t}{4}}$. Calculate

(i) the value of t when P is instantaneously at rest,

(ii) the total distance travelled by P in the first 5 seconds of its motion.

9. A particle moves in a straight line so that its displacement, s metres, from a fixed point O, is given by $s = 5 - \frac{15}{2t+3}$, where t is the time in seconds after the start of motion.

(i) Find the velocity of the particle when it is $3\frac{1}{3}$ m from O.

(ii) Show that the particle will never return to O.

(iii) Find the greatest distance from O that the particle can travel.

10. A particle moves in a straight line so that its displacement, s metres, from a fixed point O, is given by $s = k \tan nt$, where t is the time in seconds after passing O and k and n are constants. Find expressions for its velocity, v m s^{-1}, and acceleration a m s^{-2}, at time t. Hence, or otherwise, show that $ak = 2nvs$.

1. Two particles P and Q leave O at the same time and travel in a straight line. P starts from rest and moves with a velocity v_P m/s, given by $v_P = 6t - t^2$, where t is the time in seconds after passing O. Q starts with a velocity of 10 m/s and moves with an acceleration, f m/s^2, given by $f = a + bt$, where a and b are constants.

Express the velocity of Q, v_Q, at time t, in terms of a and b.
If P and Q have equal velocities at the instant when $t = 1$ and $t = 2$,

(i) prove that $a = -9$ and $b = 8$,

(ii) find the distance travelled by P during the first 7 seconds, giving your answer correct to the nearest metre.

2. The distance, s cm, moved by a particle in t seconds, is given by the equation $s = 2t \cos \frac{\pi t}{6}$. Find expressions for the velocity, v, and the acceleration, a, of the particle. Hence, show that $(36a + \pi^2 s)t^2 = 72(vt - s)$.

REVISION EXERCISE F1

1. Evaluate each of the following integrals.

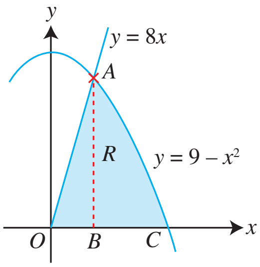
   (a) $\displaystyle\int_{-1}^{2} \frac{3x^3 + 10x^2}{x}\, dx$ (b) $\displaystyle\int_{0}^{\frac{\pi}{4}} (\sin x + \cos x)^2\, dx$ (c) $\displaystyle\int_{0}^{1} \frac{3 + e^{4x}}{e^{2x}}\, dx$

2. Prove that $\dfrac{d}{dx}(x+1)\sqrt{2x-1} = \dfrac{3x}{\sqrt{2x-1}}$. Hence, evaluate $\displaystyle\int_{5}^{13} \frac{x}{\sqrt{2x-1}}\, dx$.

3. The gradient of a curve at any point P is given by $2x - 1$. If this curve passes through the point $(2, 2)$, find the equation of the curve and the coordinates of the points where it cuts the x-axis.

4. The curve $y = ax^2 + bx + c$ passes through the points $(0, 4)$ and $(2, 7)$. If the area enclosed by the curve, the x-axis and the lines $x = 1$ and $x = 2$ is 3 units², find the equation of the curve.

5. The shaded region R shown in the diagram is bounded by the line $y = 8x$, the curve $y = 9 - x^2$ and the x-axis. Find
   (i) the coordinates of A, B and C,
   (ii) the area of R.



6. A particle moves in a straight line so that its distance, s cm, from a point O, is given by $s = 6t^2 - t^3$, where t is the time in seconds after passing O. Find
   (i) the velocity and position of the particle when the acceleration is zero,
   (ii) the acceleration and position of the particle when the velocity is 12 cm/s.

7. The velocity, v m/s, of a particle, t seconds after passing a fixed point O, is given by $v = 3t^2 - \dfrac{16}{t^2}$, where $t > 1$. Calculate
   (i) the distance travelled by the particle in its fourth second of motion,
   (ii) the acceleration of the particle when $t = 2$.

REVISION EXERCISE F2

1. Evaluate each of the following integrals.

   (a) $\int_1^4 \left(5\sqrt{x^3} - \frac{12}{x^2}\right) \mathrm{d}x$

   (b) $\int_{\frac{\pi}{12}}^{\frac{\pi}{6}} (\sin 2x + \cos 4x)\, \mathrm{d}x$

2. Given that $y = x\sqrt{4 + 3x^2}$, show that $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{4 + 6x^2}{\sqrt{4 + 3x^2}}$. Hence, evaluate $\int_0^2 \frac{2 + 3x^2}{\sqrt{4 + 3x^2}}\, \mathrm{d}x$.

3. The gradient function of a curve is given by $\frac{\mathrm{d}y}{\mathrm{d}x} = a - \frac{b}{x^2}$. Given that the gradient of the tangent at the point $A(1, 15)$ is -7 and that $B\left(\frac{4}{3}, 14\right)$ is a stationary point on the curve, find

   (i) the values of the constants a and b,

   (ii) the equation of the curve.

4. The gradient of a curve at any point is given by $2x - 5$. If the curve passes through the point $(2, -2)$, find the equation of the curve and sketch the curve for the domain $0 \leqslant x \leqslant 5$. Calculate the area enclosed by the curve and the x-axis.

5. A particle moves in a straight line so that its distance, s metres, from a fixed point O, is given by $s = t^4 - 1$, where t is the time in seconds after passing O. Find

   (i) the velocity of the particle when $t = 3$,

   (ii) the velocity and acceleration of the particle when it returns to O.

6. A particle moves along a straight line so that its velocity, v m/s, t seconds after passing a fixed point O, is given by $v = 30t - 6t^2$.

   (i) Find the time interval during which the velocity is positive.

   (ii) After how many seconds will the particle return to its initial position?

   (iii) Find the total distance covered in the first 7.5 seconds.

7. A particle starts from rest and its acceleration, t seconds after passing a fixed point P, is given by $(5 - t)$ m/s^2. Find the distance the particle moves before coming to rest, the time it takes to return to its initial position and its velocity at that moment.

ANSWERS TO PRACTISE NOW

Chapter 1

Practise Now 1

1. $x = 3, y = 1$ or $x = -5\frac{1}{2}, y = 5\frac{1}{4}$; $(3, 1)$, $\left(-5\frac{1}{2}, 5\frac{1}{4}\right)$

2. $\left(13\frac{1}{2}, -6\frac{1}{2}\right)$, $(4, 3)$

3. $x = 3, y = 4$ or $x = 4, y = 3$

Practise Now 2

5 cm, 12 cm, 13 cm

Practise Now 3

(i) $8x^3 + 3x^2 - 7x + 11$, 3 **(ii)** $3x^2 - 3x - 1$, 2

Practise Now 4

(i) $5x^5 - 15x^4 + 32x^3 - 9x^2 + 21x - 18$

(ii) -100

(iii) Degree of $P(x) \times Q(x)$ = Degree of $P(x)$ + Degree of $Q(x)$

Practise Now 5

$A = 3, B = 5, C = 7$

Practise Now 6

1. $A = 3, B = 3, C = -5$ **2.** $A = 3, B = 12, C = 17$

Practise Now 7

1. $(2x^2 + 3x + 8)(2x - 3) - 2$; -2

2. $(x + 2)(x - 2)(4x + 3)$

Practise Now 8

(a) 3 **(b)** -480

(c) $-\frac{31}{8}$

Practise Now 9

1. -2 **2.** $A = 7, B = -5$

Practise Now 10

$k = -67\frac{1}{2}$; -125

Practise Now 11

$p = -7, q = 16$; $(2x + 3)(x - 1)(x - 2)^2$

Practise Now 12

1. $(x - 2)(x - 3)(x + 7)$ **2.** $(2x - 1)(x + 2)(x + 3)$

Practise Now 13

1. $-6, -\frac{1}{2}, 1$ **2.** $-5, \frac{2}{3}, 3$

Practise Now 14

1. $-7.27, 0.27, 3$ **2.** $-7.65, 0.65, 2$

Practise Now 15

1. $6x^4 - 45x^3 + 87x^2 - 45x + 6$

2. $-36 + 54x + 4x^2 - 12x^3 + 2x^4$

Practise Now 16

(a) $(6p + 7q)(36p^2 - 42pq^2 + 49q^4)$

(b) $(1 - 2x)(37 + 20x + 4x^2)$

Practise Now 17

$\frac{4}{5}$

Practise Now 18

(a) $2x^2 + 4x + 15 + \frac{35}{x - 2}$ **(b)** $2 + \frac{13 - 18x}{(3x - 1)(x + 2)}$

Practise Now 19

1. $\frac{2}{x - 3} + \frac{5}{2x + 5}$

2. $(x - 1)(x + 4)(2x - 3)$, $\frac{2}{x - 1} + \frac{4}{x + 4} - \frac{5}{2x - 3}$

Practise Now 20

1. $-\frac{15}{2(2x + 1)} + \frac{25}{2(2x + 1)^2}$ **2.** $\frac{5}{x + 2} - \frac{42}{(x + 2)^2}$

Practise Now 21

1. $\frac{1}{x + 2} + \frac{2}{(x + 2)^2} + \frac{3}{x - 11}$

2. $(x + 2)(x - 2)^2$, $\frac{15}{4(x + 2)} - \frac{11}{4(x - 2)} + \frac{5}{(x - 2)^2}$

Practise Now 22a

1. $\frac{1}{3(x+1)}+\frac{14x+22}{3(x^2+5)}$

2. $\frac{1}{16x}-\frac{x}{16(x^2+4)}$

Practise Now 22b

1. $3x+\frac{4}{x+1}-\frac{1}{x-1}$

2. $3x-2+\frac{1}{x-3}-\frac{7}{2x-5}$

Practise Now 23

(i) $\frac{1}{n}-\frac{2}{n+1}+\frac{1}{n+2}$

(ii) $\frac{2\ 025\ 077}{4\ 050\ 156}$

Chapter 2

Practise Now 1

1. (i) $\frac{1}{6}$ (ii) $\frac{25}{4}$
   (iii) $\frac{7}{2}$ (iv) $\frac{91}{8}$
2. (i) 1 (ii) $-\frac{55}{28}$
   (iii) $-\frac{83}{64}$ (iv) $5\frac{177}{256}$

Practise Now 2

1. (i) $5x^2-8x+36=0$
   (ii) $45x^2+74x+45=0$
2. (i) $9x^2+29x+25=0$
   (ii) $5x^2-x+3=0$
   (iii) $27x^2+44x+125=0$

Practise Now 3

1. 4 or $-\frac{3}{2}$
2. 8

Practise Now 5

1. -2
2. 10

Practise Now 6

1. $p>\frac{3}{2}$
2. $k>-\frac{9}{2}$

Practise Now 7

1. $h>\frac{25}{8}$
2. $p>\frac{6}{5}$

Practise Now 9

(i) $-\frac{37}{4}$ (ii) $-\frac{5}{2}$

Practise Now 10

(a) $3(x-1)^2-6$; minimum $y=-6$, $x=1$

(b) $5\left(x+\frac{4}{5}\right)^2-\frac{26}{5}$; minimum $y=-\frac{26}{5}$, $x=-\frac{4}{5}$

(c) $-2\left(x-\frac{5}{4}\right)^2+\frac{25}{8}$, maximum $y=\frac{25}{8}$, $x=\frac{5}{4}$

(d) $-4\left(x-\frac{1}{2}\right)^2$; maximum $y=0$, $x=\frac{1}{2}$

(e) $-4\left(x-\frac{1}{2}\right)^2+4$; maximum $y=4$, $x=\frac{1}{2}$

(f) $3\left(x+\frac{1}{3}\right)^2$; minimum $y=0$, $x=-\frac{1}{3}$

Practise Now 11

1. $y=-2\left(x-\frac{3}{4}\right)^2+\frac{129}{8}$; maximum $y=\frac{129}{8}$, $x=\frac{3}{4}$
2. $a=-3$, $b=1$, $c=20$; maximum $f(x)=20$, $x=1$

Practise Now 12

(a) $\left(-\frac{1}{4}, -6\frac{1}{8}\right)$, minimum point

(b) $\left(-1\frac{3}{4}, -6\frac{1}{8}\right)$, minimum point

(c) $\left(3, \frac{1}{2}\right)$, minimum point

Practise Now 13

1. $x<-\frac{1}{3}$ or $x>\frac{5}{4}$
2. $-\frac{3}{2}<x<1$
3. $-2<x<\frac{1}{3}$

Practise Now 14

1. $k\leqslant-\frac{13}{9}$ or $k\geqslant 3$
2. $1<a<8$

Practise Now 15

1. 2
2. $k>\frac{25}{8}$

Practise Now 16

1. 3
2. $p>\frac{4}{3}$

Practise Now 17

1. $k < -1$ or $k > 1$
2. $k < 4$ or $k > 8$

Practise Now 18

1. $-3 < k < 9$; -3 or 9
2. $k > \frac{19}{2}$; $\frac{19}{2}$

Practise Now 19

(a) $\frac{20}{3}$ or -2
(b) ± 3
(c) 0 or 4

Chapter 3

Practise Now 1

$1 + 7b + 21b^2 + 35b^3 + 35b^4 + 21b^5 + 7b^6 + b^7$

(a) $1 - 7x + 21x^2 - 35x^3 + 35x^4 - 21x^5 + 7x^6 - x^7$

(b) $1 + 14y + 84y^2 + 280y^3 + 560y^4 + 672y^5 + 448y^6 + 128y^7$

Practise Now 2

(a) $1 + 10x + 45x^2 + 120x^3 + \dots$

(b) $1 - 4x + 7x^2 - 7x^3 + \dots$

Practise Now 3

1. **(a)** 56 **(b)** 15
   (c) 105
2. **(a)** $\frac{n(n-1)}{2}$ **(b)** n

Practise Now 4

9

Practise Now 5

15

Practise Now 6

(a) 126
(b) 77 520

Practise Now 7

1. **(a)** $x^4 + 8x^3y + 24x^2y^2 + 32xy^3 + 16y^4$
   (b) $243a^5 - \frac{405}{2}a^4b + \frac{135}{2}a^3b^2 - \frac{15}{4}a^2b^3 + \frac{15}{16}ab^4 - \frac{1}{32}b^5$
2. $64 - 64x + \frac{80}{3}x^2 - \frac{160}{27}x^3 + \dots$
3. **(a)** $256x^{16} - 1024x^{13} + 1792x^{10} - 1792x^7 + \dots$
   (b) $65\,536 + \frac{131\,072}{x} + \frac{122\,880}{x^2} + \frac{71\,680}{x^3} + \dots$

Practise Now 8

1. $256 + 256x^2 + 112x^4 + 28x^6 + \dots$, 140
2. $32 - 144x - 240x^2 - \dots$
3. **(i)** $1 + 15x + 90x^2 + \dots$
   (ii) $243 - \frac{405}{2}x + \frac{135}{2}x^2 + \dots$, 18 900

Practise Now 9

1. 425
2. $1 + 16x + 120x^2 + 560x^3 + \dots$, -9

Practise Now 10

(a) $\frac{11\,941\,020}{x^4}$
(b) $7920y^4$

Practise Now 11

1. **(i)** $792x^9$ **(ii)** 495
2. 5
3. 8736

Practise Now 12

1. 1.104 62
2. 1.1716

Practise Now 13

0.273

Chapter 4

Practise Now 1

(a) 5
(b) $-\frac{3}{2}$
(c) ± 1
(d) -2

Practise Now 2

1. -2

2. **(i)** $u^3 + 2u - 12 = 0$ **(iii)** 0.631

Practise Now 3

1. $x = 1, y = -1$

2. $a = 2, k = 4, p = 128$

Practise Now 4

(a) 9 **(b)** 5

(c) $2\sqrt{3}$ **(d)** $\frac{\sqrt{15}}{2}$

Practise Now 5

(a) $11\sqrt{3}$ **(b)** $2\sqrt{5}$

(c) $-\sqrt{6}$

Practise Now 6

(a) $29 + 3\sqrt{3}$ **(b)** $34 - 24\sqrt{2}$

(c) -11 **(d)** $86 + 48\sqrt{3}$

Practise Now 7

1. **(a)** $4\sqrt{3}$ **(b)** $8 - 2\sqrt{5}$

(c) $\frac{2\sqrt{6}+3}{3}$ **(d)** $\frac{6\sqrt{3}-48}{11}$

2. $16 - 10\sqrt{6}$

3. $h = \frac{47}{2}, k = \frac{21}{2}$

Practise Now 8

$p = 2, q = 3$

Practise Now 9

1. $\left(\frac{19}{11} - \frac{6}{11}\sqrt{3}\right)$ cm **2.** $(33 - 18\sqrt{3})$ cm

Practise Now 10

(a) 2 **(b)** $\frac{1}{2}$

(c) 7

Practise Now 11

(a) $3 = \log_{10} 1000$ **(b)** $x = \log_4 10$

(c) $x = \log_{10} 80$ **(d)** $k = \log_a 2$

Practise Now 12

1. **(a)** $8^x = 5$ **(b)** $3^{\frac{1}{7}} = x$

(c) $4^{-2} = \frac{1}{16}$ **(d)** $a^x = y$

2. $c = x - 4y$

Practise Now 13

(a) 0.477 **(b)** 0

(c) -3 **(d)** 5.09

Practise Now 14

(a) 20.1 **(b)** 1

(c) 1.65 **(d)** $0.006\,74$

Practise Now 15a

(a) 366 **(b)** 6

(c) 1

Practise Now 15b

(a) 3 **(b)** 14

Practise Now 16a

(a) $2 \times 3, 6$ **(b)** $20 \times 20, 400$

(c) $5 \times \frac{1}{2}, \frac{5}{2}$ **(d)** mn

Practise Now 16b

(a) $\log_2 21$ **(b)** $\ln 10pq^2$

(c) no further simplification **(d)** 1

Practise Now 17

(a) 1 **(b)** $\log_a 36$

(c) no further simplification

Practise Now 18

(a) 4 **(b)** 4

(c) $\frac{1}{2}$ **(d)** 3

Practise Now 19

(a) $\log_2 56$ **(b)** $\ln 3e^2$

(c) $\log_3 16.2$

Practise Now 20

(a) 2.84 **(b)** 0.721

(c) 0.195 **(d)** –0.732

Practise Now 21

(a) 3 **(b)** 16

(c) 4 **(d)** 15

(e) $\frac{1}{3}$ **(f)** $\frac{2}{3}$

Practise Now 22

1. **(a)** 8.55 **(b)** –3.06

(c) 0.313 **(d)** 1.10

2. **(a)** $-\frac{1}{4n}$ **(b)** $\frac{1-n}{3n}$

(c) $n - 1$ **(d)** $\frac{n+1}{2}$

Practise Now 23

–3

Practise Now 24

(a) 30 **(b)** –0.169 or 0.185

Practise Now 25

1. **(a)** 64 or $\frac{1}{64}$ **(b)** 168

(c) $\frac{3}{2}$

2. **(i)** $\frac{2}{3}$ **(ii)** $\frac{4}{3}$

(iii) $\frac{1}{2}$ **(iv)** $-\frac{5}{3}$

Practise Now 26

(a) –0.604 **(b)** –0.146

(c) 1.10 **(d)** 1.70

Practise Now 27

1. $x = 5, y = 2$

2. $x = 3, y = 5$

Practise Now 28

1. **(ii)** $y = x - 3$, 1 solution

2. **(i)** Graph A: $y = 2e^x$, Graph B: $y = e^{2x}$

(ii) **(a)** no solution

(b) –0.5 or 1.5

Practise Now 29

1. **(ii)** $y = \frac{4}{3} - \frac{1}{3}x$, 1 solution **2.** no solution

Practise Now

1. **(i)** 50.0 g **(ii)** 47.6%

(iii) 21.4 years

2. **(i)** 1.52, acidic **(ii)** 2.00×10^{-13}

(iii) 0.301

Practise Now 30

(i) \$50 000 **(ii)** \$34 500

(iii) 50 months

Chapter 5

Practise Now 1

(a) **(i)** (4, 3) **(ii)** (1, 1)

(iii) (2, 5)

(b) $x = 4, y = 3$

Practise Now 2

(i) $\left(\frac{1}{2}, \frac{11}{2}\right)$ **(ii)** (–1, 4)

(iii) $AC = 5\sqrt{2}$ units, $BD = 3\sqrt{2}$ units

Practise Now 3

(a) 2 or 3

(b) –13

Practise Now 4

1. –3

2. **(i)** (3, 2) **(ii)** 12

(iii) $3\sqrt{10}$ units

Practise Now 5

1. **(a)** $y = 3x - 5$ **(b)** $y = -7x + 29$

2. $y = 2x + 2$

Practise Now 6

(i) $2y = x + 15$, $y = -2x + 40$ **(ii)** (13, 14)

(iii) $4\sqrt{5}$ units

Practise Now 7

(i) $y = \frac{5}{2}x - \frac{27}{4}$ **(ii)** $\left(\frac{27}{10}, 0\right)$

(iii) $\left(\frac{43}{10}, 4\right)$

Practise Now 8

1. $\frac{3}{2}$ units^2 **2.** 23 units^2

Practise Now 9

53 units^2

Chapter 6

Practise Now 1

(a) (0, 0), 9 **(b)** (–4, 6), 10

(c) $\left(0, -\frac{1}{3}\right)$, 4

Practise Now 2

(a) $(x + 1)^2 + (y - 3)^2 = 16$ **(b)** $(x - 4)^2 + (y + 2)^2 = 49$

Practise Now 3

1. $(x - 1)^2 + (y - 2)^2 = 10$ **2.** $(x + 3)^2 + (y - 5)^2 = 25$

Practise Now 4

(a) (3, –4), 4 **(b)** $\left(-1, \frac{3}{4}\right)$, $\frac{3}{4}$

Practise Now 5

1. $(3x - 2)^2 + (3y - 5)^2 = 185$

2. **(i)** $A(-1, 3)$, $B(-3, -1)$ **(ii)** $y = -\frac{1}{2}x$

(iii) 2

Practise Now 6

(i) $A(4, 4)$, $B(1, -2)$, $C(2, 0)$ **(ii)** $3\sqrt{5}$ units

(iii) 2 : 1

Practise Now 7

(i) $A(4, 2)$, $B(2, 0)$ **(ii)** $y = -x + 4$

Chapter 7

Practise Now 1

There may be more than one method to do this.

(a) $\frac{y}{x} = ax + b$; $X = x$, $Y = \frac{y}{x}$, $m = a$, $c = b$

(b) $y - x^3 = ax + b$; $X = x$, $Y = y - x^3$, $m = a$, $c = b$

(c) $y\sqrt{x} = ax + b$; $X = x$, $Y = y\sqrt{x}$, $m = a$, $c = b$

(d) $\frac{x}{y} = ax^2 + b$; $X = x^2$, $Y = \frac{x}{y}$, $m = a$, $c = b$

(e) $y = \frac{ky}{x} + h$; $X = \frac{y}{x}$, $Y = y$, $m = k$, $c = h$

(f) $\frac{1}{y} = \frac{q}{px} + \frac{1}{p}$; $X = \frac{1}{x}$, $Y = \frac{1}{y}$, $m = \frac{q}{p}$, $c = \frac{1}{p}$

Practise Now 2

There may be more than one method to do this.

