

Genetic association models are robust to common population kinship estimation biases

Zhuoran Hou¹, Alejandro Ochoa^{1,2,*}

⁴ ¹ Department of Biostatistics and Bioinformatics, Duke University, Durham, NC 27705, USA

⁵ Duke Center for Statistical Genetics and Genomics, Duke University, Durham, NC 27705, USA

⁶ * Corresponding author: alejandro.ochoa@duke.edu

Abstract

Common genetic association models for structured populations, including Principal Component Analysis (PCA) and Linear Mixed-effects Models (LMM), model the correlation structure between individuals using population kinship matrices, also known as Genetic Relatedness Matrices or “GRMs”. However, the most common kinship estimators can have severe biases that were only recently determined. Here we characterize the effect of these kinship biases on genetic association. We employ a large simulated admixed family and genotypes from the 1000 Genomes Project, both with simulated traits, to evaluate key kinship estimators. Remarkably, we find practically invariant association statistics for kinship matrices of different bias types (matching all other features). We then prove using statistical theory and linear algebra that LMM association tests are invariant to these kinship biases, and PCA approximately so. Our proof shows that the intercept and relatedness effect coefficients compensate for the kinship bias, an argument that extends to generalized linear models. As a corollary, association testing is also invariant to changing the reference ancestral population of the kinship matrix. Lastly, we observed that all kinship estimators, except for popkin ROM, can give improper non-positive semidefinite matrices, which can be problematic although some LMMs handle them surprisingly well, and condition numbers can be used to choose kinship estimators. Overall, we find that existing association studies are robust to kinship estimation bias, and our calculations may help improve association methods by taking advantage of this unexpected robustness, as well as help determine the effects of kinship bias in related problems.

27 **Abbreviations:** PCA: principal component analysis; PCs: principal components; LMM: linear
28 mixed-effects model; MOR: mean of ratios; ROM: ratio of means; WG: Weir-Goudet (kinship
29 estimator); MRCA: Most Recent Common Ancestor; SRMSD_p: p-value Signed Root Mean Square
30 Deviation; AUC_{PR}: Area Under the Precision Recall Curve; GCTA: Genome-wide Complex Trait
31 Analysis (software); PSD: positive semidefinite.

32 1 Introduction

33 The goal of genetic association is to detect loci that are related to a specific trait, either causally
34 or by proximity to causal loci. When applied to structured populations with admixed individuals,
35 multiethnic cohorts, or close relatives, controlling for relatedness is crucial to avoid spurious associa-
36 tions and loss of power (Devlin and Roeder, 1999; Voight and Pritchard, 2005; Astle and Balding,
37 2009; Yao and Ochoa, 2022). The most popular association models for structured populations are
38 Linear Mixed-effects Models (LMM) and Principal Component Analysis (PCA), which are closely
39 related except LMM is capable of modeling high-dimensional structures whereas PCA is strictly a
40 low-dimensional model (Astle and Balding, 2009; Hoffman, 2013; Yao and Ochoa, 2022).

41 Various association models, including both PCA and LMM, parameterize relatedness using
42 kinship matrices, also known as Genetic Relatedness Matrices or “GRMs”. Kinship coefficients
43 are well suited for this task since they model the covariance structure of genotypes (Malécot, 1948;
44 Jacquard, 1970). Kinship is often encountered in family studies, where they reflect recent relatedness
45 and can be calculated from pedigrees (Wright, 1922; Emik and Terrill, 1949; García-Cortés, 2015).
46 However, as kinship is defined as a probability of identity by descent, it may also capture ancient
47 population relatedness (Malécot, 1948; Astle and Balding, 2009), and common non-parametric
48 kinship estimators from genotypes indeed include population structure in their estimates (Ochoa and
49 Storey, 2021). In LMMs, the kinship matrix is an explicit parameter determining the random effect
50 covariance structure (Xie et al., 1998; Yu et al., 2006; Aulchenko et al., 2007; Astle and Balding,
51 2009; Kang et al., 2008; Kang et al., 2010; Zhou and Stephens, 2012; Yang et al., 2014; Loh et al.,
52 2015; Sul et al., 2018). In PCA, the principal components (PCs) are in practice the eigenvectors
53 of an empirical genetic covariance matrix that is equivalent to the most common kinship estimator

54 (Price et al., 2006; Astle and Balding, 2009; Hoffman, 2013; Yao and Ochoa, 2022).

55 Although several kinship estimators have been used with LMMs in the past, work from the
56 last 15 years has converged on what we call the “standard” kinship estimator, which is the same
57 estimator used in PCA and other related models (Price et al., 2006; Astle and Balding, 2009;
58 Rakovski and Stram, 2009; Thornton and McPeek, 2010; Yang et al., 2010; Yang et al., 2011; Zhou
59 and Stephens, 2012; Speed et al., 2012; Yang et al., 2014; Speed and Balding, 2015; Loh et al.,
60 2015; Wang et al., 2017; Sul et al., 2018). The impetus of our work is the recent characterization
61 of a complex bias for this standard estimator, which varies for every pair of individuals (Weir
62 and Goudet, 2017; Ochoa and Storey, 2021). These recent works also produced two new kinship
63 estimators, which we are interested in characterizing in the context of association. The Weir-Goudet
64 (WG) estimator constitutes a key improvement in that it has a uniformly downward bias (Weir and
65 Goudet, 2017; Ochoa and Storey, 2021). Lastly, the popkin estimator is the only unbiased estimator
66 under arbitrary relatedness (Ochoa and Storey, 2021). To the best of our knowledge, the new WG
67 and popkin estimators have not been used in association studies before, but represent potential
68 improvements over the use of the standard estimator for association.

69 One potential confounder when comparing the above kinship estimators is that the standard
70 estimator upweights rare variants in a formulation previously called “mean-of-ratios” (MOR), whereas
71 WG and popkin do not, instead following a “ratio-of-means” (ROM) estimation strategy (Bhatia
72 et al., 2013; Ochoa and Storey, 2021). Recent work also formulated a ROM version of the standard
73 estimator, which has a more predictable bias than the widely used MOR version (Ochoa and Storey,
74 2021). Following a locus weight formulation that allows the standard estimator to weigh loci in both
75 ways (Wang et al., 2017), here we generalize the popkin and WG estimators to have both MOR
76 and ROM versions, to test estimators without confounding by locus weighing strategy.

77 In this work, we originally hypothesized that kinship estimation bias would affect association
78 testing. We perform evaluations using an admixed family simulation (Yao and Ochoa, 2022) as
79 well as real genotypes from the 1000 Genomes project (Consortium, 2010; 1000 Genomes Project
80 Consortium et al., 2012; Fairley et al., 2020), in both cases with simulated traits, to characterize type
81 I error control and power using robust statistics. Surprisingly, we find that both LMM and PCA

82 association statistics are largely invariant to kinship estimation bias. We theoretically characterize
 83 the conditions under which these kinship biases result in invariant association statistics, which
 84 encompass changing ancestral population in the kinship matrix too. As we discover that most
 85 kinship estimates are non-positive semidefinite (non-PSD), breaking a key modeling assumption, we
 86 perform additional empirical validations and discover that some LMMs can handle these improper
 87 covariance matrices surprisingly well. Overall, we find that long-used association approaches are
 88 unaffected by the most common kinship estimation biases, and develop theory that may help improve
 89 association and related approaches such as heritability estimation.

90 2 Methods

91 2.1 Genetic model

92 The following genetic model justifies the use of kinship matrices in association studies, and is the
 93 basis of all kinship estimation bias calculations that our theoretical work depends upon.

94 Suppose there are m biallelic loci and n diploid individuals. The genotype $x_{ij} \in \{0, 1, 2\}$ at a
 95 locus i of individual j is encoded as the number of reference alleles, for a preselected but otherwise
 96 arbitrary reference allele per locus. Genotypes are treated as random variables structured according
 97 to relatedness. If T is the ancestral population on which allele frequencies are conditioned, φ_{jk}^T is
 98 the kinship coefficient of two individuals j and k , and p_i^T is the ancestral allele frequency at locus
 99 i , then under the kinship model (Malécot, 1948; Wright, 1949; Jacquard, 1970; Astle and Balding,
 100 2009; Ochoa and Storey, 2021) the expectation and covariance are given by

$$E[\mathbf{x}_i|T] = 2p_i^T \mathbf{1}, \quad \text{Cov}(\mathbf{x}_i|T) = 4p_i^T(1 - p_i^T) \boldsymbol{\Phi}^T,$$

101 where $\mathbf{x}_i = (x_{ij})$ is the length- n column vector of genotypes at locus i , $\boldsymbol{\Phi}^T = (\varphi_{jk}^T)$ is the $n \times n$
 102 kinship matrix, and $\mathbf{1}$ is a length- n column vector of ones. Both $\boldsymbol{\Phi}^T$ and p_i^T are parameters that
 103 depend on the choice of ancestral population, for which the Most Recent Common Ancestor (MRCA)
 104 population is the most sensible choice (Ochoa and Storey, 2021). However, one of the results of this
 105 work is proof that the choice of ancestral population does not affect association testing.

106 **2.2 Kinship estimators**

107 Each subsection below corresponds to a kinship estimator bias type: Popkin is unbiased, while
108 Standard and WG have different bias functions (defined shortly). Each estimator bias type has two
109 locus weight types called *ratio-of-means* (ROM) and *mean-of-ratios* (MOR), a terminology that
110 follows previous convention for these and related estimators (Bhatia et al., 2013; Ochoa and Storey,
111 2021). Only ROM estimators have closed-form limits. Below $\hat{p}_i^T = \frac{1}{2n}\mathbf{x}_i^\top \mathbf{1}$ is the standard ancestral
112 allele frequency estimator, where the \top superscript denotes matrix transposition (do not confuse
113 with ancestral population superscript T), and $\hat{\Phi}^{T,\text{name}} = (\hat{\varphi}_{jk}^{T,\text{name}})$ relates the scalar and matrix
114 formulas of each named kinship estimator.

115 **2.2.1 Popkin estimator**

116 The popkin (population kinship) estimator (Ochoa and Storey, 2021), generalized here to include
117 locus weights w_i , is given by

118

$$\hat{\varphi}_{jk}^{T,\text{popkin}} = 1 - \frac{A_{jk}}{\hat{A}_{\min}}, \quad A_{jk} = \frac{1}{m} \sum_{i=1}^m w_i((x_{ij} - 1)(x_{ik} - 1) - 1), \quad (1)$$

119 where in this work $\hat{A}_{\min} = \min_{j \neq k} A_{jk}$, and w_i must be positive but need not add to 1. We consider
120 two broad forms for this estimator. The original ROM estimator has $w_i = 1$ and has an unbiased
121 almost sure limit as the number of loci m go to infinity,

$$\hat{\Phi}^{T,\text{popkin-ROM}} \xrightarrow[m \rightarrow \infty]{\text{a.s.}} \Phi^T,$$

122 under the assumption that the true minimum kinship is zero. The MOR version, introduced here,
123 upweights rare variants by using $w_i = (\hat{p}_i^T(1 - \hat{p}_i^T))^{-1}$; although it has no closed-form limit, it is
124 approximately unbiased as well (Appendix A) and it is connected to the most common estimator,
125 Standard MOR (Appendix B). The use of locus weights here is inspired by previous calculations
126 relating the standard ROM and MOR estimators (Wang et al., 2017).

¹²⁷ **2.2.2 Standard estimator**

¹²⁸ The ROM and MOR versions of the standard kinship estimator are, respectively,

$$\hat{\varphi}_{jk}^{T,\text{std-ROM}} = \frac{\sum_{i=1}^m (x_{ij} - 2\hat{p}_i^T)(x_{ik} - 2\hat{p}_i^T)}{\sum_{i=1}^m 4\hat{p}_i^T(1 - \hat{p}_i^T)}, \quad (2)$$

$$\hat{\varphi}_{jk}^{T,\text{std-MOR}} = \frac{1}{m} \sum_{i=1}^m \frac{(x_{ij} - 2\hat{p}_i^T)(x_{ik} - 2\hat{p}_i^T)}{4\hat{p}_i^T(1 - \hat{p}_i^T)}. \quad (3)$$

¹²⁹ The ROM estimator has a biased limit, which is a function of the true kinship matrix (Ochoa and
¹³⁰ Storey, 2021):

$$\hat{\Phi}^{T,\text{std-ROM}} \xrightarrow[m \rightarrow \infty]{\text{a.s.}} F^{\text{std}}(\Phi^T) = \frac{1}{1 - \bar{\varphi}^T} \left(\Phi^T + \bar{\varphi}^T \mathbf{J} - \varphi^T \mathbf{1}^\top - \mathbf{1} (\varphi^T)^\top \right), \quad (4)$$

¹³² where $\mathbf{J} = \mathbf{1}\mathbf{1}^\top$ is the $n \times n$ matrix of ones, $\varphi^T = \frac{1}{n}\Phi^T\mathbf{1}$ is a length- n vector of per-row mean
¹³³ kinship values, and $\bar{\varphi}^T = \frac{1}{n^2}\mathbf{1}^\top\Phi^T\mathbf{1}$ is the scalar overall mean kinship. The MOR estimator does
¹³⁴ not have closed-form limit, but it is well approximated by Eq. (4) in practice, especially when loci
¹³⁵ with small minor allele frequencies are excluded prior to calculating this estimate. In Appendix B
¹³⁶ we prove that, when there are no missing genotypes, the two standard estimators are functions of
¹³⁷ the corresponding popkin estimators, given by the bias function F^{std} :

$$\begin{aligned} \hat{\Phi}^{T,\text{std-ROM}} &= F^{\text{std}}(\hat{\Phi}^{T,\text{popkin-ROM}}), \\ \hat{\Phi}^{T,\text{std-MOR}} &= F^{\text{std}}(\hat{\Phi}^{T,\text{popkin-MOR}}). \end{aligned}$$

¹³⁸ **2.2.3 Weir-Goudet estimator**

¹³⁹ The ROM version of the Weir-Goudet (WG) kinship estimator is given by (Weir and Goudet, 2017;
¹⁴⁰ Ochoa and Storey, 2021)

$$\hat{\varphi}_{jk}^{T,\text{WG-ROM}} = 1 - \frac{A_{jk}}{\hat{A}_{\text{avg}}}, \quad \hat{A}_{\text{avg}} = \frac{2}{n(n-1)} \sum_{j=2}^n \sum_{k=1}^{j-1} A_{jk}, \quad (5)$$

¹⁴² where A_{jk} is as in Eq. (1). Its biased limit is also a function of the true kinship matrix:

$$\hat{\Phi}^{T,\text{WG-ROM}} \xrightarrow[m \rightarrow \infty]{\text{a.s.}} F^{\text{WG}}(\Phi^T) = \frac{1}{1 - \tilde{\varphi}^T} (\Phi^T - \tilde{\varphi}^T \mathbf{J}), \quad (6)$$

¹⁴⁴ where $\tilde{\varphi}^T$ is the mean kinship excluding the matrix diagonal:

$$\tilde{\varphi}^T = \frac{2}{n(n-1)} \sum_{j=2}^n \sum_{k=1}^{j-1} \varphi_{jk}^T. \quad (7)$$

¹⁴⁶ In Appendix C we prove that

$$0 \leq \tilde{\varphi}^T \leq \bar{\varphi}^T \leq 1,$$

¹⁴⁷ and equalities are achieved if and only if all kinship values are equal. Since the WG-ROM estimator

¹⁴⁸ closely resembles the popkin estimator in Eq. (1), it is clear that they are related by the bias function

¹⁴⁹ F^{WG} , while WG-MOR is introduced here and defined by the below formula:

$$\begin{aligned} \hat{\Phi}^{T,\text{WG-ROM}} &= F^{\text{WG}}(\hat{\Phi}^{T,\text{popkin-ROM}}), \\ \hat{\Phi}^{T,\text{WG-MOR}} &= F^{\text{WG}}(\hat{\Phi}^{T,\text{popkin-MOR}}). \end{aligned}$$

¹⁵⁰ 2.3 Association models

¹⁵¹ LMM and PCA are closely related association models (Astle and Balding, 2009; Hoffman, 2013;

¹⁵² Yao and Ochoa, 2022):

$$\text{LMM: } \mathbf{y} = \mathbf{1}\alpha + \mathbf{x}_i\beta_i + \mathbf{s} + \boldsymbol{\epsilon}, \quad (8)$$

$$\mathbf{s} \sim \text{Normal}(\mathbf{0}, 2\sigma^2 \Phi^T), \quad (9)$$

$$\text{PCA: } \mathbf{y} = \mathbf{1}\alpha + \mathbf{x}_i\beta_i + \mathbf{U}_r\boldsymbol{\gamma}_r + \boldsymbol{\epsilon}, \quad (10)$$

$$\Phi^T = \mathbf{U}\Lambda\mathbf{U}^\top, \quad (11)$$

¹⁵³ where \mathbf{y} is a length- n vector of continuous trait values, α is the intercept coefficient, β_i is the genetic

¹⁵⁴ effect (association) coefficient of locus i , \mathbf{s} is the (genetic) random effect, σ^2 is the random effect

variance factor, \mathbf{U}_r is the $n \times r$ matrix of top- r eigenvectors (PCs) of Φ^T , $\boldsymbol{\gamma}_r$ is a length- r vector of coefficients for each eigenvector, $\epsilon \sim \text{Normal}(\mathbf{0}, \sigma_\epsilon^2 \mathbf{I})$ are random independent residuals, and \mathbf{I} is the $n \times n$ identity matrix. Furthermore, Eq. (11) is the complete eigendecomposition of Φ^T , where \mathbf{U} is the $n \times n$ matrix of eigenvectors, and Λ is the $n \times n$ diagonal matrix of eigenvalues. As \mathbf{s} and \mathbf{U}_r play analogous roles in modeling the effect of relatedness in LMM and PCA, respectively, we refer to them jointly as relatedness effects, and σ^2 and $\boldsymbol{\gamma}_r$ as their coefficients.

