7 Stability regions of constrained dynamical systems

Constrained nonlinear dynamical systems have been used to model a variety of practical nonlinear systems. Constraints usually have their origin in equations of balance such as the balance of energy or power in physical systems. These balance equations are a consequence of the physical laws of conservation, such as the energy conservation principle and the Kirchoff law of currents. Usually, these constraints appear in mathematical models in the form of algebraic equations and, consequently, constrained dynamical systems are often represented by a set of differential and algebraic equations. For instance, power system transient stability models are described by a set of differential-algebraic equations. The set of algebraic equations defines a constraint set or constraint manifold that constrains the system dynamics. Hence, the system trajectories of constrained nonlinear dynamical systems are confined to the constraint manifold, making the analysis of constrained dynamical systems challenging. Depending on the nature of the constraint set, bounded trajectories might not be defined for all times. This again makes the analysis of the asymptotic behavior of solutions of constrained dynamical systems a challenge.

Constrained nonlinear dynamical systems can exhibit trajectories whose sources and sinks all lie in singularities of the constraint set. These singular sets can exist on the stability boundary, making the task of characterizing the stability boundary of these systems even more complicated than the characterization of stability boundaries of ordinary differential equations developed in Chapter 4.

A comprehensive theory of stability regions and of stability boundaries for unconstrained continuous dynamical systems was presented in Chapters 4 and 5. In this chapter, a comprehensive theory of the stability boundary and of the stability regions of constrained nonlinear dynamical systems will be developed. Two approaches for studying the stability region of constrained dynamical systems will be presented in this chapter. The first relies on an approximation of the stability region via the singular perturbation theory and the second relies on a regularization of the vector field on the singular surface. For the latter approach, we follow the developments presented in [262,264].

7.1 Constrained nonlinear dynamical systems

Consider the following class of autonomous constrained nonlinear dynamical systems:

$$(\Sigma) \quad \begin{cases} \dot{x} = f(x,y) \\ 0 = g(x,y) \end{cases} \tag{7.1}$$

where $x \in R^n$ is the vector of dynamical state variables, $y \in R^m$ is the vector of algebraic or instantaneous state variables and f: $R^n \times R^m \to R^n$ and g: $R^n \times R^m \to R^m$ are smooth functions. This system is composed of a set of differential and algebraic equations and will be called, for short, a DAE system.

The state variables of this system belong to the Euclidean space R^{n+m}, however, trajectories of this system are constrained to a subset of R^{n+m}, the *constraint set*:

$$\Gamma = \{(x, y) \in R^{n+m} | 0 = g(x, y)\}. \tag{7.2}$$

A continuous and differentiable function $\phi(t) = (x(t), y(t))$: $I \to R^{n+m}$ defined on an interval $I \subset R$ is a solution of the DAE system (7.1) if (i) $(x(t), y(t)) \in \Gamma$ for all $t \in I$ and (ii) $(x(t), y(t))$ satisfies equation (7.1) for all $t \in I$. The solution of the DAE system (7.1) starting in (x, y) at time $t = 0$ is denoted $\phi(t, (x, y))$. The maximal interval of definition of a solution will be denoted (ω_-, ω_+).

A point $(\bar{x}, \bar{y}) \in R^{n+m}$ is an equilibrium point of the DAE system (7.1) if $f(\bar{x}, \bar{y}) = 0$ and $g(\bar{x}, \bar{y}) = 0$. The definitions of stability and asymptotic stability of equilibrium points of DAE systems are analogous to those of equilibrium points for ODE systems, see Section 2.3 of Chapter 2. The only difference is that the open neighborhoods of the equilibrium points in these definitions have to be taken on the constraint set Γ and not in R^{n+m}.

The associated linearized system, in the neighborhood of the equilibrium point $(\bar{x}, \bar{y})$, is given by:

$$\begin{cases} \dot{\zeta} = D_x f(\bar{x}, \bar{y})\zeta + D_y f(\bar{x}, \bar{y})\xi \\ 0 = D_x g(\bar{x}, \bar{y})\zeta + D_y g(\bar{x}, \bar{y})\xi. \end{cases}$$

If $D_y g$, calculated at the equilibrium point $(\bar{x}, \bar{y})$, is invertible, then we can solve the algebraic equation and derive the following reduced linearized system:

$$\dot{\zeta} = [D_x f - D_y f (D_y g)^{-1} D_x g]\zeta.$$

The equilibrium point $(\bar{x}, \bar{y})$ is hyperbolic if all eigenvalues of the reduced Jacobian matrix $J_{red} = D_x f - D_y f (D_y g)^{-1} D_x g$ have real part different from zero. Moreover, a hyperbolic equilibrium point is said to be a type-k equilibrium point if J_{red} possesses exactly k eigenvalues with real part greater than zero.

If the algebraic equation $0 = g(x, y)$ can be solved, i.e. if there exists a smooth function h:$R^n \to R^m$ such that $0 = g(x, h(x))$ for all x, then the instantaneous state variables can be eliminated from (7.1), reducing the problem to the analysis of the following ordinary differential equation:

$$\dot{x} = f(x, h(x)). \tag{7.3}$$

The theory developed in Chapters 4 and 5 is applicable and sufficient to establish a complete characterization of the stability region and stability boundary of (7.3) and consequently of (7.1). However, the condition that instantaneous state variables can be completely eliminated may not hold for many dynamical systems in the form of (7.1). In spite of that, we can locally solve the algebraic equation and use the theory of ordinary differential equations to guarantee the local existence and uniqueness of solutions of DAE systems in *regular points* of the constraint set Γ.

A point $(x_0, y_0) \in \Gamma$ is regular if the derivative $D_y g(x_0, y_0)$ of g with respect to y, calculated at $(x_0, y_0) \in \Gamma$, is invertible. If $(x_0, y_0) \in \Gamma$ is a regular point, then the implicit function theorem guarantees the existence of neighborhoods U of x_0 and W of y_0, and a smooth function h: $U \to W$, with $h(x_0) = y_0$, that solves the algebraic equation $0 = g(x, h(x))$ in the neighborhood $U \times W$ of the point $(x_0, y_0) \in \Gamma$. Then the smoothness of f and g and the theory of existence of solutions of ordinary differential equations guarantee the existence and uniqueness of the solution of (7.1) passing through (x_0, y_0) in a neighborhood of $(x_0, y_0) \in \Gamma$. More precisely, there is an interval of time I containing the origin and a function $\alpha(t)$: $I \to R^n$ such that $\alpha(t) \in U$ and

$$\dot{\alpha}(t) = f(\alpha(t), h(\alpha(t))) \quad \text{for all } t \in I; \tag{7.4}$$

consequently, $(x(t) = \alpha(t), y(t) = h(\alpha(t)))$ is a solution of the constrained system (7.1) for all $t \in I$. In other words, at regular points of Γ, the local existence and uniqueness of trajectories of the DAE system (7.1) are inherited from the local existence and uniqueness of trajectories of ordinary differential equations.

Unlike ordinary differential equations (see Theorem 2.1 of Chapter 2), bounded trajectories of DAE systems cannot always be extended infinitely because of the presence of singular points in Γ. Trajectories can be extended until they reach these singularities and, therefore, the singular surface plays a role in the characterization of stability boundaries. Next, the constraint set and its singular points will be examined.

7.1.1 Constraint set and singular points

The constraint set Γ may have several connected components and, typically, each connected component of Γ is an n-dimensional submanifold of R^{n+m}. A real number α is said to be a regular value of a scalar function $h(x)$ if the derivative Dh is surjective at every point of $h^{-1}(\alpha)$. In addition, zero is said to be a regular value for the vector function g if $\text{rank}[D_x g \ D_y g] = m$ at every point of Γ. The pre-image theorem [109] ensures that each connected component of Γ is usually an n-dimensional submanifold of R^{n+m}. Define the following subset of Γ:

$$M = \{(x, y) \in \Gamma | \text{ rank } [D_x g(x, y) \, D_y g(x, y)] = m\}.$$

The derivative of g is surjective in every point of M and therefore M is an n-dimensional submanifold in R^{n+m}. Typically M is a dense set in Γ. Actually, we will show that $\Gamma - M$, the set of points of Γ at which $\text{rank}[D_x g \ D_y g] < m$, is a thin set in Γ. While the points $(x_0, y_0) \in \Gamma$ where the derivative $D_y g(x_0, y_0)$ is invertible are called *regular points* of Γ, the points where the derivative $D_y g(x_0, y_0)$ is non-invertible are called *singular points*. The collection of singular points is called the *singular surface* and will be denoted by S:

$$S = \{(x, y) \in \Gamma | \Delta(x, y) = 0\}, \tag{7.5}$$

where $\Delta(x, y) = \det D_y g(x, y)$.

The following rank condition is usually satisfied

$$\operatorname{rank}\begin{bmatrix} D_x g(x,y) \, D_y g(x,y) \\ D_x \Delta(x,y) \, D_y \Delta(x,y) \end{bmatrix} = m+1$$

for every $(x, y) \in S$, thus, from the pre-image theorem [109], it is concluded that S is usually a submanifold of dimension $n-1$ embedded in Γ.

Typically, trajectories do not exist at singular points. Physically speaking, these singular points are usually due to un-modeled or neglected dynamics, such as parasitic dynamics, and therefore these singular points signal the onset of unpredictable system behavior. The following example, which appears in [246], illustrates a singular point on a two-dimensional system:

$$\begin{aligned} \dot{x} &= y \\ 0 &= (y-1)^2 + x. \end{aligned} \tag{7.6}$$

The constraint set $\Gamma = \{(x, y) \in R^2 | 0 = (y-1)^2 + x\}$ is a parabola passing through the point (0,1). This parabola is shown in Figure 7.1. Indeed $\operatorname{rank}[D_x g \, D_y g] = \operatorname{rank}[2(y-1) \; 1] = 1$ for all $(x, y) \in \Gamma$ and therefore Γ is a manifold of dimension one in R^2. Calculating the derivative of the algebraic equation with respect to y, one obtains $D_y g = 2(y-1)$. In this example, matrix $D_y g$ is a real number and therefore $\det D_y g = 2(y-1)$. This determinant is equal to zero if and only if $y = 1$, which implies, from the algebraic equation (7.6), that x=0. It follows that the point (0,1) is the unique singular point in Γ, see Figure 7.1. At every other point of Γ, the trajectories exist locally and are unique.

Consider the trajectory that passes through the point ($x_0 = -1$, $y_0 = 2$). This trajectory moves along the constraint set Γ and approaches the singular point in finite time as illustrated in Figure 7.1. However, at the singular point (0,1), there is no function that

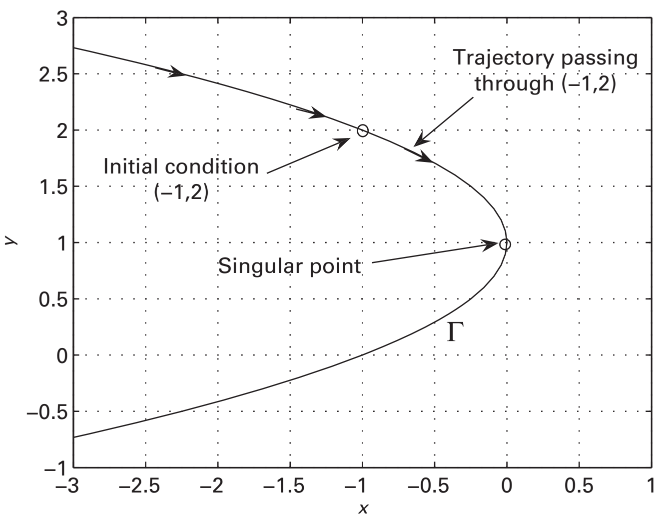

Figure 7.1 Phase portrait of system (7.6). The trajectory passing through the regular point (−1,2) reaches the singularity point (0,1) in finite time.

satisfies the conditions for a solution of (7.1). Therefore, the trajectory passing through the point (−1, 2) can only be extended up to the time when the trajectory reaches the singular point.

The constraint manifold Γ typically possesses a number of isolated connected components. Within each connected component, the singular surface S decomposes Γ into several smaller components Γ_i, such that $\Gamma - S = \bigcup_i \Gamma_i$. Since trajectories of the differential-algebraic system (7.1) cannot cross singular surfaces, they are confined to a single component Γ_i of Γ. If rank$[D_x g\ D_y g] < m$ at a point $(x_0, y_0) \in \Gamma$, then in particular rank $D_y g < m$ and thus $(x_0, y_0) \in S$. Therefore $\Gamma - S$ is always an n-dimensional submanifold of R^{n+m}. In particular, every component Γ_i of Γ is an n-dimensional submanifold of R^{n+m}.

7.2 Stability region of DAE systems

Let Γ_s be a component of Γ and $(x_s, y_s) \in \Gamma_s$ an asymptotically stable equilibrium point of a DAE system (7.1). The stability region of (x_s, y_s) is defined as

$$A(x_s, y_s) = \{(x,y) \in \Gamma_s : \lim_{t\to\infty} \phi(t,(x,y)) = (x_s, y_s)\}. \tag{7.7}$$

The continuity of solutions with respect to the initial conditions at regular points of the constraint set Γ ensures that the stability region $A(x_s, y_s)$ is an open set relative to $\Gamma_s \subset M$. The topological boundary of the stability region $A(x_s, y_s)$ will be denoted $\partial A(x_s, y_s)$. The stability boundary is a closed set. The stability boundary has maximal dimension $n-1$ at every point. However, the stability boundary may have parts with lower dimension.

