3 Energy function theory

Energy functions play an important role in characterizing stability regions and offer a practical and effective way to estimate stability regions. Energy functions are useful, for example, for global analysis of system trajectories and predicting the structure of the limit sets of trajectories. A comprehensive energy function theory for general nonlinear autonomous dynamical systems will be presented in this chapter. This energy function theory can be applied to a variety of nonlinear dynamical systems. Applications of energy functions to optimally estimate stability regions of large-scale nonlinear systems and their theoretical basis will be developed in later chapters.

3.1 Energy functions

We consider a general nonlinear autonomous dynamical system described by the following equation:

$$\dot{x}(t) = f(x(t)). \tag{3.1}$$

We say a C^r-function $V{:}R^n \rightarrow R$, with $r \geq 1$, is an **energy function** for the system (3.1) if the following three conditions are satisfied.

(E1) The derivative of the energy function $V(x)$ along any system trajectory $x(t)$ is non-positive, i.e.

$$\dot{V}(x(t)) \leq 0.$$

(E2) If $x(t)$ is a nontrivial trajectory (i.e. $x(t)$ is not an equilibrium point), then along the nontrivial trajectory $x(t)$, the set $\{t \in R : \dot{V}(x(t)) = 0\}$ has measure zero in R.

(E3) That a trajectory $x(t)$ has a bounded value of $V(x(t))$ for $t \in R^+$ implies that the trajectory $x(t)$ is also bounded for $t \in R^+$. Stating this in brief: if $V(x(t))$ is bounded then $x(t)$ itself is also bounded.

Property (E1) indicates that the energy function is non-increasing along its trajectory, but it alone does not imply that the energy function is strictly decreasing along its trajectory. There may exist a time interval $[t_1, t_2]$ such that $\dot{V}(x(t)) = 0$ for $t \in [t_1, t_2]$. However, properties (E1) and (E2) together imply that the energy function is strictly decreasing along any system trajectory. Property (E3) states that the energy function is a proper map along any system trajectory, but it need not be a proper map for the entire state space.

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Recall that a proper map is a function $f{:}X \to Y$ such that for each compact set $D \in Y$, the set $f^{-1}(D)$ is compact in X. Property (E3), which can be viewed as a "dynamic" proper map, is useful in the characterization of a stability boundary. From the above definition of energy functions, it is obvious that an energy function may not be a Lyapunov function and a Lyapunov function may not be an energy function.

Example 3-1 Consider the class of gradient systems

$$\dot{x} = -\nabla V(x) \tag{3.2}$$

where $V{:}R^n \to R$ is a scalar, proper C^1-function. We will show that V is an energy function for system (3.2). Differentiating $V(x(t))$, one obtains:

$$\dot{V}(x) = \langle \nabla V(x), f(x) \rangle = -\|\nabla V(x)\|^2 \leq 0$$

and therefore condition (E1) holds. Moreover $\dot{V}(x) = 0$ if and only if x is an equilibrium point. Thus (E2) is also satisfied. The condition that $V(x)$ is proper is a sufficient condition for the satisfaction of (E3). Consequently, $V(x)$ is an energy function for system (3.2).

Example 3-2 We consider the following classical model for transient stability analyses in power systems and derive an energy function for it. Consider a power system consisting of n generators. Let the loads be modeled as constant impedances. Under the assumption that the transfer conductance of the reduced network, after eliminating all load buses, is zero, the dynamics of the ith generator can be represented by the equations

$$\begin{aligned} \dot{\delta}_i &= \omega_i \\ M_i \dot{\omega}_i &= P_i - D_i \omega_i - \sum_{j=1}^{n+1} V_i V_j B_{ij} \sin(\delta_i - \delta_j) \end{aligned} \tag{3.3}$$

where the voltage angle at node $i + 1$ is served as the reference, i.e. $\delta_{i+1}{:=}0$. This is a version of the so-called *classical model* of the power system, where M_i and D_i are positive constants for $i = 1, \ldots, n$ and B_{ij} are the elements of the susceptance matrix. We next show that there exists an energy function $V(\delta, \omega)$ for the classical model (3.3).

Consider the following function:

$$V(\delta, \omega) = \frac{1}{2} \sum_{i=1}^{n} M_i \omega_i^2 - \sum_{i=1}^{n} P_i (\delta_i - \delta_i^s) - \sum_{i=1}^{n} \sum_{j=i+1}^{n+1} V_i V_j B_{ij} \{\cos(\delta_i - \delta_j) - \cos(\delta_i^s - \delta_j^s)\} \tag{3.4}$$

where $x^s = (\delta^s, 0)$ is the stable equilibrium point of (3.3) under consideration.

Differentiating V along the trajectory $(\delta(t), \omega(t))$ of (3.3) gives

$$\dot{V}(\delta(t),\omega(t)) = \sum_{i=1}^{n}\left(\frac{\partial V}{\partial \delta_i}\dot{\delta}_i + \frac{\partial V}{\partial \omega_i}\dot{\omega}_i\right) = -\sum_{i=1}^{n}(D_i\omega_i^2) \leq 0 \tag{3.5}$$

and therefore condition (E1) of an energy function holds.

Suppose that there is an interval $t \in [t_1, t_2]$ such that

$$\dot{V}(\delta(t),\omega(t)) = 0, \quad t \in [t_1, t_2]. \tag{3.6}$$

Hence

$$\omega(t) = 0, \; t \in [t_1, t_2]. \tag{3.7}$$

But this implies $\omega(t) = 0$ and $\delta(t) =$ constant for $t \in [t_1, t_2]$. It then follows from (3.3) that

$$P_i - \sum_{j=1}^{n+1} V_iV_jB_{ij}\sin(\delta_i - \delta_j) = 0 \tag{3.8}$$

which are precisely the equations for the equilibrium point of (3.3). Therefore, ($\delta(t)$, $\omega(t)$), $t \in [t_1, t_2]$, must be on an equilibrium point. Consequently, condition (E2) holds.

In order to prove (E3), let us first integrate (3.3) for ω_i

$$\begin{aligned}\omega_i(t) = \; & e^{-D_i/M_it}\omega_i(0) \\ & + \int_0^t e^{-D_i/M_it(t-s)}\left\{P_i - \sum_{j=1}^{n+1} V_iV_jB_{ij}\sin(\delta_i(s) - \delta_j(s))\right\}ds\end{aligned}. \tag{3.9}$$

The term in the bracket is uniformly bounded, say, by a_i, i.e.

$$\left|P_i - \sum_{j=1}^{n+1} V_iV_jB_{ij}\sin(\delta_i(s) - \delta_j(s))\right| \leq a_i. \tag{3.10}$$

Since D_i and M_i are positive numbers, we have from (3.9):

$$|\omega_i(t)| \leq |\omega_i(0)| + a_i\frac{M_i}{D_i}. \tag{3.11}$$

That is, $\omega_i(t)$ is bounded by $b_i := |\omega_i(0)| + a_iM_i/D_i$. We next show that condition (E3) of the energy function is satisfied. Suppose $V(\delta(t), \omega(t))$ is bounded below and above, say, by c_1 and c_2 respectively. Then we have

$$\begin{aligned} c_1 - \frac{1}{2}\sum_{i=1}^{n} M_ib_i^2 < & -\sum_{i=1}^{n} P_i(\delta i - \delta_i^s) \\ & - \sum_{i=1}^{n}\sum_{j=i-1}^{n+1} V_iV_jB_{ij}\left\{\cos(\delta_i - \delta_j) - \cos(\delta_i^s - \delta_j^s)\right\} < c_2 \end{aligned} \tag{3.12}$$

But the second term on the right hand side is uniformly bounded, say, by c, i.e.

$$\left|\sum_{i=1}^{n}\sum_{j=i-1}^{n+1} V_iV_jB_{ij}\{\cos(\delta_i-\delta_j)-\cos(\delta_i^s-\delta_j^s)\}\right| < c. \tag{3.13}$$

Substituting (3.13) into (3.12), we get

$$c_1-\frac{1}{2}\sum_{i=1}^{n} M_i b_i^2 - c < -\sum_{i=1}^{n} P_i(\delta_i-\delta_i^s) < c_2 + c. \tag{3.14}$$

Hence, the term $P^T\delta$ is bounded, which implies $\delta(t)$ is bounded. This result, together with Eq. (3.11), asserts that the trajectory $(\delta(t), \omega(t))$ is bounded. Hence, condition (E3) of energy function is satisfied.

