Math Review

Algorithms

Corazza

A course in Algorithms requires some familiarity with logarithms, properties of sets and operations on them, summation notation, basic number theory and probability theory, and the concept of recursion. It also requires the ability to understand, and carry out, simple arguments using mathematical induction. We review most of these concepts here (basics of probability theory will be introduced in a separate lesson) and we provide some sample problems. Students should study the samples and work as many of the exercises associated with this lesson as necessary. Answers to exercises are provided at the end.

Section L: Laws Of Logarithms

$$
\begin{aligned}
y = \log_b x &\text{ means } x = b^y \\
\log x &\text{ means } \log_2 x \\
\log^n x &\text{ means } (\log x)^n \\
\ln x &\text{ means } \log_e x \ (e \approx 2.71828) \\
\log_b(xy) &= \log_b x + \log_b y \\
\log_b(x^y) &= y \log_b x \\
\log_b(\frac{x}{y}) &= \log_b x - \log_b y \\
\log_b x &= \frac{\log_a x}{\log_a b} \\
0 < x < y &\Rightarrow \log_b(x) < \log_b(y) \\
\log_b 1 &= 0 \\
\log_b b &= 1 \\
\log_b x &< x \text{ (for } b \geq 2 \text{ and } x > 0) \\
\log 1024 &= 10 \\
\ln 2 &\approx .693 \text{ (in particular, } 0 < \ln 2 < 1) \\
\log e &\approx 1.44 \text{ (in particular, } 1 < \log e < 2)
\end{aligned}
$$

Problem L1. Show that the following is not true in general, for $k > 1$:

$$(\log n)^k = k \log n.$$

In other words, give a counterexample (find a natural number $k > 1$ for which the formula is not true).

Problem L2. Show that the following is not true in general:

$$\log_b(x + y) = \log_b x + \log_b y$$

Problem L3. Show that, for all $n > 2$,

$$n < n \log n < n^2.$$

It is also true that for all $n > 4$, $n^2 < 2^n$. This is proved by induction. See Problem MI2.

Problem L4. Solve for n:

$$2^{3n-1} = 32.$$

Problem L5. Try this one if you have had a course in calculus. Show that

$$\lim_{n \to \infty} \frac{\log n}{\ln n}$$

is a number between 1 and 2.

Section S: Sets

A. A *set* is a collection of objects (this is only approximately correct!).
   - The notation $x \in A$ signifies that x is an element of A.
   - *Set notation.* The set containing just the elements 1, 2, 3 is denoted $\{1, 2, 3\}$. Elliptical notation can be used to denote larger sets, such as $\mathbf{N} = \{1, 2, 3, \ldots\}$. Set-builder notation defines a set by specifying properties; for instance:

     $$E = \{n \mid n \text{ is a natural number and for some } x,\ n = 2 * x\}.$$

   - Two sets are *equal* if and only if they have the same elements. Therefore, duplicate elements are not allowed in a set when viewed as a data structure.

B. B is a *subset* of A, $B \subseteq A$, if every element of B is also an element of A. The empty set, denoted $\emptyset$, is a subset of every set (but is *not* an element of every set!).

C. If A and B are sets, $A \cup B$ ("the union of A and B") consists of all objects that belong to at least one of A and B; and $A \cap B$ ("the intersection of A and B") consist of all objects that belong to both A and B. Example:

$$\{1, 2, 3\} \cup \{2, 3, 4\} = \{1, 2, 3, 4\}$$
$$\{1, 2, 3\} \cap \{2, 3, 4\} = \{2, 3\}$$

D. Suppose each of A, B is a set. Then A, B are *disjoint* if A and B have no element in common (that is, $A \cap B = \emptyset$). Similarly, $A_i (i \in I)$ are disjoint if no two of the sets have an element in common.

E. The *cardinality* or *size* of a set A is denoted $|A|$. Example: $|\{2, 7, 14\}| = 3$.

F. The *power set* of a set A, denoted $\mathcal{P}(A)$, is the set whose elements are all the subsets of A. Example $\mathcal{P}(\{1, 2\}) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Note: If A has n elements, $\mathcal{P}(A)$ has 2^n elements. That is, a set with n elements has 2^n subsets.

G. If a set A having n elements is totally ordered (like the natural numbers or strings with alphabetical ordering), then a *permutation* of A is a re-arrangement of the elements of A.
   - Example: The following are two of the permuations of $\{1, 2, 3, 4\}$:

$$[1, 2, 4, 3], [4, 3, 2, 1]$$

   - The permutation of A that does not re-arrange any of the elements is called the *identity permutation*.
   - The number of permutations of an n-element set is $n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$.

