12 Estimating stability regions of discrete dynamical systems

In this chapter, a theoretical basis for using energy functions to estimate stability regions of general nonlinear discrete-time dynamical systems is developed. The results developed in this chapter are the counterpart, for discrete systems, of the results developed in Chapter 10. On the basis of the theoretical developments, an optimal scheme for estimating the stability region of a large class of nonlinear discrete systems is developed. This computational scheme explores the stability boundary characterization and the tool of energy function (a scalar fuction) to optimally estimate stability regions of discrete dynamical systems in the form of level sets of a given energy function. More precisely, we develop an algorithm to obtain the largest level set of that energy function that is entirely contained in the stability region.

It is important to mention that most of the existing methods for estimating stability regions also search for a positively invariant level set contained inside the stability region. This invariant level set is usually characterized by a scalar function (usually a Lyapunov function). However, they do not explore the stability boundary characterization. As a consequence, their results might be conservative and not optimal in the sense that the given level set might not be the largest one entirely contained in the stability region.

12.1 Energy functions and the stability boundary

We will develop the foundations for estimating stability regions of the following class of autonomous nonlinear discrete dynamical systems:

$$x_{k+1} = f(x_k) \tag{12.1}$$

where $k \in Z$, $x_k \in R^n$ and $f\colon R^n \to R^n$ is a continuous map. Let E denote the set of fixed points of (12.1). We will assume that system (12.1) admits an energy function. Recall that an energy function for the discrete system (12.1) is a continuous scalar function $V: R^n \to R$ that satisfies the following conditions.

(E1) $\Delta V(x) \leq 0$ for all $x \in R^n$.
(E2) $\Delta V(x_k) = 0$ implies x_k is a fixed point.
(E3) If $V(x_k)$ is bounded for $k \in Z_+$, then the orbit x_k is itself bounded for $k > 0$.

Here $\Delta V(x_k) = V(x_{k+1}) - V(x_k)$ is the first difference of V along the solution x_k.

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Given a C^r-energy function $V:R^n \to R$ and a real number k, we consider the following set:

$$S(k) = \{x \in R^n: V(x) < k\}. \tag{12.2}$$

We shall call the boundary of set (12.2), $\partial S(k) := \{x \in R^n: V(x) = k\}$ the *level set* (or *constant energy surface*) and k the corresponding *level value*. If k is a regular value of V (i.e. $\nabla V(x) \neq 0$, for all $x \in V^{-1}(k)$), then by the inverse function theorem, $\partial S(k)$ is a C^r $(n-1)$-dimensional submanifold of R^n. Moreover, if $r > n-1$, then by the Morse–Sard theorem, the set of regular values of V is residual; in other words "almost all" level values are regular. In particular, for almost all values of k, the level set $\partial S(k)$ is a C^r $(n-1)$-dimensional submanifold.

Generally speaking, set $S(k)$ can be very complicated with several connected components even for the two-dimensional case. Let $S^i(k)$, $i = 1, 2, \ldots, m$, be these connected components such that

$$S(k) = S^1(k) \cup S^2(k) \cup \ldots \cup S^m(k) \tag{12.3}$$

with $S^i(k) \cap S^j(k) = \varnothing$ when $i \neq j$. Each of these components is connected and disjoint from each other. Since $V(\cdot)$ is continuous, $S(k)$ is an open set. Because $S(k)$ is an open set, the level set $\partial S(k)$ is of $(n-1)$ dimensions. Actually, every connected component $S^i(k)$ is an open set whose boundary $\partial S^i(k)$ is an $(n-1)$-dimensional set.

Set $S(k)$ is positively invariant, however, unlike continuous systems, the connected components of the set $S(k)$ are not necessarily positively invariant. Moreover, unlike continuous systems, which admit a single connected component of the level set inside the stability region, discrete systems might admit more than one connected component of the level set entirely contained on the stability region.

As we increase the level value k, at least one connected component of the set $S(k)$ will appear inside the stability region. However, unlike continuous systems, multiple connected components might appear inside the stability region, even when the stability region is connected. By continually increasing the level value k, the connected components of the set S(k) inside the stability region enlarge until one of them touches the stability boundary. We will see, in the next theorem, that the point on the stability boundary that is first reached by the connected components of set S(k), as we increase k, must be an unstable fixed point (UFP).