2.4 Simulations

2.4.1 Admixed family genotype simulation

An admixed family is simulated following previous work (Yao and Ochoa, 2022), except here only $K = 3$ ancestries are simulated and $F_{ST} = 0.3$ for the admixed individuals, which more closely resembles Hispanics and African Americans. Briefly, our admixture model first simulates $n = 1000$ founder individuals with $m = 100,000$ loci. Random ancestral allele frequencies p_i^T , subpopulation allele frequencies $p_i^{S_u}$, individual-specific allele frequencies π_{ij} , and genotypes x_{ij} are drawn from this hierarchical model:

$$\begin{aligned}
 p_i^T &\sim \text{Uniform}(0.01, 0.5), \\
 p_i^{S_u} | p_i^T &\sim \text{Beta}\left(p_i^T \left(\frac{1}{f_{S_u}^T} - 1\right), (1 - p_i^T) \left(\frac{1}{f_{S_u}^T} - 1\right)\right), \\
 \pi_{ij} &= \sum_{u=1}^K q_{ju} p_i^{S_u}, \\
 x_{ij} | \pi_{ij} &\sim \text{Binomial}(2, \pi_{ij}),
 \end{aligned}$$

where this Beta is the Balding-Nichols distribution (Balding and Nichols, 1995) with mean p_i^T and variance $p_i^T (1 - p_i^T) f_{S_u}^T$. This is implemented in the R package `bnpsd`.

We also include family structure in the simulation. 20 generations are generated iteratively. Individuals in the first generation ($n = 1000$) are ordered by 1D geography, randomly assigned sex, and treated as locally unrelated. From the next generation, individuals are paired iteratively: randomly choosing males from the pool and pairing them with the nearest available female with

175 local kinship $< 1/4^3$ (to preserve the admixture structure) until there are no available males or
 176 females. Family sizes are drawn randomly ensuring every family has at least one child. Children
 177 are reordered by the average coordinates of their parents, their sex are assigned randomly, and their
 178 alleles are drawn from parents independently per locus. The simulation is implemented in the R
 179 package `simfam`.

180 **2.4.2 Trait simulation algorithm**

181 Given an $m \times n$ genotype matrix $\mathbf{X} = (\mathbf{x}_i^\top)$, traits are simulated from

$$\mathbf{y} = \mathbf{1}\alpha + \mathbf{X}^\top \boldsymbol{\beta} + \boldsymbol{\epsilon}, \quad \boldsymbol{\epsilon} \sim \text{Normal}(\mathbf{0}, (1 - h^2)\mathbf{I}).$$

182 Given a desired number of causal loci $m_1 = n/10$ and heritability $h^2 = 0.8$, the goal is to choose
 183 causal coefficients $\boldsymbol{\beta}$ and the intercept α that result in zero mean and the desired trait heritability.
 184 Here, we use the “fixed effect sizes” trait simulation model described in (Yao and Ochoa, 2022).
 185 Briefly, first m_1 causal loci are randomly selected, and for these steps only \mathbf{X} is subset to these loci
 186 and reindexed. For known p_i^T , causal coefficients are constructed as:

$$\beta_i = \sqrt{\frac{h^2}{2m_1 v_i^T}},$$

187 where $v_i^T = p_i^T (1 - p_i^T)$; for unknown p_i^T (real genotypes), the unbiased estimate $\hat{v}_i^T = \hat{p}_i^T (1 - \hat{p}_i^T) / (1 - \bar{\varphi}^T)$
 188 is used, where $\bar{\varphi}^T$ is the mean kinship estimated from `popkin`. Coefficients are made negative
 189 randomly with probability 0.5. For known p_i^T , we obtain the desired zero trait mean with $\alpha =$
 190 $-2(\mathbf{p}^T)^\top \boldsymbol{\beta}$, where here $\mathbf{p}^T = (p_i^T)$ contains causal loci only. When p_i^T are unknown, to avoid
 191 covariance distortions, the intercept coefficient is constructed as

$$\alpha = -2\hat{p}^T \mathbf{1}_{m_1}^\top \boldsymbol{\beta}, \quad \hat{p}^T = \frac{1}{m_1} \mathbf{1}_{m_1}^\top \hat{\mathbf{p}}^T,$$

192 where $\mathbf{1}_{m_1}$ is a length- m_1 column vector of ones.

193 **2.5 Real genotype data processing**

194 To evaluate different kinship estimators on a real dataset, we use the high-coverage NYGC version of
195 the 1000 Genomes Project (Fairley et al., 2020), which is processed as before (Yao and Ochoa, 2022).
196 Briefly, using `plink2` (Chang et al., 2015) we keep only autosomal biallelic SNP loci with filter
197 “PASS”, pruned for linkage disequilibrium with parameters “`--indep-pairwise 1000kb 0.3`” to
198 remove loci that have a greater than 0.3 squared correlation coefficient with other loci within 1000kb,
199 and lastly remove loci with minor allele frequencies < 0.01. The resulting data have $m = 1,111,266$
200 loci and $n = 2,504$ individuals. Traits are simulated for this dataset with $m_1 = n/10 = 250$ causal
201 loci.

202 **2.6 Evaluation of performance**

203 AUC_{PR} and $SRMSD_p$ are used to evaluate approaches as before (Yao and Ochoa, 2022). Briefly,
204 $SRMSD_p$ (Signed Root Mean Square Deviation) measures the difference between the observed null
205 p-value quantiles and the expected uniform quantiles:

$$SRMSD_p = \text{sgn}(u_{\text{median}} - p_{\text{median}}) \sqrt{\frac{1}{m_0} \sum_{i=1}^{m_0} (u_i - p_{(i)})^2},$$

206 where $m_0 = m - m_1$ is the number of null (non-causal) loci, i indexes null loci only, $p_{(i)}$ is the i th
207 ordered null p-value, $u_i = (i - 0.5)/m_0$ is its expectation, p_{median} is the median observed null p-value,
208 $u_{\text{median}} = \frac{1}{2}$ is its expectation, and sgn is the sign function (1 if $u_{\text{median}} \geq p_{\text{median}}$, -1 otherwise).
209 $SRMSD_p = 0$ corresponds to calibrated p-values, $SRMSD_p > 0$ indicate anti-conservative p-values,
210 and $SRMSD_p < 0$ are conservative p-values.

211 AUC_{PR} (Area Under the Precision and Recall Curve) is a binary classification measure that
212 reflects calibrated power (Yao and Ochoa, 2022), which is calculated from the total numbers of true

213 positives (TP), false positives (FP) and false negatives (FN) at some threshold or parameter t :

$$\text{Precision}(t) = \frac{\text{TP}(t)}{\text{TP}(t) + \text{FP}(t)},$$
$$\text{Recall}(t) = \frac{\text{TP}(t)}{\text{TP}(t) + \text{FN}(t)},$$

214 followed by calculating the area under the curve traced as t varies recall from zero to one. Higher
215 AUC_{PR} is better, with best performance at AUC_{PR} = 1 for a perfect classifier, while worst perfor-
216 mance at AUC_{PR} = $\frac{m_1}{m}$ (overall proportion of causal loci) is for random classifiers.

217 2.7 Software

218 Popkin kinship estimates are calculated with the `popkin` R package. Standard MOR kinship esti-
219 mates are calculated with GCTA (version 1.93.2beta). All other kinship estimators and limits are
220 calculated using the `popkinsupp1` R package. PCs are calculated with the `eigen` function of R.

221 GCTA is used to run all LMM associations (Yang et al., 2011; Yang et al., 2014). We pass $2\Phi^T$
222 for all kinship matrices tested (the same scale as its own kinship estimate). PCA association is
223 performed with `plink2` (Chang et al., 2015). We use $r = K - 1 = 2$ PCs for the admixed family
224 simulations, and $r = 10$ PCs for 1000 Genomes.

225 3 Results

226 3.1 Empirical analysis using admixed family simulation

227 To quantify the effect of kinship estimation bias, we simulate genotypes and traits, and calculate
228 association p-values using a factorial design that tests all kinship matrix (3 bias types, times two lo-
229 cus weight types and one limit) and association model (PCA and LMM) combinations. We simulate
230 an admixed population with $K = 3$ ancestries, who serve as founders for a 20-generation random
231 pedigree. This high-dimensional admixed family scenario yields a large difference in performance
232 between PCA and LMM (Yao and Ochoa, 2022).

233 Kinship estimates and limits for this simulation are shown in Fig. 1. The true kinship matrix

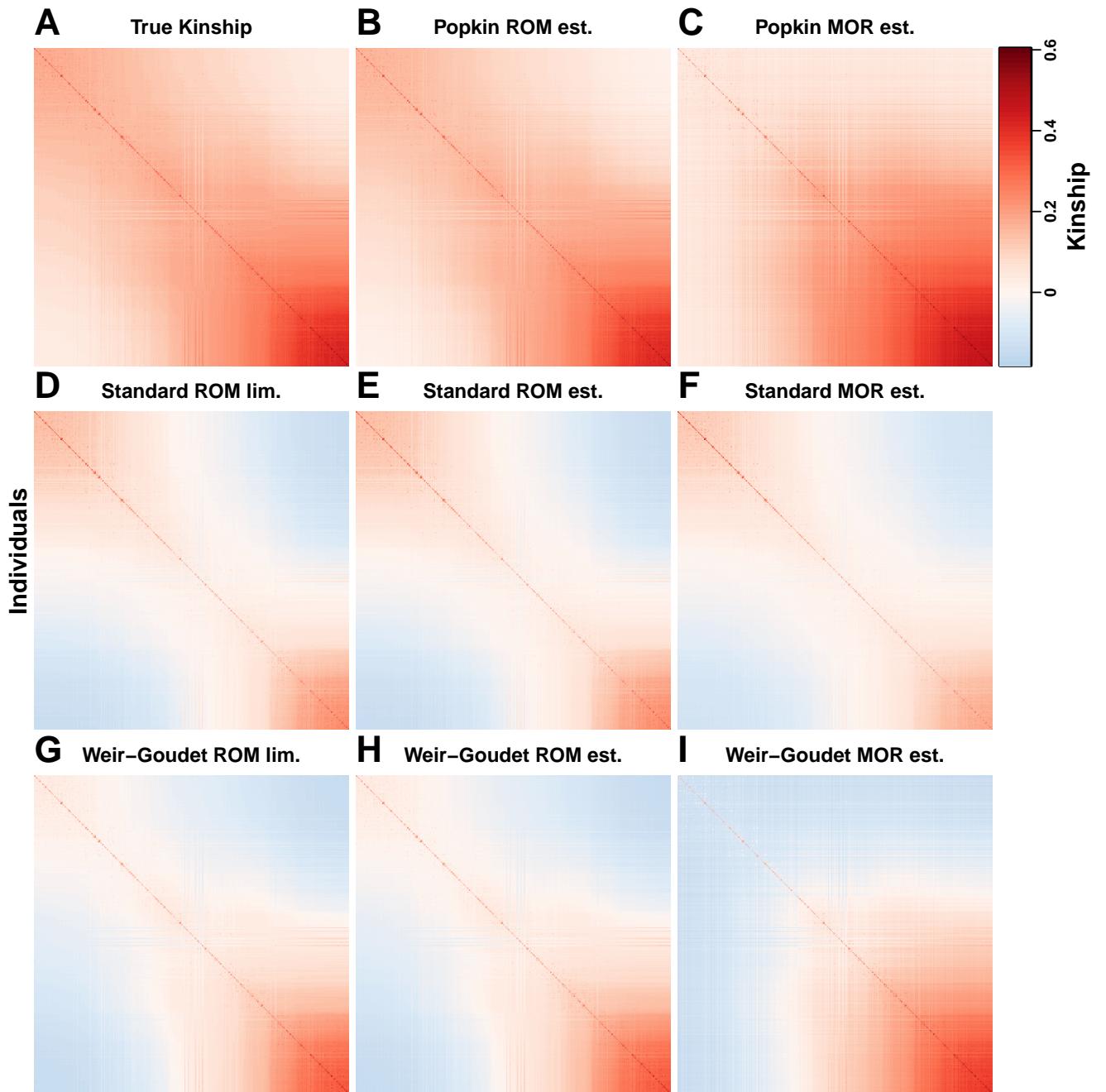


Figure 1: **Kinship estimates and limits on the admixed family simulation.** Each panel shows a kinship matrix as a heatmap, with each of the $n = 1000$ individuals along both x and y axes, color represents kinship: positive values in red, negative in blue. Diagonal contains inbreeding values. Each estimator bias type (Popkin, Standard, and Weir-Goudet; rows) has three matrices (columns): two locus weight types (ROM (ratio of means) and MOR (mean of ratios)) and limit of ROM.

234 shows the family relatedness as high values concentrated near the diagonal and the ancestry-driven
 235 population structure as the broad patterns off-diagonal. Only Popkin ROM is unbiased, while
 236 popkin MOR has a slight upward bias that varies across the matrix (Fig. S1A). In contrast, the
 237 Standard and Weir-Goudet (WG) estimates have large downward biases overall, resulting in abun-
 238 dant negative values; Standard biases vary for every pair of individuals, while WG has a uniform
 239 bias.

240 We perform LMM and PCA association tests to determine how kinship biases affect association
 241 performance. Surprisingly, we find that kinship bias type does not have a discernible effect on as-
 242 sociation performance, as summarized by AUC_{PR} (a robust proxy for power; Fig. 2) and $SRMSD_p$
 243 (measures null statistic calibration; Fig. S2). The largest differences in performance are explained

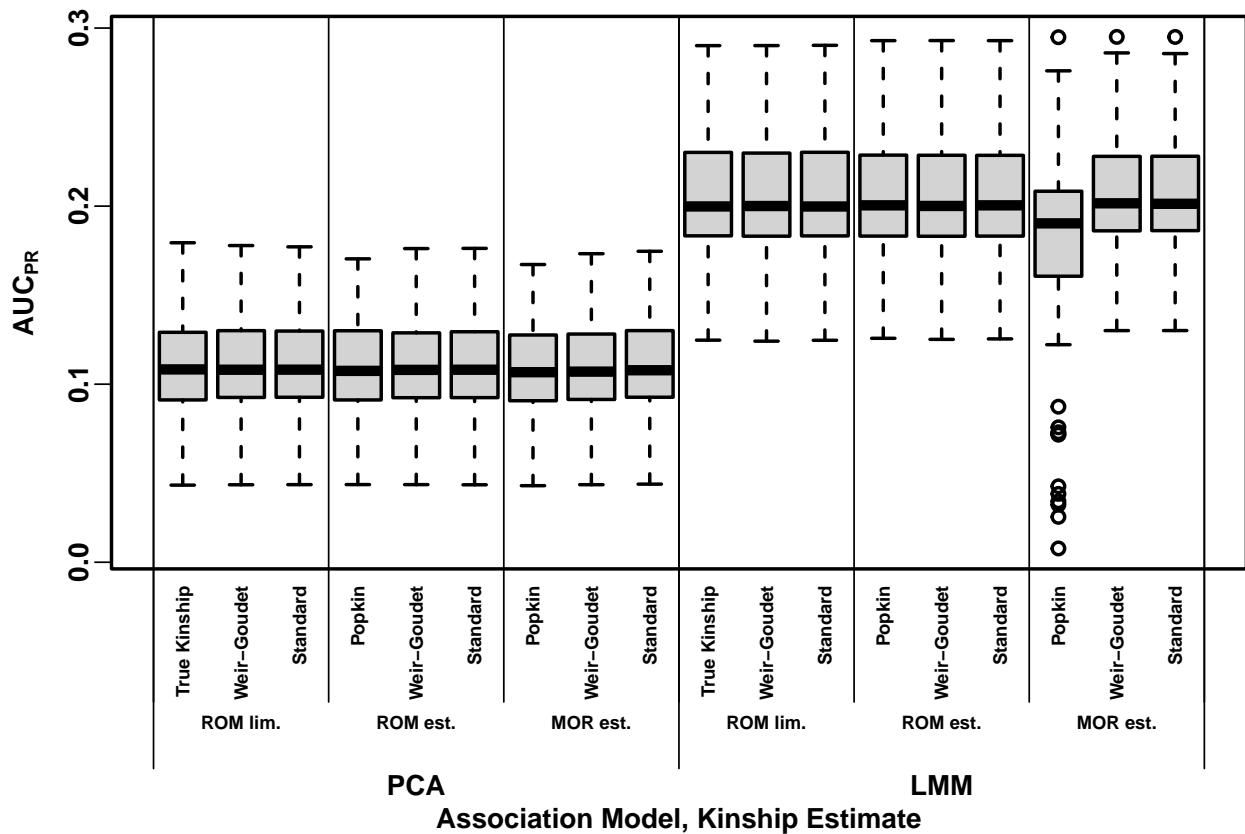


Figure 2: **Distributions of Area Under the Precision-Recall Curve (AUC_{PR}) on the admixed family simulation.** Higher AUC_{PR} is better performance. Results for 100 replicates (each a random genotype matrix and trait vector). Approaches cluster primarily by association model (LMM or PCA), and vary little across bias types.