H. The notation $C(n, m)$ is read "the number of combinations of n things taken m at a time" and can be understood to mean "the number of m-element subsets of an n-element set."
   - For small values of n, m, $C(n, m)$ can be computed by inspection. Example: Compute $C(3, 2)$. To do the computation, take any 3-element set $\{a, b, c\}$ and write out the 2-element subsets:

$$\{\{a, b\}, \{b, c\}, \{a, c\}\}.$$

     The resulting collection now contains 3 two-element subsets of $\{a, b, c\}$. Therefore, $C(3, 2) = 3$.
   - Formula for computing $C(n, m)$

$$C(n, m) = \frac{n!}{m!(n-m)!}.$$

     Example:

$$C(10, 2) = \frac{10!}{2!8!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdots 2 \cdot 1}{(2 \cdot 1)(8 \cdot 7 \cdot 6 \cdots 2 \cdot 1)} = \frac{10 \cdot 9}{2 \cdot 1} = 45.$$

I. The notation $P_{n,m}$ is read "the number of permutations of n things taken m at a time." The meaning is this: We have a set S with n elements, and we want to arrange m of the elements of S in a particular order.
   - The computation is easier to understand in a simple case. We want to compute $P_{3,2}$. Let $S = \{a, b, c\}$. We want to arrange two elements from S in a particular order. We can think that there are two "slots" to fill—positions 1 and positions 2—with elements from S:

$$\underline{\quad 1 \quad} \qquad \underline{\quad 2 \quad}$$

     To fill these slots, we perform two tasks in succession:

     Task 1: Pick a 2-element subset from S

     Task 2: Arrange it so one element is in position 1, the other in position 2.

     There are $C(3, 2)$ ways to perform Task 1. After a set has been selected, there are 2! ways to arrange that set—that is, 2! ways to place the elements into position 1 and position 2. Therefore:

$$P_{3,2} = C(3, 2) \cdot 2!$$

- The same logic gives the formula for $P_{n,m}$:

$$P_{n,m} = C(n,m)m! = \frac{n!}{(n-m)!}.$$

- Example: Compute $P_{10,2}$.

$$P_{10,2} = \frac{10!}{(10-2)!} = \frac{10!}{8!} = 10 \cdot 9 = 90.$$

Problem S1. Are the following sets equal? Explain.

$$\{1,1,2\}, \{1,2\}, \{2,1\}.$$

Problem S2. Is the following statement true or false?

$$\{1, \{2,3\}\} \subseteq \{1,2,3,4,5,\ldots\}$$

Solution. False. The first set contains an element that is *not* an element of the second set — namely, $\{2,3\}$.

Problem S3. What is the powerset of the set $\{1,2\}$?

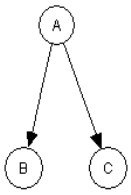
Problem S4. List all the permutations of the set $\{1,3,4\}$.

Problem S5. In how many ways can 5 students, from a group of 9 students, be seated in a row of 5 chairs?

Problem S6. A committee of three representatives is to be chosen from a larger group of 20 people. In how many ways can this committee be formed?

Section DGF: Directed Graphs and Functions

A directed graph is a set of objects (called *vertices* or *nodes*) together with a set of arrows that join some of the vertices. Here is a simple example:



A function from a set X to a set Y—written $f : X \to Y$—is a special kind of directed graph f (we usually denote functions using typical letters f, g, h, etc.) with the following characteristics:

- The objects of the graph f are the elements of X together with the objects of Y.
- Each arrow of f always starts at an element of X and points to an element of Y. If, in f, x points to y, we write $x \to y$ or $f(x) = y$.

- In f, no $x \in X$ ever points to more than one element of Y. In fact, in f, each $x \in X$ points to *exactly one* element of Y.

When $f : X \to Y$ is a function, X is called its *domain*, Y its *codomain*.

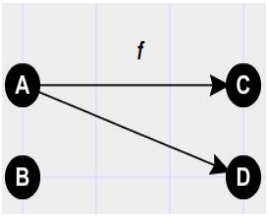
Concepts Related to Functions. Suppose $f : X \to Y$ is a function.

(1) *Onto.* A f is *onto* if for each $y \in Y$ there is an element $x \in X$ so that $x \to y$.

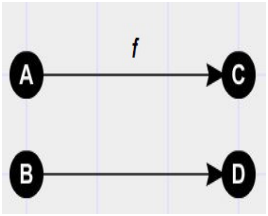
(2) *Range.* The range of f is the set of all $y \in Y$ that are pointed to by one or more x in X; the range is the set of all *output values* of f. If the range of f is Y itself, f is onto.

(3) *1-1.* A function $f : X \to Y$ is *1-1* if, whenever x and x' are distinct elements of X, and $x \to y$ and $x' \to y'$, then y and y' are also distinct elements of Y.