THEOREM 12-1 **(Energy functions and fixed points on the stability boundary)** *Consider the nonlinear discrete dynamical system (12.1) that admits an energy function. Suppose that f is a continuous function and let x_s be an asymptotically stable fixed point of (12.1) whose stability region $A(x_s)$ is not dense in R^n. If $E \cap \partial A(x_s)$ is a bounded set, then the point with the minimal value of the energy function over the stability boundary $\partial A(x_s)$ exists and must be an unstable fixed point.*

Proof If the stability region of x_s is not dense in R^n, then the stability boundary is a nonempty set of dimension $n-1$. We have also proven, in the proof of Theorem 9-16 of Chapter 9, that $V(x_s)$ is a lower bound of the energy function V over the stability

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boundary. Hence, we claim that the global minimum of V over the stability boundary exists. In order to prove this, suppose, on the contrary, the non-existence of a global minimum of V over the stability boundary. Then $\partial A(x_s)$ has to be unbounded and $V(x_s) \leq \inf_{x \in \partial A(x_s)} V(x) = b \leq \infty$. Then there is a sequence of points $x_k \in \partial A(x_s)$ with $\|x_k\| \to \infty$ as $k \to \infty$ such that $V(x_k)$ is a decreasing function in k satisfying $V(x_k) \to b$ as $k \to \infty$. On the other hand, every trajectory $\phi(t, x_k)$ is bounded and converges to a fixed point x_k^{eq} on the stability boundary. Assumptions (E1) and (E2) guarantee that $V(x_k^{eq}) \leq V(x_k)$. This implies the existence of a sequence of fixed points such that $V(x_k^{eq}) \to b$ as $k \to \infty$. The condition that $E \cap \partial A(x_s)$ is bounded ensures that the sequence of fixed points x_k^{eq} is bounded; therefore there is a convergent subsequence that converges to a fixed point p on the stability boundary with $V(p) = b$. But this contradicts the fact that the global minimum over the stability boundary does not exist. Suppose the minimum of V over $\partial A(x_s)$ is attained at a regular point x. The existence of an energy function implies that $V(f(x)) < V(x)$. But $f(x) \in \partial A(x_s)$ because $\partial A(x_s)$ is a positively invariant set (see Proposition 7-9). This contradicts the fact that the minimum of V over $\partial A(x_s)$ is attained at x. As a consequence, the minimum of the energy function over the stability boundary must be attained at an unstable fixed point. This completes the proof.

12.2 Closest unstable fixed point and characterization

Theorem 12-1 shows that the minimum of an energy function on the stability boundary of a discrete dynamical system exists and must be attained at an unstable fixed point. The point of minimum energy on the stability boundary may not be unique. However, since the property that all equilibrium points of system (12.1) have distinct energy function values is generic [46], we can affirm that the point of minimum energy on the stability boundary is generically unique. In other words, the uniqueness of the minimum point is almost always guaranteed. We call the fixed point on the stability boundary with a minimum value of energy the *closest unstable fixed point*.

DEFINITION **(Closest unstable fixed point)**
A fixed point $\bar{x}$ is the closest unstable fixed point *of a stable fixed point x_s with respect to an energy function V, if $\bar{x} \in \partial A(x_s)$ and $V(\bar{x}) = \min_{x \in E \cap \partial A(x_s)} V(x)$.*

As in the case of the closest UEP for continuous nonlinear systems having an energy function, the closest UFP for discrete nonlinear systems exists and is generically unique. Theorem 12-2 gives a geometrical and dynamic characterization of the closest fixed point. Note that this theorem holds without the transversality condition.

THEOREM 12-2 **(Geometrical and dynamical characterization)**
Consider the nonlinear discrete dynamical system (12.1) satisfying assumption (A1) and that admits an energy function. Suppose that f is a diffeomorphism. Let x_s be an asymptotically stable fixed point of (12.1) and let $\hat{x}$ be the closest UFP on $\partial A(x_s)$ with respect to the energy function V. If $\hat{x}$ is hyperbolic, then $W^u(\hat{x}) \cap A(x_s) \neq \emptyset$.

