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Brief paper

Distributed zonotopic estimation for interconnected systems: A fusing overlapping states strategy☆

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ABSTRACT

This paper is concerned with the distributed estimation problem for a class of interconnected systems with overlapping states. Under local communication, a distributed zonotopic estimator is designed by using the information from each subsystem and its neighbors. However, the obtained state estimates are also overlapping and contain redundant information that can be shared to further improve estimation performance for subsystems. To better utilize such overlapping estimates from neighbors, zonotopic fusion criteria are proposed to combine local estimates into a better result. Notice that the proposed distributed zonotopic estimator does not require knowing the correlations of estimation errors, such as the complex calculation of cross-covariances, so it can be implemented online. Moreover, robust stability conditions are provided to ensure the boundedness of estimation errors. Finally, illustrative examples are employed to show the effectiveness of the proposed methods.

1. Introduction

With the great development of communication and sensor technology, the systems we seek to design and control have become increasingly large-scale. For example, modern power networks have involved wide-area deployments for generation from distributed renewable resources (Fang, Misra, Xue, & Yang, 2012). Another example is the gene regulatory network, which represents regulation effects among molecular components to govern the gene expression level of proteins (Huang, Tienda-Luna, & Wang, 2009; Xiong & Zhou, 2014). The increased computing and storage requirements of these high-dimensional systems have prevented traditional approaches designed for a lumped system from achieving satisfactory performance. Several system decomposition techniques (Khan & Moura, 2008; Michel, Miller, & Tang, 1978; Sezer & Šiljak, 1986; Stanković, Stanković, & Stipanović, 2009) have been developed to simplify the analysis of large-scale systems by dividing a system into interconnected subsystems with lower dimensions and overlapping states (Stankovic, Stanojevic, & Siljak, 2000). Then, the solution with fewer resource requirements can be obtained by a distributed structure that solves subsystems with only local information.

Distributed estimation plays a key role in such interconnected dynamic systems (IDSs) because it helps design efficient control or decision strategies. Many results have been obtained for this important theoretical issue in systems and control. For example, a distributed moving horizon estimator was designed in Haber and Verhaegen (2013) for sparse banded IDSs by the Chebyshev approximation of a centralized solution. For the same system structure, a distributed Kalman filter is optimally obtained in Khan and Moura (2008) with the Gauss–Markov approximation. In Chen, Hu, Ho, and Yu (2019a), the distributed Kalman filtering problem was addressed for IDSs that are sequentially connected. For general IDSs without any structure constraints, the problem is more complicated due to their coupled estimation error iterations. In this case, the distributed estimation problem was addressed by assuming the estimation errors of subsystems are uncorrelated (Farina, Ferrari-Trecate, & Scattolini, 2010) or by the offline calculation of estimation error covariances (Zhang, Chen, Yu, & Ho, 2021). However, it is usually not feasible to accurately compute cross-covariances by local communication. The set-membership estimation methods can avoid this problem and provide sets that enclose the admissible values of the state, such as intervals (Tang, Wang, Wang, Raïssi, & Shen, 2019), ellipsoids (Yang & Li, 2009), zonotopes (Combastel, 2015; Kühn, 1998), and general polytopes. For example, distributed estimators with plug-and-play fashion were developed in Riverso, Farina, Scattolini, and Ferrari-Trecate (2013), Riverso, Rubini, and Ferrari-Trecate (2015) based on robust positive invariance. In Orihuela, Roshany-Yamchi, García, and Millán (2017), a distributed set-membership

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estimator was designed for IDSs under zonotopic noises, but no stability guarantee was provided. More recently, the distributed estimation problem for IDSs under bounded noises with unknown lower and upper bounds was addressed in Chen, Hu, Ho, and Yu (2022). Notice that the work mentioned above does not account for the overlap of system states, a common phenomenon in decomposed large-scale systems. For subsystems with overlapping states, their local estimates are also overlapping and contain redundant information of other subsystems (see the multi-view surveillance systems in Wang, Li, Battistelli, Chisci, & Yi, 2021, as an example). How to obtain a better state estimate for overlapping subsystems with redundant information exchange is an interesting problem to be addressed.

At the same time, multi-sensor fusion estimation provides an efficient way to deal with redundant sensor information and has attracted considerable research interest during the past few decades (Sun, Lin, Ma, & Li, 2017). The major concern of fusion estimation is to best combine multiple local estimates to produce a fused result by designed fusion criteria (Chen, Zhang, & Yu, 2014). Most fusion criteria are designed in a probabilistic setting, where local estimates are with correlated Gaussian distribution errors and are usually characterized by the first two moments. When the error correlations are known, the optimal fusion criteria were designed in the linear minimum mean-square error sense (Li, Zhu, Wang, & Han, 2003). In many situations, it is not practical to obtain error correlations, and the well-known covariance intersection (CI) algorithm in Julier and Uhlmann (1997) provided a consistent fusion estimate irrespective of the actual correlations. Furthermore, the generalized CI algorithm (Hurley, 2002; Julier, Bailey, & Uhlmann, 2006) was designed by characterizing local estimates as Gaussian densities. More recently, the generalized CI approach was extended to multi-view surveillance systems (Wang et al., 2021) and applied well in the multi-target tracking problem. However, the literature concerned with the fusion estimation problem under bounded noises is very scarce, limited to some preliminary results (Chen, Hu, Ho, & Yu, 2019b; Shen, Zhu, Song, & Luo, 2011). The fusion criteria for set-membership estimation have practical significance since the physical signals in the real world are only affected by finite energy interference.

Compared with the estimation methods under other bounded noises, zonotopic estimators are simple enough to be carried out in distributed systems (Orihuela et al., 2017). They can avoid the propagation of overestimates and the resulting wrapping effect (Combastel, 2015; Kühn, 1998). In view of these advantages, we focus on the distributed zonotopic estimation problem for IDSs by following a fusion estimation approach. The addressed problem for IDSs is more intricate than the traditional distributed fusion estimation for sensor networks (Chen et al., 2014; Wang et al., 2021) because of the complex interactions among subsystems and the unknown correlations of estimation errors. The contributions of this paper can be summarized as follows.

- **Fusion Strategy.** A distributed zonotopic estimator (DZE) with a fusion strategy is designed to better utilize the redundant information from neighbors. With the proposed fusion strategy, the local estimate in a subsystem is fused with its neighbors' overlapping state estimates to provide a better estimation performance.
- **Fusion Criteria.** New fusion criteria for zonotopes are proposed to best combine original zonotopic estimates.
- **Stability Analysis.** A robust stability analysis for the proposed DZE is provided to ensure the boundedness of local estimation errors.

Notations. The identity matrix and the zero matrix are respectively represented as $\mathbf{I}$ and $\mathbf{0}$, while $\mathbf{1}$ is defined as a column vector of ones. Let us define $\mathbb{N}_L = \{1, 2, \ldots, L\}$ with L a natural number excluding zero. Give sets A and B, $A \setminus B$ represents the set of all elements of A that are not in B, and $A \cap B$ is the intersection set of A and B. The notation $A \succ 0$ ($\prec 0$) means that A is a positive definite (negative definite) matrix, and $\mathrm{tr}(A)$ represents the trace of A. Moreover, the infinite norm of a vector x is represented as $\|x\|_\infty$, and the weight Forbenius norm of a matrix $R \in \mathbb{R}^{n\times r}$ is denoted by $\|R\|_W = \sqrt{\mathrm{tr}(R^{\mathrm{T}}WR)}$ with $W \in \mathbb{R}^{n\times n}$ being a positive definite matrix. A diagonal matrix obtained from a vector of diagonal elements is denoted by $\mathrm{diag}(\cdot)$, and the notation $|\cdot|$ represents the element-by-element absolute value operator. Given a matrix $R = [R_1, R_2, \ldots, R_r] \in \mathbb{R}^{n\times r}$, let us define the submatrix $R_{a:b} := [R_a, R_{a+1}, \ldots, R_b]$ with $1 \le a \le b \le r$, and define the matrix that sorts the columns of R in decreasing weighted vector norm order as $\vec{R} := \mathrm{sort}\,(R, \|\cdot\|_W) = \left[\vec{R}_1, \ldots, \vec{R}_j, \ldots, \vec{R}_r\right]$, where $\vec{R}_j$ is a column of R and satisfies $\left\|\vec{R}_j\right\|_W^2 \ge \left\|\vec{R}_{j+1}\right\|_W^2$.

