9 Stability regions of discrete dynamical systems

A comprehensive theory of stability regions and of stability boundaries for continuous dynamical systems was presented in Chapters 4, 5 and 6. In this chapter, a comprehensive theory for both the stability boundary and the stability regions of nonlinear discrete dynamical systems will be developed.

The analytical results to be presented in this chapter can be viewed as the counterpart of the results derived in Chapter 4 for discrete systems. Although the theory of stability regions of discrete systems is parallel to the theory of continuous dynamical systems, discrete dynamical systems do possess some peculiarities. One fundamental difference is the connectivity of trajectories of continuous systems that do not exist in discrete systems. Another important difference is that backward trajectories are not defined or are not unique for many discrete dynamical systems. These peculiarities pose some challenges for the development of a theory of stability regions for discrete dynamical systems and justify an entire chapter devoted to this subject.

9.1 Introduction

Nonlinear discrete dynamical systems have been used to model a variety of practical nonlinear systems. Stability analysis and stability region characterizations are essential in many of these applications. For instance, the dynamics of power systems with LTCs (on Load Tap Changers) are modeled by difference equations [269], iterated-map neural networks are another example of discrete-time dynamical systems [174], discrete-time dynamical systems are used to study the dynamics of ecosystems [272] and economic models [283].

Discrete dynamical models are important in the stability analysis of sample-data systems. These models typically appear in the analysis of systems that are controlled by a computer [155]. Although the majority of physical systems have variables that continuously evolve in time, the employment of digital computers to simulate, control and interact with continuous systems has stimulated the development of stability theory for discrete systems [14,130,157,159]. Discrete dynamical models also appear in the analysis of systems that cannot be continuously measured. An application of discrete-time approximate models in the analysis of HIV dynamics and treatment schedules is presented in [87]. Although the physical variables of this system evolve continuously in time, measurements, blood examinations, are made periodically at discrete times. Also, it is impractical to employ continuous varying doses of medication during the treatment. Hence a discrete model is more suitable for the analysis of these systems.

The task of determining stability regions of nonlinear discrete dynamical systems is of fundamental importance. Knowledge of the stability boundary can lead to the development of more efficient and less conservative methods for estimating stability regions. Knowledge of the stability region is also important in controlled discrete systems [159]. Networked control systems, for example, can be modeled and analyzed by approximate discrete-time nonlinear models [72]. The control of engines with fuel injection is another application of nonlinear control of discrete-time dynamical systems [104].

In this chapter, a comprehensive theory for both the stability boundary and the stability regions of nonlinear discrete dynamical systems will be developed. In particular, theoretical developments of the following topics will be presented.

Topological properties and global behaviors:

(1) topological properties of stability regions of nonlinear discrete dynamical systems,
(2) global behavior of discrete-time trajectories,
(3) global behavior of discrete-time trajectories on the stability boundary.

Characterizations of the stability boundary:

(1) local characterizations of the stability boundary,
(2) complete characterizations of the stability boundary,
(3) sufficient conditions for an unbounded stability boundary.

A complete characterization of the stability boundary for a fairly large class of nonlinear discrete dynamical systems that admit energy functions will be developed. Energy functions for discrete system will be further explored in Chapter 12 in the development of algorithms to obtain optimal estimates of the stability region of discrete systems.

9.2 Discrete dynamical systems

Consider the following class of autonomous nonlinear discrete dynamical systems:

$$x_{k+1} = f(x_k) \tag{9.1}$$

where $k \in Z$, $x_k \in R^n$ and f: $R^n \rightarrow R^n$ is a vector-valued map. The solution of (9.1) starting from $x_0 \in R^n$ at $k = 0$, denoted by $\phi(x_0,\cdot)$: $Z \rightarrow R_n$, is called an orbit (or trajectory) of (9.1). The solution of the discrete dynamical system (9.1) is an infinite sequence x_k that can be obtained by successive applications of the map f, i.e. $x_k = \phi(x_0, k) = f^k(x_0)$.

The existence and uniqueness of solutions of system (9.1) for times greater than zero ($k > 0$) is not an issue for discrete systems. If the map f is a well-defined function on R^n, then the solution starting at x_0 is defined in Z_+ and can easily be obtained by successive application of the map. On the other hand, solutions may not exist for negative times ($k < 0$), and when they exist the solution may not be unique. However, if function f is invertible, then solutions of (9.1) exist and are defined in Z.

A point x^* is a periodic point of period p if $f^p(x^*) = x^*$ and $f^k(x^*) \neq x^*$ for every k satisfying $0 < k < p$. If x^* is a periodic point of period p, the sequence $\gamma = \{x^*, f(x^*), \ldots, f^{p-1}(x^*)\}$ is a periodic orbit or closed orbit of system (9.1). If x^* has period one, i.e. $f(x^*) = x^*$, then x^* is called a fixed point of system (9.1). A state vector x is called a regular point if it is not a fixed point.

A fixed point x^* is stable if, for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that $\|x_0 - x^*\| < \delta$ implies $\|x_k - x^*\| < \varepsilon$, $\forall k \in Z_+$; it is asymptotically stable if it is stable and δ can be chosen such that $\|x_0 - x^*\| < \delta$ implies $\lim_{k\to\infty} x_k = x*$. A fixed point x^* is called unstable if it is not stable. A periodic point x^* of period p is (asymptotically) stable if x^* is an (asymptotically) stable fixed point of the map $f^p(\cdot)$.

When function f is continuous and differentiable, we say that a fixed point of (9.1) is hyperbolic if the Jacobian matrix at x^*, denoted by $Df(x^*)$, has no eigenvalues with modulus one. A hyperbolic fixed point is asymptotically stable if all the eigenvalues of its corresponding Jacobian have modulus less than one, while a hyperbolic fixed point is unstable if at least one eigenvalue of the Jacobian has modulus greater than one. A periodic or closed orbit γ of period p is hyperbolic if and only if every point of the orbit is a hyperbolic fixed point of f^p [228].

A set M is positively invariant with respect to the discrete system (9.1) if $f(M) \subset M$, which implies that every orbit x_k starting in M remains in M for all $k \geq 0$. *A* set M is invariant if $f(M) = M$ and negatively invariant if $f^{-1}(M) \subset M$. Solutions of discrete systems might be not defined in Z_-; however, if M is invariant, and x_0 is an initial condition in M, then there exists one solution x_k of system (9.1) satisfying the initial condition x_0, defined on Z, such that $x_k \in M$ for all $k \in Z$. In spite of this, and different from continuous systems, there may exist solutions starting outside M that enter M in finite time. Nevertheless, if f is injective, then invariance of M implies negative invariance. In this case, solutions starting outside of M cannot enter M in finite time.

Lemmas 9.1 and 9.2 establish a relationship between the invariance of a set with the invariance of its complement and its closure.

LEMMA 9-1 *If M is a positively invariant set with respect to the discrete system (9.1), then its complement $R^n - M$ is negatively invariant. Moreover, if f is a continuous function then:*

(a) the closure $\overline{M}$ is a positively invariant set;
(b) the complement of the closure $R^n - \overline{M}$ is a negatively invariant set.

LEMMA 9-2 *If M is an invariant set with respect to the discrete system (9.1) and f is a homeomorphism, then its closure $\overline{M}$ is also invariant.*

A point p is said to be in the ω-limit set (or α-limit set) of x_0 if for any given $\varepsilon > 0$ and $N > 0$ (or $N < 0$), there exists a $k > N$ (or $k < N$) such that $\|x_k - p\| < \varepsilon$. This is equivalent to the condition that there is a sequence $k_i \in Z$ with $k_i \to \infty$ (or $k_i \to -\infty$) as $i \to \infty$ such that $p = \lim_{i\to\infty} x_{k_i}$. The ω-limit set is closed and positively invariant. If function f is invertible, then the ω-limit set is closed and invariant. If f is continuous and the orbit $\{x_k\}$ is also bounded for $k > 0$, then the ω-limit set is non-empty, compact and invariant. Moreover,

$$x_k = f^k(x) \to \omega(x) \text{ as } k \to \infty.$$

In comparison with continuous dynamical systems, the ω-limit set of bounded orbits of discrete-time dynamical systems is usually not connected. This difference comes from the fact that orbits of discrete systems are not connected, while orbits of continuous systems are always connected. For instance, the α-limit and ω-limit of a periodic point of period p is the periodic orbit, which is formed of the union of p isolated points. Clearly, this is an example of a limit set that is not path connected.

