Efficient Quantum Circuits for Schur and Clebsch-Gordan Transforms

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The Schur basis on n d-dimensional quantum systems is a generalization of the total angular momentum basis that is useful for exploiting symmetry under permutations or collective unitary rotations. We present efficient (size $poly(n,d,\log(1/\epsilon))$ for accuracy ϵ) quantum circuits for the Schur transform, which is the change of basis between the computational and the Schur bases. These circuits are based on efficient circuits for the Clebsch-Gordan transformation. We also present an efficient circuit for a limited version of the Schur transform in which one needs only to project onto different Schur subspaces. This second circuit is based on a generalization of phase estimation to any nonabelian finite group for which there exists a fast quantum Fourier transform.

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A key component of quantum algorithms is their ability to reveal information stored in non-local degrees of freedom. In particular, one of the most important building blocks known is the quantum Fourier transform (QFT)[1], an efficient circuit construction for conversion between discrete position and momentum bases. The QFT converts a vector of 2^n amplitudes in $O(n^2)$ steps, in contrast to the $O(n2^n)$ which would be required classically.

Another elementary basis change important in quantum physics is between independent local states and those of definite total angular momentum. When two identical spins interact with a global excitation, due to their permutation symmetry they appear as a singlet or a triplet to the external interaction. Such states of definite permutation symmetry can naturally hold entanglement, a physical resource central to quantum information.

The basis transformation defined by permutation symmetry is also central to a plethora of quantum information protocols. These include methods to estimate the spectrum of a density operator[2], achieve optimal quantum hypothesis testing[3], perform universal quantum source coding[4], concentrate entanglement in a distortion-free manner[5], create decoherence-free states[6], and communicate without a shared reference frame[7].This deep connection is a natural consequence of the underlying states and random variables being independent and identically distributed. However, unlike the QFT, the complexity of this transform has been unknown, rendering protocols which use it nonconstructive. Also, to be useful in quantum algorithms, this transform must be efficient, that is, constructible for an n spin system using a quantum circuit of depth polynomial (versus exponential) in n.

Here, we resolve this problem by giving an efficient construction for performing a transformation between local and total angular momentum descriptions of a set of n d-dimensional systems (n "qudits"), for arbitrary n

and d. This is achieved using a quantum circuit of size $poly(n, d, log(1/\epsilon))$ for accuracy ϵ . We believe that this basis change, which we call the *Schur transform*, is important not only for quantum information, but also as a new building block for future quantum algorithms. This is demonstrated by a close connection between the Schur transform and a generalization of the quantum phase estimation algorithm[8] for non-abelian groups, as discussed below.

The Schur Transform — Consider a system of n qudits, each with a standard local ("computational") basis $|i\rangle$, $i=1\ldots d$. The Schur transform relates transforms on the system performed by local d-dimensional unitary operations to those performed by permutation of the qudits. Recall that the symmetric group S_n is the group of all permutations of n objects. This group is naturally represented in our system by

$$\mathbf{P}(\pi)|i_1 i_2 \cdots i_n\rangle = |i_{\pi^{-1}(1)} i_{\pi^{-1}(2)} \cdots i_{\pi^{-1}(n)}\rangle, \quad (1)$$

where $\pi \in \mathcal{S}_n$ is a permutation and $|i_1 i_2 ...\rangle$ is shorthand for $|i_1\rangle \otimes |i_2\rangle \otimes ...$ Let \mathcal{U}_d denote the group of $d \times d$ unitary operators. This group is naturally represented in our system by

$$\mathbf{Q}(U)|i_1i_2\cdots i_n\rangle = U|i_1\rangle \otimes U|i_2\rangle \otimes \cdots \otimes U|i_n\rangle, \quad (2)$$

where $U \in \mathcal{U}_d$.