2. Preliminaries

An r-order zonotope $\mathcal{Z} \in \mathbb{R}^n$ is an affine transformation of a unit hypercube $[-1, +1]^r \subset \mathbb{R}^r$. It is defined as

$$\mathcal{Z} := \{z \in \mathbb{R}^n : z = c + Ru,\ \|u\|_\infty \le 1\} = \langle c, R\rangle \tag{1}$$

where $c \in \mathbb{R}^n$ is the center of $\mathcal{Z}$ and $R \in \mathbb{R}^{n\times r}$ is the generator matrix of $\mathcal{Z}$. For zonotopes, the Minkowski sum $\oplus$ and the linear image $\odot$ by a matrix L satisfy (Kühn, 1998):

$$\begin{aligned}&\langle c_1, R_1\rangle \oplus \langle c_2, R_2\rangle = \langle c_1 + c_2, [R_1\ \ R_2]\rangle\\ &L \odot \langle c, R\rangle = \langle Lc, LR\rangle\end{aligned} \tag{2}$$

When the operations of the Minkowski sum are executed consecutively, the number of columns in R will increase rapidly, resulting in much higher demands on storage and computing resource. However, this situation can be avoided by using a box enclosure of the zonotope:

$$\mathcal{Z} \subseteq \mathrm{box}(\mathcal{Z}) = \langle c, b(R)\rangle \tag{3}$$

where the aligned box $b(R) \in \mathbb{R}^{n\times n}$ in (3) is defined as

$$b(R) := \begin{cases}\mathrm{diag}(|R|\cdot\mathbf{1}) & \text{if } r > n\\ R & \text{otherwise}\end{cases} \tag{4}$$

Given $W \succ 0$, a less conservative enclosure with a weighted reduction operator $\downarrow_{q,W}(\cdot)$ was introduced in Combastel (2015) as

$$\begin{aligned}&\langle c, R\rangle \subset \langle c, \downarrow_{q,W}(R)\rangle\\ &\downarrow_{q,W}(R) := \left[\vec{R}_{1:q-n},\ \ b(\vec{R}_{q-n+1:r})\right]\end{aligned} \tag{5}$$

where $\downarrow_{q,W}(R) \in \mathbb{R}^{n\times q}$ and q ($n \le q \le r$) specifies the number of columns for the new generator matrix after the weighted reduction operator. This operator firstly reorders the columns of R in decreasing weighted vector norm order as a new matrix $\vec{R}$, then encloses the submatrix $\vec{R}_{q-n+1:r}$, which contains less important columns of R, into the aligned box $b(\vec{R}_{q-n+1:r})$.

As a special kind of polytopes, zonotopes also have their $\mathcal{H}$-representations and $\mathcal{V}$-representations (Ziegler, 2012). Let an r-order zonotope $\mathcal{Z} \in \mathbb{R}^n$ be denoted by an intersection of closed half-spaces, then the $\mathcal{H}$-representation is in the following form:

$$\mathcal{Z} = \left\{z \in \mathbb{R}^n :\ Hz \le b\right\} \tag{6}$$

where $H = \mathrm{col}\{h_1, -h_1, \ldots, h_\xi, -h_\xi\} \in \mathbb{R}^{2\xi\times n}$ and 2ξ is the number of closed half-spaces. According to (6), the zonotope $\mathcal{Z}$ can also be represented as the intersection of ξ tight strips:

$$\mathcal{Z} = \bigcap_{j=1}^{\xi} \mathcal{S}(\eta_j, \delta_j) \tag{7}$$

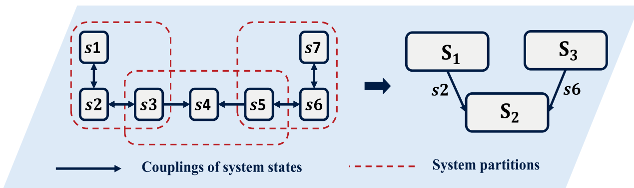


Fig. 1. An example of decomposing a large system into interconnected subsystems $\mathbf{S}_1$, $\mathbf{S}_2$ and $\mathbf{S}_3$ with overlapping states.

where $\mathcal{S}(\eta_j, \delta_j) := \{z : |\eta_j z - \delta_j| \leq 1\}$ is defined as a strip, a region enclosed by two parallel hyperplanes. Its parameters can be calculated as $\eta_j = \frac{2h_j}{b_{2j-1}+b_{2j}}$ and $\delta_j = \frac{2(b_{2j-1}-b_{2j})}{b_{2j-1}+b_{2j}}$. When the zonotope $\mathcal{Z}$ is denoted as a convex hull of a finite point set, the $\mathcal{V}$-representation is as follows:

$$\mathcal{Z} = \text{cone}(V) := \left\{ z \in \mathbb{R}^n : z = \sum_{j=1}^{\epsilon} t_j V_j,\ \ t_j \geq 0 \right\} \tag{8}$$

where $V_j \in \mathbb{R}^{n\times 1}$ is a vertex of the zonotope $\mathcal{Z}$ and $V = [V_1, \ldots, V_\epsilon] \in \mathbb{R}^{n\times\epsilon}$. Let us denote the box enclosure of a general polytope $\mathcal{P}$ as box($\mathcal{P}$). Given a $\mathcal{V}$-representation of $\mathcal{P}$ as in (8) and searching for the maximum and the minimum values in each row of V (denoted as vectors $V_{\max}$ and $V_{\min}$), then the center $c = (V_{\max} + V_{\min})/2$ and the generator matrix $R = \text{diag}\{|V_{\max} - V_{\min}|/2\}$ of box($\mathcal{P}$) can be determined.

Remark 1. Different from an ellipsoid, a zonotope can give a tighter enclosure to some complex state sets and does not suffer from the computational complexity problem of general polytopes. Therefore, when the statistical information of noises is unknown, a zonotopic description for system dynamics is more suitable for practical situations with both accurate noise description and lightweight computing requirements. $\diamond$

