Simon Fraser University

APMA 922: Numerical Solution of PDEs

Finite Volume Methods for Hyperbolic Problems

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1 Introduction

The elementary hyperbolic equation (also known as the linear advection equation) is of the form:

$$q_t(x,t) + cq_x(x,t) = 0, \tag{1}$$

where $q : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, and $c \in \mathbb{R}$. Often times $q(x,t)$ represents the concentration or density and c is the velocity of the fluid. In general, for a first-order hyperbolic system c is now a matrix say, A, where A is diagonalizable with real eigenvalues. This leads to distinct waves.
Conservation laws are of great importance since their structure is in the form of a hyperbolic system,

$$q_t(x,t) + f(q(x,t))_x = 0, \tag{2}$$

where $f(q)$ is the flux function.
Rather than working with the derivatives explicitly, an integral approach is taken:

$$\frac{d}{dt}\int_{x_1}^{x_2} q(x,t)dx = f(q(x_1,t)) - f(q(x_2,t)). \tag{3}$$

This creates a framework for problems with discontinuities in either the initial condition or solution. Attention is given to these conditions since nonlinear conservation laws can have solutions in the form of shock waves even when given smooth initial data. Finite volume methods (FVM) approximate the integral form by solving the domain in grid cells whereas finite difference methods approximate derivatives which may not be continuous for hyperbolic problems. The method conserves quantities in the grid cells which mimics the integral conservation laws. Hence, studying these methods will be the fundamental approach to solving these problems numerically.
Similar to finite difference methods (FDM), in order to determine whether a FVM is converging, we investigate the consistency and stability of the scheme. There is some discussion in the latter chapters discussing how to deal with discontinuous solutions.
There are some standard FVMs that can be derived via FDMs, however there are many more classes of schemes. Each of the schemes pose properties that help deal with difficulties such as discontinuities or potential oscillations in more standard schemes.
A comparison between FDMs and FVMs is given to demonstrate that for hyperbolic PDEs, FVMs obtain much more accurate results based on their original formulation (3). This project will highlight the main topics seen in LeVeque [3] in order to give some insight on this powerful research area.

2 Linear Equations

2.1 Advection Equation

The advection equation in one-dimension is given by

$$q_t + \bar{u}q_x = 0.$$

The name derives from the idea that this equation models the advection of a tracer along with the fluid. The solution to this problem is in the form

$$q(x,t) = \tilde{q}(x - \bar{u}t),$$

for some $\tilde{q}$ and $\bar{u}$ is the wave speed. The solution is derived via the method of characteristics, which results in the solution being constant along these characteristics.

2.2 Variable Coefficients

If we change the flux by letting the fluid velocity $\bar{u}$ depend on x, then the characteristic curves are solutions to the ordinary differential equation (ODE)

$$X'(t) = u(X(t)).$$

This leads to curves that are no longer straight lines however the problem can be reduced to solving sets of ODEs:

$$\begin{aligned} \frac{d}{dt}q(X(t),t) &= q_t(X(t),t) + X'(t)q_x(X(t),t) \\ &= q_t + u(X(t))q_x \\ &= q_t + (u(X(t))q)_x - u'(X(t))q \\ &= -u'(X(t))q(X(t),t). \end{aligned}$$

2.3 Linear System of Hyperbolic Equations

A linear system of the form

$$q_t + Aq_x = 0$$

is called hyperbolic if the $m \times m$ matrix A is diagonalizable with real eigenvalues, λ^p for $p = 1 \cdots m$ (the eigenvectors along the columns of the matrix is denoted by R). Hence we have that $A = R\Lambda R^{-1}$ and so the PDE becomes

$$R^{-1}q_t + R^{-1}ARR^{-1}q_x = 0.$$

With a change of variables, $w(x,t) \equiv R^{-1}q(x,t)$, we then have

$$w_t + \Lambda w_x = 0.$$

Notice that Λ decouples the system into m independent advection equations. Hence, the system is nothing more than solving a set of one-dimensional problems. Note this is only for a constant matrix A. If the matrix is not constant, then the analysis is not as straight forward [3].

3 Two-string and Riemann Problem

3.1 Two-string Problem

The two-string problem [6] involves solving the wave equation with variable sound speed,

$$u_{tt} = c^2(x)u_{xx}.$$

Intuitively, the variable sound speed that we investigate models a wave traveling through a piece of string that has different density connected at the red dot in Fig. 1. The variable density corresponds to the changing wave speed. For the numerical simulations, dirichlet data is given which corresponds to a reflection at the boundaries. We also see a reflection at the change of density (red dot). The function c is defined as a piecewise constant function. This problem can be studied in order to mimic the effects of acoustics in a closed tube of gas. The Lax-Wendroff FVM is applied to solve this PDE, where details of this method are discussed in the next section 4.

