Finite-Volume Methods for Hyperbolic Problems

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Introduction

Problem

To solve a given hyperbolic partial differential equation (PDE) numerically.

Thus far, we have focused primarily on parabolic and elliptic PDE's. However, we want to extend further to hyperbolic PDE's to study other physical phenomenon.

Purpose

Typical courses go through Finite Difference Methods (FDM) in a rigorous fashion, however the idea of Finite Volume Methods (FVM) is only ever briefly discussed. I wish to understand the ideas and concepts for this method.

This requires careful attention. Let us first go through some preliminary introductions.

Linear Hyperbolic Problems

Linear Hyperbolic Equation

$$q_t + aq_x = 0, \tag{1}$$

where a is the wave speed.

Linear Hyperbolic Systems

Using eigenvalue decomposition and some change of variables, we can rephrase this system into N-1D decouples problems.

$$q_t + Aq_x = 0, \tag{2}$$

where $A \in \mathbb{R}^m \times \mathbb{R}^m$ and diagonalizable with real eigenvalues.

Classical Problem

Riemann Problem

Hyperbolic equation with initial data that is piecewise constant with a single jump discontinuity.

$$\hat{q}(x) = \begin{cases} q_l & \text{if } x < 0, \\ q_r & \text{if } x > 0. \end{cases} \tag{3}$$

Finite Volume Methods

General Form

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}\left(F_{i+1/2}^n - F_{i-1/2}^n\right)$$

where Q is the discrete solution and F is some approximation to the average flux:

$$F_{i-1/2}^n \approx \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} f(q(x_{i-1/2}, t))dt.$$

When to use?

- Dealing with integral form of conservation laws.
- Excellent for hyperbolic PDE's.
- One of the standard approaches in Computational Fluid Dynamics (CFD)

Solving Linear Systems

Gudonov's Method

- Reconstruct a piecewise polynomial function defined for all x from the cell averges Q_i^n.
- evolve the hyperbolic equation exactly with this initial data to obtain q a time Δt later.
- Average this function over each grid cell to obtain new cell averages.

This was first applied to solving the nonlinear Euler equations of gas dynamics.

Convergence

Analysis of Methods

Consistency and stability theory is identical to that of finite difference methods.

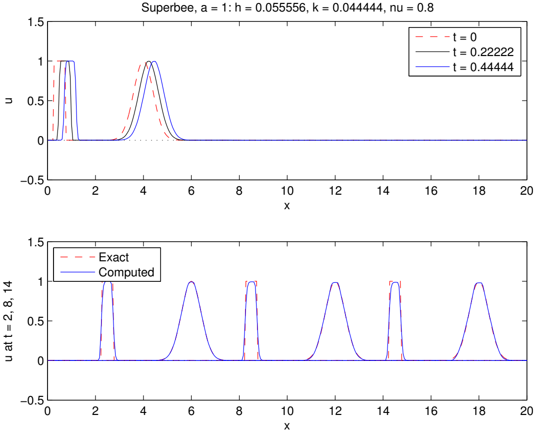
- Stability for linear constant coefficient equations is accomplished via Von Neumann stability analysis.
- Consistency follows from Taylor Series on the finite volume scheme.

Convergence continued

Dealing with discontinuities

- Taylor Series is based on smooth solutions which are no longer valid.
- There is theory for these situations, however it is not as concrete as the theory for smooth solutions.
- Looking at modified equations helps gain intuition on the expected behaviour of the method.
- i.e. Take more terms in the Taylor series expansion.

1D numerical results



(a)

Figure: Linear advection with Superbee and Lax-Wendroff finite

1D numerical results continued

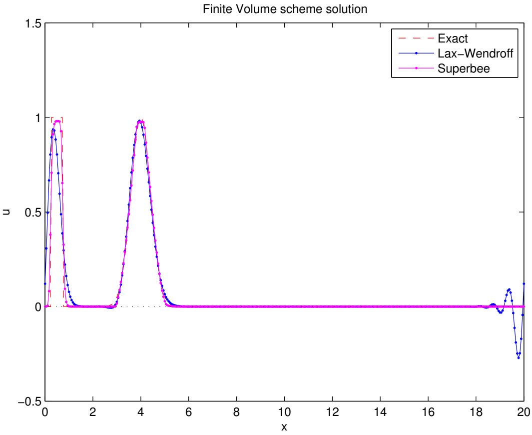


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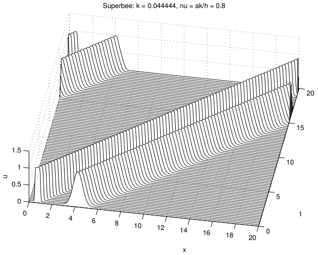
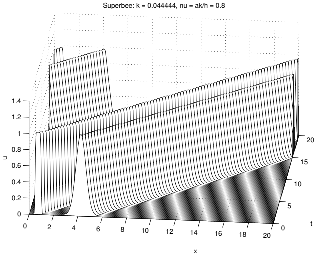
Figure: Linear advection with Superbee and Lax-Wendroff finite

1D numerical results continued



(a) Waterfall of solution



(b) Waterfall of solution

Figure: Linear advection with Superbee waterfall view.

Two Strings Problem: string with different density connected at red dot

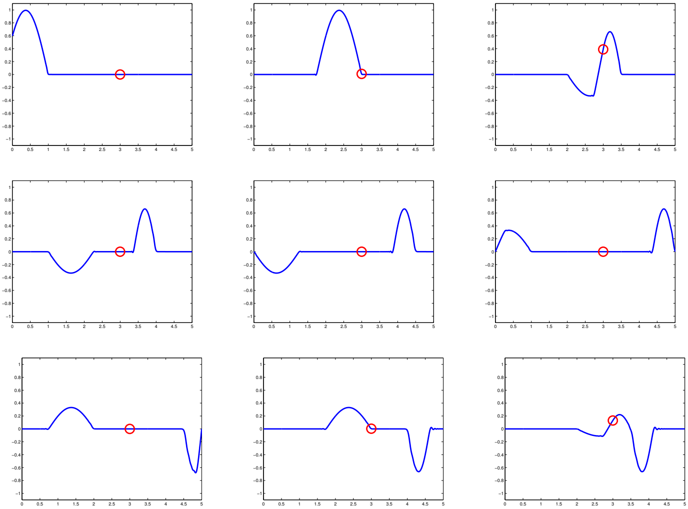


Figure: Time frames of the wave propagating through the string.

Conclusion

Finite Volume Methods are excellent for obtaining numerical solutions to hyperbolic problems.
They have similar characteristics to the Finite Difference Method.
They are based on integral formulations and conservation laws.

References

LeVeque, R.J. Finite Volume Methods for Hyperbolic Problems., 2002

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