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Overlapping Fusion Estimation for Discrete Time-Varying Interconnected Systems

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***Abstract*—This letter is concerned with the distributed estimation problem for discrete time-varying interconnected systems with overlapping states. A fusion strategy, which weights neighbors' overlapping state estimates into a better result, is used to exploit the redundant information from sensors. Under this fusion strategy, an overlapping fusion estimator is proposed with only the information from each subsystem and its neighbors. By the idea of constructing and minimizing an upper bound of the square error of the estimate, convex optimization problems are established in terms of linear matrix inequalities to obtain the fusion weights and estimator gains successively. Notice that these optimization problems can be directly solved by standard software packages, and the computational complexity of each optimization problem is low, even for large-scale systems. Moreover, conditions on coupling structures are provided to guarantee the stability of the overlapping fusion estimator. Finally, an illustrative example is employed to show the effectiveness of the proposed methods.**

***Index Terms*—Distributed estimation, interconnected system, overlapping state, fusion estimator, convex optimization.**

I. Introduction

NOWADAYS, critical infrastructures such as power networks [1], industrial manufacturing systems [2], and water distribution networks [3] are becoming increasingly dispersed, large-scale, and complex. These systems are usually regarded as interconnections of subsystems, which may be known as physical entities, or are from a purely mathematical artifice to be identified by a suitable partitioning algorithm [4]. For example, the most common system decomposition technique, epsilon decomposition, yields interconnected subsystems with overlapping states [4], [5]; that is, subsystems share a certain number of state variables. An important problem for interconnected systems is estimating the system states by using widely deployed sensors.

In this letter, how to design distributed estimators with only local communication, computation, and storage is investigated for interconnected systems with overlapping states. In fact, the obtained state estimates are also overlapping and contain redundant information that can be communicated to further improve estimation performance for subsystems. Following a fusion estimation approach [6], [7], how to best combine overlapping estimates with a fusion strategy is addressed. As a result, an overlapping fusion estimator (OFE) is designed by merging distributed estimator design and fusion strategy.

Related work: Distributed estimation is one of the most important focuses in interconnected systems. Most solutions to this problem rely on systems' structure constraints, such as the sparse banded structure [8], [9] and the sequential structure [10]. The main difficulty of distributed estimation for general interconnected systems is the iteration of coupled estimation errors. In this case, a distributed moving horizon estimator was designed in [11] by assuming the estimation errors are uncorrelated, while a distributed Kalman filter was designed in [12] with the offline calculation of error covariances. Alternative estimation methods based on bounded noises can avoid the complex calculation of covariances [13], [14], [15], [16]. For example, a distributed set-membership estimator was designed for interconnected systems under zonotopic noises in [15], while the distributed estimation problem for interconnected systems under bounded noises with unknown bounds was addressed in [16].

The above estimation approaches do not account for the overlap of system states. For subsystems with overlapping states, local estimates are also overlapping and contain redundant information to their neighbors. The fusion estimation scheme, which aims to best combine multiple local estimates to produce a fused result, provides a promising way to utilize sensor redundancy [17]. Most fusion criteria are designed in a probabilistic setting [18], [19], where local estimates are with correlated Gaussian distribution errors and are usually characterized by the first two moments. The fusion estimation problem under bounded noises has received only limited and recent attention [7], [20], [21].

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Main contributions: This letter features three main contributions.

1) A fusion strategy is introduced to the distributed estimation problem for an interconnected system with overlapping states. As a result, the redundant information provided by subsystems' sensors can be effectively utilized.
2) An OFE is designed for each subsystem under local communication. The designed estimator contains two parts. The first part aims to fuse overlapping estimates at the last instant by designing weight matrices such that an upper bound of the square error (SE) of the fusion estimate is minimal. The second part provides a distributed estimate by designing a gain matrix such that an upper bound of the SE of the estimate is minimal.
3) A stability analysis based on coupling structures is provided to ensure the boundedness of estimation errors.

Notation: The following notations are used throughout this letter. $\mathbb{R}$ is the set of real numbers, and $\mathbb{N}_L := \{1, 2, \dots, L\}$ is the set of positive integers not greater than L. The symbol $\mathbf{I}$ represents the identity matrix with appropriate dimensions. Given sets X and Y, $X \setminus Y$ represents the set of all elements of X that are not in Y, and $X \cap Y$ is the intersection set of X and Y. The notation $A \succ 0$ ($\prec 0$) means that A is a positive definite (negative definite) matrix. Meanwhile, A^{T}, $\mathrm{Tr}\{A\}$, and $\lambda_{\max}(A)$ are the transpose, the trace, and the maximum eigenvalue of the matrix A, respectively. The notation $\mathrm{col}\{a_1, \dots, a_n\}$ means a column vector whose elements are $a_1, \dots, a_n$.

