Monte-Carlo Graph Search:

the Value of Merging Similar States

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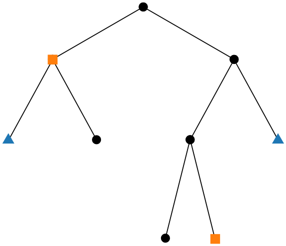
Monte-Carlo Tree Search algorithms

- rely on a tree structure to represent their value estimates.



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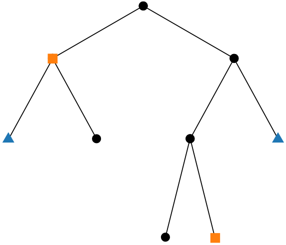
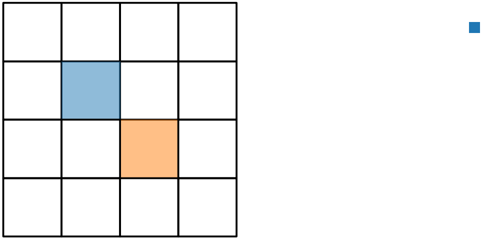


- performance independent of the size S of the state space

Tabular RL	(UCBVI)	$\sqrt{HSAn}$
MCTS	(OPD)	$n^{-\log\frac{1}{\gamma}/\log A}$

Limitation

- There can be several paths to the same state s



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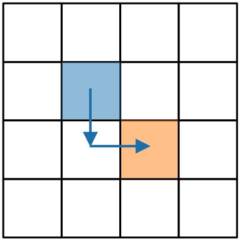

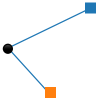

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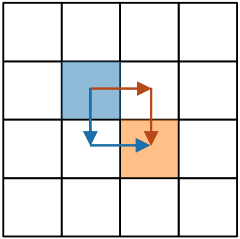

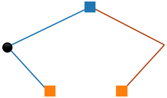

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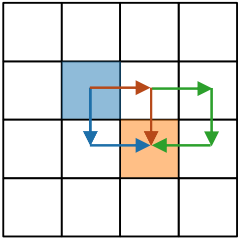

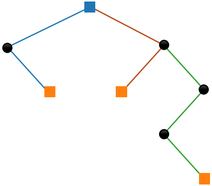

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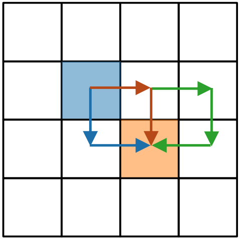

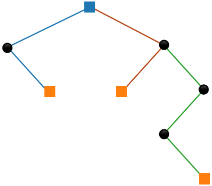

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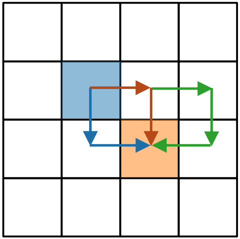

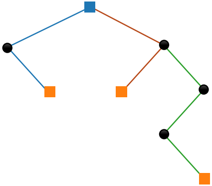

- s is represented several times in the tree
- No information is shared between these paths

Not accounting for state similarity hinders exploration

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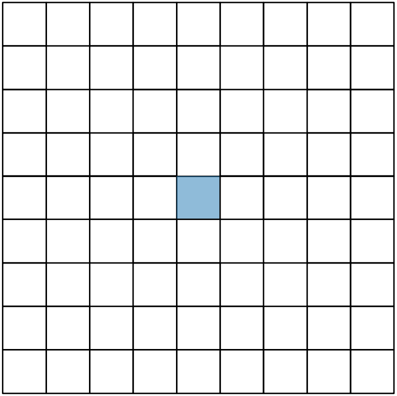
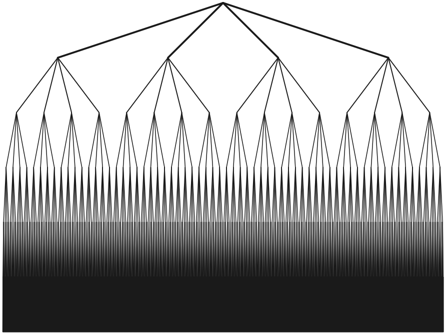
Sparse gridworld: reward of 0 everywhere



↳ Uniform planning in the space of sequences of actions



`OPD`, budget of $n = 5460$ moves

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- Does not lead to uniform exploration of the state space