244 by the association model (LMM vs PCA), as expected due to our use of a family simulation where
 245 PCA performs poorly. Within association models, there are no clear differences between the per-
 246 formance of any of the kinship matrices, in fact many appear to have identical distributions (both
 247 statistics), the only clear exception being LMM popkin MOR, which has a few outlier replicates
 248 where performance is exceedingly poor.

249 To better characterize the nearly identical performance distribution just observed, we next mea-

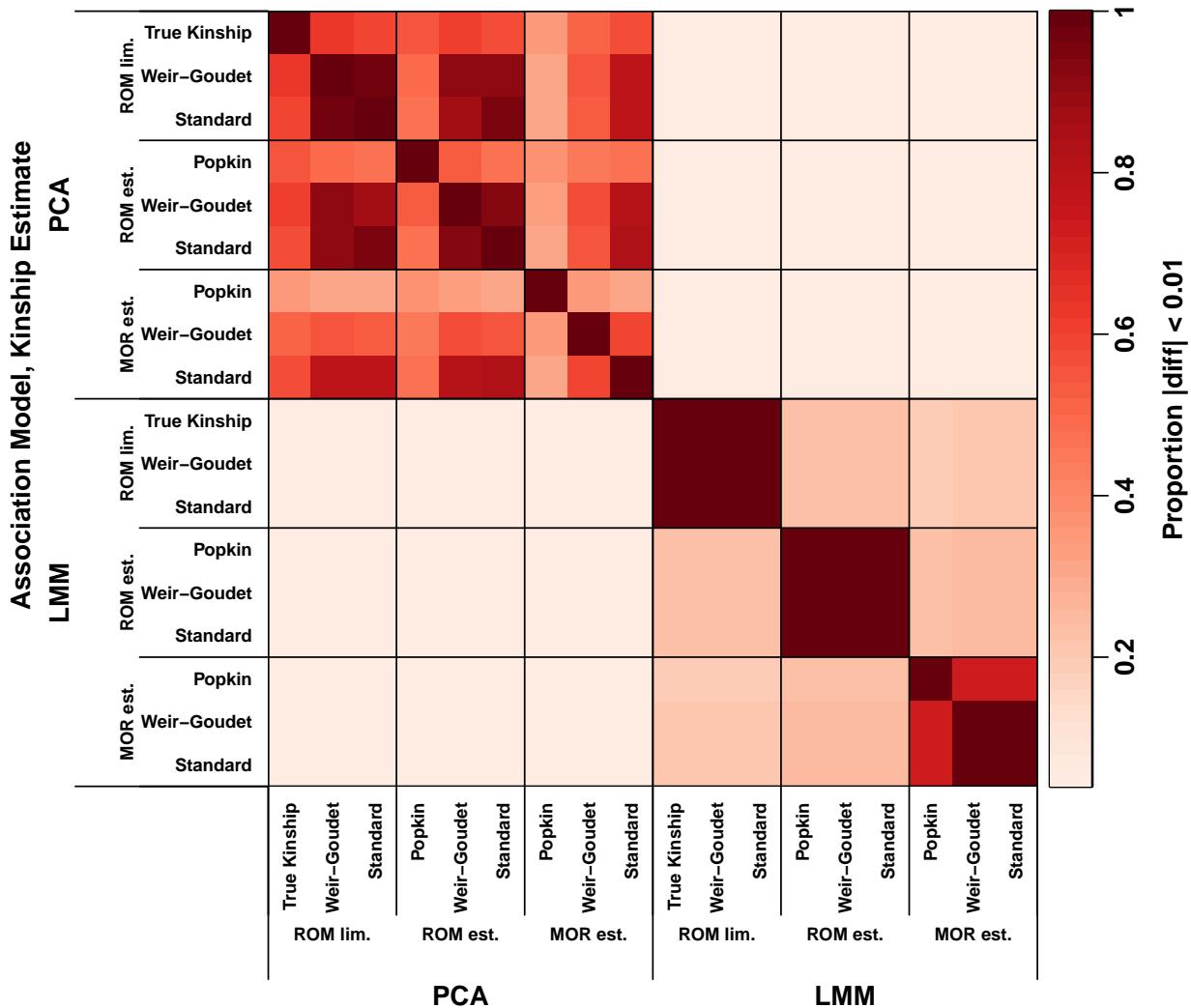


Figure 3: **Approximate agreement between p-values on the admixed family simulation.** Calculated agreement (absolute difference under 0.01) averaged over loci (color) of association p-values between association models (LMM vs PCA) and kinship matrices (x and y axes). All 100 replicates are used. Different bias types (matched for association model and locus weight type) have large proportions of nearly identical p-values.

sure the agreement between individual association p-values. We calculate the proportion of loci between two methods with p-values within 0.01 of each other, which is an approximate measure of agreement, and find a remarkably high agreement between estimators of different bias types after matching association model and locus weight type or limit (Fig. 3). This is in contrast to low agreement between PCA and LMM statistics, and between LMM statistics with different locus weight types or limits. Minimum agreements are higher across PCA methods, though here the true kinship or popkin estimates disagree more from Standard and WG matrices. Overall, kinship matrices with different bias types (otherwise matched) result in nearly identical association statistics.

3.2 Empirical analysis using 1000 Genomes

Now we repeat our analysis using the real genotypes of 1000 Genomes. Kinship estimates are shown in Fig. 4 (note real data have no true kinship or estimator limits). Popkin ROM estimates display an approximate nested block structure that arises from the tree relationships between subpopulations (Fig. 4A; trees were explicitly fit to this data in previous work (Yao and Ochoa, 2022)). However, popkin MOR estimates do not follow the nested blocks tree structure, since kinship between African and non-African populations is higher than kinship within African populations (Fig. 4B). Standard estimates have values closer to zero, and a different bias for each pair of individuals, resulting in higher relative kinship for African compared to non-African populations (Fig. 4C-D). Lastly, WG estimates are uniformly smaller than popkin's and attain large negative values (Fig. 4E-F).

Our association test conclusion are similar to our simulation study: AUC_{PR} and $SRMSD_p$ distributions are nearly identical for estimators of different bias types but same locus weight type (ROM or MOR) and association model. However, unlike the simulation, here the MOR estimates noticeably outperform ROM estimates (LMM only), in terms of both AUC_{PR} (Fig. 5) and $SRMSD_p$ (Fig. S3). P-values are again nearly identical at a large proportion of loci between approaches with matched association model and locus weight type (MOR or ROM), regardless of bias type (Fig. S4).

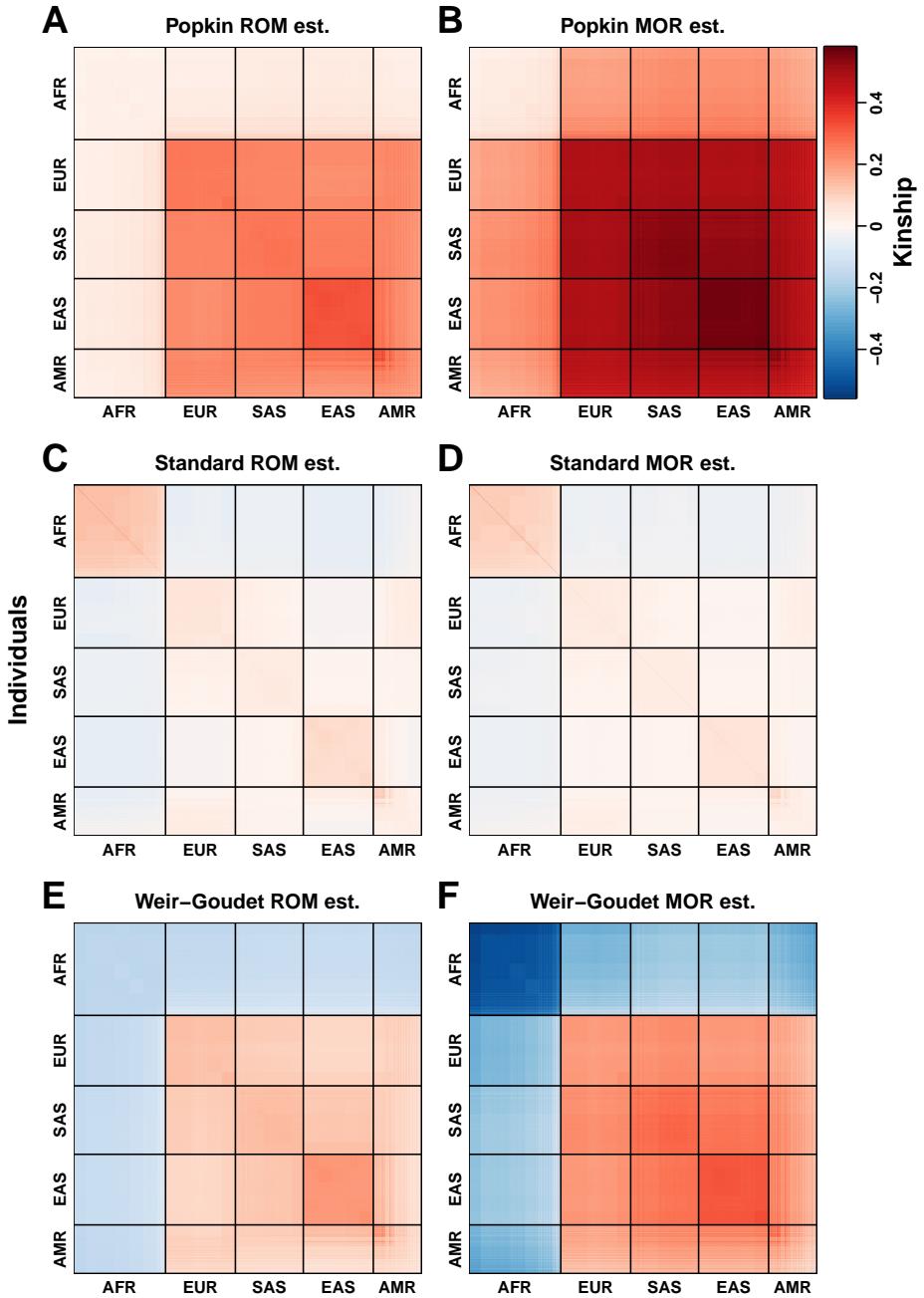


Figure 4: **Kinship estimates on 1000 Genomes.** Each panel represents a kinship matrix as a heatmap, as in Fig. 1. Superpopulation codes: AFR = African, EUR = European, SAS = South Asian, EAS = East Asian, AMR = Admixed Americans (Hispanics). Each estimator bias type (Popkin, Standard, and Weir-Goudet; rows) has two locus weight types (columns): ROM (ratio of means) and MOR (mean of ratios). In this visualization the upper range of all panels is capped to the 99 percentile of the diagonal (population inbreeding values) of the popkin MOR estimates.

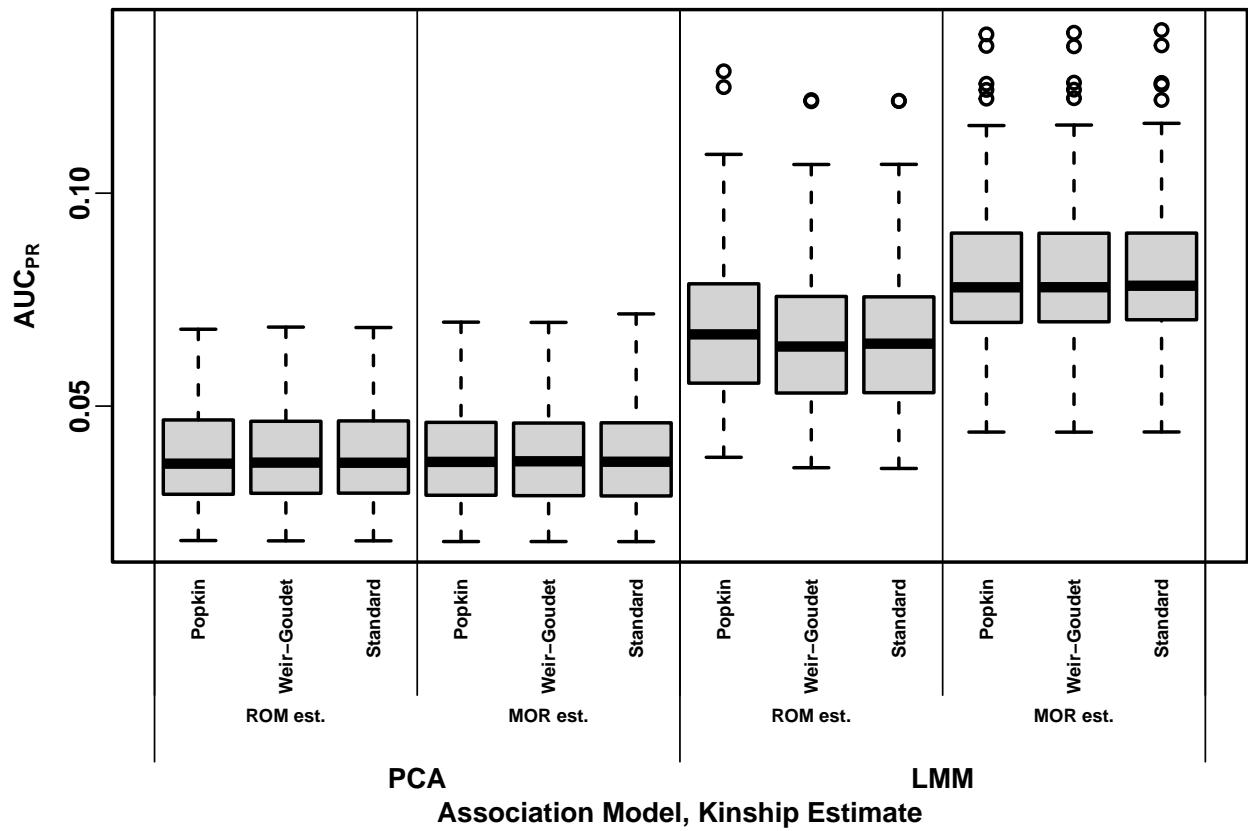


Figure 5: **Distributions of Area Under the Precision-Recall Curve (AUC_{PR}) on 1000 Genomes.** Higher AUC_{PR} is better performance. Results based on 100 simulated trait replicates (real genotype matrix is fixed). Approaches cluster primarily by association model (LMM or PCA) and locus weight type (ROM or MOR), and do not depend much at all on the bias type.

274 **3.3 Proof of association invariability to common kinship biases**

275 Our empirical observations suggest that replacing a kinship matrix with either the Standard or
276 WG-biased version does not alter association statistics (with exceptions we attribute to numerical
277 precision artifacts); here prove a more general version of these facts mathematically. Our construc-
278 tive proof shows that only a regression model with relatedness effects as covariates and an intercept
279 is required, whose coefficients adapt to the bias, and no other coefficients change. This is fortu-
280 nate, as the intercept and relatedness effect coefficients are nuisance parameters that usually go
281 unreported, while the focal genetic association coefficient and its p-value are unchanged by these
282 biases.

283 The most general form we identified of the bias function, mapping a kinship matrix to its bias-
284 transformed version, and for which association invariability holds, is

285
$$\Phi^{T'} = F(\Phi^T) = \frac{1}{c} \mathbf{B} \Phi^T \mathbf{B}^\top, \quad \mathbf{B} = \mathbf{I} - \mathbf{1}\mathbf{b}^\top, \quad (12)$$

286 where c is any positive scalar and \mathbf{b} is any length- n vector. The key property that the linear operator
287 \mathbf{B} must satisfy is that it shifts the input vector by the same scalar across its values, or

288
$$\mathbf{B}\mathbf{s} = \mathbf{s} - \mathbf{1}\eta, \quad (13)$$

289 where \mathbf{s} is any vector and the scalar $\eta = \mathbf{b}^\top \mathbf{s}$ is a function of the input vector. \mathbf{B} in Eq. (12) is the
290 only form that results in Eq. (13).