(a) $\ln y = qx + \ln p$; $X = x$, $Y = \ln y$, $m = q$, $c = \ln p$

(b) $\lg y = -n \lg x + \lg a$; $X = \lg x$, $Y = \lg y$, $m = -n$, $c = \lg a$

(c) $\lg y = -x \lg b + \lg a$; $X = x$, $Y = \lg y$, $m = -\lg b$, $c = \lg a$

(d) $\lg y = -\frac{q}{p} \lg x + \frac{1}{p}$, $X = \lg x$, $Y = \lg y$, $m = -\frac{q}{p}$, $c = \frac{1}{p}$

(e) $\lg (y - x^3) = x \lg a - h \lg a$; $X = x$, $Y = \lg (y - x^3)$, $m = \lg a$, $c = -h \lg a$

(f) $\lg y = \frac{2}{n} \lg x + \frac{\lg h}{n}$, $X = \lg x$, $Y = \lg y$, $m = \frac{2}{n}$, $c = \frac{\lg h}{n}$

Practise Now 3

1. (i) $a = 3, b = 5$ (ii) $\frac{1}{2}$
2. (i) $k = 1000, n = -\frac{1}{2}$ (ii) 100

Practise Now 4

(i) $a \approx -2.4, b \approx -7.4$ (ii) 7.6

(iii) 2.65

Practise Now 5

(ii) $a \approx 1.9, n \approx 0.5$

(iii) (a) $T = 2\pi\sqrt{\frac{\cos\theta}{g}}\ L^n$ (b) 30.6°

Chapter 8

Practise Now 1

(a) $\frac{2\sqrt{3}}{3}$ (b) $-\frac{\sqrt{2}+2}{4}$

(c) $\frac{3\sqrt{2}}{8}$

Practise Now 2

(a) 70° (b) 20°

(c) $\frac{\pi}{3}$ (d) $\frac{\pi}{5}$

Practise Now 3

(a) 325° (b) $\frac{7\pi}{4}$

(c) –495° (d) $-\frac{10\pi}{3}$

Practise Now 4

(a) $\frac{\sqrt{3}}{3}$ (b) $-\frac{1}{2}$

(c) $-\frac{\sqrt{2}}{2}$ (d) $-\sqrt{3}$

Practise Now 5

1. $-\frac{4}{5}, -\frac{3}{4}$
2. $-\frac{5}{13}, \frac{12}{13}$

Practise Now 6

(a) $-\frac{3}{5}$ (b) $\frac{3}{4}$

(c) $-\frac{\sqrt{10}}{10}$ (d) $-\frac{3\sqrt{10}}{10}$

Practise Now 7

(a) $-\sqrt{1-a^2}$ (b) $\frac{a}{\sqrt{1-a^2}}$

(c) $-a$ (d) $\sqrt{1-a^2}$

(e) $\frac{a}{\sqrt{1-a^2}}$ (f) $\sqrt{1-a^2}$

Practise Now 8

(a) –sin 80°, cos 80°, –tan 80°

(b) –sin 80°, –cos 80°, tan 80°

(c) sin 75°, –cos 75°, –tan 75°

(d) sin 60°, cos 60°, tan 60°

Practise Now 9

(i) 2 amperes, $\frac{1}{2}$ minute (ii) 2 amperes, –2 amperes

Practise Now 10

(i) 5, –1 (ii) 3, 2π, $-1 \leqslant y \leqslant 5$

Practise Now 11

(i) 360°

Practise Now 12

(i) $a = 4, b = 2$ (ii) 4, 360°

Practise Now 13

(a) 4 (b) 4

(c) 2

Practise Now 14

(i) 35 m (ii) 5 m

Practise Now 15

1. $0 \le y \le 1$
2. $0 \le y \le 2$

Practise Now 16

1. 5
2. 3

Practise Now 17

1. 4

2. 2

Practise Now 18

1. **(i)** $\frac{3}{2}$ **(ii)** $\frac{3\sqrt{5}}{5}$

(iii) $\frac{\sqrt{5}}{2}$

2. **(i)** $-\frac{5}{3}$ **(ii)** $\frac{5}{4}$

(iii) $-\frac{3}{4}$

Practise Now 19

1. 144.1°, 215.9°

2. 25.5°, 154.5°

Practise Now 20

1. 1.89, –1.25

2. 2.42

Practise Now 21

1. **(a)** 97.4°, 132.6° **(b)** 6.13

2. 60°, 120°, 240°, 300°

Practise Now 22

1. 90°, 150°, 270°, 330°

2. $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

Practise Now 23

1. 30°, 90°, 150°, 270°

2. $0, \frac{\pi}{4}, \pi, \frac{5\pi}{4}, 2\pi$

Chapter 9

Practise Now 1

1. 0.723, 5.56

2. 2.30, π, 3.98

Practise Now 2

1. 37.8°, 142.2°, 217.8°, 322.2°

2. $\frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

Practise Now 6

(a) $\frac{\sqrt{6}+\sqrt{2}}{4}$ **(b)** $\frac{\sqrt{2}}{2}$

(c) $\frac{\sqrt{3}}{3}$

Practise Now 7

1. **(i)** $-\frac{56}{65}$ **(ii)** $\frac{63}{65}$

(iii) $-\frac{56}{33}$

2. **(i)** $-\frac{24}{7}$ **(ii)** $\frac{7}{25}$

(iii) 0

Practise Now 8

(i) $\frac{p+q}{\sqrt{(1+p^2)(1+q^2)}}$ **(ii)** $\frac{1+pq}{\sqrt{(1+p^2)(1+q^2)}}$

(iii) $\frac{p+q}{1-pq}$

Practise Now 9

1. **(i)** $\frac{5}{8}$ **(ii)** $\frac{3}{8}$

(iii) $\frac{5}{3}$

2. $\tan A \tan B = \frac{1}{7}$

Practise Now 10

1. **(i)** $\frac{7}{25}$ **(ii)** $-\frac{120}{169}$

(iii) 5

2. **(i)** $\frac{84}{85}$ **(ii)** $-\frac{\sqrt{5}}{5}$

Practise Now 11

1. **(i)** $-\frac{4}{5}$ **(ii)** $\frac{\sqrt{10}}{10}$

(iii) $-\frac{13\sqrt{10}}{50}$

2. **(i)** $\frac{1}{5}$ **(ii)** $\frac{120}{169}$

Practise Now 12

(i) $2a\sqrt{1-a^2}$ **(ii)** $1 - 8a^2 + 8a^4$

(iii) $\frac{1+\sqrt{1-a^2}}{2}$

Practise Now 13

1. (a) 48.6°, 90°, 131.4°, 270°

   (b) 0°, 48.6°, 131.4°, 180°, 360°

2. (a) 1.99, 4.30

   (b) $0, \frac{\pi}{6}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{11\pi}{6}, 2\pi$

Practise Now 17

1. $\sqrt{97}\sin(\theta + 66.03°)$; 104.6°, 303.3°

2. $\sqrt{74}\cos(\theta + 0.9505)$; 0.410, 3.97

Practise Now 18

$\sqrt{113}\sin(\theta - 41.19°)$; $\sqrt{113}$, 131.2°; $-\sqrt{113}$, 311.2°

Practise Now 19

(ii) $\sqrt{193}\cos(\theta - 30.26°)$ (iii) $\sqrt{193}$, 30.3°

Chapter 10

Practise Now 3

2. (ii) $\triangle ADR$ (iii) $\triangle QCP$

Practise Now 6

1. 24°

2. (i) 52° (ii) 36°

Chapter 11

Practise Now 1

(a) $-\frac{1}{x^2}$ (b) $\frac{1}{3\sqrt[3]{x^2}}$

Practise Now 2

1. (a) $-\frac{4}{3x^4} + a$ (b) $-\frac{3}{x^2} + \frac{1}{3\sqrt[3]{x^2}}$

2. $-\frac{4}{\sqrt[3]{t^2}} + \frac{3}{2\sqrt{t}}$

Practise Now 3

1. 28 2. $\frac{1}{6}$

Practise Now 4

1. $a = 8, b = 3$ 2. $a = 5, b = -6$

Practise Now 5

1. (a) $84x^2(4x^3 - 1)^6$ (b) $\frac{4(2x+3)}{\sqrt{x^2 + 3x - 4}}$

   (c) $\frac{360x^3}{\left(2 - 5x^4\right)^4}$

2. 18

Practise Now 6

1. (a) $8(7x - 1)(2x + 7)^5$ (b) $\frac{15 - 16v - 25v^2}{2\sqrt{5v + 4}}$

2. (a) $\frac{3(a + bx^2)}{(a - bx^2)^2}$

   (b) $2(5ax - 3x^3)^3(10a^3 - 18a^2x^2 - 15ax^2 + 21x^4)$

Practise Now 7

1. $\frac{2}{3}, -\frac{5}{2}, -\frac{16}{33}$

2. $\frac{3}{2}, -5, -\frac{15}{16}$

Practise Now 8

1. (a) $-\frac{6}{(4x-3)^2}$ (b) $-\frac{7}{(3x-7)^2}$

2. (a) $\frac{6x^3 + 6x^2 + 6}{(3x+1)^3}$ (b) $\frac{8x - 16}{3(2x-5)^{\frac{4}{3}}}$

   (c) $\frac{-6x^2 - 3x - 14}{\sqrt{1 + 4x}\sqrt{\left(3x^2 - 7\right)^3}}$

3. $\frac{35}{3757}$

Practise Now 9

1. $\frac{2 - 2x^2}{\left(x^2 + 1\right)^2}$, $x < -1$ 2. $\frac{5 + 4x - x^2}{\left(x^2 + 5\right)^2}$, $2 < x < 5$

Practise Now 10

1. $y = 3x - 6$, $3y + x = -8$

2. $4y + 6x = 55$, $6y - 4x = 63$

3. $9y = 5x - 13$, $15y + 27x = 49$

Practise Now 11

1. (i) $(2, -6)$ (ii) $y = x - 8$
   (iii) $(1, -5)$
2. $y = -\frac{3}{4}x + 2$, $A\left(2\frac{2}{3}, 0\right)$, $B(0, 2)$; $\left(1\frac{1}{3}, 1\right)$; No

Practise Now 12

1. 4
2. 33.5

Practise Now 13

(i) 9.6π cm^2/s (ii) 57.6π cm^3/s

Practise Now 14

(i) $\frac{1}{32\pi}$ cm/s (ii) 5 cm^2/s

Chapter 12

Practise Now 1

1. (i) $15x^2 - 6x$, $30x - 6$ (ii) $x > \frac{2}{5}$
2. $-\frac{1}{3}$, 1

Practise Now 2

$\frac{3 - 3x^2}{\left(x^2 + 1\right)^2}$, $x < -1$ or $x > 1$

Practise Now 3

$0 \leqslant x < 2$

Practise Now 4

$(3, -22)$, minimum point; $(-1, 10)$, maximum point

Practise Now 5

$(0, 0)$, point of inflexion; $\left(\frac{3}{4}, -\frac{27}{256}\right)$, minimum point

Practise Now 6

$(2, -14)$, minimum point; $(-1, 13)$, maximum point

Practise Now 7

$(3, 1)$, point of inflexion; $\left(\frac{3}{4}, -16\frac{11}{128}\right)$, minimum point

Practise Now 8

-31.4, $x = 1.6$

Practise Now 9

$\theta = \frac{7 - 2r}{r}$, $3\frac{1}{16}$ cm^2

Practise Now 10

(iii) $r = 2.8$, minimum, $A = 65.1$ cm^2

Practise Now 11

5 m × 5 m × 10 m

Practise Now 12

$\sqrt{(22 - 4t)^2 + (14 - 3t)^2}$, $t = 5.2$, 2 m

Chapter 13

Practise Now 1

1. (a) $3\sec^2 x - 2\cos x$ (b) $3x^4(5\sin x + x\cos x)$
   (c) $-\frac{10\cos x + 2(5x + 1)\sin x}{(5x + 1)^2}$
   (d) $6(3x + 5\cos x)^5(3 - 5\sin x)$
2. $a = 1$, $b = -1$

Practise Now 2

1. (a) $-10\sin\left(2x + \frac{\pi}{4}\right)$
   (b) $12x\sec^2\left(3x - \frac{\pi}{3}\right) + 4\tan\left(3x - \frac{\pi}{3}\right)$
   (c) $\frac{10\sin 4x\cos 5x - 8\sin 5x\cos 4x}{\sin^2 4x}$
2. $\sec\pi x\,(\pi x\tan\pi x + 1)$

Practise Now 3

(a) $10\sin^4 x\cos x - 20\tan\left(2x + \frac{\pi}{2}\right)\sec^2\left(2x + \frac{\pi}{2}\right)$

(b) $36\cos^2\left(\frac{\pi}{2} - 4x\right)\sin\left(\frac{\pi}{2} - 4x\right)$

Practise Now 4

1. 0.207 **2.** –3

Practise Now 5

1. **(a)** $\dfrac{5}{5x+2}$ **(b)** $4 \cot 4x$

(c) $\dfrac{3x^2+2}{3(x^3+2x)}$ **(d)** $\dfrac{5x^4}{(x^5+4)\ln 7}$

(e) $\dfrac{1}{x} - \dfrac{x}{x^2+2}$

2. $\dfrac{15x^2+2}{5x^3+2x}$

Practise Now 6

1. 2.64 **2.** 0.198

Practise Now 7

(a) $15e^{5x}$ **(b)** $-4e^{5-2x}$

(c) $(6x-5)e^{3x^2-5x}$

(d) $2e^{5+2x}(\cos 2x - \sin 2x)$

(e) $\dfrac{e^{2+5x}(5x-1)}{3x^2}$

Practise Now 8

96

Practise Now 9

1. $y = 3x + \dfrac{\pi}{12} + \dfrac{\sqrt{3}}{2}$, $y = -\dfrac{1}{3}x + \dfrac{13\pi}{36} + \dfrac{\sqrt{3}}{2}$

2. 5 **3.** (1, 1), $y = -2x + 3$

Practise Now 10

1. **(i)** $\dfrac{23}{(3x-4)(2x+5)}$ **(ii)** $1\dfrac{22}{23}$ units/s

2. $\dfrac{\sqrt{3}}{5}$ units/s

3. $3(1-2x)e^{-2x} - 15e^{-3x}$, –10.0 units/s

Practise Now 11

1. $3 + 3\ln x$, $\left(\dfrac{1}{e}, -\dfrac{3}{e}\right)$ **2.** 1.02, maximum

Practise Now 12

1.13, maximum

Chapter 14

Practise Now 1

(a) $\dfrac{1}{4}x^4 + c$ **(b)** $\dfrac{2}{3}\sqrt{x^3} + c$

Practise Now 2

1. **(a)** $-\dfrac{2}{9x^2} + c$ **(b)** $3x + c$

(c) $\dfrac{1}{2}ax^2 + c$

2. $9\sqrt[3]{t^2} + c$

Practise Now 3

1. **(a)** $x^3 - \dfrac{21}{2}x^2 + c$ **(b)** $\dfrac{5}{3}u^3 + 4\sqrt{u} + c$

(c) $\dfrac{x^3}{12} + \dfrac{3x}{4} - \dfrac{9}{16x} + c$

2. $y = \dfrac{4}{3}x^3 - \dfrac{3}{2}x^2 + 2x - \dfrac{5}{6}$

Practise Now 4

$R = \dfrac{12}{5}x^{\frac{5}{3}} + \dfrac{1}{2}x^2$, \$740 000

Practise Now 5

(a) $\dfrac{1}{60}(5x+4)^{12} + c$ **(b)** $\dfrac{4}{3}\sqrt{3x+5} + c$

(c) $-\dfrac{4}{3(4x-7)^3} + c$

Practise Now 6

(a) $-\dfrac{5}{4}\cos 4x + \dfrac{3}{2}\sin\dfrac{1}{2}x + c$ **(b)** $\dfrac{1}{3}\tan 3x - x + c$

(c) $\dfrac{3}{2}x^2 + \dfrac{2}{21}\tan 6x + c$

Practise Now 7

1. **(a)** $\dfrac{2}{3}\sqrt{x^3} + \dfrac{1}{5}\sin(5x+3) + c$

(b) $2x - \dfrac{1}{3}\cos(3x+\pi) + c$

(c) $\dfrac{1}{2}\tan\left(2x + \dfrac{\pi}{2}\right) - x + c$

2. $y = \dfrac{x^2}{4} + \cos 2x - 1$

Practise Now 8

(a) $\frac{1}{4}\ln x + c$ **(b)** $\frac{1}{5}\ln(5x+3) + c$

(c) $\frac{3}{7}\ln(7x+1) + c$ **(d)** $-\frac{5}{9}\ln(6-9x) + c$

Practise Now 9

1. **(a)** $\frac{1}{2}e^{2x+\pi} + c$ **(b)** $-\frac{1}{4}e^{3-4x} + c$

(c) $x - \frac{1}{3e^{3x}} + c$

2. $y = \frac{2}{5}x^{\frac{5}{2}} - 2\sqrt{x} + \frac{3}{2}e^{-\frac{2}{3}x} - 1$, -1.83

Practise Now 10

$2\theta + 2\sin\theta + c$

Practise Now 11

$-\frac{3}{6x+1}, -\frac{3}{7}\ln\frac{7}{\sqrt{6x+1}} + c$

Practise Now 12

1. $\frac{1}{2}x\sqrt{4+3x^2} + c$ **2.** $\frac{2x}{\sqrt{1-x^2}} + c$

Practise Now 13

1. $\frac{2}{x-1} + \frac{3}{2x+3}$, $2\ln(x-1) + \frac{3}{2}\ln(2x+3) + c$

2. $\frac{6x-1}{(3x-2)^2}$, $\frac{1}{3}\ln(3x-2) - \frac{1}{2(3x-2)} + c$

Practise Now 14

1. $-2e^x\sin x$, $-\frac{5}{2}e^x(\cos x - \sin x) + c$

2. **(i)** $-2te^{-t^2}$

(ii) $P(t) = -\frac{3}{2}e^{-t^2} + 20\ 000 + \frac{3}{2e^4}$

Chapter 15

Practise Now 1

(a) $17\frac{1}{3}$ **(b)** 10

(c) $1\frac{77}{81}$

Practise Now 2

(a) 78 **(b)** 37.8

Practise Now 3

(a) $\frac{3}{2}$ **(b)** $\frac{\sqrt{2}}{6} + 2 - \frac{\pi}{2}$

Practise Now 4

(a) 2.92 **(b)** 16.6

Practise Now 5

1. **(a)** 19.09 **(b)** 20.09

2. 1.108

Practise Now 6

(a) -10 **(b)** 8

Practise Now 7

$\frac{15}{2x+3} - \frac{1}{x+1}$, 6.85

Practise Now 8

1. 13.76 **2.** $\frac{12}{13}$

Practise Now 9

1. 2 **2.** $\sqrt{3}+1$

Practise Now 10

1. $4xe^{2x}$, 164.79 **2.** $4xe^{2x^2}$, 22.3

Practise Now 11

1. $1 + \ln x$, 2.55

2. $5x^2 + 15x^2 \ln x$, 48.99

Practise Now 12

1. 51 units^2

2. $(\sqrt{2} - 1)$ units^2

Practise Now 13

1. 30.8 units^2

2. 6.904 units^2

Practise Now 14

1. 8 units^2

2. $4\frac{1}{2}$ units^2

Practise Now 15

$4\frac{1}{2}$ units^2

Practise Now 16

16

Practise Now 17

1. $1\frac{1}{3}$ units^2

2. **(i)** $P(0, 17), Q(8.5, 0), R(3, 11)$

(ii) $45\frac{1}{4}$ units^2

(iii) $\frac{181}{108}$

Practise Now 18

(i) (1, 1)

(ii) $1\frac{1}{6}$ units^2

Chapter 16

Practise Now 1

(i) 36 m s^{-1}

(ii) 42 m s^{-2}

(iii) 1 or 1.5

(iv) 8.5 m

(v) 8 m

Practise Now 2

(i) 6.53 cm

(ii) 7 cm s^{-1}

(iii) –14.8 cm s^{-2}

Practise Now 3

(i) 40 m

(ii) 10 m s^{-1}, –10 m s^{-1}

(iii) 0.764, 5.24

Practise Now 4

(i) 0.405 s

(ii) 58.8 m

Practise Now 5

(i) $s = \frac{4}{3}t^3 - t^2 + 2t$

(ii) 33 m

(iii) 264 m

Practise Now 6

(i) 5.31 m, 5.20 m s^{-2}

(ii) $\frac{\pi}{3}$ s

(iii) 5.27 m s^{-2}

Practise Now 7

(i) 3.38 m s^{-2}

(ii) 17.9 m

Practise Now 8

(i) $v = 2t^2 - 16t + 30$

(iii) 36 cm

(iv) 36 cm

(v) $2\frac{2}{3}$ cm

ANSWERS TO EXERCISES

Exercise 1A

1. **(a)** $(2, 1)$ **(b)** $(1, 1)$ or $\left(2\frac{1}{3}, -\frac{1}{3}\right)$
(c) $(-1, 1)$ or $(5, -3)$ **(d)** $(3, 5)$ or $\left(5\frac{2}{5}, -2\frac{1}{5}\right)$
(e) $\left(2\frac{1}{6}, -\frac{5}{6}\right)$ **(f)** $(1, 3)$ or $\left(1\frac{1}{3}, 1\frac{2}{3}\right)$
(g) $(2, 3)$ or $\left(-8\frac{1}{2}, 10\right)$
(h) $(9, 1)$ or $\left(-2\frac{2}{17}, -\frac{10}{17}\right)$

2. **(a)** $(3, 4)$ or $\left(\frac{3}{7}, \frac{1}{7}\right)$
(b) $\left(\frac{1}{2}, \frac{1}{3}\right)$ or $\left(\frac{4}{13}, \frac{6}{13}\right)$

3. 9

4. 7 cm, 12 cm

5. $x + y = 3$, $xy = 1.25$, $\left(\frac{1}{2}, 2\frac{1}{2}\right)$ or $\left(2\frac{1}{2}, \frac{1}{2}\right)$

6. $a = 2\frac{1}{4}$, $b = -\frac{1}{8}$ or $a = -2$, $b = 2$

7. $2\frac{3}{4}$, $1\frac{1}{4}$

8. $(7, -2)$, $\left(-3\frac{4}{5}, 5\frac{1}{5}\right)$

Exercise 1B

1. **(a)** $x^3 + 8x^2 - 4x + 9$ **(b)** $3x^3 - 7x^2 + 6x + 16$
(c) $x^3 - x^2 + 10x + 6$ **(d)** $3x^4 - 3x^3 + 6x^2 - x + 2$