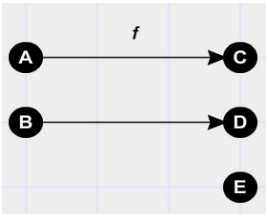
Examples



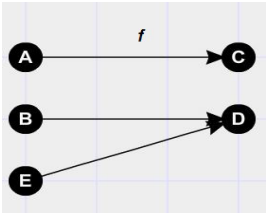
f is *not* a function



f is a 1-1 and onto function



f is a 1-1 function that is not onto



f is an onto function that is not 1-1

Example Let $A = \{1, 2, 3\}$ and $B = \{4, 5, 6\}$. Let's define $f : A \to B$ as follows:

$$\begin{array}{ccc} \underline{A} & f & \underline{B} \\ 1 & \to & 4 \\ 2 & \to & 5 \\ 3 & \to & 6 \end{array}$$

In other words, in the directed graph f, $1 \to 4, 2 \to 5, 3 \to 6$. Another way to say this is that f takes the number 1 to 4, the number 2 to 5, and the number 3 to 6. Here is notation that can be used to state this fact:

$$f(1) = 4$$
$$f(2) = 5$$
$$f(3) = 6$$

Example. We define a function g, also having domain A and codomain B (defined in the previous example), as follows:

$$g(1) = 4$$
$$g(2) = 4$$
$$g(3) = 4$$

Here g is also a function. In the previous example, the function f was *1-1*—no two elements of the domain were assigned the same value by f. Clearly, g does not have that property; in fact, all elements of the domain A of g are assigned the single value 4.

Example. Returning to the functions f and g of the previous examples, notice that the range of f is precisely equal to B, so f is onto. On the other hand, the range of g is just the singleton set $\{4\}$, and so g is *not* onto.

Problem DGF1. Consider the function $f(n) = n^2$, where the domain of f is the set $\mathbf{N}$ of all natural numbers. Is f 1-1? What is the range of f? Is f onto?

Some functions from $\mathbf{N}$ to $\mathbf{N}$ have the convenient property of being *increasing*. This means that as input values increase, output values also increase. More precisely, we have the following definition:

Definition. A function $f : \mathbf{N} \to \mathbf{N}$ is said to be *increasing* if, whenever $m < n$, we have $f(m) < f(n)$. A function $g : \mathbf{N} \to \mathbf{N}$ is said to be *nondecreasing* if, whenever $m < n$, we have $g(m) \leq g(n)$.

Example. Obviously, the identity function $f(n) = n$ is increasing. It is equally easy to see that the function $g(n) = kn$ for any integer $k > 1$ is also increasing. This can be verified by simple algebra: if $m < n$, then multiplying on both sides by k gives us $km < kn$, which establishes that $g(m) < g(n)$.

Problem DGF2. Show that the function $f(n) = n^2$, with domain $\mathbf{N}$, is increasing.

Section SUM: Summations

$$\sum_{i=1}^{N} 1 = N$$

$$\sum_{i=1}^{N} i = \frac{N(N+1)}{2}$$

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{i=0}^{N} 2^i = 2^{N+1} - 1$$

$$\sum_{i=0}^{N} a^i = \frac{a^{N+1} - 1}{a - 1}$$

$$\sum_{i=0}^{N} a^i < \frac{1}{1-a} \text{ (whenever } 0 < a < 1)$$

$$\sum_{i=1}^{N} \frac{1}{i} \approx \ln 2 \log N \text{ (the difference between these falls below 0.58 as } N \text{ tends to infinity)}$$

Problem SUM1. Rewrite the following in terms of the variable N, using the formulas above.

$$\sum_{i=1}^{N} 2i^2 + 3i - 4.$$

Section MI: Mathematical Induction

Mathematical induction is a technique for proving mathematical results having the general form "for all natural numbers n, ..." For example, suppose you would like to prove that for all natural numbers $n > 1$, $n^2 > n+1$. You might try a few values for n to see if the statement makes sense. Certainly $2^2 > 2+1, 3^2 > 3+1, 10^2 > 10+1$. These examples suggest that the statement always holds true. But how do we know for sure? It is at least conceivable that for certain very large numbers that we are unlikely to consider, the statement is no longer true. Mathematical induction is a technique for demonstrating that such a formula must hold true for every natural number > 1, without exception.

The intuitive idea behind Mathematical Induction is this: Suppose you wish to prove that some statement $\phi(n)$, which asserts something about each whole number n, is true for every n. For example, to prove that for all $n \geq 0$, $n < 2^n$, we would use "$n < 2^n$" as our statement $\phi(n)$. We wish to show that this statement holds for every n. Suppose now that we can prove two things:

(1) that $\phi(0)$ is true (in our example, this would mean that we can prove $0 < 2^0$);
(2) that, for any n, if $\phi(n)$ happens to be true, then $\phi(n+1)$ must also be true (in our example, this would mean that, if it happens to be true that $n < 2^n$, then it must be true that $n+1 < 2^{n+1}$).