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Proof The closest UFP $\hat{x}$ is a hyperbolic equilibrium point lying on the stability boundary $\partial A(x_s)$. Then, by Theorem 9-8, $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A}(x_s) \neq \emptyset$. Suppose, by contradiction, that the unstable manifold $W^u(\hat{x})$ does not intercept $A(x_s)$. Then, there exists a trajectory $x(t)$ in the unstable manifold $W^u(\hat{x})$ such that $x(t) \in \partial A(x_s)$ for all t. As a consequence of Theorem 9-15, trajectory $x(t)$ converges to a fixed point on the stability boundary. In addition, the energy function value is strictly decreasing along any nontrivial trajectory of system (12.1). These two facts imply the existence of other points on the stability boundary with an energy function value lower than $V(\hat{x})$. This contradicts the fact that $\hat{x}$ possesses the lowest energy function value within the stability boundary $\partial A(x_s)$ and the proof is complete.

12.3 Optimal scheme for estimating the stability region

In this section, we explore the concept of closest UFP to develop a computational algorithm to obtain the optimal estimate of the stability region in the form of level sets of a given energy function.

THEOREM 12-3 **(Optimal estimation of the stability region)**
Consider the nonlinear discrete dynamical system (12.1) that admits an energy function $V(x)$. Suppose that f is a continuous function satisfying assumption (A1). Let x_s be an asymptotically stable fixed point of (12.1) and $\hat{x}$ be the closest UFP on $\partial A(x_s)$ with respect to $V(x)$. If $L = V(\hat{x})$, then:

(a) *every connected component $S^i(L)$ of the set $S(L)$ has an empty intersection with the stability boundary $\partial A(x_s)$;*
(b) *there exists a non-empty collection of connected components of the set $S(L)$ entirely contained in the stability region $A(x_s)$; this collection is positively invariant, in particular, there always exists a connected component in this collection containing the fixed point x_s;*
(c) *there exists a connected component of the set $S(B)$ that has a non-empty intersection with the stability boundary $\partial A(x_s)$ for any number $B > L$.*

Proof

(a) We shall prove this theorem by contradiction. Suppose the existence of a connected component $S^i(L)$ of $S(L)$ with a non-empty intersection with the stability boundary $\partial A(x_s)$. Let $q \in S^i(L) \cap \partial A(x_s)$. Therefore, $V(q) < L$. But this contradicts the fact that L is the minimum value of V on $\partial A(x_s)$. Consequently, every connected component $S(L)$ has an empty intersection with the stability boundary.
(b) Certainly, $V(x_s) < L$, therefore there exists one connected component of the set $S(L)$ that contains the asymptotically stable fixed point x_s. Since every connected component of $S(L)$ has an empty intersection with the stability boundary, either a connected component is entirely contained on the stability region or it is entirely contained on the interior of its complement. Let $S_A(L)$ be the collection of connected

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components of $S(L)$ that are entirely contained in the stability region $A(x_s)$. Set $S_A(L)$ is non-empty because it contains the connected component of $S(L)$ that contains the asymptotically stable fixed point x_s. If $x \in S_A(L)$, then $x \in A(x_s)$ and $f(x) \in A(x_s)$ because $A(x_s)$ is positively invariant. Moreover, property (E1) of an energy function ensures that $f(x) \in S(L)$. Then $f(x) \in S(L) \cap A(x_s) = S_A(L)$. Consequently, $f(S_A(L)) \subset S_A(L)$.

(c) For any $B > L$, the closest UFP $\hat{x} \in S(B)$. In particular, $\hat{x}$ belongs to one of the connected components of $S(B)$. Therefore $S(B) \cap \partial A(x_s) \neq \varnothing$. This completes the proof.

In practice, Theorem 12-3 ensures that by calculating the energy function value of all fixed points on the stability boundary, we can obtain the optimal estimate of the stability region, in the form of a level set of the energy function $V(x)$, by picking the level set with a level value that equals the value of the energy of the fixed point on the stability boundary which has the lowest value of energy over the stability boundary.

The next example illustrates a discrete dynamical system that admits an energy function and whose optimal estimation of the stability region is composed of two connected components of the level set $S(L)$.

Example 12-1 Consider the nonlinear discrete dynamical system:

$$x_{k+1} = f(x_k) \tag{12.4}$$

with

$$f(x) = \begin{cases} bx + c & \text{if} \quad x < -1 \\ ax & \text{if} \quad -1 \leq x \leq 1 \\ dx + e & \text{if} \quad x > 1 \end{cases} \tag{12.5}$$

where $a = 1/2, b = 2, c = 1, d = -2, e = 3$. This system possesses two fixed points. The origin is an asymptotically stable fixed point while UPF $= -c/(b-1) = -1.5$ is an unstable fixed point on the stability boundary of the origin. The stability region and stability boundary are shown in Figure 12.1. The stability boundary is composed of the stable set of the unstable fixed point, UFP, which is composed of the UFP union with the point indicated by an 'X' in Figure 12.1.