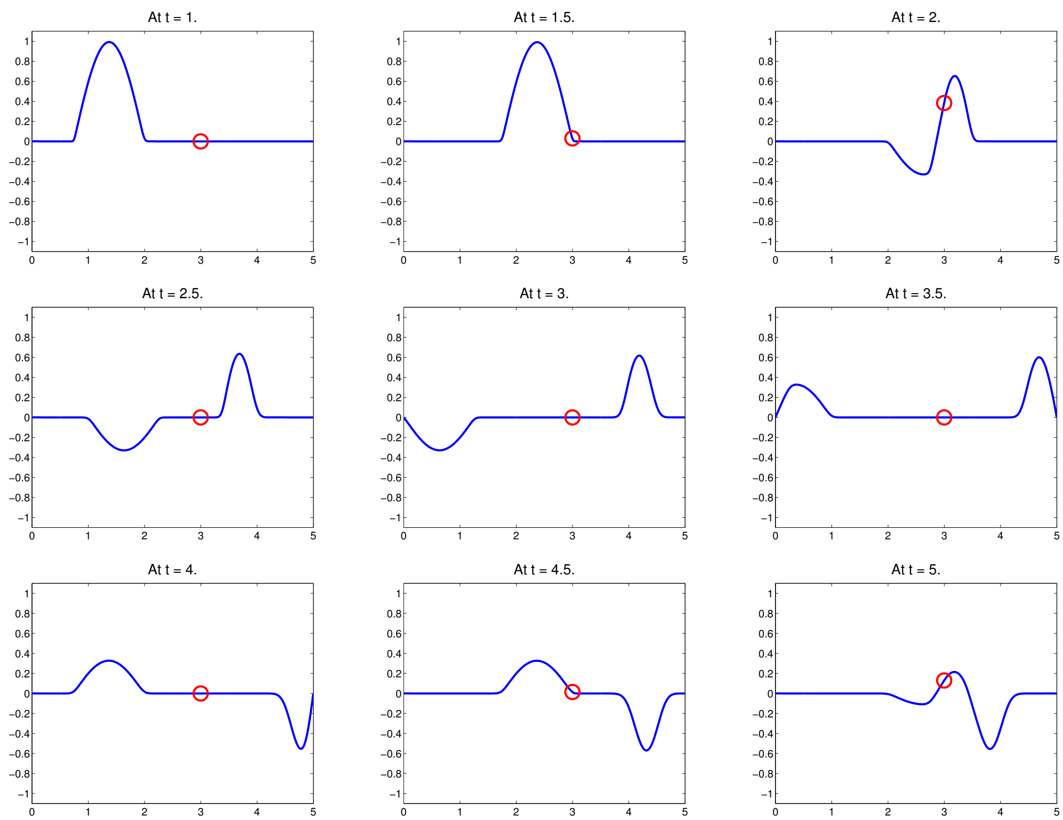

Figure 1: Time frames of the wave propagating through the string with variable density.

3.2 The Riemann Problem

The Riemann problem consists of some hyperbolic PDE with piecewise constant data with a single jump discontinuity,

$$\hat{q}(x) = \begin{cases} q_l & \text{if } x < 0, \\ q_r & \text{if } x > 0. \end{cases} \tag{4}$$

For linear systems, this involves solving a Riemann problem for each decoupled one-dimensional equation. The only caveat is that now the initial condition comes from the grid cells before or after (whether the wave is positive or negative). In order for these to match up, we impose the Rankine-Hugoniot condition which is well known in the theory of hyperbolic PDEs. It is seen from integral conservation laws along discontinuities. It is defined as,

$$s(q_r - q_l) \quad = \quad f(q_r) - f(q_l).$$

Where s is the speed of the shock, f is the flux of the PDE, l and r subscripts correspond to left and right values respectively. Sometimes written as $s[[q]] = [[f]]$ where $[[\cdot]]$ represents the jump across the shock. The Riemann problem is an excellent way to see how well the FVM behaves under some difficult (although simple) conditions.

4 Finite Volume Methods

4.1 Formulation

Finite volumes are also called *grid cells*. Denote the ith grid cell by $\mathcal{C}_i = (x_{i-1/2}, x_{i+1/2})$. The value Q_i^n will approximate the average value over the ith interval at time t_n:

$$Q_i^n \approx \frac{1}{\Delta x}\int_{x_{i-1/2}}^{x_{i+1/2}} q(x,t_n)dx \equiv \frac{1}{\Delta x}\int_{\mathcal{C}_i} q(x,t_n)dx,$$

where $\Delta x = x_{i+1/2} - x_{i-1/2}$ is the length of the cell. Now looking at the integral form of the conservation law and integrating from t_n to t_{n+1} where $\Delta t = t_{n+1} - t_n$:

$$\begin{aligned}
\frac{d}{dt}\int_{\mathcal{C}_i} q(x,t)dx \quad &= \quad f(q(x_{i-1/2},t)) - f(q(x_{i+1/2},t)) \\
\implies \int_{\mathcal{C}_i} q(x,t_{n+1})dx - \int_{\mathcal{C}_i} q(x,t_n)dx \quad &= \quad \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))dt - \int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))dt \\
\implies \frac{1}{\Delta x}\int_{\mathcal{C}_i} q(x,t_{n+1})dx \quad &= \quad \frac{1}{\Delta x}\int_{\mathcal{C}_i} q(x,t_n)dx \\
&\quad - \quad \frac{1}{\Delta x}\left[\int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))dt - \int_{t_n}^{t_{n+1}} f(q(x_{i-1/2},t))dt\right]
\end{aligned}$$

