Week 4: Linear algebra never ends…

Mathematics and Computational Methods for Complex Systems, 2023

Dhruva V. Raman

Notebook anxiety

Am I completing the notebooks? ✗

Am I learning? ✓

Suggested strategy

…for you

Plan time per week on notebooks

Skim what you haven’t covered

Don’t worry about exam

Don't worry about the exam

School is for knowledge

Cover syllabus

Get answers right

Don’t worry about the exam

School is for knowledge

Cover syllabus

Get answers right

Life is for goals

Figure out required knowledge

Quickly acquire it

You don’t have to memorise anything!

(Although some knowledge helps)

↓

Get better at solving things!

(By practicing!)

Suggested strategy

…for me

Stratify questions
by difficulty

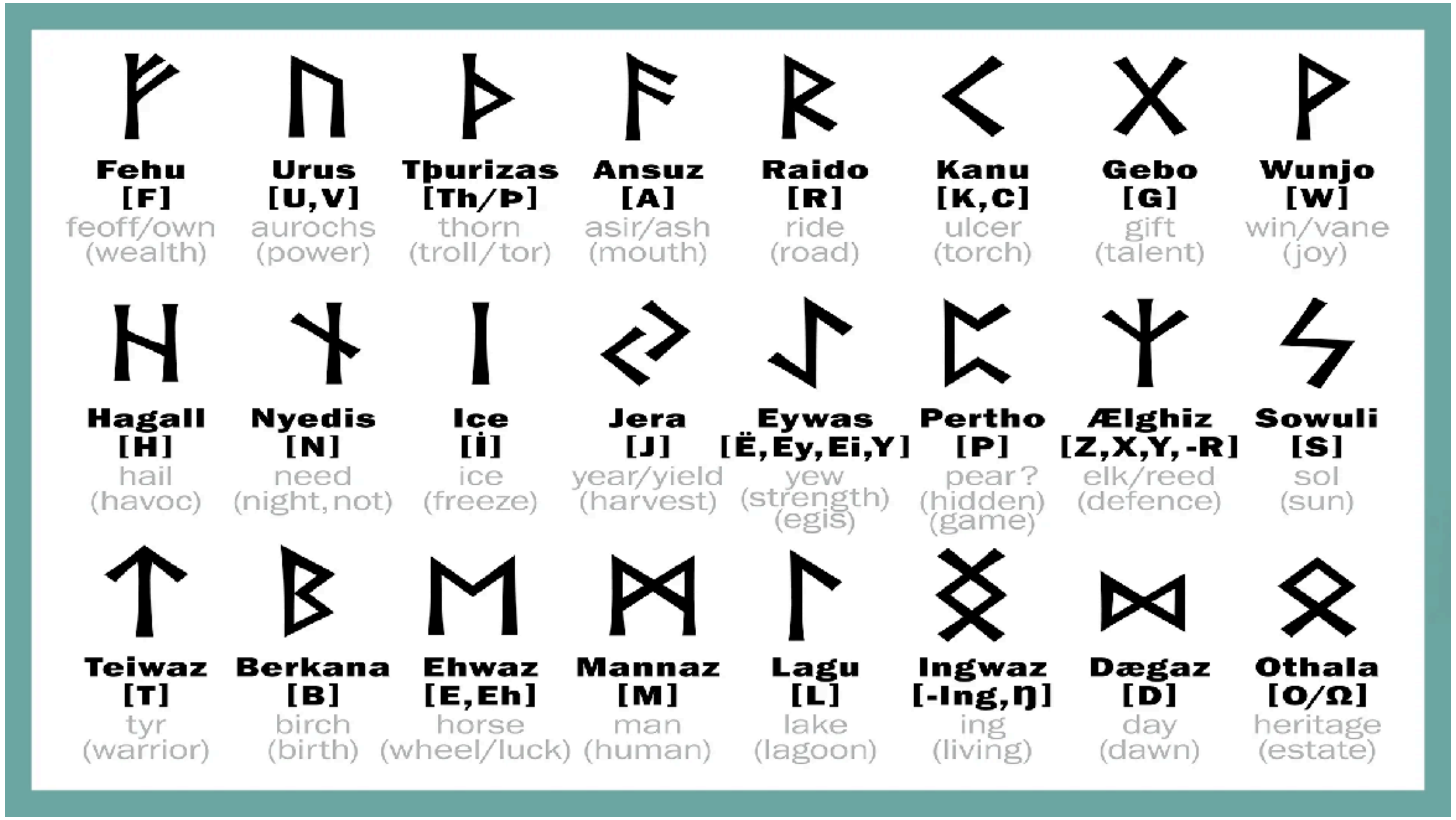
Summary of core
vs extension topics

Simultaneous
questions and answers

Although what you learn <
How quickly you learn
maths/programming topics

Notebooks are to
improve this!

Norse runes


Fehu
[F]
feoff/own
(wealth)
Urus
[U,V]
aurochs
(power)
Tþurizas
[Th/Þ]
thorn
(troll/tor)
Ansuz
[A]
asir/ash
(mouth)
Raido
[R]
ride
(road)
Kanu
[K,C]
ulcer
(torch)
Gebo
[G]
gift
(talent)
Wunjo
[W]
win/vane
(joy)
Hagall
[H]
hail
(havoc)
Nyedis
[N]
need
(night, not)
Ice
[İ]
ice
(freeze)
Jera
[J]
year/yield
(harvest)
Eywas
[Ë,Ey,Ei,Y]
yew
(strength)
(egis)
Pertho
[P]
pear?
(hidden)
(game)
Ælghiz
[Z,X,Y,-R]
elk/reed
(defence)
Sowuli
[S]
sol
(sun)
Teiwaz
[T]
tyr
(warrior)
Berkana
[B]
birch
(birth)
Ehwaz
[E,Eh]
horse
(wheel/luck)
Mannaz
[M]
man
(human)
Lagu
[L]
lake
(lagoon)
Ingwaz
[-Ing,ŋ]
ing
(living)
Dægaz
[D]
day
(dawn)
Othala
[O/Ω]
heritage
(estate)

Most soul-destroying activity:

Learning things for no reason

Are you in this position?

I’m wasting my time on Julia

(Everything I’ve said before, plus:)

You’re learning generally applicable programming patterns

Look at worksheet answers, even if you solved it yourself!

I’m not learning about complex systems

Doing ‘complex systems’ research project in 6 months. What to learn?

Core knowledge

EG Linear algebra, probability, etc

Programming maths solutions

Effectively, efficiently

Writing maths

LaTeX, general principles

(Use overleaf.com or some LaTeX editor for your masters project)

I'm not learning about complex systems

Each of these are infinitely improvable! *(except last)*

Only then, actual complex systems material

Core knowledge

EG Linear algebra, probability, etc

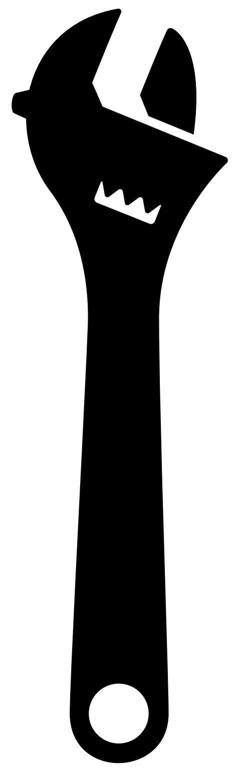
Programming maths solutions

Effectively, efficiently

Writing maths

LaTeX, general principles

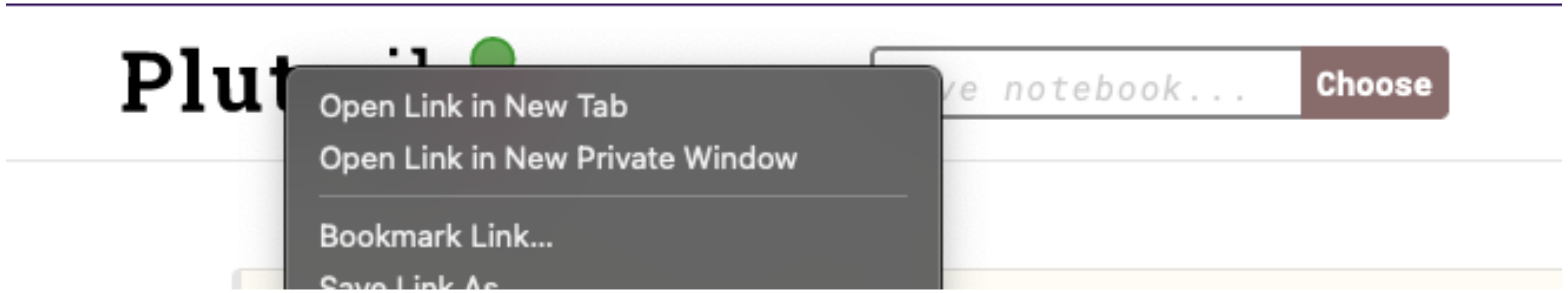
Feedback



Don’t be shy!