Characterizations of the stability boundary $\partial A(x_s, y_s)$ of DAE systems have recently been developed. It has been shown that under certain conditions, the stability boundary $\partial A(x_s, y_s)$ consists of two parts: the first part is the stable manifolds of the equilibrium points on the stability boundary, while the second part contains points whose trajectories reach singular surfaces [264]. The second part can be further delineated as a union of the stable manifolds of pseudo-equilibrium points and semi-singular points on the stability boundary and parts of singular surfaces [262,264]. These characterizations will be presented in the later part of this chapter.

The next chapter studies the stability region of constrained dynamical systems by establishing a relationship with the stability region of an associated family of unconstrained dynamical systems, the so-called singularly perturbed systems. The advantages of this approach are (i) we avoid the difficulties of dealing with DAE models and (ii) the theory of stability regions for unconstrained systems has already been developed.

7.3 Singular perturbation approach

The singular perturbation approach treats the set of algebraic equations describing a DAE system as a limit of the fast dynamics $\varepsilon\dot{y} = g(x, y)$. In other words, for ε sufficiently small, the dynamics will quickly approach the algebraic manifold Γ and the solution of this approximated system will quickly converge to a solution of the DAE system.

With a possible change of sign of function g, the component of interest of the constraint manifold Γ will be an attractor for these pseudo fast dynamics. If the points on a component Γ_i of Γ are such that the corresponding Jacobian matrix $(\partial g/\partial y)\,(x,y)$ has all the eigenvalues with negative real parts, then the component is stable; otherwise, it is an unstable component.

Therefore, for the DAE system (7.1), we can define an associated singularly perturbed system:

$$\begin{aligned} \dot{x} &= f(x,y) \\ \varepsilon\dot{y} &= g(x,y) \end{aligned} \tag{7.8}$$

where ε is a sufficiently small positive number. The state variables of system (7.8) have very different rates of dynamics and they can be separated into two distinct time scales: slow variable x and fast variable y.

The trajectory of the singularly perturbed system (7.8) starting in (x,y) at time t=0 will be denoted by $\varphi_\varepsilon(t, x, y)$. If (x_s, y_s) is an asymptotically stable equilibrium point of the singularly perturbed system (7.8), then its stability region is defined as:

$$A_\varepsilon(x_s, y_s) = \{(x,y) \in R^{n+m} : \varphi_\varepsilon(t,x,y) \to (x_s, y_s) \text{ as } t \to \infty\}.$$

A relationship exists between the stability region of the DAE system and that of the singularly perturbed system. In particular, we will see that for sufficiently small ε, their stability boundaries are close, at least in compact sets of these boundaries.

Note that trajectories of the singularly perturbed system (7.8) are not confined to the algebraic manifold Γ and are not exactly the same as those of the original DAE system (7.1). However, trajectories generated by the singularly perturbed system are still valid approximations to those of the DAE system. Indeed, a theoretical justification ensuring that the difference of trajectories between the original DAE (7.1) and the singularly perturbed system (7.8) is uniformly bounded by the order of $O(\varepsilon)$ is provided by Tikhonov's theorem. Tikhonov's theorem is formally stated in Chapter 16. Figure 7.2

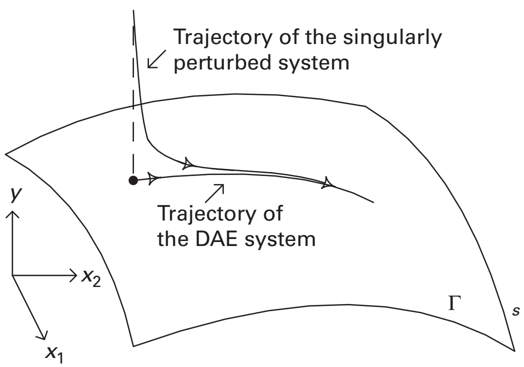

Figure 7.2 Relationship between the trajectory of the singularly perturbed system and that of the DAE system.

illustrates the relationship between the trajectory of the singularly perturbed system and that of the DAE system. In the beginning, the trajectory of the singularly perturbed system quickly approaches the constraint manifold Γ and then follows very close to a solution of the DAE system in the neighborhood of Γ.

A DAE system and its corresponding singularly perturbed system share several similar dynamical properties, including a close relationship between their stability regions. One obvious property is that they have the same set of equilibrium points, i.e. a point $(\bar{x}, \bar{y})$ is an equilibrium point of the DAE system (7.1) if and only if it is an equilibrium point of the singularly perturbed system (7.8) for all ε. In a stable component Γ_s of the constraint manifold Γ, the equilibrium points of the DAE system are the same type as those of the singularly perturbed system. The following results show this invariant topological relationship between the equilibrium points of a DAE system and those of its associated singularly perturbed system.

THEOREM 7-1 **(Invariant topological relationship)**
If an equilibrium point, say $(\bar{x}, \bar{y})$ of system (7.1) lies on one stable component Γ_s of the constraint manifold Γ, then there exists an $\varepsilon > 0$ such that for all $\varepsilon \in (0, \varepsilon)$, it follows that

(a) if $(\bar{x}, \bar{y})$ is a hyperbolic equilibrium point of the DAE system (7.1), then $(\bar{x}, \bar{y})$ is a hyperbolic equilibrium point of the singularly perturbed system (7.8), moreover
(b) if $(\bar{x}, \bar{y})$is a type-k equilibrium point of the DAE system (7.1), then $(\bar{x}, \bar{y})$ is a type-k equilibrium point of the singularly perturbed system (7.8).

Proof Since $(\bar{x}, \bar{y})$ lies on a stable component Γ_s of the constraint manifold Γ, $D_z g(\bar{x}, \bar{y})$ possesses no eigenvalues on the right hand side of the complex plane and m eigenvalues on the left hand side. Therefore, there exists a real number $\alpha > 0$ such that $|\mathrm{Re}\lambda\}| > \alpha$ for every eigenvalue λ of $D_z g(\bar{x}, \bar{y})$.

Consider the following time scale change $t = \varepsilon\tau$. In this new time scale, the singularly perturbed system (7.8) assumes the form:

$$\begin{aligned} \frac{dx}{d\tau} &= \varepsilon f(x, y) \\ \frac{dy}{d\tau} &= g(x, y). \end{aligned} \tag{7.9}$$

Consider the linearization of system (7.9) in the neighborhood of the equilibrium $(\bar{x}, \bar{y})$:

$$\begin{bmatrix} \dfrac{d\Delta x}{d\tau} \\ \dfrac{d\Delta y}{d\tau} \end{bmatrix} = J_\varepsilon^{fast} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

where

$$J_\varepsilon^{fast} = \begin{bmatrix} \varepsilon D_x f & \varepsilon D_y f \\ D_x g & D_y g \end{bmatrix}.$$

The complex number μ is an eigenvalue of matrix J_ε^{fast} if there exists a vector $(\Delta x, \Delta y) \neq 0$ satisfying:

$$\begin{bmatrix} \varepsilon D_x f - \mu I_n & \varepsilon D_y f \\ D_x g & D_y g - \mu I_m \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = 0.$$

For $\mu \neq 0$ and sufficiently small ε, the submatrix $\varepsilon D_x f - \mu I_n$ is invertible. Then, one can solve Δx as a function of Δz and write:

$$[D_y g - \varepsilon D_x g(\varepsilon D_x f - \mu I_n)^{-1} D_y f - \mu I_m]\Delta z = 0.$$

Then μ is an eigenvalue of a matrix that can be seen as a perturbation of the matrix $D_y g$. Define the function $p_\varepsilon(\mu) = \det[D_y g - \varepsilon C(\varepsilon, \mu) - \mu I_m]$ where $C(\varepsilon, \mu) = D_x g(\varepsilon D_x f - \mu I_n)^{-1} D_y f$. For $\mu \neq 0$ and sufficiently small ε, C is a continuous function of μ and ε.

Consider a simple closed curve γ in the complex plane such that all the eigenvalues of $D_y g$ are contained in the area that is delimited by this curve. Since all the m eigenvalues of $D_y g$ are located on the left hand side of the complex plane, the curve γ can be chosen such that $\gamma \subset \{\mu\text{: } \mathrm{Re}\{\mu\} < \alpha < 0\}$. Therefore $p_0(\mu) \neq 0$ for all $\mu \in \gamma$ and, as a consequence, $\inf_{\mu \in \gamma}|p_0(\mu)| = m > 0$. Using the continuity of $p_\varepsilon(\mu)$ with respect to ε, one concludes that $\inf_{\mu \in \gamma}|p_\varepsilon(\mu)| > 0$ for sufficiently small ε. Then

$$v(\varepsilon) = \frac{1}{2\pi i} \oint_\gamma \frac{p_\varepsilon{}'(\mu)}{p_\varepsilon(\mu)}.$$

is well defined and represents, according to the theory of complex variables, the number of zeros of $p_\varepsilon(\mu)$ inside γ. Since $v(\varepsilon)$ must be an integer number, we conclude, from the continuity of $v(\varepsilon)$, that $m = v(0) = v(\varepsilon)$ for sufficiently small ε. In other words, the existence of m eigenvalues of $D_y g$ with real part less than zero implies the existence of m eigenvalues of J_ε^{fast} with real part less than zero.

Suppose that $(\bar{x}, \bar{y})$ is a type-k equilibrium point of the DAE system (7.1). Then the matrix $J_{red} = D_x f - D_y f(D_y g)^{-1} D_x g$ has k eigenvalues with real part grater than zero and $n{-}k$ eigenvalues with real part less than zero. Beyond that, there exists a number $M{>}0$ such that every eigenvalue λ satisfies $|\lambda| < M$.

Consider the linearization of the singularly perturbed system (7.8) in the neighborhood of the equilibrium point $(\bar{x}, \bar{y})$:

$$\begin{bmatrix} \dfrac{d\Delta x}{d\tau} \\ \dfrac{d\Delta y}{d\tau} \end{bmatrix} = J_\varepsilon \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix}$$

where

$$J_\varepsilon = \begin{bmatrix} D_x f & D_y f \\ \dfrac{1}{\varepsilon} D_x g & \dfrac{1}{\varepsilon} D_y g \end{bmatrix}.$$

The complex number μ is an eigenvalue of matrix J_ε if there exists a vector $(\Delta x, \Delta y) \neq 0$ satisfying

$$\begin{bmatrix} D_x f - \mu I_n & D_y f \\ \frac{1}{\varepsilon} D_x g & \frac{1}{\varepsilon} D_y g - \mu I_m \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = 0$$

which is equivalent to

$$\begin{bmatrix} D_x f - \mu I_n & D_y f \\ D_x g & D_y g - \varepsilon \mu I_m \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = 0.$$

If μ is bounded and ε is sufficiently small, then $D_y g - \varepsilon\mu I_m$ is invertible. Hence, one can solve Δy as a function of Δx and write:

$$[D_x f - D_y f (D_y g - \varepsilon \mu I_m)^{-1} D_x g - \mu I_n]\Delta x = 0.$$

Using the identity $B^{-1} = A^{-1} - B^{-1}(B - A)A^{-1}$ with $B = D_y g - \varepsilon\mu I_m$ and $A = D_y g$, one obtains:

$$[J_{red} - \varepsilon \mu C(\mu, \varepsilon) - \mu I_n]\Delta x = 0,$$

where, in this case, $C(\mu, \varepsilon) = D_y f(D_y g - \varepsilon\mu I_m)^{-1}(D_y g)^{-1}D_x g$. Hence, the eigenvalue μ of J_ε can be seen as an eigenvalue of a matrix that is a perturbation of J_{red}.

Define function $q_\varepsilon(\mu) = \det[J_{red} - \varepsilon\mu C(\mu, \varepsilon) - \mu I_n]$. For bounded μ and sufficiently small ε, C is a continuous function of μ and ε. Consider again a simple closed curve γ in the complex plane such that all the eigenvalues of J_{red} with real part greater than zero are contained in the area delimited by this curve. This curve can be chosen such that $\gamma \subset \{\mu: \mathrm{Re}\{\mu\} > 0 \text{ and } |\mu| < M\}$.

Therefore $q_0(\mu) \neq 0$ for all $\mu \in \gamma$ and, as a consequence, $\inf_{\mu\in\gamma}|q_0(\mu)| = b > 0$. Using the continuity of $q_\varepsilon(\mu)$ with respect to ε, one concludes that $\inf_{\mu\in\gamma}|q_\varepsilon(\mu)| > 0$ for sufficiently small ε. Then

$$v(\varepsilon) = \frac{1}{2\pi i}\oint_\gamma \frac{q_\varepsilon{}'(\mu)}{q_\varepsilon(\mu)}$$

is well defined and represents, according to the theory of complex variables, the number of zeros of $q_\varepsilon(\mu)$ inside γ. Since $v(\varepsilon)$ must be an integer number, we conclude, from the continuity of $v(\varepsilon)$, that $k = v(0) = v(\varepsilon)$ for sufficiently small ε. In other words, the existence of k eigenvalues of J_{red} with real part greater than zero implies the existence of k eigenvalues of J_ε with real part greater than zero for sufficiently small ε.

Similar arguments can be used to show that the existence of $n-k$ eigenvalues of J_{red} with real part less than zero implies the existence of $n-k$ eigenvalues of J_ε with real part less than zero for sufficiently small ε.

Using the fact that $J_\varepsilon^{fast} = \varepsilon J_\varepsilon$, one concludes that λ is an eigenvalue of J_ε^{fast} if and only if λ/ε is an eigenvalue of J_ε. Then, for sufficiently small ε, the m eigenvalues of J_ε obtained in the fast time scale analysis via $D_y g$ have modulus sufficiently large to be different from the n eigenvalues obtained in the analysis in the slow time scale via J_{red}.