291 The Standard bias function $F = F^{\text{std}}$ of Eq. (4) can be written as Eq. (12) with $c = 1 - \bar{\varphi}^T$
292 and $\mathbf{b} = \frac{1}{n}\mathbf{1}$, in which case \mathbf{B} equals the centering matrix. Further, the generalized Standard
293 estimator studied in Ochoa and Storey (2021) has \mathbf{b} be a vector of individual weights that sum to
294 one: $\mathbf{b}^\top \mathbf{1} = 1$. These \mathbf{B} and $\Phi^{T'}$ are singular transformations (they are not invertible and have a
295 zero eigenvalue), since $\mathbf{B}\mathbf{1} = \mathbf{0}$ and $\mathbf{B}^\top \mathbf{b} = \mathbf{0}$.

296 The WG bias function $F = F^{\text{WG}}$ of Eq. (6) can be written as Eq. (12) with $c = 1 - \tilde{\varphi}^T$ and

$$\mathbf{b} = q \frac{(\boldsymbol{\Phi}^T)^{-1} \mathbf{1}}{\mathbf{1}^\top (\boldsymbol{\Phi}^T)^{-1} \mathbf{1}}, \quad (14)$$

$$q = 1 \pm \sqrt{1 - \tilde{\varphi}^T \left(\mathbf{1}^\top (\boldsymbol{\Phi}^T)^{-1} \mathbf{1} \right)}. \quad (15)$$

297 The derivation of this factorization is given in Appendix D. The determinant of the quadratic
298 solution q would be non-negative if $\tilde{\varphi}^T$ satisfied $\tilde{\varphi}^T \leq 1 / (\mathbf{1}^\top (\boldsymbol{\Phi}^T)^{-1} \mathbf{1})$. However, the actual $\tilde{\varphi}^T$
299 does not satisfy this inequality in any of our empirical cases, and in fact $1 / (\mathbf{1}^\top (\boldsymbol{\Phi}^T)^{-1} \mathbf{1}) \leq \tilde{\varphi}^T$
300 holds (proven in Appendix E; although $\tilde{\varphi}^T \leq \varphi^T$ (Appendix C), in practice those two are very close
301 while $1 / (\mathbf{1}^\top (\boldsymbol{\Phi}^T)^{-1} \mathbf{1})$ is much smaller than both), so b above is complex. This is a consequence
302 of WG estimates being non-PSD, which we elaborate in the following sections. Nevertheless, PCA
303 as well as the GCTA algorithms work for non-PSD matrices without invoking complex numbers
304 (following sections and Appendix F).

305 **3.3.1 Proof for LMM case**

306 Consider a random effect \mathbf{s} drawn using $\boldsymbol{\Phi}^T$, as given in Eq. (9). Using the affine transformation
307 property of Multivariate Normal distributions (which holds even if \mathbf{B} below is singular) and Eq. (12),
308 then

$$\mathbf{s}' = \mathbf{B}\mathbf{s} \sim \text{Normal}(\mathbf{0}, 2\sigma^{2\prime} \boldsymbol{\Phi}^{T'}) ,$$

309

$$\sigma^{2\prime} = c\sigma^2. \quad (16)$$

310

311 (This \mathbf{s}' has a degenerate distribution for Standard bias, since $\boldsymbol{\Phi}^{T'}$ is singular, but $\mathbf{s}' + \boldsymbol{\epsilon}$ is usually
312 non-degenerate, since its covariance $\mathbf{V}' = 2\sigma^{2\prime} \boldsymbol{\Phi}^{T'} + \sigma_\epsilon^2 \mathbf{I}$ is invertible as long as $\sigma_\epsilon^2 \neq 0$.) Replacing
313 $\mathbf{B}\mathbf{s}$ with the shift form in Eq. (13) shows that $\mathbf{s}' = \mathbf{s} - \mathbf{1}\eta$ are equal in distribution. Therefore, the
314 random effect \mathbf{s}' of the biased kinship matrix differs from the random effect \mathbf{s} of the original kinship
315 only by $\mathbf{1}\eta$, a difference compensated for by adjusting the intercept coefficient in Eq. (8):

$$\alpha' = \alpha + \eta. \quad (17)$$

316

317 No other regression coefficients, or the total residuals, change when Φ^T is replaced with $\Phi^{T'}$,
318 including the association coefficient β_i that is the focus of the test.

319 The above results require PSD kinship matrices $\Phi^{T'}$, which covariance matrices must be, and
320 which are characterized by non-negative eigenvalues and determinants. Nevertheless, for the non-
321 PSD WG bias (has a negative eigenvalue) combined with the generalized least squares association
322 algorithm, which is used by GCTA and other LMMs (Kang et al., 2008; Kang et al., 2010; Yang
323 et al., 2014), we find a stronger result consistent with Eq. (17), namely that $\alpha' = \alpha$, or in other
324 words, $\eta = 0$ (Appendix F).

325 The LMM association p-value does not change in several common tests, including the F-test,
326 since it only depends on the residuals and these do not change, as well as the likelihood ratio test,
327 because although covariance determinants change, they cancel out in the ratio. The Wald test used
328 by GCTA (Yang et al., 2014) is also invariant to these kinship biases given our empirical results in
329 Figs. 3 and S4 and proven explicitly for WG bias in Appendix F. Lastly, we confirmed empirically
330 that the Score test for the GCTA model is also invariant to these kinship biases (not shown).
331 These arguments hold whether variance components are fit with maximum likelihood or restricted
332 maximum likelihood (Kang et al., 2008; Kang et al., 2010; Yang et al., 2014), since multiplying the
333 estimated genetic variance component σ^2 by c and adjusting the intercept compensates for the bias
334 regardless of how $\sigma^2, \sigma_\epsilon^2$ are estimated.

335 3.3.2 Proof for PCA case

336 We present a proof for the PCA case that relies on an approximation that holds well in practice.
337 Based on the PCA model of Eqs. (10) and (11), let \mathbf{U}_r be the top eigenvectors of Φ^T , and \mathbf{U}'_r those
338 of $\Phi^{T'}$. The key approximation is that

$$\mathbf{U}'_r \approx \mathbf{B} \mathbf{U}_r, \quad (18)$$

340 which is not strictly equal (since $\mathbf{B} \mathbf{U}_r$ is not generally orthogonal, as eigenvectors must be), but we
341 have found it to be a good approximation in practice. In this case the eigenvector coefficients need
342 not change, $\gamma'_r = \gamma_r$, since the difference in scale of the kinship matrices (c in Eq. (12)) is absorbed

343 by the eigenvalues not present in this model. Applying the shift of Eq. (13) shows that

$$\mathbf{U}'_r \boldsymbol{\gamma}'_r = \mathbf{B} \mathbf{U}_r \boldsymbol{\gamma}_r = \mathbf{U}_r \boldsymbol{\gamma}_r - \mathbf{1}\eta,$$

344 where $\eta = \mathbf{b}^\top \mathbf{U}_r \boldsymbol{\gamma}_r$ is a scalar. Therefore, the relatedness effects again differ only by $\mathbf{1}\eta$, which is
345 compensated for by adjusting the intercept using Eq. (17), so the association coefficient β_i and the
346 residuals are the same in both cases. This proof works if there are small numbers of zero or negative
347 eigenvalues in $\Phi^{T'}$ (non-PSD cases), as those rank last and are simply ignored. The observations
348 from LMMs, that p-values are invariant to bias types, also hold for PCA.

349 We visualize the top PCs of our datasets in Fig. 6 to assess the validity of Eq. (18). The
350 approximation is equivalent to each biased PC (Standard or Weir-Goudet) being shifted from the
351 unbiased PC (Popkin), as described in Eq. (13). Fig. 6 indeed shows that PC1 is shifted by
352 noticeable amounts in each of these cases, while PC2 is less shifted. However, a rotation of the
353 PCs is also noticeable, particularly in the simulated data, and other large differences between MOR
354 estimators, as expected since we know the approximation cannot be exact. Also, PCs can change
355 order upon bias transformation, which we notice in the admixed family simulation, where PC2 and
356 PC3 from popkin (and true kinship) actually correspond to PC1 and PC2, respectively, in both
357 Standard and WG, and are plotted as such. No PC reordering occurs in 1000 Genomes. Overall,
358 while the approximation of Eq. (18) can be weakened to merely require that the biased PCs plus
359 intercept span the same subspace of the unbiased PCs plus intercept, the approximate PC shifts
360 better explain intuitively why the result for LMM is also observed for PCA association.

361 3.4 Proof of association invariability to change in ancestral population

362 The kinship matrices we used so far have values that depend on the choice of ancestral population
363 T . Here we consider the effect on association of changing ancestral population, and prove that it is
364 also compensated for by the relatedness and intercept coefficients.

365 Start from a kinship matrix Φ^S in terms of ancestral population S , and let T is a population
366 ancestral to S . If the inbreeding coefficient of S when T is the reference ancestral population is f_S^T ,

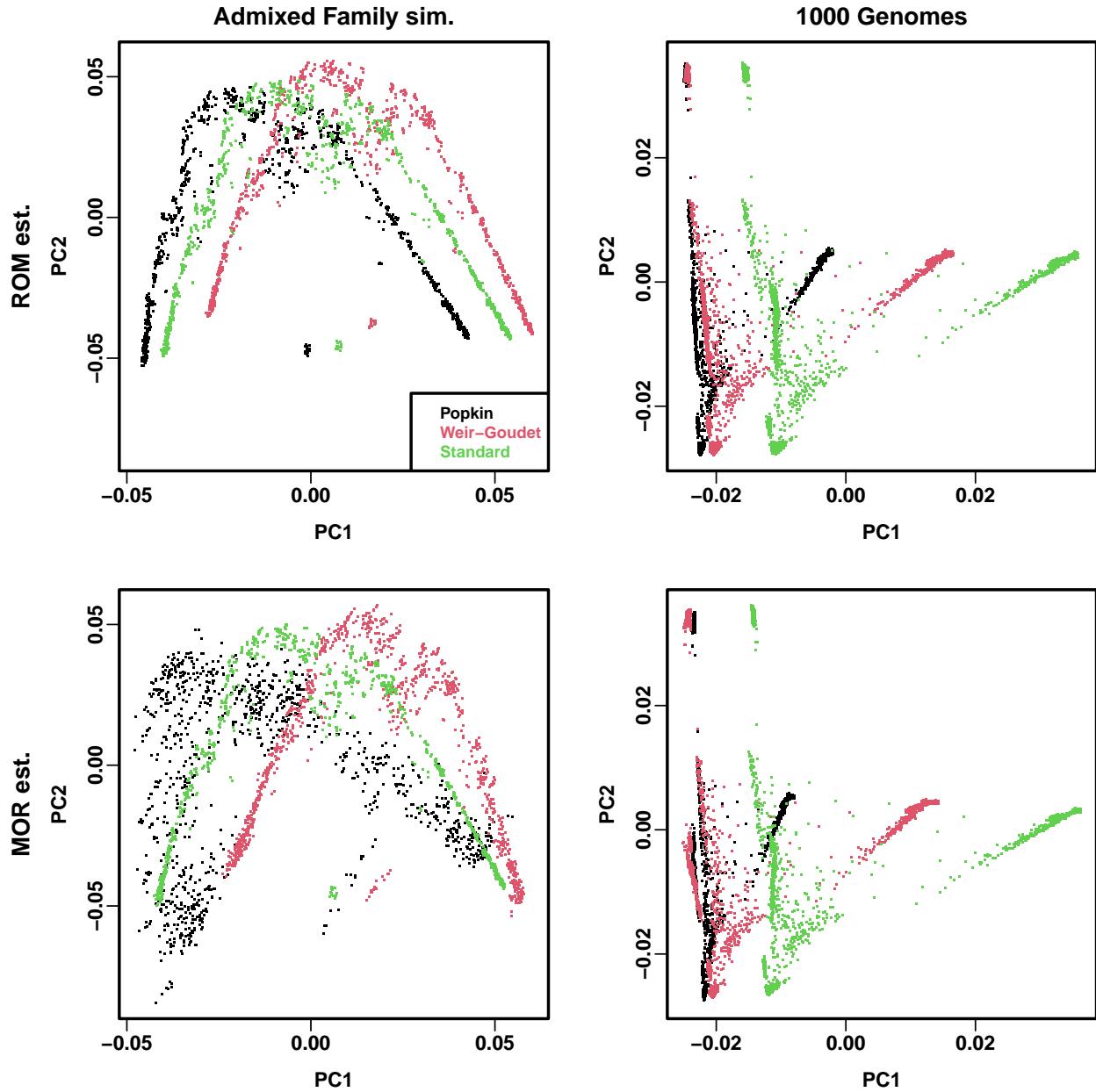


Figure 6: **Visualization of PC shift due to kinship biases.** Each panel shows three estimates (bias types): Popkin, Standard, and Weir-Goudet. ROM estimates are in first row, MOR in second row. (In admixed family, ROM limits are very similar to ROM estimates (not shown).) Columns show estimates from each dataset: admixed family simulation (first replicate) and 1000 Genomes. For popkin (both ROM and MOR estimates) in admixed family only, PC1 and PC2 are replaced with PC2 and PC3 (see text).

367 then the kinship matrix Φ^T in terms of T is given by (Ochoa and Storey, 2021)

$$(\mathbf{J} - \Phi^T) = (\mathbf{J} - \Phi^S) (1 - f_S^T).$$

368 Solving for Φ^T and simplifying results in

$$\Phi^T = (1 - f_S^T) \Phi^S + f_S^T \mathbf{J}.$$

369 This resembles WG bias but in reverse: whereas WG reduces and rescales kinship by $\tilde{\varphi}^T$, changing
370 to a more ancestral population rescales and increases kinship by f_S^T . Indeed, excluding $f_S^T = 1$, this
371 transformation can be written as Eq. (12) with $c = (1 - f_S^T)^{-1}$ and

$$\begin{aligned} \mathbf{b} &= q \frac{(\Phi^S)^{-1} \mathbf{1}}{\mathbf{1}^\top (\Phi^S)^{-1} \mathbf{1}}, \\ q &= 1 \pm \sqrt{1 + \frac{f_S^T}{1 - f_S^T} (\mathbf{1}^\top (\Phi^S)^{-1} \mathbf{1})}. \end{aligned}$$

372 The determinant of q is strictly positive, since $\mathbf{1}^\top (\Phi^S)^{-1} \mathbf{1} > 0$ (since Φ^S is positive definite, its
373 inverse is too) and $0 \leq f_S^T < 1$. Thus, our previous results apply: ancestor change is compensated
374 for by the relatedness and intercept coefficients, so the association statistics are invariant to this
375 transformation.

376 **3.5 Characterization of non-PSD and singular kinship and trait covariance es-**
377 **timators**

378 While attempting to validate and characterize the earlier factorization of the WG bias function
379 (Eqs. (12) to (15)), we discovered that it does not produce PSD matrices, which covariance matrices
380 are required to be. To characterize this problem more broadly, we calculate the eigenvalues of all
381 kinship matrices Φ^T and trait covariance matrices $\mathbf{V} = 2\sigma^2 \Phi^T + \sigma_\epsilon^2 \mathbf{I}$, the latter used by LMMs and
382 which we calculate using GCTA's estimates of σ^2 and σ_ϵ^2 .

383 We find that all WG matrices have very large negative minimum eigenvalues, and popkin MOR

384 estimates also have smaller negative minimum eigenvalues (Fig. S5). Moreover, besides all WG
 385 matrices and most popkin MOR estimates, Standard matrices are also often non-PSD but only in
 386 1000 Genomes (Fig. 7), which has missing genotypes (the admixed family simulation does not have
 387 missing genotypes). Each of these non-PSD matrices only has one negative eigenvalue. Notably, all
 388 popkin ROM estimates are PSD in every evaluation, including under missingness in 1000 Genomes.