This suggests we study numerical methods of the form:

$$Q_i^{n+1} = Q_i^n - \frac{\Delta t}{\Delta x}(F_{i+1/2}^n - F_{i-1/2}^n),$$

where $F_{i+1/2}^n = \int_{t_n}^{t_{n+1}} f(q(x_{i+1/2},t))dt$. Note that this integral is not computed explicitly but an assumption is made that $q(x_{i+1/2},t)$ is constant in time along each cell interface. This gives birth to the name"grid cells", in finite volume methods.

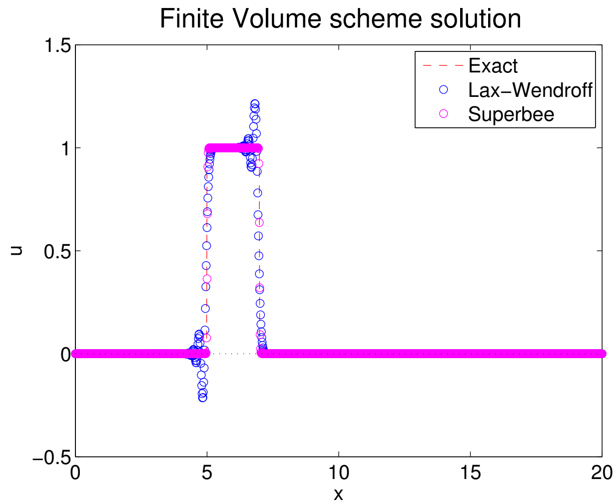

Figure 2: Oscillation forming near discontinuity for L-W but not for Superbee (FVM).

4.2 Standard Methods

Standard finite volume methods can be derived via finite difference techniques. These consist of the Upwind, Lax-Friedrichs and Lax-Wendroff methods [4]. The issue with these techniques begins to develop in more realistic problems involving Riemann data. When we apply Lax-Wendroff with rectangle data, we can see in Fig. 2 that oscillations form near the discontinuities whereas the Superbee finite volume method does an excellent job at capturing the exact solution.

4.3 Godunov's Method/ REA Algorithm

One of the standard approaches to hyperbolic problems is given by Godunov's Method. He first implemented this method for Euler's nonlinear equations. The method can be described in this three step process:

- Reconstruct a piecewise polynomial function defined for all x from the cell averges Q_i^n.
- Evolve the hyperbolic equation exactly with this initial data to obtain q a time Δt later.
- Average this function over each grid cell to obtain new cell averages.

A visual interpretation is seen in Fig. 3 from LeVeque [3]. This specific formulation of Godunov's method also has the acronym REA algorithm. The main process described above is termed Godunov's method, and certain components of the algorithm have been improved since it was first invented.

4.4 Roe's Method

In regards to a constant-coefficient linear problem, we can rewrite the flux term $F_{i-1/2}^n$ as

$$\begin{aligned} F_{i-1/2}^n &= \frac{1}{2}\left(f(Q_{i-1}^n) + f(Q_i^n)\right) - \frac{1}{2}|A|(Q_i^n - Q_{i-1}^n), \\ \text{where} & \\ |A| &= R|\Lambda|R^{-1}, \\ |\Lambda| &= \text{diag}(|\lambda^p|). \end{aligned}$$

This form demonstrates the averaging technique with a correction term to stabilize the method. This formulation is of great interest since it is a rewritten Godunov method and is often seen in nonlinear problems with extensions.