Mathematical Induction says that, if you can prove both (1) and (2), then you have proven that, for every n, $\phi(n)$ is indeed true.

Below are several forms of induction. Each provides a valid approach to proving the correctness of a statement about natural numbers. Different forms are useful in different contexts. We include an example of each.

Standard Induction. Suppose $\phi(n)$ is a statement depending on n. If

- $\phi(0)$ is true, and
- under the assumption that $n \geq 0$ and $\phi(n)$ is true, you can prove that $\phi(n+1)$ is also true,

then $\phi(n)$ holds true for all natural numbers n.

In Standard Induction, the step in the proof where $\phi(0)$ is verified is called the *Basis Step*. The second step, where $\phi(n+1)$ is proved assuming $\phi(n)$, is called the *Induction Step*. As we reason during this second step, we will typically need to make use of $\phi(n)$ as an assumption; in this context, $\phi(n)$ is called the *induction hypothesis*.

Note. Standard Induction allows you to establish that a statement $\phi(n)$ holds for all natural numbers 0,1,2, ... However, sometimes the objective is to show that $\phi(n)$ holds for all numbers n that are larger than a fixed number k. Standard Induction may still be used. Here is a precise statement:

Standard Induction (General Form). Let $k \geq 0$. Suppose $\phi(n)$ is a statement depending on n. If

- $\phi(k)$ is true, and
- under the assumption that $n \geq k$ and $\phi(n)$ is true, you can prove that $\phi(n+1)$ is also true,

then $\phi(n)$ holds true for all natural numbers $n \geq k$.

Problem MI1. Prove that, for every natural number $n \geq 1$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Problem MI2. Show that for every natural number $n > 4$, $n^2 < 2^n$.

Total Induction. Suppose $\phi(n)$ is a statement depending on n and $k \geq 0$. If

- $\phi(k)$ is true, and
- under the assumption that $n > k$ and that each of $\phi(k), \phi(k+1), \ldots, \phi(n-1)$ are true, you can prove that $\phi(n)$ is also true,

then $\phi(n)$ holds true for all $n \geq k$.

Problem MI3. Prove that if $f(n) = 2^n$, then f is increasing.

Finite Induction. Suppose $0 \leq k \leq n$, and suppose $\phi(i)$ is a statement depending on i, where $k \leq i \leq n$. If

- $\phi(k)$ is true, and
- under the assumption that $k \leq i < n$ and that $\phi(i)$ is true, you can prove $\phi(i+1)$ is true,

then $\phi(i)$ holds true for all i with $k \leq i \leq n$.

Note. Another equally valid variant of Finite Induction uses an induction hypothesis that is essentially the same as the one used for Total Induction.

Problem MI4. The following is a Java method for computing $n!$ for any n.

```
int factorial(int n) {
  if(n==0 || n==1) return 1;
  int accum = 1;
  for(int i = 2; i <= n; ++i) {
    accum *= i;
  }
  return accum;
 }
```

Prove that for every n, the output of `factorial(n)` is $n!$.

Section BNT: Basic Number Theory

We review some basics about number theory. Assume $a, b, c, \ldots$ are integers.

- [*divides*] $a \mid b$ means a divides b, i.e., for some c, $b = ac$
- [*floor and ceiling*] $\lfloor a \rfloor$ is the largest integer not greater than a ($\lfloor \cdot \rfloor$ is called the *floor function*) and $\lceil a \rceil$ is the smallest integer not less than a ($\lceil \cdot \rceil$ is called the *ceiling function*).

  Examples.
  (a) $\lfloor \frac{5}{4} \rfloor = 1$

(b) $\lceil \frac{5}{4} \rceil = 2$

Note. The floor function applied to rational numbers a/b yields the same results as Java's integer division when both a and b are positive. However, when one is negative and the other positive, the results differ:

$$-5/4 = -(5/4) = -1 \quad \text{(Java integer division)}$$
$$\lfloor -5/4 \rfloor = -2 \quad \text{(mathematics)}$$

- [*greatest common divisor*] $c = \gcd(a, b)$ means c is the largest integer that divides both a and b
- [*least common multiple*] $c = \text{lcm}(a, b)$ means c is the smallest integer for which $a \mid c$ and $b \mid c$
- [*modulus*] If $a > 0$, then $b \bmod a$ equals the (nonnegative) remainder on dividing b by a. ($b \bmod a$ is a nonnegative number less than a.)