3.2 Energy function theory

The dynamic behaviors of general nonlinear systems can be very complicated. The asymptotic behavior (i.e. the ω-limit set) of trajectories can be manifested as equilibrium points, or closed orbits, or quasi-periodic trajectories or chaotic trajectories. However, as shown in Theorem 3-1, if the underlying dynamical system admits an energy function, then the system only allows simple trajectories. For instance, every trajectory of system (3.1) admitting an energy function has only two modes of behavior: its trajectory either converges to an equilibrium point or goes to infinity (becomes unbounded) as time increases or decreases. This result is explained in the following theorem.

THEOREM 3-1 **(Global behavior of trajectories)**
If there exists a function satisfying condition (E1) and condition (E2) of the energy function for system (3.1) and all equilibrium points are isolated, then every bounded trajectory of system (3.1) converges to one of the equilibrium points.

Proof Let S be the ω-limit set of the bounded trajectory $x(t)$. This set is non-empty according to Theorem 2.2. In order to prove that S consists of only equilibrium points, we prove that (a) S is contained in the set at which the derivative of function V is zero, and (b) $\hat{x} \notin S$ if $\hat{x} \notin E$.

Suppose that $\hat{x} \in S$. Then, there exists a sequence of increasing times $\{t_n\}$ such that $x(t_n) \rightarrow \hat{x}$ as $n \rightarrow \infty$. Condition (E1) and the boundedness of $x(t)$ ensures that $V(x(t))$ is a non-increasing function bounded from below. Thus there is a real number α such that $V(x(t)) \rightarrow \alpha$ as $t \rightarrow \infty$. In particular, $V(x(t_n)) \rightarrow \alpha$ as $n \rightarrow \infty$. Therefore, by the continuity of V, $V(\hat{x}) = \alpha$ for every $\hat{x} \in S$. The invariance of S implies that $\dot{V}(\hat{x}) = 0$ for every $\hat{x} \in S$. Thus S is contained in the set at which the derivative of V is zero and (a) is true.

Suppose now that $\hat{x} \in S$ and $\hat{x} \notin E$. Since S is an invariant set and S is contained in the set at which the derivative of V is zero, there is an interval I at which the derivative of the solution passing through $\hat{x} \in S$ is zero. This contradicts condition (E2). Thus $x \in E$ and (b) is true. The connectedness of S and the fact that all equilibrium points are isolated

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guarantee that every bounded trajectory of (3.1) must converge to one of the equilibrium points. This completes the proof.

Theorem 3-1 asserts that there does not exist any limit cycle (oscillation behavior) or bounded complicated behavior (such as an almost periodic trajectory, chaotic motion, etc.) in the system. In Theorem 3-1 we have shown that the trajectory of system (3.1) either converges to one of the equilibrium points or goes to infinity. We next show a sharper result, asserting that every trajectory on the stability boundary must converge to one of the equilibrium points on the stability boundary.

THEOREM 3-2 **(Trajectories on the stability boundary)**
If there exists an energy function for system (3.1), then every trajectory on the stability boundary $\partial A(x_s)$ converges to one of the equilibrium points on the stability boundary $\partial A(x_s)$.

The significance of this theorem is that it offers an effective way to characterize the stability boundary. In fact, Theorem 3-2 asserts that the stability boundary $\partial A(x_s)$ is contained in the union of stable manifolds of the unstable equilibrium points (UEPs) on the stability boundary. One corollary of Theorem 3-2 shown below provides a characterization of stability boundaries.

THEOREM 3-3 **(Energy function and stability boundary)**
If there exists an energy function for system (3.1) which has an asymptotically stable equilibrium point x_s (but not globally asymptotically stable), then the stability boundary $\partial A(x_s)$ is contained in the set which is the union of the stable manifolds of the UEPs on the stability boundary $\partial A(x_s)$, i.e.

$$\partial A(x_s) \subseteq \bigcup_{x_i \in \{E \cap \partial A(x_s)\}} W^s(x_i).$$

The following two theorems give interesting results on the structure of the equilibrium points on the stability boundary, and present necessary condition for the existence of certain types of equilibrium points on a *bounded* stability boundary.

THEOREM 3-4 **(Structure of equilibrium points on the stability boundary)**
If there exists an energy function for the system (3.1) which has an asymptotically stable equilibrium point x_s (but not globally asymptotically stable), then the stability boundary $\partial A(x_s)$ must contain at least one type-one equilibrium point. If, furthermore, the stability region is bounded, then the stability boundary $\partial A(x_s)$ must contain at least one type-one equilibrium point and one source.

The contra-positive of Theorem 3-4 leads to the following corollary, which is useful in predicting the unboundedness of a stability region.

THEOREM 3-5 **(Sufficient condition for an unbounded stability region)**
If there exists an energy function for the system (3.1) which has an asymptotically stable equilibrium point x_s (but not globally asymptotically stable) and if $\partial A(x_s)$ contains no source, then the stability region $A(x_s)$ is unbounded.

3.3 Generalized energy functions

In this section, generalized energy functions, which are generalizations of energy functions and Lyapunov functions, are presented. One distinguishing feature of a generalized energy function is that its derivative along system trajectories can be positive in some bounded sets, while the derivative along a system trajectory of Lyapunov functions and of energy functions must be negative semi-definite. Generalized energy functions are useful in providing global information about the limit sets of general nonlinear systems, including those systems exhibiting complex behaviors in their limit sets. In addition, generalized energy functions can be explored to estimate stability regions of complex attractors.

It was shown in Section 3.2 that the existence of an energy function implies that (i) the limit set of the underlying dynamical system is composed only of equilibrium points, (ii) every bounded trajectory converges to an equilibrium point, and therefore (iii) there is no limit cycle or other complex behavior in these systems. As a consequence, nonlinear dynamical systems that exhibit complex behavior, such as closed, quasi-periodic orbits and chaos, cannot admit energy functions.

Many nonlinear system models exhibit complex behavior, such as closed orbits and chaos in their limit sets, see for example [19,70,165]. For instance, extended power system transient stability models can exhibit equilibrium points, closed orbits, quasi-periodic solutions and chaos in their limit sets [57,240,263]. Consequently, energy function theory is not applicable to this class of nonlinear model. To this end, by following the same "spirit" as energy functions, we generalize an energy function such that it has a zero derivative at every limit point. In [24] for instance, the existence of Lyapunov-like functions with derivatives equal to zero in the attracting set was studied. However, the complex structure of limit sets makes the task of finding an energy function difficult even if one can prove its existence.