2. **(a)** $2x^2 + 4x - 6$ **(b)** $-2x^3 - 2x^2 + 15x$
(c) $x^3 - 4x^2 - 5x + 20$
(d) $x^5 - 7x^4 + 5x^3 - 3x^2 + 14x - 7$

3. **(a)** $14x^2 + 13x - 12$
(b) $30x^2 + 37x - 7$
(c) $4x^3 - 27x + 5$
(d) $2x^3 + 7x^2y + xy + 3xy^2 + 3y^2$
(e) $9x^3 + 9x^2y - 7xy^2 + 5y^3$

4. **(a)** $9x^2 + 15x - 40$
(b) $7x^2 - 6x - 15$
(c) $7x^3 - 17x^2 + 7x - 1$
(d) $-x^3 - 9x^2y + 11xy^2 - 2y^3$
(e) $5x^4 - 11x^3 - 10x^2 + 20x + 33$

5. **(a)** $5x + 12$, 4 **(b)** $4x + 21$, 13
(c) $3x^2 - 3x + 7$, -12 **(d)** $x^2 + 4x - 16$, 48
(e) $5x^2 + x - 7$, 15 **(f)** $3x^2 + x + 1$, 4
(g) $6x^2 + 3x + 2$, -1 **(h)** $3x^2 + 9x + 27$, 74
(i) $x + 2$, $-2x + 5$ **(j)** $2x - 1$, $3x + 5$
(k) $5x^2 + 25x + 112$, $492x - 340$

6. $(2x-1)\left(-\frac{1}{2}x^2 + \frac{13}{4}x - \frac{3}{8}\right) - \frac{99}{8}$

7. **(a)** $A = 3$, $B = -2$ **(b)** $A = 7$, $B = -3$
(c) $A = 5$, $B = 6$

8. **(a)** $A = 4$, $B = -5$, $C = 0$
(b) $A = 2$, $B = -3$, $C = 14$
(c) $A = 3$, $B = 2$, $C = 6$ or $A = 2$, $B = 3$, $C = 6$
(d) $A = 2$, $B = -3$, $C = 5$

9. $A = 3$, $B = -1$, $C = 6$

10. $a = -3$, $b = -3$, $c = 3$

Exercise 1C

1. **(a)** 6 **(b)** -20
(c) -38 **(d)** $-8\frac{3}{4}$
(e) 51 **(f)** $5\frac{1}{4}$

2. **(a)** $\frac{1}{2}$ **(b)** 4
(c) -5 **(d)** -8

3. $k = -17$ **4.** $h = -1\frac{1}{2}$, $k = -4\frac{1}{2}$

5. $k = 0$, 4 or -4 **6.** $p = 3$, $q = -8$

7. $p = -5\frac{1}{2}$, $q = -1\frac{1}{2}$ **8.** -7

9. $p = -8$, $q = 8$ **10.** $p = \dfrac{k-b}{h-a}$

Exercise 1D

1. **(a)** 15 **(b)** $4\frac{1}{2}$
(c) -16 **(d)** $-5\frac{7}{8}$
(e) 1 or 2 **(f)** 11

2. $k = 3$; 4 **3.** $1\frac{1}{2}$ or -15

4. $p = -6$, $q = 4\frac{1}{2}$ **5.** $p = -1$, $q = 2$

6. $h = -16$, $k = 12$; $(x - 2)(x - 1)(x + 6)$

7. $k = 2$, $(2x + 1)$, $(x + 4)$

8. $p = 10$, $q = -90$,
$(x - 3)(x + 1)(x + 3)(x + 9)$

9. $a = 7$, $b = 24$, $c = 9$

10. $p = -5$, $q = -6$, $(x - 1)$, $(x - 2)$

Exercise 1E

1. **(a)** $(x - 1)(x + 1)(x - 5)$
(b) $(2x - 3)(x - 2)(x + 4)$
(c) $(x - 2)(x - 5)(2x + 3)$
(d) $(x - 2)(x - 3)(x - 4)$
(e) $(x - 2)(x + 3)(2x - 1)$
(f) $(x - 3)(x - 5)(x + 2)$
(g) $(2x - 1)(2x + 1)(3x - 2)$

2. **(a)** $(x - 2)(x^2 + 2x + 4)$
(b) $(5x - 4y)(25x^2 + 20xy + 16y^2)$
(c) $(3x^2 + 2y^3)(9x^4 - 6x^2y^3 + 4y^6)$
(d) $(7a + 6x^2)(49a^2 - 42ax^2 + 36x^4)$
(e) $(10 - 3x)(9x^2 - 6x + 28)$

3. **(a)** 1, -1 or -2 **(b)** 2, 3 or -2
(c) 2, $\frac{3}{4}$ or -3 **(d)** $\frac{1}{3}$, -2 or 7
(e) $\frac{1}{5}$, 2 or 3

4. **(a)** 2, 7.41 or –0.41 **(b)** 3, 4.79 or 0.21
(c) 4, 5.24 or 0.76 **(d)** $\frac{1}{2}$, 3.19 or –2.19
(e) $-\frac{1}{3}$, 7.32 or 0.68 **(f)** $\frac{1}{2}$, 2.78 or 0.72
(g) 1, 2, –2 or –3 **(h)** $\frac{1}{2}$, 2, 3 or 4
5. $f(x) = 3x^4 - 27x^3 + 24x^2 + 156x - 48$
6. $f(x) = -3 + 16x + 31x^2 - 24x^3 + 4x^4$
7. 10 cm, 12 cm
8. $n = 2, k = 81; (x^2 + 9)(x + 3)(x - 3)$
9. $n = 2; (x - 1)(x + 1)(x - 2)(x + 2)$
10. $a = 1$
11. **(ii)** $y = 0.05x^3 - 0.55x^2 + 2x + 57.6$
(iii) 88.8 m^3

Exercise 1F

1. **(a)** $4 + \frac{7}{2x - 1}$ **(b)** $5x + \frac{5x + 3}{3x^2 - 1}$
(c) $7 - \frac{46}{2x^2 + 7}$
2. **(a)** $15 + \frac{68x + 33}{x^2 - 5x - 2}$ **(b)** $2x - 1 + \frac{7x - 4}{(2x - 3)(x + 1)}$
(c) $3x - 1 + \frac{3x - 3}{x^2 - 9}$
3. $5x + 13 + \frac{x - 78}{x^2 - 2x + 5}$
4. $a = 3, b = 1, c = -15, d = -18$
5. $a = 2, b = 4, c = -3, d = -4$
6. $a = 6, b = 7, c = 3, d = -17$
7. $a = 3, b = 4, c = 1, d = -1$
8. $a = 3, b = -1, c = -7, d = -14$

Exercise 1G

1. **(a)** $\frac{3}{x} + \frac{4}{x + 2}$ **(b)** $\frac{2}{3x} + \frac{3}{x - 1}$
(c) $\frac{2}{x + 3} + \frac{5}{x - 2}$ **(d)** $\frac{7}{x - 1} - \frac{6}{3x + 1}$
2. **(a)** $\frac{1}{x - 1} + \frac{1}{(x - 1)^2}$ **(b)** $\frac{1}{x + 4} - \frac{1}{(x + 4)^2}$
(c) $\frac{2}{x} + \frac{3}{x - 1} + \frac{4}{(x - 1)^2}$
(d) $\frac{3}{x} + \frac{5}{x + 2} - \frac{7}{(x + 2)^2}$
(e) $\frac{2}{x + 1} - \frac{2}{x - 2} - \frac{14}{(x - 2)^2}$
(f) $\frac{5}{2x + 3} - \frac{1}{3x - 1} + \frac{1}{(3x - 1)^2}$
3. **(a)** $\frac{5}{x} + \frac{3}{x^2 + 4}$ **(b)** $\frac{2}{x + 1} + \frac{3x - 2}{x^2 + 3}$
(c) $\frac{27}{8(x + 3)} + \frac{9 - 11x}{8(x^2 + 7)}$ **(d)** $\frac{2}{3x - 2} - \frac{4x + 5}{2x^2 + 9}$

4. **(a)** $2 + \frac{2}{x - 2} - \frac{3}{x - 3}$
(b) $2x + 1 + \frac{2}{x + 1} - \frac{5}{x - 4}$
(c) $4 + \frac{23}{9(x + 2)} - \frac{14}{9(x - 1)} + \frac{11}{3(x - 1)^2}$
(d) $3x + 2 + \frac{185}{29(2x - 3)} - \frac{223x + 30}{29(x^2 + 5)}$
5. $a = 2, b = 5, c = \frac{5}{2}, d = \frac{19}{2}$
6. $3x - 2 + \frac{x - 27}{6x^2 - x - 15}$; $3x - 2 + \frac{3}{2x + 3} - \frac{4}{3x - 5}$
7. $(x + 5)(x - 5)(2x + 3)$; $\frac{1}{x + 5} + \frac{2}{x - 5} - \frac{3}{2x + 3}$
8. $(2x - 1)(4x + 3)(x - 3)$; $\frac{2}{x - 3} + \frac{6}{4x + 3} - \frac{7}{2x - 1}$
9. **(i)** $h = -2, k = 12, f(x) = (x^2 + 4)(3x - 2)$
(ii) $\frac{4x + 24}{f(x)} = \frac{6}{3x - 2} - \frac{2x}{x^2 + 4}$
10. **(i)** $\frac{10}{n} - \frac{10}{n + 2}$ **(ii)** $15 - \frac{10}{n + 1} - \frac{10}{n + 2}$

Review Exercise 1

1. **(a)** $x = -1, y = 3$ or $x = \frac{2}{3}, y = 2\frac{1}{6}$
(b) $x = 6, y = 6$ or $x = 1\frac{5}{19}, y = -1\frac{17}{19}$
(c) $x = 8, y = 2\frac{1}{2}$ or $x = -2\frac{1}{2}, y = -8$
(d) $x = \frac{1}{6}, y = \frac{1}{4}$ or $x = -\frac{1}{4}, y = -\frac{1}{6}$
2. (3, –1), (6, 3)
3. **(a)** $A = 4, B = 7, C = -5$
(b) $A = 7, B = -28, C = 6$
(c) $A = 2, B = -5, C = 10$
(d) $A = 4, B = -13, C = -8$
4. **(a)** –5 **(b)** 61
(c) –9 **(d)** 10
(e) 3 **(f)** 1 or 5
5. **(a)** $(x + 1)(2x - 3)(x - 3)$
(b) $(x - 3)^2(x + 4)$
(c) $(2x + 1)(x + 4)(3x - 4)$
(d) $(2x - 1)(2x + 5)(3x + 1)$
(e) $(2x^2 - 7)(4x^4 + 14x^2 + 49)$
(f) $(3a + 4y^2)(9a^2 - 12ay^2 + 16y^4)$
(g) $2(4 - x)(28x^2 - 26x + 19)$
(h) $2(2x + 3)(7x^2 + 24x + 21)$
6. **(a)** $2, 2\frac{1}{2}, -3$ **(b)** $\frac{2}{3}, -2, -\frac{1}{2}$
(c) $1\frac{1}{2}, -2, 4$ **(d)** $2, 4, -\frac{1}{4}, -\frac{1}{2}$
(e) 2, 5.16, –1.16 **(f)** $\frac{1}{2}$, 0.48, 14.52
(g) $\frac{1}{4}$, 8.14, 0.86 **(h)** $\frac{1}{2}$, 1.36, –2.69
7. $a = 4; 6$
8. $p = 1, q = -4, (3x + 1), (x + 1)$

9. $p = -4, q = -4, r = 4$
10. $p = -4, q = -14,$
$(x + 1)(2x - 3)(2x + 1)(2x - 1)$
11. $k = 1, A = 17; (x + 5)(x - 1)(x - 3)$
12. **(a)** $\frac{2}{5(2x-1)} + \frac{24}{5(x+2)}$
(b) $\frac{1}{3-x} + \frac{3}{2(2x-1)} + \frac{13}{2(2x-1)^2}$
(c) $\frac{x+2}{x^2+1} - \frac{2}{2x+1}$
(d) $\frac{5}{2x+3} + \frac{2}{x-1} + \frac{5}{(x-1)^2}$
(e) $1 + \frac{2}{x-3} + \frac{1}{x+4}$
(f) $\frac{1}{4x} - \frac{9}{8(x+2)} + \frac{7}{8(x-2)}$
13. $a = 1, b = -3, c = 2$
14. $a = 1, b = -1, c = -3$

Chapter 1 — Challenge Yourself

1. $-2, -1$ or $\frac{3}{2}$
2. $\frac{1}{x-2} + \frac{2}{x+3}$

Exercise 2A

1. **(a)** $-\frac{5}{3}, -\frac{7}{3}$ **(b)** $\frac{5}{2}, 1$
(c) $0, -\frac{5}{2}$ **(d)** $-\frac{7}{3}, 0$
(e) $\frac{2}{k}, \frac{1}{k}$ **(f)** $3, -7$
(g) $2, -5$ **(h)** $-1, -\frac{7}{2}$
(i) $\frac{3}{2}, 1$ **(j)** $-1, -4$
2. $2x^2 - 11x + 18 = 0$
3. **(i)** $x^2 + 5x - 14 = 0$ **(ii)** $7x^2 - 5x - 2 = 0$
(iii) $4x^2 - 53x + 49 = 0$ **(iv)** $8x^2 + 335x - 343 = 0$
4. **(i)** 4 **(ii)** $5\frac{7}{9}$
(iii) 13 **(iv)** $8\frac{2}{3}$
(v) $32\frac{40}{81}$
5. $9x^2 + 22x + 9 = 0$ **6.** -6 or -14
7. **(i)** $\frac{3}{4}$ **(ii)** ± 12
8. $\alpha = \frac{3}{4}, k = -18$ or $\alpha = -\frac{3}{4}, k = 18$
9. $h = \frac{12}{13}, k = 4$
10. **(i)** 14 **(ii)** $3x^2 - 42x - 13 = 0$
11. **(a)** 7 **(b)** 1
12. $\alpha = 2, k = 10$

13. $k = \frac{1}{3}, h = -75$ or $k = -\frac{1}{3}, h = -87$
16. $h = 16 - \frac{9}{4}k^2$
17. **(i)** $2, \frac{1}{3}$ **(ii)** $p = -6, q = 3$
18. $p = -11, q = 18$ **19.** $h = -102, k = 95$
20. **(i)** $h = 6\alpha$ **(ii)** $\alpha = 1, 2$ or -3

Exercise 2B

1. **(a)** no real roots **(b)** no real roots
(c) real and equal **(d)** real and distinct
(e) real and distinct **(f)** no real roots
(g) real and distinct **(h)** real and distinct
2. ± 6 **3.** 1 or 8
4. $k < 0$ **5.** $\frac{1}{2}$ or $-3\frac{2}{7}$
6. $a > 0$ **7.** $k > 4\frac{1}{12}$
8. $p < 6\frac{1}{4}$ **9.** $k < -2\frac{49}{60}$
10. $k \geqslant 1\frac{2}{5}$ **11.** $p < -3\frac{1}{3}$

Exercise 2C

1. **(a)** -4, min, $\frac{2}{5}$ **(b)** 2, max, $\frac{1}{2}$
(c) -2, max, $-2\frac{1}{2}$ **(d)** -3, min, $-1\frac{1}{3}$
2. **(a)** min $(-3, -16)$ **(b)** min $\left(-1\frac{1}{2}, -16\right)$
(c) min $\left(1\frac{1}{2}, -12\frac{1}{4}\right)$ **(d)** max $\left(-\frac{1}{2}, 6\frac{1}{4}\right)$
(e) max $\left(\frac{3}{8}, \frac{9}{16}\right)$ **(f)** min $\left(1\frac{1}{3}, -5\frac{1}{3}\right)$
3. **(a)** 5, min, 1 **(b)** $-4\frac{1}{8}$, min, $1\frac{1}{4}$
(c) $4\frac{11}{12}$, min, $-1\frac{1}{6}$ **(d)** 3, max, 2
(e) 16, max, -1
4. $p = 2\frac{1}{2}, q = 6\frac{3}{4}$
(i) $6\frac{3}{4}$ **(ii)** $2\frac{1}{2}$
5. $y = 4\left(x - \frac{3}{4}\right)^2 + 11\frac{3}{4}$; $11\frac{3}{4}$
6. $\frac{2}{3}$
7. $y = 2(x + 1)^2 + 15$; $15, -1$
8. $y = 16\frac{1}{4} - \left(x + 1\frac{1}{2}\right)^2$; $16\frac{1}{4}, -1\frac{1}{2}$
9. 3 m, 2 m

Exercise 2D

1. (a) $x > 3$ or $x < -2$ (b) $-7 < x < 4$
(c) $x \geqslant \frac{2}{3}$ or $x \leqslant -\frac{5}{4}$ (d) $x \geqslant \frac{1}{2}$ or $x \leqslant -4$
(e) $2 < x < 5$ (f) $x < 6$ or $x > 7$
(g) $x < -\frac{9}{2}$ or $x > 6$ (h) $x < -4$ or $x > \frac{7}{2}$
(i) $-\frac{2}{5} < x < 1$ (j) $x > 1$ or $x < -3$
(k) all real values of x (l) $-\frac{5}{3} < x < \frac{3}{2}$
2. $k < -1\frac{9}{16}$
3. (a) $x \geqslant 2$ or $x \leqslant 1\frac{1}{5}$ (b) $x > 1$ or $x < -2$
(c) $-1\frac{1}{2} \leqslant x \leqslant 1$ (d) $x > 2$ or $x < -1\frac{1}{2}$
4. $-4\frac{1}{2} < x < 8$
5. $x \leqslant 1\frac{1}{6}$ or $x \geqslant 2\frac{1}{6}$
6. $-2 < x < 2$ or $3 < x < 7$
7. $4 < x < 6$ or $1 < x < 2$
8. $k \geqslant 8$ or $k \leqslant 1$
9. $k > 3$ or $k < -1\frac{4}{9}$; 3 or $-1\frac{4}{9}$
10. $a > \frac{1}{4}$
11. 2
12. -3
13. 1.5 years
14. $3 < k < 7$
15. $-1\frac{2}{3} < x < 3\frac{1}{2}$
16. $x > 1\frac{2}{3}$ or $x < -2$
17. $a > 0$ and $b^2 - 4ac < 0$

Exercise 2E

1. ± 3
2. $k < 5$
3. $k < 0$
4. $k > -4$
5. $k > 9$
6. ± 3
7. 0 or 10
8. $k < -6$ or $k > 6$
9. $p > 1\frac{4}{5}$
12. $(a + b)^2 < 2(b - c)$

Exercise 2F

1. (a) 22 or -12 (b) 13 or -6
(c) 4 or $\frac{2}{3}$
2. (a) 13 or $-13\frac{2}{3}$ (b) $12\frac{1}{2}$ or $-5\frac{1}{2}$
(c) 1 or $\frac{1}{2}$ (d) 3
3. (a) $-2\frac{3}{7}$ or $-7\frac{2}{3}$ (b) $-3\frac{2}{5}$ or $-1\frac{1}{8}$
(c) 8 or -2 (d) $\frac{1}{8}$
4. (a) ± 2 or $\pm\sqrt{2}$ (b) 3
(c) 1 or $2\frac{1}{2}$ (d) $\frac{1}{2}$ or 3
(e) $-1\frac{2}{3}$ or $\frac{4}{5}$
5. (a) $15 + 2\pi$ (b) $23 + 3\pi$
(c) $16e - 27$
6. (a) $x = 2, y = 5$ or $x = -3, y = 0$ or $x = 0, y = 3$
(b) $x = 0, y = 3$ or $x = 12, y = -3$
(c) $x = -3, y = 10$ or $x = 3.5, y = 16.5$

Review Exercise 2

1. (a) $2x^2 - 9x - 36 = 0$ (b) $2x^2 - 15x + 23 = 0$
(c) $x^2 - 9x + 10 = 0$ (d) $4x^2 + 9x - 18 = 0$
(e) $2x^2 - 9x + 5 = 0$
2. (a) $x^2 + 5x + 6 = 0$ (b) $2x^2 + 15x + 28 = 0$
(c) $9x^2 - 13x + 4 = 0$ (d) $6x^2 + 5x + 1 = 0$
(e) $4x^2 - 13x + 9 = 0$
3. $a = -10, c = -15$
4. (ii) $x^2 + 140x - 1 = 0$
6. 2 or $-\frac{2}{3}$
7. 12
8. $k > 0$
9. $-3 < a < \frac{3}{4}$
10. (a) $4 \leqslant x \leqslant 6$ (b) 9
11. (i) 4 or 12 (ii) $m > 3$
12. 3 or -5; $(-1, 0)$, $(3, 0)$
13. (a) $1 < x < 4$ (b) $-\frac{1}{2} < k < 1\frac{1}{2}$
14. $k < -\frac{13}{9}$ or $k > 3$
15. 13
16. $2 \leqslant k \leqslant 4$
17. $a > mc$
18. $q^2 < 12q - 8$
19. $25b^2 + 4a < 0$
21. (a) 22 or -4 (b) $1\frac{1}{7}$ or $-\frac{6}{11}$

Chapter 2 — Challenge Yourself

1. 0 or 3
2. $a = b$ and $c = 0$
3. $-6 < a < 2$
4. $-\frac{1}{3} \leq k \leq 1$
5. $x < -1$ or $\frac{3}{5} < x < \frac{3}{2}$
6. $\sqrt{a^2} = |a|$
7. $x = 4, y = 5$
8. $x = \pm 3$, line of symmetry is $x = 0$.

