Metode iterative

Vrem să rezolvăm $Ax=b$

Pt metodele iterative, se vor găsi matricile N și P a.î.

$A=N-P$

Înlocuim în $Ax=b$:

$(N-P)\cdot x=b \Rightarrow Nx-Px=b \Rightarrow Nx=Px+b \quad /\cdot N^{-1}$

$\Rightarrow x=N^{-1}Px+N^{-1}b$

Pt metodele iterative, construim soluția bazându-ne pe pasul anterior:

$x^{(n+1)}=N^{-1}\cdot P\cdot x^{(n)}+N^{-1}\cdot b$, unde $x^{(n)}=x$ la pasul n

Vom partiționa matricea $A=D-L-U$.

ex:

$$A=\begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix}=\underbrace{\begin{bmatrix}1&0&0\\0&5&0\\0&0&9\end{bmatrix}}_{D}-\underbrace{\begin{bmatrix}0&0&0\\-4&0&0\\-7&-8&0\end{bmatrix}}_{L}-\underbrace{\begin{bmatrix}0&-2&-3\\0&0&-6\\0&0&0\end{bmatrix}}_{U}$$

Metoda Jacobi

$N = D$

$P = L + U$

$A = N - P = D - (L+U) = D - L - U$

$$x^{(n+1)} = D^{-1} \cdot (L+U)\, x^{n} + D^{-1} \cdot b \quad / \cdot D \ (1)$$

$$D \cdot x^{(n+1)} = (L+U) \cdot x^{n} + b$$

$\Downarrow$

$$\begin{bmatrix} a_{11} & & & 0 \\ & a_{22} & & \\ & & \ddots & \\ 0 & & & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} x_1^{(n+1)} \\ \vdots \\ \vdots \\ x_n^{(n+1)} \end{bmatrix} = \begin{bmatrix} 0 & -a_{12} & \dots & -a_{1n} \\ -a_{21} & 0 & & \vdots \\ \vdots & & \ddots & \\ -a_{n1} & \dots & \dots & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1^{(n)} \\ \vdots \\ \vdots \\ x_n^{(n)} \end{bmatrix} + \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

$$a_{ii} \cdot x_i^{(n+1)} = -a_{i1} \cdot x_1^{(n)} - a_{i2}\, x_2^{(n)} + \dots + 0\, a_{i\,i+1}\, x_{i+1}^{(n)} - \dots - a_{in} \cdot x_n^{(n)} + b_i$$

$\Downarrow$

$$x_i^{(n+1)} = \frac{-\sum_{k=1,\, k \neq i}^{n} a_{ik} \cdot x_k^{(n)} + b_i}{a_{ii}}$$

Ex Rezolvăm cu Jacobi

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Pornim cu o aproximație inițială a soluției:

$$x_1^0 = 0, \quad x_2^0 = 0, \quad x_3^0 = 0$$

$$x_1^1 = \frac{-(1\cdot 0 + (-1)\cdot 0) + 1}{2} = \frac{1}{2}$$

$$x_2^1 = \frac{-(1\cdot 0 + 1\cdot 0) + 2}{2} = 1$$

$$x_3^1 = \frac{-((-1)\cdot 0 + 2\cdot 0) + 3}{1} = 3$$

$x_n^{(k)}$

$$x_1^2 = \frac{-(1\cdot 1 + (-1)\cdot 3) + 1}{2} = \frac{3}{2}$$

$$x_2^2 = \frac{-(1\cdot \frac{1}{2} + 1\cdot 3) + 2}{2} = -\frac{3}{4}$$

$$x_3^2 = \frac{-((-1)\cdot \frac{1}{2} + 2\cdot 1) + 3}{1} = \frac{3}{2}$$

Metoda Gauss - Seidel

$N = D - L$

$P = U$

$A = N - P = (D-L) - U$

$$x^{(n+1)} = (D-L)^{-1} \cdot U x^{(n)} + (D-L)^{-1} \cdot b \quad / \cdot (D-L)$$

(1)

$$(D-L) \cdot x^{(n+1)} = U \cdot x^{(n)} + b$$

$$\begin{bmatrix} a_{11} & & & \\ a_{21} & a_{22} & & 0 \\ \vdots & & \ddots & \\ a_{n1} & \dots & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1^{(n+1)} \\ \vdots \\ x_n^{(n+1)} \end{bmatrix} = \begin{bmatrix} 0 & -a_{12} & \dots & -a_{1n} \\ 0 & 0 & -a_{23} & \dots \\ & 0 & & \ddots \\ & & & \end{bmatrix} \begin{bmatrix} x_1^{(n)} \\ \vdots \\ x_n^{(n)} \end{bmatrix} + b$$

~~$x_i^{(n+1)}$~~

Ne uităm pe rândul i:

$$a_{i1} \cdot x_1^{(n+1)} + a_{i2} \cdot x_2^{(n+1)} + \dots + a_{ii} \cdot x_i^{(n+1)} = - x_{i+1}^{(n)} \cdot a_{i\,i+1} - \dots - x_n^{(n)} \cdot a_{in} + b(i)$$

$\Downarrow$

$$x_i^{(n+1)} = \frac{b_i - \sum_{k=1}^{i-1} a_{ik} \cdot x_k^{(n+1)} - \sum_{k=i+1}^{n} a_{ik} \cdot x_k^{(n)}}{a_{ii}}$$

Dacă A e diagonal dominantă $\rightarrow$ converge sigur

A

x^0

x_1^1

x_2^1

x_3^1

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 2 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$x^0 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1^1 = \frac{1 - 0 - (1 \cdot 0 + (-1) \cdot 0)}{2} = \frac{1}{2}$$

$$x_2^1 = \frac{2 - (1 \cdot \frac{1}{2}) - (1 \cdot 0)}{2} = \frac{3}{4}$$

$$x_3^1 = \frac{3 - ((-1) \cdot \frac{1}{2} + 2 \cdot \frac{3}{4})}{1} = 2$$

$\cdot\, a_{i_n} + b(i)$

Metoda suprarelaxării (SOR)

Alegem un $\omega \in (0,2)$ care s.n factor de relaxare.

$$N(\omega) = (1-\omega)N = \frac{1}{\omega} D - L$$

$$P(\omega) = P - \omega N = \left(\frac{1}{\omega} - 1\right) D + U$$

$$x_i^{(n+1)} = \omega \cdot \frac{b_i - \sum_{k=1}^{i-1} a_{ik} \cdot x_k^{(n+1)} - \sum_{k=i+1}^{n} a_{ik} \cdot x_k^{(n)}}{a_{ii}} + (1-\omega)\, x_i^{(n)}$$

Obs Dacă $\omega = 1$, avem fix Gauss-Seidel.

Pt a face algoritmul în Octave, avem fix Gauss-Seidel, iar după ce s-a calculat $x(i)$, pur şi simplu se mai adaugă linia:

$$x(i) = w \cdot x(i) + (1-w) \cdot x0(i)$$

SOR = Successive Overrelaxation