389 In order to quantify matrix singularity, as well as numerical accuracy problems caused by mul-
 390 tiplying by inverses of nearly-singular matrices, we calculate condition numbers, which equal the
 391 maximum absolute eigenvalue divided by the minimum absolute eigenvalue of our covariance ma-
 392 trices. As expected, we see that Standard kinship matrices are singular on our admixed family
 393 simulation (which lacks missingness), as reflected by extremely high condition numbers, but their
 394 trait covariances have small condition numbers (Fig. S6). No other matrices are singular, but popkin

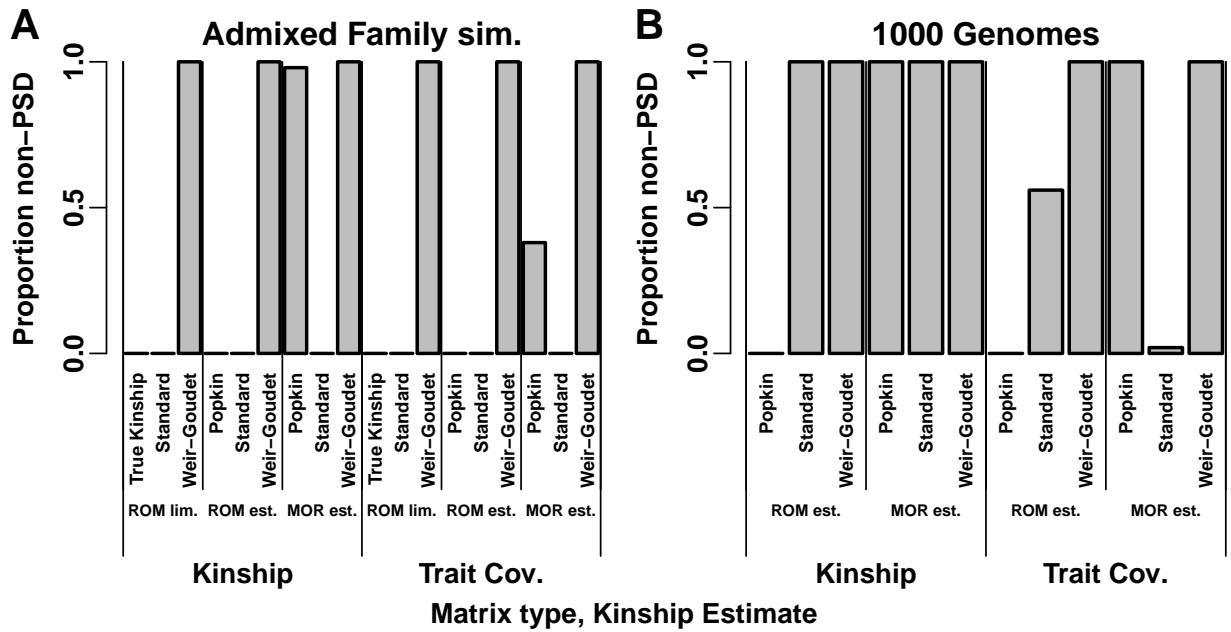


Figure 7: **Proportion of kinship and trait covariance (\mathbf{V}) matrices that are not positive semidefinite (PSD).** A matrix is non-PSD if it has negative eigenvalues (below -10^{-7} to allow for limited machine precision). Proportion is calculated over 100 replicates (1000 Genomes kinship has one value since genotypes are fixed, but \mathbf{V} varies per replicate). **A.** In admixed family simulation, which does not have missing genotypes, all WG matrices and most popkin MOR estimates are non-PSD. All non-PSD kinship matrices result in non-PSD \mathbf{V} except some popkin ROM estimates yield PSD \mathbf{V} . **B.** In 1000 Genomes, which has missingness, all kinship estimates are non-PSD except popkin ROM. Of the non-PSD kinship matrices, only some Standard estimates yield PSD \mathbf{V} .

395 MOR estimates in the admixed family simulation have relatively high condition numbers for both
396 kinship and trait covariance.

397 Consider the theoretical connection between the eigenvalues of Φ^T and those of \mathbf{V} . The eigen-
398 decomposition trick widely used to fit variance components in LMMs (Kang et al., 2008; Lippert
399 et al., 2011; Svishcheva et al., 2012; Zhou and Stephens, 2012; Sul et al., 2018) yields

$$\mathbf{V} = \mathbf{U} (2\sigma^2 \boldsymbol{\Lambda} + \sigma_\epsilon^2 \mathbf{I}) \mathbf{U}^\top,$$

400 where \mathbf{U} and $\boldsymbol{\Lambda}$ are the eigenvectors and eigenvalues of Φ^T , respectively (Eq. (11)), so the eigen-
401 vectors of \mathbf{V} are also \mathbf{U} and its eigenvalues are $2\sigma^2 \boldsymbol{\Lambda} + \sigma_\epsilon^2 \mathbf{I}$. Therefore, since $\sigma^2, \sigma_\epsilon^2 \geq 0$, then if
402 Φ^T is positive definite (all of its eigenvalues are positive) then so is \mathbf{V} , and the condition num-
403 ber of \mathbf{V} is always smaller (better) or equal than that of Φ^T . A negative kinship eigenvalue λ_k
404 may become positive for \mathbf{V} only if $\lambda_k > -\sigma_\epsilon^2/(2\sigma^2) = -(1-h^2)/(2h^2)$, so very large negative λ_k
405 values as observed for WG do not become positive in \mathbf{V} , in fact they can become more negative
406 (Fig. S5). \mathbf{V} is always invertible and well-conditioned even when Φ^T is singular PSD (has zero
407 eigenvalues), as the Standard estimator is under no missingness, since a kinship zero eigenvalue
408 becomes σ_ϵ^2 for \mathbf{V} . Conversely, the above equation explains why some non-PSD kinship matrices
409 are particularly problematic: negative eigenvectors near $-\sigma_\epsilon^2/(2\sigma^2)$ can result in ill-conditioned \mathbf{V} .
410 We see that popkin MOR estimates are non-PSD (Fig. S5) in such a way that some of their \mathbf{V} are
411 ill-conditioned (Fig. S6), and this explains its poorer performance in the admixed family evaluations
412 (Figs. 2 and S2), as shown in the next subsection.

413 3.6 Further empirical validation of theoretical predictions

414 Seeing that WG is always non-PSD, and to query other instances where predictions are not fully
415 met, here we analyze estimation accuracy of various parameters to better understand theoretically
416 and empirically how broken assumptions affect them. With PCA, no deviations from expectation
417 of AUC_{PR} and $SRMSD_p$ are observed for WG (Figs. 2 and 5, Figs. S2 and S3), which makes sense
418 since PCA simply ignores eigenvectors with negative or zero eigenvalues. Therefore, our analysis
419 focuses on LMM, where deviations are observed and clarification regarding WG is needed.

420 LMMs such as GCTA perform association testing in two steps. First is the restricted maximum
421 likelihood step used to fit variance components. Although the eigendecomposition approaches (Kang
422 et al., 2008; Lippert et al., 2011; Svishcheva et al., 2012; Zhou and Stephens, 2012; Sul et al., 2018)
423 require positive definite \mathbf{V} (lest the determinant of \mathbf{V} be negative), surprisingly the GCTA average
424 information algorithm only requires in practice that \mathbf{V} be invertible (Yang et al., 2011). Thus,
425 the relationship between WG, Standard, and True or Popkin variance components are largely as
426 expected from our theoretical prediction $\sigma^{2t} = c\sigma^2$ in Eq. (16), with the exception of popkin ROM
427 on 1000 Genomes only, whose genetic variance estimates are slightly smaller than expected (Fig. S7).

428 Next we determine the effect of WG bias on coefficient estimates. In this second step of LMM
429 association testing, once \mathbf{V} is determined, GCTA and other LMMs use generalized least squares
430 to estimate fixed effects coefficients (Kang et al., 2008; Kang et al., 2010; Yang et al., 2014).
431 Using the first replicate of the admixed family simulation and the true kinship matrix and the
432 Standard and WG limits only, we recalculate the genetic effect β_i and intercept coefficients α in R
433 for all loci, and confirm that we recover the GCTA estimates for β_i to the given precision. We then
434 compare intercept coefficients, which are not given by GCTA, and confirm our theoretical prediction
435 (Appendix F) that they are identical whether the True or WG ROM limit kinship matrices are used
436 (the mean absolute difference is below 10^{-7}). In contrast, intercepts fit using the Standard ROM
437 limit kinship matrix are different than those of the true kinship (not shown), which agrees with our
438 theoretical prediction that the intercept varies to compensate for the kinship matrix bias ($\alpha' = \alpha + \eta$
439 in Eq. (17)).

440 Lastly, we explain the largest deviations from our predictions of the performance metrics AUC_{PR}
441 and $SRMSD_p$. We find that the small performance errors of popkin ROM in 1000 Genomes (Figs. S3
442 and 5) are driven by errors in genetic variance component estimation σ^2 (Fig. S8). However, the
443 larger performance errors of popkin MOR in the admixed family simulation (Figs. 2 and S2) are
444 instead explained by the condition number of \mathbf{V} (Fig. S9). This result makes sense since the
445 condition number by definition quantifies regression coefficient estimation accuracy.

446 **4 Discussion**

447 Previous research showed that commonly used kinship estimators are biased, and that these biases
448 can be large (Ochoa and Storey (2021); Fig. 1). Our initial hypothesis was that these kinship biases
449 would affect association testing, but surprisingly found that association is unaffected. We then
450 proved theoretically that it is the intercept and relatedness effect (random effect or PCs) coeffi-
451 cients that compensate for the bias, and result in identical association coefficients and significance
452 statistics.

453 Kinship estimates depend on the choice of ancestral population, which conditions the distribu-
454 tions of allele frequencies and genotypes, but the effect of this choice of association testing was not
455 only unknown but completely disregarded. A corollary of our theoretical results is that changes
456 of ancestral population, which behave algebraically like kinship bias, are also compensated for by
457 the relatedness and intercept coefficients, so association testing is also invariant to the choice of
458 ancestral population. Thus, although a choice of ancestral population is always being made when
459 estimating kinship, this choice is fortunately inconsequential to association testing, as it ought to
460 be since relatedness is being conditioned upon in these tests.

461 Given that kinship bias type is not important for association studies, we are free to choose a
462 kinship estimator based on other properties. Ideally, kinship matrices result in well conditioned trait
463 covariance matrices, since that has the largest effect in numerical accuracy and power in LMMs.
464 Well-conditioned association is guaranteed for PSD kinship matrices, and popkin ROM is the only
465 estimator that produces PSD matrices consistently across our evaluations (Fig. 7). Popkin ROM
466 is also the only unbiased kinship estimator (Ochoa and Storey, 2021). We observed that Standard
467 kinship estimates are also not PSD when genotypes are missing, a well understood phenomenon
468 for related sample covariance estimators outside genetics (Jurczak and Rohde, 2017). Fortunately,
469 non-PSD kinship estimators often perform well for association. Nevertheless, in our admixed family
470 simulation we did see the other popkin estimator (the MOR version) perform particularly poorly
471 due to being non-PSD, which in combination with the heritability parameters of this simulation
472 results in ill-conditioned association tests and substantial loss of accuracy and power (Figs. 2, S2
473 and S9). Theory predicts that the same can happen with any non-PSD estimator, depending on

474 unknowns such as the heritability and the value of the negative eigenvalues of the kinship estimator,
475 so it is risky to use MOR estimators (all of which are non-PSD in 1000 Genomes), as well as the
476 WG estimator generally (which is non-PSD in all replicates of all of our evaluations). We also
477 observe smaller numerical inaccuracies for popkin ROM, the estimator we recommend, in 1000
478 Genomes only, although the result is mixed: performance is slightly better (Fig. 5) although null p-
479 value calibration is slightly worse (Fig. S3). The cause is variance components are poorly estimated
480 (Fig. S7), but we did not find a more fundamental explanation. Overall, our assessment suggests that
481 the popkin ROM estimator is the safest choice due to its guarantee of well-conditioned associations
482 that other estimators cannot make.

483 Despite being non-PSD, we observe better performance for MOR versus ROM estimators in
484 LMM association of 1000 Genomes (Fig. 5). Perhaps this is expected because we simulated larger
485 coefficients for rare variants, while MOR estimators upweigh rare variants. This effect is not observed
486 in the admixed family simulation, where MOR and ROM versions give similar kinship estimates
487 (Fig. 1) and performed similarly (Fig. 2), compared to 1000 Genomes where kinship estimates are
488 also strikingly different (Fig. 4). However, only popkin ROM is unbiased (Fig. 1B, Fig. S1). One
489 potential explanation is that our kinship model assumes that all variants existed in the MRCA
490 population, whereas rare variants in human data are known to be more recent mutations, and thus
491 their effective kinship matrix is different than that of ancestral variants. Therefore, despite its
492 biases, the popkin MOR estimator may better capture the covariance of rare variants and thus
493 model them better in association tests, particularly in LMMs where the effect is most pronounced.
494 Future work should focus on better approaches for upweighing rare variants or otherwise estimating
495 their covariance structure while resulting in positive definite kinship estimates.

496 Our conclusions that common kinship biases do not affect association studies extend to variations
497 of the Standard kinship estimator that weigh loci according to linkage disequilibrium (Speed et al.,
498 2017; Wang et al., 2017), which also have the Standard bias type since this bias is present in each
499 locus (Ochoa and Storey, 2021). As shown in our theoretical results, another form of the Standard
500 kinship estimator that weighs individuals to estimate ancestral allele frequencies \hat{p}_i^T , including the
501 best unbiased linear estimator in Appendix E (Astle and Balding, 2009; Thornton and McPeek,

502 2010), is also subject to the same conclusions.

503 In this study, we show empirically and theoretically that association tests are invariant to the
504 use of common kinship estimators that are biased versus a more recent unbiased estimator. The
505 underpinnings of our proof show that the same result holds for association with generalized linear
506 models, since the intercept and relatedness effects interact in the same way as for linear models (the
507 link function goes around the trait only); these models include case/control models such as logistic
508 PCA and LMM. However, heritability estimation requires unbiased estimates of the random effect
509 coefficient (σ^2), so it is biased when the standard kinship estimator is used, as it is using GCTA
510 (Yang et al., 2011; Yang et al., 2014). Nevertheless, heritability estimation is a complex problem
511 and a complete analysis is beyond the scope of this work. Overall, we have described an unexpected
512 robustness of association studies, and our theoretical understanding of this result may help guide
513 future improvements for association and other related models.

514 Declaration of interests

515 The authors declare no competing interests.

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520 Web resources

521 plink2, <https://www.cog-genomics.org/plink/2.0/>

522 GCTA, <https://yanglab.westlake.edu.cn/software/gcta/>

523 bnpsd, <https://cran.r-project.org/package=bnpsd>

524 simfam, <https://cran.r-project.org/package=simfam>

525 simtrait, <https://cran.r-project.org/package=simtrait>

526 popkin, <https://cran.r-project.org/package=popkin>
527 popkinsuppl, <https://github.com/OchoaLab/popkinsuppl>

528 Data and code availability

529 The data and code generated during this study are available on GitHub at <https://github.com/>
530 [OchoaLab/bias-assoc-paper](#). The high-coverage version of the 1000 Genomes Project was down-
531 loaded from ftp://ftp.1000genomes.ebi.ac.uk/vol1/ftp/data_collections/1000G_2504_high_
532 [coverage/working/20190425_NYGC_GATK/](#).

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630 Appendices

631 A Justification for popkin generalizations

632 The popkin estimator in Eq. (1) has been generalized in this work to include locus weights w_i . The
 633 original ROM formulation had $w_i = 1$ for all loci i (Ochoa and Storey, 2021). Recalling from that
 634 original work that

$$\mathrm{E}[(x_{ij} - 1)(x_{ik} - 1) - 1|T] = 4p_i^T(1 - p_i^T)(\varphi_{jk}^T - 1),$$

635 then for fixed w_i we get

$$\begin{aligned}\mathrm{E}[A_{jk}|T] &= v_m^T(\varphi_{jk}^T - 1), \\ v_m^T &= \frac{4}{m} \sum_{i=1}^m w_i p_i^T(1 - p_i^T).\end{aligned}$$

636 Therefore, as before all the unknowns p_i^T and now also the known weights w_i collapse into a single
 637 parameter v_m^T , which is estimated under the assumption that the minimum kinship is zero, giving
 638 $\hat{A}_{\min} = -v_m^T$, so that

$$\hat{\varphi}_{jk}^{T,\text{popkin-ROM}} = 1 - \frac{A_{jk}}{\hat{A}_{\min}} \xrightarrow[m \rightarrow \infty]{\text{a.s.}} \varphi_{jk}^T$$

639 as desired.