Tips

Separate windows for
questions and answers



Tips

Choose unique variable names

Multiple definitions for b

Combine all definitions into a single reactive cell using a `begin ... end` block.

```
begin
    b = 4
    c = 6
end
```

Tips

Don’t ignore error information

```
MethodError: no method matching +(::Set{Int64}, ::Int64)
Closest candidates are:
+(::Any, ::Any, !Matched::Any, !Matched::Any...)
@ Base operators.jl:578
+(!Matched::T, ::T) where T<:Union{Int128, Int16, Int32, Int64, I
UInt32, UInt64, UInt8}
@ Base int.jl:87
+(!Matched::Distributions.Normal, ::Real)
@ Distributions ~/.julia/packages/Distributions/Ufrz2/src/univari
/normal.jl:112
...

  1. top-level scope @ Local: 1  [inlined]

+(Set((1,)), 4)
```

Today

Matrix multiplication, inverses

Solving matrix equations

Feel free to leave. Honestly.

Recap: dot product

$$\underline{a} = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$$

$$\underline{x} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

Inner product often written as:

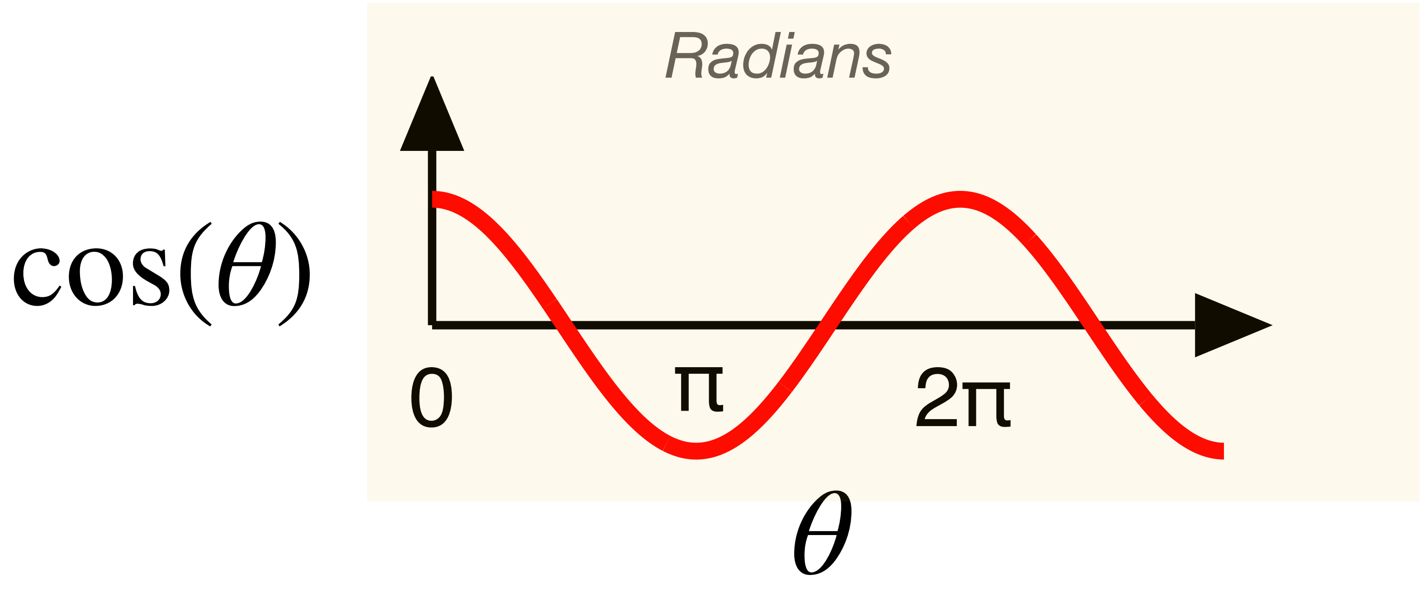
$$\langle \underline{a}, \underline{x} \rangle := \underline{a}^T \underline{x}$$

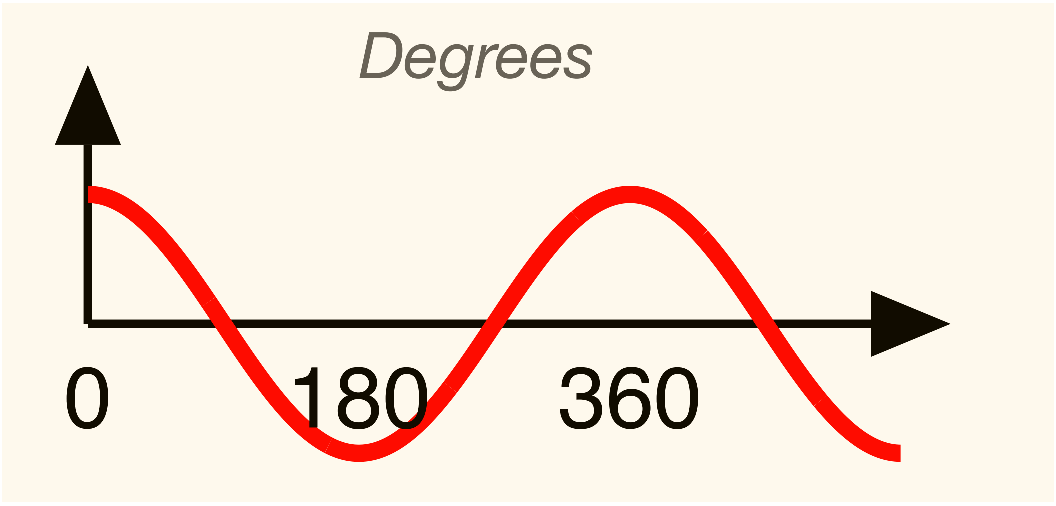
$$[a, b, c, d] \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$= aw + bx + cy + dz$$