A generalized energy function is a practical alternative for analyzing the dynamics of systems that exhibit complex behaviors in their limit sets. The feature of generalization is achieved by allowing the derivative of an energy function to have positive values in some bounded sets. This feature will be further explored in Chapter 11 to obtain estimates of the stability region of an attractor for nonlinear dynamical systems exhibiting complex behaviors such as closed orbits, quasi-periodic orbits and chaos.

Let $V: R^n \to R$ be a C^r-function, $r \geq 1$, and define the following set:

$$C : \{x \in R^n : \dot{V}(x) \geq 0\} \tag{3.15}$$

composed of the points where the derivative of $V(x)$ is positive. Set C is generally composed of several connected components, denoted by C_i, the ith connected component of C. These components are generally isolated (i.e. there exists a collection of disjoint open sets D_{is} satisfying $C_i \subset D_i$ for every i). If zero is a regular value of $\dot{V}$, then the boundary of set C is an $(n{-}1)$-dimensional C^{r-1} submanifold of R^n.

A C^r function $V{:}R^n \to R$, with $r \geq 1$, is a generalized energy function for system (3.1) if it satisfies the following three conditions:

(G1) the number of connected components C_i of C is finite;
(G2) every component C_i is bounded and
(G3) that a trajectory $x(t)$ has a bounded value of $V(x(t))$ for $t \in R^+$ implies that the trajectory $x(t)$ is also bounded.

Condition (G3) is identical to condition (E3) for an energy function. Conditions (G1) and (G2) imply that the derivative of generalized energy functions along trajectories can be positive in some bounded sets C_{is}. In contrast to energy functions, the definition of a generalized energy function does not rely on any type of limit set. It allows a variety of limit sets with complex behaviors including chaotic, quasi-periodic and closed trajectories. In the next section, the implications of the existence of a generalized energy function on global dynamics of nonlinear dynamical systems will be presented.

3.4 Generalized energy function theory

It was shown in Section 3.2 that the limit sets of nonlinear dynamical systems admitting an energy function are strictly located on the set $M = \{x \in R^n : \dot{V}(x) = 0\}$, where the derivative of the energy function is zero. The existence of generalized energy functions can also provide useful information about the location of limit sets. We next show that the existence of a generalized energy function ensures that the limit sets of the underlying system have to intersect the bounded sets C_{is} where the derivative of the generalized energy function is non-negative. Nevertheless, the existence of a generalized energy function does not preclude the possibility of the limit set being entirely located on the set $M = \{x \in R^n : \dot{V}(x) = 0\}$, where the derivative of the generalized energy function is zero.

THEOREM 3-6 **(Location of the ω-limit set)**
Let $V(x)$ be a generalized energy function for system (3.1). Suppose the trajectory $\phi(t, x_0)$ of the dynamical system (3.1) is bounded for $t \geq 0$. Then there exists at least one component C_j of C such that $\omega(x_0) \cap C_j \neq \emptyset$.

Proof Since $\phi(t, x_0)$ is bounded, the ω-limit set $\omega(x_0)$ is a non-empty, closed, invariant connected set. Suppose $\phi(t, x_0) \notin C$ for all $t \geq 0$. Then $V(t) = V(\phi(t, x_0))$ is a non-increasing function of t bounded from below. Hence, there exists a real number l such that $V(t) \to l$ as $t \to \infty$. If $p \in \omega(x_0)$, there exists a sequence of times $\{t_n\} \to \infty$ such that $\phi(t_n, x_0) \to p$ as $n \to \infty$. Therefore $V(\phi(t_n, x_0)) \to l$ as $n \to \infty$ and due to the continuity of V, $V(p) = l$. Since this is true for any point in $\omega(x_0)$, $\omega(x_0) \subset \{x \in R^n : x \in V^{-1}(L)\}$. Using the invariance of $\omega(x_0)$, we conclude that $\dot{V}(p) = 0$ for any $p \in \omega(x_0)$, and so $\omega(x_0) \subset M \subset C$. The connectedness of the limit set $\omega(x_0)$ guarantees the existence of a component C_j such that $\omega(x_0) \subset C_j$.

Suppose now the trajectory $\phi_t(x_0) := \{\phi(t, x_0) \in R^n : t \geq 0\}$ has a non-empty intersection with set C. Then there exists a connected component C_{j_1} such that either $x_o \in C_{j_1}$ or there exists a pair of times t_1 and t_1^* such that $\phi(t, x_0) \notin C$ for $0 \leq t < t_1$ and $\phi(t, x_o) \in C_{j_1}$ for

$t_1 \le t \le t_1^*$. If $\phi(t,\, x_0)$ stays inside C_{j_1} for all $t \ge t_1$, that is $t_1^* = +\infty$, then $V(t)$ is a non-decreasing function of t bounded from above for $t \ge t_1$. Using arguments similar to those used in the first part of the proof, we conclude that $\omega(x_o) \subset C_{j_1}$.

If $t_1^* < \infty$, two dynamical behaviors can occur. Either $\phi(t,\, x_0) \notin C$ for $t \ge t_1^*$ or there exists a connected component C_{j_2} and a pair of times t_2 and t_2^* such that $\phi(t,\, x_0) \notin C$ for $t_1^* < t < t_2$ and $\phi(t, x_o) \in C_{j_2}$ for $t_2 \le t \le t_2^*$. Beyond this point, the analysis is repeated. If the number of times this analysis is repeated is finite, then $\omega(x_0) \subset M$ and $\omega(x_0) \subset C_j$ for some j. Otherwise, there exists a sequence of times $\{t_n\} \to \infty$ and a sequence of connected components C_{j_n} such that $\phi(t_n, x_o) \in C_{j_n}$. Since the number of connected components C_i of C is finite, there exists at least one component C_{j_k} that is visited by the trajectory an infinite number times. In other words, there exists a subsequence of times t_{n_i} of $\{t_n\}$ such that $x_i = \phi(t_{n_i}, x_o) \in C_{j_k}$. Since C_{j_k} is a compact set, there exists a convergent subsequence $\{x_{i_v}\}$ of $\{x_i\}$ converging to some point $\widetilde{x} \in C_{j_k}$. By definition, $\widetilde{x}$ is an ω-limit point of x_0 and so $\omega(x_o) \cap C_{j_k} \neq \emptyset$. This completes the proof.

Theorem 3-6 provides great insight into the location of the limit sets of bounded trajectories for the class of nonlinear dynamical systems (3.1) that admit generalized energy functions. It asserts that the ω-limit set of bounded solutions must intersect at least one connected component C_i of C. Figure 3.1 illustrates the two possible limit sets of bounded trajectories. It is important to emphasize that the ω-limit set of complex nonlinear systems can intersect more than one connected component C_j of C as shown in Figure 3.2.

Of all the possible limit sets, the equilibrium points must lie on the following set

$$M = \{x \in R^n : \dot{V}(x) = 0\} \tag{3.16}$$

for all nonlinear dynamical systems admitting either an energy function or a generalized energy function. The generalized energy function does not preclude the possibility of other types of limit sets (such as closed orbits, quasi-periodic orbits, and chaotic orbits) lying on M, even though this can be rare given the complex nature of these sets.