  Note. Java's mod function % is the same as mod for positive inputs, but if $a, b > 0$, then $-a \,\%\, b = -(a \bmod b)$.

  Example: $8 \,\%\, 3 = 8 \bmod 3 = 2$

  Example: $-8 \bmod 3 = 1$ but $-8 \,\%\, 3 = -(8 \bmod 3) = -2$

  Example. Show that whenever $m \geq n \geq 2$ are integers,

$$m \,\%\, n < \frac{m}{2}.$$

  Proof. This is shown by considering two cases: When $n > m/2$, since $m \,\%\, n = m - n$, then

$$n > m/2 \;\Rightarrow\; -n < -m/2 \;\Rightarrow\; m - n < m - m/2 \;\Rightarrow\; m \,\%\, n < m/2.$$

  When $n \leq m/2$, then since $m \,\%\, n < n$, it follows $m \,\%\, n < m/2$.

- [*congruence*] $b \equiv a (\bmod n)$ means $b \bmod n = a \bmod n$. Equivalently $n \mid (b - a)$ (see one of the examples below for a proof of this equivalence).
- **The Division Algorithm.** For each pair of integers a, b with $a > 0$, there is a unique pair q, r such that
  - $b = aq + r$ (q is the *quotient*, r is the *remainder*), and
  - $0 \leq r < a$.

  Moreover, $q = \lfloor \frac{b}{a} \rfloor$ and $r = b \bmod a$, so we can write:

$$b = a \cdot \lfloor \frac{b}{a} \rfloor + b \bmod a.$$

  Note. The equation $b = aq + r$ also holds with $q = \frac{b}{a}$ (integer division) and $r = b \,\%\, a$, but in the case where $a > 0$ and $b < 0$, it turns out that $r < 0$ (so the inequality $0 \leq r < a$ given above fails if these computations are used).
- [primes] A positive integer p is *prime* if its only positive divisors are 1 and p. A positive integer c is *composite* if there are positive integers m, n, both greater than 1, such that $c = m \cdot n$.

Example. Show that every integer > 1 is a product of primes. (A prime itself is considered a product of primes.)

Solution. Proceed by induction on natural numbers $n \geq 2$. Since 2 is prime, 2 is a product of primes. This takes care of the base case. Proceeding with Total Induction, assume $n > 2$ and every number $< n$ is a product of primes. Consider n. If n is already prime, we are done. If n is composite, $n = m \cdot k$, then since both m, k are $< n$, by the induction hypothesis, each of m, k is a product of primes. It follows that n is a product of primes. This completes the induction and the proof.

Example. Prove that there are infinitely many primes.

Solution. Suppose there were only finitely many primes. Let $p_0, p_1, p_2, \ldots, p_m$ be a list of all primes in increasing order. Let P be the product of these primes; that is, let $P = p_0 \cdot p_1 \cdot p_2 \cdot \ldots \cdot p_m$. Since $P+1$ is larger than all the primes in the list, $P+1$ must be composite and we can write $P+1 = k \cdot n$ for some k, n. By the previous example, k must be a product of primes; in particular, some p_i divides k; it follows that p_i divides $P+1$. But p_i also divides P (recall the definition of P). Hence, p_i divides the difference $(P+1) - P$, which is impossible. Therefore, there cannot be only finitely many primes.

- *Fibonacci Numbers.* The sequence $F_0, F_1, F_2, \ldots, F_n, \ldots$ of Fibonacci numbers is defined by

$$F_0 = 0;$$
$$F_1 = 1;$$
$$F_n = F_{n-1} + F_{n-2}.$$

Problem BNT1. Let a, b be integers, not both 0.

A Suppose $g = \gcd(a, b)$. Show that there are integers x, y so that $g = ax + by$. *Hint.* Let $S = \{z \mid z \text{ is positive, having the form } ax + by \text{ for some } x, y\}$, and let $d = \min S$. First show that $d|a$ and $d|b$. Then show that $d = \gcd(a, b)$ by showing that $d \geq c$ for any common divisor c of a, b.

B Suppose there are integers x, y, c such that $c > 0$ and $ax+by = c$. Show that if $g = \gcd(a, b)$ then $g|c$. Hint: Use the Hint for part (A) and make use of the Division Algorithm.

Problem BNT2. Suppose $g = \gcd(a, b)$ and suppose d is some other common divisor of a, b; that is, suppose $d|a$ and $d|b$. Show that $d|g$.

Problem BNT3. Show that $a \equiv b \bmod n$ if and only if $n|(a - b)$. *Hint.* Try writing

$$a = \lfloor \frac{a}{n} \rfloor \cdot n + a \bmod n$$
$$b = \lfloor \frac{b}{n} \rfloor \cdot n + b \bmod n$$

Subtracting, you get

$$(*) \qquad a - b = \big(\lfloor \frac{a}{n} \rfloor \cdot n - \lfloor \frac{b}{n} \rfloor \cdot n\big) + \big(a \bmod n - b \bmod n\big).$$

This observation will help in the proof in both directions.