The concept of invariantly connected sets relies on the dynamics to fix this problem, providing a concept of connectedness that is the counterpart of path connectedness in the continuous case. A closed and invariant set M is *invariantly connected* if it is not the union of two non-empty disjoint closed invariant sets [124,156,205,277]. If f is continuous and the orbit $\{x_k\}$ is bounded for $k > 0$, then the ω-limit set is also invariantly connected.

The hyperbolic fixed point x^* is called a *type-k fixed point* if the Jacobian $Df(x^*)$ has exactly k eigenvalues with modulus greater than one. The set of points whose ω-limit set is the fixed point x^* is called the stable set of x^* and is denoted $W^s(x^*)$. Similarly, the unstable set $W^u(x^*)$ is the set of points whose α-limit set is the fixed point x^*. Under the condition that the map f is a diffeomorphism, the stable and unstable sets have the structure of a manifold, similar to the case of stable and unstable manifolds of continuous dynamical systems, see Section 2.4 of Chapter 2. The function f is a diffeomorphism if it is differentiable and invertible and its inverse f^{-1} is differentiable. If f is a C^r-diffeomorphism, with $r \geq 1$, and x^* is a hyperbolic fixed point of (9.1), then the tangent space at x^*, $T_{x^*}(R^n)$, can be uniquely decomposed as a direct sum of two subspaces denoted by E^s and E^u, which are invariant with respect to the linear operator $Df(x^*)$:

$$\begin{aligned} E^s &= \text{span}\{e_1, \ldots, e_s\}, \\ E^u &= \text{span}\{e_{s+1}, \ldots e_{s+u}\}, \end{aligned}$$

where $\{e_1, \ldots, e_s\}$ and $\{e_{s+1}, \ldots, e_{s+u}\}$ are the generalized eigenvectors of $Df(x^*)$ respectively associated with the eigenvalues of $Df(x^*)$ that have modulus less and greater than one. Vector subspaces E^s and E^u are respectively called stable and unstable subspaces or eigenspaces.

There are local manifolds $W^s_{loc}(x^*)$ and $W^u_{loc}(x^*)$ of class C^r, invariant with respect to (9.1) [123] that are tangent to E^s and E^u at x^*, respectively. $W^s_{loc}(x^*)$ and $W^u_{loc}(x^*)$ are respectively termed local stable and unstable manifolds. These local stable and unstable manifolds are unique. Every orbit x_k starting in $W^s_{loc}(x^*)$ approaches x^* as $k \to \infty$, while every orbit starting in $W^u_{loc}(x^*)$ approaches x^* as $k \to -\infty$.

The local unstable manifold can be extended via dynamics of system (9.1) to form the (global) unstable manifold:

$$W^u(x^*) = \bigcup_{k \geq 0} \phi(W^u_{loc}(x^*), k) = \bigcup_{k \geq 0} f^k(W^u_{loc}(x^*)).$$

The (global) stable manifold can also be obtained via backward iterations of system (9.1) to form the (global) stable manifold:

$$W^s(x^*) = \bigcup_{k\leq 0} \phi(W^s_{loc}(x^*), k) = \bigcup_{k\leq 0} f^k(W^s_{loc}(x^*))$$

The manifolds $W^s(x^*)$ and $W^u(x^*)$ are of class C^r. If x^* is a type-k fixed point, then the dimension of $W^u(x^*)$ is k and the dimension of $W^s(x^*)$ is $n-k$.

9.3 Stability regions and topological properties

The stability region of an asymptotically stable fixed point x_s of (9.1) is defined as

$$A(x_s) := \{x \in R^n : lim_{k\to\infty} f^k(x) = x_s\}.$$

Similar to the continuous case, the stability region of a discrete dynamical system is composed of a set of points whose trajectories approach the asymptotically stable fixed point as the discrete time k tends to infinity. In the following, we derive several topological characterizations of stability regions and their boundaries. We will start with a very general characterization and gradually move into classes of discrete vector fields to gain more refined topological properties of the stability region and the stability boundary.

PROPOSITION 9.3 **(Topological characterization 1)**
The stability region $A(x_s)$ of an asymptotically stable fixed point x_s of (9.1) is positively invariant and negatively invariant. The stability boundary $\partial A(x_s)$ is a closed set.

Proof In order to show that $A(x_s)$ is positively invariant, let $x \in A(x_s)$ be an arbitrary point in the stability region. We need to show that $x \in A(x_s) \Rightarrow f(x) \in A(x_s)$. Since $f^k(f(x)) = f^{k+1}(x)$, it is clear that $\lim_{k\to\infty} f^k(f(x)) = x_s$. Hence $f(x) \in A(x_s)$ and, as a consequence, $A(x_s)$ is a positively invariant set.

Let $y \in f^{-1}(x)$. Then $f^k(y) = f^{k-1}(x)$. Consequently, $f^k(y) \to x_s$ as $k \to \infty$. Therefore every point y that belongs to the inverse image of a point on the stability region belongs to the stability region. So $f^{-1}(A(x_s)) \subset A(x_s)$ and $A(x_s)$ is negatively invariant. Since the stability boundary $\partial A(x_s) = \overline{A(x_s)} \cap \overline{\{R^n - A(x_s)\}}$ and the intersection of two closed sets is closed, we conclude the stability boundary is a closed set. This completes the proof.

Proposition 9.3 provides general topological information regarding the stability region and stability boundary without imposing any condition on the map f. As we impose conditions on the map f, refined results regarding the topological properties of these sets can be obtained. The next proposition assumes that the map f is a continuous function.

PROPOSITION 9.4 **(Topological characterization 2)**
Let x_s be an asymptotically stable fixed point of (9.1) and suppose f is a continuous function. The stability region $A(x_s)$ is an open, positively and negatively invariant set. The stability boundary $\partial A(x_s)$ is a closed and positively invariant set formed by forward orbits. Moreover, the stability boundary $\partial A(x_s)$ is of dimension less than n and if $A(x_s)$ is not dense in R^n, then $\partial A(x_s)$ is of dimension $n-1$.

Proof In order to show that $A(x_s)$ is an open set, let p be an arbitrary point in $A(x_s)$. We will show that every point in a neighborhood of p belongs to $A(x_s)$. To that end, let $\varepsilon > 0$ be sufficiently small such that the set $\{x \in R^n : \|x - x_s\| < \varepsilon\}$ is contained in $A(x_s)$. This number ε always exists according to the definition of an asymptotically stable fixed point. Let $N > 0$ be large enough such that $\|f^N(p) - x_s\| < \varepsilon/2$.

Since f^N is a continuous function, we can choose δ small enough to ensure that for any point q in the neighborhood $\{x \in R^n : \|x - p\| < \delta\}$ of p, $\|f^N(q) - f^N(p)\| < \varepsilon/2$. Hence, $\|f^N(q) - x_s\| \le \|f^N(q) - f^N(p)\| + \|f^N(p) - x_s\| < \varepsilon$. This shows that the point $f^N(q)$ is inside $A(x_s)$ and the set $A(x_s)$ is open. Invariance of the stability region trivially follows from Proposition 9.3. From Lemma 9.1, it follows that $\overline{A(x_s)}$ is positively invariant, i.e. $f(\overline{A(x_s)}) \subset \overline{A(x_s)}$. In order to show that $\partial A(x_s)$ is positively invariant, we must show, for every $x \in \partial A(x_s)$, that $f(x) \in \partial A(x_s)$. Suppose on the contrary that $f(x) \notin \partial A(x_s)$, then the positive invariance of $\overline{A(x_s)}$ implies $f(x) \in A(x_s)$. Since $x \in f^{-1}(A(x_s))$ and $A(x_s)$ is a negatively invariant set, we conclude that $x \in A(x_s)$. This leads to a contradiction because $x \in \partial A(x_s)$ and $A(x_s)$ is an open set. Therefore, $f(x) \in \partial A(x_s)$, and this shows that $\partial A(x_s)$ is a positively invariant set and the stability boundary $\partial A(x_s)$ is formed by forward trajectories of the points lying in $\partial A(x_s)$. We have already shown in Proposition 9.3 that the stability boundary is a closed set. The dimension of $\partial A(x_s)$ is a direct consequence of the results in [131]. This completes the proof.