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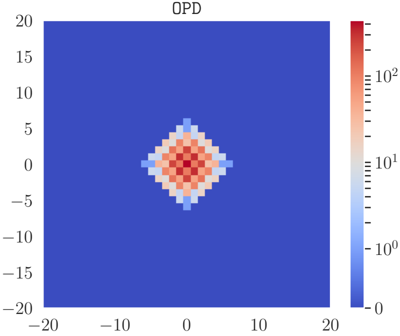
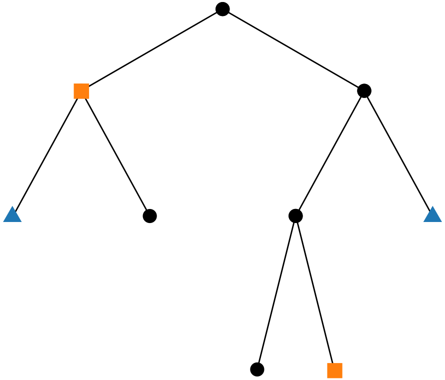


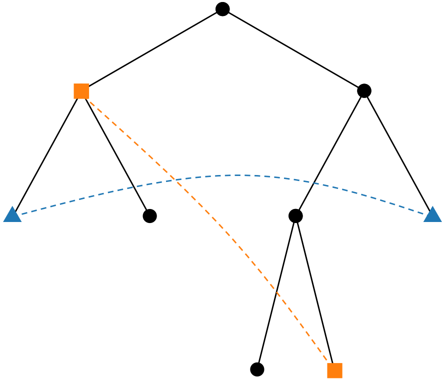
budget of 5460 samples, maximum distance $d = 6$

Better exploit this wasted information



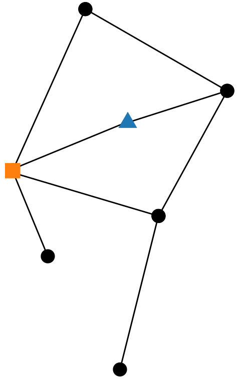
Better exploit this wasted information

- By merging similar states



Better exploit this wasted information

- By merging similar states into a graph



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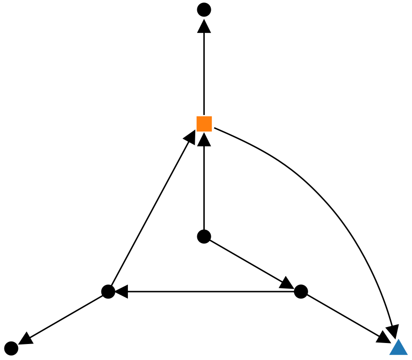
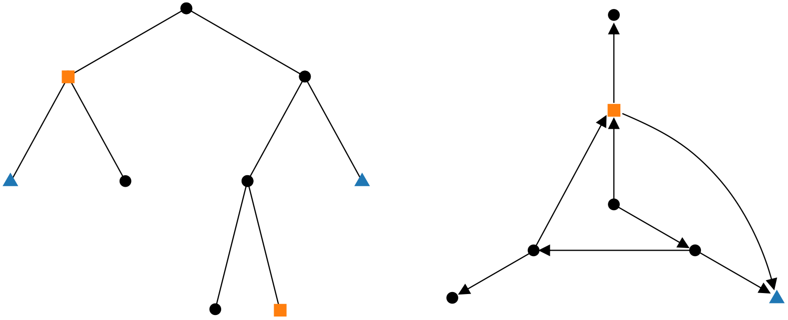
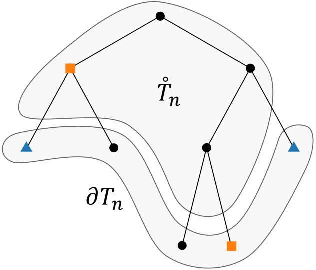


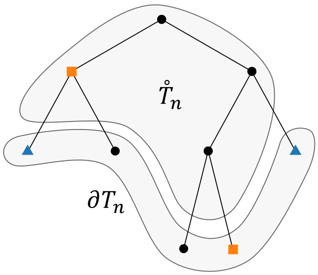
Questions



- How to adapt MCTS algorithms to work on graphs?
- Can we quantify the benefit of using graphs over trees?



Optimism in the Face of Uncertainty: `OPD`



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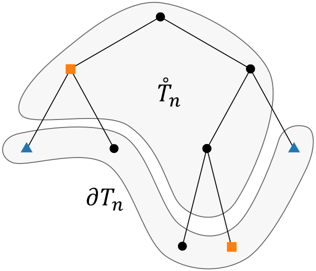

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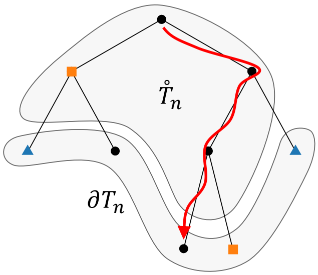

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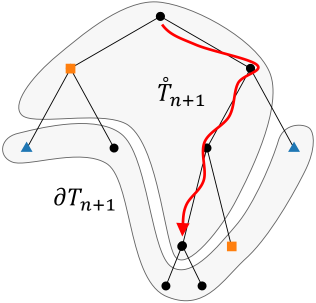