3. Problem formulation

In this paper, we consider a discrete-time interconnected system $\mathbf{S}$ with L subsystems, and the ith subsystem $\mathbf{S}_i$ ($i \in \mathbb{N}_L$) is described by the following state and measurement dynamics:

$$\mathbf{S}_i : \begin{cases} x_i(k+1) = A_i(k)x_i(k) + \displaystyle\sum_{i_\rho \in \Omega_i} A_{i,i_\rho}(k)x_{i_\rho}(k) \\ \qquad\qquad + \ \Gamma_i(k)w_i(k) \\ y_i(k) = C_i(k)x_i(k) + D_i(k)v_i(k) \end{cases} \tag{9}$$

where $x_i(k) \in \mathbb{R}^{n_i}$ and $y_i(k) \in \mathbb{R}^{m_i}$ are the state and the measurement of subsystem $\mathbf{S}_i$, respectively. The bounded matrices $A_i(k)$, $A_{i,i_\rho}(k)$, $\Gamma_i(k)$, $C_i(k)$, and $D_i(k)$ are with appropriate dimensions, and the sampling period T is omitted in the context for simplicity. The set Ω_i contains the indices of all the θ_i ($\theta_i < L$) neighbors of subsystem $\mathbf{S}_i$. Let the ρth neighbor of subsystem $\mathbf{S}_i$ be denoted as $\mathbf{S}_{i_\rho}$, then all the neighboring subsystems of $\mathbf{S}_i$ can be represented as $\mathbf{S}_{i_1}, \ldots, \mathbf{S}_{i_\rho}, \ldots, \mathbf{S}_{i_{\theta_i}}$. Here, the word neighbor means in-neighbor, but we do not make that distinction in the context. The communication is assumed to be unidirectional from neighbors to a subsystem. The system noise $w_i(k) \in \mathbb{R}^{n_{w_i}}$, the measurement noise $v_i(k) \in \mathbb{R}^{n_{v_i}}$, and the initial state $x_i(0)$ are wrapped within the following zonotopic sets:

$$\begin{cases} w_i(k) \in \mathbf{W}_i := \langle \mathbf{0}, \mathbf{I}_{n_{w_i}} \rangle \\ v_i(k) \in \mathbf{V}_i := \langle \mathbf{0}, \mathbf{I}_{n_{v_i}} \rangle \\ x_i(0) \in \langle c_{i,0}, R_{i,0} \rangle \subset \mathbb{R}^{n_i} \end{cases} \tag{10}$$

After system decomposition, the states of subsystems are often overlapping (see Fig. 1 as an example). Let the index set of the state components in $x_i(k)$ be denoted as $I_i = \{s_1^i, \ldots, s_{n_i}^i\}$. Then, the fact that the subsystem $\mathbf{S}_i$ and the subsystem $\mathbf{S}_{i_\rho}$ are with overlapping states means that $I_i \cap I_{i_\rho} \neq \emptyset$. The index sets of overlapping and non-overlapping state components between $\mathbf{S}_i$ and $\mathbf{S}_{i_\rho}$ are defined by $I_{i\cap i_\rho} := I_i \cap I_{i_\rho}$ and $I_{i\setminus i_\rho} := I_i \setminus I_{i_\rho}$. The numbers of elements for $I_{i\cap i_\rho}$ and $I_{i\setminus i_\rho}$ are $n_{i\cap i_\rho}$ and $n_{i\setminus i_\rho}$, respectively. Subsequently, we can denote the vectors containing the state components in $I_{i\cap i_\rho}$ and $I_{i\setminus i_\rho}$ as $x_{i\cap i_\rho}(k) \in \mathbb{R}^{n_{i\cap i_\rho}}$ and $x_{i\setminus i_\rho}(k) \in \mathbb{R}^{n_{i\setminus i_\rho}}$. It turns out that

$$x_{i\cap i_\rho}(k) = P_{i\cap i_\rho}x_i(k) = P_{i_\rho\cap i}x_{i_\rho}(k) = x_{i_\rho\cap i}(k), \tag{11a}$$

$$x_{i\setminus i_\rho}(k) = P_{i\setminus i_\rho}x_i(k). \tag{11b}$$

The matrices $P_{i\cap i_\rho} \in \mathbb{R}^{n_{i\cap i_\rho}\times n_i}$ and $P_{i\setminus i_\rho} \in \mathbb{R}^{n_{i\setminus i_\rho}\times n_i}$ are composed of n_i-dimensional canonical row vectors. The sth canonical row vector for $P_{i\cap i_\rho}$ ($P_{i\setminus i_\rho}$) is with all zeros but one element being 1, where the order of this non-zero element corresponds to the order of the sth state components of $x_{i\cap i_\rho}(k)$ ($x_{i\setminus i_\rho}(k)$) in the index set I_i. Without loss of generality, it is assumed that each subsystem $\mathbf{S}_i$ is in its simplest form, i.e., the coupling term $A_{i,i_\rho}(k)x_{i_\rho}(k)$ in (1) is only a linear combination of the non-overlapping state components $x_{i_\rho\setminus i}(k)$. It is also equivalent to

$$A_{i,i_\rho}(k)P_{i_\rho\cap i}^{\mathrm{T}}P_{i_\rho\cap i} = 0 \ \ (i_\rho \in \Omega_i). \tag{12}$$

A simple system transformation can be implemented to transform any interconnected system into the simplest form. Notice that both the local measurement $y_i(k)$ and the neighboring measurement $y_{i_\rho}(k)$ contain the information of $x_{i\cap i_\rho}(k)$. To improve the estimation performance, subsystems need to receive information from their neighbors and fuse local estimates to obtain more accurate ones.

Therefore, the following DZE is proposed for the interconnected system (9), which consists of four steps: zonotopic fusion, prediction, observation update, and reduction. At a given time instant $k-1$, assume that the local state is estimated to be within a zonotope $x_i(k-1) \in \hat{\mathcal{X}}_i(k-1) := \langle \hat{x}_i(k-1), R_i(k-1)\rangle$ with $R_i(k-1) \in \mathbb{R}^{n_i\times r_i}$.

Zonotopic fusion: When the distributed estimator receives the zonotopic estimates from neighbors, it turns out that the overlapping components $x_{i\cap i_\rho}(k-1)$ are simultaneously estimated by $\hat{\mathcal{X}}_i(k-1)$ and $\hat{\mathcal{X}}_{i_\rho}(k-1)$. To make full use of these redundant information, it is proposed to obtain a less conservative fusion estimate for $x_i(k-1)$ by fusing $\hat{\mathcal{X}}_i(k-1)$ and $\hat{\mathcal{X}}_{i_\rho}(k-1)$ ($i_\rho \in \Omega_i$) as

$$\begin{aligned} \hat{\mathcal{X}}_i^f(k-1) &:= \langle \hat{x}_i^f(k-1), R_i^f(k-1)\rangle \\ &= \mathcal{F}_i\left(\hat{\mathcal{X}}_i(k-1), \hat{\mathcal{X}}_{i_1}(k-1), \ldots, \hat{\mathcal{X}}_{i_{\theta_i}}(k-1)\right) \end{aligned} \tag{13}$$

Here, the function $\mathcal{F}_i(\cdot)$ is the fusion criterion to be designed, and it maps several zonotopes into an aimed zonotope. $\diamond$

Prediction: Then, given the dynamic of each subsystem in (9), the prediction for $x_i(k)$ is obtained by

$$\begin{aligned} &\hat{\mathcal{X}}_i^p(k) := \langle \hat{x}_i^p(k), R_i^p(k)\rangle \\ &\hat{x}_i^p(k) = A_i(k-1)\hat{x}_i^f(k-1) \\ &\qquad + \sum_{i_\rho\in\Omega_i} A_{i,i_\rho}(k-1)\hat{x}_{i_\rho}(k-1) \\ &R_i^p(k) = \Big[A_i(k-1)R_i^f(k-1) \ \ A_{i,i_1}(k-1)R_{i_1}(k-1) \\ &\qquad\qquad \ldots \ \ A_{i,i_{\theta_i}}(k-1)R_{i_{\theta_i}}(k-1) \ \ \Gamma_i(k-1)\Big] \end{aligned} \tag{14}$$

Observation update: After the measurement $y_i(k)$ is observed, the observation update for $x_i(k)$ can be designed as

$$\begin{aligned}
&\hat{\mathcal{X}}_i^o(k) := \langle \hat{x}_i^o(k), R_i^o(k)\rangle \\
&\hat{x}_i^o(k) = \hat{x}_i^p(k) + K_i(k)\left[y_i(k) - C_i(k)\hat{x}_i^p(k)\right] \\
&R_i^o(k) = \left[(\mathbf{I} - K_i(k)C_i(k))\, R_i^p(k) \qquad -K_i(k)D_i(k)\right]
\end{aligned} \tag{15}$$

where $R_i^o(k) \in \mathbb{R}^{n_i \times r_i^o}$. The gain matrix $K_i(k)$ is used to turn the uncertainty propagation and gives more freedom degrees to balance the influence of the system noise $w_i(k)$ and the measurement noise $v_i(k)$ (Combastel, 2015). $\diamond$