II. Problem Formulation

A. Time-Varying Interconnected Systems

Consider a large-scale system $\mathbf{S}$ and its state $x(k) \in \mathbb{R}^n$ whose components are indexed by a set $I = \{\mathbf{s}_1, \mathbf{s}_2, \dots, \mathbf{s}_n\}$. After system decomposition, the overall system $\mathbf{S}$ is partitioned into L lower-dimensional interconnected subsystems with overlapping states. The state and measurement dynamics of the ith subsystem $\mathbf{S}_i (i \in \mathbb{N}_L)$ is as follows:

$$\mathbf{S}_i\colon \begin{cases} x_i(k+1) = A_i(k)x_i(k) + \sum_{i_\omega \in \Omega_i} A_{i,i_\omega}(k)x_{i_\omega}(k) \\ +\Gamma_i(k)w_i(k) + B_i(k)u_i(k) \\ y_i(k) = C_i(k)x_i(k) + D_i(k)v_i(k) \end{cases} \tag{1}$$

The vectors $x_i(k) \in \mathbb{R}^{n_i}$, $y_i(k) \in \mathbb{R}^{m_i}$, and $u_i(k) \in \mathbb{R}^{r_i}$ are the state, the measurement, and the control input of subsystem $\mathbf{S}_i$, respectively. The matrices $A_i(k)$, $A_{i,i_\omega}(k)$, $\Gamma_i(k)$, $B_i(k)$, $C_i(k)$, and $D_i(k)$ are with appropriate dimensions. The neighbors of subsystem $\mathbf{S}_i$ are characterized by the set Ω_i with θ_i ($\theta_i < L$) elements. Let the ωth neighbor of subsystem $\mathbf{S}_i$ be denoted as $\mathbf{S}_{i_\omega}$, then all the neighboring subsystems of $\mathbf{S}_i$ can be represented as $\mathbf{S}_{i_1}, \dots, \mathbf{S}_{i_\omega}, \dots, \mathbf{S}_{i_{\theta_i}}$. Here, the word neighbor means in-neighbor, but we do not make that distinction in the context. The communication is assumed to be unidirectional from neighbors to a subsystem. The system noise $w_i(k) \in \mathbb{R}^{n_{w_i}}$ and the measurement noise $v_i(k) \in \mathbb{R}^{n_{v_i}}$ are unknown but bounded, and their upper bounds are

$$w_i^{\mathrm{T}}(k)w_i(k) \le \delta_{w_i} \quad v_i^{\mathrm{T}}(k)v_i(k) \le \delta_{v_i} \tag{2}$$

Let the index set of the state components in $x_i(k)$ be denoted as $I_i = \{\mathbf{s}_1^i, \dots, \mathbf{s}_{n_i}^i\}$. It is assumed that subsystem $\mathbf{S}_i$ and subsystem $\mathbf{S}_{i_\omega}$ are with overlapping states, which means that $I_i \cap I_{i_\omega} \neq \emptyset$. However, the results in this letter can be easily extended to include other state-overlapping situations. The index sets of overlapping and non-overlapping state components between $\mathbf{S}_i$ and $\mathbf{S}_{i_\omega}$ are defined by $I_{i\cap i_\omega} := I_i \cap I_{i_\omega}$ and $I_{i\setminus i_\omega} := I_i \setminus I_{i_\omega}$. The numbers of elements for $I_{i\cap i_\omega}$ and $I_{i\setminus i_\omega}$ are $n_{i\cap i_\omega}$ and $n_{i\setminus i_\omega}$, respectively. Subsequently, we can denote the vectors containing the state components in $I_{i\cap i_\omega}$ and $I_{i\setminus i_\omega}$ as $x_{i\cap i_\omega}(k) \in \mathbb{R}^{n_{i\cap i_\omega}}$ and $x_{i\setminus i_\omega}(k) \in \mathbb{R}^{n_{i\setminus i_\omega}}$. It turns out that

$$x_{i\cap i_\omega}(k) = P_{i\cap i_\omega}x_i(k) = P_{i_\omega\cap i}x_{i_\omega}(k) = x_{i_\omega\cap i}(k) \tag{3a}$$

$$x_{i\setminus i_\omega}(k) = P_{i\setminus i_\omega}x_i(k) \tag{3b}$$

The matrices $P_{i\cap i_\omega} \in \mathbb{R}^{n_{i\cap i_\omega}\times n_i}$ and $P_{i\setminus i_\omega} \in \mathbb{R}^{n_{i\setminus i_\omega}\times n_i}$ are composed by n_i-dimensional canonical row vectors. The sth canonical row vector for $P_{i\cap i_\omega} \in \mathbb{R}^{n_{i\cap i_\omega}\times n_i}$ ($P_{i\setminus i_\omega} \in \mathbb{R}^{n_{i\setminus i_\omega}\times n_i}$) are with all zeros, but one element is 1, where the order of this non-zero element corresponds to the order of the sth state component of $x_{i\cap i_\omega}(k)$ ($x_{i\setminus i_\omega}(k)$) in the index set I_i. Without loss of generality, it is assumed that each subsystem $\mathbf{S}_i$ is in its simplest form, i.e., the coupling term $A_{i,i_\omega}(k)x_{i_\omega}(k)$ in (1) is only a linear combination of the non-overlapping state components $x_{i_\omega\setminus i}(k)$. It is equivalent to

$$A_{i,i_\omega}(k)P_{i_\omega\cap i}^{\mathrm{T}}P_{i_\omega\cap i} = 0 \quad (i_\omega \in \Omega_i) \tag{4}$$

A simple system transformation can be implemented to transform any interconnected system into the simplest form.