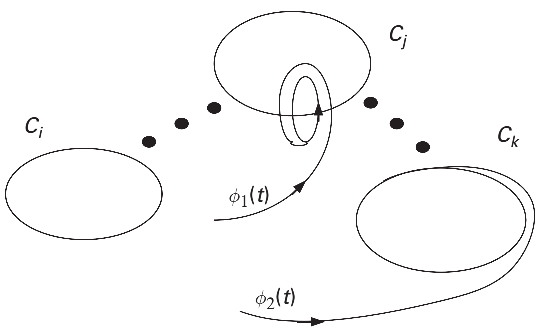

Figure 3.1 Illustration of Theorem 3-6. Two cases can occur: (i) the bounded trajectory $\phi_1(t)$ has a non-empty intersection with the interior of C_j or (ii) $\phi_2(t)$ approaches the set M, where the derivative of the generalized energy function equals zero on the boundary of component C_k.

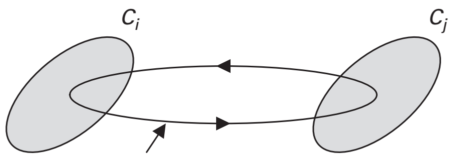

Figure 3.2 Illustration of Theorem 3-6. The ω-limit set can intersect with more than one connected component C_j of C. For instance, a limit cycle can have a non-empty intersection with two connected components C_i and C_j of C.

3.5 Energy functions for second-order dynamical systems

Energy functions can give sharp information regarding the global behavior of nonlinear dynamical systems. While the task of finding energy functions is not trivial, there is no systematic procedure to search for energy functions and for generalized energy functions. We next show how to derive an energy function for an important class of nonlinear dynamical systems.

Many physical systems are modeled by a second-order nonlinear dynamical system of the form:

$$M\ddot{x} + D\dot{x} + f(x) = 0$$

whose state space representation is:

$$\begin{aligned} \dot{x} &= y \\ M\dot{y} &= -Dy - f(x) \end{aligned} \tag{3.17}$$

where M is a diagonal matrix with positive elements, D is a symmetric, diagonally dominant matrix with positive diagonal elements and $f{:}R^n \to R^n$ is a C^1-class function. Let E denote the set of equilibrium points of (3.17) and suppose the number of equilibrium points on any stability boundary is finite. We use the notation $d(M,D)$ to denote system (3.17). We next present a sufficient condition for the existence of an energy function and also show how to derive an energy function for system (3.17).

THEOREM 3-7 **(Sufficient condition)**
If $f(x)$ is a conservative vector field, i.e. there exists a scalar C^1-function $V_p{:}R^n \to R$ such that $f = \nabla V_p$, then there exists a C^1 function $V{:}R^{2n} \to R$ for $d(M,D)$ such that

(a) $\dot{V}(x(t), y(t)) \le 0$,
(b) Let $(x(0), y(0)) \notin E$, *then the set* $\{t \subset R : \dot{V}(x(t), y(t)) = 0\}$ *has measure zero in R.*

Proof We define $V{:}R^n \times R^n \to R$ by

$$V(x,y) = \frac{1}{2}\langle y, My\rangle + V_p(x). \tag{3.18}$$

The derivative of $V(\cdot)$ along the trajectory of $d(M,D)$ is

$$\begin{aligned}\dot{V}(x,y) &= \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial y}\dot{y} \\ &= -\langle y, Dy\rangle \leq 0.\end{aligned} \tag{3.19}$$

So, part (a) is true. Suppose that part (b) is not true, then there exists an interval $T = (t_1, t_2)$ with $t_2 > t_1 \geq 0$ such that $\dot{V}(x(t), y(t)) = 0$ for $t \in T$. From (3.19), we have $y(t) = 0$ for $t \in T$. This implies that $y(t) = 0$ and $x(t) =$ constant, for $t \in (t_1, t_2)$. From (3.17), this implies that $f(x(t)) = 0$. So, we have $(x(t), y(t)) \in E$ for $t \in (t_1, t_2)$. Since (3.17) is an autonomous dynamical system, it follows that $(x(t), y(t)) \in E$ for $t \in R$. This contradicts the fact that $(x(0), y(0)) \notin E$. Therefore, part (b) is also true. The proof is complete.

Theorem 3-7 shows that function (3.18) satisfies conditions (E1) and (E2) of an energy function. Thus, as a direct consequence of Theorem 3-1, the ω-limit set of every bounded trajectory of system (3.17) consists of only equilibrium points. In other words, the class of systems in the form of (3.17) does not admit complex behavior such as limit cycles or chaos. We hence conclude that every bounded trajectory of system (3.17) converges to one of the equilibrium points.

THEOREM 3-8 **(Existence of an energy function)**
Consider the nonlinear system $d(M,D)$. If function f is bounded and $V_p(x)$ is a proper function, then there exists a C^1-energy function $V: R^{2n} \rightarrow R$ for $d(M,D)$.

Proof According to Theorem 3-7, function (3.18) satisfies conditions (E1) and (E2) of an energy function. We conclude the proof of this theorem by showing that function (3.18) also satisfies condition (E3) of an energy function under the condition that f is bounded and $V_p(x)$ is a proper function. From system (3.17), one obtains:

$$\dot{y} = -M^{-1}Dy - M^{-1}f(x). \tag{3.20}$$

As a direct application of the variation of constants formula, one has:

$$||y(t)|| \leq \left\|e^{At}\right\| ||y_o|| + \int_0^t \left\|e^{A(t-s)}\right\| \left\|M^{-1}\right\| ||f(x(s))|| ds \tag{3.21}$$

with $A = -M^{-1}D$. Since both M and D are diagonal matrices with positive elements, then $A = -M^{-1}D$ has all eigenvalues on the left side of the complex plane. Therefore, there exist real constants $C > 0$ and $\alpha > 0$ such that $\left\|e^{At}\right\| \leq Ce^{-\alpha t}$ for all $t \geq 0$. If b is a bound for the norm of f, one has:

$$||y(t)|| \leq Ce^{-\alpha t}||y_o|| + \int_0^t Ce^{-\alpha(t-s)}\left\|M^{-1}\right\| b ds. \tag{3.22}$$

Then we conclude, after some calculation, that:

$$||y(t)|| \leq C||y_o|| + \frac{C\left\|M^{-1}\right\| b}{\alpha}. \tag{3.23}$$

In other words, $y(t)$ is bounded. Suppose now that the trajectory $(x(t), y(t))$ is unbounded for $t \geq 0$ while the value of $V(x,y)$ calculated along this trajectory is bounded.

Since $y(t)$ is always bounded, we conclude: (i) that $x(t)$ must be unbounded and (ii) the term $V_p(x)$ must be bounded. This contradicts the hypothesis that $V_p(x)$ is a proper function. This completes the proof.

Consider now the following class of second-order nonlinear dynamical systems:

$$\begin{aligned} \dot{x} &= y \\ M\dot{y} &= -Dy = \frac{\partial W(x)}{\partial x} + \varepsilon g(x) \end{aligned} \tag{3.24}$$

where M and D are diagonal matrices with positive entries and ε is a small real number. Function $W{:}R^n \to R$ is C^2 and g is a uniformly bounded C^1 function. System (3.24) is a perturbed version of system (3.17) in which the perturbation $g{:}R^n \to R^n$ is a non-gradient vector field. This class of systems appears, for example, on transient stability analysis of power systems in the presence of transfer conductances [29,47].

