Reinforcement Learning
Adversarial Learning

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Aggregation of experts

- Compare to: Best arm, Best convex combinations, Best sequence, Best recurring sequence.

Adversarial bandits

- Exp3, Exp4
- Best of both world

Min-max games

- Bandits and Nash equilibrium

Risk-aversion

- CVaR, EVaR, etc.

Robust planning

- Autonomous vehicles.

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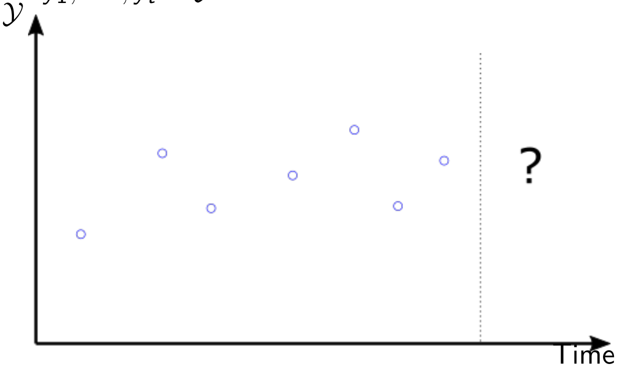
Many models?

▷ Observe a signal $y_1, \dots, y_t \in \mathcal{Y}$



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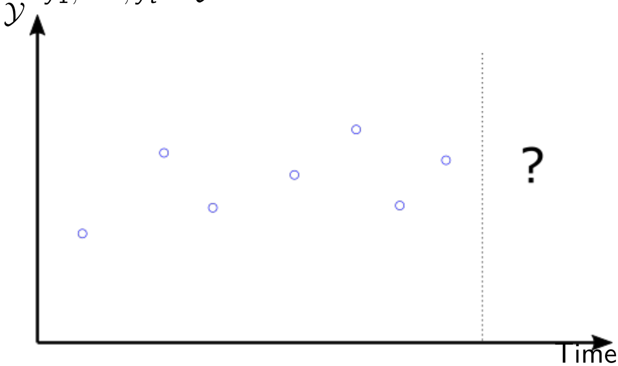

- Goal: Predict observation at time $t+1$?

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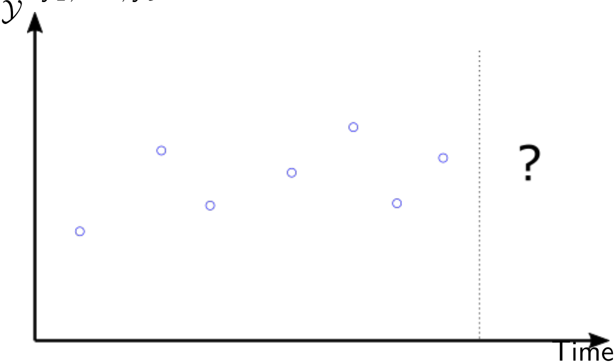

- Goal: Predict observation at time $t+1$?
- **Many** available models:
  - *I.i.d.*: $[0,1]$-bounded ?
  - *Parametric*: $y_t = \langle \theta, \varphi(t) \rangle + \xi_t$ for φ: polynomials, wavelets, etc. ?
  - *Markov*: $y_t \sim P(\cdot | y_{t-1})$, *k-order Markov*: $y_t \sim P(\cdot | y_{t-1}, \dots, y_{t-k})$?
  - *States*: representation maps $\psi(h_t) = s_t$ for observation history h_t?

Which model is best?

Parametric: $y_t = \langle \theta, \varphi(t) \rangle + \xi_t$

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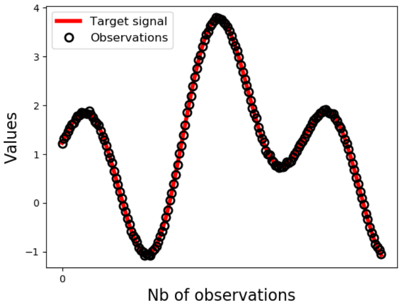
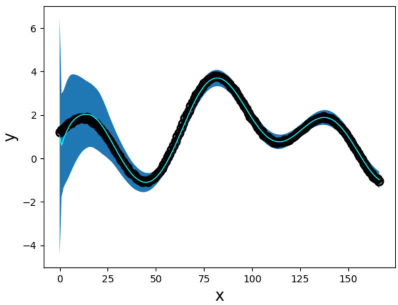
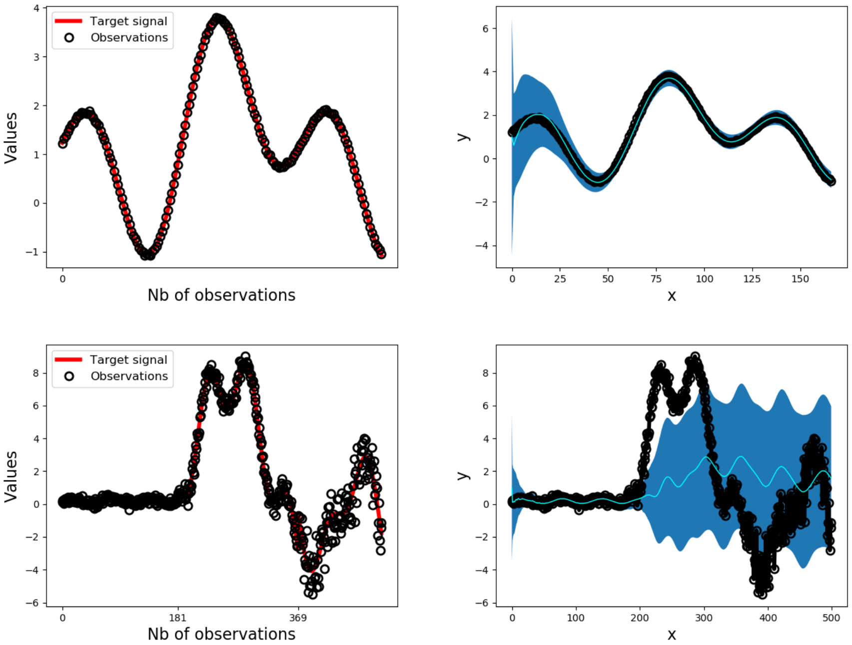

Target signal
Observations
Values
Nb of observations
4
3
2
1
0
−1
0


y
x
6
4
2
0
−2
−4
0
25
50
75
100
125
150


Target signal
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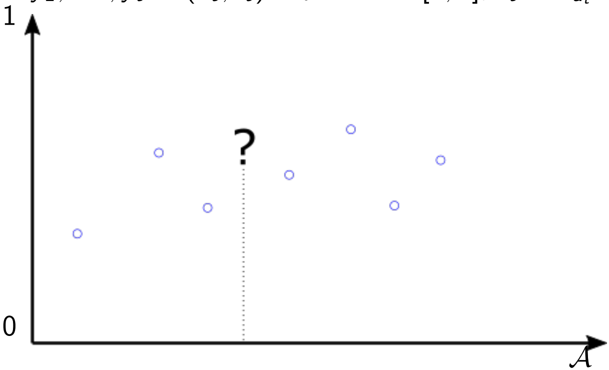
- Sample a signal $y_1, \ldots, y_t = (a_t, r_t) \in \mathcal{Y} = \mathcal{A} \times [0,1]$, $r_t \sim \nu_{a_t}$.



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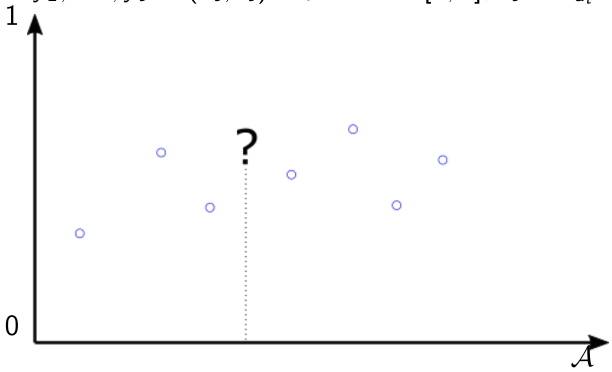

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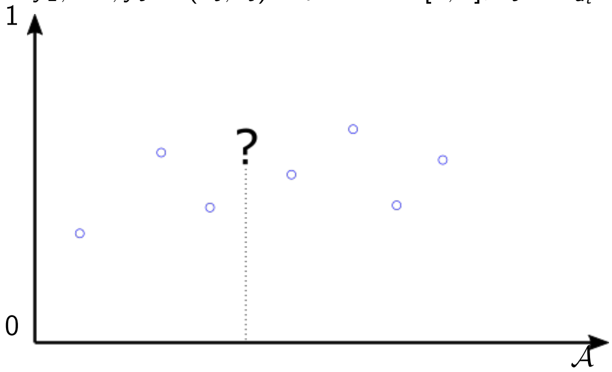

- Goal: choose $a_t \in \mathcal{A}$ to maximize rewards.
- Many available algorithms:
  - *Bandits*: UCB? UCB-V? KL-UCB? TS?
  - *Structured bandits*: OFUL, GP-UCB? IMED?
  - *MDPs*: UCRL? Q-learning? DQN?

Which algorithm is best?

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Decisions and Losses

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- All decisions evaluated via a **loss** $\ell : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}^+$
  - Quadratic: $\ell(x, y) = \frac{(x-y)^2}{2}$,
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▷ Q: in Expectation? High probability?

▷ Say after playing x_t, you **observe** y_t (more generally ℓ_t). Then, you can compute $\ell_t(x)$ for all other choice.

Full information

▷ In bandit, only $\ell_t(x_t)$ is observed, but ℓ_t is **unknown**:

Partial information: Bandit feedback

▷ Intermediate settings: e.g. Classification $\ell(x, y) = \mathbb{I}\{x \neq y\}$.
(Only) If I receive **loss** 0, then, I know y, hence I can compute $\ell(x, y)$ for all x.

