1. If the graph G is a connected then, by the Theorem from the lecture (lecture-9 p. 18 “Suppose G is a connected graph with n vertices and m edges. Then G contains a cycle if and only if m ≥ n.”) G would contain a cycle if m >= n.

If the graph G is not connected. Let us use the Theorem from the lecture (lecture-9 p.23 “Suppose G is a graph with n vertices and m edges. Let F be a spanning forest for G (F is the union of the spanning trees inside each component of G). Let mF denote the number of edges in F. Then G has a cycle if and only if m > mF”) it would contain a cycle.

Assume each tree in forest has a F1, F2, …, Fk vertices. Then n=F1 + F2 + …+ Fk, and since tree each tree Fi has Fi-1 edges mF = F1 – 1 + F2 – 1 + …+ Fk – 1 = F1 + F2 + …+ Fk – k = n – k. Hence, the graph G has m >= n >= n – k = mf property and it would contain a cycle by the theorem.

1. Since, S and T are a tree and we are connecting this 2 trees with any edge (x,y) in E for which x is in VS and y is in VT. Then the total number of the edges in U = (VS U VT, ES U ET U {(x,y)}) is |ES| + |ET| + 1 = |VS| -1 + |VT| - 1 + 1 = |VS| + |VT| -1. Now if we prove that U is connected then by the theorem we learned in lecture (lecture-9 p. 18 “Suppose G is a connected graph with n vertices and m edges. Then G contains a cycle if and only if m ≥ n.”) it will not contain any cycle and it would prove U is a tree.

Let (*a, b*) be 2 distinct vertices from U. If (*a, b*) both belongs to either VS or VT there should exists a path between them. If (*a, b*) belongs to VS, and VT, we can safely assume *a* belongs to VS and *b* belongs to VT. Then there should a path exists *a*->x->y->b which shows U is a connected graph.

1. Code attached
2. Code attached