Nonlinear dynamical systems in the form of (3.24) do not admit energy functions. For instance, the following simple power system model,

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{x}_2 &= y_2 \\ 0.053\dot{y}_1 &= 1.78 - 3.16\sin x_1 - 0.28\cos x_1 - 0.9\sin(x_1 - x_2) - \varepsilon\cos(x_1 - x_2) - 0.1y_1 \\ 0.079\dot{y}_1 &= 3.83 - 7.85\sin x_2 - 0.255\cos x_2 - 0.9\sin(x_2 - x_1) - \varepsilon\cos(x_2 - x_1) - 0.1y_2, \end{aligned} \tag{3.25}$$

can be put into the form (3.24) by choosing the following function:

$$\begin{aligned} W(x) = &-1.78x_1 - 3.83x_2 - 3.16\cos x_1 + 0.28\sin x_1 - 7.85\cos x_2 \\ &+0.255\sin x_2 - 0.9\cos(x_1 - x_2). \end{aligned}$$

It can be shown that this simple model admits a limit cycle for $\varepsilon > 3.3$. It is therefore inferred that system (3.25) cannot admit an energy function, according to Theorem 3-1. Figure 3.3 shows this limit cycle for $\varepsilon = 3.5$ and Figure 3.4 depicts a bifurcation diagram of this system. For $\varepsilon > 3.3$, this system exhibits complex behavior and cannot admit an energy function.

Although nonlinear dynamical systems in the form of (3.24) do not admit energy functions, it is possible to show the existence of a general generalized energy function for this class for sufficiently small ε. For this purpose, we assume system (3.24) has a finite number of isolated equilibrium points. We next show the following function with $\beta > 0$ is a generalized energy function of system (3.24) for sufficiently small ε:

$$V(x,y) = \frac{1}{2}y^T My + W(x) + \beta\left[\frac{\partial W(x)}{\partial x} - \varepsilon g(x)\right]^T y. \tag{3.26}$$

PROPOSITION 3-9 **(Conditions (G1) and (G2))**
Consider the nonlinear dynamical system (3.24) where $W(x)$: $R^n \to R$ is a C^2 function and $g{:}R^n \to R^n$ is a C^1 function. If g is a bounded function, then for a sufficiently small ε, function (3.26) satisfies conditions (G1) and (G2) of a generalized energy function.

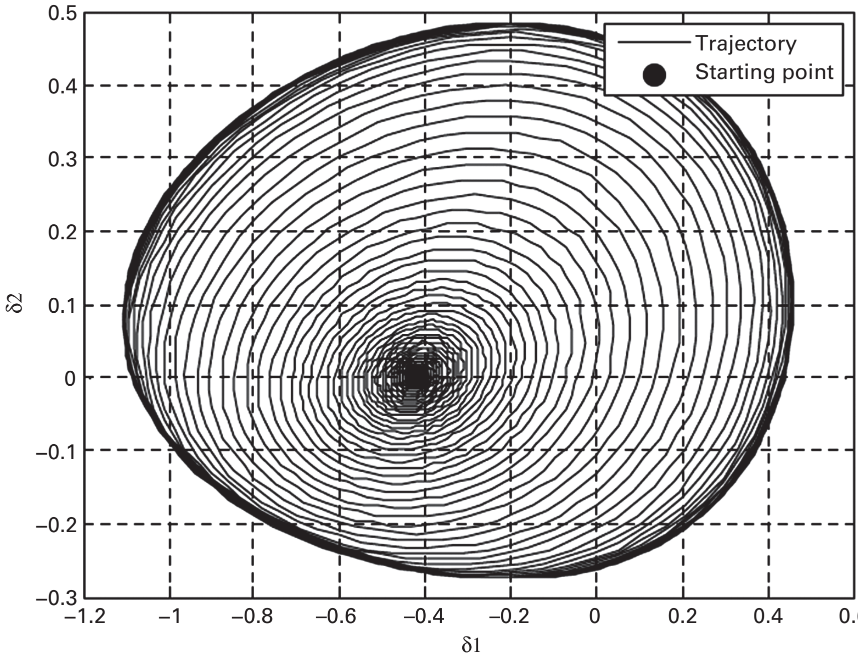

Figure 3.3 Limit cycle of system (3.25) for ε = 3.5 projected to the space $\omega_1 = \omega_2 = 0$.

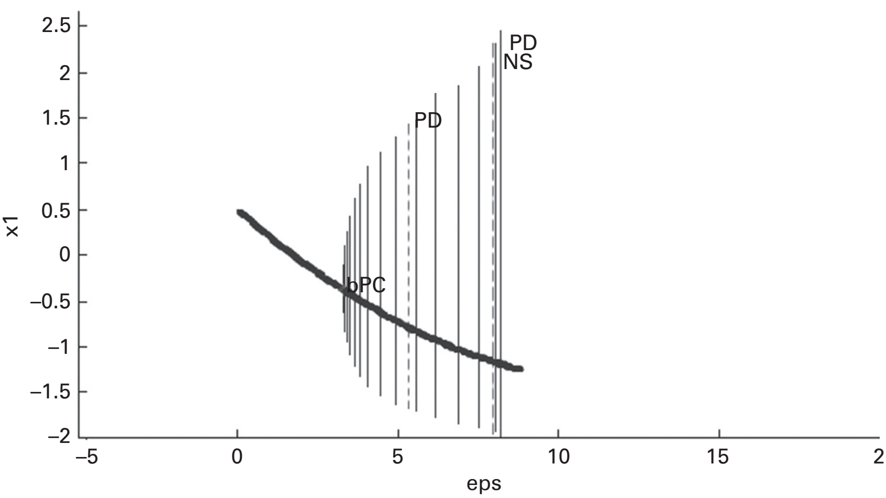

Figure 3.4 Bifurcation diagram of system (3.25). At ε =3.3 a Hopf bifurcation occurs. The asymptotic stable equilibrium point loses stability and an asymptotic stable limit cycle is created.

Proof The derivative of V along the trajectory of (3.24) is given by:

$$\dot{V}(x,y) = -\left[y^T\left[\frac{\partial W(x)}{\partial x} - \varepsilon g(x)\right]^T\right] Q \left[\frac{y}{\frac{\partial W(x)}{\partial(x)}} - \varepsilon g(x)\right] + \varepsilon g^T(x) y$$

where

$$Q = \begin{bmatrix} \left(D - \beta\dfrac{\partial^2 W(x)}{\partial x^2} + \beta\varepsilon\dfrac{\partial g}{\partial x}\right) & \left(\dfrac{1}{2}\beta D M^{-1}\right) \\ \dfrac{1}{2}\beta D M^{-1} & \beta M^{-1} \end{bmatrix}.$$

First of all, we show that matrix Q is positive definite for a small enough $\beta > 0$. By the continuity of the determinants with respect to the matrix entries, the following relationship holds for sufficiently small β,

$$A: \ = D - \beta\frac{\partial^2 W(x)}{\partial x^2} + \beta\varepsilon\frac{\partial g}{\partial x} > 0.$$

Let A_{ij} denote the minor of A obtained by deleting the ith row and jth column. Consider now the following minor of matrix Q:

$$B_1 := \begin{bmatrix} & \frac{1}{2}\beta D_1 M_1^{-1} \\ A & 0 \\ & 0 \\ & \cdots \\ & 0 \\ \frac{1}{2}\beta D_1 M_1^{-1} 0\, 0 \ldots 0 & \beta M_1^{-1} \end{bmatrix}.$$

Using the Laplacian expansion of determinants by minors, one obtains

$$\det B_1 = \beta M_1^{-1}\left[\det A - \frac{1}{4}\beta D_1^2 M_1^{-1}\det A_{11}\right].$$

Since $A > 0$ and symmetric, then $\det A > 0$ and $\det A_{11} > 0$. As a consequence, $\det B_1 > 0$ if $0 > \beta < \dfrac{4M_1 \det A}{D_1^2 \det A_{11}}$. We next consider

$$B_2 : \begin{bmatrix} & \frac{1}{2}\beta D_2 M_2^{-1} \\ B_1 & 0 \\ & 0 \\ & \cdots \\ & 0 \\ 0\,\frac{1}{2}\beta D_2 M_2^{-1} 0\, 0 \ldots 0 & \beta M_2^{-1} \end{bmatrix}.$$

It follows that

$$\det B_2 = \beta M_2^{-1}\left[\det B_1 - \frac{1}{4}\beta D^2 M_2^{-1}\det B_{1_{22}}\right].$$

Since $\det B_1 = O(\beta) > 0$ and $\det B_{1_{22}} = O(\beta) > 0$, $\det B_2 > 0$ for sufficiently small β. Repeating this procedure n times we prove that the determinant of all main minors of Q are positive for sufficiently small $\beta > 0$. Then, as an implication of Sylvester's criterion, $Q > 0$.

