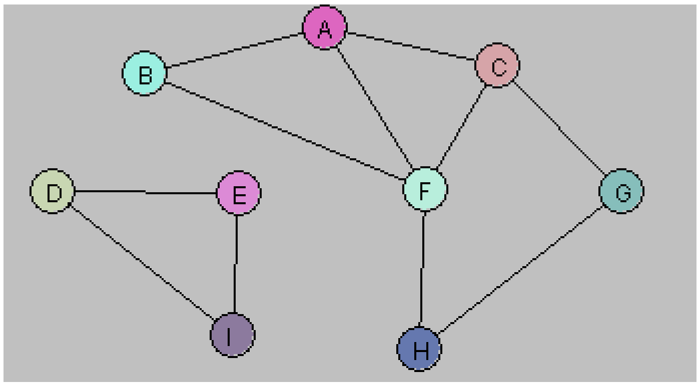
Lab 8

1. *Induced Graphs.* Answer questions about the graph $G = (V,E)$ displayed below.

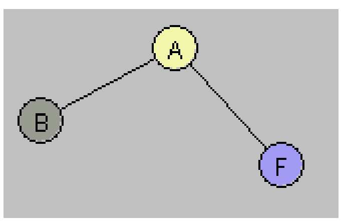


   A. Let $U = \{A, B\}$. Draw $G[U]$.

   B. Let $W = \{A, C, G, F\}$. Draw $G[W]$.

   C. Let $Y = \{A, B, D, E\}$. Draw $G[Y]$.

   D. Consider the following subgraph H of G:

   Is there a subset X of the vertex set V so that $H = G[X]$? Explain.

   E. Find a way to partition the vertex set V into two subsets V_1, V_2 so that each of the induced graphs $G[V_1]$ and $G[V_2]$ is connected and $G = G[V_1] \cup G[V_2]$.

2. *Graph Implementation.* Use the BFS class to solve the following problems. Implement by implementing the unimplemented methods in the Graph class.

   - Given two vertices, is there a path that joins them?
   - Is the graph connected? If not, how many connected components does it have?
   - Does the graph contain a cycle?

   Hint: For the third problem, you may use the following Fact (which we will prove in tomorrow's class):

   Fact: Suppose G is a graph and T is a spanning tree/forest for G. Then G has a cycle if and only if G has more edges than T.

3. *Graph Exercises.*
   A. Suppose G = (V, E) is a connected simple graph. Suppose $V_1, V_2, \ldots, V_k$ are disjoint subsets of V and that $V_1 \cup V_2 \cup \ldots \cup V_k = V$. Show that there is an edge (x,y) in E such that for some $i \neq j$, x is in V_i and y is in V_j.

   B. In class it was shown that a graph G = (V, E) is connected whenever the following is true,

   $$\epsilon > \binom{\nu - 1}{2} \quad (*)$$

   where ν is the number of vertices and ϵ is the number of edges. Is the following true or false?

   Every connected graph satisfies the inequality (*).

   Prove your answer.

   C. Suppose G is a graph with two vertices. What is the minimum number of edges it must have in order to be a connected graph? Suppose instead G has three vertices; what is the minimum number of edges it must have in order to be connected? Fill in the blank with a reasonable conjecture:

   If G has n vertices, G must have at least _____ edges in order to be connected.

4. *IsPrime Problem, Revisited.* The goal of this exercise is to devise a feasible algorithm that decides whether an input integer is prime. The key fact that you will make use of is the following:

   <u>Fact</u>: There is a function *f*, which runs in O(log *n*) (that is, O(length(*n*))), such that for any odd positive integer *n* and any *a* chosen randomly in [1, *n* - 1], if $f(a, n) = 1$, then *n* is composite, but if $f(a,n) = 0$, *n* is "probably" prime, but is in fact composite with probability < ½.

   A first try at such an algorithm would be:
   Algorithm **FirstTry**:

Input: A positive integer n
Ouptut: TRUE if n is prime, FALSE if n is composite

```
if n % 2 = 0 return FALSE
a ← random number in [1, n-1]
if f(a,n) = 1
   return FALSE
return TRUE
```

Notice that **FirstTry** runs in O(log *n*). It also produces a correct result more than half the time.

What could be done to improve the degree of correctness of **FirstTry** but still preserve a reasonably good running time? Explain.