Recap: dot product

$$\langle \underline{v}, \underline{w} \rangle = \|v\|_2 \|w\|_2 \cos(\theta)$$

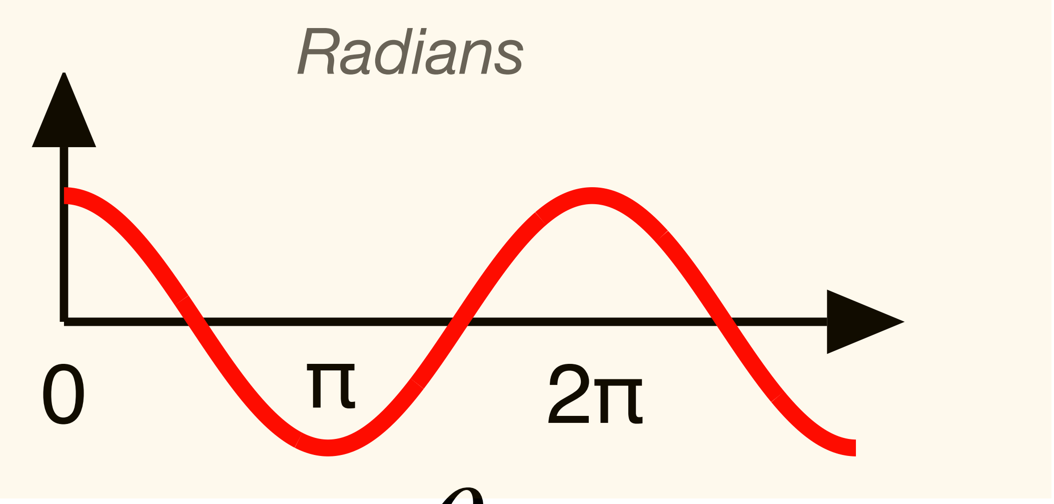




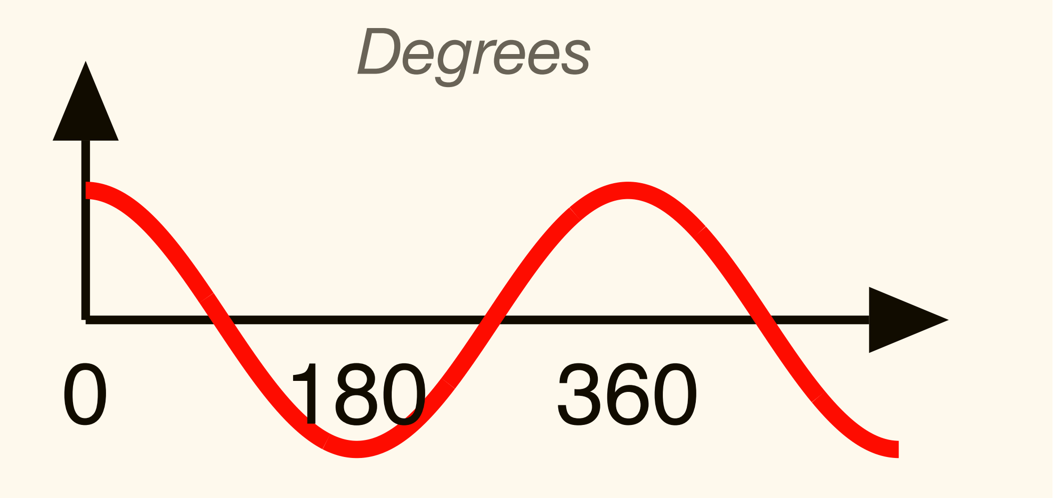
Recap: dot product

$$\langle \underline{v}, \underline{w} \rangle = \|v\|_2 \|w\|_2 \cos(\theta)$$

$\cos(\theta)$

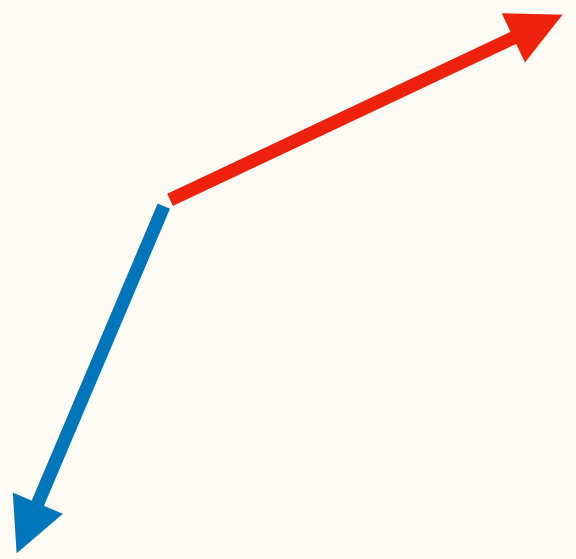


θ



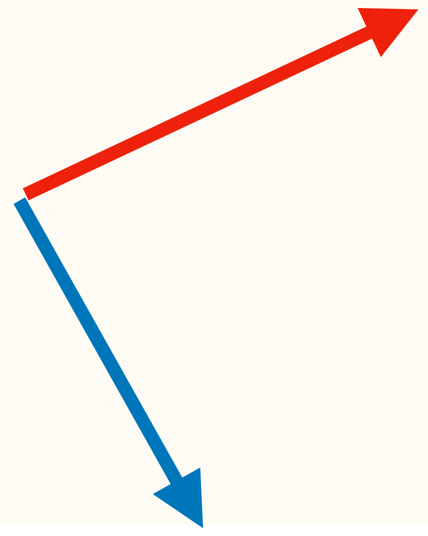
Anticorrelated

$$\langle \underline{v}, \underline{w} \rangle < 0$$



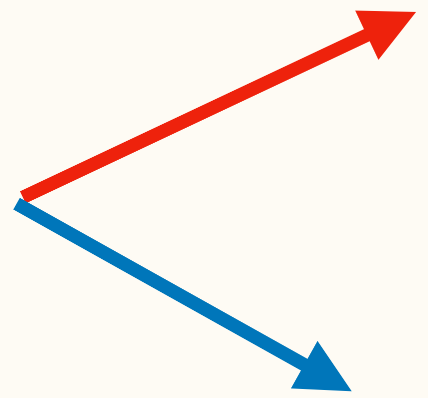
Uncorrelated

$$\langle \underline{v}, \underline{w} \rangle = 0$$



Correlated

$$\langle \underline{v}, \underline{w} \rangle > 0$$



Dot product?

$$\langle \underline{v}, \underline{w} \rangle = \|v\|_2 \|w\|_2 \cos(\theta)$$

Dot product?

$$\langle \underline{v}, \underline{w} \rangle = \|v\|_2 \|w\|_2 \cos(\theta)$$

What's the point of a matrix?

1. Data storage ✓

Stored as a matrix of pixels

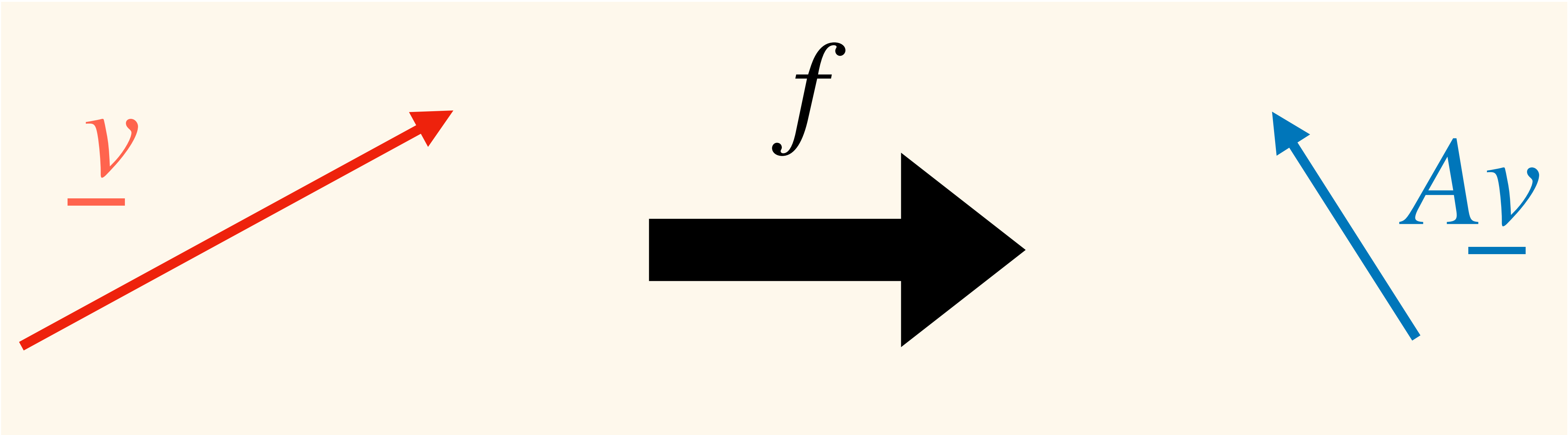
What's the point of a matrix?

2. A function that transforms vectors

$$f(\underline{v}) = A\underline{v} \qquad (= A \times \underline{v})$$

Vector

Matrix



Matrix multiplication

= *multiplying a matrix by a vector*

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = ?$$

$$A \qquad \underline{v}$$

Matrices are collections of vectors

But two options!