The Schur transform is based on Schur duality, a well known[9] and powerful way to relate the representation theory of $\mathbf{P}(\pi)$ and $\mathbf{Q}(U)$. For example, consider the case of two qubits (n=2, d=2). The two-qubit Hilbert space $(\mathbb{C}^2)^{\otimes 2}$ decomposes under \mathbf{Q} into a one-dimensional spin-0 singlet space and a three-dimensional spin-1 triplet space. Both of these are irreducible representations (irreps) of \mathcal{U}_2 , but they also happen to be irreps of \mathcal{S}_2 . The singlet state changes sign under permutation of the two spins, and the triplet states are invariant under permutation. These correspond to the sign

 \mathcal{P}_{sign} and the trivial $\mathcal{P}_{trivial}$ irreps of \mathcal{S}_2 , and thus we can write $(\mathbb{C}^2)^{\otimes 2} \cong (\mathcal{Q}_1 \otimes \mathcal{P}_{\text{trivial}}) \oplus (\mathcal{Q}_0 \otimes \mathcal{P}_{\text{sign}})$, where \mathcal{Q}_J is the spin-J irrep of \mathcal{U}_2 .

This unusual coincidence between the two representations exists for an arbitrary number of qudits, becoming quite non-trivial for larger n and d. For example, the Hilbert space of three qubits (n = 3, d = 2) decomposes into $(\mathcal{Q}_{3/2} \otimes \mathcal{P}_{\text{trivial}}) \oplus (\mathcal{Q}_{1/2} \otimes \mathcal{P}_{2,1})$, where $\mathcal{P}_{2,1}$ denotes a particular two-dimensional mixed symmetry irrep of S_3 . In terms of the original (local) basis the $\mathcal{Q}_{1/2} \otimes \mathcal{P}_{2,1}$ space contains two spin-1/2 objects, one spanned by $|110\rangle + \omega |011\rangle + \omega^* |101\rangle$ (suppressing normalization) and $|001\rangle + \omega |100\rangle + \omega^* |010\rangle$, and the other obtained by replacing $\omega = e^{2\pi i/3}$ with ω^* . These two spaces correspond to the two mixed symmetries in $\mathcal{P}_{2,1}$.

The general theorem of Schur duality states that for any (integer) d and n,

$$\left(\mathbb{C}^d\right)^{\otimes n} \cong \bigoplus_{\lambda \in \operatorname{Part}[n,d]} \mathcal{Q}_{\lambda} \otimes \mathcal{P}_{\lambda}, \qquad (3)$$

where λ is chosen from the set of possible partitions of n into $\leq d$ parts. Thus, there exists a basis for $(\mathbb{C}^d)^{\otimes n}$ with states $|\lambda, q_{\lambda}, p_{\lambda}\rangle_{Sch}$ where λ labels the subspaces $\mathcal{Q}_{\lambda} \otimes \mathcal{P}_{\lambda}$ and $q_{\lambda} \in \mathcal{Q}_{\lambda}$ and $p_{\lambda} \in \mathcal{P}_{\lambda}$ label bases for \mathcal{Q}_{λ} and \mathcal{P}_{λ} respectively. We may represent these states of generalized definite total angular momentum and permutation multiplicity by vectors $|\lambda, q, p\rangle$ in the computational basis, with bit strings λ , q, and p. Note that $\dim(\mathcal{Q}_{\lambda})$ and $\dim(\mathcal{P}_{\lambda})$ vary with λ , and so $|q\rangle$ and $|p\rangle$ are padded; this requires only constant spatial overhead.

Just as in the examples above, the Schur basis states $|\lambda, q_{\lambda}, p_{\lambda}\rangle_{\rm Sch}$ are superpositions of the n qudit computational basis states $|i_1 i_2 \dots i_n\rangle$,

$$|\lambda, q_{\lambda}, p_{\lambda}\rangle_{\mathrm{Sch}} = \sum_{i_{1}, \dots, i_{n}} \left[\mathbf{U}_{\mathrm{Sch}} \right]_{i_{1}, i_{2}, \dots, i_{n}}^{\lambda, q, p} |i_{1}i_{2}\cdots i_{n}\rangle.$$
 (4)

By the isomorphism of Eq.(3), this defines a unitary transformation $\mathbf{U}_{\mathrm{Sch}}$ (with matrix elements as given), the Schur transform we desire. U_{Sch} maps the computational basis to the $|\lambda, q, p\rangle$ representation of the Schur basis.