Revision Exercise A1

1. $(3, -1)$, $\left(-4\frac{6}{7}, 2\frac{1}{7}\right)$
2. $A = 7, B = 24, C = 9$
3. $a = 2, b = 3$
4. $x = 2$, 0.14 or -2.47
5. (a) $\frac{3}{x - 3} + \frac{5}{x - 4}$ (b) $\frac{5}{x + 3} - \frac{6}{(x + 3)^2}$
6. $10x^2 - 28x + 25 = 0$

7. $y = 2(x-2)^2 - 13$, least $y = -13$, $x = 2$
8. $k > 2$ or $k < -6$
9. $-1 < k < 3$

Revision Exercise A2

1. $x = 2, y = 3$ or $x = -4\frac{2}{5}, y = -\frac{1}{5}$
2. $A = 3, B = -5, C = -3$
3. $a = 3, b = 3; R = 6$
4. $a = 9, (x+3)(x-3)(x-1)$
5. **(a)** $\frac{5}{x-2} + \frac{3x}{x^2+3}$ **(b)** $2 + \frac{1}{x} + \frac{7}{x-4}$
6. $1\frac{4}{5}, \frac{2}{5}; 25x^2 - 180x + 283 = 0$
7. $-2 < k < 6$
8. $q < 3$
9. $\pm\sqrt{140}$

Exercise 3A

1. **(a)** $1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$
   (b) $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5$
   (c) $1 + 12x + 54x^2 - 108x^3 + 81x^4$
   (d) $1 - 6x + 12x^2 - 8x^3$
2. **(a)** $1 + 8x + 28x^2 + 56x^3 + \ldots$
   (b) $1 - 9x + 36x^2 - 84x^3 + \ldots$
   (c) $1 - 30x + 405x^2 - 3240x^3 + \ldots$
   (d) $1 + 22x + 220x^2 + 1320x^3 + \ldots$
3. **(a)** 84 **(b)** 5
   (c) 91 **(d)** 252
4. **(a)** 5040 **(b)** 362 880
   (c) 2880 **(d)** 6720
5. **(a)** 70 **(b)** 84
   (c) 252 **(d)** 120
6. **(a)** $1 + 3x + \frac{15}{4}x^2 + \frac{5}{2}x^3 + \frac{15}{16}x^4 + \frac{3}{16}x^5 + \frac{1}{64}x^6$
   (b) $1 - \frac{5}{3}x + \frac{10}{9}x^2 - \frac{10}{27}x^3 + \frac{5}{81}x^4 - \frac{1}{243}x^5$
   (c) $1 - 4ky + 6k^2y^2 - 4k^3y^3 + k^4y^4$
   (d) $1 + 3x^2 + 3x^4 + x^6$
7. **(a)** $1 - \frac{7}{2}x + \frac{21}{4}x^2 - \frac{35}{8}x^3 + \ldots$
   (b) $1 + 2x + \frac{7}{4}x^2 + \frac{7}{8}x^3 + \ldots$
   (c) $1 + 9px + 36p^2x^2 + 84p^3x^3 + \ldots$
   (d) $1 - 10x^2 + 45x^4 - 120x^6 + \ldots$
8. **(a)** $\frac{n(n-1)(n-2)(n-3)}{24}$ **(b)** $\frac{(n-1)(n-2)}{2}$
   (c) n **(d)** 1
9. **(a)** 25 920, 1 959 552 **(b)** $-1512, -13\,608$
   (c) $\frac{35}{8}, \frac{21}{32}$ **(d)** $-\frac{5}{32}, -\frac{1}{1024}$
10. 3
11. **(i)(a)** 48 **(b)** 40 320
    (ii) No **(iii)** No
12. 8
13. 6

Exercise 3B

1. **(a)** $32x^5 + 80x^4y + 80x^3y^2 + 40x^2y^3 + 10xy^4 + y^5$
   (b) $64 + 96x + 60x^2 + 20x^3 + \frac{15}{4}x^4 + \frac{3}{8}x^5 + \frac{1}{64}x^6$
   (c) $\frac{a^5}{32} - \frac{15a^4}{16b} + \frac{45a^3}{4b^2} - \frac{135a^2}{2b^3} + \frac{405a}{2b^4} - \frac{243}{b^5}$
2. **(a)** $29\,120x^{12}y^4$ **(b)** $\frac{55x^6}{128y^3}$
   (c) $210x^2y$
3. **(a)** 448 **(b)** 81 081
   (c) 448 **(d)** -448
4. **(a)** 240 **(b)** $10\frac{1}{2}$
5. 5 : 6
6. $\pm\frac{4}{3}$
7. $2\frac{1}{2}$
8. **(a)** $16 - 32x + 24x^2 - 8x^3 + x^4$
   (b) $1 + 10x + 40x^2 + 80x^3 + 80x^4 + 32x^5$; 232
9. **(a)** $64 + 96x + 60x^2 + \ldots$
   (b) $729 - 2916x + 4860x^2 + \ldots$; 74 844
10. $256x^8 - 768x^6 + 1008x^4$; 1520
11. $\frac{63}{256}x^5y^5$
12. $32 + 240x + 400x^2 + \ldots$
13. **(a)** $1 + 5x - 30x^3 + \ldots$ **(b)** $1 - 4x - 42x^2 + \ldots$
14. $2 + \frac{27}{2}x + \frac{77}{2}x^2 + \frac{119}{2}x^3 + \ldots$, 3.7945
15. $1 + 16x + 88x^2 + 112x^3 + \ldots$; 1.1689
16. 2
17. $p = 1, q = 3, k = 224$
18. $2^n - n2^{n-4}x + n(n-1)2^{n-9}x^2 + \ldots$
    (i) 8 **(ii)** $a = 512, b = -72$
19. 5
20. **(i)** 0.147 **(ii)** 12

Review Exercise 3

1. **(a)** $x^{30} + 45x^{28}y + 945x^{26}y^2 + \ldots$
   (b) $81x^4 - 216x^3y + 216x^2y^2 + \ldots$
   (c) $x^{12} + 12x^{10}y^3 + 60x^8y^6 + \ldots$
   (d) $\frac{x^8}{a^8} + \frac{8x^7y}{a^7b} + \frac{28x^6y^2}{a^6b^2} + \ldots$
2. **(a)** -20 **(b)** 1000
3. **(a)** $605\frac{5}{8}$ **(b)** $510\,743\frac{3}{4}$
   (c) 78 **(d)** 145
4. **(a)** -28 **(b)** 18 564
   (c) 84 **(d)** $113\frac{3}{4}$
5. $1 + 4x + 14x^2 + 28x^3 + \ldots$
6. 11
7. 14 or 23
8. 7 or 14
9. $x^6 - 12x^4 + 60x^2 + \ldots$; 156
10. $\frac{15}{112}$
11. $1 - 4x + 3x^2 + 10x^3 - 25x^4 + \ldots$; 0.8829
12. 16

13. $1 + 10x + 55x^2 + 210x^3 + ...; 4\frac{3}{4}$

14. $1 - 2\frac{1}{2}x + 2\frac{13}{16}x^2 - 1\frac{7}{8}x^3 + ..., -25\frac{15}{64}$

15. $1 - 5x + 10x^2 - 10x^3 + 5x^4 - x^5; -x^{10}$

16. $3\frac{5}{9}$

17. **(a)** **(i)** $1 + 8x + 28x^2 + ...$
(ii) $81 - 216x + 216x^2 + ...; 756$
(b) $a = -2, b = 3$
(c) **(i)** $1 - 24x + 240x^2 + ...$
(ii) $128 + 448x + 672x^2 + ...; 20\ 640$

18. $m = 5, n = 7, h = 229$

19. **(i)** 6 **(ii)** –624

Chapter 3 — Challenge Yourself

1. $a = 10, b = 20, c = 2, 10020.002\sqrt{10}$
2. 0.188

Exercise 4A

1. **(a)** 4 **(b)** $-\frac{3}{2}$
(c) ±2 **(d)** $-\frac{1}{2}$
2. **(a)** 2 **(b)** 3
(c) 0 **(d)** –1 or 4
3. **(a)** 0 **(b)** 0 or $\frac{1}{2}$
(c) no solution
4. **(a)** $x = -2, y = -1$ **(b)** $x = -\frac{9}{5}, y = \frac{6}{5}$
(c) $x = \frac{5}{4}, y = -\frac{1}{4}; x = -5, y = 1$
5. $a = 7, b = 3, k = 3$
6. $p = 2, q = 2$
7. **(a)** 0 or 1 **(b)** 0 or –2

Exercise 4B

1. **(a)** 8 **(b)** 7
(c) $3\sqrt{7}$ **(d)** 15
2. **(a)** $6\sqrt{7}$ **(b)** $3\sqrt{3}$
(c) $-2\sqrt{15}$ **(d)** $\frac{1}{2}$
3. **(a)** $24 - 9\sqrt{2}$ **(b)** $33 + 12\sqrt{6}$
(c) 61 **(d)** –17
4. **(a)** $\frac{3\sqrt{5}}{5}$ **(b)** $\frac{\sqrt{2}}{3}$
(c) $14 - 7\sqrt{3}$ **(d)** $\frac{4+\sqrt{6}}{5}$
5. **(a)** $\frac{5+10\sqrt{3}}{22}$ **(b)** $\frac{28-8\sqrt{6}}{25}$
(c) $\frac{4\sqrt{3}-\sqrt{15}}{22}$ **(d)** $\frac{8\sqrt{5}-36}{11}$
6. $(4 + 2\sqrt{6})$ cm
7. **(a)** $\frac{23\sqrt{2}}{12}$ **(b)** $\frac{64\sqrt{3}-27\sqrt{2}}{36}$
(c) $\frac{10}{3}$ **(d)** $-\frac{23}{2}$
8. $6\sqrt{2}$
9. **(a)** 3 **(b)** $\frac{2}{3}$
(c) $-\frac{53}{23}$
10. **(a)** $\frac{19-8\sqrt{3}}{13}$ **(b)** $\frac{76+6\sqrt{2}}{89}$
(c) $2\sqrt{2} - 3$ **(d)** $\frac{16-7\sqrt{6}}{19}$
11. **(i)** $3 + 2\sqrt{2}$ **(ii)** $4\sqrt{2}$
12. $a = 8, b = 4$
13. $(3 - \sqrt{3})$ cm

Exercise 4C

1. **(a)** $x = \log_{10} 40$ **(b)** $x = \log_5 10$
(c) $\frac{1}{2} = \log_{16} 4$ **(d)** $y = \log_a x$
2. **(a)** $2^x = 7$ **(b)** $4^{\frac{1}{5}} = x$
(c) $3^{-3} = \frac{1}{27}$ **(d)** $0.25^n = k$
3. **(a)** 0.497 **(b)** 1
(c) 2 **(d)** 9.64
(e) 54.6 **(f)** 0.368
(g) 1.65 **(h)** 0.0821
4. **(a)** 3 **(b)** 4
(c) –4 **(d)** –2
5. **(a)** 18.6 **(b)** $\frac{1}{200}$
(c) –1080 **(d)** 8
(e) $\frac{1}{625}$ **(f)** 2
(g) –1 **(h)** $\frac{1}{2}$
(i) 1 or 81
6. **(a)** 2^{6p+q} **(b)** 2^{p-2q}
7. **(a)** 512 **(b)** 22.5
(c) 5.41×10^{13} **(d)** $\frac{1}{8}$
(e) $-\frac{1}{2}$ **(f)** $\frac{1}{3}$ or –2

Exercise 4D

1. **(a)** lg 30 **(b)** 0
(c) 1 **(d)** lg p
(e) $\log_3 40$ **(f)** ln 18
(g) cannot be further simplified
(h) 0
2. **(a)** $\log_7 5$
(b) $3 - 2 \ln 3$
(c) $\lg \frac{7}{15k}$ **(d)** 1
3. **(a)** 3 **(b)** –2
(c) $\frac{1}{2}$ **(d)** $\frac{1}{2}$

4. **(a)** 1.04 **(b)** 0.263
(c) –0.715 **(d)** 3.96
5. **(a)** $5 \log_3 5$ **(b)** ln 6
(c) 1 **(d)** 4
6. **(a)** $\log_6 72$ **(b)** $\lg \frac{1}{200}$
(c) $\log_4 192$ **(d)** $\log_3 6$
7. **(i)** 3.99 **(ii)** 0.225
(iii) –3.35 **(iv)** –0.861
8. **(a)** k^3 **(b)** $3k$
(c) $2(1 + k)$ **(d)** $\frac{1}{2}k - 2$
9. **(a)** 16 **(b)** –12
(c) 3 **(d)** 12
(e) $\frac{2}{3}$ **(f)** $\frac{2}{3}$
10. **(i)** $\frac{1}{m}$ **(ii)** $\frac{2}{m}$
(iii) $-m$ **(iv)** $-2m$
11. **(i)** $-4p$ **(ii)** $-3p$
(iii) $p - 1$ **(iv)** $\frac{1-p}{2}$
12. $\frac{ab}{1-a}$
13. **(ii)** 0.301

Exercise 4E

1. **(a)** 0.975 **(b)** –0.485
(c) ±3.07 **(d)** 1.39
2. **(a)** –1 **(b)** 5
(c) 5
3. **(a)** 30 **(b)** –2.29 or 9.68
(c) $\frac{1}{2}$ **(d)** 4
4. **(a)** $x = -2, y = 3$ **(b)** $a = \frac{2}{3}, b = \frac{1}{5}$
(c) $p = 4, q = -1$
5. $x = \frac{4e^4}{e^4 - 1}$
6. **(a)** 6561 or $\frac{1}{6561}$ **(b)** 1630
(c) 0.669 or 5 **(d)** 5.59 or 0.679
7. **(a)** 0.494 **(b)** 2.27 or –0.273
(c) –0.104 **(d)** 1.48
(e) –2.14 **(f)** –0.712
8. 1
10. $y = 4x^{\frac{1}{4}}$
11. $x = 2, y = 4$ or $x = 4, y = 2$
12. **(a)** $\frac{1}{2}$ **(b)** $\frac{5}{4}$
(c) $\frac{2}{5}$ **(d)** $-\frac{9}{5}$

Exercise 4F

2. **(ii)** They are reflections of each other in the y-axis.
3. **(ii)** $y = 3 - 2x$, 2 solutions
4. **(i)** no solution **(ii)** –0.24 or 2.76
5. **(ii)** They are reflections of each other in the line $y = x$.
6. **(ii)** $y = ex - 4$, 2 solutions
7. 3.1
8. **(i)** $g(x) = -f(x)$.
(iii) They are reflections of each other in the x-axis.

Exercise 4G

1. **(i)** 10 000 **(ii)** 3680
(iii) after 12 days
2. **(i)** 0 **(ii)** 10^{-12} W/m^2
(iii) 43.0 dB **(iv)** 100 W/m^2
3. **(i)** 74 g **(ii)** 1.21×10^{-4}
(iii) 10 400 years
4. **(i)** 0.139 **(ii)** 11.6 years
5. **(i)** 0 **(ii)** 1100
(iii) It approaches zero.
6. **(i)** Bank Y
(ii) the interest rate
7. **(i)** $a = 12, k = 7000$ **(ii)** 551
(iii) 4 years **(iv)** It will approach 583.
8. **(i)** 1.4 **(ii)** 4.01×10^{15} km
(iii) 1.51 times **(iv)** 0.398, B
9. **(ii)** $y = \ln(2t + 3) + 0.5$ **(iii)** 5990

Review Exercise 4

1. $z = 2x - 6y$
2. **(a)** $-\frac{1}{2}$ **(b)** 1.95
(c) 0.301 **(d)** $\frac{11}{2}$
(e) $\frac{7}{3}$ **(f)** 729
3. $x = 5, y = -2$
4. $p = \frac{3175}{4}, q = 300$
5. $a = 5, b = 3$
6. $\left(2 - \frac{1}{2}\sqrt{6}\right)$ cm
7. **(i)** $2y^3 + y^2 + 7y - 4 = 0$ **(iii)** –1
8. $\log_7 3 = \frac{pq}{1-p}$
9. **(iii)** They are reflections of each other in the line $y = x$.
10. **(i)** 240 **(ii)** 1989
(iii) 4 billion

Chapter 4 — Challenge Yourself

2. **(i)** $a = 2, b = 4$ or $a = 4, b = 2$
3. **(ii)** $4 + 3\sqrt{2}$
4. **(i)** 44
5. **(b)** 6651

Revision Exercise B1

1. $1 + 24x + 264x^2 + \ldots$; 604
2. $n = 10, 22$
3. $1 + 6kx + 15k^2x^2 + \ldots$; $\frac{3}{5}$ or -3; $9\frac{3}{5}$ or -12
4. **(a)** 0.26 or -2.43 **(b)** 2 or -6
5. $19 - 5\sqrt{3}$
6. **(a)** 1 **(b)** 2
7. $x = 20, y = \frac{1}{5}$
8. **(a)** 1 **(b)** 125 or $\sqrt{5}$
9. $x = \frac{z^4}{y^2}$
10. Draw $y = \frac{1}{3} - \frac{2}{3}x$.

Revision Exercise B2

1. **(a)** $1 + 4x + 7x^2$
(b) $729 - 1458x + 1215x^2$; 486
2. $p = 3, q = -5$
3. $1 + nx + \left[2an + \frac{1}{2}n(n-1)\right]x^2$; $a = 1, n = 6$; 80
4. **(a)** 2 **(b)** 3
5. $k = 2$
6. **(a)** $y - x$ **(b)** $2y - 2x + 3$
(c) $2y + 2x - 4$ **(d)** $3x + 0.5 - 3y$
7. $x = 16, y = 64$ or $x = 64, y = 16$
8. **(a)** $\frac{1}{2}$ **(b)** 1.10 or 1.61
9. $b = \frac{a^2c^3}{3}$ **10.** 2.

Exercise 5A

1. **(a)** (6, 4) **(b)** (0, 2)
(c) (4, 2) **(d)** $\left(\frac{1}{2}, 2\frac{1}{2}\right)$
(e) $\left(\frac{1}{2}a + b, \frac{1}{2}a + b\right)$
(f) $\left[\frac{1}{2}a(h^2 + k^2), ah + 2ak\right]$
2. **(i)** $(4, -1)$ **(ii)** $(4, -11)$
3. $\left(4, -\frac{1}{2}\right)$, $(5, -6)$ **4.** $\left(-\frac{1}{6}, 2\frac{7}{12}\right)$
5. **(i)** $\left(2\frac{5}{6}, 2\frac{5}{6}\right)$ **(ii)** $\frac{\sqrt{26}}{3}$ units
6. $P(-11, -3)$, $Q(7, 9)$, $R(3, -11)$

Exercise 5B

1. **(a)** ± 1 **(b)** $\pm\frac{1}{3}$
2. $\frac{26}{3}$ **3.** 1
4. Yes
5. **(i)** (4, 0) or (8, 0) **(ii)** (0, 24)
6. **(i)** (4, 2) **(ii)** No
7. $-\frac{1}{3}$
8. **(a)** -9 **(b)** 17
9. Yes **10.** Yes

Exercise 5C

1. $3y + x - 9 = 0$
2. **(a)** $y = 5$ **(b)** $x = -1$
(c) $y + 3x = 20$ **(d)** $2y = x - 6$
3. **(i)** -9 **(ii)** 45 units^2 **(iii)** $7\frac{1}{2}$ units
4. $2y - x + 6 = 0$ **5.** $y + 1 = 0$
6. **(i)** 1 **(ii)** $\frac{4}{7}$ **(iii)** 29.7°
7. **(i)** (5, 6) **(ii)** 7 **(iii)** Yes
8. $26y + 13x - 25 = 0$
9. **(i)** $-\frac{4}{3}$ **(iii)** (5, 7) **(iv)** 50 units^2
10. **(i)** $y + 2x = 7$ **(ii)** (2, 3)
(iii) $\sqrt{5}$ units, $2\sqrt{5}$ units, $5\sqrt{2}$ units
(iv) 5 units^2 **(v)** $\sqrt{2}$ units
11. **(a)** $1\frac{2}{3}$ **(b)** -55
(c) $1 \pm 2\sqrt{19}$ **(d)** $\frac{5}{3}$
12. **(a)** $y + x = 8$ **(b)** $5y = x + 16$, (4, 4)
13. **(i)** $4y = 2x + 27$ **(ii)** 15.1
(iii) $\left(14\frac{1}{2}, 14\right)$
14. **(i)** $\left(\frac{7}{2}, \frac{9}{2}\right)$ **(ii)** $\frac{7}{3}$
(iii) (5, 8)

Exercise 5D

1. **(a)** 12 units^2 **(b)** 27 units^2
2. **(a)** 52 units^2 **(b)** 88.5 units^2
3. **(i)** 38.5 units^2
4. **(i)** $(1, -2)$ **(ii)** $6y = x + 24$
(0, 4) $y + 6x = 41$
(iii) (6, 5) **(iv)** $55\frac{1}{2}$ units^2
5. **(i)** $3y + x = 13$, $y = 3x + 11$
(ii) $C(9, 8)$, $D(0, 11)$
(iii) $\left(-3\frac{2}{3}, 0\right)$
(iv) 30 units^2, 25 units^2
6. **(i)** $3y - 2x - 8 = 0$, $2y + 3x = 34$
(ii) $Q(-4, 0)$, $R\left(0, 2\frac{2}{3}\right)$, $S\left(6\frac{8}{13}, 7\frac{1}{13}\right)$
(iii) $47\frac{1}{13}$ units^2
7. **(i)** $13\frac{1}{2}$ units^2 **(ii)** 0
(iii) $(6, -4)$ **(iv)** 27 units^2

8. **(i)** $\left(\frac{5}{2}, 3\right)$ **(ii)** $\frac{3\sqrt{5}}{2}$ units
9. **(i)** $3y = -2x + 25$ **(ii)** $P(0, 4)$, $Q\left(12\frac{1}{2}, 0\right)$
(iii) $(20, -5)$ **(iv)** 94.25 units^2

Review Questions 5

1. $\left(-1\frac{1}{2}, \frac{1}{2}\right)$ 2. $33y = 55x - 41$
3. $y = 2x - 1$
4. **(i)** $(-1, 3)$, $(-4, -1)$, $(1, -1)$
(ii) $(4, 3)$
5. $\left(1\frac{3}{4}, 6\frac{1}{2}\right)$ 6. 5.4
7. **(i)** $4y + 5x = 11$ **(ii)** $5y = 4x + 24$
(iii) $(4, 8)$; $(12, -2)$, $(7, -6)$, 82 units^2
8. **(a)** **(i)** $(1, 7)$, $(-6, 0)$ **(ii)** $y = x + 6$
(iii) 84 units^2
9. **(i)** $(4, 3)$ **(ii)** $y + 4x = 19$
(iii) $\sqrt{17}$ **(iv)** $(5, -1)$, $(3, 7)$
(v) 34 units^2
10. **(a)** **(i)** $(5, 6)$ **(ii)** $(10, 4)$
(iii) $5y + 2x = 40$ **(iv)** 52

Chapter 5 — Challenge Yourself

1. $A(-5, 12)$, $B(0, 16)$, $C(7, 10)$, $D(2, 6)$,
58 units^2
2. **(i)** $A\left(\frac{9}{5}, \frac{12}{5}\right)$, $B\left(\frac{16}{5}, \frac{12}{5}\right)$, $C(5, 0)$, $D\left(\frac{5}{2}, \frac{15}{8}\right)$
(ii) $7\frac{5}{16}$ units^2
3. $(3, 7)$, $(-5, 5)$, $(5, -5)$
Centres: $\left(\frac{3}{2}, \frac{7}{2}\right)$, $\left(-\frac{1}{2}, 3\right)$, $\left(2, \frac{1}{2}\right)$