Problem BNT4. Find the unique q and r guaranteed by the Division Algorithm, where $a = 7$ and $b = -20$; that is, find q, r with $b = aq + r$ and $0 \leq r < a$. Then obtain values q' and r' such that $b = aq' + r'$ that makes use of Java's mod function.

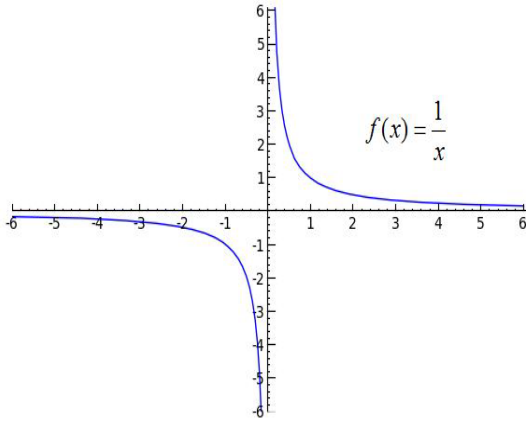
Problem BNT5—Extra Credit. Prove the following: For all nonzero a, b, $\gcd(a, b) \cdot \text{lcm}(a, b) = |ab|$. If you want to try this one, proceed by proving the following lemmas:

Lemma 1. Show that if a, b are nonzero integers and $\ell = \text{lcm}(a, b)$, then the rational number $\frac{ab}{\ell}$ is an integer.

Lemma 2. Show that if a, b are nonzero integers and $\ell = \text{lcm}(a, b)$, then the integer $\frac{ab}{\ell}$ divides both a and b.

Section LIM: Limits at Infinity

Consider the following graph of $f(x) = \frac{1}{x}$:



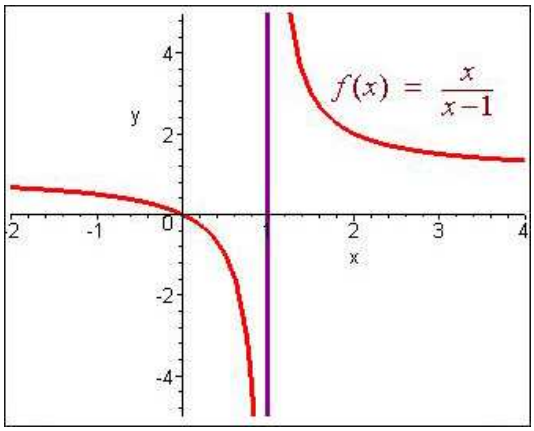
As x gets bigger and bigger, $f(x)$ gets closer and closer to 0. We write

$$\lim_{x \to \infty} \frac{1}{x} = 0.$$

Since we will be working with the set **N** of natural numbers, instead of the set **R** of real numbers, we will express this limit as

$$\lim_{n \to \infty} \frac{1}{n} = 0.$$

The following is the graph of $f(x) = \frac{x}{x-1}$:



Here, as x gets large, the graph approaches the line $y = 1$. We write:

$$\lim_{x\to\infty} \frac{x}{x-1} = 1$$

or when we are dealing only with natural numbers:

$$\lim_{n\to\infty} \frac{n}{n-1} = 1$$

We can compute this limit algebraically by factoring from numerator and denominator the reciprocal of the highest power of x that occurs in the expression:

$$\begin{aligned}\lim_{n\to\infty} \frac{n}{n-1} &= \lim_{n\to\infty} \left(\frac{n}{n-1} \cdot \frac{1/n}{1/n}\right) \\ &= \lim_{n\to\infty} \frac{1}{1-\frac{1}{n}} \\ &= 1.\end{aligned}$$

This computation makes use of one of several useful formulas that we will need in this course:

Facts About Limits

(1) $\lim_{n\to\infty} \frac{1}{n^r} = 0$ for any $r > 0$
(2) $\lim_{n\to\infty} \frac{1}{r^n} = 0$ for any $r > 1$
(3) $\lim_{n\to\infty} a \cdot \frac{f(n)}{g(n)} = a \cdot \lim_{n\to\infty} \frac{f(n)}{g(n)}$ (whenever the limits of f, g exist)
(4) $\lim_{n\to\infty} f(n) + g(n) = \lim_{n\to\infty} f(n) + \lim_{n\to\infty} g(n)$ (whenever the limits of f, g exist)
(5) $\lim_{n\to\infty} n = \infty$
(6) $\lim_{n\to\infty} 2^n = \infty$
(7) Suppose $p(n)$ and $q(n)$ are polynomials. If $\deg(p(n)) < \deg(q(n))$ then $\lim_{n\to\infty} \frac{p(n)}{q(n)} = 0$ and $\lim_{n\to\infty} \frac{q(n)}{p(n)} = \infty$