Thus, J_ε possesses k eigenvalues with real part greater than zero and $m+n-k$ eigenvalues with real part less than zero. Consequently, $(\bar{x}, \bar{y})$ is a type-k hyperbolic equilibrium point of the singularly perturbed system (7.8) for sufficiently small ε. This completes the proof.

The above result shows that the type of equilibrium point of the singularly perturbed system is the same as the type of the corresponding equilibrium point of the DAE system, provided ε is sufficiently small. The converse of this theorem is not true. The following example illustrates an equilibrium point of a DAE system that does not have the same type of equilibrium points as the singularly perturbed system.

Example 7-1 Consider the following DAE system:

$$\begin{aligned} \dot{x} &= -x - z \\ \dot{y} &= y + z \\ 0 &= -x - y - z \end{aligned} \tag{7.10}$$

and the associated singularly perturbed linear system

$$\begin{aligned} \dot{x} &= -x - z \\ \dot{y} &= y + z \\ \varepsilon\dot{z} &= -x - y - z. \end{aligned} \tag{7.11}$$

The origin is an equilibrium point of these systems. In this case the constraint manifold is the plane $\Gamma = \{(x, y, z) \in R^3 : z = -x - y\}$. In this simple DAE system, the algebraic equation can easily be solved and the variable z can be eliminated. The reduced system is the harmonic oscillator, and the eigenvalues of the DAE system calculated at the origin are purely imaginary. On the other hand, the origin is a type-two equilibrium point of the singularly perturbed system for all $\varepsilon > 0$. Figure 7.3 presents the eigenvalues of the full system for the range $0.01 < \varepsilon < 0.5$. Observe that the eigenvalues of the singularly perturbed system approach the imaginary axis as ε approaches zero.

Nevertheless, under the condition that eigenvalues of the singularly perturbed system do not approach the imaginary axis for sufficiently small ε, the converse implication of Theorem 7.1 can be proven. We show this result in the next theorem.

THEOREM 7-2 **(Invariant topological relationship)**
Let $(\bar{x}, \bar{y})$ be a type-k hyperbolic equilibrium point of the singularly perturbed system (7.8) on a stable component Γ_s of the constraint manifold Γ for every sufficiently small ε. If for every sufficient small $\varepsilon > 0$ there exists a real number $\alpha > 0$ such that $|\mathrm{Re}\{\lambda\}| > \alpha$ for every eigenvalue λ of the Jacobian matrix of the singularly perturbed system (7.8) calculated at $(\bar{x}, \bar{y})$, then $(\bar{x}, \bar{y})$ is a type-k hyperbolic equilibrium point of the DAE system (7.1).

Proof Suppose, by contradiction, that $(\bar{x}, \bar{y})$ is not a hyperbolic equilibrium point of the DAE system (7.1). Then there are eigenvalues of $J_{red} = D_x f - D_y f (D_y g)^{-1} D_x g$ lying on

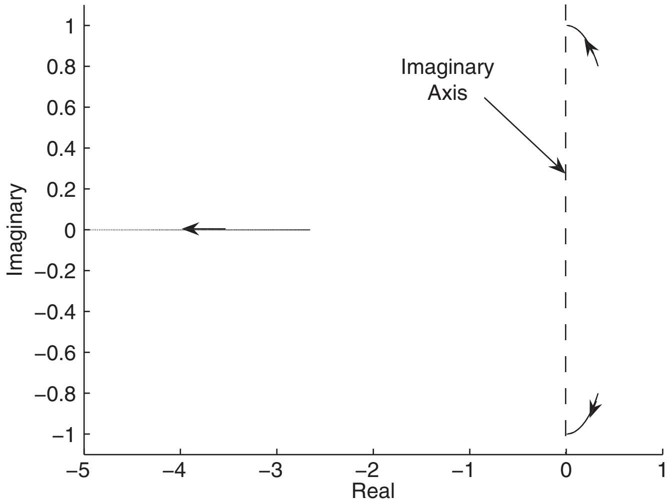

Figure 7.3 Root locus of system (7.11) for $0.01 < \varepsilon < 0.5$. Arrows indicate the movement of eigenvalues in the complex plane in the direction of decreasing ε. A pair of complex conjugate eigenvalues approaches the imaginary axis as $\varepsilon \to 0$.

the imaginary axis. Consider a simple closed curve γ in the complex plane such that all the eigenvalues of J_{red} on the imaginary axis belong to the area delimited by this curve. The curve γ can be chosen such that $\gamma \subset \{\mu{:}\mathrm{Re}\{\mu\} < \alpha \text{ and } |\mu| < M\}$ for some sufficiently large $M>0$. Using arguments similar to those employed in the proof of Theorem 7.1, one proves the existence of eigenvalues of J_ε in the set $\{\mu{:}\ \mathrm{Re}\{\mu\} < \alpha \text{ and } |\mu| < M\}$ for sufficiently small ε. This leads us to a contradiction and, therefore, $(\bar{x}, \bar{y})$ is a hyperbolic equilibrium point of the DAE system (7.1) for sufficiently small ε. Suppose now that the type of hyperbolic equilibrium $(\bar{x}, \bar{y})$ of the DAE system (7.1) is $r \neq k$, then Theorem 7.1 implies that $(\bar{x}, \bar{y})$ is a type-r hyperbolic equilibrium of the singularly perturbed system (7.8). Thus we reach a contradiction. Consequently, $(\bar{x}, \bar{y})$ is a type-k hyperbolic equilibrium of the DAE system (7.1) for sufficiently small ε. This completes the proof.

Theorem 7-1 and Theorem 7-2 assert, under certain conditions, that a point $(\bar{x}, \bar{y})$ is a hyperbolic type-k equilibrium point of the DAE system if and only if $(\bar{x}, \bar{y})$ is a hyperbolic type-k equilibrium point of the singularly perturbed system for all small $\varepsilon > 0$.

These local results have been extended into global results to establish the relationship between the stability boundaries of the DAE system and the singularly perturbed system. It will be shown that the stability boundaries of these two systems contain the same set of equilibrium points on the stable components of the constraint manifold.

THEOREM 7-3 **(Dynamic relationship)**
Let (x_s, y_s) and (x_u, y_u) be a hyperbolic asymptotically stable and an unstable equilibrium point of the DAE system (7.1) on the stable component Γ_s of Γ respectively. Suppose that for each $\varepsilon > 0$, the associated singularly perturbed system (7.8) has an energy function. Then there exists an $\varepsilon > 0$ such that if (x_u, y_u) lies on the stability boundary $\partial A(x_s, y_s)$ of

the DAE system (7.1), then (x_u, y_u) *lies on the stability boundary* $\partial A(x_s, y_s)$ *of the singularly perturbed system (7.8) for all* $\varepsilon \in (0, \varepsilon)$.

Proof Let N be a neighborhood in R^{n+m} of the asymptotically stable equilibrium point (x_s, y_s) of the singularly perturbed system (7.8) and M a neighborhood in R^{n+m} of (x_u, y_u). The asymptotically stability property of (x_s, y_s) ensures that the neighborhood N can be chosen sufficiently small such that every trajectory of the DAE system starting in $N \cap \Gamma_s$ stays bounded and approaches (x_s, y_s) as $t \to \infty$. Let B be an open ball strictly contained in N. If (x_u, y_u) belongs to the stability boundary of the DAE system (7.1), then there exists a point (x^*, y^*) in the induced neighborhood $M \cap \Gamma_s$ such that the trajectory $\varphi_{DAE}(t, x^*, y^*)$ of the DAE system starting at (x^*, y^*) enters B in finite time. More precisely, there is a time $T > 0$ such that $\varphi_{DAE}(T, x^*, y^*) \in B$. Using Tikonov's theorem for finite interval time, one can prove that trajectories of the singularly perturbed system will enter the neighborhood N in finite time for sufficiently small ε. In other words, $\varphi_\varepsilon(T, x^*, y^*) \in N$ for sufficiently small ε. Inside N, we can apply Tikonov's theorem for infinite time intervals to conclude that every trajectory of the singularly perturbed system starting in N is bounded and stays close to the asymptotically stable equilibrium point (z_s, y_s) for sufficiently small ε. The existence of an energy function implies that the trajectory approaches the asymptotically stable equilibrium point (x_s, y_s) as $t \to \infty$. As a result, we have proven that (x^*, y^*) is a point in the neighborhood M that belongs to the stability region of the singularly perturbed system for every sufficiently small ε. This completes the proof.

In addition, the converse of this theorem is not true. However, by imposing the condition that the eigenvalues of the equilibriums of the singularly perturbed system do not approach the imaginary axis as ε approaches zero, the converse holds as well. The next theorem formally states this result.

THEOREM 7-4 **(Invariant property)**
Let (x_s, y_s) *and* (x_u, y_u) *be a hyperbolic asymptotically stable equilibrium point and a hyperbolic unstable equilibrium point, respectively, of the singularly perturbed system (7.8) for all sufficiently small* ε. *Suppose the existence of a constant* $\alpha > 0$ *such that every eigenvalue* λ *of the Jacobian matrix of the singularly perturbed system (7.8) satisfies* $|\mathrm{Re}\{\lambda\}| > \alpha$ *for all* $\varepsilon \le \bar{\varepsilon}$. *Then, if* (x_u, y_u) *lies on the stability boundary of the singularly perturbed system (7.8) for* $\varepsilon \le \bar{\varepsilon}$, *then* (x_u, y_u) *lies on the stability boundary of the DAE system (7.1).*

Proof According to Theorem 7-3, (x_s, y_s) is a hyperbolic asymptotically stable equilibrium point and (x_u, y_u) is a hyperbolic unstable equilibrium point of the DAE system. For a given neighborhood $M \subset R^{n+m}$ of (x_u, y_u) and every sufficiently small $\varepsilon > 0$, there exists a point $(x_\varepsilon, y_\varepsilon) \in W_\varepsilon^u(x_u, y_u) \cap M$ such that $(x_\varepsilon, y_\varepsilon) \in A_\varepsilon(x_s, y_s)$. Consider a monotonically decreasing sequence $\{\varepsilon_j\}$ with $\varepsilon_j \to 0$ as $j \to \infty$. Since $\overline{M}$ is compact, the sequence $\{(x_{\varepsilon_j}, y_{\varepsilon_j})\}$ possesses a convergent subsequence, i.e. there exists (x_0, y_0) such that $(x_{\varepsilon_{j_k}}, y_{\varepsilon_{j_k}}) \to (x_0, y_0)$ as $\varepsilon_{j_k} \to 0$. We can also chose this sequence such that $(x_0, y_0) \neq (x_u, y_u)$. We know that $W_\varepsilon^u(x_u, y_u) \cap \overline{M}$ is ε-close to $W^u(x_u, y_u) \cap \overline{M}$. Then $d\Big((x_{\varepsilon_{j_k}}, y_{\varepsilon_{j_k}}), W^u(x_u, y_u)\Big) \to 0$ as $\varepsilon_{j_k} \to 0$. Then $d((x_0, y_0), W^u(x_u, y_u)) \le d((x_0, y_0), (x_{\varepsilon_{j_k}}, y_{\varepsilon_{j_k}})) + d\Big((x_{\varepsilon_{j_k}}, y_{\varepsilon_{j_k}}), W^u(x_u, y_u)\Big) \to 0$ as $\varepsilon_{j_k} \to 0$ and $(x_0, y_0) \in \overline{W^u(x_u, y_u)} \subset \Gamma$.

Suppose, by contradiction, that $(x_0, y_0) \notin A(x_s, y_s)$. Then, there exists a number $\rho > 0$ such that the solution of the DAE system $\phi(t, x_0, y_0) \notin B_\rho\ (x_s, y_s)$ for all $t > 0$, where $B_\rho(x_s, y_s)$ is an open ball of radius ρ centered at (x_s, y_s). According to Tikhonov's theorem for finite time intervals, for every $T > 0$, there exists an $\varepsilon(T)$ such that $\phi_\varepsilon(t, x_\varepsilon, y_\varepsilon) \notin B_{\rho/2}(x_s, y_s)$ for all $t \in [0, T]$ and $\varepsilon \in (0, \varepsilon)$. But this contradicts the fact that $\phi_\varepsilon(t, x_\varepsilon, y_\varepsilon) \to (x_s, y_s)$ as $t \to \infty$. Hence, $(x_0, y_0) \in A(x_s, y_s) \cap W^u(x_u, y_u)$ and $(x_u, y_u) \in \partial A(x_s, y_s)$. This completes the proof.

The above theorems provide a theoretical basis for characterizing an unstable equilibrium point on the stability boundary of the DAE system (7.1) by characterizing the same equilibrium in the stability boundary of the singularly perturbed system (7.8). In other words, we can check whether an equilibrium lies on the stability boundary of a DAE system, checking whether it lies on the stability of the singularly perturbed system for small ε, which is a unconstrained dynamical system.

Although the stability boundaries of the singular perturbed system and DAE system share the same unstable equilibrium points, the stability region and the stability boundary lie on completely different spaces. While the stability region of the singularly perturbed system is a set of dimension $n+m$ in the space R^{n+m}, the stability region of the DAE system is a set of dimension n on the set Γ. However, it can be proved that the intersection of the stability boundary of the singularly perturbed system with the constraint set Γ is a good approximation of the stability region of the DAE system for small ε. The next theorem establishes this property.

THEOREM 7-5 **(Stability boundary approximation)**
Let (x_s, y_s) be an asymptotically stable equilibrium point of the DAE system (7.1) in a stable component Γ_s of Γ. Suppose that for each $\varepsilon > 0$, the associated singularly perturbed system (7.8) has an energy function. If (x, y) belongs to the stability region $A(x_s, y_s)$ of the DAE system (7.1) then there exists $\bar{\varepsilon} > 0$ such that (x, y) belongs to the stability region $A_\varepsilon(x_s, y_s)$ of the singularly perturbed system for all $\varepsilon \in (0, \bar{\varepsilon}]$.