B. Problem of Interest

The major concern of distributed estimation for interconnected systems is to estimate the state of each subsystem under local communication and computation. Assume that neighboring subsystems transmit their estimates $\hat{x}_{i_\omega}^o(k)$ ($i_\omega \in \Omega_i$) to subsystem $\mathbf{S}_i$ at a given time instant k, then the overlapping state components $x_{i\cap i_\omega}(k)$ are simultaneously estimated by $\hat{x}_i^o(k)$ and $\hat{x}_{i_\omega}^o(k)$. It means that the available information from neighbors' estimates is redundant. To make use of this redundant information, the idea of fusion estimation is introduced, and a less conservative overlapping fusion estimate $\hat{x}_i(k)$ for subsystem $\mathbf{S}_i$ is obtained by fusing $\hat{x}_i^o(k)$ and $\hat{x}_{i_\omega}^o(k)$ ($i_\omega \in \Omega_i$). Thus, the available information for subsystem $\mathbf{S}_i$ at the instant k includes its local measurement $y_i(k)$, the estimates from its neighbors $\hat{x}_{i_\omega}^o(k-1)$ ($i_\omega \in \Omega_i$), and the calculated overlapping fusion estimate $\hat{x}_i(k-1)$. Therefore, the following OFE with three steps of prediction, observation update, and fusion is proposed:

$$\hat{x}_i^p(k) = A_i(k-1)\hat{x}_i(k-1) + B_i(k-1)u_i(k-1) + \sum_{i_\omega \in \Omega_i} A_{i,i_\omega}(k-1)\hat{x}_{i_\omega}^o(k-1) \tag{5a}$$

$$\hat{x}_i^o(k) = \hat{x}_i^p(k) + K_i(k)\big[y_i(k) - C_i(k)\hat{x}_i^p(k)\big] \tag{5b}$$

$$\hat{x}_i(k) = f_i(\hat{x}_i^o(k), \hat{x}_{i_1}^o(k), \dots, \hat{x}_{i_{\theta_i}}^o(k)) \tag{5c}$$

Here, $\hat{x}_i^p(k)$, $\hat{x}_i^o(k)$, and $\hat{x}_i(k)$ are the one-step prediction, the estimate, and the overlapping fusion estimate for $x_i(k)$, respectively. The gain matrix $K_i(k)$ is used to balance the uncertainty propagation induced by the system noise $w_i(k)$ and the measurement noise $v_i(k)$, while the function $f_i(\cdot)$ is a fusion criterion to be designed.

Consequently, the problems to be solved in this letter are described as follows:

1) Design a fusion criterion $f_i(\cdot)$ such that an upper bound of the SE of the fusion estimate is minimal, and design a gain matrix $K_i^{\text{opt}}(k)$ such that an upper bound of the SE of the estimate is minimal.
2) Analyze the stability of the proposed OFE such that estimation error is ultimately bounded.

Remark 1: Different from [8], [9], [10], [11], [12], [13], [14], [15], [16], the interconnected systems considered in this letter are more general in that they not only have no structural constraint of subsystem couplings but also take the phenomenon of state overlap among subsystems into account. Notice that if no state overlap exists between subsystems, the considered interconnected system in (1) degenerate into the original interconnected system in [12], [16]. In addition, the designed OFE in (5) is fully distributed. Only local information and information from neighbors are required in the estimator structure and the subsequent design for estimator gain and fusion criterion.

III. Main Results

A. Overlapping Fusion Estimator Design

In this section, a gain matrix $K_i^{\text{opt}}(k)$ and a fusion criterion $f_i(\cdot)$ will be designed. Let us define the one-step prediction error, the estimation error, and the fusion estimation error as $e_i^p(k) := x_i(k) - \hat{x}_i^p(k)$, $e_i^o(k) := x_i(k) - \hat{x}_i^o(k)$, and $e_i(k) := x_i(k) - \hat{x}_i(k)$, respectively. According to (1) and ((5a)-(5b)), the estimation error $e_i^o(k)$ can be derived as follows:

$$\begin{aligned} e_i^o(k) &= \bar{K}_i(k)A_i(k-1)e_i(k-1) \\ &\quad + \sum_{i_\omega \in \Omega_i} \bar{K}_i(k)A_{i,i_\omega}(k-1)e_{i_\omega}^o(k-1) \\ &\quad + \bar{K}_i(k)\Gamma_i(k-1)w_i(k-1) - K_i(k)D_i(k)v_i(k) \end{aligned} \tag{6}$$

where $\bar{K}_i(k) = \mathbf{I} - K_i(k)C_i(k)$. Equivalently, the estimation error of $\hat{x}_i^o(k)$ can be rewritten as