Exercise 6A

1. **(a)** $(0, 0)$, 7 **(b)** $(0, 0)$, $\frac{\sqrt{10}}{2}$
(c) $(-2, 3)$, 4 **(d)** $(1, 0)$, 3
(e) $\left(\frac{1}{2}, \frac{5}{2}\right)$, $\frac{\sqrt{10}}{2}$ **(f)** $\left(\frac{3}{4}, -\frac{5}{4}\right)$, $\frac{3}{2}$
2. **(a)** $x^2 + y^2 = 9$
(b) $x^2 + y^2 + 4x - 6y - 3 = 0$
(c) $25x^2 + 25y^2 - 25x + 20y - 46 = 0$
(d) $x^2 + y^2 - 8x + 2y - 20 = 0$
(e) $x^2 + y^2 - 6x + 8y - 11 = 0$
3. $x^2 + y^2 - 2x - 2y - 3 = 0$
4. $x^2 + y^2 = 13$
5. $(-4, 0)$, $(3, 7)$ 6. $y + 2x = 5$
7. $x^2 + y^2 - 10x - 8y + 25 = 0$
8. $x^2 + y^2 - 4x + 4y - 12 = 0$
9. $3\sqrt{2}$ units
10. **(i)** $x^2 + y^2 = 65$ **(ii)** $4\sqrt{5}$ units
11. **(i)** $x^2 + y^2 + 3x - y - 10 = 0$
(ii) 5 units
12. **(i)** $x^2 + y^2 - 4x - 4y - 2 = 0$
(ii) $y + 2x = 6$
13. $x^2 + y^2 - 2x + 2y - 3 = 0$
14. $x^2 + y^2 + 6x - 2y - 15 = 0$
15. $a = 4$, $b = 1$
17. **(i)** $A(-3, 4)$, $B(4, 3)$ **(ii)** $y = 7x$
(iii) $\left(\pm\frac{\sqrt{2}}{2}, \pm\frac{7\sqrt{2}}{2}\right)$
18. **(i)** $\left(2\frac{1}{2}, 4\frac{1}{2}\right)$ **(ii)** $6y + 2x = 27$

Exercise 6B

3. **(i)** $(2, 2)$, $(8, -4)$ **(ii)** 8.49 units
4. **(i)** $(1, 2)$, $(-2, -1)$ **(ii)** 1.5 units^2
5. $a = -1$, $b = \pm 1$, $c = 2$
6. **(i)** $(9, 3)$, $(3, 0)$ **(ii)** $2y = 27 - 4x$
7. **(i)** $(9, 0)$, $(5, 2)$ **(ii)** 30 units^2
8. **(i)** 12 **(ii)** $(6, 36)$, $(0, 0)$, $(12, 0)$
9. **(i)** $a = 16$, $b = 8$ **(ii)** 4.12 units
10. **(i)** $(8, 2)$, $(-8, -2)$ **(iii)** 6 units^2

Review Questions 6

1. **(a)** $x^2 + y^2 - 4x - 14y + 28 = 0$
(b) $x^2 + y^2 + 4x - 18y + 40 = 0$
2. $(8, 6)$, 5 km
3. **(a)** $(2, 5)$, 3 **(b)** $(-2, 2)$, 5
(c) $(0, 0)$, $\frac{4\sqrt{3}}{3}$ **(d)** $(2, 3)$, $\sqrt{10}$
4. $x^2 + y^2 - 4y - 12 = 0$, $(0, 6)$, $(0, -2)$, $(\pm 2\sqrt{3}, 0)$
5. **(i)** $x^2 + y^2 - 4x - 45 = 0$
(ii) 7, yes
6. $x^2 + y^2 - 2x + 2y - 3 = 0$
7. **(i)** $x^2 + y^2 = 9$ **(ii)** Yes
8. **(i)** $x^2 + y^2 - 2y - 9 = 0$ **(ii)** Yes, yes
9. $a = -4$, $b = 1$
10. **(i)** $(2, 4)$ **(ii)** $y = x + 2$
(iii) $a = 1$, $b = 1$
11. $a = 4$, $m = 2$, $c = -4$, $k = -2$
12. $\frac{1}{4}$ 13. $(1, 4)$

Chapter 6 — Challenge Yourself

1. $x^2 + y^2 - 10x + 14y + 49 = 0$
or $x^2 + y^2 - 10x + 2y + 1 = 0$

Exercise 7A

1. *There may be more than one method to do this.*
   (a) $\frac{y}{x} = ax^2 - b$, $X = x^2$, $Y = \frac{y}{x}$, $m = a$, $c = -b$
   (b) $y - x^2 = -bx + a$, $X = x$, $Y = y - x^2$, $m = -b$, $c = a$
   (c) $y = \frac{h}{x^2} + k$, $X = \frac{1}{x^2}$, $Y = y$, $m = h$, $c = k$
   (d) $xy = hx^3 + k$, $X = x^3$, $Y = xy$, $m = h$, $c = k$
   (e) $\frac{\sqrt{x}}{y} = qx + p$, $X = x$, $Y = \frac{\sqrt{x}}{y}$, $m = q$, $c = p$
   (f) $x = b\left(\frac{x}{y}\right) - a$, $X = \frac{x}{y}$, $Y = x$, $m = b$, $c = -a$
   (g) $\lg y = n \lg x + \lg p$, $X = \lg x$, $Y = \lg y$, $m = n$, $c = \lg p$
   (h) $\lg y = ax + b$, $X = x$, $Y = \lg y$, $m = a$, $c = b$
2. *There may be more than one method to do this.*
   (a) $\ln y = -bx + \ln a$, $X = x$, $Y = \ln y$, $m = -b$, $c = \ln a$
   (b) $\lg y = (\lg h)x - \lg a$, $X = x$, $Y = \lg y$, $m = \lg h$, $c = -\lg a$
   (c) $\ln y = -\frac{a}{b} \ln x + \frac{1}{b}$, $X = \ln x$, $Y = \ln y$, $m = -\frac{a}{b}$, $c = \frac{1}{b}$
   (d) $\lg y = \frac{3}{n} \lg x + \frac{2 \lg h}{n}$, $X = \lg x$, $Y = \lg y$, $m = \frac{3}{n}$, $c = \frac{2 \lg h}{n}$
   (e) $\lg\left(\frac{1}{y} - x^2\right) = (-\lg a)x + b \lg a$
   (f) $x \lg y = (\lg q)x^2 + \lg p$, $X = x^2$, $Y = x \lg y$, $m = \lg q$, $c = \lg p$
   (g) $x = h\left(\frac{x^2}{y}\right) - k$, $X = \frac{x^2}{y}$, $Y = x$, $m = h$, $c = -k$
   (h) $\frac{1}{y} - x^3 = hx - k^2$, $X = x$, $Y = \frac{1}{y} - x^3$, $m = h$, $c = -k^2$
3. (a) $y = x^2 - x - 2$ (b) $y = 10^{\frac{4x-3}{5}}$
   (c) $y = e^{15 - 2\sqrt{x}}$
4. $a = 4$, $b = 20$
5. (i) $y = \frac{3x^2}{x - 4}$ (ii) $x = 8$
6. $a = 15.8$, $b = 1.5$
7. $a = \frac{10}{3}$, $b = -1$
8. $A = 100$, $B = 1$, $x = \frac{2}{3}$
9. $a = \frac{3}{2}$, $b = -\frac{7}{3}$, 6

Exercise 7B

1. $a \approx 8.0$, $b \approx 2.0$
2. (i) $a \approx -2.0$, $b \approx 7.9$ (ii) $x \approx 3.0$
3. (i) $k \approx 12$, $b \approx 2.5$ (ii) $y \approx 126$
   (iii) Incorrect value of y is 550.
   Correct value of $y \approx 500$.
4. (i) $n \approx 0.52$, $c \approx 150$
   (ii) $y \approx 90$
5. (i) $a \approx 2.2$, $b \approx 3.5$
   (ii) Incorrect value of y is 25.2.
   Correct value of $y \approx 18$.
   $x \approx 4.2$
6. (i) $y = e^{3.2 - 0.5x}$ (ii) $y \approx 5.0$
7. (i) $y = \frac{x}{1.3 + 0.15x}$ (ii) $y \approx 0.11$
8. (i) $\ln y = k \ln x + \ln h$ (ii) $h \approx 2.4$, $k \approx 1.5$
   (iii) $y \approx 180$
9. Plot $\ln y$ against $\ln x$.
   (i) $a \approx 2.5$, $b \approx 2.0$
   (ii) Incorrect value of y is 465.
   Correct value of $y \approx 540$
10. (i) $\ln C = -kt + \ln A$ (ii) $A \approx 99$, $k \approx 0.015$
    (iii) 99 mg/l (iv) 40 mg/l
11. (i) $\ln (T - 25) = (-\ln a)t + \ln k$
    (ii) $a \approx 1.2$, $k \approx 67$ (iii) 92°C
    (iv) 8.9 min (v) 25°C

Review Exercise 7

1. (i) $y = \frac{1}{2}x^2 + \frac{3}{2}x$ (ii) 5
2. $k = 1000$, $n = -0.6$
3. $y = \frac{1}{3}x + \frac{6}{x}$
4. $y = \frac{1}{2}(x^{\frac{7}{3}} - x^2)$
5. $p = q = 3$
6. $A = 61.1$, $b = 4.39$
7. $k = 46.4$, $n = -\frac{2}{3}$
8. $a \approx 2.0$, $b \approx -6.9$
9. (ii) (a) $h \approx 1.5$, $k \approx 14$ (b) $x \approx 4.4$
10. (ii) $k \approx 0.45$, $N_0 \approx 25$ (iii) 570 minutes
11. (i) Plot $\ln \sqrt[3]{y}$ against x.
    (ii) $p \approx 4.1$, $q \approx 0.80$ (iii) $x \approx 2.8$
12. (i) $a \approx -2.3$, $b \approx -7$ (ii) $x \approx 1.5$

Chapter 7 — Challenge Yourself

$y = -0.8x^2 + 2.5x + 0.3$

Revision Exercise C1

1. (i) $y + 3x = 3$ (ii) $C(6, 0)$, $M\left(3, 1\frac{1}{2}\right)$
   (iii) $3y = x + 9$
   (iv) $B\left(4\frac{1}{2}, 4\frac{1}{2}\right)$, $D\left(1\frac{1}{2}, -1\frac{1}{2}\right)$
2. 15 units^2; 6 units
3. $x^2 + y^2 + 4x - 10y + 4 = 0$; $4\sqrt{5}$
4. (i) (4, 6), (5, –1) (ii) $7y = x + 13$
   (iii) $\left(\frac{2 \pm 7\sqrt{2}}{2}, \frac{4 \pm \sqrt{2}}{2}\right)$
5. $y = e^{\frac{4x}{3} + \frac{2}{3}}$
6. $a \approx 120$, $b \approx 10$

Revision Exercise C2

1. (i) $t = 1$ (ii) $4y + 3x = 54$
   (iii) $S(6, 9)$ (iv) $11y + 2x = 48\frac{1}{2}$
   (v) 50 units^2
2. (5, 6)
3. $x^2 + y^2 - 4x - 10y + 4 = 0$
4. (i) $x^2 + y^2 = 25$ (ii) -3
5. $y = \dfrac{6}{5x^2 + 21}$
6. Plot lg y against x; $a \approx 4.5$, $b \approx 6.4$

Exercise 8A

1. (a) 1 (b) $\frac{1}{2}$
   (c) $\frac{\sqrt{3}}{2}$ (d) $-\frac{1}{2}$
2. (a) 55° (b) 28°
   (c) 46° (d) 44°
3. (a) 3rd (b) 4th
   (c) 3rd (d) 3rd
4. (a) $-\sin 20°$ (b) $-\cos 55°$
   (c) $-\sin \frac{\pi}{4}$ (d) $-\tan \frac{\pi}{3}$
5. (a) 2nd (b) 4th
   (c) 3rd (d) 4th
   (e) 2nd (f) 4th
6. (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$
   (c) $\frac{\sqrt{3}}{2}$ (d) 1
   (e) $-\frac{\sqrt{3}}{2}$ (f) $-\frac{1}{2}$
   (g) $-\frac{\sqrt{2}}{2}$ (h) 1
7. (a) $\frac{1}{2}$ (b) -1
   (c) $-\frac{1}{3}$ (d) $\frac{5 - 3\sqrt{3}}{2}$
8. (a) 18° (b) 55°
   (c) 35° (d) 20°
   (e) $\frac{\pi}{3}$ (f) $\frac{\pi}{4}$
   (g) $\frac{\pi}{6}$ (h) $\frac{\pi}{3}$
9. (a) 35°, 145° (b) 42°, 222°
   (c) 65°, 115°, 245°, 295° (d) $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$
   (e) $\frac{\pi}{5}, \frac{4\pi}{5}, \frac{6\pi}{5}, \frac{9\pi}{5}$
10. (a) cos 40° (b) $-\tan 64°$
    (c) $-\sin \frac{\pi}{3}$ (d) $\tan \frac{\pi}{4}$
11. $-\frac{3}{5}, \frac{3}{4}$
12. $-\frac{3}{5}, -\frac{4}{5}$
13. $\frac{12}{13}, -\frac{5}{12}$
14. (a) $\frac{5}{13}$ (b) $-\frac{12}{13}$
    (c) $\sqrt{\frac{2}{3}}$ (d) $-\sqrt{2}$
15. (a) $\frac{3}{5}$ (b) $-\frac{4}{5}$
    (c) $-\frac{2\sqrt{5}}{5}$ (d) $-\frac{\sqrt{5}}{5}$
16. (a) $-p$ (b) $\sqrt{1 - p^2}$
    (c) $\frac{p}{\sqrt{1 - p^2}}$ (d) $\sqrt{1 - p^2}$
17. (a) q (b) $\sqrt{1 - q^2}$
    (c) $-\frac{\sqrt{1 - q^2}}{q}$ (d) $\sqrt{1 - q^2}$
18. $-\frac{3 + \sqrt{3}}{9}$
19. (i) 3rd (ii) $\sqrt{3}$
    (iii) $\frac{4\pi}{3}, -\frac{1}{2}$

Exercise 8B

1. (a) 3, –3 (b) 2, –2
   (c) 7, –7 (d) 11, –5
   (e) 11, 1
2. (a) 4, 360° (b) 5, 360°
   (c) 1, 360° (d) 2, 360°
   (e) 3, 360°
3. (a) 2, π (b) 3, π
   (c) 3, 4π (d) 2, π
   (e) 1, 6π
5. $a = 2, b = 5$
6. $p = 7, q = 3$
7. (a) $-1 \leqslant y \leqslant 5$ (b) $-4 \leqslant y \leqslant 2$
   (c) $1 \leqslant y \leqslant 5$ (d) $1 \leqslant y \leqslant 5$
8. $1 \leqslant y \leqslant 3$
9. 6
10. 2
11. (i) $a = 3, p = 2, q = 4$ (ii) $b = -1$
12. (i) 30 m (ii) 6 m
13. (ii) (a) C (b) A
14. 5
15. (i) $a = 90, b = 80$
    (iii) (a) 170 m (b) 10 m

Exercise 8C

2. 4
3. 2
4. (a) 4 (b) 2
5. (ii) 3

Exercise 8D

1. (i) $1\frac{2}{3}$ (ii) $1\frac{1}{4}$ (iii) $\frac{3}{4}$
2. (i) $-\frac{\sqrt{5}}{2}$ (ii) $\sqrt{5}$ (iii) -2
3. (i) -4 (ii) $-\frac{4\sqrt{15}}{15}$ (iii) $\frac{\sqrt{15}}{15}$
4. (i) $\frac{\sqrt{3}}{2}$ (ii) $\frac{2\sqrt{3}}{3}$ (iii) $\frac{\sqrt{3}}{3}$

5. **(i)** $\frac{1}{3}$ **(ii)** $2\sqrt{2}$ **(iii)** $2\sqrt{2}$

6. **(i)** $\frac{2\sqrt{29}}{29}$ **(ii)** $\frac{5\sqrt{29}}{29}$ **(iii)** $\frac{\sqrt{29}}{2}$

Exercise 8E

1. **(a)** 46.4°, 133.6° **(b)** 64.3°, 295.7°
(c) 132.4°, 227.6° **(d)** 126.1°, 306.1°
2. **(a)** 13.4°, 76.6° **(b)** 27.5°, 117.5°
(c) 102.6°, 167.4° **(d)** 68.9°, 111.1°
3. **(a)** 337.8° **(b)** 113.0°
(c) 241.9° **(d)** 339.3°
4. **(a)** 43°, 137° **(b)** 38.6°, 321.4°
(c) 45.0°, 225.0° **(d)** 78°, 102°, 258°, 282°
(e) 18.8°, 78.8°, 138.8°, 198.8°, 258.8°, 318.8°
5. **(a)** 108.0°, 162.0°, 288.0°, 342.0°
(b) 44.5°, 75.5°, 164.5°, 195.5°, 284.5°, 315.5°
(c) 16.8°, 58.2°, 196.8°, 238.2°
(d) 56.7°, 138.3°, 236.7°, 318.3°
(e) 10.2°, 100.2°, 190.2°, 280.2°
(f) 127.7°, 340.3°
6. **(a)** –0.723, 0.723 **(b)** –2.04, 1.11
(c) –2.84, –0.305 **(d)** –1.66, 1.48
7. **(a)** 4.14, 5.29 **(b)** 0.630, 2.61, 3.77, 5.75
(c) 2.95, 6.09 **(d)** 1.58, 2.73, 4.73, 5.87
9. **(a)** 0°, 25°, 180°, 205°, 360°
(b) 0°, 45°, 180°, 225°, 360°
(c) 0°, 66.0°, 180°, 246.0°, 360°
(d) 90°, 270°
10. **(a)** $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$
(b) $\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}$
(c) 0, 0.111, 3.03, π, 2π
(d) $0, \frac{\pi}{8}, \frac{\pi}{2}, \frac{5\pi}{8}, \pi, \frac{9\pi}{8}, \frac{3\pi}{2}, \frac{13\pi}{8}, 2\pi$
11. **(a)** 18.4°, 198.4°
(b) 54.7°, 125.3°, 234.7°, 305.3°
(c) 0°, 180°, 360°
(d) 30°, 90°, 150°, 270°

Review Exercise 8

1. **(a)** $\frac{3}{4}$ **(b)** 0 **(c)** $3\sqrt{3} - 5$
2. **(a)** $-3 \leqslant f(x) \leqslant 7$, 5, $\frac{2\pi}{3}$
(b) $0 \leqslant f(x) \leqslant 2$, 1, 2π
(c) $1 \leqslant f(x) \leqslant 3$, 2, 4π
4. 4 **5.** 4
6. **(a)** 22.5°, 82.5°, 202.5°, 262.5°
(b) 60°, 90°, 240°, 270°
(c) 62.6°, 117.4°, 242.6°, 297.4°
(d) 53.1°, 90°, 126.9°, 270°
(e) 0°, 121.0°, 180°, 301.0°, 360°
(f) 71.6°, 251.6°

7. **(a)** $q = 3, m = 3$ **(b)** $p = -1$
8. **(i)** 2
(iii) $a = 2, b = 2, c = -1$; 30°, 150°, 210°, 330°
9. **(i)** $a = \pm 3$ **(ii)** $b = 2$
(iii) 4

Chapter 8 — Challenge Yourself

1. $\sin x = \frac{2p}{1+p^2}$, $\cos x = \frac{p^2-1}{p^2+1}$
2. **(a)** $\frac{4}{5} \leqslant y \leqslant 4$ **(b)** $1 \leqslant y \leqslant 2$
3. $\frac{183}{340}$

Exercise 9A

1. **(a)** 45°, 153.4°, 225°, 333.4°
(b) 270°
(c) 33.7°, 153.4°, 213.7°, 333.4°
(d) 0°, 360°
2. **(a)** 0.340, 2.80
(b) $\frac{\pi}{3}, \frac{5\pi}{3}$, 1.82, 4.46
(c) 0.766, 5.52
(d) 0.464, 1.11, 3.61, 4.25
3. **(a)** 26.6°, 153.4°, 206.6°, 333.4°
(b) 72.3°, 147.5°, 252.3°, 327.5°
(c) 66.4°, 180°, 293.6°
(d) 63.4°, 146.3°, 243.4°, 326.3°
4. **(a)** $\frac{\pi}{2}, \frac{3\pi}{2}$ **(b)** $\frac{\pi}{6}, \frac{5\pi}{6}$
(c) 0, 1.77, 4.51, 2π **(d)** 0.927, 2.21
5. $6\sin^2 x + 13\sin x - 5 = 0$
$x = 19.5°, 160.5°$

Exercise 9B

5. $\sin x = \frac{2p}{p^2+1}$, $\cos x = \frac{p^2-1}{p^2+1}$

Exercise 9C

1. **(a)** sin 62° **(b)** cos 40°
(c) sin 103° **(d)** cos 51°
(e) tan 40° **(f)** tan 25°
2. **(a)** $\frac{1}{4}(\sqrt{6} - \sqrt{2})$ **(b)** $\frac{1}{4}(\sqrt{6} + \sqrt{2})$
(c) $\frac{1}{4}(\sqrt{6} + \sqrt{2})$ **(d)** $\frac{1}{4}(\sqrt{2} - \sqrt{6})$
(e) $\frac{1}{4}(\sqrt{6} + \sqrt{2})$ **(f)** $2 + \sqrt{3}$
(g) $2 + \sqrt{3}$ **(h)** $-2 - \sqrt{3}$
3. **(i)** $\frac{56}{65}$ **(ii)** $\frac{33}{65}$ **(iii)** $\frac{16}{63}$
4. **(i)** $-\frac{16}{65}$ **(ii)** $-\frac{33}{65}$ **(iii)** $\frac{16}{63}$
5. **(i)** $-\frac{33}{65}$ **(ii)** $\frac{16}{65}$ **(iii)** $\frac{33}{56}$
6. **(i)** $-\frac{36}{85}$ **(ii)** $-\frac{13}{85}$ **(iii)** $-\frac{36}{77}$

7. **(i)** $\frac{84}{85}$ **(ii)** $-\frac{13}{85}$ **(iii)** $\frac{36}{77}$

8. 130°, 310°

9. **(i)** $\frac{4}{13}$ **(ii)** $\frac{56}{65}$ **(iii)** $5\frac{2}{5}$

10. **(i)** $\frac{1+a}{1-a}$ **(ii)** $\frac{\sqrt{3}+a}{2\sqrt{1+a^2}}$ **(iii)** $\frac{\sqrt{3}+a}{2\sqrt{1+a^2}}$

11. **(i)** $\frac{b+a\sqrt{1-b^2}}{\sqrt{1+a^2}}$ **(ii)** $\frac{ab+\sqrt{1-b^2}}{\sqrt{1+a^2}}$