The previous proposition asserts that the existence of at least two asymptotically stable fixed points is sufficient to guarantee that the dimension of the stability boundary of each stable fixed point is $n-1$. In the next proposition, the map f is assumed to be surjective. The stability region of surjective maps is shown to be invariant.

PROPOSITION 9.5 **(Topological characterization 3)**
Let x_s be an asymptotically stable fixed point of system (9.1) and suppose f is surjective, then the stability region $A(x_s)$ is an invariant set.

Proof From Proposition 9.3, the stability region is positively invariant, i.e. $f(A(x_s)) \subset A(x_s)$. In order to complete the proof, we have only to show that $A(x_s) \subset f(A(x_s))$. Let x be an arbitrary point in $A(x_s)$. Since f is surjective, there exists a y such that $x = f(y)$. Moreover, y belongs to $A(x_s)$. As a consequence, $x \in f(A(x_s))$ and the proof is complete.

Another topological property of the stability region of fixed points is that it may not be path connected. The following example illustrates a one-dimensional discrete system that contains a fixed point with a stability region that is not path connected.

Example 9-1 The following discrete system

$$x_{k+1} = x_k - 0.8\,x_k(x_k - 1)(x_k - 2)(x_k - 3) \tag{9.2}$$

possesses four fixed points, $x^1 = 0$, $x^2 = 1$, $x^3 = 2$ and $x^4 = 3$. These fixed points are indicated at the intersections of a graph of the discrete map and the straight line $x_{k+1} = x_k$. The fixed point $x^2 = 1$ is an asymptotically stable fixed point. A trajectory starting at $x_0 = 0.5$ and converging to the asymptotically stable fixed point x^2 is depicted in

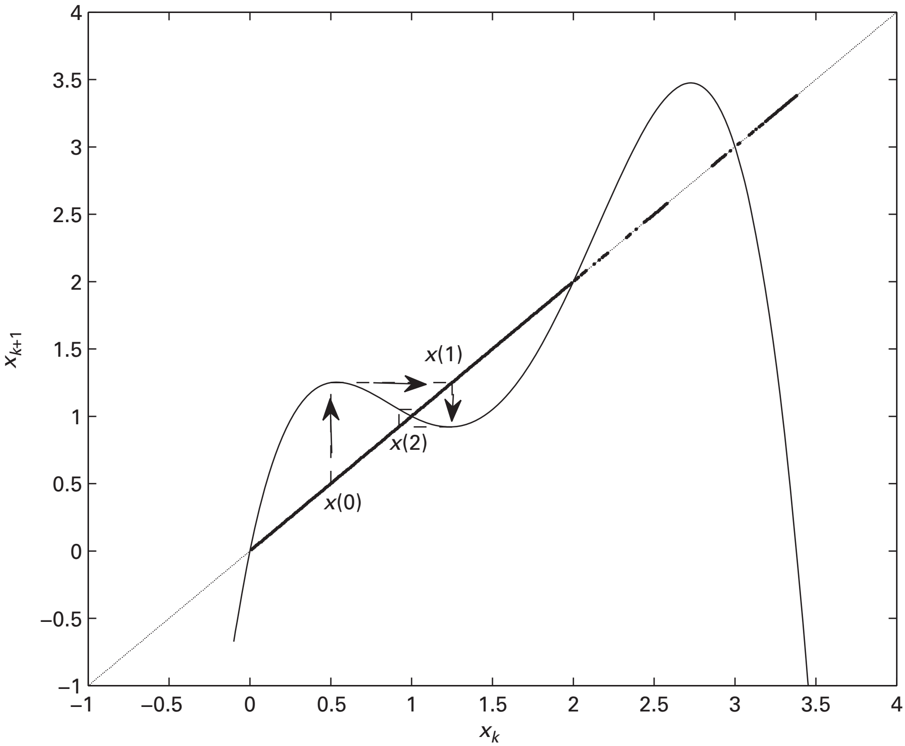

Figure 9.1 The stability region of system (9.2) is indicated by a thick diagonal line. The stability region is not a path connected set and it is not invariant.

Figure 9.1. Its stability region is highlighted by a thick black line on the diagonal line. The "shape" of this stability region is quite complex, even for this one-dimensional dynamical system. In particular, the stability region is not a connected set and it is not invariant.

However, if f is invertible and the inverse is a continuous function, then the stability region is path connected, as shown in Proposition 9.6.

PROPOSITION 9.6 **(Topological characterization 4)**
Let x_s be an asymptotically stable fixed point of the discrete system (9.1). If function f is invertible and its inverse is a continuous function, then the stability region is path connected.

Proof Since x_s is an asymptotically stable fixed point, there exists a path connected neighborhood U of x_s that is entirely contained in $A(x_s)$. The stability region can be written as $A(x_s) = \bigcup_{k>0} f^{-k}(U)$. Since f^{-k} is a continuous function and U is path connected, the set $f^{-k}(U)$ is also path connected. Moreover, set $f^{-k}(U)$ contains the fixed point x_s for every $k > 0$. Since arbitrary unions of open path connected sets sharing a common point are path connected, we conclude that the stability region is a path connected set. This completes the proof.

If function f is a homeomorphism, i.e. f is continuous, invertible and the inverse is also continuous, then all properties of the stability region and stability boundary that have been shown in Propositions 9.3–9.6 are valid. The next theorem asserts these properties and shows that the stability boundary is also an invariant set.

THEOREM 9-7 **(Topological characterization 5)**
Let x_s be an asymptotically stable fixed point of the discrete system (9.1). If function f is a homeomorphism, then the stability region $A(x_s)$ has the following topological and dynamic properties; it is

(a) open,
(b) positively and negatively invariant,
(c) invariant, and
(d) path connected;
and the stability boundary $\partial A(x_s)$ is:
(a) closed, and
(b) invariant.

Proof All the four properties of the stability region and property (a) of the stability boundary are a direct consequence of Propositions 9.3–9.6. The invariance of the stability boundary comes from the fact that the closure of an invariant set is also invariant (see Lemma 9.2). This completes the proof.

9.4 Characterization of stability regions

Our aim in this section is to present a comprehensive characterization of stability regions for the nonlinear discrete dynamical system (9.1). Our approach starts from a local characterization of the stability boundary and progresses towards a global characterization of the stability boundary.

We first derive a complete characterization for a fixed point lying on the stability boundary, which is a key step in the characterization of the stability region $A(x_s)$. We do this in two steps. First we impose only one generic assumption, namely, that fixed points are hyperbolic, and derive characterizations for a fixed point that lies on the stability boundary. These characterizations are expressed in terms of both the stable and unstable manifolds. Additional conditions are then imposed on the discrete dynamical system and the results are further sharpened.

Let x be a hyperbolic fixed point. Let U be a neighborhood of x in $W^s(x)$ whose boundary ∂U is transversal to the vector field f. We call ∂U a *fundamental domain* of $W^s(x)$. Any neighborhood $V \subset R^n$ of a fundamental domain ∂U that is disjoint from $W^u(x)$ is called a *fundamental neighborhood* for $W^s(x)$. Similarly we define a fundamental domain and a fundamental neighborhood for the unstable manifold $W^u(x)$.

If the vector field of the nonlinear discrete system (9.1) is a diffeomorphism, then we have already shown that the stability region is a path connected, open and invariant set and the stability boundary is a closed and invariant set. The following two theorems characterize all the fixed points lying on the stability boundary and prepare the terrain for developing a complete characterization of the stability boundary in terms of the invariant manifolds of the fixed points that lie on the stability boundary.

THEOREM 9-8 **(Fixed points on the stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable fixed point x_s of (9.1). Let $\hat{x} \neq x_s$ be a hyperbolic fixed point of system (9.1) and suppose that f is a diffeomorphism. Then,

(a) $\hat{x} \in \partial A(x_s)$ if and only if $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A}(x_s) \neq \emptyset$;
(b) if $\hat{x}$ is not a source, then $\hat{x} \in \partial A(x_s)$ if and only if $\{W^s(\hat{x}) - \hat{x}\} \cap \partial A(x_s) \neq \emptyset$.