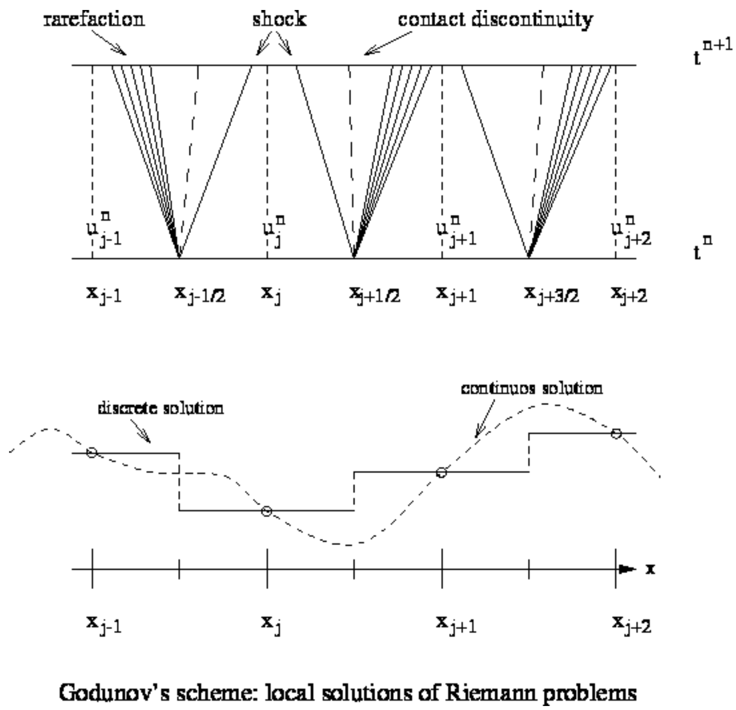

Figure 3: Godunov Implementation

5 High-Resolution Methods

5.1 TVD

Total Variation Diminishing (TVD) methods place a constraint on the slope of the solution to avoid oscillations in the solution. These methods are a remedy to the oscillations as seen in the Lax-Wendroff method for Heaviside initial data for the linear advection equation, for example.

Definition 1. *A two-level method is called total variation diminishing (TVD) if, for any set of data A^n, the values Q^{n+1} computed by the method satisfy*

$$TV(Q^{n+1}) \leq TV(Q^n).$$

For a grid function Q we define $TV(Q) = \sum_{i=-\infty}^{\infty} |Q_i - Q_{i-1}|$. There are other definitions for the total variation in LeVeque [3].

In order to truly understand the strength of this method, we will look at some applications and numerical experiments.

5.1.1 Burgers equation

A numerical simulation showing the TVD method applied to Burgers equation with periodic boundary conditions is implemented. We use the Superbee flux limiter with CFL 0.3 and neumann boundary conditions. See [1] (TVD-Burgers1Da.gif).

5.1.2 2D scalar advection equation

We solve the 2D scalar advection equation using the TVD algorithm. We use the van Leer flux limiter with CFL 0.4 and neumann boundary conditions. See [1] (TVD-2D.gif).

5.2 Flux Limiters

Flux-Limiter methods have the general form:

$$\begin{aligned}
Q_i^{n+1} &= Q_i^n - v(Q_i^n - Q_{i-1}^n) \\
&- \frac{1}{2}v(1-v)\left[\phi(\theta_{i+1/2}^n)(Q_{i+1}^n - Q_i^n) - \phi(\theta_{i-1/2}^n)(Q_i^n - Q_{i-1}^n)\right] \\
\text{if} &\quad \bar{u} > 0 \text{ or} \\
Q_i^{n+1} &= Q_i^n - v(Q_{i+1}^n - Q_i^n) \\
&- \frac{1}{2}v(1+v)\left[\phi(\theta_{i+1/2}^n)(Q_{i+1}^n - Q_i^n) - \phi(\theta_{i-1/2}^n)(Q_i^n - Q_{i-1}^n)\right] \\
\text{if} &\quad \bar{u} < 0.
\end{aligned}$$

where

$$\begin{aligned}
\theta_{i-1/2}^n &= \frac{\Delta Q_{I-1/2}^n}{\Delta Q_{i-1/2}^n} \\
I &= \begin{cases} i-1 \text{ if } \bar{u} > 0, \\ i+1 \text{ if } \bar{u} < 0. \end{cases}
\end{aligned}$$

Now we can redefine some old methods using this new formulation along with creating some new methods as follows.
Linear Methods:

$$\begin{aligned}
\text{upwind :} &\qquad \phi(\theta) = 0, \\
\text{Law-Wendroff :} &\qquad \phi(\theta) = 1, \\
\text{Beam-Warming :} &\qquad \phi(\theta) = \theta, \\
\text{Fromm :} &\qquad \phi(\theta) = \frac{1}{2}(1+\theta),
\end{aligned}$$

High-resolution limiters:

$$\begin{aligned}
\text{minmod :} &\qquad \phi(\theta) = minmod(1, \theta), \\
\text{superbee :} &\qquad \phi(\theta) = \max(0, \min(1, 2\theta), \min(2, \theta)), \\
\text{MC :} &\qquad \phi(\theta) = \max(0, \min(1, \theta)/2, 2, 2\theta)), \\
\text{van Leer :} &\qquad \phi(\theta) = \frac{\theta + |\theta|}{1 + |\theta|},
\end{aligned}$$

An excellent advantage of this TVD formulation is that we can rewrite standard FVMs and define new ones that are better suited for riemann problems per se. As well the code is reusable with very little modification required.

6 Convergence

Standard techniques for studying convergence of a finite volume method is identical to the methods proposed for finite differences [4]. Consistency is studied through Taylor Series expansions of the discrete points to determine the local truncation error. For linear constant coefficient problems, Von Neumann analysis is the ideal candidate. Together from the Lax-Richtmyer (or Lax equivalence) Theorem ties the two concepts together to guarantee convergence.
As previously mentioned, hyperbolic problems can start or lead to discontinuous data and we ask ourselves, how do we study convergence in these situations? Firstly, we note that the choice of norm is an

important one. The max norm in hyperbolic problems is no longer a good indication since discontinuous can lead to large errors. However, the one norm is very useful for conservation purposes. Moreover, the analysis using the 1-norm is feasible.
In regards to consistency, by taking more terms in the Taylor Series expansion, for the most part, we can see how the solution will behave for long times due to dispersive, diffusive terms, or etc. This leads to the modified equations, which simply refers to the problem that the numerical method is more closely related too (normally in part of extra terms in the Taylor Series). Rather than study the convergence where discontinuities exist, we look at problems with smooth solutions.