Applying the Schur transform extracts λ , q, and pvalues for a given state, allowing the values be manipulated like any other quantum data. For example, since maximally entangled states are invariant under permutation, if $|\psi\rangle_{AB}$ is a bipartite partially entangled state, then a universal scheme for entanglement concentration from $|\psi\rangle^{\otimes n}$ is given by both parties performing the Schur transform $\mathbf{U}_{\mathrm{Sch}}$, measuring $|\lambda\rangle$, discarding $|q\rangle$ and noting that $|p\rangle$ contains a maximally entangled state of Schmidt rank dim $(\mathcal{P}_{\lambda})[5]$. It is then straightforward to map $|p\rangle$ reversibly onto the integers $\{1, \ldots, \dim(P_{\lambda})\}$ to obtain the state $\frac{1}{\sqrt{d}}\sum_{i=1}^{d}|ii\rangle_{AB}$ for $d=\dim(\mathcal{P}_{\lambda})$ [10]. The defining property of $\mathbf{U}_{\mathrm{Sch}}$ is that it reduces the

action of **Q** and **P** into irreps. For any $\pi \in S_n$ and any

 $U \in \mathcal{U}_d$, $\mathbf{P}(\pi)$ and $\mathbf{Q}(U)$ commute, so we can express both reductions at once as

$$\mathbf{U}_{\mathrm{Sch}}\mathbf{Q}(U)\mathbf{P}(\pi)\mathbf{U}_{\mathrm{Sch}}^{\dagger} = \sum_{\lambda \in \mathrm{Part}(d,n)} |\lambda\rangle\langle\lambda| \otimes \mathbf{q}_{\lambda}(U) \otimes \mathbf{p}_{\lambda}(\pi),$$
(5)

where \mathbf{q}_{λ} and \mathbf{p}_{λ} are irreps of \mathcal{U}_d and \mathcal{S}_n respectively.

Quantum Circuit for the Schur Transform — We construct a quantum circuit[11] for U_{Sch} in two stages, first for d=2, then generalizing to d>2. Each of these constructions follows an iterative structure, in which the Schur transform on n qudits is realized using O(n) elementary steps, each of which adds a single qudit to an existing Schur state of the form $|\lambda, q, p\rangle$.

For d=2, this elementary step is familiar from basic quantum mechanics, because it involves simple addition of angular momentum, following the prescription for calculation of the Clebsch-Gordan (CG) coefficients[12]. In this case, λ and q can be conveniently denoted by half-integers J and m (with $|m| \leq J \leq n/2$) which give the total angular momentum and the z-component of angular momentum respectively. And in terms of J, the CG transform takes as input $|J,m\rangle$ and a single spin $|s\rangle$, and outputs a linear combination of the states $|J\pm 1/2, m\pm 1/2\rangle$. The amplitudes of the linear combination are readily computed using the usual ladder operators for raising and lowering angular momenta. In addition, however, we must distinguish between multiple distinct pathways which add up to give the same total J, as demonstrated by the three qubit example above. In fact, it is the permutation symmetry of these pathways which give rise to \mathcal{P}_J , and thus we track the pathway with another output label p = J' - J.

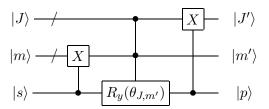


FIG. 1: Quantum circuit implementing \mathbf{U}_{CG} to convert between the $|J, m\rangle|s\rangle$ and $|J', m', p\rangle$ bases, for the d=2 (qubit) case. Following standard conventions[11], time goes from left to right, the $|J\rangle$ and $|m\rangle$ wires hold multiple qubits, and $|s\rangle$ is one qubit. The controlled X operation C_X adds the control to the target qubits, i.e. $C_X|s\rangle|m\rangle = |s\rangle|m+s\rangle$. The doubly controlled $R_y(\theta_{J,m'})$ gate implements the rotation given by Eq. (6) using the J and m' qubits.