Proof (a) We first prove the if condition. Suppose $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A}(x_s) \neq \emptyset$. Then, there exists $y \in \{W^u(\hat{x}) - \hat{x}\} \cap \overline{A}(x_s)$. Moreover $\lim_{n\to\infty} f^{-n}(y) = \hat{x}$. Since $\bar{A}(x_s)$ is an invariant set, it follows $f^{-n}(y) \in \bar{A}(x_s)$ for all $n \in Z$. Consequently, $\hat{x} \in \overline{A}(x_s)$. Since $\hat{x} \notin A(x_s)$, one has $\hat{x} \in \partial A(x_s)$.

Next, we prove the only if condition. Suppose $\hat{x} \in \partial A(x_s)$. Then every neighborhood U of $\hat{x}$ has a non-empty intersection with the stability region $A(x_s)$. Let D be a fundamental domain of $W^u(\hat{x})$ and D_ε be an ε-neighborhood of D. According to the λ-lemma [201], there is a neighborhood U of $\hat{x}$ such that $\bigcup_{k>0} f^{-k}(D_\varepsilon)$ contains the set $\{U - W^s(\hat{x})\}$. Since $W^s(\hat{x}) \cap A(x_s) = \emptyset$ and $U \cap A(x_s) \neq \emptyset$, there exists a point $q \in \{U - W^s(\hat{x})\}$ such that $q \in A(x_s)$. Moreover, the positive invariance of $A(x_s)$ guarantees the existence of a point $p \in D_\varepsilon$ such that $p \in A(x_s)$. Since D_ε can be chosen arbitrarily small, we can find a sequence of points $\{p_j\}$ with $p_j \in A(x_s)$ for every $j = 1,2, \ldots$, such that $d(p_j, D) \to 0$ as $j \to \infty$. Since the sequence $\{p_j\}$ is bounded, there exists a subsequence $\{p_{ji}\}$ that converges to p. By construction, $p \in \overline{A(x_s)} \cap \{W^u(\hat{x}) - \hat{x}\}$ and the proof of this part is completed.

(b) We prove the if condition. Now suppose $\{W^s(\hat{x}) - \hat{x}\} \cap \partial A(x_s) \neq \emptyset$. Then, there exists $y \in \partial A(x_s) \cap \{W^s(\hat{x}) - \hat{x}\}$. Moreover, $\lim_{n\to\infty} f^n(y) = \hat{x}$. Since $\partial A(x_s)$ is an invariant set, one has that $f^n(y) \in \partial A(x_s)$ for all $n \in Z$. Hence, $\hat{x} \in \partial A(x_s)$. The proof of the only if condition is very similar to that of part (a) and is hence omitted. This completes the proof.

The above characterization of a fixed point lying on the stability boundary can be extended to another critical element, i.e. closed orbit. The stable and unstable manifolds of a hyperbolic closed orbit γ are defined as follows:

$$W^s(\gamma) = \{x \in R^n : f^k(x) \to \gamma \text{ as } k \to \infty\}$$

$$W^u(\gamma) = \{x \in R^n : f^{-k}(x) \to \gamma \text{ as } k \to \infty\}.$$

A characterization of the closed orbit on the stability boundary is as follows.

THEOREM 9-9 **(Characterization of closed orbit on the stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable fixed point x_s of (9.1). Let γ be a hyperbolic closed orbit. Then

(a) $\gamma \subseteq \partial A x_s$ if and only if $\{W^u(\gamma) - \gamma\} \cap \overline{A(x_s)} \neq \emptyset$;
(b) suppose $\{W^s(\gamma) - \gamma\} \neq \emptyset$, then $\gamma \subseteq \partial A(x_s)$ if and only if $\{W^s(\gamma) - \gamma\} \cap \partial A(x_s) \neq \emptyset$.

Proof The proof of Theorem 9-9 is very similar to the proof of Theorem 9-8 and will be omitted.

As a corollary to Theorem 9-8, if $\{W^u(\hat{x}) - \hat{x}\} \cap A(x_s) \neq \emptyset$, then $\hat{x}$ must be on the stability boundary. Since any orbit in $A(x_s)$ approaches x_s, a sufficient condition for $\hat{x}$ to be on the stability boundary is the existence of an orbit in $W^u(\hat{x})$ which approaches x_s. One favorable property of this sufficient condition is that it is numerically checkable. From a practical point of view, one would like to see when this sufficient condition is also necessary. We will show that this sufficient condition becomes necessary under two additional assumptions.

So far we have assumed only that the critical elements are hyperbolic. This is a generic property for dynamical systems. Roughly speaking, we say a property is generic for a class of systems if that property is true for *almost all* systems in the class. A formal definition is given in [203]. It has been shown that among C^r $(r \geq 1)$ vector fields, the following properties are generic: (i) all fixed points and closed orbits are hyperbolic and (ii) the intersections of the stable and unstable manifolds of critical elements satisfy the transversality condition.

Theorem 9-8 can be sharpened under two conditions, one of which is generic for a nonlinear dynamical system (9.1). That is the transversality condition. The other condition requires that every orbit on the stability boundary approaches one of the fixed points. Hence, we consider the following assumptions.

(A1) All the fixed points on $\partial A(x_s)$ are hyperbolic.
(A2) The stable and unstable manifolds of fixed points on $\partial A(x_s)$ satisfy the transversality condition.
(A3) Every orbit on $\partial A(x_s)$ approaches one of the fixed points as $k \to \infty$.

Assumptions (A1) and (A2) are generic properties of dynamical discrete systems whose vector fields are diffeomorphisms, while assumption (A3) is not a generic property.

Now, we present the key theorem which characterizes a fixed point being on the stability boundary in terms of both its stable and unstable manifolds. From a practical point of view, this result is more useful in the numerical verification of fixed points on the stability boundary than Theorem 9-8.

THEOREM 9-10 **(Fixed points on the stability boundary)**
Let $A(x_s)$ be the stability region of an asymptotically stable fixed point x_s of system (9.1). Let $\hat{x}$ be a hyperbolic fixed point, suppose that f is a diffeomorphism and assumptions (A1)–(A3) are satisfied. Then, the following characterizations hold:

(a) $\hat{x} \in \partial A(x_s)$ if and only if $W^u(\hat{x}) \cap A(x_s) \neq \emptyset$;
(b) if $\hat{x}$ is not a source, then $\hat{x} \in \partial A(x_s)$ if and only if $W^s(\hat{x}) \subseteq \partial A(x_s)$.

Proof (a) We first prove the if condition. Suppose that $W^u(\hat{x}) \cap A(x_s) \neq \emptyset$. Since $\hat{x} \notin A(x_s)$, it follows that $\{W^u(\hat{x}) - \hat{x}\} \cap A(x_s) \neq \emptyset$. Moreover, $\{W^u(\hat{x}) - \hat{x}\} \cap A(x_s) \subset \{W^u(\hat{x}) - \hat{x}\} \cap \overline{A(x_s)}$, therefore, $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A(x_s)} \neq \emptyset$. By applying Theorem 9.8, it follows that $\hat{x} \in \partial A(x_s)$. We next prove the only if condition. Suppose $\hat{x} \in \partial A(x_s)$. Then, by Theorem 9.8, it follows that $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A(x_s)} \neq \emptyset$. We next show, under assumptions (A1)–(A3), that $\{W^u(\hat{x}) - \hat{x}\} \cap \overline{A(x_s)} \neq \emptyset$ implies $W^u(\hat{x}) \cap A(x_s) \neq \emptyset$. Let $p \in \{W^u(\hat{x}) - \hat{x}\} \cap \overline{A(x_s)}$. There are two possibilities: (1) $p \in A(x_s)$ and (2) $p \in \partial A(x_s)$. If $p \in A(x_s)$, then the result follows. Suppose that $p \in \partial A(x_s)$. From assumption (A3), there exists a fixed point $x^* \in \partial A(x_s)$ such that $f^k(p) \to x^*$ as $k \to \infty$, i.e. $p \in W^s(x^*)$. We note that both fixed points x^* and $\hat{x}$ are hyperbolic according to assumption (A1). Moreover, they are fixed points of type k with $k \geq 1$. The point p is a point of intersection between the manifolds $W^u(\hat{x})$ and $W^s(x^*)$ which, according to assumption (A2), is a point of transversal intersection.

Suppose that $\hat{x}$ is a hyperbolic fixed point of type one. Then, according to the transversality intersection, x^* has to be a fixed point of type zero. This leads to a contradiction since x^* must be a fixed point of type $k \geq 1$. Hence, it follows that $W^u(\hat{x}) \cap A(x_s) \neq \emptyset$ for all fixed points of type one on the stability boundary.