The proof of this theorem is a direct consequence of Tikhonov's results. It is also very similar to the proof of Theorem 7-3 and for this reason will be omitted. Theorem 7-5 asserts that $A_\varepsilon(x_s, y_s) \cap \Gamma_s$ is a good candidate for approximating the stability region of the DAE system.

The following example, illustrates the results of this section and how the stability region of the singularly perturbed system can approximate the stability region of the DAE system.

Example 7-2 (Simple DAE system) The following dynamical system models a power system composed of a single generator and one load bus:

$$\begin{aligned} \dot{\omega} &= -\frac{1}{M_g} D_g \omega - \frac{1}{M_g} f(\alpha, V) \\ \dot{\alpha} &= -\frac{1}{D_l} f(\alpha, V) + \omega \\ 0 &= -g(\alpha, V) \end{aligned} \tag{7.12}$$

where $f(\alpha, V) = B_{12}\, V \sin \alpha - P_l$ and $g(\alpha, V) = \frac{1}{V}(Q_l - B_{12} V \cos \alpha - B_{22} V^2)$.

For the following set of parameters, $M_g = 20$, $D_g = 9$, $D_l = 50$, $P_l = 4$, $Q_l = -0.5$, $B_{12} = 10$ and $B_{22} = -10$, system (7.12) possesses the stable equilibrium point (0, 0.4291, 0.9613) and the following two unstable equilibrium points (0, 1.2660, 0.4193) and (0, −5.0172, 0.4193). These equilibrium points lie on the constraint manifold $\Gamma = \{(\omega, \alpha\ V){:}g(\alpha, V) = 0\}$. Figure 7.4 illustrates a stable component of this constraint manifold, which contains these three equilibrium points.

Both unstable equilibrium points lie on the stability boundary of the stable equilibrium point of the DAE system and as a consequence of Theorem 7-3, they also lie on the stability boundary of the singularly perturbed system for sufficiently small ε. The stability boundary of the DAE system is composed of the union of the stable manifolds of these two UEPs, as indicated in Figure 7.4. We compare the stability region of the DAE system with the stability region of the corresponding singularly perturbed system with different values of ε. The comparison is made on the subspace of (ω, α); i.e. the projection of the stability region in the subspace of (ω, α). As can be seen from Figure 7.5–Figure 7.7, the intersection of the stability region of the corresponding singularly perturbed system with the constraint manifold approaches the stability region of the DAE system as the values of ε approach zero, and this observation is in agreement with our theoretical development. When ε becomes smaller at the value of 0.1, the stability region of the corresponding singularly perturbed system captures that of the DAE system more accurately as we can see in Figure 7.7.

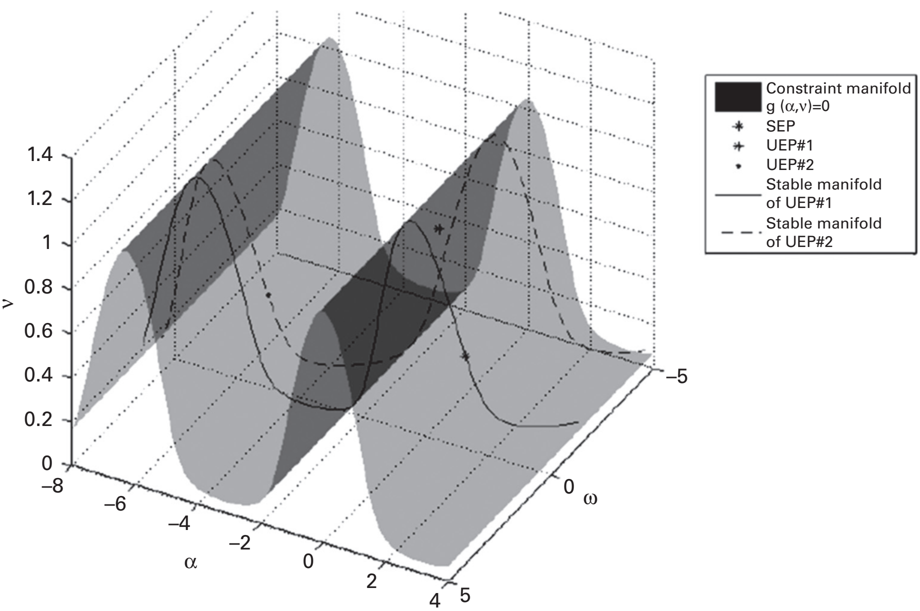

Figure 7.4 The constraint manifold of the DAE system (7.12). Both unstable equilibrium points (0, 1.2660, 0.4193) and (0, −5.0172, 0.4193), marked with the symbols * and °, lie on the stability boundary of the stable equilibrium point (0, 0.4291, 0.9613), marked with the symbols *.

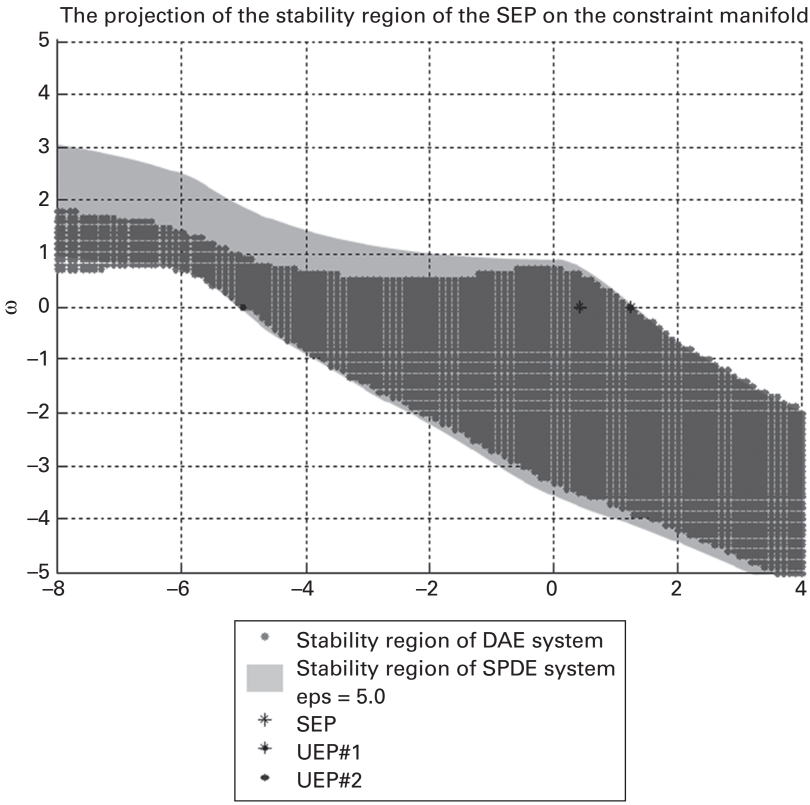

Figure 7.5 The stability region (on $\omega = 0$ and $g(x,y) = 0$) of the singularly perturbed system with $\varepsilon = 5.0$ is close to that of the corresponding DAE system.

7.4 Regularization of the DAE vector field

Another approach to studying the properties of the stability region and stability boundary of the DAE system (7.1) is to construct a transformed vector field that is well defined in the singular surface and that is equivalent to the DAE system (7.1) in $\Gamma - S$. A close relationship between these two systems, and in particular between their stability regions, exists. Exploring this relationship, the regular tools of ordinary differential equations can be employed to study the stability region of the transformed system and consequently the stability region of the DAE system (7.1).

To avoid the problem of singularities, we will extend the vector field of the DAE system (7.1) defined on the constraint set $\Gamma - S$ to a vector field that is globally defined, smooth and equivalent to system (7.1) when restricted to $\Gamma - S$. This approach allows the application of the regular theory of ODEs to study dynamical behaviors of DAE systems. We will achieve this extension in two steps. First we will extend the vector field of the DAE system to the set R^{n+m} with the exception of singular points, and then we will regularize the vector field at singular points by a convenient change of time scale.

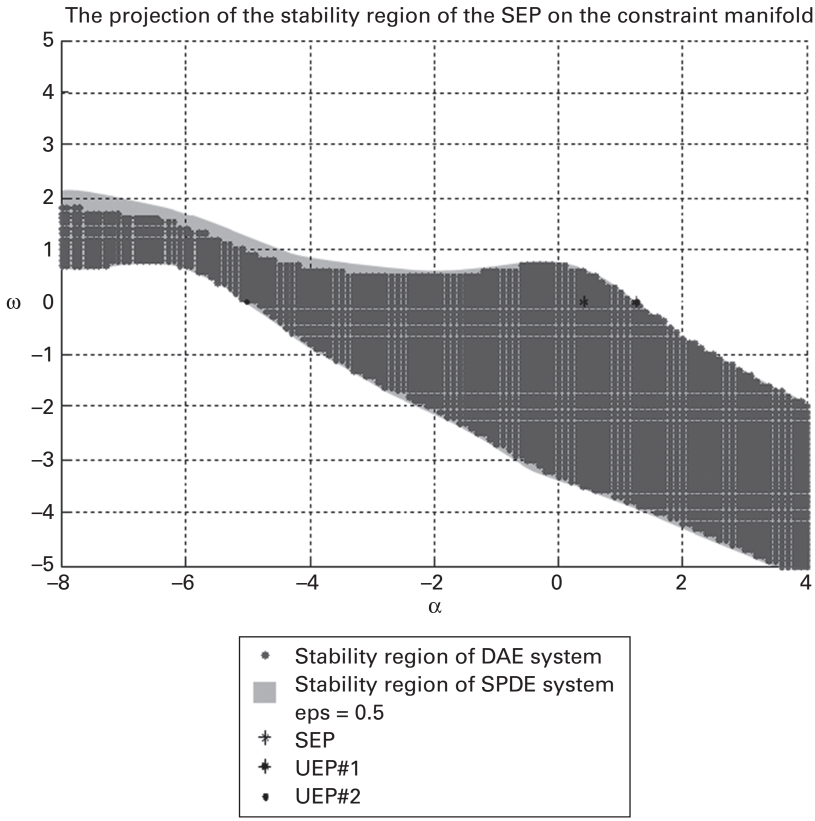

Figure 7.6 The stability region (on $\omega = 0$ and $g(x,y) = 0$) of the singularly perturbed system with $\varepsilon = 0.5$ is close to that of the corresponding DAE system.

The system of differential-algebraic equations (7.1) can be interpreted as a dynamical system on the manifold Γ. The vector field of this dynamical system is a function that take values from the manifold Γ to vectors in the tangent space $T\Gamma$ of Γ. This vector field can be calculated by implicit differentiation of the algebraic equation of (7.1):

$$D_x g(x,y) f(x,y) + D_y g(x,y)\dot{y} = 0. \tag{7.13}$$

If (x,y) is a regular point of Γ, then $D_y g(x,y)$ is invertible and one obtains:

$$\dot{y} = -(D_y g(x,y))^{-1} D_x g(x,y) f(x,y). \tag{7.14}$$

Thus the vector field on $\Gamma - S$ assumes the form:

$$(\Sigma') \begin{cases} \dot{x} = f(x,y) \\ \dot{y} = -(D_y g(x,y))^{-1} D_x g(x,y) f(x,y). \end{cases} \tag{7.15}$$

Actually, this vector field is defined for all points in R^{n+m} excluding those where $D_y g(x,y)$ is not invertible. The manifold Γ is invariant to (Σ') and its restriction to $\Gamma - S$ is

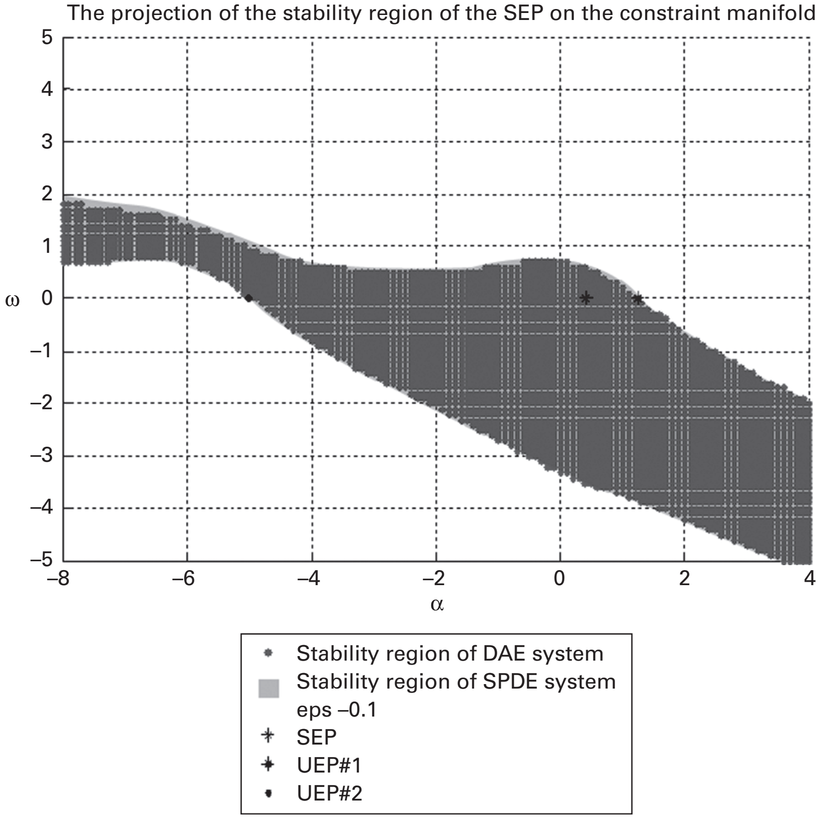

Figure 7.7 The stability region (on ω =0 and $g(x,y)$=0) of the singularly perturbed system with $\varepsilon = 0.1$ is close to that of the corresponding DAE system.

completely equivalent to the dynamical system (7.1), in the sense that every trajectory of (7.1) is also a trajectory of (7.15).