$$e_i^o(k) = \Xi_{i,1}(k)e_i(k-1) + \Xi_{i,2}(k)\xi_i(k-1) \tag{7}$$

where

$$\begin{cases} \xi_i(k-1) := \text{col}\left\{e_{i_1}^o(k-1), \ldots, e_{i_{\theta_i}}^o(k-1), w_i(k-1), v_i(k)\right\} \\ \Xi_{i,1}(k) := \bar{K}_i(k)A_i(k-1) \\ \Xi_{i,2}(k) := \Big[\bar{K}_i(k)A_{i,i_1}(k-1), \ldots, \bar{K}_i(k)A_{i,i_{\theta_i}}(k-1), \\ \qquad\qquad \bar{K}_i(k)\Gamma_i(k-1), -K_i(k)D_i(k)\Big] \end{cases} \tag{8}$$

Then, a gain matrix is designed by minimizing an upper bound of the SE of the estimate $\hat{x}_i^o(k)$. This upper bound is constructed by introducing three matrices $\Psi_{i,1}(k) \succ 0$, $\Psi_{i,12}(k)$, and $\Psi_{i,2}(k) \succ 0$ such that [20]:

$$\left[e_i^o(k)\right]^{\mathrm{T}} e_i^o(k) < \underbrace{\begin{bmatrix} e_i(k-1) \\ \xi_i(k-1) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Psi_{i,1}(k) & \Psi_{i,12}(k) \\ * & \Psi_{i,2}(k) \end{bmatrix} \begin{bmatrix} e_i(k-1) \\ \xi_i(k-1) \end{bmatrix}}_{U_{i,1}(k)} \tag{9}$$

The following theorem shows how to obtain such a gain matrix.

Theorem 1: The gain matrix $K_i^{\text{opt}}(k)$ that minimizes the upper bound of the SE of the estimate $\hat{x}_i^o(k)$ in (9) can be designed by the following optimization problem:

$$\begin{aligned} \min_{\substack{K_i(k), \Psi_{i,1}(k) \succ 0 \\ \Psi_{i,12}(k), \Psi_{i,2}(k) \succ 0}} \quad & \text{Tr}\left\{\Psi_{i,1}(k)\right\} + \text{Tr}\left\{\Psi_{i,2}(k)\right\} \\ \text{s.t.} \quad & \begin{bmatrix} -\mathbf{I} & \Xi_{i,1}(k) & \Xi_{i,2}(k) \\ * & -\Psi_{i,1}(k) & -\Psi_{i,12}(k) \\ * & * & -\Psi_{i,2}(k) \end{bmatrix} \prec 0 \end{aligned} \tag{10}$$

Proof: To guarantee $U_{i,1}(k)$ in (9) is an upper bound of $[e_i^o(k)]^{\mathrm{T}} e_i^o(k)$, the following inequality must be satisfied:

$$\begin{bmatrix} e_i(k-1) \\ \xi_i(k-1) \end{bmatrix}^{\mathrm{T}} Z_i(k) \begin{bmatrix} e_i(k-1) \\ \xi_i(k-1) \end{bmatrix} < 0 \tag{11}$$

where

$$Z_i(k) = \begin{bmatrix} \Xi_{i,1}^{\mathrm{T}}(k)\Xi_{i,1}(k) - \Psi_{i,1}(k) & \Xi_{i,1}^{\mathrm{T}}(k)\Xi_{i,2}(k) - \Psi_{i,12}(k) \\ * & \Xi_{i,2}^{\mathrm{T}}(k)\Xi_{i,2}(k) - \Psi_{i,2}(k) \end{bmatrix} \tag{12}$$

The inequality (11) means that $Z_i(k) \prec 0$. By the Schur complement lemma, $Z_i(k) \prec 0$ turns out to be the inequality constraint in (10). Moreover, the upper bound $U_{i,1}(k)$ in (9) is a scalar, which satisfies $U_{i,1}(k) = \text{Tr}\{U_{i,1}(k)\}$ and

$$\begin{aligned} U_{i,1}(k) &= \text{Tr}\left\{\begin{bmatrix} e_i(k-1) \\ \xi_i(k-1) \end{bmatrix}\begin{bmatrix} e_i(k-1) \\ \xi_i(k-1) \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Psi_{i,1}(k) & \Psi_{i,12}(k) \\ * & \Psi_{i,2}(k) \end{bmatrix}\right\} \\ &\le \lambda_{\max}\left\{\begin{bmatrix} e_i(k-1) \\ \xi_i(k-1) \end{bmatrix}\begin{bmatrix} e_i(k-1) \\ \xi_i(k-1) \end{bmatrix}^{\mathrm{T}}\right\} \\ &\quad \times \left(\text{Tr}\left\{\Psi_{i,1}(k)\right\} + \text{Tr}\left\{\Psi_{i,2}(k)\right\}\right) \end{aligned} \tag{13}$$

The estimator gain matrix is designed by minimizing this upper bound. Hence, "min(Tr{$\Psi_{i,1}(k)$} + Tr{$\Psi_{i,2}(k)$})" is chosen as the optimization objective in (10). ■