Reduction: Finally, the reduction operation in (5) is used at each time instant to reduce the dimension of the generator matrix $R_i^o(k)$ as

$$\begin{aligned}
\hat{\mathcal{X}}_i(k) &:= \langle \hat{x}_i(k), R_i(k)\rangle \\
&= \langle \hat{x}_i^o(k), \downarrow_{r_i,W} (R_i^o(k))\rangle
\end{aligned} \tag{16}$$

where $W \succ 0$ is a weight matrix. By keeping the generator matrix low-dimensional, the weighted reduction operator can significantly simplify the calculation for the designed distributed estimator. Afterward, each local estimator transmits $\hat{\mathcal{X}}_i(k)$ to its out-neighbors. $\diamond$

It will be proved in the next section that the inclusion property $x_i(k) \in \hat{\mathcal{X}}_i(k)$ holds for the proposed distributed estimator. Notice that the accuracy of a zonotopic estimate is determined by the size of the wrapped region of the zonotope and can be characterized by its generator matrix. One purpose of distributed estimation is to design an optimal gain matrix $K_i^{\mathrm{opt}}(k)$ under a certain performance index $J(\cdot)$ of the generator matrix $R_i^o(k)$. Consequently, the aims of this paper are described as follows.

(1) Design zonotopic fusion criteria such that the obtained fusion estimate $\hat{\mathcal{X}}_i(k)$ can satisfy inclusion property $x_i(k) \in \hat{\mathcal{X}}_i(k)$ and provide less conservative fusion results.
(2) Find an optimal gain matrix $K_i^{\mathrm{opt}}(k)$ for the designed DZE under the following performance index

$$J\left(R_i^o(k)\right) = \mathrm{tr}\left(WR_i^o(k)\left[R_i^o(k)\right]^{\mathrm{T}}\right) = \|R_i^o(k)\|_W^2 \tag{17}$$

(3) Derive stability conditions for the designed DZE with the fusion strategy.

Remark 2. The proposed zonotopic estimator in (13)–(16) is fully distributed. The iteration for $\hat{x}_i(k)$ only uses the local information (i.e., $y_i(k)$ and $\hat{x}_i(k-1)$) and the information from neighbors (i.e., $\hat{x}_{i_\rho}(k-1)$). Meanwhile, the calculation of $R_i(k)$ only requires the generator matrices $R_i(k-1)$ and $R_{i_\rho}(k-1)$. In this case, the optimal gain matrix $K_i^{\mathrm{opt}}(k)$ can be determined by local communication, and the proposed zonotopic estimator can be independently implemented in subsystems. $\diamond$

4. Main results

4.1. Zonotopic fusion criteria

Before a zonotopic fusion step, the useful information for $x_i(k)$ is $x_i(k) \in \hat{\mathcal{X}}_i(k)$ and

$$x_{i_\rho \cap i}(k) \in \langle P_{i_\rho\cap i}\hat{x}_{i_\rho}(k), P_{i_\rho\cap i}R_{i_\rho}(k)\rangle \tag{18a}$$

$$x_{i\setminus i_\rho}(k) \in \langle P_{i\setminus i_\rho}\hat{x}_i(k), P_{i\setminus i_\rho}R_i(k)\rangle \tag{18b}$$

where (18a) is from $x_i(k) \in \hat{\mathcal{X}}_{i_\rho}(k)$ and (18b) is from $x_i(k) \in \hat{\mathcal{X}}_i(k)$. Notice that $x_{i\cap i_\rho}(k) = x_{i_\rho\cap i}(k)$ by (11), then an induced estimate $\hat{\mathcal{X}}_{i,i_\rho}(k)$ can be obtained from (18) as

$$\begin{aligned}
&x_i(k) \in \hat{\mathcal{X}}_{i,i_\rho}(k) := \langle \hat{x}_{i,i_\rho}(k), R_{i,i_\rho}(k)\rangle \\
&\hat{x}_{i,i_\rho}(k) = P_{i\cap i_\rho}^{\mathrm{T}} P_{i_\rho\cap i}\hat{x}_{i_\rho}(k) + P_{i\setminus i_\rho}^{\mathrm{T}} P_{i\setminus i_\rho}\hat{x}_i(k) \\
&R_{i,i_\rho}(k) = \left[P_{i\cap i_\rho}^{\mathrm{T}} P_{i_\rho\cap i}R_{i_\rho}(k),\ \ P_{i\setminus i_\rho}^{\mathrm{T}} P_{i\setminus i_\rho}R_i(k)\right]
\end{aligned} \tag{19}$$

The operations between the generator matrices $R_i(k)$ and $R_{i,i_\rho}(k)$ in (19) require them to be of the same dimension. This can be achieved by simply setting the same reduction parameter as $r_i = r$ $(i \in \mathbb{N}_L)$ or adding zero column vectors into these matrices. By now, we have known that $x_i(k)$ is wrapped within several zonotopic estimates $\hat{\mathcal{X}}_i(k)$ and $\hat{\mathcal{X}}_{i,i_\rho}(k)$ $(i_\rho \in \Omega_i)$. In the zonotopic fusion step, the key technique for fusing several zonotopic estimates lies in finding a zonotopic enclosure for their common estimation region. In what follows, two zonotopic enclosures are constructed for fusion estimation.

Let the $\mathcal{H}$-representations of $\hat{\mathcal{X}}_i(k)$ and $\hat{\mathcal{X}}_{i,i_\rho}(k)$ $(i_\rho \in \Omega_i)$ be denoted as

$$\hat{\mathcal{X}}_i(k) = \{x_i(k) : H_i(k)x_i(k) \le b_i(k)\} \tag{20a}$$

$$\hat{\mathcal{X}}_{i,i_\rho}(k) = \left\{x_i(k) : H_{i,i_\rho}(k)x_i(k) \le b_{i,i_\rho}(k)\right\} \tag{20b}$$

where $H_i(k) \in \mathbb{R}^{2\xi_i \times n_i}$ and $H_{i,i_\rho}(k) \in \mathbb{R}^{2\xi_{i,i_\rho} \times n_i}$. Then, the common region of these zonotopes can be represented as

$$\mathcal{O}_i(k) = \left\{ x_i(k) : \underbrace{\begin{bmatrix} H_i(k) \\ H_{i,i_1}(k) \\ \vdots \\ H_{i,i_{\theta_i}}(k) \end{bmatrix}}_{H_i^o(k)} x_i(k) \le \underbrace{\begin{bmatrix} b_i(k) \\ b_{i,i_1}(k) \\ \vdots \\ b_{i,i_{\theta_i}}(k) \end{bmatrix}}_{b_i^o(k)} \right\} \tag{21}$$

Here, one can rewrite $H_i^o(k)$ and $b_i^o(k)$ as

$$\begin{cases} H_i^o(k) = \mathrm{col}\left\{h_i^1(k), -h_i^1(k), \ldots, h_i^{\xi_i^o}(k), -h_i^{\xi_i^o}(k)\right\} \\ b_i^o(k) = \mathrm{col}\left\{b_i^1(k), b_i^2(k), \ldots, b_i^{2\xi_i^o}(k)\right\} \end{cases} \tag{22}$$

where $\xi_i^o := \xi_i + \sum_{i_\rho\in\Omega_i}\xi_{i,i_\rho}$. It is obvious that $\mathcal{O}_i(k)$ is still a polytope, and it has a $\mathcal{V}$-representation

$$\mathcal{O}_i(k) = \mathrm{cone}(V_i(k)) \tag{23}$$

where $V_i(k) = \left[V_{i,1}(k), \ldots, V_{i,\epsilon_i}(k)\right]$. The first zonotopic fusion estimate is proposed to be the box enclosure for the common estimation region

$$\hat{\mathcal{X}}_i^f(k) = \mathrm{box}(\mathcal{O}_i(k)) = \langle \hat{x}_i^f(k), R_i^f(k)\rangle \tag{24}$$

where $\hat{x}_i^f(k)$ and $R_i^f(k)$ can be obtained as in the calculation procedure after (8). Then, the following result is derived.