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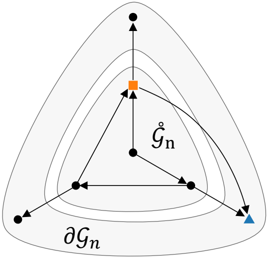
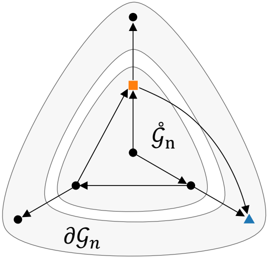


Same principle: `GBOP-D`



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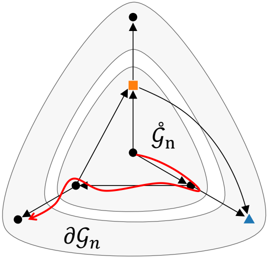

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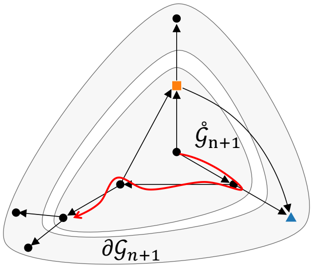

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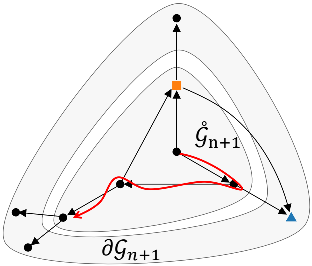

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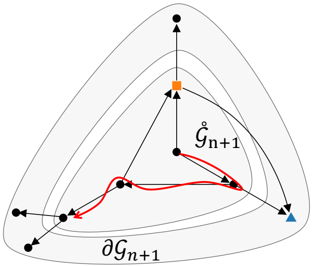

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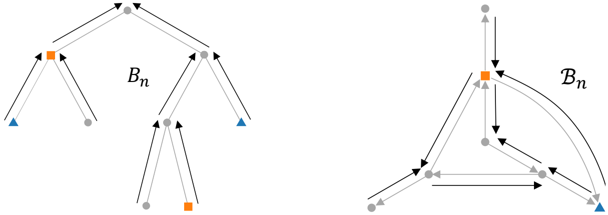

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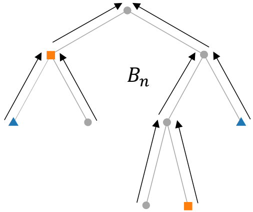

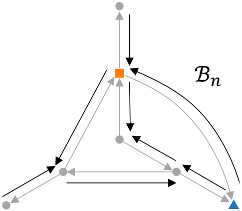

- **Trees**: Converges in d_n steps

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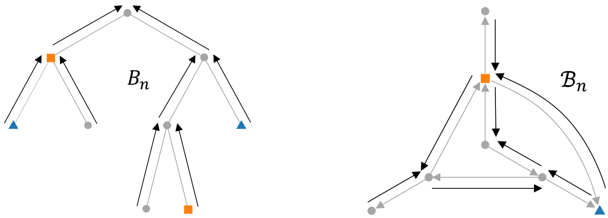

- **Trees**: Converges in d_n steps
- **Graphs**: May converge in ∞ steps when there is a loop

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Theorem (Sample complexity of OPD, Hren and Munos, 2008)

$$r_n = \widetilde{\mathcal{O}}\left(n^{-\log\frac{1}{\gamma}/\log\kappa}\right),$$

where κ is a problem-dependent difficulty measure.

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Theorem (Sample complexity of `GBOP-D`)

$$r_n = \widetilde{\mathcal{O}}\left(n^{-\log\frac{1}{\gamma}/\log\kappa_\infty}\right),$$

where κ_∞ is a tighter problem-dependent difficulty measure:

$$\kappa_\infty \leq \kappa$$

- $\kappa_\infty = \kappa$ if the MDP has a tree structure

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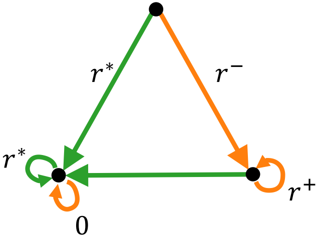
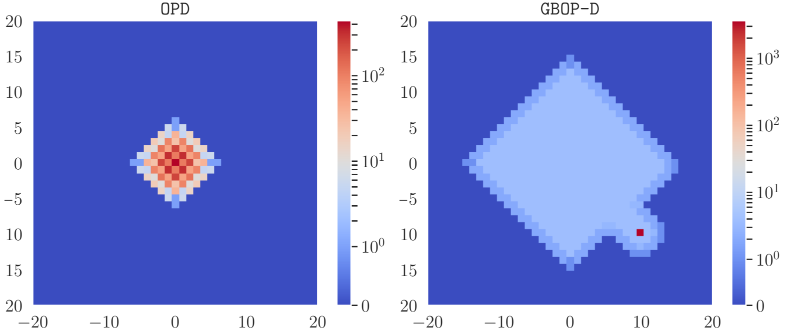
Illustrative example: 3 states, $K > 2$ actions