The quadratic term of V equals zero only at the set of equilibrium points E. Since g is uniformly bounded, for ε sufficiently small, the regions where the derivative of V is positive are contained in small bounded connected sets C_{i_s} close to the equilibrium points. Since the equilibrium points are isolated, for sufficiently small ε, the sets C_{i_s} are isolated; every two sets C_i and C_j have an empty intersection and the distance between them is greater than zero. Therefore, condition (G2) is satisfied. A finite number of equilibrium points on the stability boundary proves condition (G1). This completes the proof.

To examine condition (G3), the vector field for this class of dynamical systems is explored. The next theorem provides some sufficient conditions to ensure that condition (G3) is satisfied.

PROPOSITION 3-10 **(Condition (G3))**
If W is a proper function and both $\partial W/\partial x$ and g are bounded, then the function $V(x)$ of (3.26) satisfies condition (G3).

Proof Since both $\partial W/\partial x$ and g are bounded, as a direct application of the variation of constants formula, we prove that $y(t)$ is always bounded. Suppose by contradiction that $\sup_{t\geq 0}\|V\| < \infty$ and $\phi(t, x_0) = (x(t), y(t))$ is unbounded for $t \geq 0$. Since $y(t)$ is bounded, then $x(t)$ must be unbounded for $t \geq 0$. This implies that $\sup_{t\geq 0}\|W(x)(t)\| = \infty$. Since $\|y(t)\|$ is bounded for $t \geq 0$, $\sup_{t\geq 0}\|W(x(t))\| = \infty$ and both $\partial W/\partial x$ and g are uniformly bounded, we conclude that $\sup_{t\geq 0}\|V\| = \infty$. This is a contradiction, and so $\sup_{t\geq 0}\|V\| < \infty$ implies that $\phi(t, x_0) = (x(t), y(t))$ is bounded for $t \geq 0$. This concludes the proof.

3.6 Numerical studies

We illustrate the analytical results of generalized energy function developed in this chapter on several simple numerical examples taken from power system models, nonlinear control models and Lorenz systems.

Example 3-3 (Power system stability model) The following system of equations was obtained from the power system literature and was proposed to model the dynamical behavior of a two-generator system versus an infinite bus with transfer conductance [29]:

$$\begin{cases} \dot{x}_1 = y_1 \\ \dot{x}_2 = y_2 \\ M_1\dot{y}_1 = P_1 - G_1 \sin x_1 - B_1 \cos x_1 \\ \qquad - G_{12} \sin(x_1 - x_2) - \varepsilon \cos(x_1 - x_2) - D_1 y_1 \\ M_2\dot{y}_2 = P_2 - G_2 \sin x_2 - B_2 \cos x_2 \\ \qquad - G_{12} \sin(x_2 - x_1) - \varepsilon \cos(x_2 - x_1) - D_2 y_2. \end{cases} \tag{3.27}$$

Parameter ε represents the transfer conductance. For $\varepsilon = 0$, this system admits a general energy function:

$$W(x_1,x_2) := -P_1x_1 - G_1\cos x_1 + B_1\sin x_1 - P_2x_2$$
$$-G_2\cos x_2 + B_2\sin x_2 - G_{12}\cos(x_1 - x_2) + \alpha$$

where α is an arbitrary constant. We leave it for the reader to show that W is an energy function for system (3.27) if $\varepsilon = 0$. However, it has been shown that a general energy function does not exist for this system when $\varepsilon \neq 0$ [47]. With the development of generalized energy functions, a generalized energy function that satisfies conditions (G1)–(G3) can be derived.

The previous set of differential equations can be put in the general form (3.24) by choosing $g(x_1, x_2)$: $\cos(x_1 - x_2)$. It is straightforward to check that (i) both $\partial W/\partial x$ and g are uniformly bounded, and (ii) although function W is not proper, condition (G3) is generically satisfied. Hence, the following function

$$\begin{aligned} V(x_1,x_2,y_1,y_2) = M_1\frac{y_1^2}{2} + M_2\frac{y_2^2}{2} + W(x_1,x_2) \\ -\beta y_1[P_1 - G_1\sin x_1 - B_1\cos x_1 - G_{12}\sin(x_1 - x_2), \\ -\varepsilon\cos(x_1 - x_2)] \\ -\beta y_2[P_2 - G_2\sin x_2 - B_2\cos x_2 - G_{12}\sin(x_2 - x_1) \\ -\varepsilon\cos(x_2 - x_1)] \end{aligned}$$

is a generalized energy function provided both $\beta > 0$ and ε are small enough.

An estimate for β can be obtained by computing the derivative of this function along the system orbits:

$$\begin{aligned} -\dot{V} = & \begin{bmatrix} P_{l_1}(x_1,x_2) \\ y_1 \\ P_{l_2}(x_1,x_2) \\ y_2 \end{bmatrix}^T A \begin{bmatrix} P_{l_1}(x_1,x_2) \\ y_1 \\ P_{l_2}(x_1,x_2) \\ y_2 \end{bmatrix} + \\ & \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}^T B \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \varepsilon[\cos(x_1 - x_2) - 1(y_1 + y_2)] \end{aligned}$$

where

$$A = \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \end{bmatrix}, \quad B = \begin{bmatrix} \dfrac{D_1}{2} & \beta G_{12}\cos(x_1 - x_2) \\ \beta G_{12}\cos(x_1 - x_2) & \dfrac{D_2}{2} \end{bmatrix},$$

$$A_{11} = \begin{bmatrix} \dfrac{\beta}{M_1} & -\dfrac{\beta D_1}{2M_1} \\ -\dfrac{\beta D_1}{2M_1} & \begin{array}{c} \dfrac{D_1}{2} + \beta[-G_1\cos x_1 + B_1\sin x_1 \\ -G_{12}\cos(x_1 - x_2) + \varepsilon\sin(x_1 - x_2)] \end{array} \end{bmatrix}$$

and

$$A_{22} = \begin{bmatrix} \dfrac{\beta}{M_2} & -\dfrac{\beta D_2}{2M_2} \\ -\dfrac{\beta D_2}{2M_2} & \begin{array}{c} \dfrac{D_2}{2} + \beta[-G_2\cos x_2 + B_2\sin x_2 \\ -G_{12}\cos(x_2 - x_1) + \varepsilon\sin(x_2 - x_1)] \end{array} \end{bmatrix}$$