6.1 Stability

As mentioned, the stability analysis in the 2-norm is identicaly for FDMs since both involve Von Neumann analysis. Moreover, the 1-norm is easier to compute analytically, giving drive to the investigation. For the Upwind method,

$$\begin{aligned}
\|Q^{n+1}\|_1 &= \Delta x \sum_i |Q_i^{n+1}| \\
&= \Delta x \sum_i |(1-v)Q_i^n + vQ_{i-1}^n| \\
&\leq \Delta x \sum_i | \big((1-v)|Q_i^n| + v|Q_{i-1}^n|\big) \\
\implies &\leq (1-v)\|Q^n\|_1 + v\|Q^n\|_1 \\
& \quad (\text{ if } 0 \leq v \leq 1) \\
&= \|Q^n\|_1.
\end{aligned}$$

Hence we have stability in the 1-norm.

6.2 Consistency

In order to determine the local truncation error of a FVM, we investigate the Taylor series of a scheme. This is a straight forward process and so we simply state some results for select FVMs.

$$\begin{aligned}
\text{The Upwind method: } \tau^n &= \frac{1}{2}\bar{u}\Delta x(1-v)q_{xx}(x_i,t_n) + \mathcal{O}(\Delta t^2) \\
\text{Lax-Friedrichs: } \tau^n &= \frac{1}{2}\bar{u}\Delta x(1/v-v)q_{xx}(x_i,t_n) + \mathcal{O}(\Delta t^2) \\
\text{Lax-Wendroff: } \tau^n &= -\frac{1}{6}\bar{u}(\Delta x)^2(1-v^2)q_{xxx}(x_i,t_n) + \mathcal{O}(\Delta t^3).
\end{aligned}$$

6.3 Modified Equations

Modified equations are derived from taking extra terms in the Taylor expansion to determine what equation the scheme truly approximates at higher order. This process mimics the manner in which we determine consistency, and so we simply state the results.
The Upwind method has modified equation:

$$q_t + \bar{u}q_x = \frac{1}{2}\bar{u}\Delta x(1-v)q_{xx}.$$

The Lax-Wendroff method has modified equation:

$$q_t + \bar{u}q_x = -\frac{1}{6}\bar{u}(\Delta x)^2(1-v^2)q_{xxx}.$$

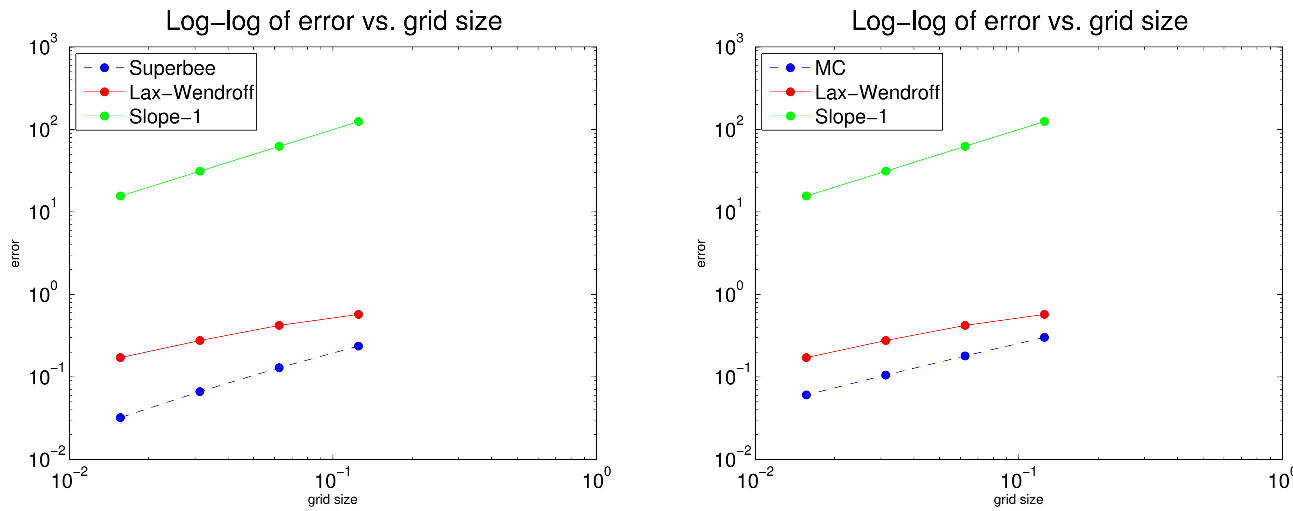

Figure 4: Error plots for the linear advection equation with Heaviside data.