This system admits an energy function:

$$V(x) = \begin{cases} x + f & \text{if} \quad x < \text{UFP} \\ -x & \text{if} \quad \text{UFP} \leq x \leq 0 \\ 2x & \text{if} \quad 0 < x \leq 1 \\ gx + h & \text{if} \quad 1 < x \leq -\dfrac{e}{d} \\ -gx + j & \text{if} \quad -\dfrac{e}{d} < x \leq -\dfrac{c}{d(b-1)} - \dfrac{e}{d} \\ -x + n & \text{if} \quad x > -\dfrac{c}{d(b-1)} - \dfrac{e}{d} \end{cases} \tag{12.6}$$

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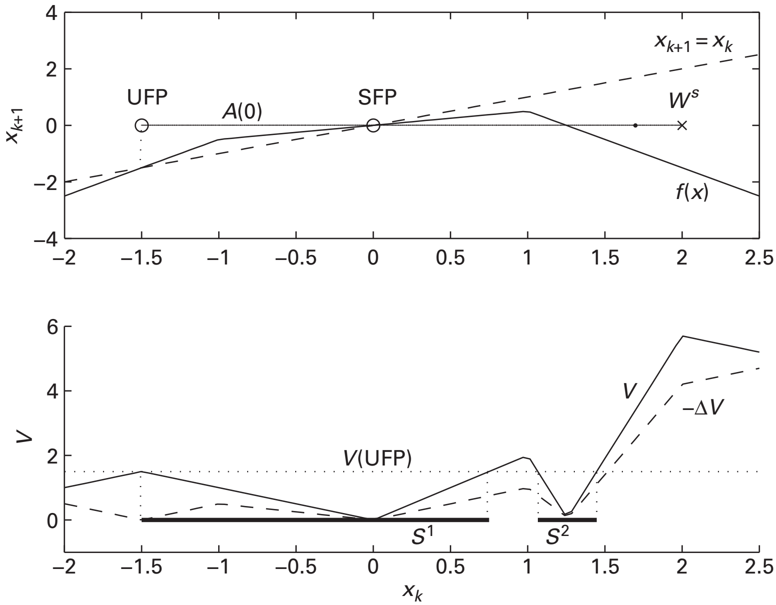

Figure 12.1 The stability region and stability region estimation of system (12.4).

where $f = 2c/(b-1)$, $g = 1.9/(1+e/d)$, $h = 2-g$, $j = -ge/d$, $n = (-g+1)(-c/d/(b-1) - e/d)+j$. The graphic of function $V(x)$ and its first difference ΔV is plotted in Figure 12.1. Clearly, $\Delta V = 0$ only at the fixed points. Since the UFP is the only unstable fixed point on the stability boundary, UFP is also the closest UFP. Using the value of the energy function calculated at the closest UEP we obtain the level curve set depicted in Figure 12.1. This set is composed of two connected components S^1 and S^2 entirely contained on the stability region of the origin. The connected component S^2 is not positively invariant. But both S^1 and the union $S^1 \cup S^2$ are positively invariant.

Based on Theorem 12-3, we propose the following scheme to estimate the stability region $A(x_s)$ of the nonlinear discrete dynamical system (12.1) that admits an energy function.

Scheme (Optimal estimation of the stability region $A(x_s)$ via an energy function $V(\cdot)$)

A Determining the type-one critical level value of an energy function.

Step 1: Find all fixed points.

Step 2: Identify those whose unstable manifold (or unstable set) intersects the stability boundary $\partial A(x_s)$, say $x_1, x_2, \ldots$.

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Step 3: Compute the value of the energy at these fixed points, $V(x_1)$, $V(x_2)$, ... and select the one with the smallest value of energy, say $L = \min_j\{V(x_j)\}$.

B Estimating the stability region $A(x_s)$.

Step 4: The connected component of the set $S(L)$ containing the fixed point x_s is an estimate of the stability region.

We comment on the estimation scheme. The analytical basis for Steps 1 and 2 is Theorem 12-2 while the analytical basis for Steps 3 and 4 is Theorem 12-3.