Example. Compute

$$\lim_{n\to\infty} \frac{2n^2-3n+5}{3n^2+1}.$$

Solution.

$$\begin{aligned}\lim_{n\to\infty} \frac{2n^2-3n+5}{3n^2+1} &= \lim_{n\to\infty} \frac{2n^2-3n+5}{3n^2+1}\cdot\left(\frac{1/n^2}{1/n^2}\right)\\ &= \lim_{n\to\infty} \frac{2-\frac{3}{n}+\frac{5}{n^2}}{3+\frac{1}{n^2}}\\ &= \frac{2}{3}.\end{aligned}$$

Example. Compute

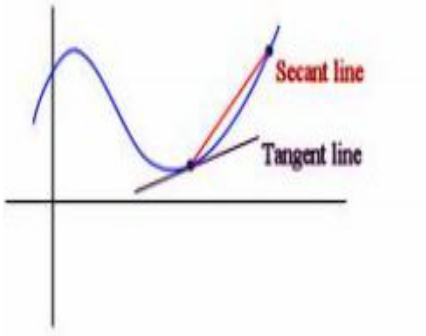
$$\lim_{n\to\infty} \frac{2n^2-1}{3n+1}.$$

Solution.

$$\begin{aligned}\lim_{n\to\infty} \frac{2n^2-1}{3n+1} &= \lim_{n\to\infty} \frac{2n^2-3n+5}{3n^2+1}\cdot\left(\frac{1/n^2}{1/n^2}\right)\\ &= \lim_{n\to\infty} \frac{2-\frac{1}{n}}{\frac{3}{n}+\frac{1}{n^2}}\\ &= \frac{2}{0} = \infty.\end{aligned}$$

Section D: Derivatives

The derivative of a function $f(x)$, which is written in any of these ways: $f'(x)$, $\frac{d}{dx}f(x)$, $\frac{dy}{dx}$, represents the *slope of the line tangent to the graph of* f *at the point* (x, y). For example:



There are a number of convenient formulas for computing derivatives of familiar functions:

(1) $\frac{d}{dx}\, a = 0$ for any real number a.
(2) $\frac{d}{dx}\, x^r = rx^{r-1}$, for any real number $r \neq 0$.
(3) $\frac{d}{dx}\, 2^x = 2^x \,\ln 2$

(4) $\frac{d}{dx}\log x = \frac{1}{x}\cdot\log e$

(5) For any functions $f(x), g(x)$ (whose derivatives exist) and real numbers a, b:

(a) (Linearity Rule) $\frac{d}{dx}(af(x)+bg(x)) = a\frac{d}{dx}f(x) + b\frac{d}{dx}g(x)$

(b) (Product Rule) $\frac{d}{dx}(f(x)\cdot g(x)) = f(x)\cdot\frac{d}{dx}g(x) + g(x)\cdot\frac{d}{dx}f(x)$

(c) (Reciprocal Rule) $\frac{d}{dx}(\frac{1}{f(x)}) = \frac{-f'(x)}{[f(x)]^2}$

(d) (Quotient Rule) $\frac{d}{dx}(\frac{f(x)}{g(x)}) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$.

(e) (Chain Rule) $\frac{d}{dx}(f(g(x)) = f'(g(x))g'(x)$.

(6) [L'Hopital's Rule] Suppose f and g have derivatives (at least when x is large) and their limits as $x \to \infty$ are either both 0 or both infinite. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}$$

as long as these limits exist.

For example:

(a) $\frac{d}{dx}ax = a$

(b) $\frac{d}{dx}ax^2 = 2ax$

(c) $\frac{d}{dx}ax^3 = 3ax^2$

(d) $\frac{d}{dx}\frac{1}{x} = \frac{-1}{x^2}$.

(e) $\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$.