The proof will be completed by induction. Suppose that $W^u(x^*) \cap A(x_s) \neq \emptyset$ for all fixed points of type k or less on the stability boundary. Let $\hat{x}$ be a hyperbolic fixed point of type k+1 on the stability boundary. Then, according to the transversality intersection, $h = \dim\{W^u(x^*)\} \leq k$. Therefore, $W^u(x^*) \cap A(x_s) \neq \emptyset$. Let $y \in W^u(x^*) \cap A(x_s)$ and let $B_\varepsilon(y)$ be an open ball of radius ε centered at y, where ε is a sufficiently small real number. Since the stability region is an open set, $B_\varepsilon(y) \subset A(x_s)$ for ε sufficiently small. Let D_ε be a neighborhood of y in $W^u(x^*)$ (a disk of dimension h) induced by $B_\varepsilon(y)$, i.e. $D_\varepsilon = B_\varepsilon(y) \cap W^u(x^*)$. Let N be a neighborhood of p in $W^u(\hat{x})$. This neighborhood is immersed on a manifold of dimension k+1. This neighborhood contains a section of dimension h transversal to $W^s(x^*)$ at the point p. A direct application of the λ-lemma shows the existence of a point z in N and an integer $K > 0$ such that $f^K(z) \in B_\varepsilon(y)$. The invariance of $A(x_s)$ guarantees that $z \in A(x_s)$, therefore $W^u(\hat{x}) \cap A(x_s) \neq \emptyset$.

(b) We prove the if condition. Suppose that $W^s(\hat{x}) \subset \partial A(x_s)$. Since $\hat{x} \in W^s(\hat{x})$, it follows that $\hat{x} \in \partial A(x_s)$. We next prove the only if condition. Suppose that $\hat{x} \in \partial A(x_s)$. Then, from the proof of (a), one has that $W^u(\hat{x}) \cap A(x_s) \neq \emptyset$. Let $y \in \{W^u(\hat{x}) - \hat{x}\} \cap A(x_s)$ and $B_\varepsilon(y)$ be an open ball of radius ε, centered at y, where ε is an arbitrarily small number. Since the stability region is an open set, $B_\varepsilon(y) \subset A(x_s)$ for ε sufficiently small. Let D_ε be a neighborhood of y in $W^u(x^*)$ (a disk of dimension h) induced by $B_\varepsilon(y)$, i.e. $D_\varepsilon = B_\varepsilon(y) \cap W^u(x^*)$. Let p be an arbitrary point in $W^s(\hat{x})$ and let S be a section transversal to $W^s(\hat{x})$ at the point p. A direct application of the λ-lemma shows the existence of a point z in S and an integer K > 0 such that $f^K(z) \in B_\varepsilon(y)$. The invariance of $A(x_s)$ guarantees that $z \in A(x_s)$. Since ε and the section S can be chosen arbitrarily small, then there exist points of $A(x_s)$ arbitrarily close to p. This means $p \in \overline{A(x_s)}$. Since $W^s(\hat{x})$ cannot contain points in $A(x_s)$, $p \in \partial A(x_s)$. The arbitrariness of the choice of p in $W^s(\hat{x})$ guarantees that $W^s(\hat{x}) \subset \partial A(x_s)$. This completes the proof.

We are now in a position to develop a complete characterization of the stability boundary of a class of nonlinear discrete dynamical systems satisfying assumptions (A1)–(A3).

THEOREM 9-11 **(Stability boundary characterization)**
Let x_s be an asymptotically stable fixed point of the discrete system (9.1), suppose that f is a diffeomorphism and assumptions (A1)–(A3) are satisfied. Let $x_1, x_2, \ldots$ be the hyperbolic unstable fixed points on the stability boundary $\partial A(x_s)$. Then the stability boundary $\partial A(x_s)$ is completely characterized by the following:

$$\partial A(x_s) = \bigcup_i W^s(x_i)$$

Proof From Theorem 9-10,

$$\bigcup_i W^s(x_i) \subseteq \partial A(x_s). \tag{9.3}$$

From assumption (A3), we have

$$\partial A(x_s) \subset \bigcup_i W^s(x_i). \tag{9.4}$$

Combining Eqs. (9.3) and (9.4) leads to $\partial A(x_s) = \bigcup_i W^s(x_i)$ and the proof is complete.

Theorem 9-11 asserts that the stability boundary is composed of the union of the stable manifolds of all fixed points that lie on the stability boundary. Theorem 9-11 can be generalized to allow closed orbits to exist on the stability boundary.

9.5 Conceptual algorithms for exact stability regions

Theorem 9-11 leads to the following conceptual algorithm for determining the stability boundary of a stable fixed point of the nonlinear discrete dynamical systems (9.1) that satisfies assumptions (A1) to (A3).

Conceptual algorithm (To determine the stability boundary $\partial A(x_s)$)
Step 1: Find all the fixed points.
Step 2: Identify those fixed points whose unstable manifolds contain orbits approaching the stable fixed point x_s.
Step 3: The stability boundary of x_s is the union of the stable manifolds of the fixed points identified in Step 2.
Step 2 can be accomplished numerically. The following procedure is suggested.

(i) Find the Jacobian at the fixed point (say, $\hat{x}$).
(ii) Find many of the generalized unstable eigenvectors of the Jacobian that have unit length.
(iii) Find the intersection of each of these normalized, generalized, and unstable eigenvectors (say, y_i) with the boundary of an ε-ball of the fixed point (the intersection points are $\hat{x} + \varepsilon y_i$ and $\hat{x} - \varepsilon y_i$).
(iv) Iterate the vector field backward (reverse time) from each of these intersection points up to some specified time. If the orbit remains inside this ε-ball, then go to the next step. Otherwise, we replace the value ε by $\alpha\varepsilon$ and also the intersection points $\hat{x} \pm \varepsilon y_i$ by $\hat{x} \pm \alpha\varepsilon y_i$, where $0 < \alpha < 1$. Repeat this step.

(v) Numerically iterate the vector field starting from these intersection points.
(vi) Repeat steps (iii) through (v). If any of these orbits approaches x_s, then the fixed point is on the stability boundary.

For a planar system, the fixed point on the stability boundary is either a type-one fixed point or a type-two fixed point, which is a source. The stable manifold of a type-one fixed point in this case has dimension one, which can easily be determined numerically as follows.

(i) Find a normalized stable eigenvector y of the Jacobian at the fixed point $\hat{x}$.
(ii) Find the intersection of this stable eigenvector with the boundary of an ε-ball of the fixed point $\hat{x}$ (where the intersection points are $\hat{x} + \varepsilon y$ and $\hat{x} - \varepsilon y$).
(iii) Iterate the vector field from each of these intersection points after some specified time. If the orbit remains inside this ε-ball, then go to next step. Otherwise, we replace the value ε by $\alpha\varepsilon$ and also the intersection points $\hat{x} \pm \varepsilon y_i$ by $\hat{x} \pm \alpha\varepsilon y_i$, where $0 < \alpha < 1$. Repeat this step.
(iv) Numerically iterate the vector field backward (reverse time) starting from these intersection points.
(v) The resulting orbits are in the stable manifold of the fixed point.

This conceptual algorithm may have difficulty in finding exact stability regions of higher dimensional systems. This is because the numerical procedure similar to that described above can only provide a set of orbits on the stable manifold. Finding the stable manifold and unstable manifold of a fixed point is a nontrivial problem and advanced numerical methods for computing stable and unstable manifolds must be developed. We next use a simple example to illustrate this conceptual algorithm.

Example 9-2 Consider the nonlinear discrete system

$$\begin{aligned} x_{k+1} &= dx_k + x_k^3 + ey_k^2 \\ y_{k+1} &= cy_k \end{aligned} \tag{9.5}$$

with $c = d = e = 0.5$. It is straightforward to show that $f(x_k, y_k)$ is a diffeomorphism. The fixed points of system (9.5) are (0, 0), which is an asymptotically stable fixed point, and $x_1 = (\sqrt{1-d}, 0)$ and $x_2 = (-\sqrt{1-d}, 0)$, which are type-one fixed points. All fixed points are hyperbolic and, as a consequence, assumption (A1) is satisfied. We make the following observations.