The parameter β can be chosen to make the quadratic term positive definite. By applying Silvester's criterion one can easily find that the positive definiteness is guaranteed if

$$\beta^2 < \frac{D_1 D_2}{4G_{12}^2}, \quad 0 < \beta < \frac{D_1}{2\left(\frac{D_1^2}{4M} + G_1 + B_1 + G_{12} + \varepsilon\right)}$$

and

$$0 < \beta \frac{D_2}{2\left(\frac{D_2^2}{4M} + G_2 + B_2 + G_{12} + \varepsilon\right)}.$$

The set with a positive derivative of $V(x_1, x_2, y_1, y_2)$ is composed of two small bounded sets C_1 and C_2. These sets intersect the stability boundary, and they are close to the unstable equilibrium points (0.91,0,2.72,0) and (2.71,0,0.6,0). Their intersections with the subset $\{(x_1, x_2, y_1, y_2) : x_1 \in R, x_2 \in R, y_1 = y_2 = 0.8\}$ are depicted in Figure 3.5 with the following parameters: $P_1 = 1.78$, $P_2 = 3.83$, $G_1 = 3.16$, $G_2 = 7.85$, $B_1 = 0.28$, $B_2 = 0.255$, $G_{12} = 0.9$, $\varepsilon = 0.1$, $D_1 = D_2 = 0.1$, $M_1 = 0.053$, $M_2 = 0.079$, $\alpha = 13.017$ and $\beta = 0.005$.

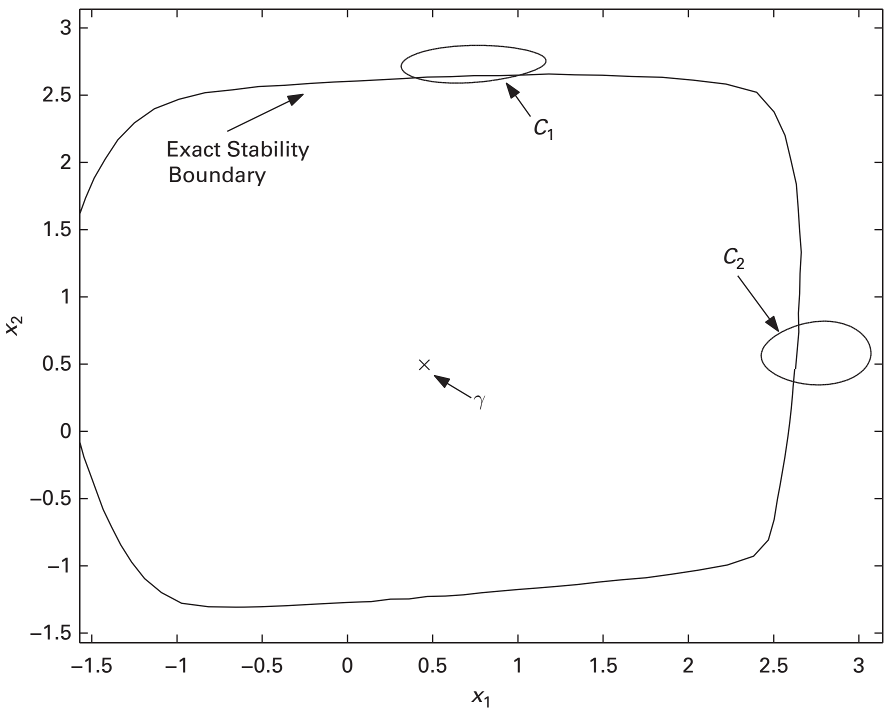

Figure 3.5 The set with a positive derivative of $V(x_1, x_2, y_1, y_2)$ is composed of two small bounded sets C_1 and C_2. These sets intersect the stability boundary, and they are close to the unstable equilibrium points (0.91,0,2.72,0) and (2.71,0,0.6,0).

Example 3-4 (Lorenz systems) Consider the Lorenz system:

$$\begin{cases} \dot{x} = -\sigma x + \sigma y \\ \dot{y} = -y - xz + rx \\ \dot{z} = -bz + xy \end{cases}$$

where, $\sigma = 10$, $r = 28$ and $b = 8/3$. The Lorenz system possesses a globally stable chaotic attractor and as a consequence it does not admit an energy function. Let us show that the following candidate:

$$V(x, y, z) = rx^2 + 4\sigma y^2 + 4\sigma(z - 5/4r)^2$$

is a generalized energy function for the Lorenz systems. It is obvious that function $V(x, y, z)$ is radially unbounded; hence, condition (G3) is satisfied. The derivative of function $V(x, y, z)$ along the system trajectory is given by:

$$\dot{V}(x, y, z) = -2\sigma(r\,x^2 + 4\,y^2 + 4\,bz^2 - 5\,rbz).$$

Set C is given by $C : \{x \in R^3 : r\,x^2 + 4y^2 + 4\,bz^2 - 5\,rbz < 0\}$. It is straightforward to verify that the boundary of set C is an ellipsoid centered at ($x = 0$, $y = 0$, $z = 5/8r$). Consequently, set C is composed of a single bounded component and conditions (G1) and (G2) are also satisfied. Thus function $V(x, y, z)$ is a generalized energy function for the Lorenz system. Every trajectory of the Lorenz system is bounded and must intersect set C. Figure 3.6 illustrates this intersection.

Example 3-5 (Nonlinear control of DC motor) Consider the nonlinear system:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -k_d x_2 - x_1 - g x_1^2\left(\frac{x_2}{k_d} + x_1 + 1\right) + u \end{aligned} \tag{3.28}$$

with $k_d = 1$ and $g = 6$. Function u is the control input. This model comes from the problem of speed control in a direct current machine using the field voltage as a control input and taking into account some nonlinearities related to the back electromotive force [92].

It is easily verified that the origin is an asymptotically stable equilibrium point of the open loop system for every $k_d > 0$. However, for $g > 4$ the system has three equilibrium points, indicating that the origin is not a globally stable equilibrium point. In this example, the concept of a generalized energy function and some peculiarities of this system will be explored to design a feedback control law $u = h(x_1, x_2)$ so that global stability of the origin is achieved. To this end, we consider the following candidate for a generalized energy function:

$$V(x, u) = k_1 x_1^2 + k_2 x_2^2 + k_3 x_1 x_2 + k_4 g \frac{x_1^4}{4} + k_5 g \frac{x_1^3}{3} \tag{3.29}$$

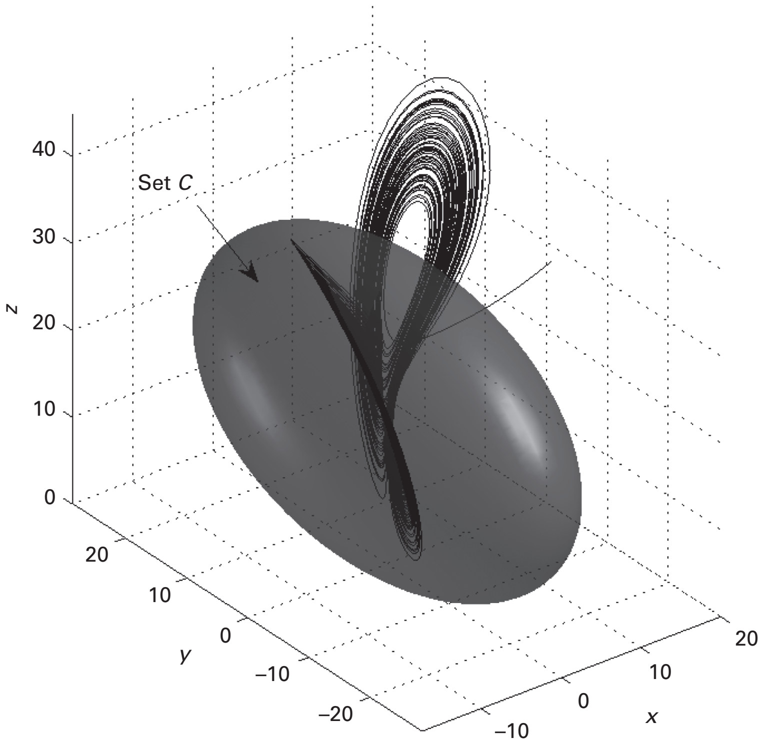

Figure 3.6 The Lorenz attractor intersects set C where the derivative of the generalized energy function is positive.