On the other hand, the fusion rules weighted by matrices [18], [19] are used for designing the fusion criterion $f_i(\cdot)$. In this case, the fusion estimate is a weighted sum of the estimates $\hat{x}_i^o(k)$ and $\hat{x}_{i_\omega}^o(k)$ ($i_\omega \in \Omega_i$). However, only the estimates of the overlapping state components $P_{i_\omega \cap i}\hat{x}_{i_\omega}^o(k)$ are used in the fusion process as

$$\hat{x}_i(k) = W_{i,0}(k)\hat{x}_i^o(k) + \sum_{\omega=1}^{\theta_i} P_{i\cap i_\omega}^{\mathrm{T}} W_{i,i_\omega}(k) P_{i_\omega \cap i}\hat{x}_{i_\omega}^o(k) \tag{14}$$

where $W_{i,0}(k) \in \mathbb{R}^{n_i \times n_i}$ and $W_{i,i_\omega}(k) \in \mathbb{R}^{n_{i\cap i_\omega} \times n_{i\cap i_\omega}}$ are weight matrices and satisfy

$$W_{i,0}(k) + \sum_{\omega=1}^{\theta_i} P_{i\cap i_\omega}^{\mathrm{T}} W_{i,i_\omega}(k) P_{i_\omega \cap i} = \mathbf{I} \tag{15}$$

According to (6) and (14), the fusion estimation error of $e_i(k)$ can be derived as follows:

$$\begin{aligned} e_i(k) &= W_{i,0}(k)\Xi_{i,1}(k)e_i(k-1) + W_{i,0}(k)\Xi_{i,2}(k)\xi_i(k-1) \\ &\quad + \sum_{\omega=1}^{\theta_i} P_{i\cap i_\omega}^{\mathrm{T}} W_{i,i_\omega}(k) P_{i_\omega \cap i} e_{i_\omega}^o(k) \end{aligned} \tag{16}$$

Equivalently, the fusion estimation error of $e_i(k)$ can be rewritten as

$$e_i(k) = W_{i,0}(k)\Xi_{i,1}(k)e_i(k-1) + \Xi_{i,3}(k)\eta_i(k) \tag{17}$$

where

$$\begin{cases} \eta_i(k) := \mathrm{col}\left\{e^o_{i_1}(k), \ldots, e^o_{i_{\theta_i}}(k), \xi_i(k-1)\right\} \\ \Xi_{i,3}(k) := \Big[P^{\mathrm{T}}_{i\cap i_1}W_{i,i_1}(k)P_{i_1\cap i}, \ldots, \\ P^{\mathrm{T}}_{i\cap i_{\theta_i}}W_{i,i_{\theta_i}}(k)P_{i_{\theta_i}\cap i}, W_{i,0}(k)\Xi_{i,2}(k)\Big] \end{cases} \tag{18}$$

Then, the problem is how to find the weight matrices such that an upper bound of the SE of the overlapping fusion estimate $\hat{x}_i(k)$ is minimal. The construction of this upper bound is as follows:

$$\begin{aligned} e_i^{\mathrm{T}}(k)e_i(k) < U_{i,2}(k) = {}& e_i^{\mathrm{T}}(k-1)\Phi_{i,1}(k)e_i(k-1) \\ &+ \eta_i^{\mathrm{T}}(k)\Phi_{i,2}(k)\eta_i(k) \end{aligned} \tag{19}$$

where $\Phi_{i,1}(k) \succ 0$ and $\Phi_{i,2}(k) \succ 0$. The following theorem shows how to obtain such weight matrices.

Theorem 2: Given the designed gain matrix $K_i^{\mathrm{opt}}(k)$ by (10), a family of weight matrices that minimizes the SE of the overlapping fusion estimate $\hat{x}_i(k)$ can be obtained by solving the following convex optimization problem:

$$\min_{W_{i,0}(k), W_{i,i_\omega}(k), \Phi_{i,1}(k)\succ 0, \Phi_{i,2}(k)\succ 0} \mathrm{Tr}\{\Phi_{i,2}(k)\}$$

$$\text{s.t.} \begin{bmatrix} -\mathbf{I} & W_{i,0}(k)\Xi_{i,1}(k) & \Xi_{i,3}(k) \\ * & -\Phi_{i,1}(k) & 0 \\ * & * & -\Phi_{i,2}(k) \end{bmatrix} \prec 0 \tag{20a}$$

$$\Phi_{i,1}(k) - \alpha_i(k)\mathbf{I} \prec 0 \tag{20b}$$

$$0 < \alpha_i(k) < 1 \tag{20c}$$

$$W_{i,0}(k) + \sum_{\omega=1}^{\theta_i} P^{\mathrm{T}}_{i\cap i_\omega}W_{i,i_\omega}(k)P_{i_\omega\cap i} = \mathbf{I} \tag{20d}$$