A 2x3 matrix

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix}$$

- *Two row vectors*

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix}$$

- *Three column vectors*

Matrix multiplication

Inner product of each row vector
in the matrix, with the input vector

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 8 + 8 + 15 \\ -6 + 24 + 3 \end{bmatrix}$$

Matrix multiplication

Inner product of each row vector in the matrix, with the vector

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 8+8+15 \\ -6+24+3 \end{bmatrix}$$

Requirement

number of matrix columns = length of input vector

Outcome

number of matrix rows = length of output vector

Square matrix properties

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \\ 1 & 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \\ \bullet \end{bmatrix}$$

Input and output vectors have same shape

Non-square matrices

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} \qquad \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}$$

Shape?

Non-square matrices

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} \bullet \\ \bullet \end{bmatrix}$$

$$f(v) = Av$$

$$f : \mathbb{R}^3 \to \mathbb{R}^2$$

f is a function between vector spaces

Matrices can multiply matrices!

Equal!

Number of columns

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -2 \\ 3 & -3 \end{bmatrix}$$

Number of rows

How to multiply matrices

1. Figure out the **shape** of the output

(m x n) matrix multiplied by (n x p) matrix = (m x p) matrix

n=3 columns

p=2 rows

m=2 rows

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 4 & -2 \\ 3 & -3 \end{bmatrix}$$

n=3 rows

***Requirement**: Number of columns (A) = Number of rows (B)*

How to multiply matrices

a) Figure out the shape of the output

$$\underset{A}{\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix}} \underset{B}{\begin{bmatrix} 2 & 1 \\ 4 & -2 \\ 3 & -3 \end{bmatrix}} = \underset{C}{\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \end{bmatrix}}$$

***Outcome**: Number of rows (A) x Number of columns (B)*

C_{ij} *Depends on i^{th} row and j^{th} column*

How to multiply matrices

b) Inner product of each row vector of A with each column vector of B

$$\overset{A}{\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix}} \overset{B}{\begin{bmatrix} 2 & 1 \\ 4 & -2 \\ 3 & -3 \end{bmatrix}} = \overset{C}{\begin{bmatrix} (8+8+15) & (4-4-15) \\ (-6+24+3) & (-3-12-3) \end{bmatrix}}$$

$$= \begin{bmatrix} 31 & -19 \\ 21 & -18 \end{bmatrix}$$

$$C_{ij} = \langle A_{i,\bullet}, B_{\bullet,j} \rangle$$

i^{th} row *j^{th} column*

How to multiply matrices

Question: Is this allowable?

$$[2,3] \quad \begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix}$$

How to multiply matrices

Question: Is this allowable?

$$[2,3] \begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} = [\,\bullet\,,\,\bullet\,,\,\bullet\,]$$

Transposition

(= swap rows and columns)

$$\begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix}^T = \begin{bmatrix} 4 & -3 \\ 2 & 6 \\ 5 & 1 \end{bmatrix}$$

$$[2,3]^T = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\left(A^T\right)^T = ?$$

Law of matrix transposition

Verify yourself:

$$\left([2,3] \begin{bmatrix} 4 & 2 & 5 \\ -3 & 6 & 1 \end{bmatrix} \right)^T = \begin{bmatrix} 4 & -3 \\ 2 & 6 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Law of matrix transposition

$$\left(A\underline{v}\right)^T = \underline{v}^T A^T$$

$$(ABCD)^T = D^T C^T B^T A^T$$

Transposition swaps the multiplication order

Important: non-commutativity of multiplication

i.e. $AB \neq BA$ *unlike scalars*

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = ?$$

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = ?$$

Different row/column vectors!

The identity matrix

I_n **is an** $(n \times n)$ **matrix with ones on the diagonal**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_n = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & \dots & \dots & 1 \end{bmatrix}$$

Exercise:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = ? \qquad \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = ?$$

The identity matrix

I_n **is an** $(n \times n)$ **matrix with ones on the diagonal**

$$AI = IA = A$$

- analogous to the number 1 in scalar multiplication

$$x(1) = 1(x) = x$$

The scalar inverse

$$f(x) = 4x$$

$$g(x) = \frac{1}{4}x$$

$$f \circ g(x) = \frac{1}{4}(4x) = 1x = x$$

The scalar inverse

$$f(x) = 4x$$

$$g(x) = \frac{1}{4}x$$

$$\rightarrow$$

$$f \circ g(x) = \frac{1}{4}(4x) = 1x = x$$

$\frac{1}{4}$ *is the* *inverse* *of* 4

$$\Updownarrow$$

$f \circ g$ *is the* *identity function:* $f \circ g(x) = 1(x)$

The matrix inverse (for square matrices)

```
numpy.linalg.inv(A)
```

Inverse of A is A^{-1} implies:

$$AA^{-1} = A^{-1}A = I$$
$$\Rightarrow AA^{-1}\underline{x} = \underline{x}$$

Example (verify yourself)

$$\begin{bmatrix} -2.5 & 1.5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} -2.5 & 1.5 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The matrix inverse

…is useful for matrix equations

Problem: find x

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$$

$$Ax = b$$

Known

Unknown

Known

The matrix inverse

…is useful

General problem: find $\underline{x}$, given $\underline{b}$

$$A\underline{x} = \underline{b}$$

$$\Rightarrow A^{-1}A\underline{x} = \underline{x} = A^{-1}b$$

The matrix inverse

…is useful

Problem: find x

$$\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 13 \end{bmatrix}$$

Cancels to identity!

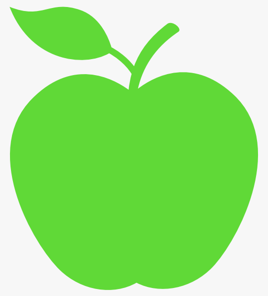
$$\boxed{\begin{bmatrix} -2.5 & 1.5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2.5 & 1.5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 13 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -2.5 & 1.5 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ 13 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

The matrix inverse

toy application

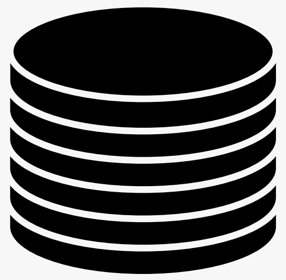
What combination of fruit do I buy to fill my bag?



$= £3.00/kg$

$= £2.00/kg$

holds $5kg$

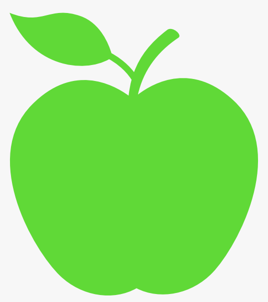


$£12.00$

Simultaneous equations

$$3x_1 + 2x_2 = 12$$

$$x_1 + x_2 = 5$$



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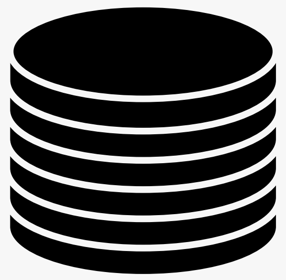
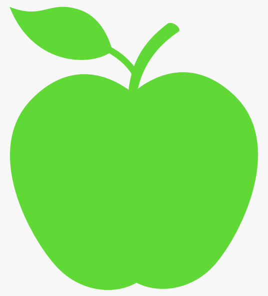


£12.00

Equivalent to a matrix problem

$$\begin{array}{ccc} A & \underline{x} & \underline{y} \\ \begin{bmatrix} £3.00/kg & £2.00/kg \\ 1 & 1 \end{bmatrix} & \begin{bmatrix} x_1 \; kg \\ x_2 \; kg \end{bmatrix} = & \begin{bmatrix} £12 \\ 5kg \end{bmatrix} \end{array}$$

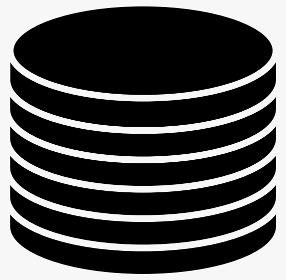
- *Rows are constraints*
- *Columns are free variables*



$= £3.00/kg$

$= £2.00/kg$

holds $5kg$



£12.00

Algebraic solution of matrix problem

…with inverses

Cancels to identity!

$$\overset{A^{-1}}{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}} \overset{A}{\begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \overset{A^{-1}}{\begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix}} \overset{b}{\begin{bmatrix} 12 \\ 5 \end{bmatrix}}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 12 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

Algebraic solution of matrix problem

…with inverses

Is a bad computational solution!

```
A = 2×2 Matrix{Int64}:
 3  2
 1  1
A = [3;1;;2;1]
```

```
xx = [2, 3]
xx = [2;3]
```

```
b = [12, 5]
b = [12;5]
```

```
[2.0, 3.0]
inv(A)*b
```

✗

```
[2.0, 3.0]
A \ b
```

✓

The matrix inverse

…doesn't always exist!