Semi-bandit Feedback

In the sequel, we first consider **full information**.

$$\text{Minimize} \quad \sum_{t=1}^{T} \ell_t(x_t) \; \ldots$$

w.r.t.

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▷ Goal 1: **best model** (Model **selection**) ?

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▷ ... ?

- You are given a **set** $\mathcal{M}$ of models.

At each time step,

- You maintain some **distribution** $p_t \in \mathcal{P}(\mathcal{M})$ on the set of models.
- You receive **recommendation** $x_{t,m}$ from each model $m \in \mathcal{M}$.
- You use them in order to output some **decision** x_t.
- You incur the corresponding **loss** $\ell_t(x_t)$, an receive **feedback**.

A first approach

▷ Choose x_t as a convex combination of the $(x_{t,m})_{m\in\mathcal{M}}$? or sample $x_t \sim p_t$?

$$x_t = \sum_{m\in\mathcal{M}} p_t(m) x_{t,m} \text{ where } p_t \in \mathcal{P}(\mathcal{M})\,.$$

$\Longrightarrow$ Assuming that $\ell_t(\cdot) = \ell(\cdot, y_t)$ is **convex**, convex combination is better:

$$\ell_t(x_t) \leqslant \sum_{m\in\mathcal{M}} p_t(m)\ell_t(x_{t,m}) = \mathbb{E}_{M\sim p_t}[\ell_t(x_{t,M})]$$

Technical property (Hoeffing Lemma for bounded random variables)

Let r.v. X s.t. $a \leqslant X \leqslant b$ a.s. then

$$\forall \eta \in \mathbb{R}^+, \quad \mathbb{E}[X] \leqslant -\frac{1}{\eta}\log \mathbb{E}[\exp(-\eta X)] + \eta\frac{(b-a)^2}{8}\,.$$

$\Longrightarrow$ Assuming that ℓ is **bounded** by 1, then

$$\mathbb{E}_{M\sim p_t}[\ell_t(x_{t,M})] \leqslant -\frac{1}{\eta}\log \sum_{m\in\mathcal{M}} p_t(m) e^{-\eta\ell_t(x_{t,m})} + \frac{\eta}{8}\,.$$

For **Bounded, convex** loss:

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- Hence $$\sum_{t=1}^{T}\ell_t(x_t) \leqslant \sum_{t=1}^{T}\ell_t(x_{t,m^\star}) + \frac{\log(|\mathcal{M}|)}{\eta} + \frac{\eta T}{8}.$$

This leads to the following strategy

1: Let $\forall m \in \mathcal{M}, w_1(m) = 1$
2: **for** $t = 1, \ldots$ **do**
3: Receive $x_{t,m}$ from each model $m \in \mathcal{M}$.
4: Let $p_t(m) = \frac{w_t(m)}{\sum_{m \in \mathcal{M}} w_t(m)}$.
5: Choose $x_t = \sum_{m \in \mathcal{M}} p_t(m) x_{t,m}$
6: Receive loss function ℓ_t.
7: **Update** $w_{t+1}(m) = w_t(m) e^{-\eta \ell_t(x_{t,m})}$ for each m,
Equivalently, $w_{t+1}(m) = \exp(-\eta L_{t,m})$
8: **end for**

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Theorem (Cesa-Bianchi,Lugosi 2006)

Assume that ℓ_t is **convex** and **bounded** by 1, then this strategy satisfies:

$$\underbrace{\sum_{t=1}^{T}\ell_t(x_t)}_{L_T} - \min_{m\in\mathcal{M}}\underbrace{\sum_{t=1}^{T}\ell_t(x_{t,m})}_{L_{T,m}} \leqslant \frac{\log(|\mathcal{M}|)}{\eta} + \frac{\eta T}{8}$$

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- In particular for the choice of parameter $\eta = \sqrt{8\log(|\mathcal{M}|)/T}$,

$$L_T - \min_{m\in\mathcal{M}} L_{T,m} \leqslant \sqrt{\frac{T\log(|\mathcal{M}|)}{2}}$$

Aggregation with exponential weights?

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- Simplify this assumption, cf. Technical property ??

We only used this:

$$\ell_t\big(\underbrace{\mathbb{E}_{M\sim p_t}[x_{t,M}]}_{x_t}\big) \leqslant -\frac{1}{\eta}\log \mathbb{E}_{M\sim p_t}\exp\big(-\eta\ell_t(x_{t,M})\big) + \frac{\eta}{8}$$

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  Ok for **quadratic** loss, pb for **self-information**: not bounded when x small!
- ▷ What about dropping $\eta/8$ term?
  Equivalent to $\exp(-\eta\ell_t(\cdot))$ is concave: **η-exp-concavity**.
  - ◇ **Self-information** loss is 1-exp-concave (with $=$ instead of $\leqslant$)
  - ◇ **Quadratic** loss is η-exp-concave for $\eta \leqslant \frac{1}{2(b-a)^2}$ on $\mathcal{X}=\mathcal{Y}\subset[a,b]$.
  - ◇ **Absolute** loss $\ell(x,y)=|x-y|$ is not exp-concave for any η.

A SECOND LOOK AT ASSUMPTIONS

- Interpretation of $-\frac{1}{\eta}\log \mathbb{E}_{M\sim p_t} \exp\big(-\eta \ell_t(x_{t,M})\big)$?
  Entropy formula:

$$-\frac{1}{\eta}\log \mathbb{E}_{M\sim p} \exp\big(-\eta X_M\big) = \inf_{q\in\mathcal{P}(\mathcal{M})} \mathbb{E}_{M\sim q}[X_M] + \frac{1}{\eta}\mathtt{KL}(q,p)\,.$$

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▷ Hence, η-exp-concavity becomes:

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A loss ℓ is η-exp-concave if $\forall\mathbf{x}\in\mathcal{X}^{\mathcal{M}}, p\in\mathcal{P}(\mathcal{M}), \forall y\in\mathcal{Y}$,

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▷ Further, infimum obtained for $q(m) = \frac{\exp(-\eta X_m)p(m)}{\sum_{m'\in\mathcal{M}}\exp(-\eta X_{m'})p(m')}$.

Generalization: we don't need that $x_t = \mathbb{E}_{M\sim p_t}[x_{t,M}]$.

η-mixability

A loss ℓ is η-mixable if $\forall \mathbf{x} \in \mathcal{X}^{\mathcal{M}}, p \in \mathcal{P}(\mathcal{M}), \exists x_{\mathbf{x},\mathbf{p}} \forall y \in \mathcal{Y}$,

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$[\mathbf{x}], \mathbf{p} \mapsto \mathbf{x}_{\mathbf{x},\mathbf{p}}$ is called the **substitution function**.

- η-exp-concave loss is η-mixable with $x_{\mathbf{x},\mathbf{p}} = \mathbb{E}_{M\sim p}\mathbf{x}_{\mathbf{M}}$.
- **Quadratic** loss is η-exp-concave for $\eta \leqslant \frac{1}{2}$ on $\mathcal{X} = \mathcal{Y} \subset [0,1]$, but η-mixable for η up to $\eta \leqslant 2$!

- Consider an η-**mixable** loss ℓ, and let $p_1 = \text{Uniform}(\mathcal{M}) \in \mathcal{P}(\mathcal{M})$.

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Theorem

Assume that ℓ_t is η-**mixable**, then after T time steps, this strategy satisfies:

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Aggregation of experts revisited

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- Still for arbitrary $y_t \in \mathcal{Y}$.
- **Independent** on T !
- Only for **specific**, possibly small η (all $\eta' \leqslant \eta$, but not larger).

We can actually get a stronger result:

Theorem (Aggregation of experts)

Assume that ℓ_t is η-mixable, then after T time steps, the aggregation strategy with $p_1 = \pi$, satifies

$$\forall q \in \mathcal{P}(\mathcal{M}) \quad L_T - \mathbb{E}_{M\sim q}\Big[L_{T,M}\Big] \leqslant \frac{1}{\eta}\Big(\mathtt{KL}(q,\pi) - \mathtt{KL}(q, p_{T+1})\Big)\,.$$

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- We can move from finitely many to **countably** many experts:

$$\pi(m) = \frac{1}{m(m+1)}, \qquad \pi(m) = \log(2)\left(\frac{1}{\log(m+1)} - \frac{1}{\log(m+2)}\right).$$

▷ **Bregman** divergence generalizes KL:

$$\mathcal{B}(p, q) = \psi(p) - \psi(q) - \langle p - q, \nabla\psi(q)\rangle$$

($\psi(p) = \sum_i p_i \log(p_i)$ gives KL as a special case.)

▷ Assumption: ℓ is **η-Bregman-mixable** w.r.t. Bregman divergence $\mathcal{B}$:

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where $\ell_{\mathbf{x}}$ denotes the vector $(\ell(x_1), \ldots, \ell(x_M))$.

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- Other interpretation: Use Legendre-Fenchel dual objective function, perform gradient descent!

When the best expert has **small loss**, we may prefer to express regret bounds on terms of this loss:

▷ Consider a loss **convex and bounded** in $[0,1]$, then:

$$L_T - L_T^\star \leqslant \left(\frac{\eta}{1-\exp(-\eta)} - 1\right) L_T^\star + \frac{\log(M)}{1-\exp(-\eta)}$$

where $L_T^\star = \min_{m\in\mathcal{M}} L_{t,m}$

Proof: One can show that any loss ℓ convex and bounded in $[0,1]$ satisfies the following extension of η-mixability property:

$$\ell(\mathbb{E}_{M\sim q}(x_M)) \leqslant -\frac{\eta}{1-\exp(-\eta)}\frac{1}{\eta}\ln\left(\mathbb{E}_{m\sim q}\exp(-\eta\ell(x_M))\right).$$

(almost η-mixable!) The rest is obtained by following the initial derivation.