[1] The unstable manifolds of $x_1 = (\sqrt{1-d}, 0)$ and $x_2 = (\sqrt{1-d}, 0)$ converge to the asymptotically stable fixed point (0, 0). According to Theorem 9-10, these two type-one fixed points lie on the stability boundary of the stable fixed point (0, 0).
[2] Figure 9.2 displays the stability region $A(0, 0)$ of the fixed point (0, 0). The unstable fixed points x_1 and x_2 lie on the stability boundary $\partial A(0, 0)$. Assumptions (A2) and (A3) are also satisfied and therefore the stability boundary is composed of the union of the stable manifolds $W^s(x_1)$ and $W^s(x_2)$, in accordance with the results of Theorem 9-11.

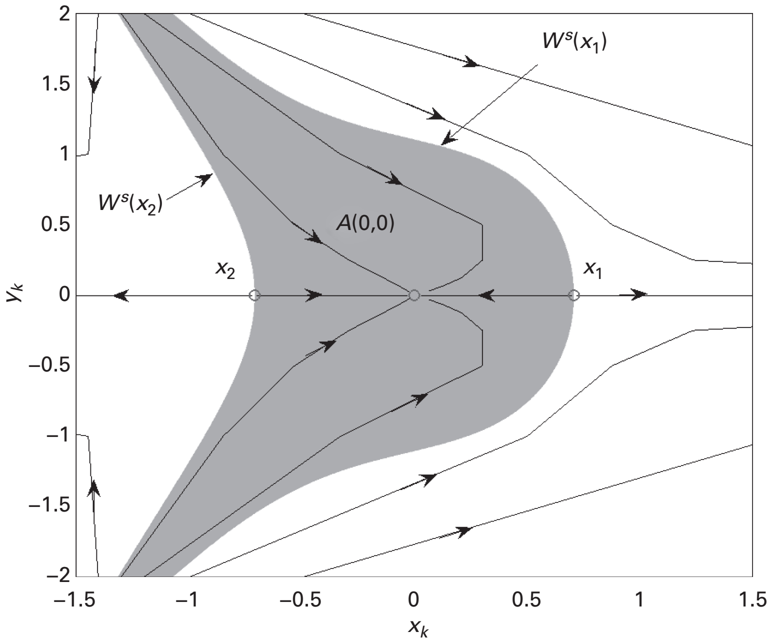

Figure 9.2 Stability region of the fixed point (0, 0) of system (9.5). The stability boundary is composed of the union of the stable manifolds $W^s(x_1)$and $W^s(x_2)$ of the type-one fixed points $x_1$and x_2.

Example 9.3 Consider the following discrete version of the nonlinear pendulum equation:

$$\begin{aligned} x_{k+1} &= x_k + hy_k \\ y_{k+1} &= (1 - dh)y_k - hc \sin x_k. \end{aligned} \tag{9.6}$$

For sufficiently small h, the vector field is a diffeomorphism. All the fixed points are hyperbolic and assumptions (A1), (A2) and (A3) are satisfied. Hence, the characterization of the stability boundary of Theorem 9-11 is valid. Figure 9.3 displays the stability region and stability boundary of the system for $h = 0.1$, $k = 1$ and $d = 0.5$. There are two hyperbolic fixed points whose unstable manifolds converge to the asymptotically stable fixed point (0,0). Hence, we have the following results.

[1] According to Theorem 9-10, one can conclude that these two hyperbolic fixed points lie on the stability boundary of the asymptotically stable fixed point (0, 0), as shown in Figure 9.3.

[2] It can be observed that the stability boundary of the asymptotically stable fixed point (0, 0) is composed of the union of the stable manifolds of these two fixed points lying on the stability boundary, confirming the complete characterization derived in Theorem 9-11.

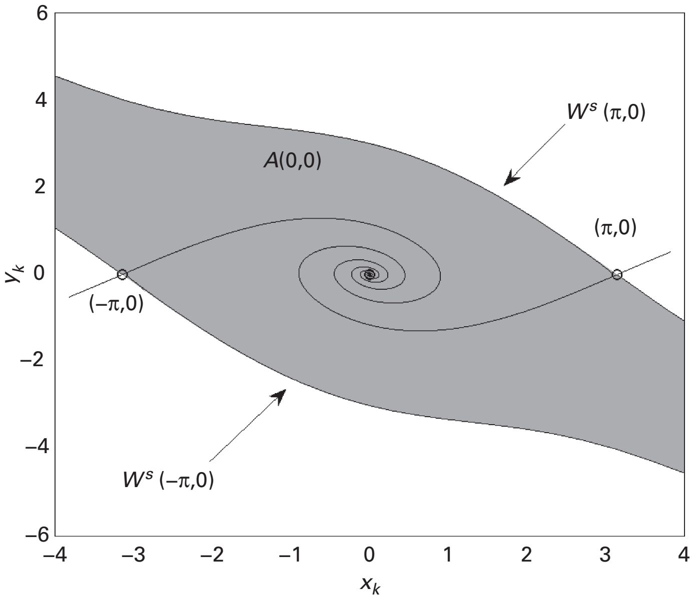

Figure 9.3 Stability region of system (9.6) for $k = 1$, $d = 0.5$ and $h = 0.1$. Two hyperbolic fixed points lie on the stability boundary of the asymptotically stable fixed point (0, 0), as asserted by Theorem 9-10. The stability boundary of the asymptotically stable fixed point (0, 0) is composed of the union of the stable manifolds of these two fixed points lying on the stability boundary, as asserted by Theorem 9-11.

While the stability boundary $\partial A(x_s)$ is of dimension $n-1$, the stable manifolds of non-type-one fixed points are thin sets on $\partial A(x_s)$. Exploring this property, the next theorem offers a complete characterization of the boundary of $\bar{A}$ in terms of the stable manifolds of type-one fixed points that lie on the stability boundary.

THEOREM 9-12 **(Another complete characterization)**
Let x_s be an asymptotically stable fixed point of the discrete system (9.1) and suppose that f is a diffeomorphism and assumptions (A1)–(A3) are satisfied. Let $x_1^1, x_2^1, \ldots$ be the type-one hyperbolic unstable fixed points on the stability boundary $\partial\bar{A}(x_s)$ of x_s. Then

$$\partial\overline{A}(x_s) = \bigcup_i \overline{W^s(x_i^1)}.$$

Proof Let $x_1^1, x_2^1, \ldots$be the type-one hyperbolic unstable fixed points on the stability boundary $\partial\bar{A}(x_s)$. The dimension of the stable manifold of a type-one fixed points is $n-1$, whereas the dimension of other fixed points is less than $n-1$, i.e. the interior of the stable manifolds of the fixed points that are not type one on $\partial A(x_s)$ is empty. The Baire category theorem [124] implies that $\partial A(x_s)$ cannot be written as a countable union of closed subsets having empty interiors, therefore we can affirm that $\partial\overline{A}(x_s) = \bigcup_i \overline{W^s(x_i^1)}$ where $x_1^1, x_2^1, \ldots$ are the type-one fixed points on $\partial\bar{A}(x_s)$. This completes the proof.

Theorem 9-12 asserts that a complete characterization of the stability boundary can be obtained by computing only the type-one fixed points that lie on the stability boundary. The following theorem gives an interesting result on the structure of the fixed points on the stability boundary. Moreover, it presents a necessary condition for the existence of certain types of fixed points on a *bounded* stability boundary.

THEOREM 9-13 **(Structure of fixed points on the stability boundary)**
Let x_s be an asymptotically stable fixed point of the discrete system (9.1) and suppose that f is a diffeomorphism and assumptions (A1)–(A3) are satisfied. If the stability region $A(x_s)$ is not dense in R^n then $\partial A(x_s)$ must contain at least one type-one fixed point. Moreover, if $A(x_s)$ is bounded, then $\partial A(x_s)$ must contain at least one type-n fixed point (i.e. a source).

Proof Since the stability region $A(x_s)$ is not dense in R^n, then Proposition 9.4 guarantees that the dimension of $\partial A(x_s)$ is $n-1$. Since $\partial A(x_s) = \bigcup_i W^s(x_i)$ where x_i, i=1,2, . . . are the fixed points on the stability boundary $\partial A(x_s)$, then at least one of the fixed points, say x_1, must be a type-one fixed point so that the dimension of $\bigcup_i W^s(x_i)$ is $n-1$. Repeating the same argument, if $\partial W^s(x_1)$ is non-empty, then the dimension of $\partial W^s(x_1)$ is $n-2$. The application of Theorem 9-11 yields $\partial W^s(x_1) = \bigcup_j W^s(x_j)$, where x_j are the fixed points in $\partial W^s(x_1)$. In order to guarantee that $\bigcup_j W^s(x_j)$ is of dimension $n-2$, at least one of the fixed points x_j on $\partial W^s(x_1)$ must be a type-two fixed point. If the stability region is bounded, the same argument can be repeated until we reach a type-n fixed point (a source). The proof is complete.