Lemma 1. *The zonotopic fusion estimate in (24) satisfies* $\left\|R_i^f(k)\right\|_W \le \|b(R_i(k))\|_W$. $\diamond$

Proof. According to some basic properties of sets, one has that

$$\begin{aligned}
\mathcal{O}_i(k) &= \hat{\mathcal{X}}_i(k) \cap \left(\bigcap_{i_\rho\in\Omega_i} \hat{\mathcal{X}}_{i,i_\rho}(k)\right) \\
&\subseteq \mathrm{box}(\hat{\mathcal{X}}_i(k)) \cap \left(\bigcap_{i_\rho\in\Omega_i} \mathrm{box}(\hat{\mathcal{X}}_{i,i_\rho}(k))\right) \\
&\subseteq \mathrm{box}(\hat{\mathcal{X}}_i(k))
\end{aligned} \tag{25}$$

Then, apply the box enclosure operation on both sides of the equation in (25), and it turns out that

$$\hat{\mathcal{X}}_i^f(k) = \text{box}(\mathcal{O}_i(k)) \subseteq \text{box}(\hat{\mathcal{X}}_i(k)) \tag{26}$$

Thus, the inequality $\left\|R_i^f(k)\right\|_W \le \|b(R_i(k))\|_W$ holds. $\diamond$

Another zonotopic fusion estimate is proposed by the set approximation approach in Vicino and Zappa (1996). Let us solve the following discrete optimization problem by enumeration:

$$\min_{s_1, s_2 \in \mathbb{N}_{\epsilon_i}} \left|h_i^s(k)(V_{i,s_1}(k) - V_{i,s_2}(k))\right| \tag{27}$$

then two supporting hyperplanes for $\mathcal{O}_i(k)$ with a normal vector $h_i^s(k)$ $(s \in \mathbb{N}_{\xi_i^o})$ can be obtained as

$$\begin{aligned} \mathscr{P}_i^{s1}(k) &:= \left\{z \in \mathbb{R}^{n_i} : h_i^s(k)z = h_i^s(k)V_{i,s_1}(k)\right\} \\ \mathscr{P}_i^{s2}(k) &:= \left\{z \in \mathbb{R}^{n_i} : h_i^s(k)z = h_i^s(k)V_{i,s_2}(k)\right\} \end{aligned} \tag{28}$$

By the parallel supporting hyperplanes $\mathscr{P}_i^{s1}(k)$ and $\mathscr{P}_i^{s2}(k)$ defined in (28), one can conclude that $\mathcal{O}_i(k)$ is contained in the following intersection of tight strips:

$$\mathcal{O}_i(k) \subseteq \bigcap_{s=1}^{\xi_i^o} \mathcal{S}(\eta_i^s(k), \delta_i^s(k)) \tag{29}$$

where $\eta_i^s(k)$ and $\delta_i^s(k)$ can be determined as in the calculation procedure after (7). By a combination of n_i out of ξ_i^o strips, there are $\begin{pmatrix}\xi_i^o \\ n_i\end{pmatrix}$ n_i-order zonotopes can be generated. Let us denote the set of all possible combinations as $\mathcal{L}$, and each combined zonotope from n_i strips is defined by

$$\begin{cases} \mathcal{Z}_i^\ell(k) := \bigcap_{j=1}^{n_i} \mathcal{S}\left(\eta_i^{l_j}(k), \delta_i^{l_j}(k)\right) = \langle \hat{x}_i^\ell(k), R_i^\ell(k)\rangle \\ \hat{x}_i^\ell(k) = R_i^\ell(k)\begin{bmatrix}\delta_i^{l_1}(k) & \cdots & \delta_i^{l_n}(k)\end{bmatrix}^{\mathrm{T}} \\ R_i^\ell(k) = \begin{bmatrix}\eta_i^{l_1}(k) \\ \vdots \\ \eta_i^{l_n}(k)\end{bmatrix}^{-1} \end{cases} \tag{30}$$

where $\ell = \{l_1, l_2, \ldots, l_n\} \in \mathcal{L}$ is a set of indices such that $1 \le l_1 < l_2 < \cdots < l_n \le \xi_i^o$. Then, the zonotopic fusion estimate is proposed to be

$$\hat{\mathcal{X}}_i^f(k) = \mathcal{Z}_i^{\ell_{\text{opt}}(k)}(k) \tag{31}$$

where $\ell_{\text{opt}}(k)$ is the optimal index set. Then, the following lemma is proposed:

Lemma 2. *The optimal combined zonotope $\mathcal{Z}_i^{\ell_{\text{opt}}(k)}(k)$ and its index set $\ell_{\text{opt}}(k)$ can be determined by the following discrete optimization problem:*

$$\ell_{\text{opt}}(k) = \arg\min_{\ell \in \mathcal{L}} \left\|R_i^\ell(k)\right\|_W^2 \tag{32}$$

By providing the above fusion strategy, the inclusion property $x_i(k) \in \hat{\mathcal{X}}_i(k)$ is proved in the following theorem for the proposed DZE.

Theorem 1. *Given $x_i(0) \in \hat{\mathcal{X}}_i(0)$ and recursively calculate $\hat{\mathcal{X}}_i(k)$ using (13)–(16), then $x_i(k) \in \hat{\mathcal{X}}_i(k)$ $(\forall k \ge 0)$ holds.* $\diamond$

Proof. By mathematical induction, it only needs to show that if $x_i(k-1) \in \hat{\mathcal{X}}_i(k-1)$ then $x_i(k) \in \hat{\mathcal{X}}_i(k)$ holds. For the proposed zonotopic fusion estimates, $x_i(k-1) \in \hat{\mathcal{X}}_i(k-1)$ implies $x_i(k-1) \in \mathcal{O}_i(k-1) \in \hat{\mathcal{X}}_i^f(k-1)$ by (24) and (29). Then, take the state dynamics in (9) into consideration, one has that

$$\begin{aligned} x_i(k) \in{} & \left(A_i(k-1) \odot \hat{\mathcal{X}}_i^f(k-1)\right) \oplus (\Gamma_i(k-1) \odot \mathbf{W}_i) \\ & \oplus_{i_\rho \in \Omega_i} \left(A_{i,i_\rho}(k-1) \odot \hat{\mathcal{X}}_{i_\rho}(k-1)\right) = \hat{\mathcal{X}}_i^p(k) \end{aligned} \tag{33}$$

By the measurement dynamics in (9), it is found that

$$\begin{aligned} x_i(k) &= x_i(k) + K_i(k)\left[y_i(k) - C_i(k)x_i(k) - D_i(k)v_i(k)\right] \\ &\in \left((\mathbf{I} - K_i(k)C_i(k)) \odot \hat{\mathcal{X}}_i^p(k)\right) \\ &\quad \oplus (-K_i(k)D_i(k) \odot \mathbf{V}_i) \oplus (K_i(k) \odot \langle y_i(k), \mathbf{0}\rangle) \\ &= \hat{\mathcal{X}}_i^o(k) \end{aligned} \tag{34}$$

After the reduction operation in (16), one can conclude that $x_i(k) \in \hat{\mathcal{X}}_i^o(k) \in \hat{\mathcal{X}}_i(k)$. $\diamond$

Remark 3. Notice that the discrete optimization problems (28) and (32) are unconstrained and can be solved by enumeration. However, the computational complexity is proportional to the cardinality of $\mathcal{L}$, which is affordable only when n_i and ξ_i^o are not too large. One possible way to reduce the computational burden is to eliminate strips whose two parallel hyperplanes do not coincide with any half-space of the common region. This simplification is very useful when fusing multiple zonotopes. Another possible way is to eliminate some most distant strips in advance. On the other hand, the computation of vertices and half-spaces of a zonotope is a mature technique, which can be achieved by "cddmex" or "cdd+" toolbox (Fukuda, 1997). $\diamond$

4.2. Optimal gain matrix

The optimal gain matrix $K_i^{\text{opt}}(k)$ that minimizes the performance index $J\left(R_i^o(k)\right)$ in (17) can be calculated in the following theorem.