Different objectives

$$\text{Minimize} \quad \sum_{t=1}^{T} \ell_t(x_t) \ \ldots$$

w.r.t.

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- best **combination** of models (Model aggregation)?

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- Left: best **combination of losses** Right: **loss of best combination**.

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- ▷ ! From set of experts $\mathcal{M}$ (finite) to set of experts $\mathcal{P}(\mathcal{M})$ (continuous) !

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- ! From set of experts $\mathcal{M}$ (finite) to set of experts $\mathcal{P}(\mathcal{M})$ (continuous) !
- If ℓ is η-exp-concave on $\mathcal{X}$, then $\overline{\ell} : q \to \ell_t(\mathbf{q}\cdot\mathbf{x}_t)$ is η-exp-concave on $\mathcal{P}(\mathcal{M})$.

- $\overline{p}_1(q) = \frac{1}{\mathrm{vol}(\mathcal{P}(\mathcal{M}))} = M!$, $p_1 = \frac{1}{|\mathcal{M}|}\mathbf{1}$.

Aggregation over $\mathcal{P}(\mathcal{M})$: Strategy

- $\overline{p}_1(q) = \frac{1}{\mathrm{vol}(\mathcal{P}(\mathcal{M}))} = M!$, $p_1 = \frac{1}{|\mathcal{M}|}\mathbf{1}$.
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- Get the next observation y_t from which we can compute $\ell_t(\mathbf{q}\cdot\mathbf{x}_t)$ for all $\mathbf{q}$.

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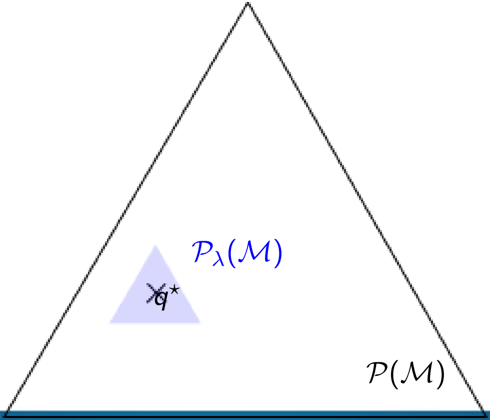
- Proof technique: Similar +



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Example of Universal prediction

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  - Yields a **fully explicit** solution:

$$q_t(1) = \frac{t\widehat{\theta}_t + 1/2}{t + 1}$$

Efficient computation despite aggregation of continuum of models.

Example of Universal prediction

- Consider Binary prediction and self-information loss ℓ.
- Aggregation over all Bernoulli $\mathcal{B}(\theta)$, $\theta \in [0,1]$.
- KT-predictor: Use prior $g(\theta) = \frac{1}{\sqrt{\theta(1-\theta)}}$ on each parameter.
- Yields a **fully explicit** solution:

$$q_t(1) = \frac{t\widehat{\theta}_t + 1/2}{t+1}$$

Efficient computation despite aggregation of continuum of models.

- Called "Universal prediction". Extends to all Markov models of arbitrary order.

- So far, we only considered **fixed** experts:

$$\min_{m\in\mathcal{M}}\sum_{t=1}^{T}\ell_t(x_{t,m}),\qquad \min_{q\in\mathcal{P}(\mathcal{M})}\sum_{m\in\mathcal{M}}q(m)L_{T,m}\qquad \min_{q\in\mathcal{P}(\mathcal{M})}\sum_{t=1}^{T}\ell_t(\sum_{m\in\mathcal{M}}q(m)x_{t,m})$$

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$$\min_{m_1,\dots,m_T\in\mathcal{S}_k(\mathcal{M})} \sum_{t=1}^{T} \ell_t(x_{t,m_t}) \text{ where } \mathcal{S}_k(\mathcal{M}) : \text{at most } k \text{ switches.}$$

  - Difficulty: Concentrating mass **exponentially fast** to a single expert means putting near 0 on others.
  - When switching to other best expert, **need to catch-up**!
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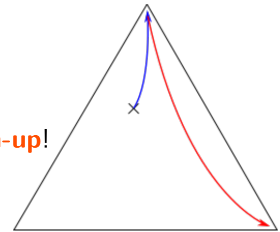

Fixed-share solution

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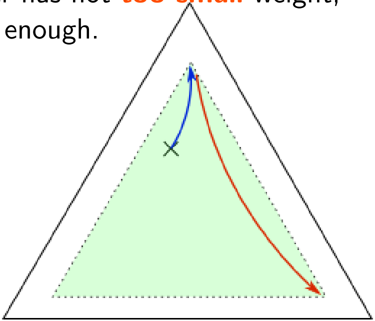
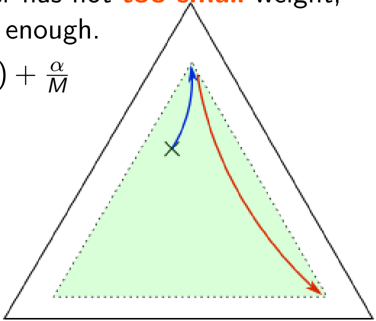
▷ Guarantees each m never has not **too small** weight, hence can catch-up fast enough.



Fixed-share solution

- Guarantees each m never has not **too small** weight, hence can catch-up fast enough.
- $\tilde{p}_{t+1}(\cdot) = (1-\alpha)p_{t+1}(\cdot) + \frac{\alpha}{M}$



Fixed-share performance

For all sequence $q_1, \dots, q_T \in \mathcal{P}(\mathcal{M})$ with at most k switches,

$$L_T - \sum_{t=1}^{T} q_t \ell_t \leqslant \frac{\log(M)}{\eta} + \frac{k}{\eta} \log\left(\frac{M}{\alpha}\right) + \frac{T-k-1}{\eta} \log\left(\frac{1}{1-\alpha}\right).$$

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- α going to 0 but not exponentially fast.

Let us consider $\tilde{p}_t$ obtained from p_t as $\tilde{p}_{t+1}(\cdot) = \sum_{m' \in \mathcal{M}} \theta(\cdot|m') p_{t+1}(m')$, from a Markov chain with initial low ω and **transition matrix** θ.
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- Fixed share: $\theta(m'|m) = (1-\alpha)\mathbb{I}\{m = m'\} + \alpha/M$.
- Variable share, sleeping experts, etc.

Note: even though huge amount of experts $O(M^T)$ they share a **rich structure**.
This enables to have an efficient strategy maintaining only few quantities $O(MT)$.

- Best **sequence** of experts:

$$\min_{m_1,\dots,m_T \in \mathcal{S}_k(\mathcal{M})} \sum_{t=1}^{T} \ell_t(x_{t,m_t}) \text{ where } \mathcal{S}_k(\mathcal{M}) : \text{at most } k \text{ switches.}$$

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- Best sequence of experts with **few good** experts:

$$\min_{m_1,\dots,m_T \in \mathcal{S}_k(\mathcal{M}_0)} \sum_{t=1}^{T} \ell_t(x_{t,m_t}) \text{ where } \mathcal{M}_0 \subset \mathcal{M} \text{ unknown but small}\,.$$

  - Intuition: the good experts should be good in the recent past.

- Ensure that experts good in the recent past have large enough weight and catch-up.

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- Ensure that experts good in the recent past have large enough weight and catch-up.
- **Mixing past posterior** $\tilde{p}_{t+1}(\cdot) = \sum_{s=0}^{t} \beta_{t+1}(s) p_s(\cdot)$
- In particular:
  - Hedge: $\beta_{t+1}(t') = \begin{cases} 1 & \text{if } t' = t \\ 0 & \text{else} \end{cases}$
  - Fixed share: $\beta_{t+1}(t') = \begin{cases} 1-\alpha & \text{if } t' = t \\ \alpha & \text{if } t' = 0 \\ 0 & \text{else} \end{cases}$
  - ...

Assume ℓ is η-mixable. For all sequence $(q_t)_{t\in\mathcal{T}}$ with k switches between at most n values,

$$L_T - \sum_{t=1}^{T} q_t \cdot \ell_t \leqslant \frac{n}{\eta}\log\left(|\mathcal{M}|\right) + \frac{1}{\eta}\sum_{t=1}^{T}\log\left(\frac{1}{\beta_t(\tau_t)}\right).$$

where τ_t is last $\tau < t$ such that $q_\tau = q_t$ (or 0 if first occurrence).

- **Sleeping experts** (Koolen et al. 2012): When experts are not available at all rounds.

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- ...

Most results are minimax-optimal, valid for any input sequence.
This contrasts with typical results for bandits: instance-optimal, for stochastic sequence.

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- Increasingly challenging targets:
  - **Constant** expert, **combination of loss** of experts. Convex and bounded or η-mixable loss.
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- Computationally efficient algorithms, leveraging structure of experts.

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Adjusting for the differences:

- Decision are arms $\mathcal{X} = \mathcal{A}$. Consider one expert per arm $\mathcal{M} = \mathcal{A}$.
- Losses $(\ell_{t,m})_{m\in\mathcal{M}}$ become rewards $(r_{t,a})_{a\in\mathcal{A}}$
- Can only output an arm $A_t \in \mathcal{A}$ (not a combination):
  $x_t = \sum_{m\in\mathcal{M}} p_{t,m} x_{t,m}$ becomes $x_t = x_{t,m_t}$ with $m_t \sim p_t$.
  - Less good, but ok as long as $\mathbb{E}$ performance.

Problem: we only observe the reward of A_t (i.e., only r_{t,A_t}) !!
Partial information: We don't observe $r_{t,a}$ for all arms.

Terminology: Adversarial setup. We want guarantees against arbitrary (bounded) sequence of rewards/losses.