12.4 Numerical studies

Example 12-2 This example illustrates the proposed optimal scheme. Consider the two-dimensional nonlinear discrete dynamical system:

$$\begin{aligned} x_{k+1} &= x_k^3 + \frac{3}{4}x_k \\ y_{k+1} &= \alpha\frac{y_k + \beta y_k^2 + y_k^3}{y_k^2 + 1}. \end{aligned} \tag{12.7}$$

It is not difficult to check that the vector field is a diffeomorphism if $\beta^2 < \alpha(3 - \alpha)$. System (12.7) possesses six hyperbolic fixed points. The origin is a hyperbolic asymptotically stable fixed point while the other five fixed points lie on the stability boundary of (0, 0). Consider the following function:

$$V(x,y) = \begin{cases} |x| + |y| & \text{if} \quad |x| \le \frac{1}{2} \quad \text{and} \quad y \le \frac{1-\alpha}{\beta} \\ 1 - |x| + |y| & \text{if} \quad |x| > \frac{1}{2} \quad \text{and} \quad y \le \frac{1-\alpha}{\beta} \\ |x| + 2\left(\frac{1-\alpha}{\beta}\right) - y & \text{if} \quad |x| \le \frac{1}{2} \quad \text{and} \quad y > \frac{1-\alpha}{\beta} \\ 1 + 2\left(\frac{1-\alpha}{\beta}\right) - |x| - y & \text{if} \quad |x| > \frac{1}{2} \quad \text{and} \quad y > \frac{1-\alpha}{\beta}. \end{cases} \tag{12.8}$$

It is straightforward to verify that function $V(x, y)$ satisfies conditions (E1), (E2) and (E3). As a consequence, $V(x, y)$ is an energy function for system (12.7) and all conditions of Theorem 9-18 are satisfied. Thus the stability boundary $\partial A(0, 0)$ is composed of the union of the stable manifolds of every unstable fixed point that lies on the stability boundary. Figure 12.2 illustrates the fixed points and the stability boundary $\partial A(0, 0)$ for $\alpha = 1/3$ and $\beta = 2/3$. Applying Steps 1 and 2 of the optimal estimation scheme presented in the previous section, one can conclude that the fixed points (0.5, 0), (0.5, 1), (0, 1), (−0.5, 1) and (−0.5, 0) lie on the stability boundary $\partial A(0, 0)$.

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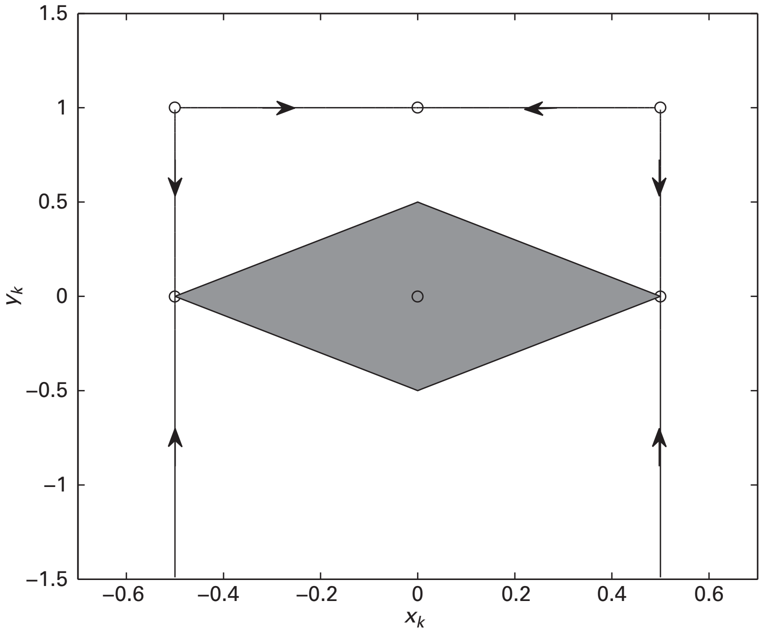

Figure 12.2 Stability region and stability region estimate of system (12.7) for $\alpha = 1/3$ and $\beta = 2/3$.