Proof: Substituting (17) into (19), it yields that

$$\begin{bmatrix} e_i(k-1) \\ \eta_i(k) \end{bmatrix}^{\mathrm{T}} \underbrace{\begin{bmatrix} \Lambda_1(k) & \Lambda_{12}(k) \\ * & \Lambda_2(k) \end{bmatrix}}_{\Lambda(k)} \begin{bmatrix} e_i(k-1) \\ \eta_i(k) \end{bmatrix} < 0 \tag{21}$$

where

$$\begin{cases} \Lambda_1(k) = \left[\Xi_{i,1}(k)W_{i,0}(k)\right]^{\mathrm{T}}\Xi_{i,1}(k)W_{i,0}(k) - \Phi_{i,1}(k) \\ \Lambda_{12}(k) = \left[\Xi_{i,1}(k)W_{i,0}(k)\right]^{\mathrm{T}}\Xi_{i,3}(k) \\ \Lambda_2(k) = \Xi^{\mathrm{T}}_{i,3}(k)\Xi_{i,3}(k) - \Phi_{i,2}(k) \end{cases} \tag{22}$$

According to the Schur complement lemma, the inequality (20a) is equivalent to $\Lambda(k) \prec 0$. In this case, the SE of the fusion estimate is bounded by $U_{i,2}(k)$. If the inequality (20b) holds, it turns out that $\lambda_{\max}(\Phi_{i,1}(k)) < \alpha_i(k)$. Hence, with a similar derivation as in (13), one has the following inequality:

$$\begin{aligned} U_{i,2}(k) &< \alpha_i(k)e_i^{\mathrm{T}}(k-1)e_i(k-1) + \eta_i^{\mathrm{T}}(k)\Phi_{i,2}(k)\eta_i(k) \\ &< \alpha_i(k)e_i^{\mathrm{T}}(k-1)e_i(k-1) \\ &\quad + \lambda_{\max}\left(\eta_i^{\mathrm{T}}(k)\eta_i(k)\right)\mathrm{Tr}\left\{\Phi_{i,2}(k)\right\} \end{aligned} \tag{23}$$

Notice that $\alpha_i(k) < 1$ in (20c) reduces the impact of the initial estimation errors on $e_i^{\mathrm{T}}(k)e_i(k)$ to be negligible. Therefore, to minimize the upper bound $U_{i,2}(k)$, "$\min \mathrm{Tr}\{\Phi_{i,2}(k)\}$" is chosen as the optimization objective to search for weight matrices. ■

Algorithm 1 Overlapping Fusion Estimation

1: **for** $i = 1 : L$ **do**
2: Subsystem $\mathbf{S}_i$ collects its local measurement $y_i(k)$, local estimate $\hat{x}_i^o(k-1)$, and its neighbors' estimates $\hat{x}^o_{i_\omega}(k-1)$;
3: Determine the fusion weights $W_{i,0}(k-1)$ and $W_{i,i_\omega}(k-1)$ by (20);
4: Calculate the fusion estimate $\hat{x}_i(k-1)$ by (5c);
5: Determine the gain matrix $K_i^{\mathrm{opt}}(k)$ by (10);
6: Perform the prediction step in (5a) and the observation update step in (5b) to obtain $\hat{x}_i^o(k)$;
7: Subsystem $\mathbf{S}_i$ transmits $\hat{x}_i^o(k)$ to its out-neighbors;
8: **end for**
9: Return to Step 1 and implement Steps 1-8 for calculating $\hat{x}_i^o(k+1)$ $(i \in \mathbb{N}_L)$.

Then, the computation procedures for the proposed OFE are summarized in Algorithm 1.

Remark 2: The optimization problems (10) and (20) are established in terms of linear matrix inequalities and can be solved by the function "mincx" of MATLAB LMI Toolbox [22]. However, the computation complexity of these optimization problems in subsystem $\mathbf{S}_i$ is related to the number of neighboring subsystems θ_i and their dimensions n_{i_ω} $(i_\omega \in \Omega_i)$. To reduce the computation burden, the decoupling methods in [16] can be used to reduce the number of involved neighboring subsystems during the optimization.

B. Stability Analysis

The stability of an interconnected system or its interconnected estimation error system (17) is highly dependent on the coupling structure of the system. In the subsequent stability analysis, we will focus on an important class of interconnected systems that are sequentially connected.

Assumption 1 [12]: The coupling structure of the interconnected system (1) is of a direct acyclic graph. Then, the system can be transformed into a sequentially connected system by reordering subsystems.

Theorem 3: For a sequentially interconnected system, the proposed OFE is ultimately bounded, i.e., there must exist a positive scalar $p_i > 0$ such that

$$\lim_{k\to\infty} e_i^{\mathrm{T}}(k)e_i(k) < p_i \tag{24}$$

Proof: By (19) and the first inequality in (23), the SE of the fusion estimate $\hat{x}_i(k)$ satisfies

$$\begin{aligned} e_i^{\mathrm{T}}(k)e_i(k) < {}& \left(\prod_{\mu=0}^{k-1}\alpha_i(k-\mu)\right)\|e_i(0)\|_2^2 \\ &+ \sum_{\tau=0}^{k-1}\left\{\left(\prod_{\mu=0}^{\tau-1}\alpha_i(k-\mu)\right)\eta_i^{\mathrm{T}}(k-\tau)\Phi_{i,2}(k-\tau)\eta_i(k-\tau)\right\} \end{aligned} \tag{25}$$