Proof by contradiction

Suppose the counterfactual is true

Matrix inverses always exist

$$\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}2\\-1\end{bmatrix}=\begin{bmatrix}1 & 2\\2 & 4\end{bmatrix}^{-1}\begin{bmatrix}0\\0\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

Contradiction!!

The matrix inverse

…doesn't always exist!

Square matrices

Invertible matrices

$$A\underline{x} = \underline{0} \Rightarrow \underline{x} = \underline{0}$$

Non-invertible matrices

$A\underline{x} = \underline{0}$ for nonzero $\underline{x}$

Array Algebra

$$A\underline{x} = \underline{y}B$$

$$\Rightarrow B^{-1}A\underline{x} = yBB^{-1}$$
$$\Rightarrow \underline{x} = \left(B^{-1}A\right)yBB^{-1}$$

Important note on array algebra

$$A\underline{x} = \underline{y}B$$

$$\Rightarrow B^{-1}A\underline{x} = \underline{y}BB^{-1}$$

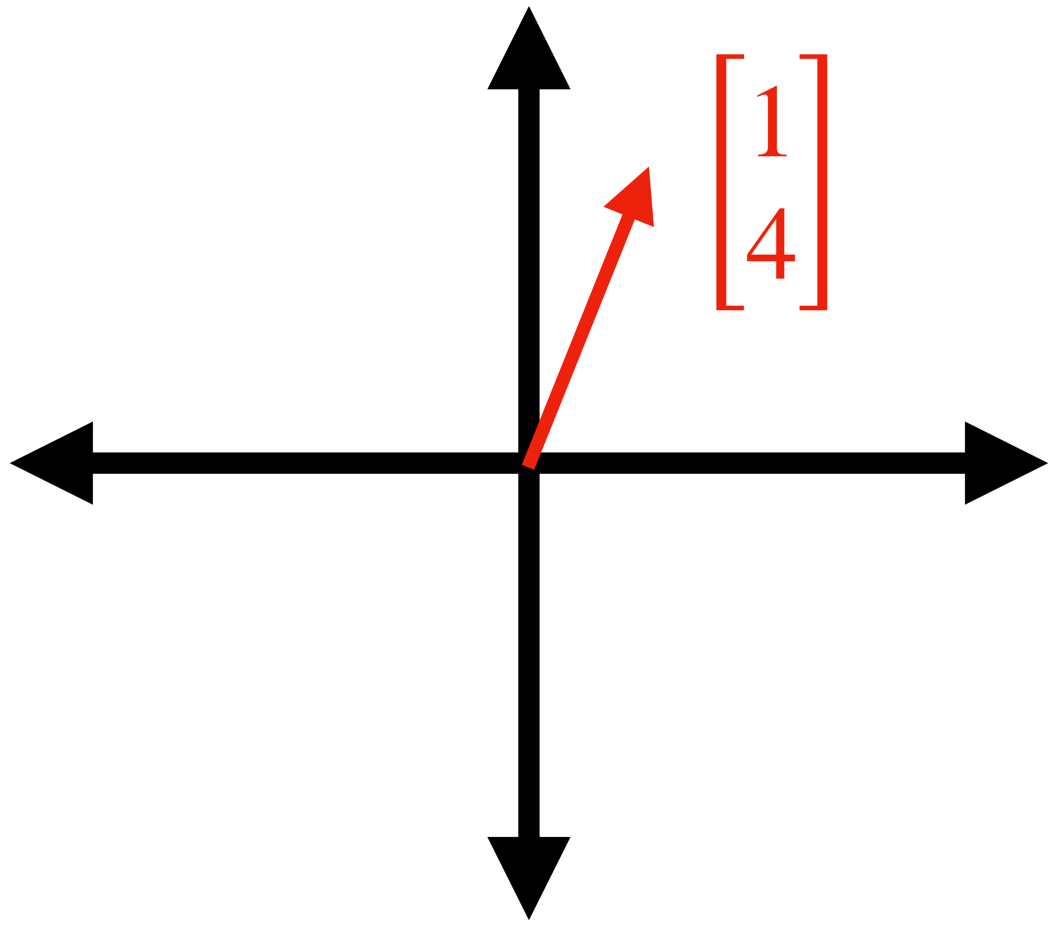
$$A^{-1}A\underline{x} = \underline{x} \neq A\underline{x}A^{-1}$$

Equation manipulation:

- *Can do same thing to both sides of the equation (like scalar algebra)*
- *Except, left multiplication and right multiplication are different*