The Beam-Warming method has modified equation:

$$q_t + \bar{u}q_x = \frac{1}{6}\bar{u}(\Delta x)^2(2 - 3v + v^2)q_{xxx}.$$

6.4 Numerics

Taking a closer look at some numerical results, we can see the order of accuracy on the simple linear advection PDE. We are looking to both identify numerically both old and new results.
As mentioned earlier, problems arise when we look at discontinuous solutions. From Fig. 4 we can see that the previous Superbee method which is normally of order 2, loses an order of accuracy when we include the discontinuous data. Similarly the same occurs for the MC flux limiter method. Figure 4 also demonstrates the order of the finite difference Lax-Wendroff method. Although this appears to illustrate a flaw in the finite volume methods, since both the FVM and FDM have the same accuracy we must recall some key components.
The Lax-Wendroff method is also a FVM in itself, so this is not so surprising that the two methods have similar behaviours. Moreover, although we lose accuracy, the solution for high-resolution limiters or TVD implementations give very accurate numerical solutions that avoid oscillations that occur in standard FDMs. Under these conditions, in order to obtain some accurate results via FDMs would require a very fine mesh size. However even then it would not completely account for the discontinuity. Using an adaptive mesh is another possible approach to solving this issue.
Lastly, it is important to realize that although we wish to have a high order of accuracy and very small error, when dealing with discontinuities, there has to be a compromise. Specifically, we can approximate the true solution closely, but unless we know the solution (which is usually not the case for more realistic nonlinear problems) then are expectations are not as ambitious.

7 Comparison

Although this report is foregoing the path of finite volume methods, there are other possible methods for solving hyperbolic problems. The two which will be highlighted are finite differences and spectral methods.

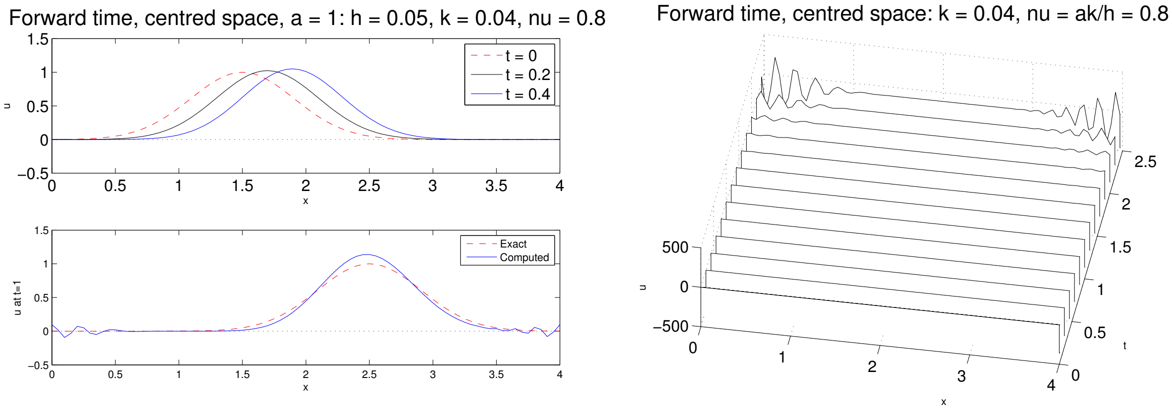

Figure 5: Advection equation using forward time and center space.

7.1 Linear Advection

For the problem of linear advection 1, we complete a comparison by highlighting both the proficiency and deficiency of each method. We take the Lax-Wendroff scheme as the candidate FDM. Taking a centered difference scheme for the linear advection equation leads to instabilities in the solution in Fig. 5. This phenomenon arises from the fact that the hyperbolic PDE depends on values from the "past" and so by using a centered difference formula, we are taking information that is not known. From Fig. 6 and Fig. 7 we see that both methods are order-2 and that the errors for the Superbee (FVM) method is smaller than the errors in the Lax-Wendroff method. This concludes that although some FDMs are equivalent to FVMs, it is only a subset of FDMs that are ideal candidates for hyperbolic PDEs.

8 Shallow Water Equations

An example of a multidimensional hyperbolic problem are the shallow water equations which are modelled by the following set of PDEs:

$$\begin{aligned} h_t + (hu)_x + (hv)_y &= 0 \\ (hu)_t + \left(hu^2 + \frac{1}{2}gh^2\right)_x + (huv)_y &= 0 \\ (hv)_t + (huv)_x + \left(hv^2 + \frac{1}{2}gh^2\right)_y &= 0. \end{aligned}$$

(u, v) is the velocity vector, h is the depth, and hu and hv are the momenta in two directions. The ideas for solving multidimensional problems are merely extensions of the one dimensional case. The code uses a first-order accurate Godunov-type finite volume method with Roe's approximate Riemann solver to estimate mass and momentum fluxes. See [1] for the numerical solution (shallowWater-2D.gif).

9 WENO

Weakly essentially non-oscillatory (WENO) methods are another apporach stemming from Finite Volume methods and hyperbolic problems. It uses a flux-splitting technique.