According to "$\alpha_i(k) < 1$" in (20c), one has that

$$\lim_{k\to\infty}\prod_{\mu=0}^{k-1}\alpha_i(k-\mu)=0,\quad \lim_{\substack{k\to\infty\\ \tau\to\infty}}\prod_{\mu=0}^{\tau-1}\alpha_i(k-\mu)=0 \quad (26)$$

Suppose estimators are employed for the reordered subsystems $\tilde{\mathbf{S}}_1, \tilde{\mathbf{S}}_2, \ldots, \tilde{\mathbf{S}}_L$, then the analysis is as follows. The first reordered subsystem has no neighbor, and $\eta_1(k)$ only contains system noises $w_1(k-1)$ and $v_1(k)$. By the boundedness of system noises, the boundedness of the SE of $\hat{x}_1(k)$ can be ensured by (25) and (26) [13], [16]. The second reordered subsystem has only one neighbor $\tilde{\mathbf{S}}_1$, and $\eta_2(k)$ contains $e_1^o(k)$, $e_1^o(k-1)$, $w_2(k-1)$ and $v_2(k)$. By the boundedness of $e_1^o(k)$ and system noises, the boundedness of the SE of $\hat{x}_2(k)$ can be ensured. Therefore, the same deduction can be performed [12], and the conclusion that the SE of $\hat{x}_i(k)$ $(i \in \mathbb{N}_L)$ are ultimately bounded holds. ■

Remark 3: The assumption of a direct acyclic graph for coupling structure can cover more interconnected systems than the sparse structure [8], [9] or the sequential structure [10]. By using the decoupling methods in [16], a large part of actual interconnected systems can be transformed into the structure in Assumption 1.

IV. Numerical Examples

To verify the effectiveness of the proposed OFE, a simulation with a set of four inverted pendulums observed by internal and external sensors (see Fig. 1) is conducted. This example is motivated by the cooperative navigation problem of sensor networks with inter-node measurements [23]. Each inverted pendulum consists of a moving cart with mass M_i $(i \in \mathbb{N}_4)$, a pendulum rod with length l, and a pendulum ball with mass m_i. To keep the inverted pendulums in balance, an external force u_i is applied to the cart. The dynamic equation of the ith inverted pendulum is discretized with a sampling period T by the forward Euler method as

$$s_i(k+1)=\underbrace{\begin{bmatrix}1 & T & 0 & 0\\ 0 & 1 & \frac{-m_i gT}{M_i} & 0\\ 0 & 0 & 1 & T\\ 0 & 0 & \frac{(M_i+m_i)gT}{M_i l} & 1\end{bmatrix}}_{A_{i,d}} s_i(k)+\underbrace{\begin{bmatrix}0\\ \frac{T}{M_i}\\ 0\\ \frac{-T}{M_i l}\end{bmatrix}}_{B_{i,d}} u_i(k) \quad (27)$$

where $s_i := \mathrm{col}\{d_i, \dot{d}_i, \vartheta_i, \dot{\vartheta}_i\}$ is the state, d_i, and $\dot{d}_i$ are the displacement and velocity of the i-th cart, while ϑ_i and $\dot{\vartheta}_i$ are the angular and angular velocity of the i-th rod. Each inverted pendulum is deployed with an internal sensor to observe the displacements of the carts and the angles of the rods. Moreover, two external sensors are deployed in the second and third inverted pendulums to measure the relative distance to the next inverted pendulum ($d_{23} := d_3 - d_2$ and $d_{34} := d_4 - d_3$). The overall system is decomposed into three subsystems $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$ with their states defined as $x_1 := \mathrm{col}\{s_1, s_2, d_{23}\}$, $x_2 := \mathrm{col}\{s_2, s_3, d_{34}\}$, and $x_3 := s_4$. The dynamics of subsystems with noise disturbances are

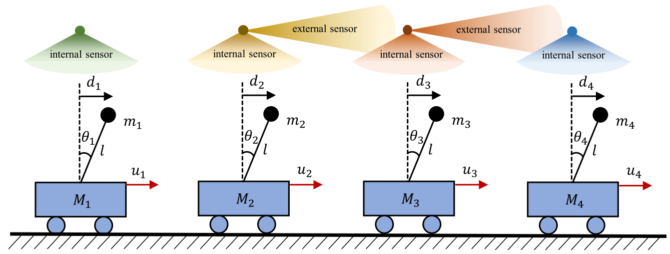

Fig. 1. A system consisting of four inverted pendulums observed by internal and external sensors.