In order to analyze the solutions of the DAE system (7.1) near singular points, we employ a singular transformation that was suggested by Takens [246] and extensively explored in [262,264]. More precisely, we multiply the vector field (7.15) with $\Delta(x, y) = \det(D_y g(x, y))$, the determinant of $D_y g(x,y)$, and exploit the following property of the adjoint matrix

$$\operatorname{adj}(D_y g) D_y g = D_y g \operatorname{adj}(D_y g) = \det(D_y g) I_n \tag{7.16}$$

to obtain the following transformed vector field:

$$(\Sigma'') \begin{cases} \dot{x} = f(x,y)\Delta(x,y) \\ \dot{y} = -\operatorname{adj}(D_y g(x,y)) D_x g(x,y) f(x,y). \end{cases} \tag{7.17}$$

The transformed vector field is globally defined and smooth and leaves the constraint manifold Γ invariant. The advantage of using this transformed vector field is that it allows study of the differential-algebraic dynamical system (7.1) using the standard tools for analysis of the set of ordinary differential equations (7.17). The relationship between system (7.17) and the original system (7.1) will be now investigated.

7.4.1 Relationship between (Σ), (Σ') and (Σ'')

The continuity of the determinant of a matrix with respect to its entries ensures that the sign of $\Delta(x, y)=\det D_y g(x,y)$ is constant in each component Γ_i of Γ. Exploring these properties, we show that the vector field (Σ'), in each component Γ_i, is equivalent either to (Σ'') or $(-\Sigma'')$, where $(-\Sigma'')$ is the same vector field (Σ'') but with opposite sign (i.e. its trajectories are equal to the trajectories of (Σ'') but they flow in the reverse direction).

THEOREM 7-6 **(Equivalency between (Σ') and (Σ''))**
We consider the system of differential-algebraic equations (*7.1*) *and its vector field on* $\Gamma - S$ *expressed in (7.15). We consider the transformed vector field of the differential-algebraic equations* (*7.1*) *expressed in (7.17), which is globally defined and smooth. Define:*

$$(\Gamma - S)_+ = \{(x,y) \in \Gamma - S \mid \Delta(x,y) > 0\} \tag{7.18}$$

$$(\Gamma - S)_- = \{(x,y) \in \Gamma - S \mid \Delta(x,y) < 0\}. \tag{7.19}$$

Then the following results hold.

(a) The vector fields (Σ') *and* (Σ'') *are equivalent in* $(\Gamma - S)_+$.
(b) The vector fields (Σ') *and* $(-\Sigma'')$ *are equivalent in* $(\Gamma - S)_-$.

Proof The vector field (Σ'') is obtained from (Σ') by a pointwise scaling of time. If t denotes the time parametrizing the orbits of system (Σ'), it is straightforward to see that the transformed vector field (Σ'') can be obtained by the following change of time scale:

$$\frac{d\tau}{dt} = \frac{1}{\Delta(x(t), y(t))} \tag{7.20}$$

where τ denotes the time parametrizing orbits of system (Σ''). Consequently, (Σ') and (Σ'') are equivalent when $\Delta(x, y) > 0$, and (Σ') and $(-\Sigma'))$ are equivalent when $\Delta(x, y) < 0$. This completes the proof.

Theorem 7-6 shows that systems (Σ') and (Σ'') possess, besides orientation and time scale, the same set of orbits in $\Gamma - S$. In particular, the vector fields (Σ') and (Σ'') have the same set of equilibrium points in the set $\Gamma - S$. If x_s is an asymptotically stable equilibrium point of (Σ) and $x_s \in \Gamma_s \subset (\Gamma - S)_+$, then x_s is also an asymptotically stable equilibrium point of (Σ'') restricted to Γ. Consequently, to study the stability region of the DAE system (Σ), it is sufficient to study the stability region of the transformed system (Σ'') restricted to the component Γ_s of the constraint set Γ. More precisely, we can define the stability region of the DAE system (Σ) in terms of trajectories of the transformed system (Σ'') as:

$$A(x_s) = \{x \in \Gamma_s | \varphi_{(\Sigma'')}(t,x) \in \Gamma_s \quad \text{for all} \quad t \geq 0, \varphi_{(\Sigma'')}(t,x) \to x_s \quad \text{as} \quad t \to \infty\}. \tag{7.21}$$

The vector field (Σ') is not defined in S while (Σ'') is a smooth vector field in all Γ, consequently, the stability region of the transformed system (Σ'') may contain points in Γ_s whose trajectories leave the set Γ_s, by crossing the singular surface S, to return later on to Γ_s, by crossing again the singular surface S. These points on the stability region of the transformed system do not belong to the stability region of the DAE system. Hence, the stability region of the DAE system $A(x_s)$ is not the intersection of the stability region of the transformed system $A_{(\Sigma'')}(x_s)$ with the component Γ_s of Γ. This explains why we have to restrict the trajectories of system (Σ'') to the set Γ_s in the definition (7.21).

The previous observation regarding orbits of (Σ'') also indicates the possibility of the existence of singular points on the boundary of $A(x_s)$. Thus, understanding the structure of the singular surface is important for the characterization of the stability boundary of constrained dynamical systems.

7.4.2 Structure of the singular surface

The dynamics of the transformed system (Σ'') in the neighborhood of the singular surface suggests a decomposition of the singular surface into three disjoint sets: the pseudo-equilibrium surface, the semi-singular surface and the remaining points of S. Every equilibrium of (Σ) is also an equilibrium of (Σ''); however, in S, the vector field (Σ'') possesses an additional set of equilibrium points. Define the function

$$k(x,y) = \text{adj}(D_y g(x,y)) D_x g(x,y) f(x,y) \tag{7.22}$$

and consider the set:

$$\Psi = \{(x,y) \in S | k(x,y) = 0\}. \tag{7.23}$$

We note that $\Delta(x, y)=0$ for every point in S and hence every point in Ψ is an equilibrium point of the transformed system (7.17). These points are called *pseudo-equilibriums* and the set Ψ is called the *pseudo-equilibrium surface*. Typically, set Ψ is an $(n-2)$-dimensional submanifold embedded in S [262]. In the pseudo-equilibrium surface, the transformed vector field (7.17) is null. Moreover, we can prove that $\Gamma - M \subset \Psi$. In other words, the points of Γ where Γ cannot be locally described as a submanifold of R^{n+m} is a subset of the pseudo-equilibrium surface, which is usually a very thin subset of Γ.

Another important subset of S is the one composed of points at which the vector field is not null but tangent to S. Let us first investigate the tangent spaces of manifolds. Consider a point (x_0,y_0) in the manifold Γ, i.e. $g(x_0,y_0)=0$. Now consider a small variation $(\Delta x, \Delta y)$ such that the perturbed point $(x_0 + \Delta x, y_0 + \Delta y)$ belongs to Γ, i.e. $g(x_0 + \Delta x, y_0 + \Delta y) = 0$. Expanding this equation in a power series and truncating at the first order we obtain

$$D_x g(x_0, y_0)\Delta x + D_y g(x_0, y_0)\Delta y = 0, \tag{7.24}$$

which is equivalent to

$$\left[D_x g(x_0, y_0) \;\; D_y g(x_0, y_0)\right] \begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = 0. \tag{7.25}$$

Therefore, the vector $v = [\Delta x^T \; \Delta y^T]^T$ belongs to the tangent space of Γ at (x_0,y_0) if and only if v belongs to the kernel of the linear operator $[D_x g(x_0, y_0), D_y g(x_0, y_0)]$. In addition, the transformed vector field (7.17) belongs to the kernel of the linear operator $[D_x g(x_0, y_0), D_y g(x_0, y_0)]$ and therefore is tangent to Γ and leaves Γ invariant.

Now consider a point (x_0,y_0) in the submanifold $S \subset \Gamma$, i.e. $g(x_0,y_0) = 0$ and $\Delta(x_0, y_0) = 0$. It is straightforward to see that the tangent space of S at (x_0,y_0) is a vector subspace of the tangent space of Γ at (x_0,y_0). Thus, a vector v belongs to the tangent space of S at (x_0,y_0) if v belongs to the kernel of the following linear operators $[D_x g(x_0, y_0), D_y g(x_0, y_0)]$ and $[D_x \Delta(x_0, y_0), D_y \Delta(x_0, y_0)]$. Now, the points in the set S where the transformed vector field is tangent to S satisfy:

$$\left[D_x \Delta(x_0, y_0) \;\; D_y \Delta(x_0, y_0)\right] \begin{bmatrix} f(x_0, y_0)\Delta(x_0, y_0) \\ -k(x_0, y_0) \end{bmatrix} = 0. \tag{7.26}$$

Since $\Delta(x, y)$=0 in every point of S, the condition (7.26) is reduced to:

$$D_y \Delta(x_0, y_0) k(x_0, y_0) = 0. \tag{7.27}$$

Consequently, the following set:

$$\Xi = \{(x, y) \in S - \psi | D_y \Delta(x_0, y_0) k(x_0, y_0) = 0\} \tag{7.28}$$

is the set of points in S that are not pseudo-equilibrium points but at which the vector field is tangent to S. These points will be called *semi-singular points* and the set Ξ the *semi-singular surface*. The reason for this name is that solutions of the DAE system at semi-singular points are somewhat well defined. At these points, the solutions of the system (Σ'') intersect the singular surface tangentially and, although they are not real solutions of the DAE system (Σ), they can be continuously extended to the singular surface. The semi-singular surface is typically an $(n-2)$-dimensional manifold embedded in S.

Consider now the following set,

$$T = S - (\Psi \cup \Xi)$$

which is composed of points of the singular surface that are neither pseudo-equilibrium points nor semi-singular points. Then, the singular surface is composed of the union of three disjoint subsets:

$$S = \Psi \cup \Xi \cup T.$$

In T, the vector field (Σ'') is not null and is not tangent to S. Consequently, the vector field is necessarily transversal to T at every point (x,y) in T, which is an $(n-1)$-dimensional submanifold embedded in S [262].

Since both Ψ and Ξ are typically $(n-2)$-dimensional submanifolds embedded in S, T is usually dense in S. Hence, S is typically an $(n-1)$-dimensional submanifold embedded in Γ. Usually, set T possesses several connected $(n-1)$-dimensional components in S that are separated by (n−2)-dimensional components of either the pseudo-surface or the semi-singular surface.

The following examples illustrate the concepts of singular surface, pseudo-equilibrium surface and semi-singular surface.

Example 7-3 Consider the following system of differential-algebraic equations:

$$\begin{cases} \dot{x}_1 = -2x_1 + 1 + y^2 \\ \dot{x}_2 = -x_2 \\ 0 = y^2 - x_1. \end{cases} \tag{7.29}$$

The constraint set Γ is given by:

$$\Gamma = \{(x_1, x_2, y) \in R^3 \,|\, 0 = y^2 - x_1\}.$$

Calculating the derivative of the algebraic equation, one obtains $[D_x g \; D_y g] = [-1 \; 0 \; 2y]$, which has complete rank independently of y. Thus $\Gamma = M$ is a two-dimensional submanifold embedded in R^3. Figure 7.8 illustrates the constraint manifold Γ. The derivative of the algebraic equation with respect to the instantaneous variable y is given by $D_y g = 2y$. Therefore $\Delta(x_1, x_2, y) = 2y$ equals zero if and only if $y = 0$. Consequently, the singular surface is given by:

$$S = \{(x_1, x_2, y) \in R^3 | x_1 = 0, y = 0\}.$$

The singular surface is depicted in Figure 7.8. It splits the constraint set Γ into two connected components Γ_s and Γ_u, where

$$\Gamma_s = \{(x_1, x_2, y) \in R^3 \,|\, 0 = y^2 - x_1, y > 0\}$$

$$\Gamma_u = \{(x_1, x_2, y) \in R^3\} \,|\, 0 = y^2 - x_1, y < 0,$$

such that $\Gamma - S = \Gamma_s \cup \Gamma_u$. The set Γ_s contains an asymptotically equilibrium point $x_s = (1,0,1)$ of (7.29).

By implicit differentiation, we calculate:

$$\dot{y} = \frac{-2x_1 + 1 + y^2}{2y}.$$

Then, the system (Σ') assumes the form:

$$(\Sigma') \begin{cases} \dot{x}_1 = -2x_1 + 1 + y^2 \\ \dot{x}_2 = -x_2 \\ \dot{y} = \dfrac{-2x_1 + 1 + y^2}{2y}. \end{cases}$$

Multiplying the vector field by $\Delta(x_1, x_2, y) = 2y$, one obtains the transformed system:

$$(\Sigma'') \begin{cases} \dot{x}_1 = (-2x_1 + 1 + y^2)2y \\ \dot{x}_2 = (-x_2)2y \\ \dot{y} = -2x_1 + 1 + y^2. \end{cases}$$

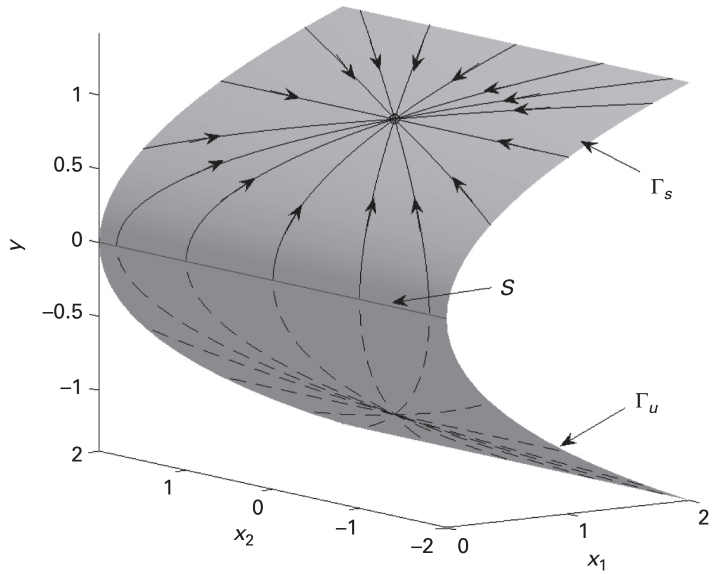

Figure 7.8 The phase portrait of system (7.29). The singular surface is composed of points of the set T. Trajectories of the transformed system (Σ'') are transversal to T.