(iii) $\frac{a\sqrt{1-b^2}-b}{ab+\sqrt{1-b^2}}$

12. **(i)** $xy+\sqrt{(1-x^2)(1-y^2)}$

(ii) $-(y\sqrt{1-x^2}+x\sqrt{1-y^2})$

(iii) $\frac{2x\sqrt{1-x^2}}{2x^2-1}$ **(iv)** $-2y\sqrt{1-y^2}$

(v) $2y^2-1$

13. **(i)** $\frac{1}{7}$ **(ii)** $\frac{3}{4}$ **(iii)** $\frac{\sqrt{5}}{5}$

14. $-\frac{3}{5}, \frac{4}{5}, \frac{5}{13}, \frac{12}{13}, -\frac{4}{7}$

Exercise 9D

1. **(a)** sin 50° **(b)** cos 54°
(c) tan 70° **(d)** cos 50°
(e) cos 26°

2. **(i)** $\frac{120}{119}$ **(ii)** $\frac{5}{12}$

(iii) $\frac{12}{13}$ **(iv)** $\frac{5}{13}$

3. $1\frac{1}{3}, 5\frac{1}{2}$

4. **(i)** $\frac{1}{3}\sqrt{5}$ **(ii)** $\frac{1}{6}\sqrt{30}$

(iii) $\frac{1}{5}\sqrt{5}$ **(iv)** $\sqrt{6}-\sqrt{5}$

5. **(i)** $\frac{7}{25}$ **(ii)** $-\frac{24}{7}$

(iii) $\frac{3}{10}\sqrt{10}$ **(iv)** $-\frac{527}{625}$

6. **(i)** $-\frac{240}{289}$ **(ii)** $\frac{3}{34}\sqrt{34}$ **(iii)** $-\frac{7}{25}$

7. **(i)** $\frac{5}{26}\sqrt{26}$ **(ii)** $\frac{7}{25}$ **(iii)** $\frac{119}{169}$

8. **(i)** $-\frac{24}{25}$ **(ii)** $-\frac{336}{625}$ **(iii)** 3

9. **(i)** $-\frac{3}{5}$ **(ii)** $1\frac{1}{3}$ **(iii)** $\frac{24}{25}$

10. **(a)** 0°, 82.8°, 180°, 277.2°, 360°
(b) 90°, 197.5°, 270°, 342.5°
(c) 38.7°, 141.3°, 270°
(d) 0°, 57.3°, 122.7°, 180°, 237.3°, 302.7°

11. **(a)** 0.340, $\frac{\pi}{2}$, 2.80, $\frac{3\pi}{2}$
(b) 0, 1.79, π, 4.50, 2π
(c) $\frac{\pi}{6}$, $\frac{5\pi}{6}$, 0.644, 2.50
(d) 0, π, 2π, 0.464, 2.68, 3.61, 5.82

12. **(i)** $\frac{1-x^2}{x^2}$ **(ii)** $2x\sqrt{1-x^2}$

(iii) $1-8x^2+8x^4$ **(iv)** $\sqrt{\frac{1-x}{2}}$

13. **(a)** $45° < A < 90°$ **(b)** $30° < A < 90°$
(c) $45° < A < 90°$

14. **(i)** $\frac{7}{25}$ **(ii)** $\frac{3\sqrt{10}}{10}$ **(iii)** $-\frac{7}{24}$

Exercise 9F

1. **(a)** $\sqrt{29}\sin(\theta+68.2°)$; $\pm\sqrt{29}$; $\theta = 21.8°, 201.8°$
(b) $2\sin(\theta+30°)$; ± 2; $\theta = 60°, 240°$

2. **(a)** $2\sqrt{2}\sin(\theta-20.7°)$; $\pm 2\sqrt{2}$; $\theta = 110.7°, 290.7°$
(b) $2\sqrt{2}\sin(\theta-37.8°)$; $\pm 2\sqrt{2}$, $\theta = 127.8°, 307.8°$

3. **(a)** $\sqrt{13}\cos(\theta+33.7°)$; $\pm\sqrt{13}$; $\theta = 326.3°, 146.3°$
(b) $\sqrt{7}\cos(\theta+40.9°)$; $\pm\sqrt{7}$; $\theta = 319.1°, 139.1°$

4. max = $\sqrt{29}$ cm, t = 1.19 s; min = $-\sqrt{29}$ cm, t = 4.33 s

5. **(a)** 8.1°, 203.8° **(b)** 309.2°
(c) 119.6°, 346.7° **(d)** 82.2°, 207.2°

6. **(a)** 1.82, 5.64 **(b)** 0.830, 3.24
(c) 0.537, 4.46 **(d)** 2.50, 6.17

7. **(a)** 53.8°, 164.9°, 233.8°, 344.9°
(b) 16.6°, 129.7°, 196.6°, 309.7°
(c) 41.0°, 114.5°, 161.0°, 234.5°, 281.0°, 354.5°
(d) 21.6°, 65.4°, 141.6°, 185.4°, 261.6°, 305.4°

8. **(i)** $13\cos(x+22.6°)$
(ii) max = 13, θ = 337.4°, min = –13, θ = 157.4°

9. $13\sin(\theta-67.4°)$
(i) 100°, 214.8° **(ii)** 169, 0

10. $5\sin(x+53.1°)$
(a) max = 6, min = –4
(b) max = 26, min = 1
(c) max = 126, min = –124
(d) max = 626, min = 1

11. **(ii)** $\sqrt{23\,125}\sin(\theta+80.5°)$
(iii) 152 cm, θ = 9.5°
(iv) 63.2°

12. **(ii)** $\sqrt{2.32}\cos(\theta-23.2°)$
(iii) θ = 23.2° **(iv)** 67.0°

13. $\sqrt{13}$ amperes, 0.588 s

14. max = 17, min = 7; θ = 45°, 171.9°, 225°, 351.9°

15. **(ii)** $R = \sqrt{89}$, α = 32.0°
(iii) 10.1°
(iv) $SP = \sqrt{89}\sin(\theta+32.0°)$
(vi) θ = 13.0°

Review Exercise 9

1. **(i)** $\frac{1}{3}$ **(ii)** $-\frac{\sqrt{2}}{4}$

(iii) $-\frac{7}{8}\sqrt{2}$ **(iv)** $-\frac{56}{81}\sqrt{2}$

2. **(i)** $-\frac{16}{65}$ **(ii)** $\frac{\sqrt{10}}{10}$

(iii) $\frac{25}{24}$ **(iv)** $-\frac{625}{527}$

3. **(i)** $\frac{2p}{p^2+1}$ **(ii)** $\frac{p}{\sqrt{p^2+1}}$

(iii) $\frac{1}{p}$ **(iv)** $\frac{4p^3-4p}{(p^2+1)^2}$

4. **(a)** 15°, 75°, 195°, 255°
(b) 30°, 120°, 210°, 300°
(c) 90°, 120°, 270°, 300°
(d) 20°, 70°, 200°, 250°
(e) 20.9°, 69.1°, 200.9°, 249.1°
(f) 101.5°, 258.5°
(g) 27.9°, 152.1°
(h) 11.3°, 45°, 191.3°, 225°
(i) 100.4°
(j) 221.8°, 318.2°
(k) 26.9°, 230.4°
(l) 42.5°, 195.6°
(m) 74.3°, 173.1°, 254.3°, 353.1°
(n) 7.9°, 85.0°, 127.9°, 205.0°, 247.9°, 325.0°

6. **(i)** $-\frac{33}{56}$ **(ii)** $-\frac{25}{7}$ **(iii)** $-\frac{24}{25}$

7. **(i)** $\frac{1}{5}$ **(ii)** $\frac{120}{169}$ **(iii)** $\frac{17}{26}\sqrt{2}$

(iv) $\frac{26}{12-5\sqrt{3}}$ or $\frac{312+130\sqrt{3}}{69}$

(v) $\frac{37}{55}$

8. **(i)** $2a\sqrt{1-a^2}$ **(ii)** $\frac{\sqrt{1-a^2}-a}{\sqrt{1-a^2}+a}$

(iii) $\frac{2}{\sqrt{3(1-a^2)}-a}$

9. **(i)** $\frac{1}{2}$ **(ii)** $\frac{2\sqrt{5}}{25}$

(iii) $\frac{25}{24}$ **(iv)** $\frac{576}{625}$

(v) $\frac{9}{25}$

10. **(i)** $-2\sqrt{2}$ **(ii)** $-\frac{7}{9}$ **(iii)** $\frac{\sqrt{3}}{3}$

12. **(a)** 0, π, 2π, 1.19, 5.10 **(b)** 0.848, $\frac{\pi}{2}$, 2.29, $\frac{3\pi}{2}$
(c) 0, 2.56, 3.73, 2π **(d)** 0.442, 3.99
(e) 1.77, 5.96

13. **(a)** 0°, 90°, 180°
(b) 22.5°, 67.5°, 112.5°, 157.5°

14. 13 sin $(x - 22.6°)$
(i) 57.8°, 167.4°
(ii) max = 4, θ = 112.6°, min = –22, θ = 292.6°

15. **(ii)** 4252.1 cos $(\theta - 0.852)$
(iii) $\theta = 0.852$ **(iv)** $\theta = 1.198$

16. **(ii)** $\sqrt{865}$ sin $(\theta + 35.3°)$
(iii) $\sqrt{865}$, $\theta = 54.7°$ **(iv)** 16.1°

17. **(i)** $\sqrt{1280}$ sin $(2\theta + 1.11)$
(ii) 0.629 **(iii)** $\sqrt{1280}$; 0.230

Chapter 9 — Challenge Yourself

1. $\frac{1}{4}, -\frac{7}{8}, -\frac{11}{16}$

Revision Exercise D1

1. **(i)** $-\frac{11}{15}$ **(ii)** $\frac{15}{16}$

2. **(a)** 60°, 170°, 240°, 350°
(b) 30°, 150°, 270°
(c) 22.6°, 90°
(d) 0°, 30°, 180°, 210°, 360°

4. **(a)** 0.251, 6.03 **(b)** 0, $\frac{\pi}{12}$, $\frac{5\pi}{12}$, π, $\frac{13\pi}{12}$, $\frac{17\pi}{12}$, 2π

5. $\sqrt{10} + \sqrt{6}$ **6.** 4 solutions

7. $3\sqrt{5}$ sin $(\theta + 63.4°)$
(i) 108.0°, 305.1°
(ii) max = $3\sqrt{5}$, θ = 26.6°
min = $-3\sqrt{5}$, θ = 206.6°

Revision Exercise D2

1. **(i)** $\frac{7}{10}\sqrt{2}$ **(ii)** $-\frac{1}{7}$ **(iii)** $3\frac{3}{7}$

2. **(a)** 130.5°, 349.5°
(b) 56.3°, 135°, 236.6°, 315°
(c) 15°, 75°, 195°, 255°
(d) 30°, 109.5°, 150°, 250.5°

4. **(a)** 2.30 or 5.84 **(b)** $\frac{2\pi}{3}$, $\frac{4\pi}{3}$, π

5. 0°, 30°, 150°, 180°, 210°, 330°, 360°

6. 4 solutions

7. **(i)** $P = 4\sqrt{5}$ sin $(\theta + 0.464)$
$\theta = 0.791$ or 1.42
(ii) max = $4\sqrt{5}$, $\theta = 1.11$

Exercise 11A

1. **(a)** $3x^2$ **(b)** $50x^4 - 12x^2 + 7$

(c) $20x^3 - \frac{1}{x^2} + \frac{2}{x^3}$ **(d)** $14x^3 - \frac{20}{3x^5} + 1$

(e) $12x^3 - \frac{4}{3x^5} - \frac{3}{x^3}$ **(f)** $8x + 12$

(g) $4x^3 - 4x$ **(h)** $5x^4 - 3x^2 + 4x$

(i) $4x^3 - \frac{8}{x^3} + 4$ **(j)** $1 - \frac{5}{x^2}$

(k) $\frac{9}{2}\sqrt{x} + \frac{1}{2\sqrt{x}}$ **(l)** $15x^2 - \frac{3}{x^2} + \frac{1}{2\sqrt{x^3}}$

2. **(a)** -3 **(b)** $30\frac{13}{16}$
(c) 31 **(d)** $4\frac{16}{27}$

3. **(a)** 8 **(b)** $4\frac{25}{27}$
(c) 2 **(d)** 1
(e) 3

4. $(4, 3)$

5. **(a)** $10ax + 9bx^2$ **(b)** $2a^2 + 3c^2x^2$
(c) $-\frac{2b}{x^3}$ **(d)** $\frac{3}{2}a\sqrt{x} + \frac{b}{2\sqrt{x}}$

6. **(a)** $-\frac{3}{t^2}$ **(b)** $\frac{2}{\sqrt{t^3}}$
(c) $\frac{1}{2\sqrt{t}} - \frac{1}{2\sqrt{t^3}}$ **(d)** $1 - \frac{4}{t^2}$
(e) $-\frac{14}{t^5} - \frac{1}{\sqrt{t^3}} + \frac{4}{3\sqrt[3]{t}}$ **(f)** $4 - \frac{1}{3\sqrt{t^3}}$

7. $7\frac{13}{16}$ **8.** -5

9. $a = 3, b = -8$ **10.** $a = -1, b = -6$

11. $a = -5, b = 2$ **12.** $-\frac{4}{9}$

13. $x < 1$ or $x > 3$ **14.** $6t + 5, t < -\frac{5}{6}$

Exercise 11B

1. **(a)** $14(2x + 5)^6$ **(b)** $-32(3 - 4x)^7$
(c) $\frac{5}{2}\left(\frac{1}{2}x - 4\right)^4$ **(d)** $\frac{15}{4}(5x - 3)^5$
(e) $15(x + 4)^4$ **(f)** $\frac{4}{9}\left(\frac{x}{6} - 1\right)^3$

2. **(a)** $\frac{-3}{(3x + 2)^2}$ **(b)** $\frac{-6x}{(x^2 + 2)^4}$
(c) $\frac{60}{(3 - 4x)^4}$ **(d)** $\frac{30}{(2 - 5x)^2}$
(e) $-\frac{72x}{(2 + 3x^2)^2}$ **(f)** $\frac{9x}{2(5 - 3x^2)^2}$

3. **(a)** $\frac{1}{\sqrt{2x + 3}}$ **(b)** $\frac{3}{2\sqrt{4 + 3x}}$
(c) $\frac{5x}{\sqrt{5x^2 + 6}}$ **(d)** $\frac{9x^2}{2\sqrt{3x^2 - 4}}$
(e) $\frac{2(x - 1)}{\sqrt{2x^2 - 4x + 5}}$ **(f)** $\frac{4x}{3(2x^2 - 5)^{\frac{2}{3}}}$
(g) $-\frac{1}{\sqrt{(x - 3)^3}}$ **(h)** $-\frac{5x}{\sqrt{18 - 5x^2}}$

4. 3 **5.** -18

6. -9 **7.** -144

8. **(a)** $5\left(x^2 + \frac{3}{x}\right)^4\left(2x - \frac{3}{x^2}\right)$
(b) $-\frac{4x}{3\sqrt{(2x^2 - 5)^3}}$
(c) $\frac{1}{3}\left(x + \frac{2}{x}\right)^{-\frac{2}{3}}\left(1 - \frac{2}{x^2}\right)$
(d) $\frac{-3(4x + 7)}{\sqrt{(2x^2 + 7x)^3}}$

9. $(2, 2)$

10. $a = 1\frac{1}{4}, b = -\frac{3}{4}$; $a = 4, b = 2$

11. **(a)** $5(ax^2 + bx^3 + c)^4(2ax + 3bx^2)$
(b) $\frac{2ax + b}{2\sqrt{ax^2 + bx + c}}$ **(c)** $\frac{1}{\sqrt{(a^2 - 2x)^3}}$
(d) $-\frac{2x}{(x^2 - 3a^2 + 2b)^{\frac{3}{2}}}$
(e) $5\left(ax + \frac{b}{x}\right)^4\left(a - \frac{b}{x^2}\right)$
(f) $8\left(a^2x^2 + \frac{b^3}{x^2}\right)^3\left(a^2x - \frac{b^3}{x^3}\right)$

Exercise 11C

1. **(a)** $6 - 4x - 9x^2$ **(b)** $\frac{3x + 5}{2\sqrt{x + 4}}$
(c) $2(5x + 3)(x - 3)^3$
(d) $2(4x + 7)(x + 1)^2(x + 3)^4$
(e) $9(x + 2)(x + 5)^2(x - 4)^5$

2. $21, 85$ **3.** 18

4. $(x + 3)^3(x - 5)^6(11x + 1)$; $-3, 5$ or $-\frac{1}{11}$

5. $\frac{3}{2}, -5$ or $-\frac{15}{16}$

6. **(a)** $\frac{24 - 105x^3}{2\sqrt{4 - 7x^3}}$ **(b)** $(x + 5)^5(8x^2 - 11x - 15)$
(c) $x(x - 2)^6(10x^2 + 3x - 4)$
(d) $\frac{12x^2 - 2x + 9}{\sqrt{2x^2 + 3}}$ **(e)** $(21x - 2)(3x - 1)^{\frac{3}{2}}$

7. $\frac{3x - 6}{\sqrt{2x - 1}}$; 2 **8.** $\frac{8}{(2 - x)^2}$, 1 or 3

9. $\frac{14 - 3x}{2\sqrt{9 - x}}$; $\frac{14}{3}$

10. **(a)** $\frac{a^2x - 3x^3}{\sqrt{a^2 - x^2}}$
(b) $a(3ax - b)^2(2ax - b^3)(30ax - 4b - 9b^3)$
(c) $(2bx - x^3)(20a^2bx^3 - 18a^2x^5 + 18bx - 21x^3)$
(d) $\frac{6a^3 + 8ax + 3b}{3(a^2 + x)^{\frac{2}{3}}}$ **(e)** $\frac{2x^2 + 3abx + a^2}{\sqrt{a^2 + x^2}}$
(f) $(x + a)^2(x^2 - b)^3(11x^2 + 8ax - 3b)$

Exercise 11D

1. **(a)** $\frac{11}{(1 - 4x)^2}$ **(b)** $\frac{5}{(3 - x)^2}$
(c) $-\frac{9}{(2x - 3)^2}$ **(d)** $\frac{4(2x^2 + 7x + 1)}{(1 - 2x^2)^2}$
(e) $-\frac{2(3x^2 + 24x + 2)}{(3x^2 - 2)^2}$ **(f)** $\frac{25}{(2x + 3)^2}$
(g) $\frac{2x(x - 1)}{(2x - 1)^2}$ **(h)** $\frac{1 - 2x - x^2}{(x^2 + 1)^2}$
(i) $\frac{3}{(2 - x)^2}$ **(j)** $\frac{3 - x^2}{(x^2 + 3)^2}$

2. 20 **3.** 0 or -2

4. $(0, 0), \left(\frac{1}{2}, \frac{3}{4}\right)$

5. (a) $\frac{3-x}{2\sqrt{x}(3+x)^2}$ (b) $\frac{x-6}{(x-3)^{\frac{3}{2}}}$
(c) $\frac{2x+11}{2(x+1)^{\frac{3}{2}}}$ (d) $\frac{4-3x^2}{2\sqrt{x}(x^2+4)^2}$

6. $\frac{8}{13}$ 7. $\frac{5+4x-x^2}{(x^2+5)^2}$; $2 < x < 5$

8. 14.21 or –0.21 9. $\frac{1}{2}$, $\frac{8}{81}$

10. (a) $\frac{4a^2x}{(a^2-x^2)^2}$
(b) $\frac{4ab^2x+b^3+6ax^4+6bx^3}{(b^2-3x^3)^2}$
(c) $\frac{-2ax(bx+2c)}{(ax^2-bx-c)^2}$ (d) $\frac{b(a+c)}{(cx+b)^2}$

11. (a) $\frac{3a}{2\sqrt{a+x}\sqrt{(2a-x)^3}}$ (b) $\frac{abx+2ac+b^2}{2\sqrt{(bx+c)^3}}$
(c) $-\frac{a^2}{\sqrt{(2ax-x^2)^3}}$ (d) $\frac{a^{\frac{5}{2}}-a^{\frac{1}{2}}x^2}{2\sqrt{x}(a^2+x^2)^{\frac{3}{2}}}$

Exercise 11E

1. (a) $y = 5x - 7$ (b) $y = 6x - 8$
(c) $y = -6x - 1$ (d) $2y = x - 1$
(e) $y + 16x = 0$
2. (a) $2y + x = 7$ (b) $10y + x = 91$
(c) $3y + 27x = 56$ (d) $y = x + 2$
(e) $8y + x = 34$
3. $2 - \frac{1}{x^2}$, $7y + 4x = 39\frac{1}{2}$ 4. $\left(0, 19\frac{9}{23}\right)$
5. $y + 5x = 10$ 6. $\left(\frac{1}{2}, 4\right)$, $\left(-\frac{1}{2}, -4\right)$
7. $y = x - 1$, $(0, -1)$ 8. $a = 2$, $b = -3$, $c = 5$
9. $y = 2x - 7$, $1\frac{3}{4}$ 10. $y = 3x - 8$, $(0, -8)$
11. $y = 3x - 5$, $\left(\frac{1}{3}, -\frac{11}{9}\right)$ 12. $k = 1$, $\left(\frac{2}{3}, 2\frac{5}{9}\right)$
13. (i) 6 (ii) $x = 4$

Exercise 11F

1. $\frac{3\sqrt{5}}{25}$
2. $6t^2 - 12t + 4$; $t \leqslant 0$ or $t \geqslant 2$
3. 4π cm^2/s 4. 120 cm^2/s
5. $\frac{1}{12}$ cm/s 6. $\frac{1}{5\pi}$ cm/s
7. (i) 3.6π cm^2/s (ii) No
8. $76\frac{2}{3}$ cm^3/s
9. –45 units/s 10. $-\frac{13}{20}$ units/s
11. 4 cm^3/s, 2.6 cm^2/s 12. $200h\pi$ cm^3/s
13. (i) $\frac{5}{8\pi}$ cm/s (ii) Yes
14. (i) $0.45(3x+2)^2$ (ii) 0.02 m/s
15. $\frac{25}{16\pi k}$ m/s 16. $\frac{64a}{45\pi}$ cm/s
17. (i) $10\sqrt{\frac{2}{\pi}}$ cm (ii) $\frac{\sqrt{2\pi}}{5\pi}$ cm/s
18. (i) $\frac{125}{36\pi}$ cm/s (ii) 0.251 cm/s
19. –7.5 cm/s