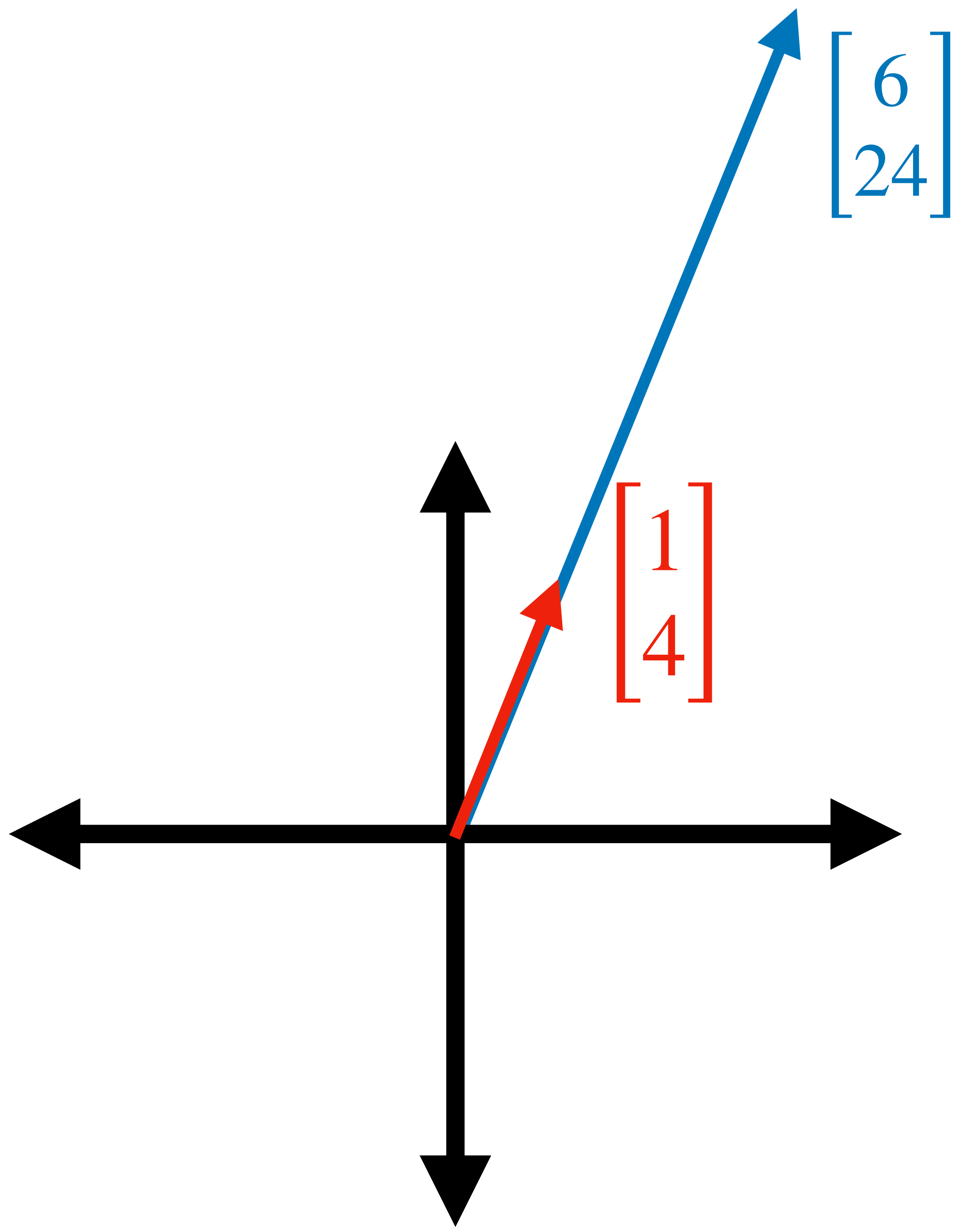
Worked examples

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = ?$$



Worked examples

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$



- *Pure scaling, no rotation*

Worked examples

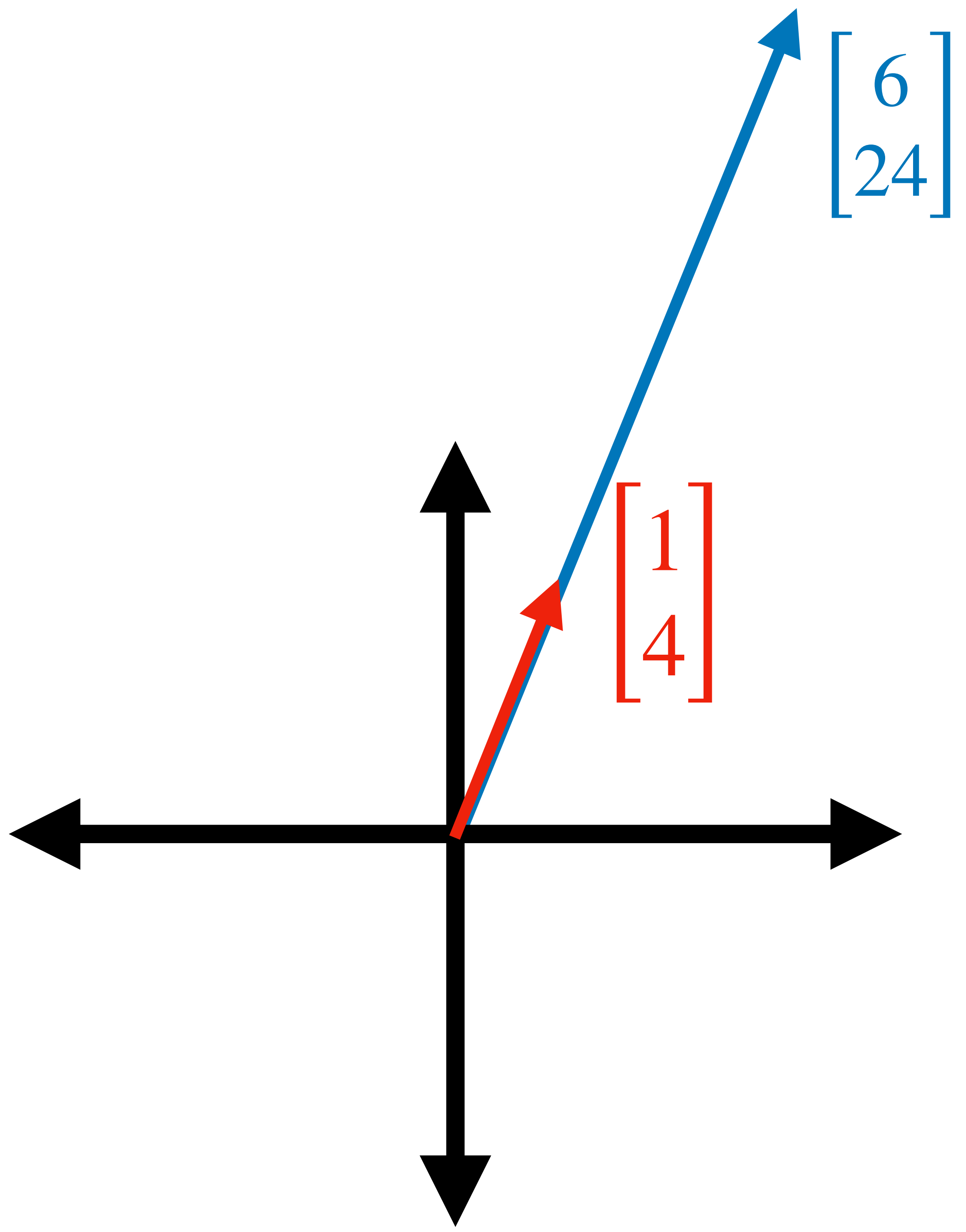
- *Eigenvectors of a matrix are those that don't rotate under transformation*

$$\begin{bmatrix} -6 & 3 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ 24 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$A\underline{v} = \lambda \underline{v}$$

Eigenvalue

Eigenvector



`numpy.linalg.eig(A)` `eigen(A)`

Worked examples

Exercise: find eigenvectors of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

- *Read: which vectors don't change direction under these transformations?*

Worked examples

- *Don't worry, more on eigenstuff coming*

Exercise: find eigenvectors of

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}$$

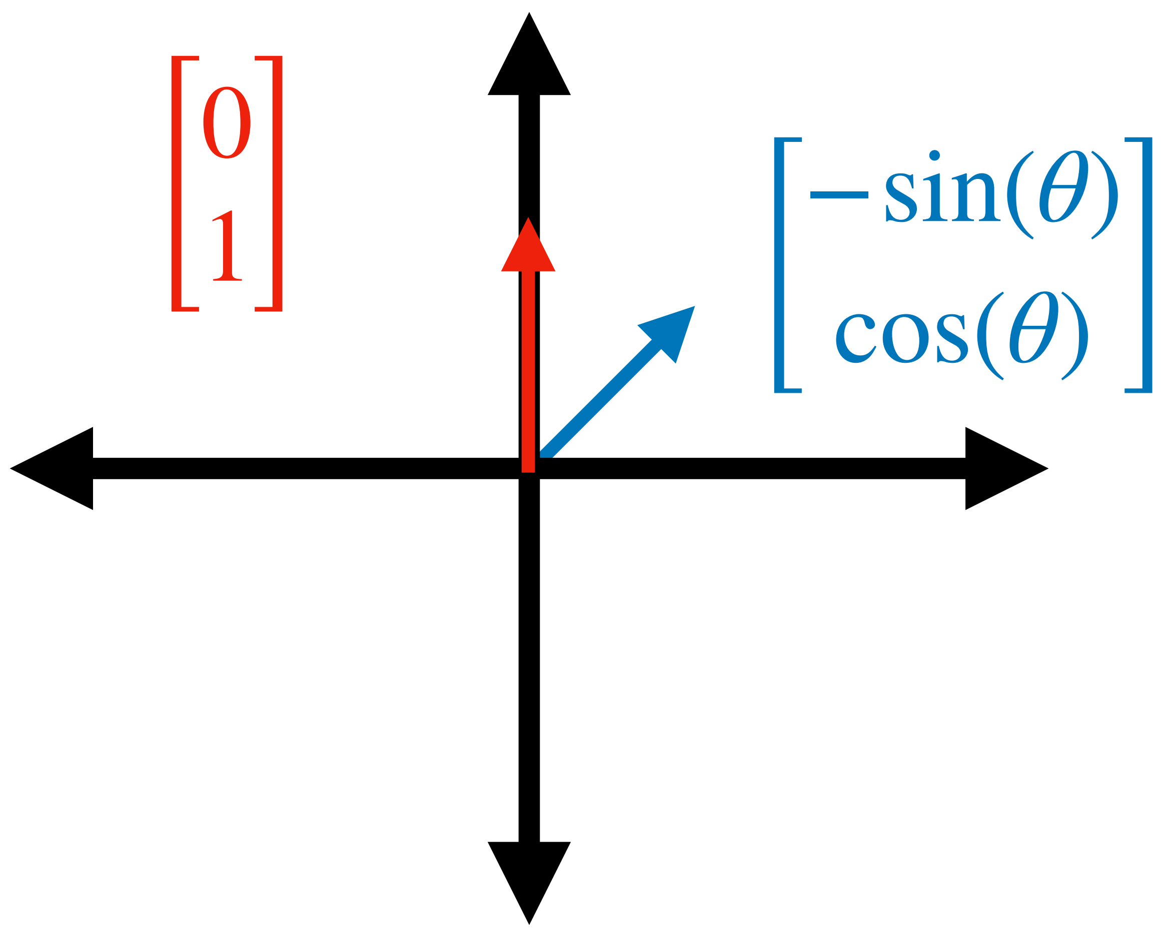
Everything! $I\underline{v} = \underline{v}$