In the pseudo-equilibrium surface, the following conditions must be satisfied: $x_1 = 0$, $y = 0$ and $-2x_1+1+y^2 = 0$, therefore, the pseudo-equilibrium surface is an empty set. The semi-singular surface is also an empty set because the following conditions must be satisfied simultaneously in this surface: $x_1 = 0$, $y = 0$ and $D_y\Delta(x, y)k(x, y) = 2(-2x_1 + 1+y^2) = 0$. As a consequence, $S = T$ is a one-dimensional submanifold embedded in Γ. Figure 7.8 illustrates the vector field (Σ'') transversal to T in Γ. Since $\Delta(x_1, x_2, y) > 0$ in Γ_s, systems (Σ) and (Σ'') are topologically equivalent in Γ_s. Figure 7.8 illustrates the orbits of (Σ'') in Γ.

Example 7-4 Consider now the following system of differential-algebraic equations, which is a slight variation of the previous example:

$$\begin{cases} \dot{x}_1 = -2x_1 + x_2 + 1 + y^2 \\ \dot{x}_2 = -x_2 \\ 0 \ = y^2 - x_1. \end{cases} \tag{7.30}$$

The algebraic equation is equal to that of the previous example, therefore the constraint set Γ and the singular surface are equal to those of the previous example and are given by:

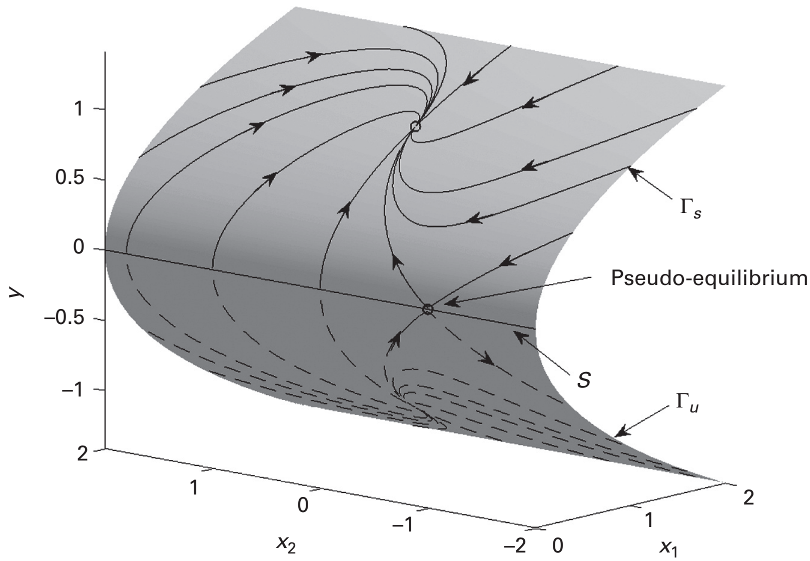

Figure 7.9 The phase portrait of system (7.30). A pseudo-saddle equilibrium point appears in the singular surface, which is composed by two connected components of set T separated by the pseudo-saddle.

$$\Gamma = \{(x_1, x_2, y) \in R^3\}|0 = y^2 - x_1\}$$
$$S = \{(x_1, x_2, y) \in R^3 | x_1 = 0, y = 0\}.$$

Figure 7.9 illustrates the constraint manifold Γ and the singular surface S. Again, the singular surface splits the constraint set Γ into two connected components Γ_s and Γ_u, where

$$\Gamma_s = \{(x_1, x_2, y) \in R^3 \big| 0 = y^2 - x_1, y > 0\}$$
$$\Gamma_u = \{(x_1, x_2, y) \in R^3 \big| 0 = y^2 - x_1, y < 0\},$$

such that $\Gamma - S = \Gamma_s \cup \Gamma_\mathrm{u}$. The set Γ_s contains an asymptotically equilibrium point x_s=(1, 0, 1) of (7.30).

By implicit differentiation, we calculate:

$$\dot{y} = \frac{-2x_1 + x_2 + 1 + y^2}{2y}.$$

Then, the system (Σ') assumes the form:

$$(\Sigma') \begin{cases} \dot{x}_1 = -2x_1 + 1 + y^2 \\ \dot{x}_2 = -x_2 \\ \dot{y} \; = \dfrac{-2x_1 + x_2 + 1 + y_2}{2y}. \end{cases}$$

Multiplying the vector field by $\Delta(x_1, x_2, y) = 2y$, one obtains the transformed system:

$$(\Sigma'') \begin{cases} \dot{x}_1 = (-2x_1 + 1 + y^2)2y \\ \dot{x}_2 = (-x_2)2y \\ \dot{y} \; = -2x_1 + x_2 + 1 + y^2. \end{cases}$$

In the pseudo-equilibrium surface, the following conditions must be satisfied: $x_1 = 0$, $y = 0$ and $-2x_1 + x_2 + 1 + y^2 = 0$, therefore, the pseudo-equilibrium surface is the set

$$\Psi = \{(0, -1, 0)\}.$$

The pseudo-equilibrium point (0,−1,0) is a type-one hyperbolic equilibrium point of system (Σ'') on the constraint manifold Γ. This pseudo-equilibrium point and the dynamics of system (Σ'') on Γ are depicted in Figure 7.9.

The semi-singular surface is empty because the following conditions must be satisfied on this surface: $x_1 = 0$, $y = 0$ and $D_y\Delta(x, y)k(x, y) = 2(-2x_1 + x_2 + 1 + y_2) = 0$. The only point that satisfies these conditions is the pseudo-equilibrium point. Consequently, there is no point in the set $S - \Psi$ that satisfies these conditions and the set of semi-singular points is empty. Set $T = S - \Psi$ is a one-dimensional submanifold embedded in Γ. The pseudo-equilibrium splits T into two connected components. Figure 7.9 illustrates the vector field (Σ'') transversal to T in Γ. Since $\Delta(x_1, x_2, y) > 0$ in Γ_s, systems (Σ) and (Σ'') are topologically equivalent in Γ_s. Figure 7.9 illustrates the orbits of (Σ'') in Γ.

Example 7-5 Consider the following system of differential-algebraic equations:

$$\begin{cases} \dot{x}_1 &= 1 - x_1 \\ \dot{x}_2 &= 2 - x_2 \\ 0 &= y^3 + x_1 y - x_2. \end{cases} \tag{7.31}$$

The constraint set Γ is given by:

$$\Gamma = \{(x_1, x_2, y) \in R^3 \,|\, 0 = y^3 + x_1 y - x_2\}.$$

Calculating the derivative of the algebraic equation, one obtains $[D_x g \; D_y g] = [y - 1 \; 3y^2 + x_1]$, which has complete rank independently of (x_1, x_2, y). Thus $\Gamma = M$ is a two-dimensional submanifold embedded in R^3. Figure 7.10 illustrates the constraint manifold Γ. The derivative of the algebraic equation with respect to the instantaneous variable y is given by $D_y g = 3y^2 + x_1$. Therefore $\Delta(x_1, x_2, y) = 3y^2 + x_1$. On the singular surface, $\Delta(x_1, x_2, y) = 0$, which is equivalent to $x_1 = -3y^2$. Substituting this condition into the algebraic constraint, we conclude that the singular surface is given by:

$$S = \{(x_1, x_2, y) \in R^3 \,|\, x_1 = -3y^2, x_2 = -2y^3, y \in R\}.$$

The singular surface is the thick line depicted in Figure 7.10. It splits the constraint set Γ into two connected components Γ_s and Γ_u. The set Γ_s contains an asymptotically equilibrium point x_s=(1, 2, 1) of (7.31).

By implicit differentiation, we calculate:

$$\dot{y} = \frac{-x_2 + 2 - y + x_1 y}{(x_1 + 3y^2)}.$$

Then, the system (Σ') assumes the form:

$$(\Sigma') \begin{cases} \dot{x}_1 = 1 - x_1 \\ \dot{x}_2 = 2 - x_2 \\ \dot{y} \; = \dfrac{-x_2 + 2 - y + x_1 y}{(x_1 + 3y^2)}. \end{cases}$$

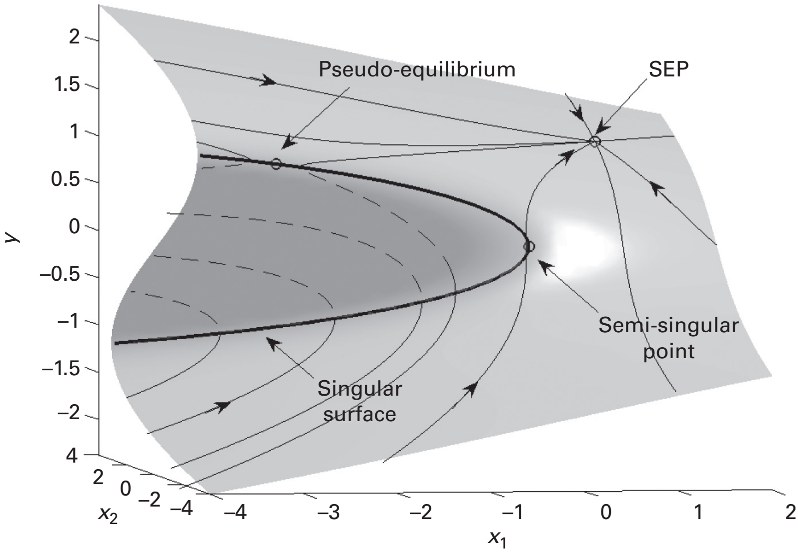

Figure 7.10 Phase portrait of system (7.31).

Multiplying the vector field by $\Delta(x_1, x_2, y) = (x_1 + 3y^2)$, one obtains the transformed system:

$$(\Sigma'') \begin{cases} \dot{x}_1 = (1 - x_1)(x_1 + 3y^2) \\ \dot{x}_2 = (2 - x_2)(x_1 + 3y^2) \\ \dot{y} = -x_2 + 2 - y + x_1 y. \end{cases}$$

On the pseudo-equilibrium surface, the following conditions must be satisfied: x_1=$-3y^2$, x_2=$-2y^3$ and $-x_2$+2$-y$+x_1y=0. Substituting the first two equations in the third equation, variables x_1 and x_2 are eliminated and we obtain the following equation: $-y^3$$-y$+2=0, which has a unique solution y=1. Therefore, the point (−3, −2, 1) is a pseudo-equilibrium point and the pseudo-equilibrium surface is the set $\Psi = \{(-3, -2, 1)\}$. In the semi-singular surface, the following conditions must be satisfied: x_1=$-3y^2$, x_2=$-2y^3$ and $D_y\Delta(x, y)k(x, y) = -6y(-x_2 + 2 - y + x_1y) = 0$. Thus the origin is a semi-singular point. Figure 7.10 illustrates the phase portrait of the vector field (Σ'') in Γ. Since $\Delta(x_1, x_2, y) > 0$ in Γ_s, systems (Σ) and (Σ'') are topologically equivalent in Γ_s. A trajectory of (Σ'') touches the semi-singular point tangent to the singular surface S.

7.4.3 Dynamics near the singular surface

The behavior of the DAE system in the neighborhood of the semi-singular surface is illustrated by a three-dimensional example in Figure 7.10 and also in the diagram of Figure 7.11.

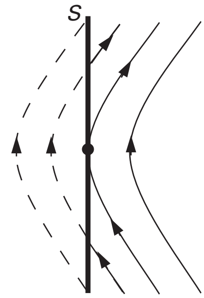

Figure 7.11 The dynamical behavior of the constrained system (Σ) in the neighborhood of the semi-singular surface. Trajectories tangentially touch the singular surphace at the semi-singular point. On the right side of S, the dynamical behavior in the neighborhood of the semi-singular point resembles the dynamics of a saddle-node equilibrium point. On the left side, the dynamical behavior resembles the dynamics of a focus.

Let us call Γ_s the component of interest (usually the component that contains the asymptotically stable equilibrium point of interest) and suppose, without loss of generality, that $\Delta(x, y) > 0$ in Γ_s. The singular surface S and in particular the semi-singular surface Ξ are on the boundary of Γ_s. In each side of the singular surface, the dynamical behavior is different. On one side, the dynamics resembles the dynamics of a saddle-node, while on the other side, the dynamics resembles the dynamics of a focus. Depending on the case, the behavior of the dynamics in Γ_s will be equivalent to one of these. This observation suggests the following subdivision of the semi-singular surface [262]:

$$\Xi'_{sa} = \{(x,y) \in \Xi | D_y\{(D_y\Delta)k\}k > 0\} \tag{7.32}$$

$$\Xi'_{fo} = \{(x,y) \in \Xi' | D_y\{(Dy\Delta)k\}k > 0\}. \tag{7.33}$$

In the neighborhood of the semi-focus surface Ξ'_{fo}, orbits which are born in the singular surface S circle around the semi-singular set dying at the singular surface in finite time, see Figure 7.11. Consequently, the semi-focus surface Ξ'_{fo} cannot lie on the stability boundary of any asymptotically stable equilibrium point in Γ_s. Due to the lack of importance of the semi-focus surface in the stability boundary characterization, this surface will not be analyzed in detail.