Theorem 9-13 offers a sufficient condition to check whether the stability region is unbounded. This condition is formally stated in the next corollary.

COROLLARY 9.14 **(Unbounded stability regions)**
If the stability boundary $\partial A(x_s)$ of the asymptotically stable fixed point x_s of the discrete system (9.1) has no source and assumptions (A1)–(A3) are satisfied, then $A(x_s)$ is unbounded.

We next illustrate Theorem 9-12, Theorem 9-13 and Corollary 9.14 on the simple test systems (9.5) and (9.6). We note that the stability boundaries of both system (9.5) and system (9.6) contain type-one fixed points on the stability boundaries as asserted by Theorem 9-13. We also note that both system (9.5) and system (9.6) contain no type-two fixed points (i.e. no source) on the stability boundaries of both systems. Hence, by Corollary 9.14, the stability region is unbounded. In addition, according to Theorem 9-12, the stability boundaries of these two simple systems (9.5) and (9.6) are composed of the union of the closure of the stable manifolds of two type-one fixed points.

9.6 Characterization via an energy function

In this section, we study energy functions which can be viewed as an extension of the Lyapunov functions. We focus on how to characterize the stability boundary of a class of nonlinear discrete systems that admit an energy function.

Before defining the concept of energy function, it is important to understand the concept of first difference, which plays the role of derivative for discrete systems.

Consider the scalar function $V: R^n \to R$. The first difference of V relative to (9.1), or to map f, at a point $x \in R^n$ is given by:

$$\Delta V(x) = V(f(x)) - V(x). \tag{9.7}$$

If x_k is a solution of (9.1) for $k \geq 0$, then the first difference of V along the solution x_k is given by:

$$\Delta V(x_k) = V(x_{k+1}) - V(x_k),\ k \geq 0. \tag{9.8}$$

DEFINITION **(Energy function)**
A continuous function $V: R^n \to R$ is called an energy function for the discrete system (9.1) if it satisfies the following conditions.

(E1) $\Delta V(x) \leq 0$ *for all* $x \in R^n$*;*
(E2) $\Delta V(x_k) = 0$ *implies* x_k *is a fixed point.*
(E3) If $V(x_k)$ *is bounded for* $k \in Z_+$*, then the orbit* x_k *is itself bounded for* $k > 0$*.*

Property (E1) indicates that the energy function is non-increasing along any orbit, but it alone does not imply that the energy function is strictly decreasing along nontrivial orbits. There may exist a discrete time k such that $\Delta V(x_k) = 0$. Properties (E1) and (E2) together imply that the energy function is strictly decreasing along any nontrivial system orbit. Property (E3) states that the energy function is a dynamic proper map along any system orbit but need not be a proper map for the entire state space. Recall that a proper map is a function $f: X \to Y$ such that for each compact set $D \in Y$, the set $f^{-1}(D)$ is compact in X. We note that if function V is proper or radially unbounded, then assumption (E3) is satisfied. From the above definition of energy function, it is obvious that an energy function may not be a Lyapunov function.

Energy functions are useful for global analysis of system orbits and for estimating stability regions and quasi-stability regions, among others. We next present a global analysis of system orbits of nonlinear discrete dynamical systems that have energy functions.

THEOREM 9-15 **(Energy functions and limit sets)**
If the nonlinear discrete dynamical system (9.1) admits an energy function and the map f is continuous with all fixed points being isolated, then every bounded trajectory x_k converges to a fixed point as $k \to \infty$.

Proof Let x_k be a bounded trajectory of (9.1) starting in x_0 at time $k=0$. Assumption (E1) implies that $V(x_k + 1) \leq V(x_k)$ for every $k \geq 0$. The non-increasing sequence $V(x_k)$ is bounded from below, since V is a continuous function. Hence, $V(x_k)$ converges to a certain value p as $k \to \infty$. On the other hand, the set $\omega(x_0)$ of a bounded orbit is non-empty and the orbit x_k approaches $\omega(x_0)$ as $k \to \infty$. Hence $V(x) = p$ for every $x \in \omega(x_0)$. The invariance of $\omega(x_0)$ implies that $\Delta V(x) = 0$ for every $x \in \omega(x_0)$. Suppose now that $x \in \omega(x_0)$ is not a fixed point. Then $\Delta V(x) = V(f(x)) - V(x) = p - p = 0$. This fact is in contradiction with assumption (E2). Therefore $\omega(x_0)$ is composed exclusively of fixed points. Since all fixed points are isolated and the limit set $\omega(x_0)$ is invariantly connected,

$\omega(x_0)$ is composed of a single fixed point. Therefore, every bounded trajectory converges to a fixed point as $k \to \infty$. This completes the proof.

Theorem 9-15 asserts that nonlinear discrete systems admitting an energy function do not present complex behavior, i.e. their limit sets are exclusively composed of fixed points, in particular, the system has no periodic orbits and consequently no limit cycles. The state space of this class of nonlinear systems does not admit nontrivial periodic solutions, quasi-periodic solutions and chaos. It will be shown in the next theorem that orbits on the stability boundary of nonlinear discrete dynamical systems admitting an energy function are bounded for $k > 0$, although the stability boundary itself can be unbounded.

THEOREM 9-16 **(Boundedness of orbits on the stability boundary)**
Let x_s be an asymptotically stable fixed point of the discrete system (9.1) that admits an energy function and suppose that the map f is continuous. Then every trajectory x_k on the stability boundary $\partial A(x_s)$ is bounded for $k > 0$.

Proof For an arbitrary $x \in \partial A(x_s)$, let $\{x_i\}$ be a sequence of points in $A(x_s)$ converging to x as $i \to \infty$. The trajectory x_{i_k} of system (9.1) starting in x_i converges to x_s as $k \to \infty$. Assumption (E1) implies $V(x_i) \geq V(x_s)$ for every x_i, i=1,2, ... The continuity of V implies that $V(x) \geq V(x_s)$. Hence, $V(x_s)$ is a lower bound of V on the stability boundary. The positive invariance of $\partial A(x_s)$ and assumption (E3) imply that x_k, the solution of system (9.1) starting in x, is bounded for $k > 0$. This completes the proof.

Combining Theorem 9-15 and Theorem 9-16 leads to a sufficient condition for the satisfaction of assumption (A3) as shown in the following.

COROLLARY 9.17 **(Sufficient condition for (A3))**
Let x_s be an asymptotically stable fixed point of the discrete system (9.1) that admits an energy function, suppose that f is continuous and all fixed points are isolated. Then assumption (A3) is satisfied, i.e. every trajectory on $\partial A(x_s)$ converges to a fixed point.

Proof Theorem 9-16 asserts that every orbit on the stability boundary $\partial A(x_s)$ is bounded and Theorem 9-15 proves that every bounded orbit converges to a fixed point. This concludes the proof.

We are now in a position to present a complete characterization of the stability boundary for a class of discrete nonlinear dynamical systems that admits energy functions.

THEOREM 9-18 **(Stability boundary characterization 1)**
Let x_s be an asymptotically stable fixed point of the nonlinear discrete system (9.1) that admits an energy function. Suppose that f is a continuous map satisfying assumption (A1) and let $x_1, x_2, \ldots$ be the unstable fixed points on the stability boundary $\partial A(x_s)$. Then, the stability boundary is contained in the union of the stable sets of all the fixed points on the stability boundary

$$\partial A(x_s) \subseteq \bigcup_i W^s(x_i)$$

where $W^s(x_i)$ is the stable set of the unstable fixed point x_i.

Proof Assumption (A1) requires that every fixed point on the stability boundary is hyperbolic and therefore isolated. Corollary 9-17 guarantees that assumption (A3) is satisfied, i.e. every trajectory on the stability boundary converges to a fixed point on $\partial A(x_s)$. Hence, every trajectory on the stability boundary must lie in the stable set of a fixed point on the stability boundary.