Choosing $k_1 = (k_3 k_d/_2) + k_2, k_4 = 2k_2 + k_3/k_d$ and $k_5 = 2k_2$ and, after calculating the derivative of function $V(x, u)$ along trajectories of (3.28), it follows that

$$\dot{V}(x,u) = -k_3 g x_1^4 - k_3 g x_1^3 - k_3 x_1^2 + (k_3 - 2k_2 k_d) x_2^2 - 2g\frac{k_2}{k_d} x_1^2 x_2^2 + 2k_2 x_2 u + k_3 x_1 u. \tag{3.30}$$

The constants can be chosen such that $2k_2 k_d > k_3$. In this example, we choose $k_3 = k_d$ and $k_2 = 1$. With this choice, the quadratic term for variable x_2 becomes negatively definite, and it is not difficult to check that $V(x, u)$ is an energy function of the open loop system (3.28) if $g < 4$. However, for $g > 4$ condition (E1) of the energy function is not satisfied.

Next, we design a feedback control law and show that $V(x, u)$ is a generalized energy function of the closed loop system. Let $u = h(x_1, x_2) = g x_1^2$ and observe that $h(0, 0) = 0$. Thus, the origin is an equilibrium point of the closed loop system and the origin is the unique equilibrium point of the closed loop system. In order to show that $V(x, u)$ is a generalized energy function of the closed loop system, we substitute the feedback law u into the expression of $\dot{V}$ and obtain the following:

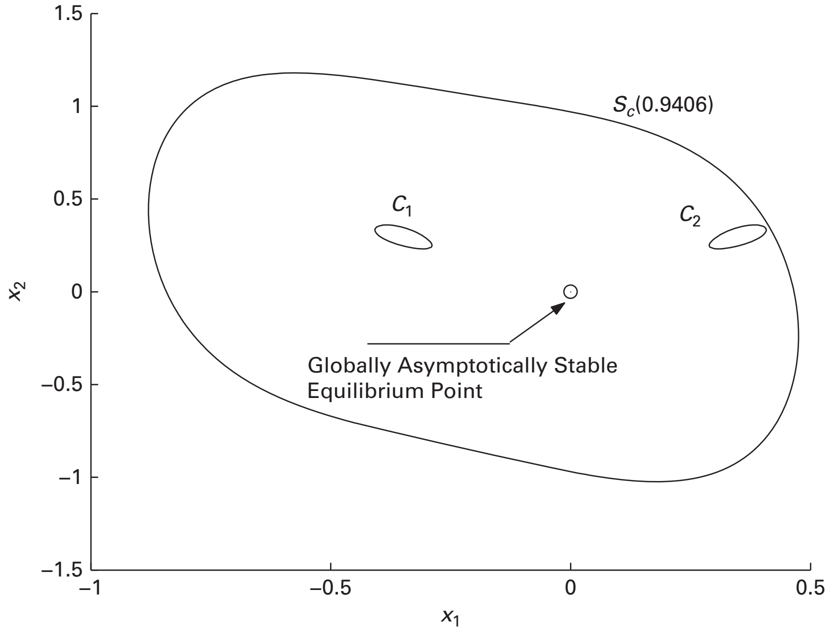

Figure 3.7 Level set $S_c(0.9406)$ of the generalized energy function $V(x, u)$ associated with the closed loop system for $k_3 = k_d$ and $k_2 = 1$. Using the generalized energy function and exploring some properties of system (3.28), it is shown that the origin is a globally asymptotically stable equilibrium point.

$$\dot{V}(x) = -k_3 g x_1^4 - k3x_1^2 - (2k_2k_d - k_3)x_2^2 - 2g\frac{k_2}{k_d}x_1^2x_2^2 + 2k_2\, g x_1^2\, x_2. \tag{3.31}$$

We then choose the constants such that $2k_2k_d - k_3 > 0$ and all the terms of equation (3.31) except the term $2k_2gx_1^2x_2$ are non-positive. The regions where the derivative of $V(x, u)$ is positive are shown in Figure 3.7. Set C is composed of three connected and bounded components: they are the sets C_1, C_2 and the origin. Therefore, conditions (G1) and (G2) of generalized energy functions are satisfied. If the parameters are chosen such that $2k_2k_d > k_3$, then the function $V(x, u)$ becomes radially unbounded and, consequently, condition (G3) of generalized energy functions is also satisfied. Therefore, $V(x, u)$ is a generalized energy function of the closed loop system.

By choosing $l = \max_{x \in C} V = 0.9406$ and using the fact that $V(x, u)$ is a radially unbounded function, we conclude that all the trajectories of the closed loop system enter the positively invariant bounded set $S_c(0.9406) = \{x \in R^2\colon (V(x) < 0.9406)\}$ for some positive time. Hence, every trajectory of the closed loop system enters the set $S_c(0.9406)$ and approaches its limit set, which has a non-empty intersection with set C.

Now, some distinguishing features of the closed loop system will be explored to show that every trajectory approaches the origin, which is an asymptotically stable equilibrium point of the closed loop system. According to the Poincaré–Bendixson theorem

[107], if the ω-limit set does not contain equilibrium points, then it is a closed orbit. One can verify that

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x2} = -k_d - \frac{g}{k_d} x_1^2 < 0 \tag{3.32}$$

for $k_d > 0$. Therefore, according to Bendixson's criterion [107], there are no closed orbits inside the set $S_c(0.9406)$. Hence, the limit sets contain the unique equilibrium point of the closed loop system that is the origin. Therefore, the origin is a globally asymptotic stable equilibrium point of the closed loop system.

3.7 Concluding remarks

In this chapter, we have presented a comprehensive energy function theory and generalized energy function theory for general nonlinear autonomous dynamical systems. Analytical results on the structure of ω-limit sets and on the global behavior of trajectories using energy functions and generalized energy functions have been presented.

The dynamic behaviors of general nonlinear systems can be very complicated. It has been shown that, if the underlying dynamical system admits an energy function, then the system only allows simple trajectories: every trajectory either converges to an equilibrium point or goes to infinity (becomes unbounded). Moreover, the stability boundary is contained in the set which is the union of the stable manifolds of the UEPs on the stability boundary, which gives a characterization of the stability boundary.

It is well recognized that there is no systematic procedure for deriving Lyapunov functions for general nonlinear systems. Likewise, there is no systematic procedure for deriving energy functions or generalized energy functions for general nonlinear dynamical systems. In order to avoid searching for an energy function for each individual nonlinear dynamical system, it is beneficial to develop energy functions for a class of nonlinear dynamical systems. We have derived an energy function for an important class of nonlinear dynamical systems: second-order nonlinear dynamical system.

Both energy function theory and generalized energy function theory will be further explored in later chapters to develop optimal schemes for estimating stability regions of general nonlinear systems. It will be shown that the stability regions estimated via energy functions or generalized energy functions using the schemes to be developed are optimal in a certain sense.