9.1 Numerical Results

WENO methods first proposed by Osher et al. [5] have led to a new class of numerical solvers for hyperbolic PDEs. There methods are also applicable to nonlinear PDEs such as Euler's equations, or

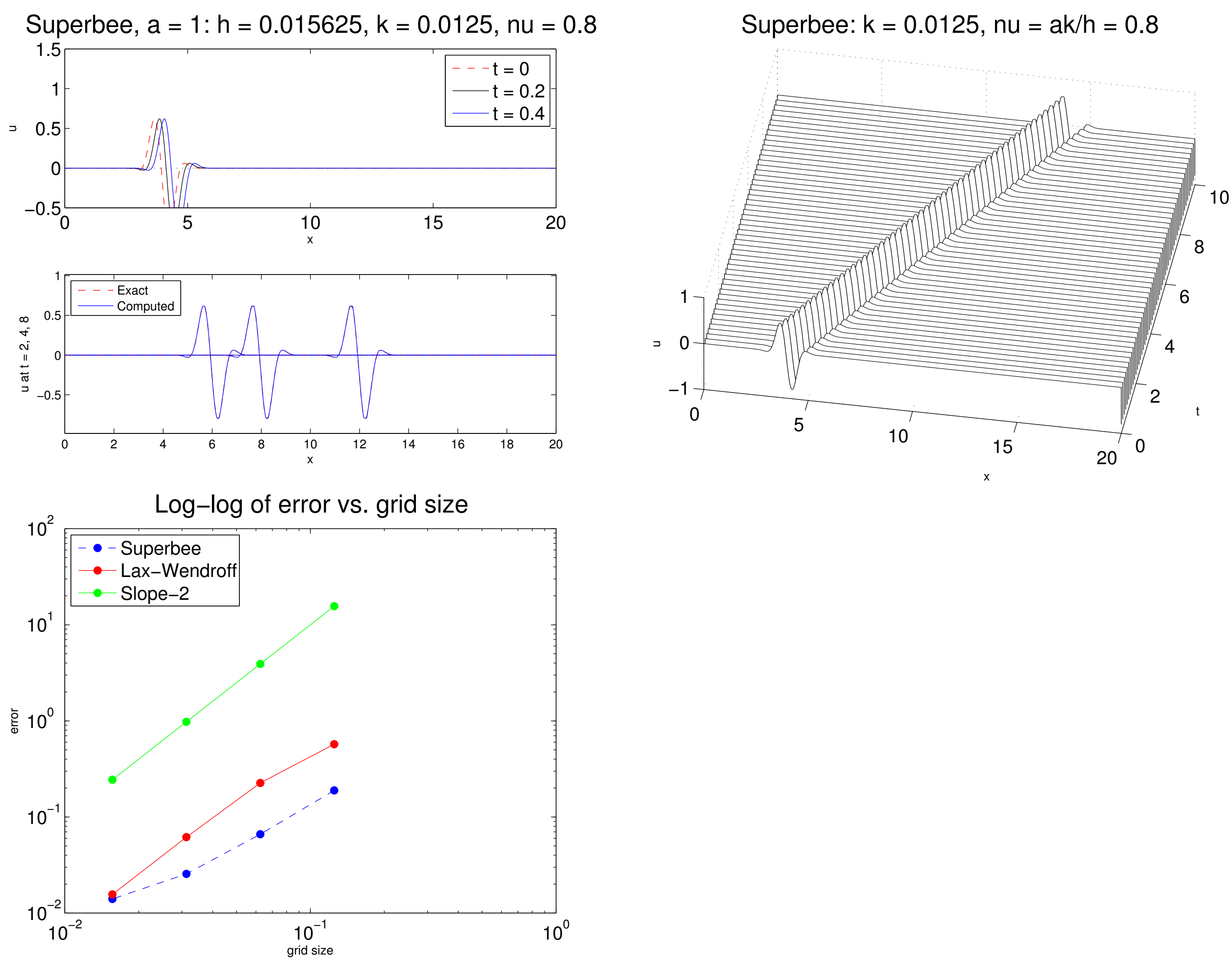

Figure 6: Advection equation using Superbee method and wavepacket data.