$$\kappa_\infty = 1 < \kappa = K - 1$$

Rewards in a ball around $(10, 10)$ of radius 5, with quadratic decay

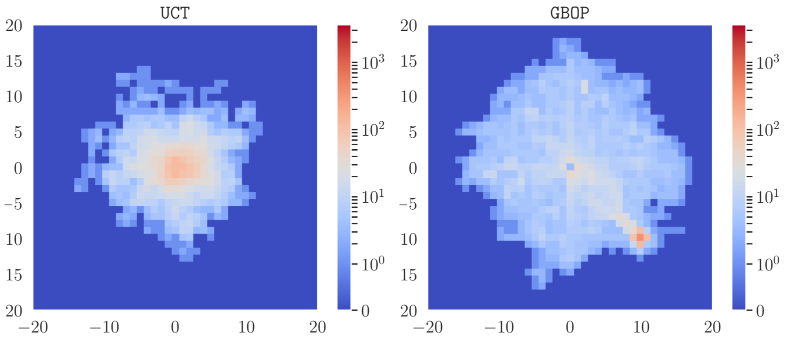


$n = 5640$ samples

Extension to stochastic MDPs

- Use state similarity to tighten the bounds $L \leq V \leq U$.
- We adapt `MDP-GapE` (Jonsson et al., 2020) to obtain `GBOP`

Noisy transitions with probability $p = 10\%$



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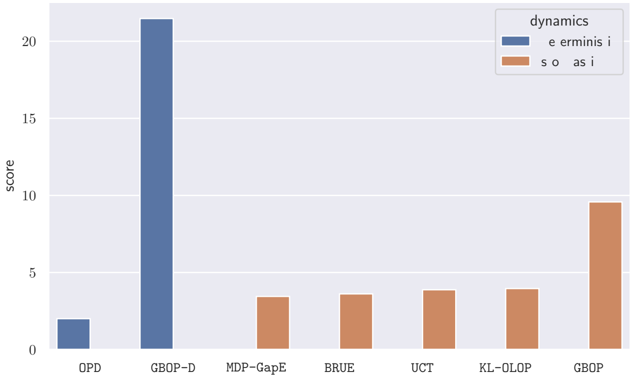
Exploration-Exploitation score

$$S = \sum_{t=1}^{n} \underbrace{d(s_t, s_0)}_{\text{Exploration}} - \underbrace{d(s_t, s_g)}_{\text{Exploitation}}$$



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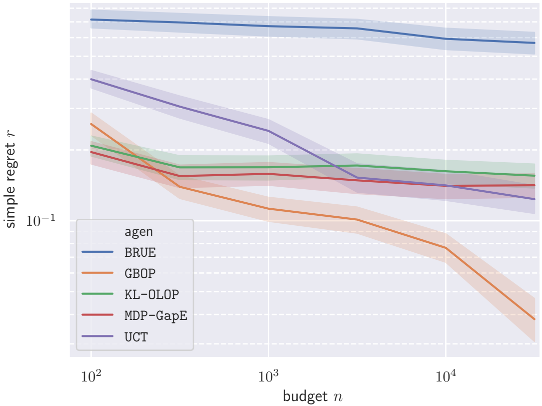
Sailing Domain (Vanderbei, 1996)



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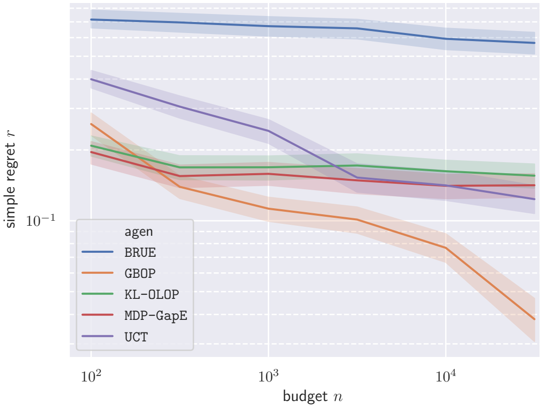

Effective branching factor κ_e:

- $\kappa_e \approx 3.6$, for BRUE, KL-OLOP, MDP-GapE, UCT
- $\kappa_e \approx 1.2$ for GBOP, which suggests our results may still hold