640 The MOR case of $w_i = (\hat{p}_i^T(1 - \hat{p}_i^T))^{-1}$ does not fit the previous case because this w_i is a
 641 random variable (it is a function of the genotypes). The term of interest $w_i((x_{ij} - 1)(x_{ik} - 1) - 1)$
 642 is a ratio of random variables whose expectation does not have a closed form. In this case, we rely

643 on the first-order approximation to this expectation, namely

$$\begin{aligned} \mathbb{E} \left[\frac{(x_{ij} - 1)(x_{ik} - 1) - 1}{\hat{p}_i^T (1 - \hat{p}_i^T)} \middle| T \right] &\approx \frac{\mathbb{E} [(x_{ij} - 1)(x_{ik} - 1) - 1 | T]}{\mathbb{E} [\hat{p}_i^T (1 - \hat{p}_i^T) | T]} \\ &= \frac{4p_i^T (1 - p_i^T) (\varphi_{jk}^T - 1)}{p_i^T (1 - p_i^T) (1 - \bar{\varphi}^T)} \\ &= \frac{4 (\varphi_{jk}^T - 1)}{1 - \bar{\varphi}^T}, \end{aligned}$$

644 where the expectation of $\hat{p}_i^T (1 - \hat{p}_i^T)$ was calculated previously (Ochoa and Storey, 2021). In this
 645 case the expectation of A_{jk} , summing across loci, is also approximated by

$$\mathbb{E} [A_{jk} | T] \approx \frac{4 (\varphi_{jk}^T - 1)}{1 - \bar{\varphi}^T}.$$

646 The same strategy as before applies to estimate the unknown factor $4/(1 - \bar{\varphi}^T)$, namely that if the
 647 minimum kinship is zero then $\hat{A}_{\min} \approx -4/(1 - \bar{\varphi}^T)$, resulting in

$$\hat{\varphi}_{jk}^{T, \text{popkin-MOR}} = 1 - \frac{A_{jk}}{\hat{A}_{\min}} \approx \varphi_{jk}^T.$$

648 B Connection between popkin and standard kinship estimator

649 Since the connection we discovered holds when data are complete, but not under missingness, to
 650 determine necessary conditions we introduce more complete forms of the estimators that handle
 651 missingness. Popkin (with locus weights) has the following parts updated:

$$\begin{aligned} A_{ijk} &= I_{ij} I_{ik} ((x_{ij} - 1)(x_{ik} - 1) - 1), \\ A_{jk} &= \frac{1}{m_{jk}} \sum_{i=1}^m w_i A_{ijk}, \\ m_{jk} &= \sum_{i=1}^m I_{ij} I_{ik}, \end{aligned}$$

652 where $I_{ij} = 1$ if x_{ij} is not missing, 0 otherwise (this way missing x_{ij} can have any value and not
 653 contribute to the estimator). Only loci with both genotypes (x_{ij} and x_{ik}) non-missing are included
 654 in the above average, and m_{jk} counts the total number of such loci. The ancestral allele frequency
 655 estimator with missingness is

$$\hat{p}_i^T = \frac{1}{2n_i} \sum_{j=1}^n I_{ij}x_{ij},$$

$$n_i = \sum_{j=1}^n I_{ij},$$

656 which averages over individuals rather than loci, so its denominator is the number of non-missing
 657 individuals at this locus. Let us compute some averages of the popkin estimator. Since the result
 658 we want holds at every locus separately, let us formulate the averages of interest at locus i only:

$$\bar{A}_{ij} = \frac{1}{n} \sum_{k=1}^n A_{ijk} = I_{ij} \frac{n_i}{n} ((x_{ij} - 1)(2\hat{p}_i^T - 1) - 1),$$

$$\bar{A}_i = \frac{1}{n} \sum_{k=1}^n \bar{A}_{ij} = - \left(\frac{n_i}{n} \right)^2 4\hat{p}_i^T (1 - \hat{p}_i^T).$$

659 Therefore, the combination of interest is:

$$A_{ijk} + \bar{A}_i - \bar{A}_{ij} - \bar{A}_{ik} = I_{ij}I_{ik} (x_{ij} - 2\hat{p}_i^T) (x_{ik} - 2\hat{p}_i^T)$$

$$+ \frac{n_i}{n} \left(I_{ij} - \frac{n_i}{n} \right) 4\hat{p}_i^T + \left(\left(\frac{n_i}{n} \right)^2 - I_{ij}I_{ik} \right) 4(\hat{p}_i^T)^2$$

$$+ I_{ij} \left(I_{ik} - \frac{n_i}{n} \right) x_{ij} (2\hat{p}_i^T - 1) + I_{ik} \left(I_{ij} - \frac{n_i}{n} \right) x_{ik} (2\hat{p}_i^T - 1).$$

660 For the above to equal $I_{ij}I_{ik} (x_{ij} - 2\hat{p}_i^T) (x_{ik} - 2\hat{p}_i^T)$, which is the first term above, the rest of the
 661 terms must vanish for arbitrary values of \hat{p}_i^T , x_{ij} , and x_{ik} . Since $n_i > 0$ (there is at least one
 662 non-missing individual at every locus), the term $\frac{n_i}{n}(I_{ij} - \frac{n_i}{n})4\hat{p}_i^T$ vanishes if and only if $I_{ij} = \frac{n_i}{n}$, and
 663 since $I_{jk} = 0$ does not solve this equation (because $n_i > 0$), then $I_{jk} = 1$, which requires $n_i = n$,
 664 so no individuals can have missing data at this locus (the rest of the terms vanish when this is so).

665 Thus,

$$A_{ijk} + \bar{A}_i - \bar{A}_{ij} - \bar{A}_{ik} = I_{ij}I_{ik} (x_{ij} - 2\hat{p}_i^T) (x_{ik} - 2\hat{p}_i^T)$$

666 if and only if there is no missing data at locus i . The other desired result of

$$\bar{A}_i = -4\hat{p}_i^T (1 - \hat{p}_i^T)$$

667 also requires $n_i = n$.

668 Assuming now no missingness, transforming the popkin estimates using the Standard bias func-
669 tion of Eq. (4) gives

$$\begin{aligned} \frac{\hat{\varphi}_{jk}^{T,\text{popkin}} + \bar{\varphi}^{T,\text{popkin}} - \bar{\varphi}_j^{T,\text{popkin}} - \bar{\varphi}_k^{T,\text{popkin}}}{1 - \bar{\varphi}^{T,\text{popkin}}} &= \frac{A_{jk} + \bar{A} - \bar{A}_j - \bar{A}_k}{-\bar{A}} \\ &= \frac{\sum_{i=1}^m w_i (A_{ijk} + \bar{A}_i - \bar{A}_{ij} - \bar{A}_{ik})}{-\sum_{i=1}^m w_i \bar{A}_i} \\ &= \frac{\sum_{i=1}^m w_i (x_{ij} - 2\hat{p}_i^T) (x_{ik} - 2\hat{p}_i^T)}{\sum_{i=1}^m w_i 4\hat{p}_i^T (1 - \hat{p}_i^T)}. \end{aligned}$$

670 Therefore, if popkin ROM is input ($w_i = 1$), this transformation yields Standard ROM. On the
671 other hand, if popkin MOR is used ($w_i^{-1} = \hat{p}_i^T (1 - \hat{p}_i^T)$), the transformation yields Standard MOR.

672 C Mean kinship inequalities

673 Denote the mean of the diagonal kinship terms as $\bar{\delta}^T = \frac{1}{n} \sum_{j=1}^n \varphi_{jj}^T$. Here we prove that

$$0 \leq \tilde{\varphi}^T \leq \bar{\varphi}^T \leq \bar{\delta}^T \leq 1,$$

674 with each of $\tilde{\varphi}^T = \bar{\varphi}^T$ and $\bar{\varphi}^T = \bar{\delta}^T$ if and only if all kinship values are equal.

675 The inequalities $0 \leq \bar{\varphi}^T \leq \bar{\delta}^T \leq 1$ follow directly from previous work, applied to a kinship
676 matrix rather than a coancestry matrix as done originally, as the proof required solely a covariance
677 matrix with values between 0 and 1 (Ochoa and Storey, 2021). $\tilde{\varphi}^T$ is defined in Eq. (7). $0 \leq \tilde{\varphi}^T$

678 follows since every kinship value is non-negative. $\bar{\varphi}^T$ and $\tilde{\varphi}^T$ are related by

$$679 \quad \bar{\varphi}^T = \frac{\tilde{\varphi}^T(n-1) + \bar{\delta}^T}{n}. \quad (19)$$

680 Applying $\bar{\varphi}^T \leq \bar{\delta}^T$ to Eq. (19) and simplifying yields $\tilde{\varphi}^T \leq \bar{\delta}^T$. Lastly, since $\bar{\varphi}^T - \tilde{\varphi}^T = (\bar{\delta}^T - \tilde{\varphi}^T)/n$
681 (from rearranging Eq. (19)), it also follows that $\tilde{\varphi}^T \leq \varphi^T$, as desired. Furthermore, $\tilde{\varphi}^T = \varphi^T$ holds
682 if and only if all $\varphi_{jk}^T = \bar{\delta}^T$, since that is necessary and sufficient for $\bar{\varphi}^T = \bar{\delta}^T$.

683 D Derivation of WG bias factorization

684 Here we rewrite the WG bias function of Eq. (6) as a factorization of the form of Eq. (12). It is
685 easy to see that $c = 1 - \tilde{\varphi}^T$. Expanding Eq. (12) gives

$$\begin{aligned} \mathbf{B}\Phi^T\mathbf{B}^\top &= (\mathbf{I} - \mathbf{1}\mathbf{b}^\top)\Phi^T(\mathbf{I} - \mathbf{b}\mathbf{1}^\top) \\ &= \Phi^T - \mathbf{1}(\Phi^T\mathbf{b})^\top - (\Phi^T\mathbf{b})\mathbf{1}^\top + \mathbf{J}(\mathbf{b}^\top\Phi^T\mathbf{b}), \end{aligned}$$

686 where $\mathbf{b}^\top\Phi^T\mathbf{b}$ is a scalar and $\Phi^T\mathbf{b}$ a vector. Equating the above to Eq. (6) and rearranging, we
687 obtain

$$\mathbf{J}(\tilde{\varphi}^T + (\mathbf{b}^\top\Phi^T\mathbf{b})) = \mathbf{1}(\Phi^T\mathbf{b})^\top + (\Phi^T\mathbf{b})\mathbf{1}^\top.$$

688 Since $\tilde{\varphi}^T + (\mathbf{b}^\top\Phi^T\mathbf{b})$ is a scalar and $\mathbf{J} = \mathbf{1}\mathbf{1}^\top$, we can see that the solution requires the right side to
689 also be a constant matrix, which is only achieved if $\Phi^T\mathbf{b} \propto \mathbf{1}$. We choose the scaling factor for the
690 last $\mathbf{1}$ to be $q(\mathbf{1}^\top(\Phi^T)^{-1}\mathbf{1})^{-1}$ as this simplifies notation later, and solving for \mathbf{b} results in Eq. (14).
691 To solve for q , we replace \mathbf{b} from Eq. (14) into the above equation, which after rearranging results
692 in

$$q^2 - 2q + \tilde{\varphi}^T \left(\mathbf{1}^\top(\Phi^T)^{-1}\mathbf{1} \right) = 0.$$

693 The solution to the above quadratic equation is given by Eq. (15), as desired.

694 **E Minimum weighted mean kinship**

695 Consider the weighted mean kinship value $\mathbf{w}^\top \Phi^T \mathbf{w}$, where \mathbf{w} are weights that sum to one ($\mathbf{w}^\top \mathbf{1} = 1$).

696 The ordinary mean kinship $\bar{\varphi}^T$ is the special case with $\mathbf{w} = \frac{1}{n} \mathbf{1}$. The weights that minimize the

697 weighted mean kinship are the solution of the Lagrangian multiplier problem

$$G = \mathbf{w}^\top \Phi^T \mathbf{w} + \lambda(\mathbf{w}^\top \mathbf{1} - 1).$$

698 The derivatives are the constraint and $\frac{dG}{d\mathbf{w}} = 2\Phi^T \mathbf{w} + \lambda \mathbf{1} = \mathbf{0}$. The optimal weights thus satisfy

699 $\mathbf{w} = \frac{-\lambda}{2} (\Phi^T)^{-1} \mathbf{1}$. Multiplying by $\mathbf{1}^\top$, since $\mathbf{1}^\top \mathbf{w} = 1$, allows us to solve for $\lambda^{-1} = -\frac{1}{2} \mathbf{1}^\top (\Phi^T)^{-1} \mathbf{1}$.

700 Thus, the optimal weights are

$$\mathbf{w} = \frac{(\Phi^T)^{-1} \mathbf{1}}{\mathbf{1}^\top (\Phi^T)^{-1} \mathbf{1}},$$

701 a solution that recurs in related settings, and applied to genotypes as $\hat{p}_i^T = \mathbf{w}^\top \mathbf{x}_i / 2$ yields the

702 best linear unbiased estimator of p_i^T (Altschul et al., 1989; Astle and Balding, 2009; Thornton and

703 McPeek, 2010). Therefore, the minimum weighted mean kinship is, and satisfies,

$$\mathbf{w}^\top \Phi^T \mathbf{w} = \frac{1}{\mathbf{1}^\top (\Phi^T)^{-1} \mathbf{1}} \leq \bar{\varphi}^T \approx \tilde{\varphi}^T.$$

704 **F Proof that WG bias results in zero intercept shift under LMM
705 generalized least squares estimation**

706 For this section suppose that variance components have been estimated, so $\mathbf{V} = 2\sigma^2 \Phi^T + \sigma_\epsilon^2 \mathbf{I}$ is

707 given, assume it is invertible, and rewrite the LMM as

$$\mathbf{y} = \mathbf{Z}\boldsymbol{\beta} + \boldsymbol{\epsilon}_V, \quad \boldsymbol{\epsilon}_V \sim \text{Normal}(\mathbf{0}, \mathbf{V}),$$

708 where the design matrix $\mathbf{Z} = (\mathbf{1}, \mathbf{x}_i, \dots)$ contains the intercept, genotype and now additional covari-

709 ates, and $\boldsymbol{\beta} = (\alpha, \beta_i, \dots)$ are their coefficients. The generalized least squares coefficients estimate,

⁷¹⁰ used by GCTA and other LMMs, is

$$\hat{\beta} = (\mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{Z})^{-1} \mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{y}.$$

⁷¹¹ Now suppose \mathbf{V} corresponds to some kinship matrix Φ^T while \mathbf{V}' corresponds to $\Phi^{T'} = F^{\text{WG}}(\Phi^T)$,
⁷¹² and \mathbf{V}' is also invertible. Our strategy involves repeated application of the Sherman-Morrison for-
⁷¹³ mula for calculating inverses of matrices after a rank-1 update, which for a symmetric update of a
⁷¹⁴ matrix \mathbf{A} with a vector \mathbf{z} and a scalar b takes the form (Sherman and Morrison, 1950)

$$(\mathbf{A} + b\mathbf{z}\mathbf{z}^\top)^{-1} = \mathbf{A}^{-1} - \frac{b}{1 + b(\mathbf{z}^\top \mathbf{A}^{-1} \mathbf{z})} (\mathbf{A}^{-1} \mathbf{z}) (\mathbf{A}^{-1} \mathbf{z})^\top.$$

⁷¹⁵ Since $F^{\text{WG}}(\Phi^T)$ is a rank-1 update of Φ^T by Eq. (6), then \mathbf{V}' is also a rank-1 update of \mathbf{V} :

$$\begin{aligned} \mathbf{V}' &= 2\sigma^{2\prime} \Phi^{T'} + \sigma_\epsilon^2 \mathbf{I} \\ &= 2\sigma^2 (\Phi^T - \tilde{\varphi}^T \mathbf{1} \mathbf{1}^\top) + \sigma_\epsilon^2 \mathbf{I} \\ &= \mathbf{V} - d \mathbf{1} \mathbf{1}^\top, \end{aligned}$$

⁷¹⁶ where $d = 2\sigma^2 \tilde{\varphi}^T$ and we used $\sigma^{2\prime} = (1 - \tilde{\varphi}^T) \sigma^2$. Therefore,

$$(\mathbf{V}')^{-1} = \mathbf{V}^{-1} + e \mathbf{V}^{-1} \mathbf{1} (\mathbf{V}^{-1} \mathbf{1})^\top,$$

⁷¹⁷ where $e = d / (1 - d(\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1}))$. Therefore the following remains a rank-1 update,

$$\mathbf{Z}^\top (\mathbf{V}')^{-1} \mathbf{Z} = \mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{Z} + e \mathbf{u} \mathbf{u}^\top,$$

⁷¹⁸ where $\mathbf{u} = \mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{1}$ is a column vector the length of the number of covariates (including intercept
⁷¹⁹ and genotype). Therefore,

$$(\mathbf{Z}^\top (\mathbf{V}')^{-1} \mathbf{Z})^{-1} = (\mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{Z})^{-1} - g \mathbf{v} \mathbf{v}^\top,$$

720 where $\mathbf{v} = (\mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{Z})^{-1} \mathbf{u}$ and $g = e/(1 + e(\mathbf{u}^\top \mathbf{v}))$. Noting that $\mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{1}$ is the first column of
 721 $\mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{Z}$, then \mathbf{v} is the first column of the identity matrix:

$$\mathbf{v} = (\mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{Z})^{-1} \mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{1} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix},$$

722 where $\mathbf{0}$ is a vector the length of the number of covariates minus one (exclude the intercept). As a
 723 consequence, $\mathbf{Z}\mathbf{v} = \mathbf{1}$, so $\mathbf{u}^\top \mathbf{v} = \mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1}$ and

$$\begin{aligned} g &= \frac{e}{1 + e(\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1})} \\ &= \frac{\frac{d}{1 - d(\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1})}}{1 + \frac{d}{1 - d(\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1})}(\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1})} \\ &= d. \end{aligned}$$

724 The final step yields the coefficient estimates as a rank-1 update:

$$\begin{aligned} \hat{\beta}' &= \left(\mathbf{Z}^\top (\mathbf{V}')^{-1} \mathbf{Z} \right)^{-1} \mathbf{Z}^\top (\mathbf{V}')^{-1} \mathbf{y} \\ &= \left((\mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{Z})^{-1} - d \mathbf{v} \mathbf{v}^\top \right) \mathbf{Z}^\top (\mathbf{V}^{-1} + e \mathbf{V}^{-1} \mathbf{1} (\mathbf{V}^{-1} \mathbf{1})^\top) \mathbf{y} \\ &= \hat{\beta} + e \mathbf{v} (\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{y}) - d \mathbf{v} (\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{y}) - d e \mathbf{v} (\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1}) (\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{y}) \\ &= \hat{\beta} + \mathbf{v} (\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{y}) (e - d - d e (\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1})). \end{aligned}$$

725 The last factor above vanishes:

$$\begin{aligned} e - d - d e (\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1}) &= \frac{d}{1 - d(\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1})} - d - d \frac{d}{1 - d(\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1})} (\mathbf{1}^\top \mathbf{V}^{-1} \mathbf{1}) \\ &= 0. \end{aligned}$$

726 Therefore, $\hat{\beta}' = \hat{\beta}$, which shows that all fixed effect coefficients, including the intercept, are invariant
 727 to using a WG-biased kinship matrix instead of the unbiased one when the coefficients are estimated
 728 with generalized least squares.

