Math (part of Lab 1)

Algorithms

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Solve the following problems and show all steps. Write answers neatly and clearly. You will probably need to review the material in MathReview.pdf.

Concepts You Should Know/Learn As You Do This Lab:

(1) computations using base-2 logarithms

(2) set algebra (union, intersection, set difference, power set)

(3) computation of combinations and permutations

(4) functions (and directed graphs)

- when a directed graph is not a function
- 1-1
- onto
- range

(5) functions that are increasing/decreasing/nondecreasing/nonincreasing

(6) summation notation

(7) proofs by mathematical induction

(8) number theory concepts

- $a \mid b$ ("a divides b")
- prime number
- floor ($\lfloor a \rfloor$) and ceiling ($\lceil a \rceil$) operations
- $\gcd(a, b)$ (greatest common divisor)
- $m \bmod n$ or $m \,\%\, n$ (modulus)
- Division Algorithm formula:

$$b = a \cdot \left\lfloor \frac{b}{a} \right\rfloor + b \bmod a.$$

- Fibonacci numbers $F_0, F_1, F_2, \ldots = 0, 1, 1, 2, 3, 5, \ldots$ computed by

$$F_0 = 0, F_1 = 1, \; F_n = F_{n-1} + F_{n-2}.$$

(9) computing derivatives

(10) computing limits at infinity

(11) L'Hopital's Rule

Problem 1. Which of the following functions are increasing? eventually nondecreasing? If you remember techniques from calculus, you can make use of those.

(1) $f(x) = -x^2$

(2) $f(x) = x^2 + 2x + 1$

(3) $f(x) = x^3 + x$

Problem 2. Compute the following limits at infinity:

(1) $\lim_{n\to\infty} (2n^2 + 3n)/(n^3 - 4)$.

(2) $\lim_{n\to\infty} n^2/2^n$.

Problem 3. Suppose A and B are sets (they are given and you do not know what the sets contain). Define a subset C of A (defined in terms of A and B) having the property that $C \cap B = \emptyset$ and $A \cup B = C \cup B$.

Problem 4. Two TAs are to be chosen from a group of five applicants. Applicants will be ranked according to background experience, math skills, and performance during interviews. The applicants that are ranked first and second by these criteria will be chosen for the job. In how many different ways can two of the five applicants be ranked first and second?

Problem 5. Use induction to show that for all $n > 4$, $2^n < n!$.

Problem 6. Write a Java method

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double log2(double x)
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that outputs $\log_2(x)$ for any input x. (Note that Java does not have a base-2 logarithm function in its Math library).

Problem 7. Compute the following derivative (recall that in this course, $\log(x)$ means $\log_2(x)$).

$$\frac{d}{dx}\left(\frac{3\log(x)}{2^x}\right)$$

Problem 8. Let X be a set having n elements, and write $X = \{x_1, x_2, \ldots, x_n\}$. Let $\mathcal{P}(X)$ denote the power set of X—the set of all subsets of X. Let $X^- = \{x_1, x_2, \ldots, x_{n-1}\}$; that is, X^- is the set you obtain by removing x_n from X.

(A) Show that $P(X) = P(X^-) \bigcup Y$ for some set Y having size 2^{n-1} and having no elements in common with $\mathcal{P}(X^-)$.

(B) Describe the set Y from (A)—what is an easy way to obtain Y from $\mathcal{P}(X^-)$?

Hint. To see how this works, try to prove it first in a special case, such as when $X = \{1, 2, 3\}$.

Problem 9. Suppose m, n are integers with $m \geq n \geq 1$.

(A) Suppose $d \mid m$ and $d \mid n$. Show that $d \mid (m \mod n)$.

(B) Suppose $d \mid n$ and $d \mid (m \mod n)$. Show that $d \mid m$.

(C) Show that

$$\gcd(m, n) = \gcd(n, m \mod n).$$

Hint. For (A) and (B), use the fact (discussed in the document MathReview.pdf), that whenever $m \geq n \geq 1$,

$$m = n \cdot \left\lfloor \frac{m}{n} \right\rfloor + m \mod n.$$

Problem 10. Use limits to prove the following:

(A) $4n^3 + n$ is $\Theta(n^3)$.

(B) $\log n$ is $o(n)$ (little-oh)

(C) 2^n is $\omega(n^2)$

(D) 2^n is $o(3^n)$

(E) 2^n is $\Theta(2^{n-1})$

(F) $\log n$ is $\Theta(\log_3 n)$.