The Exponentially Weighted Average Forecaster

- Output $m_t \sim p_t$ where $p_t(m) = \frac{w_t(m)}{\sum_{m\in\mathcal{M}} w_t(m)}$,
  - $\forall m \in \mathcal{M}, w_1(m) = 1$ and $w_{t+1}(m) = w_t(m)\exp(-\eta\ell_{t,m})$.

$\ell_{t,m}$ is not available for all arms!

$\ell_{t,m} = 1 - r_{t,a}$?

We can use **importance sampling**

$$\widehat{\ell}_{t,m} = \begin{cases} \frac{\ell_{t,m}}{p_t(m)} & \text{if } m = m_t \\ 0 & \text{otherwise} \end{cases}$$

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Why it may be a bad idea:

- $p_{t,m}$ typically small for bad arms, hence this estimates has large variance for bad arms!

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- Exp3 has a small regret **in expectation**
- Exp3 might have large deviations with **high probability** (ie, from time to time it may **concentrate $\widehat{p}_t$ on the wrong arm** for too long and then incur a large regret)

Fix: add some extra uniform exploration

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Theorem

If Exp3 is run with $\gamma = \eta$, then it achieves a regret

$$R_T = \max_{a\in\mathcal{A}} \sum_{t=1}^{T} r_{t,a} - \mathbb{E}\Big[\sum_{t=1}^{T} r_{t,A_t}\Big] \leqslant (e-1)\gamma G_{\max} + \frac{A\log A}{\gamma}$$

with $G_{\max} = \max_{a\in\mathcal{A}} \sum_{t=1}^{T} r_{t,a}$.

Theorem

If Exp3 is run with

$$\gamma = \eta = \sqrt{\frac{A\log A}{(e-1)T}}$$

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$$R_T \leqslant O(\sqrt{TA\log A})$$

Comparison with online learning (convex, bounded):

$$R_T(Exp3) \leqslant O(\sqrt{TA\log A})$$

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Intuition: in online learning at each round we obtain A feedbacks, while in bandits we receive 1 feedback.

$$R_T(Exp3) = \mathbb{E}\Big(\sum_{t=1}^{T} r_{t,a} - r_{t,a_t}\Big) \leqslant \frac{\log(A)}{\eta} + \frac{A}{2}\eta T\,.$$

Further, For any non-increasing sequence $(\eta_t)_t$:

$$R_T(Exp3) = \mathbb{E}\Big(\sum_{t=1}^{T} r_{t,a} - r_{t,a_t}\Big) \leqslant \frac{\log(A)}{\eta_T} + \frac{A}{2}\sum_{t=1}^{T}\eta_t\,.$$

Step 1. $\mathbb{E}_{a\sim p_{t,\eta}}\tilde{\ell}_t(a) = 1 - r_{t,a_t}$ and $\mathbb{E}_{a_t\sim p_{t,\eta}}\tilde{\ell}_t(a) = 1 - r_{t,a}$. Thus:

$$\forall a \in \mathcal{A}, \quad \sum_{t=1}^{T} r_{t,a} - r_{t,a_t} = \sum_{t=1}^{T} \mathbb{E}_{a\sim p_{t,\eta}}\tilde{\ell}_t(a) - \sum_{t=1}^{T} \mathbb{E}_{a_t\sim p_{t,\eta}}\tilde{\ell}_t(a)\,.$$

Step 2. The random variable $X = \tilde{\ell}_t(a)$, is positive. By Hoeffding's lemma,

$$\begin{aligned}
\mathbb{E}_{a\sim p_{t,\eta}}(\tilde{\ell}_t(a)) &\leqslant -\frac{1}{\eta}\log\left(\mathbb{E}_{a\sim p_{t,\eta}}\left[\exp(-\eta\tilde{\ell}_t(a))\right]\right) + \frac{\eta}{2}\mathbb{E}_{a\sim p_{t,\eta}}(\tilde{\ell}_t(a)^2) \\
&= -\frac{1}{\eta}\log\left(\frac{\sum_{a\in\mathcal{A}} e^{-\sum_{s=1}^{t}\eta\tilde{\ell}_s(a)}}{\sum_{a\in\mathcal{A}} e^{-\sum_{s=1}^{t-1}\eta\tilde{\ell}_s(a)}}\right) + \frac{\eta}{2}\mathbb{E}_{a\sim p_{t,\eta}}(\tilde{\ell}_t(a)^2)\,.
\end{aligned}$$

Step 3. Thus,

$$\sum_{t=1}^{T} \mathbb{E}_{a\sim p_{t,\eta}}(\tilde{\ell}_t(a)) \leqslant -\frac{1}{\eta}\log\Big(\frac{1}{A}\sum_b \exp\big(-\sum_{t=1}^{T}\eta\tilde{\ell}_t(b)\big)\Big) + \sum_{t=1}^{T}\frac{\eta}{2}\mathbb{E}_{a\sim p_{t,\eta}}(\tilde{\ell}_t(a)^2)\,.$$

Since the reward function is bounded by 1 we have:

$$\mathbb{E}_{a\sim p_{t,\eta}}(\tilde{\ell}_t(a)^2) = \mathbb{E}_{a\sim p_{t,\eta}}\Big(\frac{(1-r_{t,A_t})^2}{p_t^2(A_t)}\mathbb{I}\{A_t = a\}\Big) \leqslant \frac{1}{p_t(a_t)}.$$

Step 4. Using the fact that the sum of positive terms is bigger than any of its term,

$$-\frac{1}{\eta}\log\Big(\sum_b \exp\big(-\sum_{t=1}^{T}\eta\tilde{\ell}_t(b)\big)\Big) \quad\leqslant\quad \sum_{t=1}^{T}\tilde{\ell}_t(a) \text{ for each } a\in\mathcal{A}\,.$$

Taking expectations, it comes for all $a\in\mathcal{A}$,

$$\mathbb{E}\Big[\sum_{t=1}^{T} r_{t,a} - r_{t,a_t}\Big] \leqslant \frac{\log(A)}{\eta} + \sum_{t=1}^{T}\frac{\eta}{2}\underbrace{\mathbb{E}\Big[\frac{1}{p_t(a_t)}\Big]}_{A}\,.$$

The Improved-Exp3 Algorithm

Using importance sampling is bad as generates large variance, especially for arms with low probability of being chosen (bad arms).

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- Many other variants.

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- **Loss** of expert $m \in \mathcal{M}$: $\ell_{t,m} = \sum_{a\in\mathcal{A}} \xi_{t,m}(a) r_t(a)$ (Instead of reward)
- Case when $|\mathcal{M}| \gg |\mathcal{A}|$?

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Exp4

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- Output $a_t \sim p_t \in \mathcal{P}(\mathcal{A})$ where $p_t(a) = (1-\gamma)\frac{w_t(m)\xi_{t,m}(a)}{\sum_{m\in\mathcal{M}} w_t(m)} + \frac{\gamma}{|\mathcal{A}|}$

Exp4

Exponential-weight algorithm for exploration and exploitation using expert advice.

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- Receive r_{t,a_t}, build $\widehat{\ell}_t(a) = \begin{cases} \frac{1-r_t(a)}{p_t(a)} & \text{if } a = a_t \\ 0 & \text{else} \end{cases}$

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- Update $\forall m \in \mathcal{M}, w_{t+1}(m) = w_t(m)\exp(-\eta\widehat{\ell}_{t,m})$. where $\widehat{\ell}_{t,m} = \sum_{a\in\mathcal{A}} \xi_{t,m}(a)\widehat{\ell}_t(a)$.

Theorem

If Exp4 is run with $\gamma \in [0,1]$, then it achieves a regret

$$R_T = \max_{a\in\mathcal{A}} \sum_{t=1}^{T} r_{t,a} - \mathbb{E}\Big[\sum_{t=1}^{T} r_{t,A_t}\Big] \leqslant (e-1)\gamma G_{\max} + \frac{A\log M}{\gamma}$$

with $G_{\max} = \max_{a\in\mathcal{A}} \sum_{t=1}^{T} r_{t,a}$.

- $\Phi : \mathcal{H} \rightarrow \mathcal{D}$, mapping from set of histories to some set $\mathcal{D}$, such that $h_1 \sim h_2$ iff $\Phi(h_1) = \Phi(h_2)$ defines **equivalence relation**; let $[h]$ the equivalence class of h.

Constrained opponent/comparison class

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- Examples:
  - $\Phi(h) = 1$ gives constant experts.
  - $\Phi(h) = (a_{-1}, \dots, a_{-m})$ last m actions, gives experts depending on last m actions only.
  - $\Phi(h) = |h| \mod k$ gives periodic experts.

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- We define the **Φ-constrained** regret:

$$\mathcal{R}_T^{\Phi} = \sup_{\pi:\mathcal{H}/\Phi\to\mathcal{A}} \mathbb{E}\Big[\sum_{t=1}^{T} r_{t,\pi([h_t])}\Big] - \mathbb{E}\Big[\sum_{t=1}^{T} r_{t,a_t}\Big]$$

More challenging than best constant expert.

- We can define a version of Exp4 for Φ-constrained policies.

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- We simply **contextualize** Exp4 by indexing losses, weights, parameters η by the equivalence classes, and computing the current active class $c_t = \Phi(h_t)$.
- Result (M. Munos, 2011)

$$\mathcal{R}_T^{\Phi} \leqslant \sum_{c \in \mathcal{H}/\Phi} \mathbb{E}\left[\frac{A\eta_c}{2} T_c + \frac{\log(A)}{\eta_c}\right].$$

where T_c is number of activation times of class c until time T.

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POOL OF CONSTRAINED STRATEGIES?

- We consider we have a **set** $(\Phi_\theta)_{\theta\in\Theta}$ of **constrained strategies**.
- One Φ_θ-Exp3 strategy for each θ: see them as different **experts**?
- Run Exp4 with all these base experts: Φ_1-Exp3, ..., Φ_P-Exp3 ?