729 Furthermore, since the diagonal values of $(\mathbf{Z}^\top (\mathbf{V}')^{-1} \mathbf{Z})^{-1}$, which correspond to $\text{Var}(\hat{\beta}'_k)$ for each
730 k , are the same as those of $(\mathbf{Z}^\top \mathbf{V}^{-1} \mathbf{Z})^{-1}$ except for the first one corresponding to the intercept, then
731 the Wald test statistic of the k th covariate coefficients, given by $\hat{\beta}_k^2 / \text{Var}(\hat{\beta}_k)$, and their p-values,
732 are also the same for $k \neq 1$ for WG bias as for the unbiased kinship matrix.

Supplemental figures

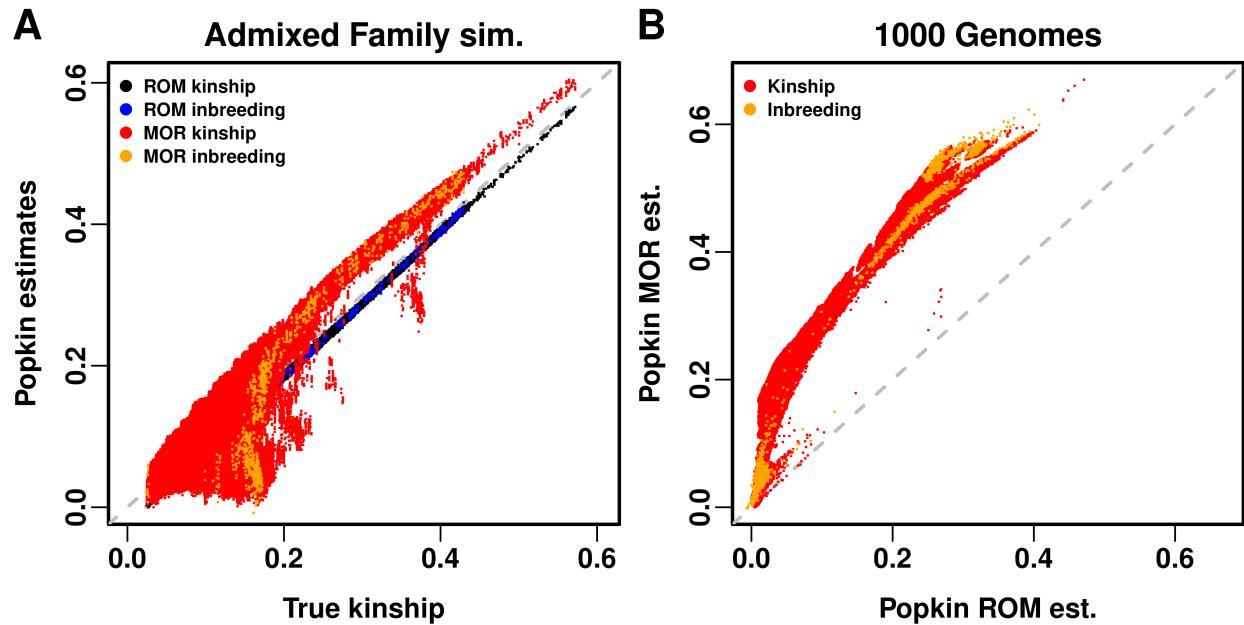


Figure S1: **Comparison of popkin ROM and MOR estimates.** Kinship (off-diagonal of matrix) and inbreeding (transformed diagonal) are plotted in different colors, which shows that their biases (if any) overlap. **A.** In admixed family simulation, both estimates are compared against true kinship. Popkin ROM has a negligible bias, due to the minimum true kinship of the simulation being slightly larger than zero. Popkin MOR has considerable biases, tending to be upward though not always. **B.** In 1000 Genomes, since true kinship is unknown, popkin ROM takes its place. Popkin MOR biases take on a similar shape as panel A.

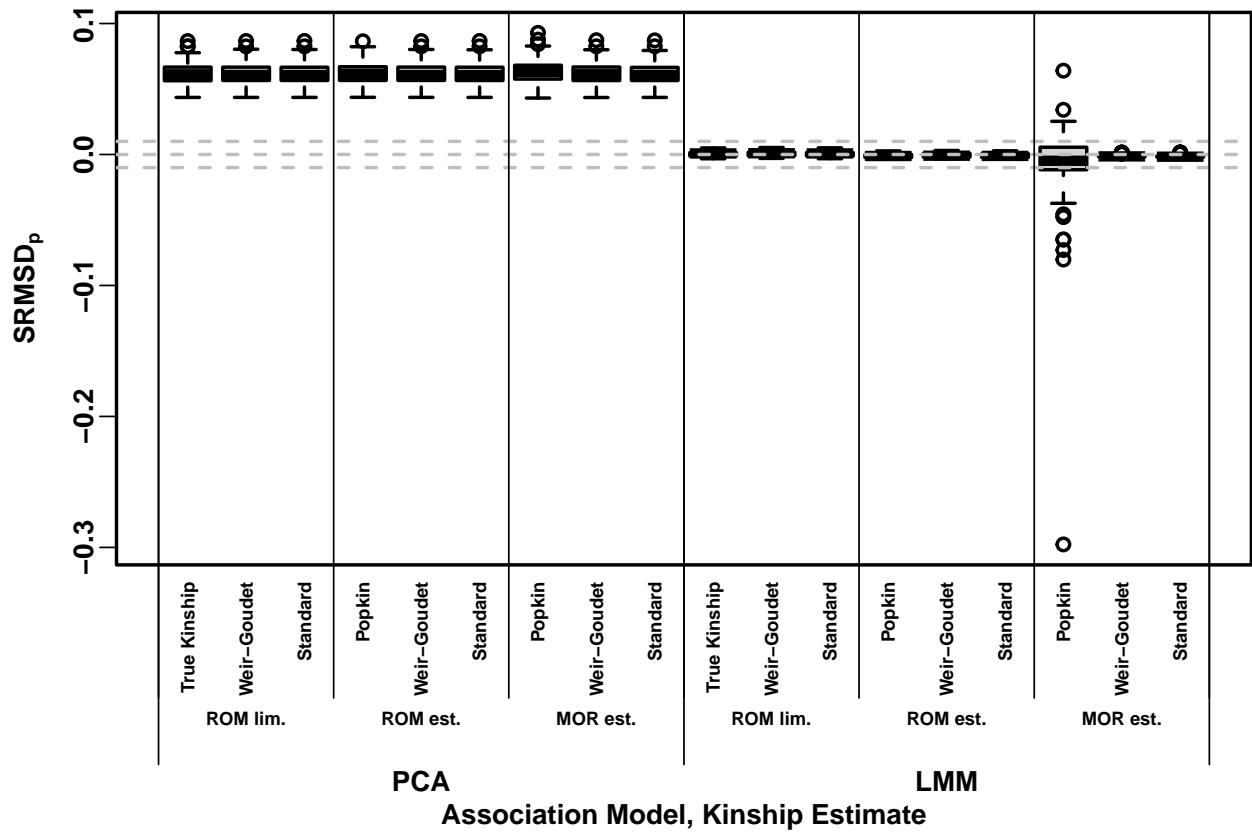


Figure S2: **Signed Root Mean Square Deviation of null p-values (SRMSD_p) on the admixed family simulation.** Same methods and simulation as Fig. 2, see that for more information. $|\text{SRMSD}_p| < 0.01$ (area between gray dashed lines) is considered calibrated. All PCA runs are miscalibrated by similar amounts, whereas most LMM runs are calibrated with few exceptions.

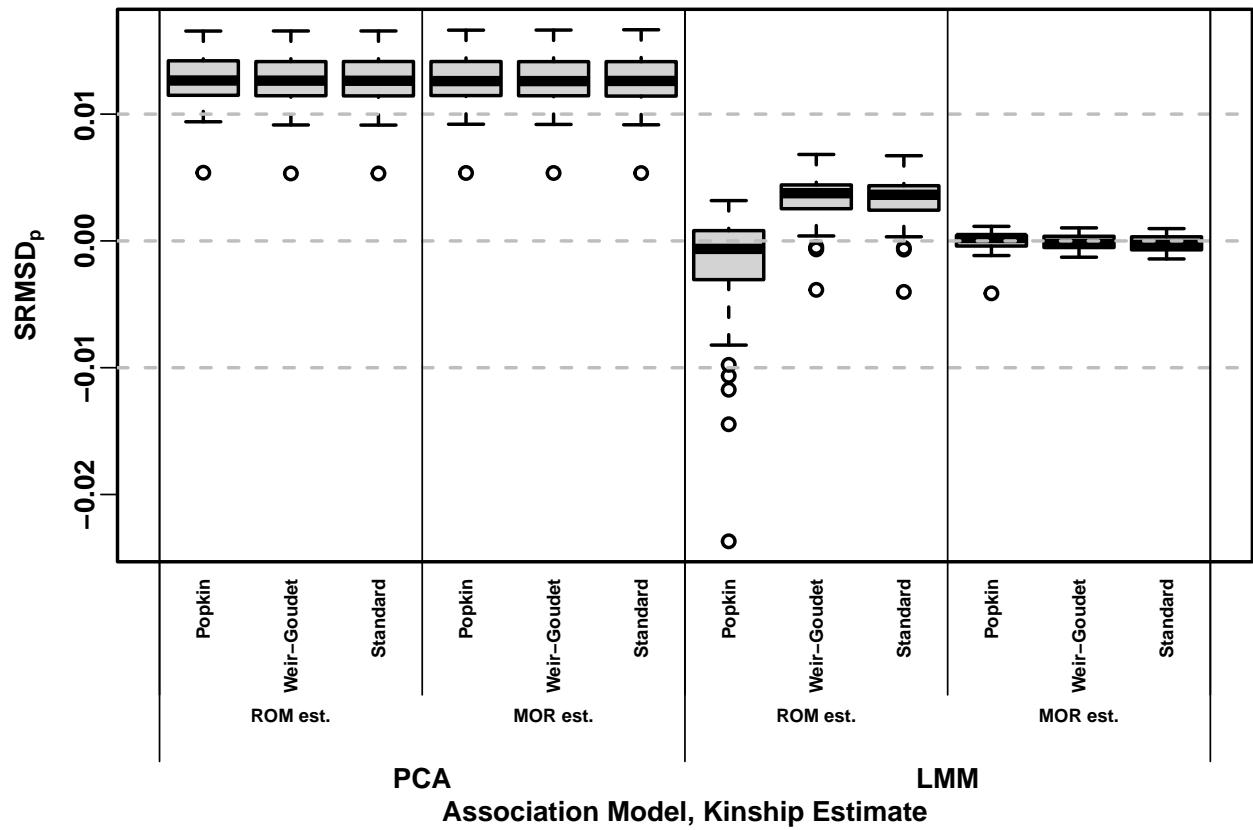


Figure S3: **Signed Root Mean Square Deviation of null p-values (SRMSD_p) on 1000 Genomes.** Same methods and simulation as Fig. 5, and y-axis statistic and conclusions of Fig. S2, see those for more information.

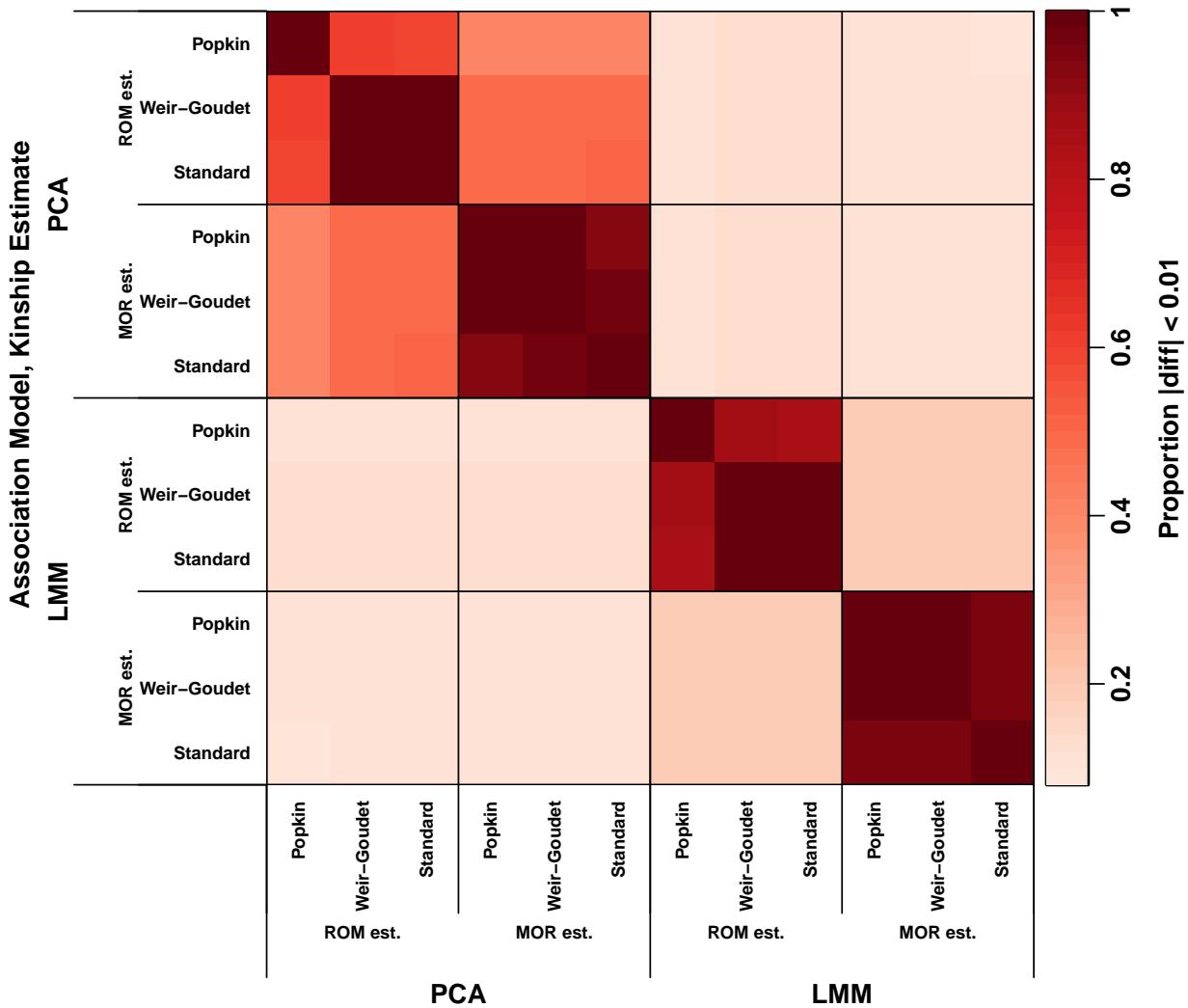


Figure S4: Approximate agreement between p-values on 1000 Genomes. See Fig. 3 for more details.