Theorem 2. *Given a matrix $W \succ 0$, the optimal gain matrix $K_i^{\text{opt}}(k) = \arg\min_{K_i(k)} \|R_i^o(k)\|_W^2$ for the designed distributed estimator is calculated as*

$$\begin{aligned} K_i^{\text{opt}}(k) ={} & R_i^p(k)\left[R_i^p(k)\right]^{\mathrm{T}} C_i^{\mathrm{T}}(k) \\ & \times \left\{C_i(k)R_i^p(k)\left[R_i^p(k)\right]^{\mathrm{T}} C_i^{\mathrm{T}}(k) + D_i(k)D_i^{\mathrm{T}}(k)\right\}^{-1} \end{aligned} \tag{35}$$

Proof. Notice that the performance index $\|R_i^o(k)\|_W^2$ is convex with respect to $K_i(k)$. Let $\partial\|R_i^o(k)\|_W^2/\partial K_i(k) = 0$, then the gain matrix $K_i(k) = K_i^{\text{opt}}(k)$ is obtained by the derivation formula of trace. Obviously, the performance index is smaller under $K_i(k) = K_i^{\text{opt}}(k)$ than it is under $K_i(k) = \mathbf{0}$. Thus, the gain matrix $K_i^{\text{opt}}(k)$ can achieve the minimum performance index. $\diamond$

Then, the computation procedures for the proposed DZE with the fusion strategy can be summarized by Algorithm 1.

Remark 4. Different from the estimation methods based on Gaussian noises in Farina et al. (2010) and Zhang et al. (2021), the proposed DZE only needs some operations on zonotopic estimation errors and does not require knowing the correlations of estimation errors; thus it can avoid the complex calculation of cross-covariances. Therefore, the calculation for the zonotopic estimates represented as vectors and matrices is simple enough to be carried out in IDSs with limited computation capabilities. On the other hand, the calculated optimal gain matrix in (35) is independent of the weight matrix W, which means the obtained gain matrix is always optimal irrespective of W. $\diamond$

Algorithm 1 Distributed Zonotopic Estimation with A Fusion Strategy.

1: Give $x_i(0) \in \hat{\mathcal{X}}_i(0)$ $(i \in \mathbb{N}_L)$
2: **for** $i = 1 : L$ **do**
3: Subsystem $\mathbf{S}_i$ collects its local measurement $y_i(k)$, local estimate $\hat{\mathcal{X}}_i(k-1)$ and its neighbors' estimates $\hat{\mathcal{X}}_{i_\rho}(k-1)$ $(i_\rho \in \Omega_i)$;
4: Determine $\hat{\mathcal{X}}_{i,i_\rho}(k-1)$ by (19);
5: Calculate the zonotopic fusion estimate $\hat{\mathcal{X}}_i^f(k-1)$ by (24) or (31);
6: Obtain the prediction $\hat{\mathcal{X}}_i^p(k)$ by (14);
7: Calculate the optimal gain $K_i^{\mathrm{opt}}(k)$ by (35);
8: Obtain the observation update $\hat{\mathcal{X}}_i^o(k)$ by (15);
9: Perform the reduction step in (16) to get $\hat{\mathcal{X}}_i(k)$;
10: Subsystem $\mathbf{S}_i$ transmits $\hat{\mathcal{X}}_i(k)$;
11: **end for**
12: Return to Step 2 and implement Steps 2–11 for calculating $\hat{\mathcal{X}}_i(k+1)$ $(i \in \mathbb{N}_L)$.

4.3. Stability analysis

In this section, a robust stability analysis for the proposed DZE will be presented. Before deriving the stability conditions, the following lemmas and assumptions are required.

Lemma 3 (*Combastel, 2015*)**.** *The discrete time-varying linear system $x(k) = A(k-1)x(k-1)$ is robust γ-stable (i.e., Lyapunov stable with decay rate γ) if there exists a bounded sequence of matrices $W(k) \succ 0$ and $\gamma \in [0, 1]$ such that, $\forall k \geq 0$,*

$$\begin{bmatrix} \gamma W(k-1) & A^{\mathrm{T}}(k-1)W(k) \\ * & W(k) \end{bmatrix} \succ 0 \tag{36}$$

Assumption 1. Bounded sequences of matrices $K_i^*(k)$ and $W_i^*(k) \succ 0$ have been designed such that the system $x_i(k) = K_{C_i}^*(k)A_i(k-1)x_i(k-1)$ is robust γ_i-stable, where $K_{C_i}^*(k) := \mathbf{I} - K_i^*(k)C_i(k)$, $0 \leq \gamma_i < 1$, and $W_i^*(k)$ have their eigenvalues all within the interval $\left[\underline{\lambda}_i, \ \overline{\lambda}_i\right]$, $\underline{\lambda}_i > 0$. ⋄

Remark 5. Notice that the condition for such gains $K_i^*(k)$ to exist is not strict. Suppose that $\gamma_i = 1$, the existence of $K_i^*(k)$ in Assumption 1 is a much weaker condition than the stability condition of a standard Kalman filter (uniformly observable and controllable). On the other hand, if the parameter γ_i is determined, the bounded sequences of matrices $K_i^*(k)$ and $W_i^*(k)$ can be obtained by transforming the following matrix inequality

$$\begin{bmatrix} \gamma_i W_i^*(k-1) & \left(K_{C_i}^*(k)A_i(k-1)\right)^{\mathrm{T}} W_i^*(k) \\ * & W_i^*(k) \end{bmatrix} \succ 0 \tag{37}$$

into a linear matrix inequality (LMI) problem (Combastel, 2015, Proposition 13) and solving it with the function "mincx" of MATLAB LMI toolbox (Boyd, El Ghaoui, Feron, & Balakrishnan, 1994). ⋄

Assumption 2 (*Zhang et al., 2021*)**.** The coupling structure of the interconnected system (9) is of a directed acyclic graph. Then, the system can be transformed into a sequentially connected system by reordering the subsystems. ⋄

Lemma 4 (*Combastel, 2015*)**.** *Given a zonotope $\langle \mathbf{0}, R \rangle$ with the generator matrix $R \in \mathbb{R}^{n \times r}$ and its box enclosure $\langle \mathbf{0}, b(R) \rangle$, the following inequality holds:*

$$\|b(R)\|_W^2 \leq \left(\overline{\lambda}/\underline{\lambda}\right) r \|R\|_W^2 \tag{38}$$

where $W \succ 0$ has all its eigenvalues in $\left[\underline{\lambda}, \overline{\lambda}\right] \subset \mathbb{R}$. ⋄