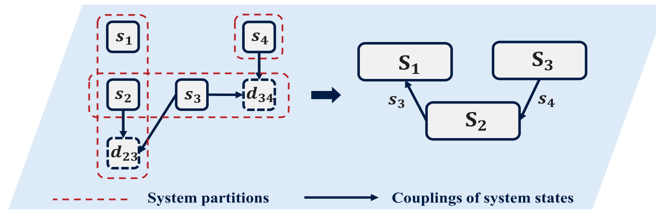

Fig. 2. Overall system partitions and subsystem couplings.

characterized by

$$x_i(k+1)=\begin{bmatrix}A_{i,d} & 0 & 0\\ 0 & A_{i+1,d} & 0\\ 0 & -M_T & 1\end{bmatrix}x_i(k)+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & M_T & 0\end{bmatrix}x_{i+1}(k)$$
$$+\begin{bmatrix}B_{i,d}u_i(k)\\ B_{i+1,d}u_{i+1}(k)\\ 0\end{bmatrix}+w_i(k)\qquad (i=1,2)$$
$$x_3(k+1)=A_{4,d}x_3(k)+B_{4,d}u_4(k)+w_3(k) \quad (28)$$

where $M_T := \begin{bmatrix}0 & T & 0 & 0\end{bmatrix}$. The internal and external sensor measurements for each subsystem can be modeled as

$$y_i(k)=\begin{bmatrix}C_d & 0 & 0\\ 0 & C_d & 0\\ 0 & 0 & 1\end{bmatrix}x_i(k)+v_i(k)\qquad (i=1,2)$$
$$y_3(k)=C_d x_3(k)+v_3(k) \quad (29)$$

where $C_d := \begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}$. The partitions of the overall system and the couplings among decomposed subsystems are depicted in Fig. 2. It is found that the decomposed subsystems are sequentially connected as in Assumption 1. In the simulation, the rod length is set as $l = 1$, and the elastic coefficients are given by $k_{i,j} = 5$, while the values of m_i and M_i are randomly extracted from the interval $[5, 15]$ and $[15, 25]$, respectively. To keep the displacement d_i and the angular ϑ_i at the equilibrium point $\vartheta_i^* = 0$ and $d_i^* = 5i - 5$, a linear quadratic regulator (LQR) controller is designed and employed in each subsystem. The system and measurement noises of the ith subsystem are taken as $w_i(k) = 0.2\rho_{w_i}(k) - 0.1$ and $v_i(k) = 0.4\rho_{v_i}(k) - 0.2$, where $\rho_{w_i}(k) \in [0\ 1]^{n_i}$ and $\rho_{v_i}(k) \in [0\ 1]^{m_i}$ are random variables that can be generated by the function "rand" of MATLAB. Then, the proposed OFE is applied to each subsystem to estimate their real states. The fusion estimates for the displacement and angle in subsystem $\mathbf{S}_1$ are plotted in Fig. 3. It can be seen from this figure that the proposed OFE can track the real states d_1 and ϑ_1 well

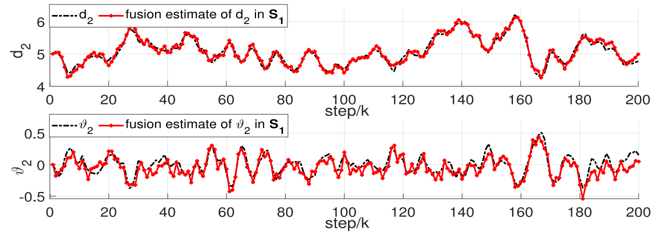

Fig. 3. The displacement of the cart and the angle of the rod in subsystem $\mathbf{S}_1$ and their fusion estimates.

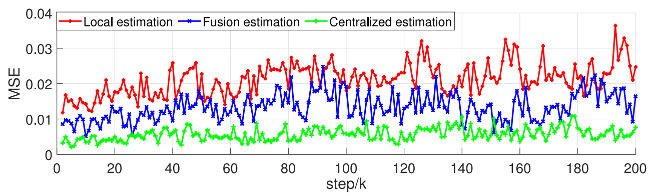

Fig. 4. The comparison of MSEs between the proposed overlapping fusion estimation, local estimation without fusion step, and centralized estimation.

and does not diverge with time. Due to the randomness of noises, the performance of estimators is assessed by the mean square error (MSE) of the state components of displacement and angular, which is calculated by the Monte Carlo method with an average of 100 runs. The comparison of the estimation performance for subsystem $\mathbf{S}_1$ between the proposed OFE and the local estimator without a fusion step is shown in Fig. 4. As is shown in this figure, the proposed OFE has better estimation performance than the local estimator. This can be attributed to the fact that the fusion step can make use of sensor redundancy information. On the other hand, the fusion estimate is also compared with the centralized estimate of [7] for the considered system. The MSE comparison is provided in Fig. 4. It is shown that the OFE is not much worse than the centralized approach.

V. Conclusion

In this letter, the distributed estimation problem for general interconnected systems with overlapping states was investigated. To better utilize the redundant information from multiple sensors, a fusion strategy was proposed by weighting the overlapping components of estimates from neighboring subsystems. Then, an OFE was designed for interconnected systems with overlapping states. The proposed estimator is fully distributed and only requires local communication to propagate the information. Moreover, the stability of the proposed estimator was analyzed with the given conditions on coupling structures. Finally, the proposed OFE was employed in an interconnected inverted pendulum system to show its effectiveness.

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