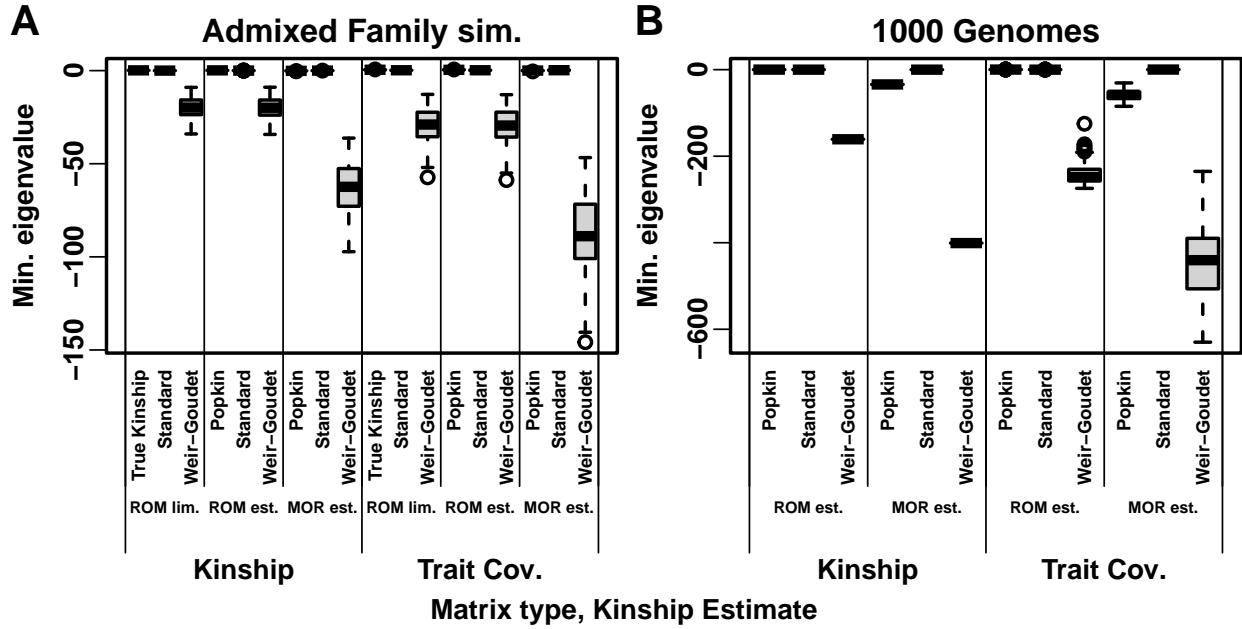


Figure S5: Minimum eigenvalue of kinship and trait covariance (\mathbf{V}) matrices. Each distribution is over 100 replicates (1000 Genomes kinship has one value since genotypes are fixed, but \mathbf{V} varies per replicate). All WG matrices has very large negative eigenvalues, and Popkin MOR has negative eigenvalues as well; in these cases \mathbf{V} always has negative eigenvalues too.

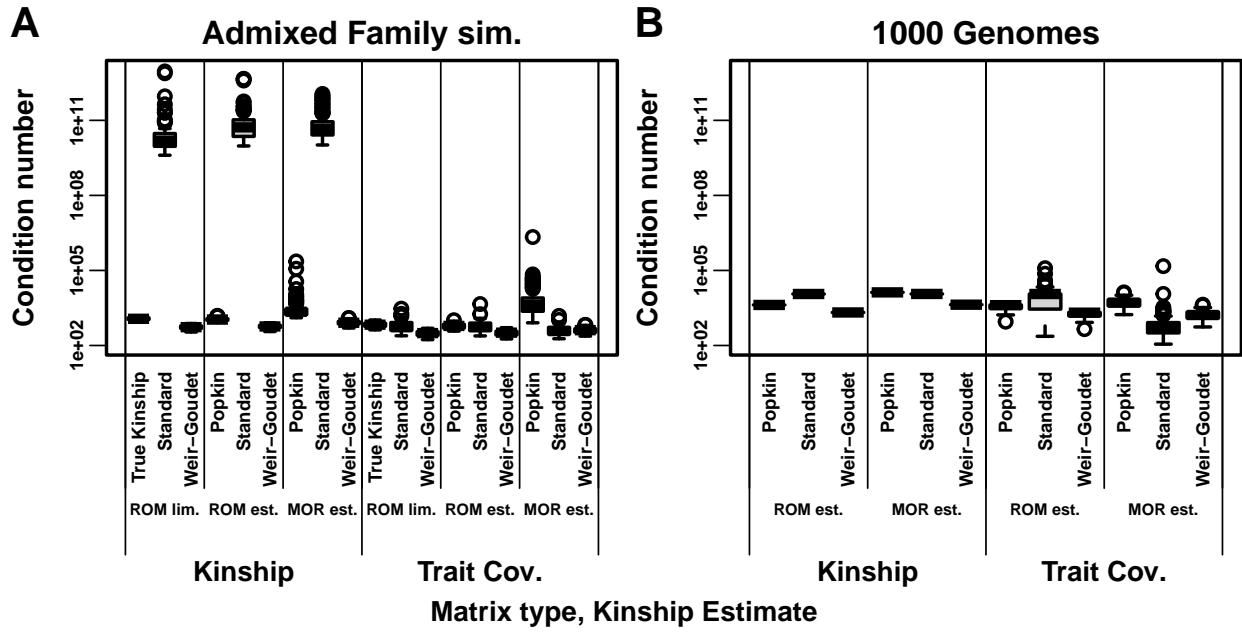


Figure S6: Condition numbers of kinship and trait covariance (\mathbf{V}) matrices. Larger condition numbers reflect ill-conditioned problems such as near singularity. Each distribution is over 100 replicates (1000 Genomes kinship has one value since genotypes are fixed, but \mathbf{V} varies per replicate).

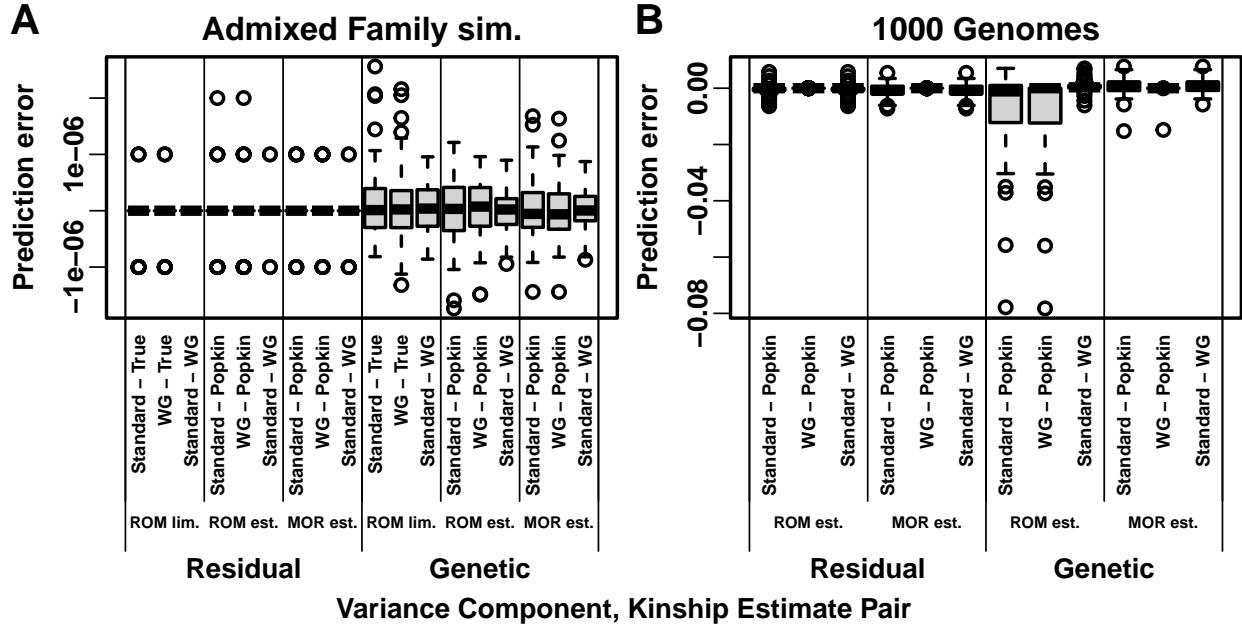


Figure S7: **Variance component prediction errors across evaluations.** Here we test the predictions that $\sigma_{\epsilon}^{2l} = \sigma_{\epsilon}^2$ and $\sigma_{\epsilon}^{2l} = c\sigma^2$ in Eq. (16). For Residual, prediction error (y-axis) is $\sigma_{\epsilon}^{2l} - \sigma_{\epsilon}^2$ between pairs of estimates as listed. For Genetic, prediction error is $\sigma_{\epsilon}^{2l} - c\sigma^2$: The biased-unbiased pairs use $c = 1 - \bar{\varphi}^T$ for Standard, $c = 1 - \tilde{\varphi}^T$ for WG, σ_{ϵ}^{2l} is their estimate and σ^2 is True or Popkin; The Standard-WG pair uses σ^2 for WG and $c = (1 - \bar{\varphi}^T) / (1 - \tilde{\varphi}^T)$. Each distribution is over the 100 replicates of each simulation. **A.** In admixed family simulation, all errors are zero within machine precision. Excess perfect zero residual prediction errors are due to limited precision of GCTA outputs. **B.** In 1000 Genomes, popkin ROM estimates has large errors compared to Standard and WG.

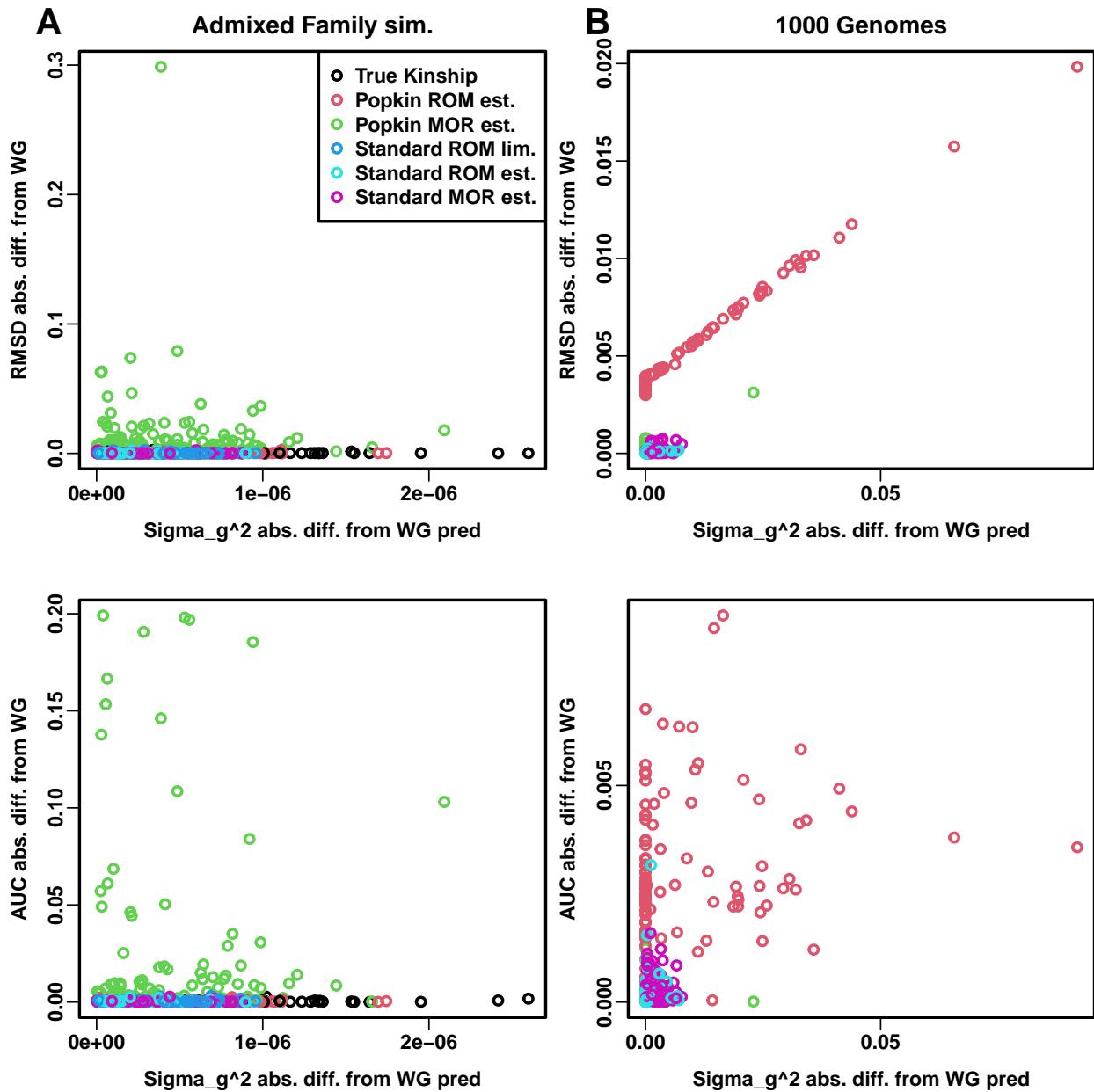


Figure S8: **AUC_{PR}** and SRMSD_p prediction errors explained by variance component errors. Genetic variance component (σ^2) absolute error is calculated with the formulas in Fig. S7 using WG as reference since its \mathbf{V} had the lowest condition numbers (Fig. S6). AUC_{PR} and SRMSD_p are expected to be the same between WG, Standard, and True or Popkin (within each locus weight type). **A.** Large errors in the admixed family simulation are not explained by high σ^2 error. **B.** Smaller popkin ROM prediction errors in 1000 Genomes are explained by high σ^2 error.

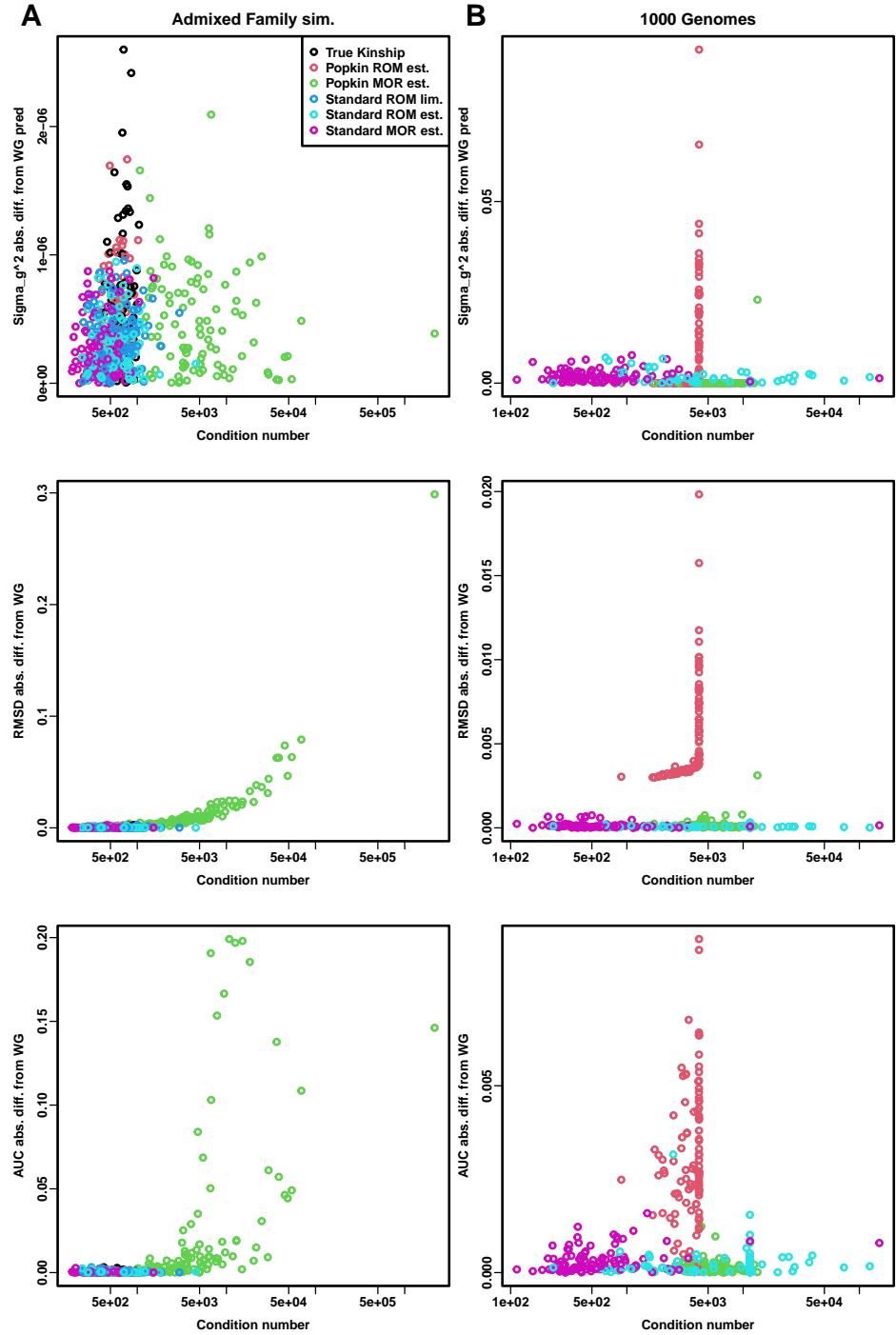


Figure S9: **AUC_{PR}** and **SRMSD_p** prediction errors explained by the condition number of \mathbf{V} . AUC_{PR} and SRMSD_p are expected to be the same between WG, Standard, and True or Popkin (within each locus weight type). WG was used as reference since its \mathbf{V} had the lowest condition numbers (Fig. S6). **A.** The large popkin MOR prediction errors (AUC_{PR}, SRMSD_p, but not σ^2) in the admixed family simulation are explained by the condition number of \mathbf{V} . **B.** Smaller errors in 1000 Genomes are not explained by the condition number of \mathbf{V} .