If $(x,y) \in \Xi'_{sa}$, then there exists a solution of the DAE system (7.1) in the component Γ_s that reaches the point $(x,y) \in \Xi'_{sa}$ in a finite time and immediately leaves the point, returning to the same component Γ_s as illustrated in Figure 7.11. These points in the semi-singular surface will be called semi-saddles.

7.4.4 Dynamics near pseudo-saddles

Pseudo-equilibriums are real equilibrium points of the transformed system (Σ''). Every point in Ψ is actually a non-hyperbolic equilibrium point of the transformed system (Σ'')

[262] with, at maximum, two eigenvalues different from zero. The center manifold of these non-hyperbolic equilibrium points is the set Ψ itself, which possesses dimension $n-2$ [262]. Depending on the signs of the nonzero eigenvalues, we have different types of behavior in the neighborhood of Ψ. We will divide set Ψ into three disjoint subsets according to the stability type of the pseudo-equilibrium points. To this end, define:

$$\Psi'_{so} = \{(x,y) \in \Psi' \,|\, \text{the Jacobian of } (\Sigma'') \text{ has two eigenvalues in } C^{+}\}, \tag{7.34}$$

$$\Psi'_{sa} = \{(x,y) \in \Psi' \,|\, \text{the Jacobian of } (\Sigma'') \text{ has one eigenvalue in } C^{+} \text{ and one in } C^{-}\}, \tag{7.35}$$

$$\Psi'_{si} = \{(x,y) \in \Psi' \,|\, \text{the Jacobian of } (\Sigma'') \text{ has two eigenvalues in } C^{-}\}. \tag{7.36}$$

The points in Ψ'_{so}, Ψ'_{sa} and Ψ'_{si} will be called respectively pseudo-sources, pseudo-saddles and pseudo-sinks.

The dynamical behavior of trajectories in the neighborhood of pseudo-sinks indicates that they cannot lie on the stability boundary of any asymptotically stable equilibrium point. Even though pseudo-sources may exist on the stability boundary, since $W^s(\Psi'_{so}) - \Psi'_{so}$ is an empty set, they are not important in the characterization of stability boundaries. Pseudo-saddles are relevant to the characterization of stability boundaries and will be studied in some detail.

The behavior of the DAE system in the neighborhood of a pseudo-saddle can be represented in a convenient coordinate system as sketched in Figure 7.12.

A pseudo-saddle is called *transverse* if neither the stable nor the unstable manifold is tangent to the singular surface S. The set of transverse pseudo-saddles will be denoted Ψ'_{trsa}. Transverse saddles are generic in the set Ψ'_{sa}. The stable and unstable manifolds of transverse pseudo-saddles are one dimensional and $W^s(\Psi'_{trsa})$ and $W^u(\Psi'_{trsa})$ are $(n-1)$-dimensional manifolds [262]. (Figure 7.12 illustrates these manifolds.)

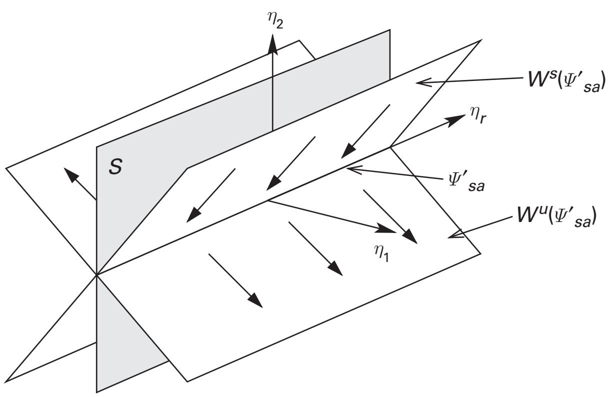

Figure 7.12 A sketch of the dynamical behavior of a system (Σ) in the neighborhood of a pseudo-saddle surface.

7.5 Stable and unstable manifolds

In this section, we will study the properties of stable and unstable manifolds of hyperbolic equilibrium points, semi-singular points and pseudo-equilibrium points. It will be shown that the stable and unstable manifolds of these points can be written in terms of trajectories of both systems: the DAE system (Σ) and the transformed system (Σ''). As a consequence, the characterization of the stability boundary of the DAE system (Σ) can be represented in terms of the stable and unstable manifolds of certain critical points of the transformed system (Σ'').

We will consider the following generic assumption for the DAE system (Σ).

(A0) Equilibrium points do not lie on the singular set S.

We will also assume, without loss of generality, that $\Delta(x, y) > 0$ in the component Γ_s of Γ that contains the asymptotically stable equilibrium point of interest.

For a hyperbolic equilibrium point $z_0 = (x_0, y_0) \in \Gamma_s$ of the DAE system (Σ), the stable and unstable manifolds are respectively defined as:

$$W^s(z_0) = \{z \in \Gamma_s | \varphi(t,z) \in \Gamma_s \text{ for all } t \geq 0 \text{ and } \varphi(t,z) \to z_0 \text{ as } t \to +\infty\} \tag{7.37}$$

$$W^u(z_0) = \{z \in \Gamma_s | \varphi(t,z) \in \Gamma_s \text{ for all } t \leq 0 \text{ and } \varphi(t,z) \to z_0 \text{ as } t \to -\infty\}. \tag{7.38}$$

The stable and unstable manifolds are subsets of the component Γ_s of the constraint set Γ. Points on the singular set will not be considered to lie on the stable or unstable manifolds of any hyperbolic equilibrium point in Γ_s. If $z_0 = (x_0, y_0) \in \Gamma_s$ is a type-k hyperbolic equilibrium point of the DAE system (Σ), then $W^u(z_0)$ is a k-dimensional manifold while $W^s(z_0)$ is an $(n-k)$-dimensional manifold. These manifolds intersect transversally at the equilibrium z_0.

We can also define the stable and unstable manifolds of pseudo-equilibrium points and of semi-singular points. However, these definitions are different from the definition of invariant manifolds of hyperbolic equilibrium points because solutions of the DAE system (Σ) reach these points in finite time. More precisely, if $z_0 = (x_0, y_0) \in S$ is either a pseudo-equilibrium point or a semi-singular point, then the stable and unstable manifolds are respectively defined as:

$$W^s(z_0) = \{z \in \Gamma_s | \exists t_0 > 0 \text{ such that } \varphi(t,z) \in \Gamma_s \text{ for } 0 \leq t \leq t_0 \text{ and } \varphi(t,z) \to z_0 \text{ as } t \to t_0\} \tag{7.39}$$

$$W^u(z_0) = \{z \in \Gamma_s | \exists t_0 < 0 \text{ such that } \varphi(t,z) \in \Gamma_s \text{ for } 0 \geq t \geq t_0 \text{ and } \varphi(t,z) \to z_0 \text{ as } t \to t_0\}. \tag{7.40}$$

Since the DAE system (Σ) and the transformed system (Σ'') have the same set of trajectories in Γ_s, we can define the stable and unstable manifolds of hyperbolic equilibrium points, pseudo-equilibrium points and semi-singular points of the DAE system (Σ) in terms of trajectories of the transformed system (Σ'').

If $z_0 = (x_0, y_0) \in \Gamma_s$ is a hyperbolic equilibrium point of the DAE system (Σ), then it is also a hyperbolic equilibrium point of the transformed system (Σ'') and the stable and unstable manifolds of z_0 of the DAE system (7.1) are respectively defined as:

$$W^s(z_0) = \{z \in \Gamma_s \big| \varphi_{(\Sigma'')}(t,z) \in \Gamma_s \text{ for all } t \geq 0 \text{ and } \varphi_{(\Sigma'')}(t,z) \to z_0 \text{ as } t \to +\infty\} \quad (7.41)$$

$$W^u(z_0) = \{z \in \Gamma_s \big| \varphi_{(\Sigma'')}(t,z) \in \Gamma_s \text{ for all } t \leq 0 \text{ and } \varphi_{(\Sigma'')}(t,z) \to z_0 \text{ as } t \to -\infty\}. \quad (7.42)$$

Observe that (7.41) and (7.37) are two representations of the same set. The same is true for (7.42) and (7.38). Note that the stable and unstable manifolds of the transformed system (Σ''), respectively denoted $W^s{}_{(\Sigma'')}$ (z_0) and $W^u{}_{(\Sigma'')}$ (z_0), are not necessarily restricted to Γ_s. Moreover, $W^s{}_{(\Sigma'')}$ $(z_0) \cap \Gamma_s$ is not necessarily equal to $W^s(z_0)$. Trajectories of the transformed system (Σ'') might leave the component Γ_s, by crossing the singular surface, and return to this component later on.

Since pseudo-equilibrium points are equilibrium points of the transformed system (Σ''), the stable and unstable manifolds of a pseudo-equilibrium z_0 are well defined for the transformed system (Σ''). Restricting the trajectories to the component Γ_s, we can define the stable and unstable manifolds of a pseudo-equilibrium z_0 for the DAE system (7.1) in terms of trajectories of the transformed system (Σ''). More precisely, the stable and unstable manifolds of a pseudo-equilibrium z_0 are also respectively defined by (7.41) and (7.42). Observe that trajectories of the DAE system (Σ) reach pseudo-equilibrium points in finite time while the trajectories of the transformed system (Σ'') approach them asymptotically as time tends to infinity. Observe that for a pseudo-equilibrium point z_0, (7.41) and (7.39) represent the same set. It is also important to note that just a subset of the stable and unstable manifolds of the transformed system contains the stable and unstable manifolds of the DAE system (7.1). Figure 7.13 illustrates these stable manifolds of the pseudo-saddle of system (7.30).

Semi-singular points are regular points of the transformed system (Σ''), therefore trajectories of the transformed system also reach these points in finite time. If $z_0 = (x_0, y_0) \in S$ is a semi-singular point, then the stable and unstable manifolds of the DAE system (7.1) are respectively defined as:

$$\begin{aligned} W^s(z_0) = \{z \in \Gamma_s \big| \exists t_0 > 0 \text{ such that } & \varphi_{(\Sigma'')}(t,z) \in \Gamma_s \text{ for } 0 \leq t \leq t_0 \\ & \text{and } \varphi_{(\Sigma'')}(t,z) \to z_0 \text{ as } t \to t_0\} \end{aligned} \quad (7.43)$$

$$\begin{aligned} W^u(z_0) = \{z \in \Gamma_s \big| \exists t_0 > 0 \text{ such that } & \varphi_{(\Sigma'')}(t,z) \in \Gamma_s \text{ for } 0 \leq t \leq t_0 \\ & \text{and } \varphi_{(\Sigma'')}(t,z) \to z_0 \text{ as } t \to t_0\}. \end{aligned} \quad (7.44)$$

Observe that, for a semi-singular point, (7.43) and (7.39) represent the same set. The same is true for (7.44) and (7.40). Now consider the set of semi-saddle-points Ξ'_{sa}, which is an $(n-2)$-dimensional manifold. We define the stable manifold

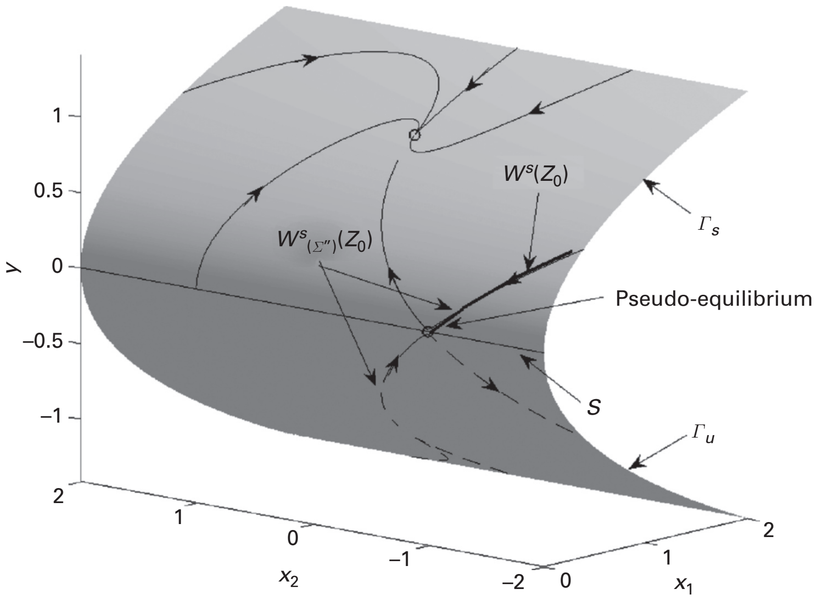

Figure 7.13 The stable manifold of a transverse pseudo-saddle.

$$W^s(\Xi'_{sa}) = \bigcup_{z_0 \in \Xi'_{sa}} W^s(z_0)$$

and the unstable manifold

$$W^u(\Xi'_{sa}) = \bigcup_{z_0 \in \Xi'_{sa}} W^u(z_0)$$

of the semi-saddle surface as the union of the manifolds of the semi-saddle-points. It is clear from the analysis of dynamics in the neighborhood of a semi-saddle that both $W^s(\Xi'_{sa})$ and $W^u(\Xi'_{sa})$ are $(n-1)$-dimensional manifolds embedded in the component Γ_s. Figure 7.12 illustrates these manifolds.

7.6 Generalized critical points on the stability boundary

The approach to developing a characterization of the stability boundary of DAE systems is similar to the approach that was employed to develop a characterization of the stability boundary of ordinary differential equations in Chapter 4. More precisely, we start from a local characterization of the stability boundary and then we extend it to a global one. The local characterization of the stability boundary is developed in this section while the global characterization is developed in the following section.