Review Exercise 11

1. (a) $-\frac{1}{4}$ (b) 18
(c) $\frac{3}{16}$ (d) 0
(e) –5 (f) –13, 13
2. (a) $25(5x-4)^4$ (b) $\frac{3x^2}{\sqrt{2x^3+5}}$
(c) $15x(5x^2+3)^{\frac{1}{2}}$ (d) $\frac{6x}{(2+9x^2)^{\frac{2}{3}}}$
(e) $-\frac{27}{(3x-5)^4}$ (f) $\frac{3x+34}{(2x+9)^{\frac{3}{2}}}$
3. (a) $6(ax^2+bx)^5(2ax+b)$
(b) $(2ax+b)^4(5x^2-ab)^5(170ax^2+60bx-10a^2b)$
(c) $-\frac{2x^2(abx^2+b^2x+ac)}{(bx^3-c)^2}$
(d) $\frac{-a^3-acx^2-2bcx}{(cx^2-a^2)^2}$
4. $\left(\frac{1}{2}, -1\frac{1}{2}\right)$, $\left(-\frac{1}{2}, 1\frac{1}{2}\right)$
5. $a = \frac{4}{27}$, $b = -1$
6. $k = 8$, $p = 13$, $q = -2$, $c = 12$
7. $12r^2 - 6r - 18$, $-1 < r < \frac{3}{2}$
8. $\frac{5}{18}$, $1\frac{3}{5}$
9. –14, $14y - x = 304$
10. $a = 5$, $b = -3$; $9y = 268x - 396$
11. (a) –4.8 units/s (b) –0.3 units/s
12. (a) –0.3 units/s (b) –0.849 units/s
13. –8 cm/s 14. $\frac{1}{6\pi}$ cm/s
15. $P(0, -2)$
16. $V = 4\sqrt{3}\ h^2$; –4.99 cm^3/s
17. $T = \frac{1}{2}(6-x)(2x^2+3)$,
$-3x^2 + 12x - 1\frac{1}{2}$, $\frac{8}{105}$ unit/s
18. (i) $V = \frac{16\pi}{675}h^3$ (ii) $\frac{125}{256\pi}$ cm/s

Chapter 11 — Challenge Yourself

1. (a) $A = x^3 - x$ (b) $\frac{47}{48}$ units^2/s
2. (i) $\frac{9}{64}$ cm/s (ii) 1 cm^2/s

Exercise 12A

1. **(a)** $12x^3$, $36x^2$ **(b)** 5, 0
(c) $6x + 5$, 6 **(d)** $-\frac{1}{x^2}$, $\frac{2}{x^3}$
(e) $30(3x + 2)^9$, $810(3x + 2)^8$
(f) $\frac{1}{2\sqrt{x-4}}$, $-\frac{1}{4}(x-4)^{-\frac{3}{2}}$

2. **(a)** $1 - \frac{1}{x^2}$; $\frac{2}{x^3}$
(b) $-\frac{1}{x^2} + \frac{2}{x^3}$; $\frac{2}{x^3} - \frac{6}{x^4}$
(c) $\frac{15}{2}x^{\frac{3}{2}} - \frac{1}{2}x^{-\frac{1}{2}}$; $\frac{45}{4}x^{\frac{1}{2}} + \frac{1}{4}x^{-\frac{3}{2}}$
(d) $-\frac{1}{(x-1)^2}$; $\frac{2}{(x-1)^3}$
(e) $\frac{-7}{(x-1)^2}$; $\frac{14}{(x-1)^3}$
(f) $\frac{x^2 + 2x}{(x+1)^2}$, $\frac{2}{(x+1)^3}$

3. **(i)** $\left(\frac{3}{4}, \frac{25}{8}\right)$, $\left(-\frac{1}{4}, -\frac{13}{8}\right)$
(ii) 152

4. **(a)** $3ax^2 + 2bx$; $6ax + 2b$
(b) $(4ax + 2b)(ax^2 + bx + c)$; $2(6a^2x^2 + 6abx + 2ac + b^2)$
(c) $\frac{3}{2}\sqrt{x}$; $\frac{3}{4\sqrt{x}}$
(d) $\frac{5a}{2}x^{1\frac{1}{2}} + \frac{3b}{2}x^{\frac{1}{2}}$; $\frac{15a}{4}x^{\frac{1}{2}} + \frac{3b}{4}x^{-\frac{1}{2}}$
(e) $\frac{a}{c} - \frac{b}{cx^2}$; $\frac{2b}{cx^3}$
(f) $\frac{-(bc + ad)}{(cx - d)^2}$; $\frac{2c(bc + ad)}{(cx - d)^3}$

Exercise 12B

1. **(a)** $x > -5$ **(b)** $x > 2$
(c) $x < -1$ **(d)** $x > 4$ or $x < -3$
(e) $x > 2\frac{1}{2}$ or $x < -1$ **(f)** $x > 5$ or $x < 3$

2. **(a)** $x < 1$ **(b)** $x > -1\frac{3}{4}$
(c) $-1 < x < 2$ **(d)** $-1\frac{1}{2} < x < 4$
(e) $-1 < x < 1\frac{1}{2}$ **(f)** $-4 < x < 2$

3. $x > 3$ **4.** $2 < x < 5$
5. $x > 6$ or $0 < x < 3$

Exercise 12C

1. **(a)** $\left(-1\frac{1}{2}, -1\frac{1}{4}\right)$ min **(b)** (1, –2) max
(c) (1, –6) pt. of inflexion
(d) $\left(-\frac{2}{3}, 8\frac{1}{9}\right)$ max, $\left(1\frac{1}{3}, -3\frac{8}{9}\right)$ min
(e) (0, 0) min, (2, 4) max
(f) (2, –64) min, (–2, 64) max

2. **(a)** $\left(\frac{1}{3}, 5\right)$ min, $\left(-\frac{1}{3}, -3\right)$ max
(b) (3, 0) min, (–3, –12) max
(c) (0, 4) pt. of inflexion
(d) $\left(\frac{2}{3}, 2\frac{5}{27}\right)$ min, $\left(-1, 4\frac{1}{2}\right)$ max

3. **(a)** (1, 6) max, (2, 5) min
(b) (–3, 34) max, (1, 2) min
(c) (1, 4) min, (0, 5) pt. of inflexion
(d) (5, 150) min

4. **(a)** $\left(2\frac{1}{2}, 171\frac{7}{8}\right)$ max, (3, 171) min, $\left(-2\frac{1}{2}, -328\frac{1}{8}\right)$ min
(b) (1, 0) min, (–1, 4) max
(c) (4, 48) min
(d) (2, 3) min, (0, –1) max

5. (2, 35) max, (3, 34) max

6. **(i)** $\frac{-7}{(2x-1)^2}$
(ii) No, $\frac{dy}{dx} \neq 0$ for all x.

7. **(i)** $\frac{x^2 - 6x + 5}{(x-3)^2}$ **(ii)** (1, 0) max, (5, 8) min

8. **(ii)** (0, 0) min

9. **(i)** $-\frac{3}{\sqrt{(2x-1)^3}}$ **(ii)** $\frac{dy}{dx} \neq 0$ for all x

10. $-\frac{1}{(3x-5)^2}$, $\frac{dy}{dx} \neq 0$ for all x

11. **(i)** $a = 1, b = 16$ **(ii)** min

12. **(i)** $a = 3, b = 4$ **(ii)** (0, 5) pt. of inflexion

13. **(i)** $a = 16, b = 1$ **(ii)** min
(iii) $x > \frac{1}{2}$

14. **(i)** $p = -9, q = 24$
(ii) (4, 17)
(iii) (4, 17) min, (2, 21) max
(iv) min gradient = –3 occurs when $x = 3$

15. **(i)** $p = -4, q = -1$ **(ii)** $\left(-1.5, -34\frac{3}{4}\right)$
(iii) $\left(-1.5, -34\frac{3}{4}\right)$ min, (2, 51) max
(iv) $-1\frac{1}{2} < x < 2$

16. **(i)** (1, –2) min, (4, 5) max, (6, 2) min

Exercise 12D

1. 1681
2. 2
3. 12 800 m^2
4. $V = x^2(15 - x)$, 500 cm^3
5. $\theta = \dfrac{80 - 2r}{r}$; $r = 20$, max $A = 400$ cm^2
6. $h = r = 6$
7. $x = 100$, $y = 133\frac{1}{3}$
8. $l = 2\sqrt[3]{2}$ m, $h = \sqrt[3]{2}$ m
9. 8.26 cm, 6.74 cm
10. $x = 2.61$, $y = 3.91$, height $= 2.94$
11. $h = \dfrac{600}{\pi r^2}$, $r = 4.57$, $h = 9.14$
12. $r = \dfrac{12 - 2x}{\pi}$, $x = \dfrac{24}{\pi + 4}$
13. $AP = 44.6$ m
14. $r = \frac{2}{3}a$, $h = \frac{1}{3}a$
15. **(i)** $r = \sqrt{24h - h^2}$
    (iii) $8\sqrt{2}$; max 2140 cm^3
16. 4.62, 1240 cm^3

Review Exercise 12

1. **(a)** $(-3, 83)$ max, $\left(\frac{1}{2}, -2\frac{3}{4}\right)$ min
   (b) $(1, 3)$ min, $\left(-1\frac{1}{3}, 15\frac{19}{27}\right)$ max
   (c) $\left(\frac{1}{2}, 2\frac{5}{8}\right)$ min
   (d) $(0, 7)$ max, $(2, 3)$ min
   (e) $(-2, -5)$ max, $(2, -5)$ max
   (f) $(0, 5)$ pt. of inflexion, $(1, 4)$ min
2. $2\frac{1}{2}$, $-1\frac{1}{5}$
3. $\left(\frac{1}{4}, -\frac{27}{128}\right)$ min, $(1, 0)$ pt. of inflexion
4. $-13\frac{1}{2}$
5. $78\frac{1}{8}$
6. 150 m × 300 m
7. 6.51 cm
8. 5
9. $r = \dfrac{40}{\pi} - \dfrac{2x}{\pi}$, $A = x^2 + \dfrac{(40 - 2x)^2}{\pi}$; 11.2 min
10. 70.71 km/h
11. $h = \dfrac{64}{x^2}$ **(ii)** 2.68 min
12. **(i)** $y = 60 - 2x$
    (iii) $x = \dfrac{240}{13} + \dfrac{60}{13}\sqrt{3}$
13. **(i)** $A = 14\pi x^2 + \dfrac{1440\pi}{x}$
    (ii) 3.72
14. $h = 10.0$, $r = 4.43$
15. 9.27 km
16. $r = 8$, $V = 682\frac{2}{3}\pi$ cm^3
17. $A = \dfrac{3\sqrt{3}}{2}\left(54 - 12x + x^2\right)$, , $x = 6$, $A = 27\sqrt{3}$ cm^2
18. **(ii)** 15, \$24 000
19. 12 noon

Chapter 12 — Challenge Yourself

(ii) $k = 4$

Exercise 13A

1. **(a)** $5\cos x + 3\sec^2 x$ **(b)** $6x + 4\sec^2 x$
   (c) $-6\sin 2x + 4\sec^2 4x$
   (d) $28\cos 7x - 1$
   (e) $-7\sin 7x + 3\cos 3x$
   (f) $7\sec^2\left(7x + \frac{\pi}{4}\right)$
   (g) $\frac{1}{\sqrt{2}}\cos\left(\frac{x}{\sqrt{2}} - \frac{\pi}{3}\right)$
   (h) $\frac{3}{2}\sin\left(\frac{\pi}{3} - \frac{3}{2}x\right)$
2. **(a)** $3\sin^2 x\cos x - 4\sin x$
   (b) $2\cos 2x + 6\cos x\sin x$
   (c) $10\sin^4(2x - 3\pi)\cos(2x - 3\pi)$
   (d) $3(2\sin x - \cos x)^2(2\cos x + \sin x)$
   (e) $3x^2\sec^2 x^3$
   (f) $4\sin 4x\sec^2 4x$
3. **(a)** $4x + 3x\cos x + 3\sin x$
   (b) $12x^2\cos 2x - 8x^3\sin 2x$
   (c) $6x^2\cos x - 2x^3\sin x + 5\sec^2 x$
   (d) $3\cos 5x - 5\tan 3x - 15x\sin 5x - 15x\sec^2 3x$
   (e) $-2\cot 2x\operatorname{cosec} 2x$
   (f) $-3\operatorname{cosec}^2 3x$
4. **(a)** $4(5x\cos x + 6x^2)^3(12x + 5\cos x - 5x\sin x)$
   (b) $\cos x\cos 2x - 2\sin x\sin 2x$
   (c) $4\cos 3x\sec^2 4x - 3\tan 4x\sin 3x$
   (d) $6x\tan(x + \pi)[x\sec^2(x + \pi) + \tan(x + \pi)]$
   (e) $\dfrac{3(2x + 1)\cos x - 6\sin x}{(2x + 1)^2}$
   (f) $\dfrac{\sin 2x - 2(2 + x)\cos 2x}{\sin^2 2x}$
   (g) $\dfrac{2}{(\sin x + \cos x)^2}$
5. **(a)** $-\dfrac{4}{x^2}\cos\dfrac{4}{x}$
   (b) $\dfrac{3}{x^2}\sin\dfrac{3}{x}$
   (c) $-\dfrac{5}{x^2}\sec^2\dfrac{5}{x}$
   (d) $12x\sin 3x^2\cos 3x^2 - \dfrac{1}{x^2}\sin\dfrac{1}{x}$
   (e) $6\sec^3 3x\sin 3x$
   (f) $-9\cot(3x + \pi)\operatorname{cosec}(3x + \pi)$
   (g) $-12\sin(3x - \pi)$
   (h) $24\sin\left(4x + \frac{\pi}{2}\right)\sec^4\left(4x + \frac{\pi}{2}\right)$

6. **(a)** $pq(\cos px - \sin qx)$
(b) $(a + b)\cos x - bx \sin x$
(c) $\sin \frac{a}{x} - \frac{a}{x} \cos \frac{a}{x}$
(d) $3p \tan^2 (px + q) \sec^2 (px + q)$
(e) $\frac{p \cos qx \cos px + q \sin px \sin qx}{\cos^2 qx}$
(f) $\frac{m \tan px \cos mx - p \sin mx \sec^2 px}{\tan^2 px}$

7. -3.28 8. -4.99
9. -3.78 10. $h = 8, k = 6$
11. $a + b = 8$ 12. $p = 2, q = 3$

Exercise 13B

1. **(a)** $\frac{1}{x-5}$ **(b)** $\frac{3x^2}{x^3-3}$
(c) $\frac{2(x-1)}{x(x-2)}$ **(d)** $\frac{10}{5x-1}$
(e) $\frac{6}{2x+5}$ **(f)** $\frac{1}{2(1+x)}$
(g) $-\frac{1}{x}$ **(h)** $\frac{2}{5-2x}$
2. **(a)** $-\tan x$ **(b)** $\frac{1}{\sin x \cos x}$
(c) $\frac{3}{2(3x-1)} - \frac{1}{2x+1}$ **(d)** $\frac{2}{3x} - \frac{5}{3(5x+4)}$
3. 1
4. **(a)** $3x + 6x \ln 5x$
(b) $\frac{(1+x^2)^3}{x} + 6x(1+x^2)^2 \ln x$
(c) $\frac{1 - \ln x}{x^2}$
(d) $\frac{1}{x - \sqrt{x}}$
5. **(a)** $x + 2x \ln x$ **(b)** $2x^2 + 6x^2 \ln x$
(c) $2x^2 + \frac{1}{x} + 6x^2 \ln x$ **(d)** $\frac{1 - \ln x}{5x^2}$
6. 0.307, 4
7. **(a)** $\frac{1}{x \ln 10}$ **(b)** $\frac{2}{(2x+3)\ln 5}$
(c) $\frac{\cot x}{\ln a}$ **(d)** $\frac{3x^2}{(x^3+1)\ln a}$
(e) $\frac{2x}{x^2+2}$ **(f)** $\frac{6x^2+7}{2x^3+7x}$
(g) $-2 \tan 2x$ **(h)** $\tan x$
9. $-0.009\ 00$

Exercise 13C

1. **(a)** $5e^x$ **(b)** $3e^{3x+1}$
(c) $-4e^{-4x}$ **(d)** $2x\, e^{x^2+7}$
2. **(a)** $\cos x\, e^{\sin x}$ **(b)** $-2 \sin 2x\, e^{\cos 2x}$
(c) $4 \sec^2 x\, e^{4 \tan x}$
(d) $3(\cos x + \sin x)e^{\sin x - \cos x}$
3. **(a)** $(2px + q)e^{px^2 + qx}$ **(b)** $e^{r^2 + x}$
4. **(a)** $\frac{e^{\sqrt{x}}}{2\sqrt{x}}$ **(b)** $-\frac{5e^{\frac{5}{x}}}{x^2}$
(c) $-\frac{1}{2\sqrt{e^x}}$ **(d)** $e^x - \frac{1}{e^x}$
5. **(a)** $xe^x(x + 2)$ **(b)** $e^x(\cos x - \sin x)$
(c) $e^x(2 \cos 2x + \sin 2x)$ **(d)** $-2e^x \sin x$
(e) $xe^{\sin x}\left(\frac{1}{2}x \cos x + 1\right)$ **(f)** $\frac{1}{2\sqrt{x}} e^{-x} - \sqrt{x}\, e^{-x}$
(g) $2xe^{-x^2}(1 - x^2)$ **(h)** $\frac{e^x}{\sqrt{x}} - \frac{e^x}{2\sqrt{x^3}}$
6. 5
7. $\frac{2 \cos 2x - 3 \sin 2x}{e^{3x+1}}$
8. $\frac{\cos x - x \sin x - x \cos x}{e^x}$

Exercise 13D

1. $2y = 10x + 5\sqrt{3} - \frac{5\pi}{3}$
$10y + 2x = 25\sqrt{3} + \frac{\pi}{3}$
2. $-3 \sin x - 5 \cos x$, 2.11, min
3. 2, $y = 2x - 2$
4. $y = 2x + 1$
5. (4, 0)
6. 1.39 units/s
7. **(a)** $\left(e^{-\frac{1}{2}}, -\frac{1}{2e}\right)$ **(b)** $\left(\frac{1}{e}, -\frac{2}{e}\right)$
8. **(i)** 4 **(ii)** $4y + x = 8$
9. $\frac{\pi}{6} + \sqrt{3}$
10. 0, $\frac{3\sqrt{3}}{16}$, $\frac{-3\sqrt{3}}{16}$
11. -0.520
12. -3 units/s
13. $3 \cos x - 6 \cos^3 x$, $\left(\frac{\pi}{2}, -1\right)$ max, $\left(\frac{\pi}{4}, -\sqrt{2}\right)$ min, $\left(\frac{3\pi}{4}, -\sqrt{2}\right)$ min.
15. $2e^{-x} \cos x$, $\frac{\pi}{2}$, max.
16. -54.0 units/month, -1.17 units/month
17. **(i)** $\frac{3}{3x+1}$ **(ii)** $\frac{3}{7}$
(iii) 0
18. 4 Watts/s
19. $2e^{x^2 - 3x}(2x^2 - 3x + 1)$, $\frac{1}{2} < x < 1$
20. **(i)** 28.3 g **(ii)** 29.2
(iii) 0.593 g/day
21. $\frac{1}{3454} - \frac{12}{1727}\sqrt{3}$
22. 1.08, max
23. **(i)** $50(2\theta + \sin 2\theta)$ **(ii)** 15 cm^2/s
25. $-\frac{4kr^3}{5}$

Review Exercise 13

1. **(a)** $2\pi - 2\pi \sin \pi x$ **(b)** $2 \cos 2x$
   (c) $3 \sin (6x - 2)$ **(d)** $-4 \sin x \cos^3 x$
   (e) $\frac{2}{3} \tan \frac{1}{3}x \sec^2 \frac{1}{3}x$ **(f)** $\frac{5}{5x-2}$
   (g) $\frac{4x}{2x^2+3}$ **(h)** $4 \cot 4x$
   (i) $6e^{2x} + e^{3+x}$
   (j) $-\frac{12}{5}e^{3-4x} + \frac{7}{2}e^{\frac{7}{2}x - \frac{3}{2}}$
2. **(a)** $-3 \sin 3x \sin^2 x + \sin 2x \cos 3x$
   (b) $\cos^3 x \sin^4 x (5 \cos^2 x - 4 \sin^2 x)$
   (c) $\frac{x \cos x - 2 \sin x}{x^3}$
   (d) $\frac{1}{x} \sin x + \cos x \ln 5x$
   (e) $\frac{6x^2}{2x-5} + 6x \ln (2x - 5)$
   (f) $e^x(\cos 3x - 3 \sin 3x)$
   (g) $x^2e^{2x}(2x + 3) + 2e^{4x}(4x + 1)$
   (h) $e^{4x}(\sec^2 x + 4 \tan x)$
   (i) $\frac{12x^3}{2x^2+3} + 6x \ln (2x^2 + 3)$
   (j) $e^{3x} \sin^3 2x (8 \cos 2x + 3 \sin 2x)$
3. $3y = 2x + \sqrt{3} - \frac{2\pi}{3}$
   $6y + 9x = 2\sqrt{3} + 3\pi$
5. **(a)** $(2, 0)$ **(b)** $\left(\frac{1}{2}, \ln \frac{5}{4}\right)$
6. $-\frac{\sqrt{3}}{10}$ unit/s
7. max $y = \frac{3\sqrt{3}}{4}$, min $y = -\frac{3\sqrt{3}}{4}$
8. **(i)** $\frac{4 \sin x - 1}{(\sin x - 4)^2}$ **(ii)** 0.253 or 2.89
9. $(1 + \sqrt{6}, \ln (2\sqrt{6} + 2))$, min
11. $2\sqrt{2}, \frac{1}{4} \ln 2$, min
12. $x = 1.02$ or 2.59
13. $(3, 0)$, $6y + x = 3$
14. $\frac{-3 \sin x - \cos x}{e^{2x}}$, 2.82 or 5.96
15. $x < \frac{5}{6}$
16. $\frac{10x}{5x^2-4}$, $1\frac{7}{8}$ units/s
17. **(i)** \$128 247.47 **(ii)** \$9090.91/year
18. **(i)** $21 \sin \theta$ cm^2, $2.5x \sin \theta$ cm^2
    (ii) $\frac{\pi}{2}$, 21 cm^2
    (iii) $2\sqrt{15}$
19. $\frac{\pi}{6}$, max
20. **(i)** 0.0808% **(ii)** –0.0120%/h
    (iii) 0.0920%
21. $V = \frac{3\pi h^2}{h-6}$, 12, min

Chapter 13 — Challenge Yourself

3. $a = -2, b = 4, c = 2$

Revision Exercise E1

1. **(a)** $\frac{9x+2}{2\sqrt{x}}$ **(b)** $\frac{3x^2}{2\sqrt{x^3+2}}$
   (c) $\cos x + \cos 2x$ **(d)** $\frac{\sec^2 x}{3x+1} - \frac{3 \tan x}{(3x+1)^2}$
   (e) $20e^{4x} - \frac{1}{2}e^{-\frac{1}{2}x}$ **(f)** $\frac{2}{\cos 2x}$
2. $(1, 1)$
3. $\frac{16}{15\pi}$ cm/s
4. $\left(\frac{\pi}{6}, \frac{1}{8}\right)$ max, $\left(\frac{\pi}{2}, -1\right)$ min
5. $A = 3, B = -1, C = 2$
6. $a = 2, b = 0$
7. **(a)** 2 **(b)** $-\frac{1}{4}(b - c)^2$
8. $AB = 40$ m, $BC = 120$ m