Difficulty: The experts are **learning** algorithms. Their performance depends on the observations they received.
We are in **partial feedback**: When Φ_p-Exp3 recommends to play action a, Exp4 may **instead** play (and received reward from) action b. Hence Φ_p-Exp3 not only faces **partial feedback**, but also it does **not** observe the reward corresponding to what it decides.

Double-bandit feedback.

Theorem (M. Munos, 2011)

In the double-bandit feedback setup, Exp4, run on $(\Phi_\theta\text{-Exp3})_{\theta\in\Theta}$ strategies with appropriate parameter tuning satisfies

$$\mathcal{R}_T = O\Big(T^{2/3}(A\log(A)C)^{1/3}\log(|\Theta|)^{1/2}\Big) \text{ with } C = \max_{\theta\in\theta}|\mathcal{H}/\Phi_\theta|.$$

- Strategies for **Stochastic** bandits: UCB, KL-UCB, etc.
  $\log(T)$ regret bounds when stochastic model, but strong assumptions on signal.

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  $\log(T)$ regret bounds when stochastic model, but strong assumptions on signal.
- Strategies for **Adversarial** bandits: Exp3, Exp4, etc.
  $\sqrt{T}$ regret bounds with little assumption on model, but perhaps too conservative.

Can we have the best of both worlds?

Several works on the topic

- Bubeck&Slivkins 2012, Auer&Chiang, 2016.
- Zimmert-Seldin 2018.

Idea: **Online Mirror Descent** regularized by **Tsallis Entropy**.
α-**Tsallis** entropy:

$$H_\alpha(x) = \frac{1}{1-\alpha}(1 - \sum_{a\in\mathcal{A}} x_a^\alpha)$$

◇ $\lim_{\alpha\to 1} H_\alpha(x) = \sum_{a\in\mathcal{A}} x_a \log(x_a)$
◇ $\lim_{\alpha\to 0} H_\alpha(x) = -\sum_{a\in\mathcal{A}} \log(x_a)$

Let us consider the potential:

$$\Psi_{t,\alpha}(q) = -\sum_{a\in\mathcal{A}} \frac{q^{\alpha}(a)}{\alpha}$$

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▷ Choose

$$p_t = \operatorname*{argmin}_{q\in\mathcal{P}(\mathcal{A})} \langle q, \widehat{L}_{t-1}\rangle + \frac{1}{\eta_t}\Psi_{\alpha}(q)$$

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  (This is gradient of dual of $\Psi_{t,\alpha}/\eta_t$ at position $\widehat{L}_{t-1}$)
- Sample $a_t \sim p_t$
- Observe ℓ_{t,a_t} then build $\widehat{\boldsymbol{\ell}}_t$ as unbiased estimate of $\boldsymbol{\ell}_t$, then $\widehat{L}_t = \widehat{L}_{t-1} + \widehat{\boldsymbol{\ell}}_t$.

	Regime	$\frac{\text{Upper bound}}{\text{Lower bound}}$	Learning rate
$\lim_{\alpha\to 0}$	Sto	$O(1)$	$\Theta(\Delta_a)$
	Adv	$O(\sqrt{\ln(T)})$	$\Theta\left(\frac{\ln(t)}{\sqrt{t}}\right)$
$\alpha = \frac{1}{2}$	Sto&Adv	$O(1)$	$\frac{1}{\sqrt{t}}$
$\lim_{\alpha\to 1}$	Sto	$O(\ln(T))$	$\Theta\left(\frac{\ln(t)}{\Delta_a t}\right)$
	Adv	$O(\sqrt{\ln(A)})$	$\Theta\left(\frac{1}{\sqrt{t}}\right)$.

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- Mixed world bandit: Some arms are stochastic, others are arbitrary bounded?

Table of contents

▷ A two–player **zero–sum** game

	A	B	C
1	30, -30	-10, 10	20, -20
2	10, -10	-20, 20	-20, 20

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A set of strategies is a **Nash equilibrium** if **no player** can do better by **unilaterally changing** his strategy.

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Red: take action 1 with **prob. 1**
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Value of the game: $V = -10$ (reward of Red at the equilibrium)

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Nash equilibrium:
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At each round t

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Value of the game

$$V = \max_{\mathbf{q}} \min_{\mathbf{p}} \bar{\ell}(\mathbf{p}, \mathbf{q})$$

with

$$\bar{\ell}(\mathbf{p}, \mathbf{q}) = \sum_{i=1}^{N} \sum_{j=1}^{M} p_i q_j \ell(i, j)$$

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$$R_n(\text{col}) = \sum_{t=1}^{n} \ell_{I_t,J_t} - \min_{j=1,\dots,M} \sum_{t=1}^{n} \ell_{I_t,j}$$

Theorem

If both the row and column players play according to an **Hannan-consistent** strategy, then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) = V$$

Theorem

The **empirical distribution** of plays

$$\widehat{p}_{i,n} = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I} I_t = i \quad \widehat{q}_{j,n} = \frac{1}{n}\sum_{t=1}^{n} \mathbb{I} J_t = j$$

induces a product distribution $\widehat{\mathbf{p}}_n \times \widehat{\mathbf{q}}_n$ which converges to the **set of Nash equilibria** $\mathbf{p} \times \mathbf{q}$.

Since $\bar{\ell}(\mathbf{p}, J_t)$ is linear, over the simplex, the minimum is at one of the corners

[math]

$$\min_{i=1,\ldots,N} \frac{1}{N} \sum_{t=1}^{n} \ell(i, J_t) = \min_{\mathbf{p}} \frac{1}{n} \sum_{t=1}^{n} \bar{\ell}(\mathbf{p}, J_t)$$

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Elaborating on it **[math]**

$$\begin{aligned} \min_{\mathbf{p}} \frac{1}{n} \sum_{t=1}^{n} \bar{\ell}(\mathbf{p}, J_t) &= \min_{\mathbf{p}} \sum_{j=1}^{M} \widehat{q}_{j,n} \bar{\ell}(\mathbf{p}, j) \\ &= \min_{\mathbf{p}} \bar{\ell}(\mathbf{p}, \widehat{\mathbf{q}}_n) \\ &\leqslant \max_{\mathbf{q}} \min_{\mathbf{p}} \bar{\ell}(\mathbf{p}, \mathbf{q}) = V \end{aligned}$$

By definition of Hannan's consistent strategy **[def]**

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{t=1}^{n} \ell(I_t, J_t) = \min_{i=1,\dots,N} \frac{1}{n} \sum_{t=1}^{n} \ell(i, J_t)$$

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Then

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If we do the same for the other player **[zero–sum game]**

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} \ell(I_t, J_t) \geqslant V$$

Question: how fast do they converge to the Nash equilibrium?

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Answer: it depends on the specific algorithm. For EWA(η), we now that

$$\sum_{t=1}^{n} \ell(I_t, J_t) - \min_{i=1,\ldots,N} \sum_{t=1}^{n} \ell(i, J_t) \leqslant \frac{\log N}{\eta} + \frac{n\eta}{8} + \sqrt{\frac{n}{2} \log \frac{1}{\delta}}$$

Generality of the results

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- Players do not know the payoff matrix
- Players do not observe the loss of the other player
- Players do not even observe the action of the other player

External (expected) regret

$$R_n = \sum_{t=1}^{n} \bar{\ell}(\widehat{\mathbf{p}}_t, y_t) - \min_{i=1,\ldots,N} \sum_{t=1}^{n} \ell(i, y_t)$$

$$= \max_{i=1,\ldots,N} \sum_{t=1}^{n} \sum_{j=1}^{N} \widehat{p}_{j,t} \left(\ell(j, y_t) - \ell(i, y_t)\right)$$

External (expected) regret

$$
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R_n &= \sum_{t=1}^{n} \bar{\ell}(\widehat{\mathbf{p}}_t, y_t) - \min_{i=1,\dots,N} \sum_{t=1}^{n} \ell(i, y_t) \\
&= \max_{i=1,\dots,N} \sum_{t=1}^{n} \sum_{j=1}^{N} \widehat{p}_{j,t} \big(\ell(j, y_t) - \ell(i, y_t)\big)
\end{aligned}
$$

Internal (expected) regret

$$
R_n^I = \max_{i,j=1,\dots,N} \sum_{t=1}^{n} \widehat{p}_{j,t} \big(\ell(i, y_t) - \ell(j, y_t)\big)
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Internal (expected) regret

$$R_n^I = \max_{i,j=1,\dots,N} \sum_{t=1}^{n} \widehat{p}_{j,t} \big(\ell(i, y_t) - \ell(j, y_t)\big)$$

Intuition: an algorithm has **small internal regret** if, for each pair of experts (i, j), the learner does not regret of not having followed expert j each time it followed expert i.

Theorem

Given a K–person game with a set of correlated equilibria $\mathcal{C}$. If all the players are internal–regret minimizers, then the **distance** *between the* **empirical distribution** *of plays and the set of* **correlated equilibria** $\mathcal{C}$ *converges to 0.*

A powerful model for **sequential** games

- Checkers / Chess / Go
- Poker
- Bargaining
- Monitoring
- Patrolling
- ...