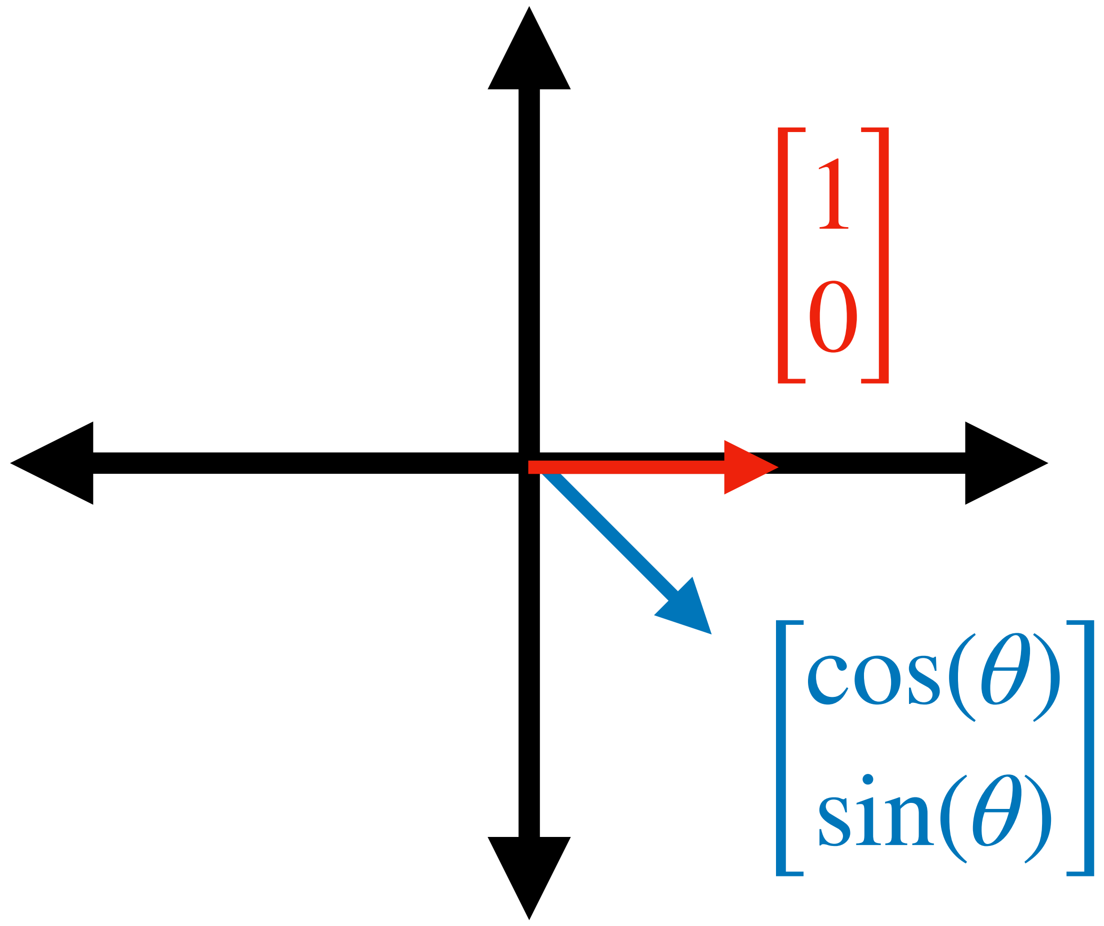
$$\begin{bmatrix} a \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ b \end{bmatrix}$$

Worked examples

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -\sin(\theta) \\ \cos(\theta) \end{bmatrix}$$



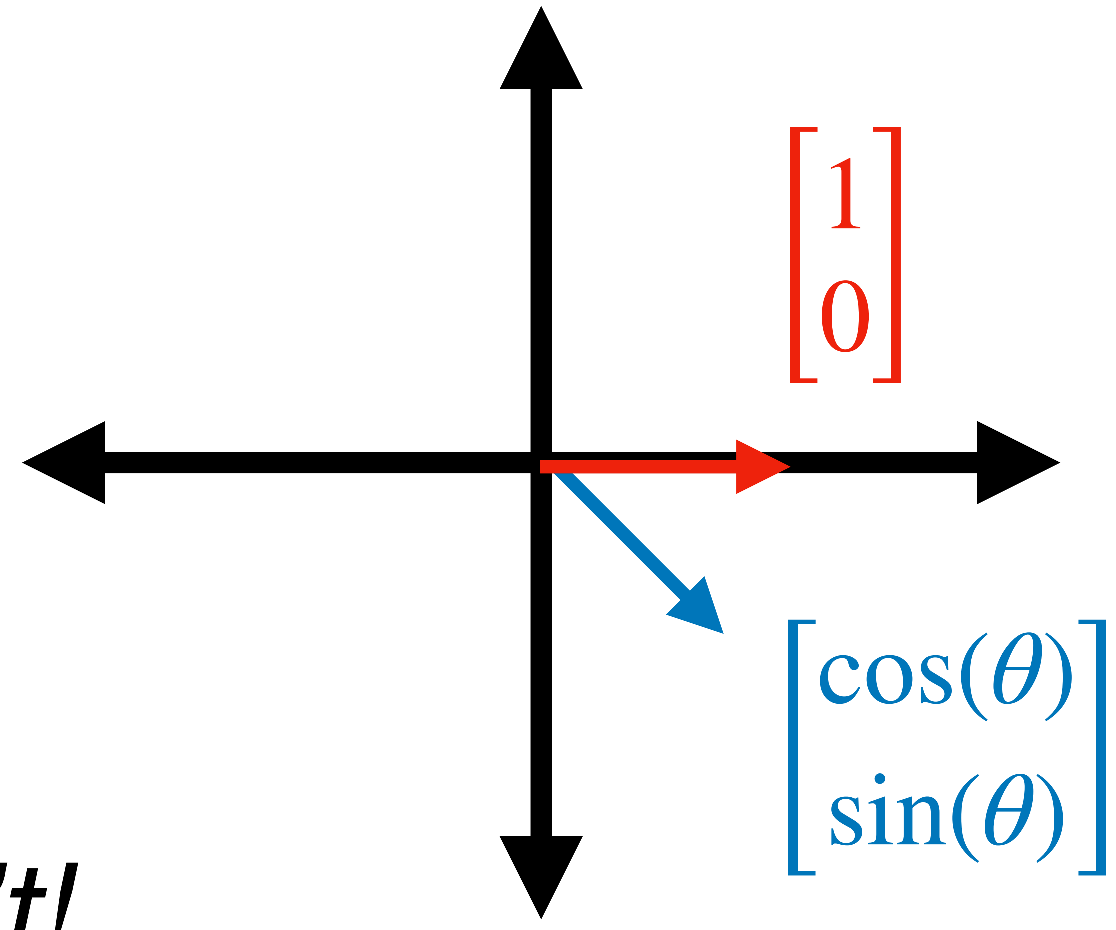


Worked examples

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

Check:

$$\left\| \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\|_2 = \left\| \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} \right\|_2$$