Computing the value of the energy function at these fixed points, as suggested in Step 3, one obtains $L = 0.5$. This minimum value of energy is attained at the fixed points (0.5, 0) and (−0.5, 0). Set $S(L)$, the optimal estimate of the stability region in the form of a level set of V, is the gray area of Figure 12.2. It can be seen from this figure that the critical level 0.5 determined by the proposed scheme is indeed the optimal one for estimating the stability region $A(0, 0)$.

Example 12-3 Consider the following discrete-time nonlinear system that resembles the model of a recurrent neural network:

$$y_i(k+1) = \sigma\left(\sum_{j=1}^{3} \mu_i \omega_{ij} y_j(k) + \mu_i s_i\right) \quad i = 1, 2, 3. \tag{12.9}$$

Function σ is the activation function of the neural network. Matrix $W = [w_{ij}]_{3\times3}$ is the synaptic weight matrix. The vector $s = [s_i]_{3\times1}$ is the input of the network and $\mu = [\mu_i]_{3\times1}$ is the activation gain of the network. The following energy function has been suggested in [174]:

$$E(y) = -\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\omega_{ij} y_i y_j - \sum_{i=1}^{3} s_i y_i + \sum_{i=1}^{3}\frac{1}{\mu_i}\int_0^{y_i}\sigma^{-1}(\tau)d\tau. \tag{12.10}$$

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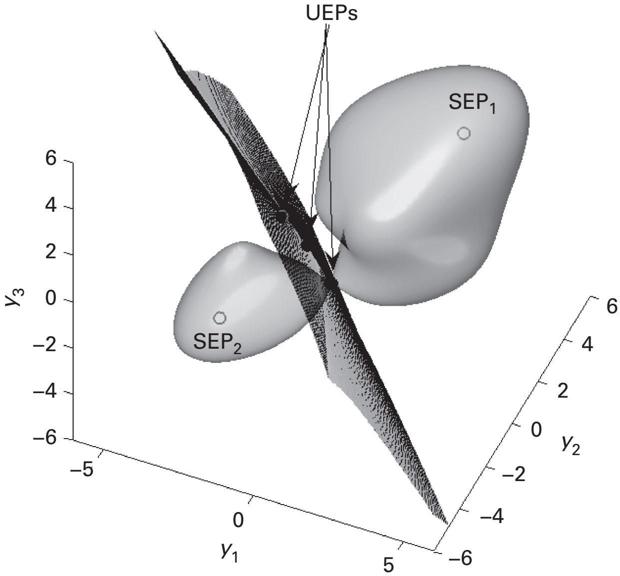

Figure 12.3 Stability region estimation of system (12.9). The dark surface containing the UEPs is the stability boundary while the surfaces involving the SEPs represent the optimal stability region estimations obtained with energy function (12.10).

In this example, $\sigma(z) = z^{1/3}$. For the following set of parameters, $\mu_1 = \mu_2 = \mu_3 = 10$, $\omega_{11} = 0.3$, $\omega_{22} = 0.2$, $\omega_{33} = 0.4$, $\omega_{12} = \omega_{21} = 0.4$, $\omega_{13} = \omega_{31} = 0.2$, $\omega_{23} = \omega_{322} = 0.1$, $s_1 = 0.5$, $s_2 = 0.2$ and $s_3 = 0.1$, the network possesses two asymptotically stable fixed points, they are SEP_1 =[3.15, 2.85, 2.77] and SEP_2 =[−2.63, −2.53, −2.58]. There are three fixed points on the stability boundary of these fixed points. The circles of Figure 12.3 display the location of these fixed points. The exact stability boundary is the black surface in this figure. By applying the optimal scheme for obtaining optimal estimates of the stability region of these asymptotically stable fixed points, we conclude that the unstable fixed point [0.17, −0.43, −1.88] is the one with the lowest energy function value on the stability boundary. The optimal stability region estimate $S(L)$, with L=−0.2152, of both fixed points is depicted in Figure 12.3.

12.5 Conclusions

A theoretical basis for using energy functions to estimate stability regions of general nonlinear discrete dynamical systems has been presented in this chapter. In addition, an optimal scheme for estimating the stability region was developed. A topological characterization and a dynamical characterization for the closest fixed

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point of a stable fixed point with respect to an energy function have been derived. Generically speaking, the closest fixed point is a type-one fixed point and the one-dimensional unstable manifold of the closest fixed point converges to the stable fixed point. This computational scheme can optimally estimate stability regions in the form of level sets of an energy function. Several examples have been given to illustrate the optimal estimation scheme.

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