The characterization of the stability boundary of systems that admit energy functions, asserted in Theorem 9-18, only requires continuity of the vector field. A sharper characterization of the stability boundary of systems having energy functions can be obtained by assuming that all fixed points are hyperbolic and f is a diffeomorphism. We can gain more structure on the stable set that in this case is a manifold at the expense of more conditions on the vector field.

THEOREM 9-19 **(Stability boundary characterization 2)**
Let x_s be an asymptotically stable fixed point of the nonlinear discrete system (9.1) that admits an energy function. Suppose that f is a diffeomorphism satisfying assumption (A1). Let $x_1, x_2, \ldots$ be the hyperbolic unstable fixed points on the stability boundary $\partial A(x_s)$ of x_s. Then

$$\partial A(x_s) \subseteq \underset{i}{\cup} W^s(x_i)$$

Moreover, if assumption (A2) is satisfied, then

$$\partial A(x_s) = \underset{i}{\cup} W^s(x_i)$$

where $W^s(x_i)$ is the stable manifold of the hyperbolic unstable fixed point x_i.

Proof The first part trivially follows from Theorem 9-18. Since assumptions (A1) and (A2) are satisfied and the existence of an energy function implies (A3), then the result follows from a direct application of Theorem 9-11. This proof is completed.

Example 9-4 Consider the following two-dimensional nonlinear discrete system:

$$\begin{aligned} x_{k+1} &= x_k^3 + \frac{3}{4}x_k \\ y_{k+1} &= y_k^3 + \frac{3}{4}y_k. \end{aligned} \tag{9.9}$$

The vector field is a diffeomorphism and the system possesses nine hyperbolic fixed points: (0,0), an asymptotically stable fixed point, and $\left(0, \pm\frac{1}{2}\right)$, $\left(\pm\frac{1}{2}, 0\right)$, $\left(\pm\frac{1}{2}, \pm\frac{1}{2}\right)$, which are unstable fixed points. Consider the following candidate energy function:

$$V(x,y) = \begin{cases} |x| + |y| & \text{if} \quad |x| \le \frac{1}{2} \quad \text{and} \quad |y| \le \frac{1}{2} \\ 1 - |x| + |y| & \text{if} \quad |x| > \frac{1}{2} \quad \text{and} \quad |y| \le \frac{1}{2} \\ 1 + |x| - |y| & \text{if} \quad |x| \le \frac{1}{2} \quad \text{and} \quad |y| > \frac{1}{2} \\ 2 - |x| - |y| & \text{if} \quad |x| > \frac{1}{2} \quad \text{and} \quad |y| > \frac{1}{2}. \end{cases}$$

It is straightforward to show that $\Delta V(x, y) \le 0$ for any (x,y) and $\Delta V(x, y) = 0$ if and only if (x,y) is a fixed point. Hence, assumptions (E1) and (E2) are satisfied. Assumption (E3) is also satisfied because $V(x,y)$ is a proper function. We conclude that V is an energy function for system (9.9) and all the conditions of Theorem 9-19 are satisfied. Thus the stability boundary $\partial A(0, 0)$ is composed of the union of the stable manifolds of every fixed point that lies on the stability boundary. Figure 9.4 illustrates the stability region and stability boundary of system (9.9). The stability boundary contains eight unstable fixed points since their unstable manifolds converge to the asymptotically stable fixed point, as asserted by Theorem 9-10. The stability boundary $\partial A(0, 0)$ equals the union of the stable manifolds of eight unstable fixed points, as asserted by Theorem 9-19.

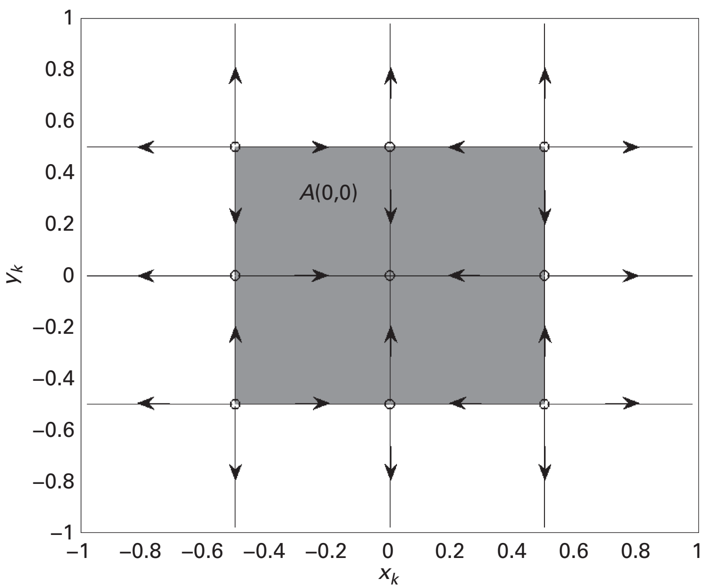

Figure 9.4 Stability region of (0, 0) which is an asymptotically stable fixed point of system (9.9). The stability boundary contains eight unstable fixed points since their unstable manifolds converge to (0, 0) (cf. Theorem 9-10). The stability boundary of (0, 0) is composed of the union of the stable manifolds of the eight unstable fixed points lying on the boundary.

9.7 Concluding remarks

A comprehensive theory of stability regions of general nonlinear autonomous discrete dynamical systems has been developed in this chapter. Several topological properties of the stability boundary and characterizations of limit sets lying on the stability boundary for general nonlinear discrete dynamical systems have been derived. Our approach starts from a local characterization of the stability boundary and progresses towards a global characterization of the stability boundary.

We have derived a complete characterization for a fixed point lying on the stability boundary, which is a key step in the characterization of the stability region $A(x_s)$. These characterizations are expressed in terms of both the stable and unstable manifolds. A complete characterization of stability boundaries for a fairly large class of nonlinear discrete dynamical systems has been obtained. For this class of nonlinear discrete systems, it was shown that the stability boundary of an asymptotically stable fixed point consists of the stable manifolds of all the fixed points on the stability boundary. Several necessary and sufficient conditions were derived to determine whether a given fixed point (or closed orbit) lies on the stability boundary.

A method to determine the exact stability region based on these results was proposed. The method, when feasible, will find the exact stability region, rather than a subset of the stability region as in the Lyapunov theory approach. For high-dimensional systems, an optimal scheme for estimating the stability region will be developed in Chapter 13.

Regarding the topological properties of stability regions of nonlinear discrete dynamical systems, the following properties have been derived.

- The stability region $A(x_s)$ is positively invariant and negatively invariant. The stability boundary $\partial A(x_s)$ is a closed set.
- If the vector field f is surjective, then the stability region $A(x_s)$ is an invariant set.
- If the vector field f is a continuous function, then the stability region $A(x_s)$ is an open, positively and negatively invariant set. The stability boundary $\partial A(x_s)$ is a closed and positively invariant set formed by forward orbits. Moreover, the stability boundary $\partial A(x_s)$ is of dimension less than n and if $A(x_s)$ is not dense in R^n, then $\partial A(x_s)$ is of dimension $n-1$.
- If the vector field f is a homeomorphism, then the stability region $A(x_s)$ has the following topological and dynamic properties. It is (i) open, (ii) positively and negatively invariant, (iii) invariant, and (iv) path connected; the stability boundary $\partial A(x_s)$ is (i) closed and (ii) invariant.

There has been significant work on the analysis of stability and asymptotic behavior of discrete-time dynamical systems in the literature. LaSalle proved an invariance principle for discrete-time systems in [156,157]. An extension of the invariance principle for discrete dynamical systems was independently derived in [219] and [5]. LaSalle theory seems to be the first practical tool to estimate the stability region of nonlinear discrete dynamical systems in the form of positive invariance sets. A survey on the theory of positively invariant sets in the analysis and control of discrete-time nonlinear dynamical systems can be found in [25]. The relationship between stability and the existence of smooth Lyapunov functions was studied in [140]. In spite of the enormous amount of work done in analysis of the asymptotic behavior of solutions of discrete-time nonlinear dynamical systems, only a few papers on the theory or estimation of stability regions of discrete dynamical systems exist [188].

The theory presented in this chapter is an extension of the theory of the stability region of continuous dynamical systems and provides a complete characterization of the stability boundary of these systems. This complete characterization allows the determination of the exact stability region and will provide the basis for the development of algorithms for estimating stability regions in Chapter 13.