Putting this together, we can define an elementary Clebsch-Gordan transform step U_{CG} as a rotation between two specific basis states,

$$\begin{bmatrix} |J'_{-}, m', p = -\frac{1}{2}\rangle \\ |J'_{+}, m', p = +\frac{1}{2}\rangle \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta_{J,m'} & -\sin\theta_{J,m'} \\ \sin\theta_{J,m'} & \cos\theta_{J,m'} \end{bmatrix} \begin{bmatrix} |J, m_{+}\rangle|s = -\frac{1}{2}\rangle \\ |J, m_{-}\rangle|s = +\frac{1}{2}\rangle \end{bmatrix} (6)$$

where $J'_{\pm} = J \pm 1/2$, $m_{\pm} = m' \pm 1/2$, and $\cos \theta_{J,m'} = \sqrt{\frac{J+m'+1/2}{2J+1}}$. \mathbf{U}_{CG} can be realized with three gates in a quantum circuit, as shown in Fig. 1, using as one gate a controlled rotation about \hat{y} by angle $\theta_{J,m'}$. This angle is computed using usual quantum and reversible circuit techniques[11] with error ϵ , using poly(log(1/ ϵ)) standard circuit elements.

The full Schur transform is implemented by cascading \mathbf{U}_{CG} as shown in Fig. 2. The complexity of this circuit is thus $\mathcal{O}(n \cdot \mathrm{poly} \log(1/\epsilon))$. In the circuit, $p_1 p_2 \cdots p_n$ is an encoding in Young's orthogonal basis[13] for p, the label capturing the permutation symmetry of the state. This basis is an example of a subgroup adapted basis, a useful type of basis used by Beals in the fast quantum Fourier transform over the symmetric group[13]. For d=2, the total angular momentum of the state, J, gives the partition λ .

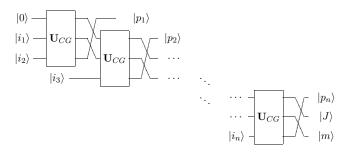


FIG. 2: Quantum circuit for the Schur transformation $\mathbf{U}_{\mathrm{Sch}}$, transforming between $|i_1i_2\cdots i_n\rangle$ and $|J,m,p\rangle$.

Construction of the Schur transform for d > 2 follows the same ideas as for d=2, but is complicated by the necessary notation; the principle challenge is showing that the elementary U_{CG} steps for d > 2 can be computed in poly(d) steps. U_{Sch} is constructed as a cascade of O(n) U_{CG} transforms, just as for d=2. Each \mathbf{U}_{CG} combines an arbitrary irrep of \mathcal{U}_d , a multi-qudit state $|\lambda, q\rangle$, with a single qudit state $|i_k\rangle$, to obtain a multi-qudit superposition of new irreps of \mathcal{U}_d , $|\lambda', q'\rangle$. Simultaneously, the permutation labels $|p\rangle$ are constructed; equivalently we could save the values of λ that we generate in each step. U_{CG} can be computed efficiently because of a recursive relationship between \mathbf{U}_{CG} for $\mathcal{U}_d \times \mathcal{U}_d$ and that of $\mathcal{U}_{d-1} \times \mathcal{U}_{d-1}$ in terms of reduced Wigner coefficients[14]. Crucially, there is an efficient classical algorithm for the computation of the reduced Wigner coefficients[15] needed for $U_{\rm CG}$. Specific details of this calculation are given elsewhere[10]. The complexity of the full Schur transform is thus found to be polynomial in n, d, and $\log(\epsilon^{-1})$.