Nash Equilibria in Extensive Form Games

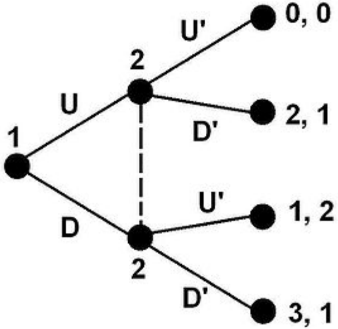
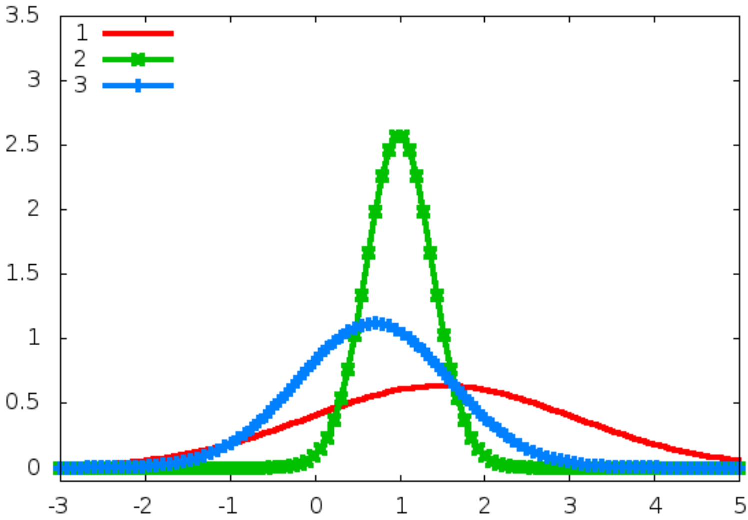


Table of contents

- We considered adversarial setup. One way to address risk.
- Other ways: **Risk-aversion** (model), **Robust** strategies (min-max).

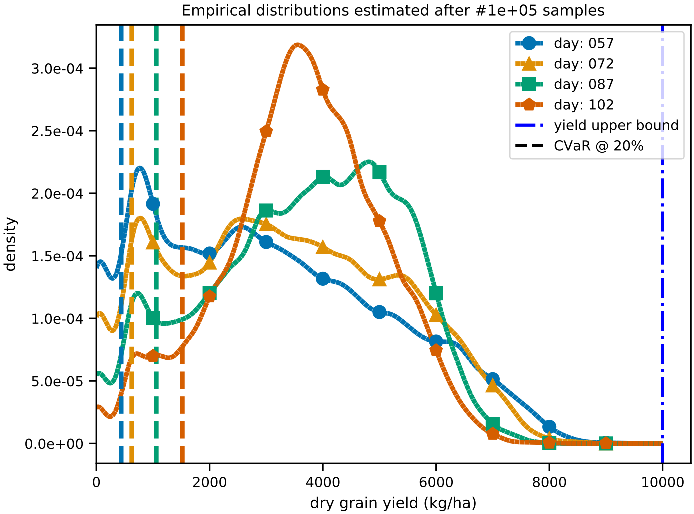
Illustration of risk-aversion.



▷ Choice for 1 sample ? For 1000 samples?

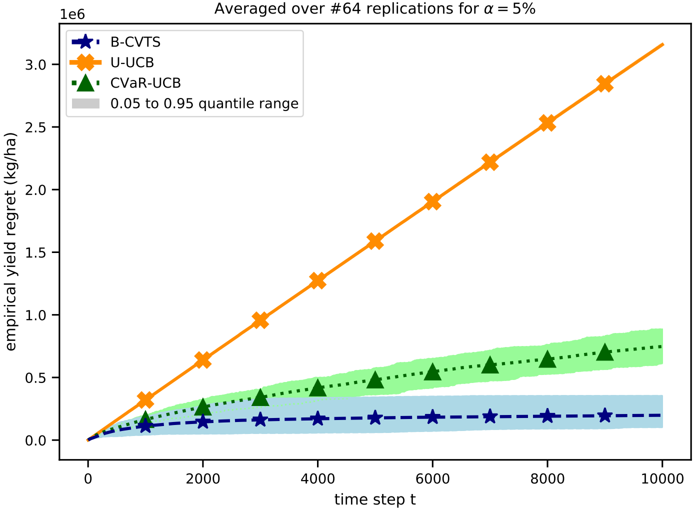
Bandit strategy for risk aversion

- DSSAT simulator: 30y of agronomy expertise, climate, ground, plant growth, etc.
- Distribution of yields for 4 different **planting date** (action) using **DSSAT**



- May not want best expectation, but rather **risk-averse** criterion.

- Novel bandit strategy for **Conditional Value at Risk** (CVaR)
- **Provably** optimal (regret bound matches lower bound).
- Based on novel statistical estimation tools. Performance (blue) against Sota:



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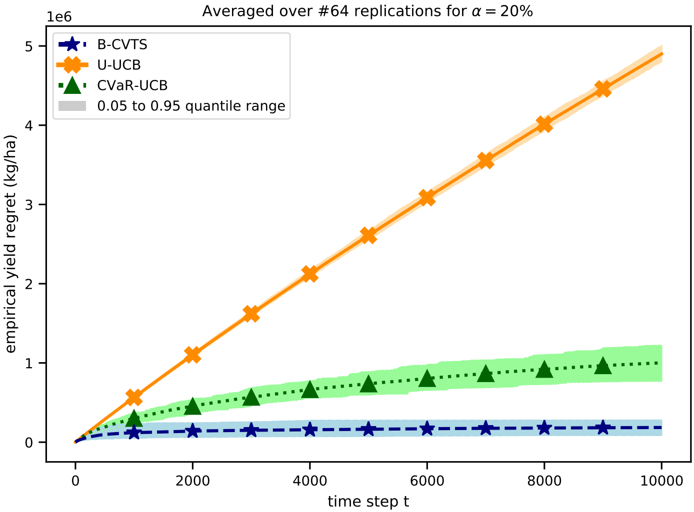

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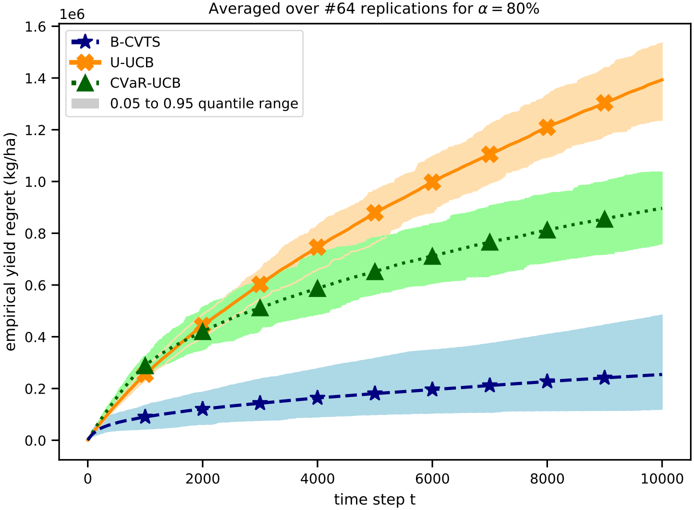

Conditional Value at Risk

▷ Consider a **Gain** (reward): We are interested in **risky (low) gains**.

Maximize gain in worst-case situations

▷ Formally, for given $\alpha \in [0,1]$:

$$\mathtt{CVaR}_\alpha(\nu_k) = \sup_{x\in\mathbb{R}} \left\{ x - \frac{1}{\alpha}\mathbb{E}_{X\sim\nu_k}\left[(x-X)^+\right]\right\} . \quad (1)$$

▷ For **continuous** distributions, $\mathtt{CVaR}_\alpha(\nu_k) = \mathbb{E}_{X\sim\nu_k}[X|X \leqslant q_\alpha(\nu_k)]$, where $q_\alpha(\nu_k) = \inf\{x : \mathbb{P}(X \leqslant x) > \alpha\}$ is the quantile at level α.

- $\alpha = 1$ is the **expectation**, $\alpha = 0$ is very risk-averse (extreme).
- It is a **coherent** risk measure (Rockafellar, Acerbi et al.): many good properties.
- Rich litterature on CVaR in finance.
- Parameter α is **easy to interpret** for many practitioners.

CVaR Regret and bandits

- Unknown arm distributions $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_K)$, given risk-level α.
- We write $c_k^\alpha = \mathtt{CVaR}_\alpha(\nu_k)$.
- Best arm is the one with the **largest** CVaR.
- The CVaR regret of a sequential sampling strategy $\mathcal{A} = (A_t)_{t\in\mathbb{N}}$ is

$$\mathcal{R}_{\boldsymbol{\nu}}^\alpha(T) = \mathbb{E}_{\boldsymbol{\nu}}\left[\sum_{t=1}^{T}\left(\max_k c_k^\alpha - c_{A_t}^\alpha\right)\right] = \sum_{k=1}^{K} \Delta_k^\alpha \mathbb{E}_{\boldsymbol{\nu}}[N_k(T)],$$

where $\Delta_k^\alpha = \max_{k'} c_{k'}^\alpha - c_k^\alpha$ is the CVaR gap.

- Algorithms: UCB style, we need upper confidence bounds for CVaR; TS, we need sampling scheme.

- **Concentration**: Brown 2007, Thomas and Learned-Miller 2019.
- **Bandits**: Agrawal et al. 2020, Galichet 2013, Tamkin et al. 2020, etc.
- **MDPs**:

Optimizing the CVaR via Sampling, Tamar et al. 2014

Risk-Sensitive and Robust Decision-Making: a CVaR Optimization Approach, Chow et al. 2015

Definition

For any $\nu \in \mathcal{C}$ and $c \in \mathbb{R}$, we define

$$\mathcal{K}_{\inf}^{\alpha,\mathcal{C}}(\nu, c) := \inf\left\{\mathrm{KL}(\nu, \nu') : \nu' \in \mathcal{C}, \mathtt{CVaR}_\alpha(\nu') \geqslant c\right\}.$$

Theorem (Regret Lower Bound in CVaR bandits)

Let $\alpha \in (0, 1]$. Let $\mathcal{F} = \mathcal{F}_1 \times \dots \times \mathcal{F}_K$ be a set of bandit models $\boldsymbol{\nu} = (\nu_1, \dots, \nu_K)$ where each ν_k belongs to the class of distribution $\mathcal{F}_k$. Let $\mathcal{A}$ be a strategy satisfying $\mathcal{R}_{\boldsymbol{\nu}}^{\alpha}(\mathcal{A}, T) = o(T^\beta)$ for any $\beta > 0$ and $\nu \in \mathcal{F}$. Then for any $\nu \in \mathcal{D}$, for any sub-optimal arm k, under the strategy $\mathcal{A}$ it holds that

$$\lim_{T \to +\infty} \frac{\mathbb{E}_{\boldsymbol{\nu}}[N_k(T)]}{\log T} \geqslant \frac{1}{\mathcal{K}_{\inf}^{\alpha,\mathcal{F}_k}(\nu_k, c^\star)},$$

where $c^\star = \max_{i \in [K]} \mathtt{CVaR}_\alpha(\nu_i)$.