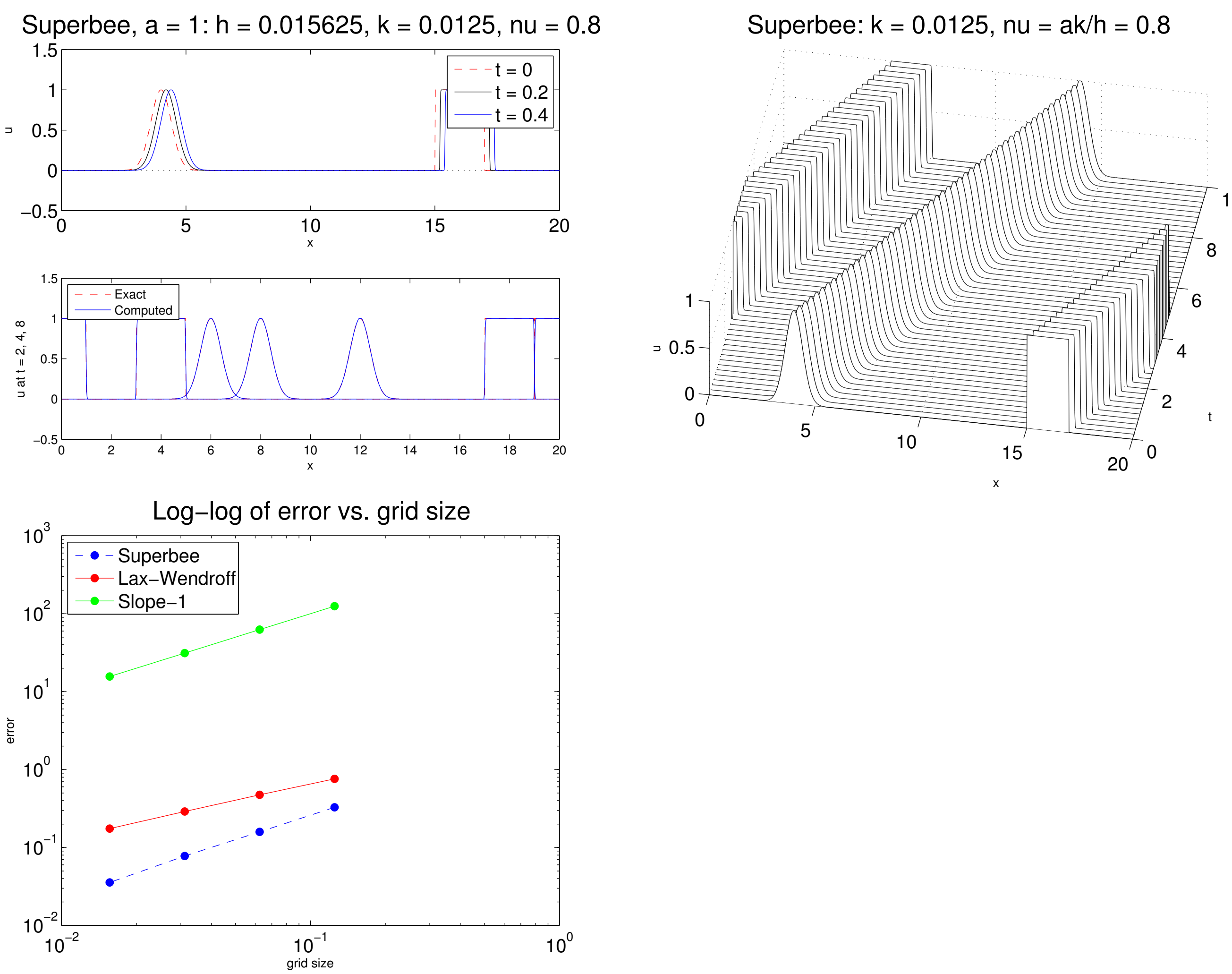

Figure 7: Advection equation using Superbee method with gaussian + rectangle data.

problems with shock waves which are of great importance. WENO methods are based on cell averages and a TVD Runge-Kutta time discretization. WENO methods have also been used for interpolation however this is not the focus of the report. A demonstration of the WENO method is shown through test problems.

9.1.1 Inviscid Burgers equation

Solving the equation $u_t + \left(\frac{u^2}{2}\right)_x = 0$ with gaussian and then Riemann data. Both of which have periodic boundary conditions. See [1] (Burgers-WENO3-1D-gaussA.gif, Burgers-WENO3-1D-riemannA.gif).

9.1.2 Scalar advection equation

The scalar advection is solved with periodic boundary conditions with gaussian and then Riemann data. See [1] (Advec-WENO3-1D-gaussA.gif, Advec-WENO3-1D-riemannA.gif).

9.1.3 Scalar advection in 2D

Now we look at the advection equation in two dimensions. Neumann boundary conditions are used with an order 3 WENO scheme. See [1] (WENO3-2D.gif).

10 Conclusion

The main concepts of the FVMs with applications in hyperbolic problems have been expressed in this project. Standard hyperbolic problems were investigated such as the linear advection equation. Both the theory and implementation of the FVMs were accomplished. Highlights of the accurate solutions were given and comparison to some similar FDMs such as the Lax-Wendroff scheme. A quick illustration of analytic stability was given in order to show the common techniques for FVMs. Otherwise, the analysis is equivalent to that of FDMs which are well studied in LeVeque [4]. Illustration of the TVD methods have shown to be powerful and accurate schemes for these hyperbolic PDEs. Lastly, some numerical results were given using an extension of the finite volume methods, i.e. WENO methods. A growing area with various applications, it is well worth mentioning (even just quickly) the WENO methods since they have been recently a commonly used technique. Unfortunately, this report could not go into more detail on the topic, however I hope this does encourage the reader to venture into the topic further.

The purpose of this report is to give a brief overview of the topic and be able to give a fellow scholar insight. The references given are the key sources to truly understanding the complete picture. I would like to thank Dr. LeVeque for writing an outstanding text on the subject matter and for his open source code [2] which was of great aid in understanding the implementation of the finite volume methods.

References

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