Circuit for Schur Basis Measurement — Unitary transforms enable measurements in alternate bases; for such applications, simpler quantum circuits can be employed, as we now demonstrate for measuring λ in the Schur basis. Fascinatingly, the quantum circuit we construct uti-

lizes the same structure as that of the quantum factoring algorithm[1], and its generalizations to non-abelian groups[13, 16–18].

Measurement of λ projects a state onto the different spaces $Q_{\lambda} \otimes \mathcal{P}_{\lambda}$, as defined in Eq. (3). For any representation of a group for which there exists a quantum Fourier transform, we give a circuit for performing this projective measurement onto irreps in Fig. 3; when specialized to the symmetric group and the representation $\mathbf{P}(\pi)$, this can be used to measure λ .

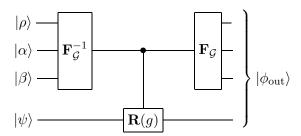


FIG. 3: Quantum circuit used in measurement of an irrep of a group \mathcal{G} .

The circuit uses the following gate elements. Let \mathcal{G} be an arbitrary finite group over which there exists an efficient quantum circuit for the quantum Fourier transform [16], $\mathbf{F}_{\mathcal{G}}$. Define $\hat{\mathcal{G}}$ to be a representative set of inequivalent irreps of \mathcal{G} and let d_{ρ} denote the dimension of the irrep $\rho \in \hat{\mathcal{G}}$. $\mathbf{F}_{\mathcal{G}}$ is then a unitary transform from a space spanned by group elements, $|g\rangle$, $g \in \mathcal{G}$ to a space spanned by irrep labels and row and column indices, $|\rho$, α , β , $\rho \in \hat{\mathcal{G}}$, and α , $\beta = 1 \dots d_{\rho}$. Specifically,

$$\mathbf{F}_{\mathcal{G}} = \sum_{\rho \in \hat{\mathcal{G}}} \sum_{\alpha, \beta = 1}^{d_{\rho}} \sum_{g \in \mathcal{G}} \sqrt{\frac{d_{\rho}}{|\mathcal{G}|}} \left[\mathbf{r}_{\rho}(g) \right]_{\alpha\beta} |\rho, \alpha, \beta\rangle \langle g| \qquad (7)$$

where $\left[\mathbf{r}_{\rho}(g)\right]_{\alpha\beta}$ is the entry of the α th row and β th column of the irrep ρ evaluated at $g \in \mathcal{G}$. Finally, let $\mathbf{R}(g), g \in \mathcal{G}$ be a generic representation of this group \mathcal{G} and suppose we can efficiently enact the controlled $\mathbf{R}(g)$ operation, $\mathbf{C}_{\mathbf{R}} = \sum_{g \in \mathcal{G}} |g\rangle\langle g| \otimes \mathbf{R}(g)$. $\mathbf{F}_{\mathcal{G}}$, its inverse, and $\mathbf{C}_{\mathbf{R}}$ are the circuit elements used to construct the circuit in Fig. 3.

This circuit is employed in the following manner, to measure an irrep of \mathcal{G} . Consider the action of the circuit in Fig. 3 when a generic input state $|\psi\rangle$ is fed into the space upon which the generic representation of the group $\mathbf{R}(g)$ acts. Since $\mathbf{R}(g)$ is a generic representation of the group \mathcal{G} , the space upon which this representation acts can be decomposed into different irreps of \mathcal{G} . Let the multiplicity of the μ th irrep ($\mu \in \hat{\mathcal{G}}$) in $\mathbf{R}(g)$ be n_{μ} . It is possible that an irrep μ does not appear at all, in which case $n_{\mu} = 0$. Then

$$|\psi\rangle = \sum_{\mu \in \hat{G}} \sum_{j=1}^{n_{\mu}} \sum_{k=1}^{d_{\mu}} c_{\mu,j,k} |\mu, j, k\rangle_{\mathcal{R}}$$
 (8)

meaning we can expand the generic input state $|\psi\rangle$ over a basis labeled by the irrep labels μ , multiplicity labels for the irrep j, and a label for the spaces upon which the irrep acts k. Since this basis fully reduces $\mathbf{R}(g)$,

$$\mathbf{R}(g)|\psi\rangle = \sum_{\mu\in\hat{\mathcal{G}}} \sum_{j=1}^{n_{\mu}} \sum_{k=1}^{d_{\mu}} c_{\mu,j,k} |\mu,j\rangle \mathbf{r}_{\mu}(g) |k\rangle_{\mathbf{R}}.$$
 (9)