▷ One can rewrite the $\texttt{CVaR}$ in terms of the **CDF** $F(x) = \mathbb{P}(X \leqslant x)$.

$$\texttt{CVaR}_\alpha(\nu) = \frac{1}{\alpha}\int g_\alpha(F_\nu(x)))dx$$

for some monotonic function g_α. Also, it holds

$$|\texttt{CVaR}_\alpha(\nu) - \texttt{CVaR}_\alpha(\nu')| \leqslant \frac{1}{\alpha}\|F_\nu - F_{\nu'}\|_\infty$$

▷ Main tool is Massart's version of Dvoretzky-Kiefer-Wolfowitz (DKW) inequality:

$$\forall \delta_0 \in [0, 0.5) \quad \mathbb{P}\bigg(\sup_{x\in\mathbb{R}} F_\nu(x) - F_n(x) > \sqrt{\frac{\ln(1/\delta_0)}{2n}}\bigg) \leqslant \delta_0\,.$$

where F_n is empirical CDF.

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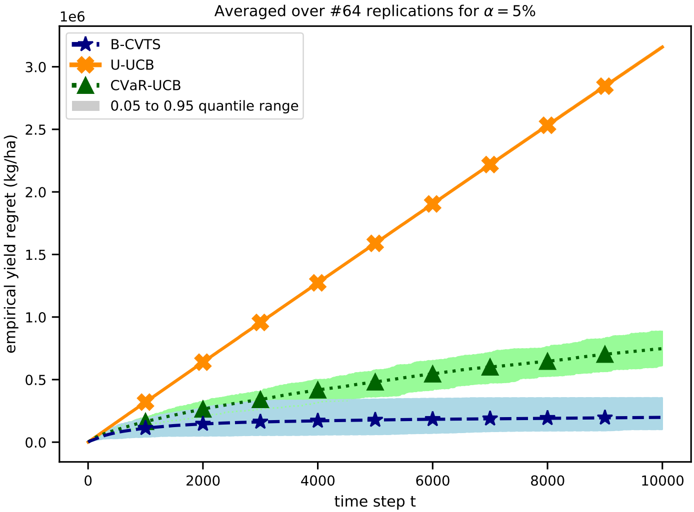

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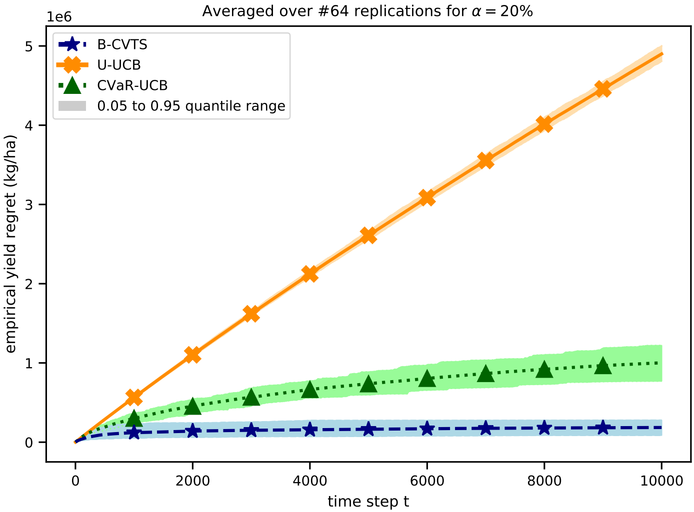

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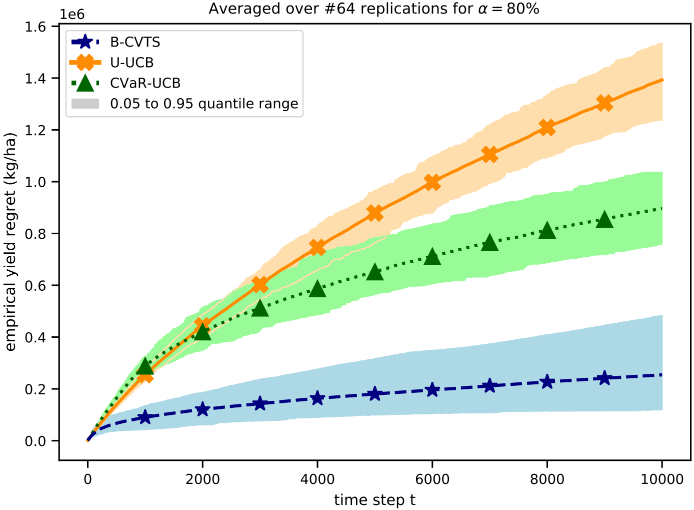

- We want to **control** how big/small can a random variable be

$$\mathbb{P}\Big[X \geqslant \dots\Big] \leqslant \delta \tag{2}$$

$$\mathbb{P}\Big[X \leqslant \dots\Big] \leqslant \delta \tag{3}$$

- Quantiles, expectiles, expected shortfall, value at risk.

- We want to **control** how big/small can a random variable be

$$\mathbb{P}\Big[X \geqslant \inf_{\lambda>0}\Big\{\frac{1}{\lambda}\log\mathbb{E}\exp(\lambda X) + \frac{\log(1/\delta)}{\lambda}\Big\}\Big] \leqslant \delta \tag{4}$$

$$\mathbb{P}\Big[X \leqslant \sup_{\lambda>0}\Big\{-\frac{1}{\lambda}\log\mathbb{E}\exp(-\lambda X) - \frac{\log(1/\delta)}{\lambda}\Big\}\Big] \leqslant \delta \tag{5}$$

(by Markov's inequality, whenever $\log\mathbb{E}\exp$ is defined near 0)

For all $\lambda > 0$,

$$\begin{aligned}\mathbb{P}[X \geqslant \varepsilon] &= \mathbb{P}[\exp(\lambda X) \geqslant \exp(\lambda\varepsilon)] \\ &\leqslant \mathbb{E}[\exp(\lambda X)]\exp(-\lambda\varepsilon) \\ &= \exp\left(-\lambda\left(\varepsilon - \frac{1}{\lambda}\log\mathbb{E}\exp(\lambda X)\right)\right)\end{aligned}$$

For $\varepsilon = \frac{1}{\lambda}\log\mathbb{E}\exp(\lambda X) + \frac{\log(1/\delta)}{\lambda}$, we get

$$\mathbb{P}\left[X \geqslant \frac{1}{\lambda}\log\mathbb{E}\exp(\lambda X) + \frac{\log(1/\delta)}{\lambda}\right] \leqslant \delta\,.$$

$$\kappa_{\lambda,\nu} \stackrel{\text{def}}{=} \frac{1}{\lambda} \log \mathbb{E}_\nu \exp\Big(\lambda X\Big), \tag{6}$$

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- more then one century year old,
- at the heart of many key-results and tools in statistical theory (Cramer-Chernoff method, Chernoff transform, log-Laplace transform)
- $\kappa_{-\lambda,\nu}$ is a key quantity to control the **probability that X is small**.

- Let $\{Z_k\}_{k=1,\dots,t}$ i.i.d. from $\mathcal{N}(\mu, \sigma^2)$.

Example: Gaussian distributions.

- Let $\{Z_k\}_{k=1,\dots,t}$ i.i.d. from $\mathcal{N}(\mu, \sigma^2)$.
- Let $X = \sum_{k=1}^{t} Z_k$ (thus $\mathcal{N}(\mu t, \sigma^2 t)$).

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- We recover in the **Gaussian** case the **mean-variance**

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- $\lambda = \sqrt{\frac{2\log(1/\delta)}{\sigma^2 t}}$ optimizes (4) and (5) and gives the familiar

$$\mathbb{P}\left(\frac{1}{t}\sum_{k=1}^t Z_k - \mu \geqslant \sigma\sqrt{\frac{2\log(1/\delta)}{t}}\right) \leqslant \delta$$

$$\mathbb{P}\left(\mu - \frac{1}{t}\sum_{k=1}^t Z_k \geqslant \sigma\sqrt{\frac{2\log(1/\delta)}{t}}\right) \leqslant \delta\,.$$

- We introduce the **mixability gaps** (always non negative):

$$m^+_{\lambda,\nu} = \kappa_{\lambda,\nu} - \mathbb{E}_\nu\Big[X\Big] \text{ and } m^-_{\lambda,\nu} = \mathbb{E}_\nu\Big[X\Big] - \kappa_{-\lambda,\nu}\,.$$

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- Now equations (4) and (5) rewrite more compactly as

$$\mathbb{P}\Big[X - \mathbb{E}_{\nu}\Big[X\Big] \geqslant \inf_{\lambda>0}\Big\{m^{+}_{\lambda,\nu} + \frac{\log(1/\delta)}{\lambda}\Big\}\Big] \leqslant \delta \,, \tag{7}$$

$$\mathbb{P}\Big[\mathbb{E}_{\nu}\Big[X\Big] - X \geqslant \inf_{\lambda>0}\Big\{m^{-}_{\lambda,\nu} + \frac{\log(1/\delta)}{\lambda}\Big\}\Big] \leqslant \delta \,. \tag{8}$$

Entropic Value At Risk

- Control of the upper/lower tails involves $\kappa_{\lambda,\nu}/\kappa_{-\lambda,\nu}$.