Example. Compute the following:

$$\frac{d}{dx}\left(x^2 - \frac{1}{x^2-1}\right)$$

Solution.

$$\begin{aligned}\frac{d}{dx}\left(x^2 - \frac{1}{x^2-1}\right) &= \frac{d}{dx}(x^2) - \frac{d}{dx}\left(\frac{1}{x^2-1}\right)\\ &= 2x - \frac{-\frac{d}{dx}(x^2-1)}{[x^2-1]^2}\\ &= 2x + \frac{2x}{[x^2-1]^2}\end{aligned}$$

Appendix: More On Mathematical Induction

In this Appendix, we will clear up common confusions about induction by building up the idea of induction from a more basic context. For this purpose, we will try to prove, as an illustrative but basic example, that for any $n \in \{1, 2, 3, 4\}$, we have $2^n > n$. One way to do it is to prove each of the following four statements directly.

$$2^1 > 1$$
$$2^2 > 2$$
$$2^3 > 3$$
$$2^4 > 4$$

We could reduce the number of statements that we need to prove to just two if we do it like this: First, we prove $2^1 > 1$. Then, we prove the statement:

$$\text{for any } n \in \{1, 2, 3\}, \text{ if } 2^n > n, \text{ then } 2^{n+1} > n + 1.$$

Now let's check that this new approach would actually establish the result. We can break this second statement into three in the following way."

$$\text{if } 2^1 > 1, \text{ then } 2^2 > 2$$
$$\text{if } 2^2 > 2, \text{ then } 2^3 > 3$$
$$\text{if } 2^3 > 3, \text{ then } 2^4 > 4$$

In our first step we established that $2^1 > 1$. If we combine this with the first conditional statement that we just put on the board, if $2^1 > 1$, then $2^2 > 2$, we can conclude that $2^2 > 2$.

Therefore, assuming once again that we have proved if $2^2 > 2$ then $2^3 > 3$, we may conclude $2^3 > 3$. And then, for the final step, we can combine $2^3 > 3$ with the conditional if $2^3 > 3$ then $2^4 > 4$ to conclude $2^4 > 4$.

These verifications show that this approach to the proof—beginning with a direct proof of $2^1 > 1$—followed by proof of those three conditional statements does indeed give a proof that, for all $n \in \{1, 2, 3, 4\}$, $2^n > n$. What actually has to be proved, using this approach, is, first, that $2^1 > 1$, and then each conditional statement—that if $2^1 > 1$, then $2^2 > 2$, and so forth.

The approach so far is cumbersome. What makes it worthwhile is that we will be able to prove all three conditional statements simultaneously. We will do this by proving the following statement:

$(*)$ $$\text{for any } n \in \{1, 2, 3\}, \text{ if } 2^n > n, \text{ then } 2^{n+1} > n + 1.$$

We discuss how to create a proof for such a statement. We begin by noticing that, for all $n \in \mathbf{N}$, $2n \geq n + 1$: Since $n \in \mathbf{N}$, we know $n \geq 1$. We can add n to both sides, and the equality still holds. So we get $n + n \geq n + 1$. Since $2n = n + n$, we get $2n \geq n + 1$.

To prove $(*)$, suppose $n \in \{1, 2, 3\}$ and assume that $2^n > n$. Then

$$(+) \qquad 2^{n+1} = 2 \cdot 2^n > 2 \cdot n \geq n + 1$$

The statement $2^{n+1} = 2 \cdot 2^n$ follows by the definition of exponentiation. The statement $2 \cdot 2^n > 2 \cdot n$ is obtained by multiplying both sides of $2^n > n$ (which we are assuming is true) by 2. And the final step, $2 \cdot n \geq n + 1$, was what we proved first.

In this argument, we have established three conditionals in one argument. The point of all these efforts is to make it clear that the same thing works when we attempt to prove $2^n > n$ for *all* $n \in \mathbf{N}$. We can first show that $2^1 > 1$. And for the second step, we can show for all $n \in \mathbf{N}$, if $2^n > n$, then $2^{n+1} > n + 1$. In this case, establishing that second step establishes *infinitely many* conditional statements in a single argument. And the proof of that second step is exactly the same as the argument we just gave in $(+)$.

From these considerations, we restate the Principle of Mathematical Induction, which tells us that reasoning of this kind produces valid arguments in proofs of statements of the form 'for all $n \in \mathbf{N}, \phi(n)$.'

Principle of Mathematical Induction Suppose $\phi(n)$ is a formula, where the parameter n stands for a natural number. Suppose the following two statements have been proved:
(1) (*Basis Step*) $\phi(1)$ is true.
(2) (*Induction Step*) For each $n \in \mathbf{N}$, if $\phi(n)$ is true, then $\phi(n + 1)$ is also true.

Then, for all $n \in \mathbf{N}$, $\phi(n)$ is true.

The Principle of Mathematical Induction is just a formal expression of the procedure we just went through to prove that for every $n \in \mathbf{N}, 2^n > n$. In the argument we did, the first step, which we now call the *Basis Step*, was to establish $2^1 > 1$. The next step, which is called the *Induction Step*, is to prove, for any $n \in \mathbf{N}$, that, assuming $2^n > n$, then $2^{n+1} > n + 1$ as well. As we perform our reasoning during the Induction Step, each time we make use of the assumption that $2^n > n$, we are *using the induction hypothesis* at that point.