The output of the circuit in Fig. 3 is thus

$$|\phi_{\text{out}}\rangle = \sum_{g \in \mathcal{G}} \sum_{\rho,\alpha,\beta} \frac{\sqrt{d_{\rho}}}{|\mathcal{G}|} \left[\mathbf{r}_{\rho}(g) \right]_{\alpha,\beta} |\rho,\alpha,\beta\rangle \otimes \mathbf{R}(g) |\psi\rangle.$$

Using the reducible action of $\mathbf{R}(g)$ on $|\psi\rangle$ given by Eq.(9) along with the orthogonality relationships for irreps[9], Eq.(10) can be reexpressed as

$$|\phi_{\text{out}}\rangle = \sum_{\mu \in \hat{\mathcal{G}}} \sum_{j=1}^{n_{\mu}} \sum_{\alpha,\beta=1}^{d_{\alpha}} \frac{c_{\mu,j,\alpha}}{\sqrt{d_{\mu}}} |\mu,\alpha,\beta\rangle \otimes |\mu,j,\beta\rangle_{\text{R}} .$$
 (11)

The output $|\phi_{\text{out}}\rangle$ has multiple interesting properties which we can now exploit. Measuring the first register (the irrep label index) produces outcome μ with probability $\sum_{j=1}^{n_{\mu}} \sum_{k=1}^{d_{\mu}} |c_{\mu,j,k}|^2$. This is exactly the probability we would obtain if we were to measure the irrep label of $|\psi\rangle$. Remarkably, this is achieved independent of the basis in which $C_{\mathbf{R}}$ is implemented. For the Schur basis, taking $\mathcal{G} = \mathcal{S}_n$ and letting $\mathbf{R} = \mathbf{P}$ denote the natural representation of the symmetric group given by Eq.(1) so that $\mu = \lambda$, the measurement circuit gives exactly the probabilities we would obtain if we were to compute $\mathbf{U}_{\mathrm{Sch}}|\psi\rangle$ then measure λ . This works for the Schur transform over not just gubits, but also over gudits. Furthermore, for large qudit dimension d, the new circuit is exponentially faster than the full Schur transform $\mathbf{U}_{\mathrm{Sch}}$ followed by measurement; in terms of d, U_{Sch} has size poly(d), while this measurement circuit is polylog(d).

The circuit in Fig. 3 is a generalization of the phase estimation circuit introduced by Kitaev[19], which uses a QFT over an abelian group (for phase estimation, a U(1) phase is approximated by the cyclic group C_N). In contrast, our circuit is applied to non-abelian groups; and in addition to allowing measurement of group irreps, this circuit also allows operations to be performed in the irreducible basis of $\mathbf{R}(g)$, for arbitrary groups \mathcal{G} . This cannot be accomplished directly with $\mathbf{U}_{\mathrm{Sch}}$, which is specialized to the permutation and unitary groups.

Conclusion—We have shown how to efficiently perform the Schur transform and shown how a generalization of Kitaev's circuit can be used to perform a useful group representation transformation. Whenever we have identical copies of a quantum system or are averaging over the diagonal action of the unitary group this first transform is often essential. Whenever the symmetry of a quantum state is described by a finite group, the generalized Kitaev transform comes into play. One of the most fundamental problems in quantum computation is the search for new quantum algorithms. In this respect, there are few unitary transforms which have both an efficient quantum circuit and interpretations which might allow these transforms to be useful in an algorithm. We are hopeful that our circuits will be useful exactly because they have such clear group representation theory interpretations.

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