Theorem 3. *Under Assumptions 1–2 and the fusion criterion (24), the generator matrix of the proposed DZE under the optimal gain matrix $K_i^{\mathrm{opt}}(k)$ in (35) is ultimately bounded (i.e., $\lim_{k\to\infty} \|R_i^o(k)\|_{W_i^*(k)}^2 < \varphi_i$, and φ_i is a finite positive number) if the condition $\gamma_i \left(\overline{\lambda}_i/\underline{\lambda}_i\right) r_i^o < 1$ holds.* ⋄

Proof. According to Lemmas 1 and 4, it is known that

$$\begin{aligned} \left\|R_i^f(k-1)\right\|_{W_i^*(k-1)}^2 &\leq \|b(R_i(k-1))\|_{W_i^*(k-1)}^2 \\ &= \left\|b(R_i^o(k-1))\right\|_{W_i^*(k-1)}^2 \\ &\leq \left(\overline{\lambda}_i/\underline{\lambda}_i\right) r_i^o \left\|R_i^o(k-1)\right\|_{W_i^*(k-1)}^2 \end{aligned} \tag{39}$$

By Assumption 1, the matrices $W_i^*(k)$ and $K_i^*(k)$ satisfy (37), and the Schur complement of (37) turns out to be

$$\begin{aligned} &A_i^{\mathrm{T}}(k-1)\left[K_{C_i}^*(k)\right]^{\mathrm{T}} W_i^*(k)K_{C_i}^*(k)A_i(k-1) \\ &- \gamma_i W_i^*(k-1) \prec \mathbf{0} \end{aligned} \tag{40}$$

By the property that the trace of a negative definite matrix is negative, (40) turns out to be

$$\left\|K_{C_i}^*(k)A_i(k-1)R_i^f(k-1)\right\|_{W_i^*(k)}^2 < \gamma_i \left\|R_i^f(k-1)\right\|_{W_i^*(k-1)}^2 \tag{41}$$

Moreover, the following upper bound exists due to the boundedness of system's matrices:

$$\begin{aligned} &\|K_{C_i}^*(k)\Gamma_i(k-1)\|_{W_i^*(k)}^2 + \|K_i^*(k)D_i(k)\|_{W_i^*(k)}^2 \\ &\leq \overline{\lambda}_i \left(\|K_{C_i}^*(k)\Gamma_i(k-1)\|^2 + \|K_i^*(k)D_i(k)\|^2\right) < \phi_i \end{aligned} \tag{42}$$

where ϕ_i is a finite positive number. Under the optimal gain matrix $K_i^{\mathrm{opt}}(k)$ designed in (35), one has that

$$\begin{aligned} \left\|R_i^o(k)\right\|_{W_i^*(k)}^2 \leq{} & \gamma_i \left(\overline{\lambda}_i/\underline{\lambda}_i\right) r_i^o \left\|R_i^o(k-1)\right\|_{W_i^*(k-1)}^2 + \phi_i \\ &+ \sum_{i_\rho \in \Omega_i} \left\|K_{C_i}^*(k)A_{i,i_\rho}(k-1)R_{i_\rho}(k-1)\right\|_{W_i^*(k)}^2 \end{aligned} \tag{43}$$

Under Assumption 2, the order of each subsystem can be rearranged to form a sequentially interconnected system with acyclic ordering, i.e., $i_\rho (\in \Omega_i) < i$ or $\Omega_i = \emptyset$. Following from the analysis in Zhang et al. (2021), the boundedness of each generator matrix can be derived from the condition $\gamma_i \left(\overline{\lambda}_i/\underline{\lambda}_i\right) r_i^o < 1$ after reordering of subsystems. ⋄

Remark 6. Notice that Assumption 2 is more realistic than the assumption of sequentially connected systems in Chen et al. (2019a). By the decoupling approach in Zhang et al. (2021), a large part of actual IDSs can be transformed into an interconnected structure described as a directed acyclic graph. When the interconnected system is sequentially reordered, the stability analysis can be achieved one by one with only the local existence condition of $K_i^*(k)$ and $W_i^*(k)$ such that $x_i(k) = K_{C_i}^*(k)A_i(k-1)x_i(k-1)$ is robust γ_i-stable. Therefore, the stability of the proposed estimator for an interconnected system can be guaranteed in advance if the matrices $K_i^*(k)$ and $W_i^*(k)$ are determined by Remark 5. ⋄

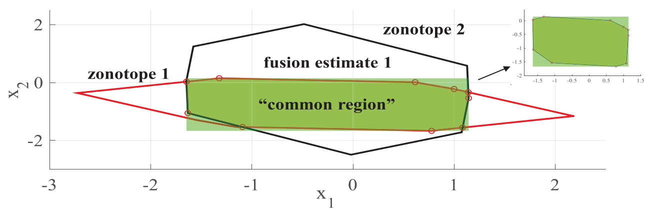
5. Numerical examples

In this section, two illustrative examples are given to show the effectiveness of the proposed zonotopic fusion criteria and DZE.

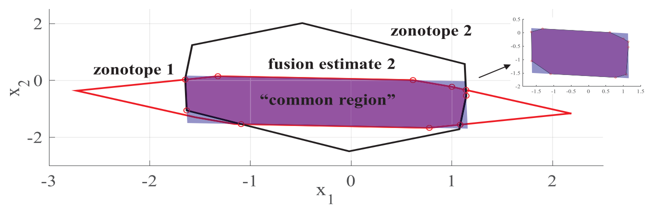
Example 1. Two zonotopes $\langle c_1, R_1 \rangle$ and $\langle c_2, R_2 \rangle$ are used for zonotopic fusion by the proposed fusion criteria, where the elements

$$
A_c := \begin{bmatrix}
0 & 1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
-\frac{k_{1,2}}{m_1} & -\frac{c_{1,2}}{m_1} & \frac{k_{1,2}}{m_1} & \frac{c_{1,2}}{m_1} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\
\frac{k_{1,2}}{m_2} & \frac{c_{1,2}}{m_2} & -\frac{k_{1,2}+k_{2,3}}{m_2} & -\frac{c_{1,2}+c_{2,3}}{m_2} & \frac{k_{2,3}}{m_2} & \frac{c_{2,3}}{m_2} & \cdots & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\
0 & 0 & \frac{k_{2,3}}{m_3} & \frac{c_{2,3}}{m_3} & -\frac{k_{2,3}+k_{3,4}}{m_3} & -\frac{c_{2,3}+c_{3,4}}{m_3} & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & \cdots & -\frac{k_{6,7}}{m_7} & -\frac{c_{6,7}}{m_7}
\end{bmatrix}, x := \begin{bmatrix} d_1 \\ \dot{d}_1 \\ d_2 \\ \dot{d}_2 \\ d_3 \\ \dot{d}_3 \\ \vdots \\ d_7 \\ \dot{d}_7 \end{bmatrix}, u := \begin{bmatrix} 0 \\ \frac{F_1}{m_1} \\ 0 \\ \frac{F_2}{m_2} \\ 0 \\ \frac{F_3}{m_3} \\ \vdots \\ 0 \\ \frac{F_7}{m_7} \end{bmatrix} \quad (44)
$$

Box I.



(a) Zonotopic fusion estimate 1 by (24).



(b) Zonotopic fusion estimate 2 by (31).