We will call equilibrium points, pseudo-equilibrium points and semi-singular points on the stability boundary generalized critical points because they play a key role in the stability boundary characterization. Without imposing any condition on the vector field, we will investigate these critical points on the stability boundary. In particular, we will derive characterizations of these critical points in the stability boundary in terms of their stable and unstable manifolds. Additional conditions on the vector field are then imposed in order to obtain sharper results on the characterizations of these critical points on the stability boundary. These characterizations will be derived in the next section.

THEOREM 7-7 **(Characterization of critical points on the stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable equilibrium point of the DAE system (7.1) in the component Γ_s of the constraint set Γ.
For a hyperbolic equilibrium point $z \in \Gamma_s$

$$z \in \partial A(x_s) \Leftrightarrow \{W^u(z) - \{z\}\} \cap \overline{A(x_s)} \neq \varnothing \tag{7.45}$$

and if z is not a source

$$z \in \partial A(x_s) \Leftrightarrow \{W^s(z) - \{z\}\} \cap \partial A(x_s) \neq \varnothing. \tag{7.46}$$

For a transverse pseudo-saddle or a semi-saddle v:

$$v \in \partial A(x_s) \Leftrightarrow W^u(v) \cap \overline{A(x_s)} \neq \varnothing. \tag{7.47}$$

Theorem 7-7 offers conditions to check whether a critical point lies on the stability boundary in terms of the properties of their unstable manifolds. More precisely, if the unstable manifold of a critical point intersects with the stability region, then the critical point lies on the stability boundary. Points in the surface T can also lie on the stability boundary. The following theorem offers a practical condition to check whether a point of T lies on the stability boundary.

THEOREM 7-8 **(Points of surface on the stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable equilibrium point of the DAE system (7.1) in the component Γ_s of the constraint set Γ. A point $v \in T$ lies on $\partial A(x_s)$ if and only if there exists a time $t_0 > 0$ such that $\varphi_{(\Sigma'')}(t, v) \cap A(x_s) \neq \varnothing$ for all $t \in (0, t_0]$.

Theorem 7-8 offers a means of checking whether a point of the surface T lies on the stability boundary. It asserts that it is sufficient to check whether the trajectory of the transformed system (Σ'') starting from this point enters the stability region for a sufficiently small time.

7.7 Characterization of the stability boundary of DAE systems

Characterizations of the stability boundary $\partial A(x_s, y_s)$ of DAE systems have recently been developed. It has been shown that under certain conditions, the stability boundary $\partial A(x_s, y_s)$ consists of two parts: the first part is the stable manifolds of the equilibrium points on the stability boundary while the second part contains points whose trajectories reach singular surfaces [262]. The second part can be further delineated as a union of the stable manifolds of pseudo-equilibrium points and semi-singular points on the stability boundary and parts of the singular surface [262,264].

A complete characterization of the stability boundary $\partial A(x_s,)$ of the DAE system (Σ) will be developed in this section. It will be shown that the stability boundaries of constrained dynamical systems are composed of stable manifolds of generalized critical points on the stability boundary and pieces of the singular surface. To this end, consider the following assumptions.

(A1) Every equilibrium point on the stability boundary $\partial A(x_s)$ is hyperbolic and, except for a set of dimension $n-3$, every pseudo-saddle in $\partial A(x_s,)$ is transverse.
(A2) Stable manifolds of equilibrium points and of connected components of Ψ, Ξ, intersect transversally with unstable manifolds of the same elements.
(A3) Every trajectory in $\bar{A}$ converges to an equilibrium point, a pseudo-equilibrium point or a semi-singular point.

Assumptions (A1) and (A2) are generic while assumption (A3) is not generic. The existence of an energy function for the DAE system (Σ) is a sufficient condition for the satisfaction of assumption (A3).

Assumption (A1) guarantees that every equilibrium point on the stability boundary is isolated. Thus there is a countable collection of equilibrium points z_i, i=1,2, . . ., on the stability boundary. Pseudo-equilibrium points and semi-singular points are not isolated. The connected components of pseudo-equilibrium points will be denoted Ψ'_j, j=1,2, ... The connected components of semi-singular points will be denoted Ξ'_l, l=1,2, ...

The stability boundary has maximal dimension $n-1$ and the quasi-stability boundary possesses a dense ($n-1$)-dimensional set. Connected components of the set T have dimension $n-1$ and therefore they are important pieces in the characterization of the stability boundary. Generalized critical elements that possess ($n-1$)-dimensional stable manifolds also have an important contribution to the characterization of the stability boundary. They are type-one hyperbolic equilibrium points, ($n-2$)-dimensional connected components of transverse pseudo-saddles (a component of the set Ψ'_{trsa}) and ($n-2$)-dimensional connected components of semi-saddles (a component of the set Ξ'_{sa}).

Next we will study further local characterization of these critical points on the quasi-stability boundary.

THEOREM 7-9 **(Equilibrium points on the quasi-stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable equilibrium point of the DAE system (7.1) in the component Γ_s of the constraint set Γ and $z \in \Gamma_s$, with $z \neq x_s$, be a hyperbolic type-one equilibrium point. If assumptions (A0)−(A3) are satisfied, then

$$z \in \partial\overline{A(x_s)} \Leftrightarrow W^u(z) \cap A(x_s) \neq \varnothing \quad and \quad \mathrm{W}^u(z) \cap \overline{A(x_s)}^c \neq \varnothing \tag{7.48}$$

$$z \in \partial\overline{A(x_s)} \Leftrightarrow W^s(z) \subset \partial\overline{A(x_s)}. \tag{7.49}$$

Now let us consider transverse pseudo-saddles on the stability boundary. Isolated transverse pseudo-saddles do not have an important contribution to the characterization of the stability boundary for systems with $n>2$. However, an ($n-2$)-dimensional component of Ψ'_{trsa}, composed of continuous transverse pseudo-saddles, has an important contribution to the characterization of the stability boundary. Consider an $(n-2)$-dimensional connected component Ψ'_j of the set of pseudo-equilibrium points such that $\Psi'_j \subset \Psi'_{trsa}$. This component might intersect with the stability boundary but it might not be entirely contained on the stability boundary. Thus, we consider the subset

$$N\Psi'_j = \{v \in \Psi'_j \cap \partial\overline{A(x_s)} \,\Big|\, \exists \text{ a } \delta\text{-neighborhood } \delta(v) \text{ in } \Psi'_j \text{ such that } \delta(v) \subset \partial\overline{A(x_s)}\} \tag{7.50}$$

of points of Ψ'_j that are entirely contained on the stability boundary. The following theorem offers necessary and sufficient conditions to guarantee that a pseudo-transverse saddle belongs to $N\Psi'_j$.

THEOREM 7-10 **(Transverse pseudo-saddles on the quasi-stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable equilibrium point of the DAE system (7.1) in the component Γ_s of the constraint set Γ and $v \in \Psi'_j$, with $\Psi'_j \subset \Psi'_{trsa}$ an $(n-2)$-dimensional component of pseudo-equilibrium points composed exclusively of transverse pseudo-saddles. If assumptions (A0) − (A3) are satisfied, then $v \in N\Psi'_j$ if and only if there exists δ_0 such that $W^u(\delta(v)) \cap A(x_s)$ is dense in $W^u(v)$ for every δ-neighborhood $\delta(v)$ of v in Ψ'_j with $\delta < \delta_0$. Moreover, $v \in N\Psi'_j$ if and only if $W^s(v) \subset \partial\overline{A(x_s)}$.

A similar result can be stated for semi-singular points. We have already observed that semi-foci cannot belong to the stability boundary of any asymptotically stable equilibrium point. However, sets of semi-saddles have an important contribution to the characterization of stability boundaries. Again, consider an $(n-2)$-dimensional connected component Ξ'_l of the semi-singular surface such that $\Xi'_l \subset \Xi'_{sa}$. This component might intersect with the stability boundary but it might not be entirely contained on the stability boundary. Thus, we consider the subset

$$N\Xi'_l = \{v \in \Xi'_l \cap \partial\overline{A(xs)} \exists \,\Big|\, \text{a } \delta\text{-neighborhood } \delta(v) \textit{ in } \Xi'_l \text{ such that } \delta(v) \subset \partial\overline{A(x_s)}\} \tag{7.51}$$

of points of Ξ'_l that are contained on the stability boundary. The following theorem offers necessary and sufficient conditions to guarantee that a semi-saddle belongs to $N\Xi'_l$.

THEOREM 7-11 **(Semi-saddles on the quasi-stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable equilibrium point of the DAE system (7.1) in the component Γ_s of the constraint set Γ and $v \in \Xi'_l$, with $\Xi'_l \subset \Xi'_{sa}$ an $(n-2)$-dimensional component of the semi-singular surface composed exclusively of semi-saddles. If assumptions (A0)–(A3) are satisfied, then $v \in N\Xi'_l$ if and only if there exists δ_0 such that $W^u(\delta(v)) \cap A(x_s)$ is dense in $W^u(v)$ for every δ-neighborhood $\delta(v)$ of v in Ξ'_l with $\delta < \delta_0$. Moreover, $v \in N\Xi'_l$ if and only if $W^s(v) \subset \partial A(x_s)$.

Now we are in a position to establish a complete characterization of the quasi-stability boundary of constrained dynamical systems.

THEOREM 7-12 **(Characterization of the quasi-stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable equilibrium point x_s in the component Γ_s of Γ. Under assumptions (A0)–(A3), the boundary of the closure of the stability region $\partial\overline{A(x_s)}$ is composed of:

(a) the stable manifolds of type-one hyperbolic equilibrium points;
(b) stable manifolds of transverse pseudo-saddles;
(c) stable manifolds of semi-saddles;
(d) pieces of singular points.

More precisely, if z_i, $i=1,2,\ldots$, are the type-one hyperbolic equilibrium points on the stability boundary, $N\Psi'_j$, $j=1,2,\ldots$, are the connected components of the set of transverse pseudo-equilibrium points on the stability boundary and $N\Xi'_l$, $l=1,2,\ldots$, are the connected components of the set of semi-saddles on the stability boundary, then:

$$\partial\overline{A(x_s)} = \overline{\bigcup_i W^s(z_i) \bigcup_i W^s(N\Psi'_j) \bigcup_l W^s(N\Xi'_l) \cup (S \cap \partial\overline{A(x_s)})}. \tag{7.52}$$

Proof Assumption (A3) guarantees that

$$\partial\overline{A(x_s)} \subseteq \overline{\bigcup_i W^s(z_i) \bigcup_i W^s(N\Psi'_j) \bigcup_l W^s(N\Xi'_l) \cup (S \cap \partial\overline{A(x_s)})}.$$

Theorem 7-9, Theorem 7-10 and Theorem 7-11 prove the other inclusion and the theorem is proven.

Example 7-6 Consider again the DAE system (7.31). This system possesses an asymptotically stable equilibrium point $(x_s, y_s) = (1, 2, 1)$. The stability boundary of this equilibrium point is illustrated in Figure 7.14. The stability boundary is composed of three main pieces, the stable manifold of a pseudo-saddle, the stable manifold of a semi-saddle and a piece of set T on the singular surface.

7.8 Concluding remarks

A comprehensive theory of stability regions and of stability boundaries for constrained continuous dynamical systems has been developed in this chapter. Two approaches have been used in the development of a complete characterization for both the stability boundary and the stability regions of constrained nonlinear dynamical systems. The first approach explores an approximation of the stability region via the singular perturbation theory while the second approach is based on a regularization of the vector field on the singular surface.

It has been shown that under certain conditions, the stability boundary of DAE systems is composed of two parts: the first part is the stable manifolds of the equilibrium

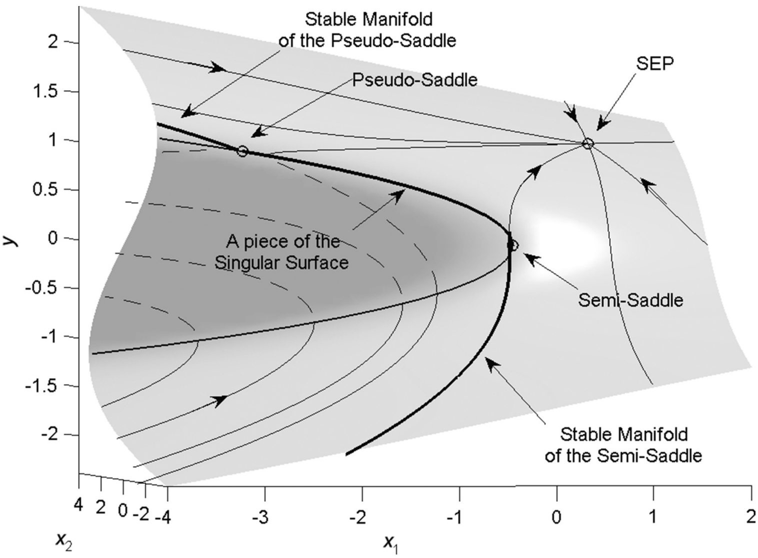

Figure 7.14 The stability boundary of the asymptotically stable equilibrium point of the DAE system (7.31). The stability boundary, the thick black curve, is composed of three main pieces: the stable manifold of the pseudo-saddle, a piece of the set T in the singular surface and the stable manifold of a semi-saddle. Points of the constraint manifold to the right of this curve belong to the stability region of the SEP.

points on the stability boundary while the second part contains points whose trajectories reach singular surfaces. The second part can be further delineated as a union of the stable manifolds of pseudo-equilibrium points and semi-singular points on the stability boundary and parts of the singular surface.