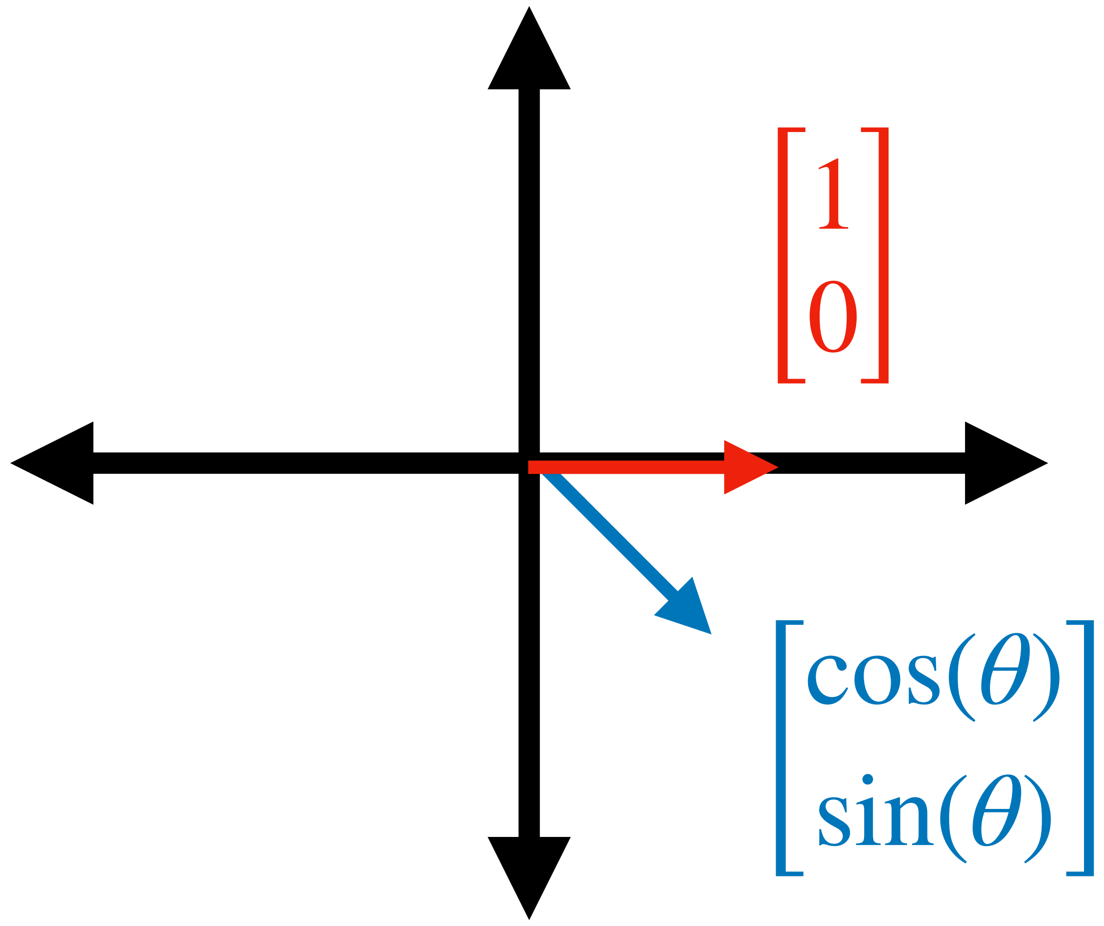


- *Google trigonometric identities if you can't!*

Worked examples

$$\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$$

- *Norm preserved, direction changed -> rotation*
- *Does this matrix rotate all vectors though?*



Worked examples

Properties of rotation:

Rotate pair => angle, size preserved

=> inner product preserved!

$$\langle x, y \rangle = \langle Ax, Ay \rangle$$

$$x^T y = x^T A^T A y$$

for all x, y

…but that means:

$$A^T A = I$$

Worked examples

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Worked examples

What's the inverse of a rotation matrix?

$$\langle x, y \rangle = \langle Ax, Ay \rangle$$

$$x^T y = x^T A^T A y$$

for all x, y

…but that means:

$$A^T A = I$$

Worked examples

What’s the inverse of a rotation matrix?

$$A^T A = I$$

$$AA^T = (A^T A)^T = I^T = I$$

$$A^T!$$

$$\langle x, y \rangle = \langle Ax, Ay \rangle$$

$$x^T y = x^T A^T A y$$

for all x, y

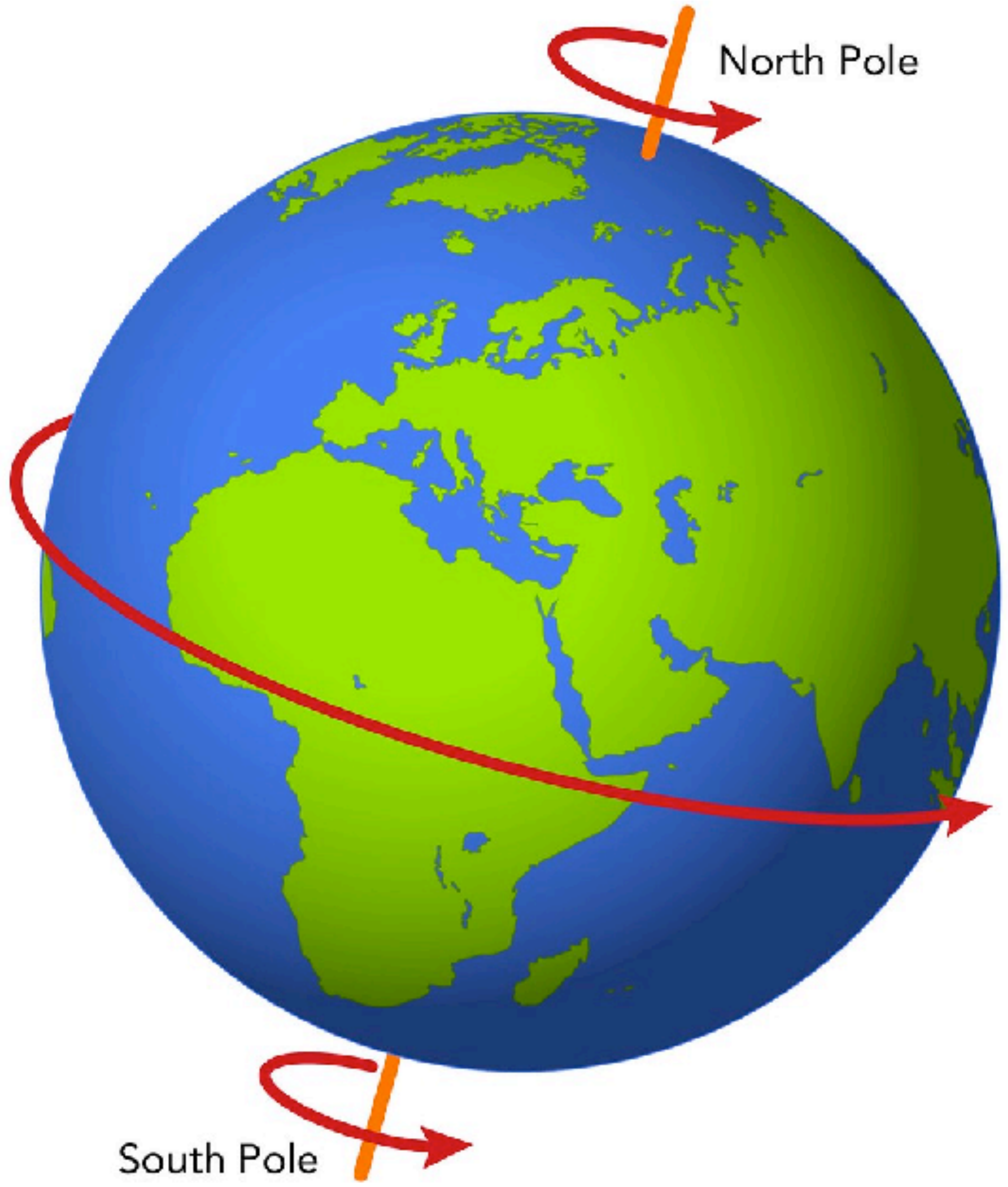
…but that means:

$$A^T A = I$$

Question

Geometrically, what are the eigenvectors of a rotation?

What are their eigenvalues?



More generally

…but don't need to learn this thoroughly!

Any matrix is a composition of
- **rotation 1**
- **scaling,**
- **rotation 2**

$$A = UDV^T$$
$$A\underline{x} = UDV^T\underline{x}$$

$U^TU = I$, $V^TV = I$, D diagonal

Rotation Rotation Scaling

Singular value decomposition

Any matrix is a composition of
- **rotation 1**
- **scaling,**
- **rotation 2**

$$A = UDV^T$$
$$A\underline{x} = UDV^T\underline{x}$$