Fig. 2. Zonotopic fusion estimates for two randomly generated original zonotopes.

of c_1, c_2, $R_1 \in \mathbb{R}^{2\times 4}$ and $R_2 \in \mathbb{R}^{2\times 4}$ are randomly generated within $[-1, 1]$ by the function "rand" of MATLAB. The zonotopic fusion results are plotted in Figs. 2(a) and 2(b). It can be seen that the common region of the two zonotopes is entirely within the proposed zonotopic estimates. In most situations, the carefully constructed zonotopic enclosure in (30) is tighter than the box enclosure. $\diamond$

Example 2. Consider an interconnected system consisting of seven mass–spring–damper systems (see Fig. 3), where masses are connected one by one by springs and dampers and are pulled forward by F_i ($i \in \mathbb{N}_7$) and its displacement d_i and velocity $\dot{d}_i$ are regarded as state variables. The state dynamics of the overall system is characterized by $\dot{x}(t) = A_c x(t) + u(t)$, where the matrix A_c, the state variable x, and the input signal u are defined in (44) (see equation in Box I). To preserve the sparse structure of A_c, the block-wise discretization (Farina, Colaneri, & Scattolini, 2013) with a sampling period T is applied to obtain its corresponding discrete-time model $x(k+1) = A_d x(k) + B_d u(k)$. This system is decomposed into three subsystems $\mathbf{S}_1$ (masses 1–3), $\mathbf{S}_2$ (masses 3–5), and $\mathbf{S}_3$ (masses 5–7) with overlapping state components, and each subsystem is monitored by a sensor. Considering the influence of noises and defining $x_i = \left[d_{2i-1}, \dot{d}_{2i-1}, d_{2i}, \dot{d}_{2i}, d_{2i+1}, \dot{d}_{2i+1}\right]^{\mathrm{T}}$ and $u_i = \left[0, F_{2i-1}/m_{2i-1}, 0, F_{2i}/m_{2i}, 0, F_{2i+1}/m_{2i+1}\right]^{\mathrm{T}}$, the state and measurement dynamics of these subsystems can be described as

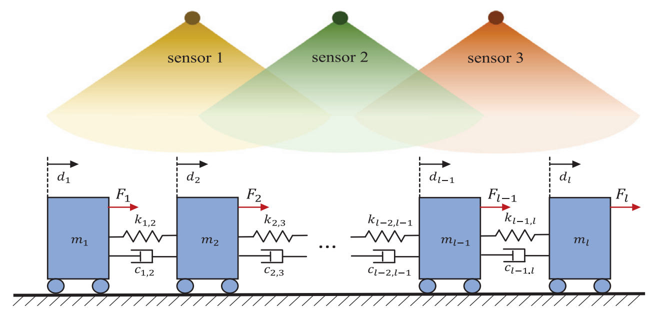


Fig. 3. An interconnected system composed of seven mass–spring–damper systems.

follows:

$$
\begin{cases}
x_1(k+1) = A_1 x_1(k) + A_{12} x_2(k) + B_1 u_1(k) + w_1(k) \\
y_1(k) = C_1 x_1(k) + v_1(k) \\
x_2(k+1) = A_2 x_2(k) + A_{21} x_1(k) + A_{23} x_3(k) \\
\qquad\qquad + B_2 u_2(k) + w_2(k) \\
y_2(k) = C_2 x_2(k) + v_2(k) \\
x_3(k+1) = A_3 x_3(k) + A_{32} x_2(k) + B_3 u_3(k) + w_3(k) \\
y_3(k) = C_3 x_3(k) + v_3(k)
\end{cases} \quad (45)
$$

where A_i, A_{ij} and B_i are submatrices in A_d or B_d, and C_i ($i = 1, 2, 3$) are identity matrices. In the simulation, the values of m_i are randomly extracted from the interval $[10, 15]$, and the external forces F_i are randomly given in the interval $[5, 10]$ at different instants. The spring and damping coefficients are set identically as 0.5, while the sampling period during the discretization is $T = 0.1$. The system noises $w_i(k)$ and measurement noises $v_i(k)$ are wrapped within $\langle \mathbf{0}, 0.05\mathbf{I} \rangle$.

By deploying the proposed DZE with the fusion criterion in (31), the state variables of each subsystem are estimated within zonotopes by only local communication. The discrete optimization problem in (32) is solved by enumeration and the simplification method in Remark 3. The velocity variables in the second subsystem and their estimated intervals are plotted in Fig. 4. Notice that the interval of a zonotope $\langle c, R \rangle$ is a loose enclosure calculated by $\langle c, |R|\mathbf{1} \rangle$, and the state is estimated to be within the interval $\left[c - |R|\mathbf{1}, c + |R|\mathbf{1}\right]$. It can be observed that the estimated intervals can track the trajectories of actual velocities, and the widths of these intervals do not diverge as time goes by. Moreover, the local estimates without a fusion step are also plotted in Fig. 4. As is shown in the figure, the distributed estimation with a fusion strategy can provide more accurate interval estimates. The estimation performance is evaluated more accurately by the index in (17), and the performance comparison between the proposed distributed estimator with and without the

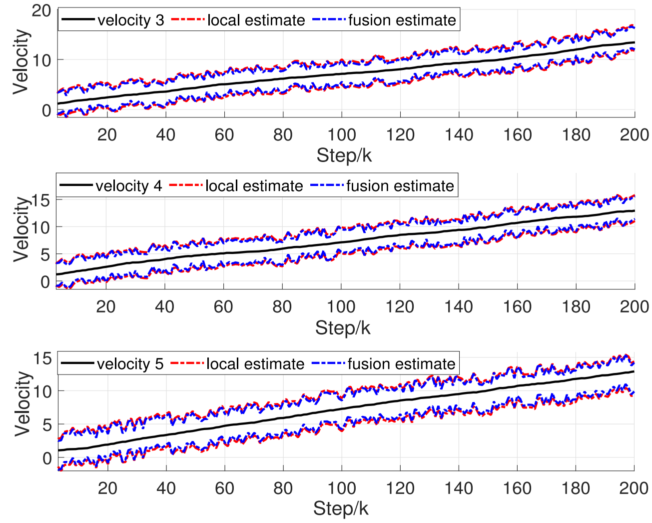


Fig. 4. The velocities for masses 3−5 and their estimated intervals in the second subsystem by the local estimation without a fusion step and the fusion estimate under the optimal gain matrix $K_i^{\mathrm{opt}}(k)$.

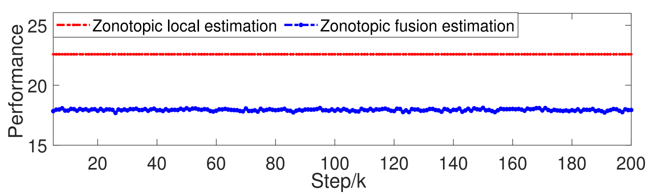


Fig. 5. The performance of the zonotopic local estimation and the zonotopic fusion estimation in the second subsystem.

fusion strategy is then plotted in Fig. 5. Although sensors with the same precision are used in different subsystems, the estimation with fusion strategy can still improve the performance to some extent. ◇

6. Conclusion

In this paper, a new method for the distributed estimation problem has been developed for IDSs with overlapping states. Based on the estimates from neighbors, a DZE was first designed. Then, the optimal gain matrix was designed for such a zonotopic estimator by minimizing a performance index of the estimation error. Notice that the proposed DZE can be implemented online and does not require knowing the correlations of estimation errors, such as the complex calculation of cross-covariances. To improve the estimation performance from overlapping state estimates, novel zonotopic fusion criteria were designed to find a tight zonotopic enclosure for the common region described by several local estimates. By solving an optimization problem, the best fusion estimate can be obtained. Moreover, the robust stability conditions, which can be further simplified into a LMI problem, were provided such that the local estimation errors can be ultimately bounded. Finally, three illustrative examples were employed to show the effectiveness of the proposed zonotopic fusion criteria and DZE.

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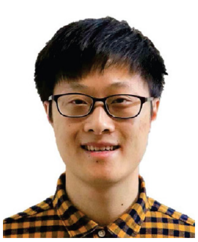

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