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$$\kappa_{-\lambda,\nu} = \inf_{\nu' \in \mathcal{M}(\mathbb{R})} \left\{ \mathbb{E}_{\nu'}(X) + \frac{1}{\lambda}\mathtt{KL}(\nu'||\nu) \right\} \leqslant \mathbb{E}_{\nu}\left[X\right] . \tag{9}$$

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- Natural measure of risk-aversion.

Setting: Unknown real-valued distributions $\{\nu_a\}_{a=1,\dots,A}$. At each t, we choose $A_t \in \{1,\dots,A\}$, receive reward $Y_t \sim \nu_{A_t}$.

The **expected regret** $\overline{\mathcal{R}}_T$ gives **no information on the risk** of the strategy and of pulling one arm (no control on the tails):

$$\overline{\mathcal{R}}_T = \sum_{a' \in \mathcal{A}} \Big(\max_{a \in \mathcal{A}} \mathbb{E}_{\nu_a}[X] - \mathbb{E}_{\nu_{a'}}[X] \Big) \mathbb{E}\Big[N_{T,a'} \Big],$$

where $N_{T,a'} = \sum_{t=1}^{A} \mathbb{I}\{A_t = a'\}$.

We are given some λ.
We define the optimal arm $a^\star$ as the one maximizing the **risk aversion** at level λ

$$a^\star \in \operatorname*{argmax}_{a=1,\dots,A} \kappa_{-\lambda,\nu_{a^\star}}.$$

Example: For $\mathcal{N}(\mu,\sigma^2)$ distributions $\kappa_{-\lambda,\nu_{a^\star}} = \mu_a - \frac{\lambda\sigma_a^2}{2}$.
In general it holds $\kappa_{-\lambda,\nu_{a^\star}} \leqslant \mathbb{E}_{\nu_a}[X]$.

- The empirical regret $\mathcal{R}_T(\lambda)$ of π with respect to the strategy $\star$ that constantly pulls arm $a^\star$ is:

$$\mathcal{R}_T(\lambda) \stackrel{\text{def}}{=} \sum_{i=1}^{T} X_{i,a^\star} - \sum_{a=1}^{A} \sum_{i=1}^{N^\pi_{T,a}} X_{i,a}\,, \qquad (10)$$

where $X_{i,a}$ denotes the i^{th} (i.i.d) sample from arm a.

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- The risk-averse regret $\overline{\mathcal{R}}_T(\lambda)$ is defined by

$$\begin{aligned} \overline{\mathcal{R}}_T(\lambda) &= \sum_{a\in\mathcal{A}} \Big(\kappa_{-\lambda,\nu_{a^\star}} - \kappa_{-\lambda,\nu_a}\Big)\mathbb{E}\Big[N_{T,a}\Big] \qquad (11)\\ &= \sum_{a\in\mathcal{A}} \boldsymbol{\Delta}_{\mathbf{a}}\mathbb{E}\Big[N_{T,a}\Big] \end{aligned}$$

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$$= \sum_{a\in\mathcal{A}} \boldsymbol{\Delta}_{\mathbf{a}}\mathbb{E}\Big[N_{T,a}\Big]$$

- We study both (10) and (12) since they offer interesting and easy interpretations.

Tradeoff between
being risk-averse versus **targeting high reward**:

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- If **"not enough"** risk-averse (protect against light lower tails only but arms have fat lower tails),
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- If **“not enough”** risk-averse (protect against light lower tails only but arms have fat lower tails),
  $\Longrightarrow$ we may get **high-regret**.
- If **“too much”** risk-averse (protect against fat lower tails but all arms have light lower tails),
  $\Longrightarrow$ a less cautious algorithm can (e.g. UCB) get better rewards.

λ defines the **risk-aversion** of the problem, irrespectively of the actual distributions of the environment.

If we design an optimal algorithm for risk-averse level λ,

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- Others will be **"harder"** (a more risk-averse algorithm, e.g. Exp3, gets better rewards).

We want to defeat e.g. not-enough cautious algorithms in hard environments.

Risk-aversion for a fixed λ (often justified in practical applications).

1. Decompose empirical regret with number of pulls of sub-optimal arms (allows **robust** analysis).
2. Introduce RAUCB for risk-aversion.
3. Get numerically efficient dual formulation.
4. Control both risk-averse and empirical regret.

Theorem (Generic decomposition of the empirical regret)

Let the event that strategy π **does not pull sub-optimal arms too often** be (for some non-negative constants $\{u_a\}_{a=1,\dots,A}$):

$$\Omega \stackrel{\text{def}}{=} \Big\{\exists a \neq a^\star \;:\; N_{T,a} > u_a\Big\}.$$

For all $\delta \in (0,1)$, with probability higher than $1-\delta-\mathbb{P}(\Omega)$, the empirical regret of π is upper bounded by

$$\mathcal{R}_T(\lambda) \leqslant \sum_{a\neq a^\star} \Delta_a u_a + \Big(\dots \mathbf{m}^-_{\lambda,\nu_{a^\star}} + \frac{\dots}{\lambda}\Big) + \inf_{\lambda'>0}\Big\{\dots \mathbf{m}^+_{\lambda',\nu_{a^\star}} + \frac{\dots}{\lambda'}\Big\}.$$

- First term: essentially risk-averse regret.
- Other terms: tails.

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$$U_t(a) \stackrel{\textbf{def}}{=} \sup_{\nu\in\mathcal{P}(\mathbb{R}_B)} \left\{ \kappa_{-\lambda,\nu} \ : \ \mathbf{K}(\widehat{\nu}_{\mathbf{t}}(\mathbf{a}), \kappa_{-\lambda,\nu}) \leqslant \frac{f(t)}{N_{t,a}} \right\},$$

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with parameter $f(t) \simeq \log(t)$ and where we introduced

$$\mathrm{K}(\widehat{\nu}_{\mathrm{t}}(\mathrm{a}), r) \stackrel{\mathrm{def}}{=} \inf_{\nu\in\mathcal{M}(\mathbb{R}_B)} \left\{ \mathrm{KL}(\widehat{\nu}_t(a)||\nu) \;:\; \kappa_{-\lambda,\nu} \geqslant r \right\}.$$

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- **Note 1:** Using mean-based confidence bounds is useless here.

2. Risk-averse algorithm: RAUCB

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with parameter $f(t) \simeq \log(t)$ and where we introduced

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- **Note 1:** Using mean-based confidence bounds is useless here.
- **Note 2:** Similarly to bandits, we do not need to estimate $\kappa_{-\lambda,\nu_a}$ (+ it would too loose here).

$\mathbf{K}(\widehat{\nu}_t(a), r)$ and then $U_t(a)$ can be **solved numerically** (deeply linked to numerically efficient dual formulation considered for standard MAB, e.g. Borwein-Lewis, 91, Harari-Kermadec, 06):

Lemma (Dual formulation)

Let $\widehat{\nu}_n$ be an empirical distribution built with n atoms $\{x_i\}_{1\leqslant i\leqslant n}$. Then the following dual formulation holds

$$\mathbf{K}(\widehat{\nu}_n, r) = \max_{0\leqslant\gamma^\star\leqslant\frac{\lambda}{1-e^{-\lambda(B-r)}}} \left\{\frac{1}{n}\sum_{i=1}^{n}\log\left(1-\frac{\gamma^\star}{\lambda}\left(1-e^{-\lambda(x_i-r)}\right)\right)\right\}.$$

Theorem (Regret of RAUCB)

The expected regret of RAUCB (for suitable f), is bounded by

$$\overline{\mathcal{R}}_T(\lambda) \leqslant 5 \sum_{a \neq a^\star} \frac{(1+\varepsilon_a)\Delta_a}{\mathbf{K}_a} \log(T) + O\Big(1\Big).$$

The empirical regret of RAUCB is bounded with high probability, for sub-Gaussians distributions of rewards (includes bounded as special case) and risk-aversion $\lambda = \Theta(\log(T)^{-1/2})$ as

$$\mathcal{R}_T(\lambda) \leqslant 5 \sum_{a \neq a^\star} \frac{(1+\varepsilon_a)\Delta_a}{\mathbf{K}_a} \log(T) + O\Big(\sqrt{\log(T)}\Big).$$

- `RAUCB` tuned with **known horizon** T (not anytime).

Comments about the result

- RAUCB tuned with **known horizon** T (not anytime).
- Ratio $\frac{\Delta_a}{\mathbf{K}_a}\log(T)$ similar to best known bounds for the expected regret Burnetas-Katehakis, 96; Cappe et al, 2013,

- RAUCB tuned with **known horizon** T (not anytime).
- Ratio $\frac{\Delta_a}{\mathbf{K}_a}\log(T)$ similar to best known bounds for the expected regret Burnetas-Katehakis, 96; Cappe et al, 2013,
- Choice of λ not too critical: still get $O(\log(T))$ for any λ not depending on T.

Further references

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- Baudry, Dorian, et al. *Optimal Thompson Sampling strategies for support-aware CVaR bandits.* International Conference on Machine Learning, 2021.

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Slides Edouard:

`https://eleurent.github.io/robust-beyond-quadratic/paper/oral`

Self-check

- **Full** information vs **Partial** information (Bandit, semi-bandit)
- Objectives: Best model vs Best combination of losses vs Best combination of models vs Best sequence
- Convex losses
- Exponential weights, Hedge strategy
- Self-information loss
- Exp-concavity, mixability
- Entropy formula
- Bregman agggregation
- Fixed-share strategy
- Markov-Hedge
- Mixing past posteriors
- Exp3, Importance sampling, Exp3-P, Exp3-IX, Exp4
- Tsallis entropy
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- Conditional value at risk, mixability gap
- Robust learning

"The more applied you go, the stronger theory you need"

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