Revision Exercise E2

1. **(a)** $-\frac{5}{(x-3)^2}$ **(b)** $5\left(2x^2 - \frac{3}{x}\right)^4\left(4x + \frac{3}{x^2}\right)$
   (c) $(x^2 + x) \sec^2 x + (2x + 1) \tan x$
   (d) $\frac{2x \sin 2x - 2x^2 \cos 2x}{\sin^2 2x}$
   (e) $2e^{4x}[\cos (2x + 3) + 2 \sin (2x + 3)]$
   (f) $2 - \frac{1}{x}$
2. $\sqrt{3}y + 2x = \frac{\pi}{3} + \frac{\sqrt{3}}{2}$
3. $-1\frac{1}{2}$ cm^3/s
4. $\left(\frac{1}{3}, 6\right)$ min, $\left(-\frac{1}{3}, -6\right)$ max
5. $a = 2, b = -4, c = 0$
7. $e^x (\cos 2x - 2 \sin 2x)$; $-e^x(3 \cos 2x + 4 \sin 2x)$
   (i) –4.81 **(ii)** 14.4
8. $y = 9 - \frac{3}{2}x$; $x = 4.22$, $A = 19.0$ cm^2

Exercise 14A

1. **(a)** $\frac{1}{6}x^6 + c$ **(b)** $\frac{2}{9}x^{\frac{3}{2}} + c$
   (c) $-\frac{1}{3x^2} + c$ **(d)** $\frac{6}{5}x^{\frac{2}{3}} + c$
2. **(a)** $\frac{3}{4}x^4 - \frac{8}{3}x^{\frac{3}{2}} + 3x + c$
   (b) $2x^3 + \frac{4}{x} + c$
   (c) $5x - 2\sqrt{x} - \frac{1}{2x^2} + c$
   (d) $\frac{1}{4}x^2 - \frac{5}{2x} + c$
3. $30\frac{1}{6}$

4. (a) $\frac{15}{16}x^{\frac{8}{5}} + c$

(b) $-\frac{9}{5x} + \frac{3}{5x^2} - \frac{1}{15x^3} + c$

(c) $\frac{9}{46}x^{\frac{23}{3}} + \frac{3}{16}x^{\frac{8}{3}} + c$

(d) $\frac{1}{3}x^3 - \frac{12}{5}x^{\frac{5}{2}} + \frac{9}{2}x^2 + c$

(e) $x - \frac{2}{3}x^{\frac{3}{2}} + c$

(f) $\frac{3}{5}x^{\frac{5}{3}} + 4x + 12x^{\frac{1}{3}} + c$

5. $y = \frac{2}{3}x^3 + \frac{7}{2}x^2$

6. $A = 2r^2 + 7r + 3$

7. $s = t^3 - 7t + 6$

8. $5y = 4x^{\frac{4}{3}} + 16$

9. $y = x + \frac{4}{x} + 3$

10. (i) 4 (ii) $y = 2x^2 + 3x - 8$

(iii) $\left(-\frac{3}{4}, -9\frac{1}{8}\right)$

11. \$4000

Exercise 14B

1. (a) $\frac{1}{15}(3x-7)^5 + c$ (b) $\frac{2}{15}(5x+1)^{\frac{3}{2}} + c$

(c) $\frac{2}{25}(5x-6)^5 + c$ (d) $\frac{1}{5}(2x+7)^{\frac{5}{2}} + c$

(e) $-\frac{10}{7}\sqrt{2-7x} + c$ (f) $\frac{3}{2\sqrt{5-4x}} + c$

(g) $-\frac{16}{9(3x-1)^3} + c$ (h) $\frac{3}{16(5-2x)^2} + c$

2. $y = \frac{1}{8}(2x+1)^4 + \frac{31}{8}$

3. $y = \frac{20}{3} - \frac{1}{3}(7-2x)^{\frac{3}{2}}$

4. $m = \frac{2}{9}(3t+6)^{\frac{3}{2}} + 6994$

5. (a) $\sqrt{6x-5} + c, x > \frac{5}{6}$

(b) $\frac{10}{21}(3x+4)^{\frac{7}{2}} + \frac{2}{9}(3x+4)^{\frac{3}{2}} + c, x > -\frac{4}{3}$

6. (a) $\frac{1}{6p}(px+q)^6 + c$

(b) $\frac{(px+q)^{m+1}}{p(m+1)} + c$

(c) $\frac{m}{p(m+1)}(px+q)^{\frac{1+m}{m}} + c$

(d) $\frac{(p^m x - q^2)^{m+1}}{p^m(m+1)} + c$

Exercise 14C

1. (a) $\frac{2}{3}\sin 3x + \frac{1}{5}\cos 5x + c$

(b) $\frac{1}{2}x^2 + 2\sin\frac{x}{2} + c$

(c) $4\tan\frac{1}{2}x + c$

(d) $\frac{1}{3}\sin(3x-2) + c$

(e) $-\frac{7}{5}\cos(5x-3) + c$

(f) $\tan x + c$

2. $y = \sin x - \cos x + 2$

3. $y = x + \frac{1}{2}\cos 2x + 1$

4. $y = x + \frac{1}{4}\sin 2x + \frac{11}{4} - \frac{\pi}{4}$

5. (a) $2\sqrt{x} - \frac{2}{3}\cos\frac{3x}{2} + c$

(b) $-\frac{1}{x} - \frac{1}{3}\tan 3x + c$

(c) $\frac{1}{3}\sin 3x - \frac{1}{4}\tan 4x + c$

(d) $\frac{3}{4}\tan x + \frac{1}{2}x + c$

(e) $-4\cos\left(\frac{\pi x}{4} - \frac{\pi}{4}\right) + c$

(f) $\sin\left(\pi x - \frac{\pi}{2}\right) + c$

6. (a) $\frac{2q}{p}\tan\frac{px}{q} + c$

(b) $\frac{1}{p}\sin(px-q) + c$

(c) $\frac{1}{p}\tan(px+q) - x + c$

(d) $p\tan x - \frac{1}{2}qx^2 + c$

7. $y = \tan x - x + \frac{3\pi}{4} - 1$

8. $y = 2\tan\frac{x}{2} + \frac{13}{3}\sqrt{3}$

9. $\frac{1}{10}\sin\left(10x - \frac{\pi}{2}\right) - \frac{1}{4}\sin 2x + \frac{3}{2}x - \frac{1}{2}x^2 + c$

10. $\frac{4}{3}$

11. $y = \frac{x}{\cos x} + 2$

Exercise 14D

1. (a) $2x + 3\ln x + c$ (b) $4x + 4\ln x - \frac{1}{x} + c$

(c) $\frac{1}{3}\ln(3x-7) + c$ (d) $-\ln(5-x) + c$

2. (a) $3e^x + c$ (b) $8e^{\frac{x}{2}} + c$

(c) $-\frac{1}{5e^x} + c$ (d) $\frac{1}{2}e^{2x+7} + c$

3. (a) $\frac{5}{2}e^{2x+\pi}+c$ (b) $\frac{1}{4}e^{4x}-e^{2x}+x+c$

(c) $-2e^{-x-1}+\frac{1}{2}e^{2x-1}+c$

(d) $-\frac{1}{9x}+\frac{1}{4}e^{2x}+c$

4. (a) $3x+5\ln x+c$ (b) $\frac{1}{2}x^2+2\ln x+c$

(c) $4x+12\ln x-\frac{9}{x}+c$ (d) $\frac{5}{2}\ln x-\frac{\pi}{2x}+c$

(e) $9x-24\ln x-\frac{16}{x}+c$ (f) $-\frac{3}{5}\ln(4-5x)+c$

5. (a) $\frac{4}{3}x^3+4\ln x-\frac{1}{3x^3}+c$ (b) $2\ln(x-2)+c$

(c) $\frac{3}{2}e^x-\frac{1}{2e^x}+c$ (d) $e^x+6x-\frac{9}{e^x}+c$

Exercise 14E

1. $x\sqrt{3x+1}+c$ 2. $\frac{4x}{\sqrt{3x+4}}+c$

3. $\frac{1}{4}x\sqrt{3x^2+7}+c$

4. $\frac{6x}{\sqrt{(3x-2x^2)^3}}$; $\frac{14x}{3\sqrt{3x-2x^2}}+c$

5. $5x+10x\ln x$; $2x^2\ln x-x^2+c$

6. $10xe^{2x}-3e^{2x}$; $\frac{3}{4}e^{2x}(2x-1)+c$

7. $\frac{1}{6}e^{3x}(\sin 3x+\cos 3x)+c$

8. $6x+\frac{1}{2}\sin 2x+c$

9. $\frac{3\sin x}{7(1+\cos x)}+c$

10. $\frac{1}{6}\sin 2x+c$

11. $6\cos^2 x-3$; $\frac{7}{2}(\sin x\cos x+x)+c$

12. $\frac{3}{x+2}-\frac{5}{2x-3}$, $2\ln(x+2)-\frac{5}{3}\ln(2x-3)+c$

13. $\frac{6}{2x+3}-\frac{3}{x-1}+\frac{4}{(x-1)^2}$; $\frac{3}{5}\ln\frac{2x+3}{x-1}-\frac{4}{5(x-1)}+c$

14. (i) $\frac{2x}{x^2+3}$

(ii) $\frac{6}{2x+5}-\frac{3x}{x^2+3}$, $\ln(2x+5)-\frac{1}{2}\ln(x^2+3)+c$

15. $\frac{1}{4}\cos 3x-\frac{9}{4}\cos x+c$

16. $6\tan^2 x\sec^2 x$; $\frac{5}{3}\tan^3 x+5\tan x+c$

Review Exercise 14

1. (a) $\frac{1}{2}t^2-\frac{1}{t}+c$

(b) $-\frac{1}{t}-\frac{2}{3t^3}+c$

(c) $\frac{1}{12}(2x+9)^6+c$

(d) $\frac{2}{25}(5x-7)^{\frac{5}{2}}+c$

(e) $-\frac{14}{3}\sqrt{2-3x}+c$

(f) $-\frac{7}{2}\cos 2x+c$

(g) $\frac{2}{7}\sin(7x+5)+c$

2. (a) $\frac{1}{3}e^{3x+1}+c$

(b) $\frac{5}{2}e^{2x+3}+c$

(c) $e^{3x}-4e^x+c$

(d) $\frac{1}{4}x+\frac{3}{4}e^x+c$

(e) $-\frac{3}{2x}+\frac{1}{2}\ln x+c$

(f) $\frac{25}{3}\ln x-\frac{10}{3}x+\frac{1}{6}x^2+c$

(g) $\frac{2}{3}\ln(3x-4)+c$

(h) $-2e^{-x-3}+e^{3x-3}+c$

3. (i) 4 (ii) $y=2x^2-3x-3$

4. $y=x^2-x^4+3$

5. $y=\frac{2}{3}x^3+3x^2+4x+5$

6. $y=2x+\frac{1}{x}+2$

7. 1.10 units

8. $y=\frac{1}{3}(e^{3x}-4)$

9. $\frac{2x-3}{4\sqrt{4x+3}}+c$

10. $\cot x$, $3\ln\sin x+c$

11. $3x+6x\ln 4x$, $\frac{1}{4}x^2(2\ln 4x-1)+c$

12. (i) $\frac{6x}{3x^2+5}$

(ii) $\frac{1}{2x-3}+\frac{7x}{3x^2+5}$, $\frac{1}{2}\ln(2x-3)+\frac{7}{6}\ln(3x^2+5)+c$

Chapter 14 – Challenge Yourself

1. (i) $\frac{24x^2}{8x^3-27}$ (ii) $\frac{27}{24}\ln(8x^3-27)+c$

2. $-\frac{4}{3(1+2x)^3}+c$

3. $y=\frac{1}{3}x^3-\ln x+\frac{5}{3}$

Exercise 15A

1. (a) 50 (b) –2
2. (a) 194.4 (b) $272\frac{1}{7}$
   (c) $\frac{1}{6}$ (d) $18\frac{2}{3}$
   (e) $\frac{2}{3}$ (f) 2
   (g) $-3\frac{1}{15}$ (h) $-61\frac{1}{2}$
   (i) $-\frac{3}{7}$ (j) $-\frac{1}{6}$
   (k) –1
3. (a) –10 (b) $4\frac{2}{3}$
   (c) $37\frac{1}{3}$ (d) 8
   (e) $\frac{8}{55}$ (f) $\frac{4}{9}$
4. (a) $\frac{\pi^2}{4}+1$ (b) $\frac{\sqrt{2}-1}{2}$
   (c) $2\sqrt{2}$ (d) $\sqrt{3}-1-\frac{\pi}{12}$
   (e) $\frac{2\sqrt{3}}{3}$ (f) $\frac{1}{3}$
5. (a) $3(e^{\frac{1}{3}}-1)$ (b) $\frac{1}{6}(e^2-1)$
   (c) $1-\frac{1}{e}$ (d) $\frac{1}{3}(e^3-1)$
   (e) $5-5e^{-4}$ (f) $2-2e^{-4}$
6. (a) 73.89 (b) 2.09
   (c) 4.70 (d) 4.64
   (e) 11.67 (f) 51.88
7. (a) 8.05 (b) 0.46
   (c) 0.47 (d) 3.30
   (e) 11.48 (f) 21.48
8. $W=\frac{1}{2}k\left(x_2^2-x_1^2\right)$
9. $\frac{1}{2}\ln 2$

Exercise 15B

1. $1\frac{5}{9}$
2. $\frac{2}{3}$
3. $\frac{\pi}{4}+\frac{1}{2}$
4. (i) $\frac{1}{x-1}+\frac{2}{x-3}$ (ii) 2.71
5. 4.47
6. (iii) $-2-\frac{\pi}{2}$
7. (i) $\cos x - x\sin x$ (ii) 1
8. $4\pi+\frac{3}{2}\sqrt{3}$
9. (iii) $\frac{\pi}{3}$
10. $\frac{e^{2\sqrt{x}}}{\sqrt{x}}$; 53.6
11. $6xe^{3x}+2e^{3x}$; 4.57
12. $x-3+2x\ln x$; 7.34
13. $4\cos 4x-2\sin 4x$

Exercise 15C

1. (a) $21\frac{1}{3}$ units^2 (b) 2.72 units^2 (c) $\left(\frac{3\pi}{2}+1\right)$ units^2
2. (a) 5.21 units^2 (b) 0.386 units^2 (c) 2.51 units^2
3. 9 units^2
4. $10\frac{2}{3}$ units^2
5. $68\frac{1}{3}$ units^2
6. $\frac{1}{2}$, $\frac{1}{2}$ m^2
7. $1\frac{1}{3}$ units^2
9. 8.79 units^2
10. (i) $2\sec^2\left(2x-\frac{\pi}{4}\right)$ (ii) $\left(\frac{\pi}{2}-1\right)$ units^2
11. (i) $Q(1, 1+e^2)$, $R(0, 2)$ (ii) 4.19 units^2; 4.19 units^2
12. 17

Exercise 15D

1. $10\frac{2}{3}$ units^2
2. (1, 0), (–2, –3), $4\frac{1}{2}$ units^2
3. (i) $P(0, 15)$, $Q(3, 9)$, $R\left(7\frac{1}{2}, 0\right)$
   (ii) 12 : 13
4. (a) $(2-\sqrt{2})$ units^2 (b) $(\sqrt{2}-1)$ units^2
5. (ii) $\frac{7}{12}$ units^2
6. (i) $P(0, 4)$, $Q(3, 3)$ (ii) $6\frac{1}{6}$ units^2
7. (i) $P(4, 8)$, $Q(6, 0)$ (ii) 8 : 19
8. 22.1 units^2
9. $1\frac{1}{4}$ units^2

Review Exercise 15

1. (a) 3 (b) $\frac{1}{2}$
   (c) 34.1 (d) 66.0
   (e) 14.5 (f) 0.332
2. 4 or 6
3. $\frac{\pi}{2}$ or $\frac{3\pi}{2}$
4. 2
6. 0.305 units^2
7. $2\frac{2}{3}$ units^2
8. (i) $\frac{x}{2+x}+\ln(2+x)$ (ii) $1-\frac{2}{2+x}$
   (iii) 7.621 units^2
9. (i) $P(4, 2)$, $Q(2, 8)$ (ii) $9\frac{1}{3}$ units^2
10. $2-\frac{5}{x+3}$; 0.562
11. $\frac{1}{8}(\pi-2)$
12. $\frac{3x^3+4x^2-4x+1}{4x^5-4x^4+x^3}$; $1\frac{3}{8}$
13. 62

Chapter 15 — Challenge Yourself

1. $\frac{1}{6}$ units^2
2. 44.9 units^2

Exercise 16A

1. (i) 11 s (ii) 452 cm (iii) 131 cm (iv) 2 cm s^{-2}
2. (i) $v = 6t^2 - 6t - 12$, $a = 12t - 6$ (ii) 2 (iii) $-13\frac{1}{2}$ m s^{-1}
3. (i) 0.243 s (ii) $2 - \frac{3\sqrt{2}}{2}$ m s^{-1}, $-\frac{9\sqrt{2}}{2}$ m s^{-2}
4. (i) 5 m s^{-1} (ii) 4 s, 45 m s^{-1} (iii) 100 m
5. (i) $V = k \sin 2t$, $a = 2k \cos 2t$ (ii) (a) $0, \frac{\pi}{2}, \pi$ (b) $\frac{\pi}{4}$
6. (i) 24 s, –576 cm s^{-1} (ii) 2048 cm
7. (i) 4 m (ii) 3 m s^{-1} (iii) $-6\sqrt{5}$ m s^{-2} (iv) 4.72 m
8. (i) 1.61 s (ii) –2 m s^{-2}
9. (i) 0.405 s (ii) $-177\frac{7}{9}$ m s^{-2}
10. $\frac{\pi}{8}$ s
11. (i) ln 3 m (ii) $\frac{2}{13}$ m s^{-1}
12. (i) $\frac{1}{3}$ or 1 (ii) $0 < t < \frac{1}{3}$ or $t > 1$ (iii) $\frac{1}{3} < t < 1$
13. $v = 2 \cos t - \sin t$, $a = -2 \sin t - \cos t$

Exercise 16B

1. (i) $s = \frac{1}{3}t^3 + t^2 + t$ (ii) (a) $8\frac{2}{3}$ m (b) $71\frac{2}{3}$ m
2. (i) 16 cm s^{-1} (ii) $25\frac{1}{2}$ cm
3. (i) $66\frac{1}{3}$ (ii) 64 cm
4. (i) $8\frac{1}{2}$ m s^{-1} (ii) $14\frac{1}{6}$ m
5. (i) $10\frac{2}{3}$ m (ii) 6 s
6. (i) $4t + 1 - 10k$ (ii) 2 (iii) $0 < t < 2.5$ or $t > 7$ (iv) 2.5 or 7 (v) $s = \frac{2}{3}t^3 - \frac{19}{2}t^2 + 35t$
7. (i) $-4 \leqslant v \leqslant 12$ (ii) $\left(\frac{\pi}{3} + 2\sqrt{3} - 4\right)$ m
8. (i) $\frac{4}{3}\pi$ (ii) 9.02 m
9. (i) 120 m s^{-2} (ii) 270 m
10. (i) 0.408 (ii) 17.9 m (iii) 24.4 m s^{-1}
11. $h = -2$, $k = 12$
12. (i) 7 m s^{-1} (ii) 8 (iii) 0.496 cm s^{-2} (iv) 31.50 m

Review Exercise 16

1. (i) $v = 3t^2 - 20t + 25$, $a = 6t - 20$ (ii) 5 m (iii) 5 (iv) $1\frac{2}{3}$ or 5 (v) $37\frac{1}{27}$ m
2. $b = -24$, $c = \frac{1}{2}$; $-22\frac{1}{2}$ m s^{-1}, 3 m s^{-2}
3. (i) $t_1 = 2\frac{1}{2}$, $t_2 = 7$ (ii) $38\frac{13}{24}$ m (iii) $68\frac{11}{12}$ m (iv) 9 m s^{-2}
4. (i) 40 m s^{-1} (ii) 80 m (iii) ±36 m s^{-1}
5. (i) –2.94 m s^{-2} (ii) 5.42 m
6. (i) $2e^2$ m s^{-1} (ii) 28.2 m
7. (i) 1 h (ii) –1.11 km h^{-2}
8. (i) 4 (ii) 1.51 m
9. (i) $\frac{10}{27}$ m s^{-1} (iii) 5 m
10. $V = kn \sec^2 nt$; $a = 2kn^2 \tan nt \sec^2 nt$

Chapter 16 — Challenge Yourself

1. (ii) 39 m
2. $v = 2\cos\frac{\pi t}{6} - \frac{\pi t}{3}\sin\frac{\pi t}{6}$,
   $a = -\frac{\pi^2 t}{18}\cos\frac{\pi t}{6} - \frac{2\pi}{3}\sin\frac{\pi t}{6}$

Revision Exercise F1

1. (a) 24 (b) $\frac{\pi}{4} + \frac{1}{2}$ (c) 4.49
2. $17\frac{1}{3}$
3. $y = x^2 - x$; (0, 0) or (1, 0)
4. $y = \frac{39}{8}x^2 - \frac{33}{4}x + 4$
5. (i) $A(1, 8)$, $B(1, 0)$, $C(3, 0)$ (ii) $13\frac{1}{3}$ units^2
6. (i) 12 cm s^{-1}, 16 cm (ii) 0 cm s^{-1}, 16 cm
7. (i) $35\frac{2}{3}$ m (ii) 16 m s^{-2}

Revision Exercise F2

1. (a) 53 (b) $\frac{\sqrt{3}-1}{4}$
2. 4
3. (i) $a = 9$, $b = 16$ (b) $y = 9x + \frac{16}{x} - 10$
4. $y = x^2 - 5x + 4$; $4\frac{1}{2}$ units^2
5. (i) 108 m s^{-1} (ii) 4 m s^{-1}; 12 m s^{-2}
6. (i) $0 < t < 5$ (ii) 7.5 s (iii) 250 m
7. $83\frac{1}{3}$ m, 15 s, $-37\frac{1}{2}$ m s^{-1}

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New Syllabus Additional Mathematics (NSAM) is an MOE-approved textbook which includes valuable learning experiences to engage the hearts and minds of students sitting for the GCE O-level examination in Additional Mathematics. It covers the MOE Syllabus for Additional Mathematics implemented from 2013.

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