Week 7

Mathematics and Computational Methods for Complex Systems, 2023-2024

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Outline of today

Differential quantities

What are they?

Operations and algebra

How do we compute them?

Differential equations

Optimisation

Machine learning

Course structure

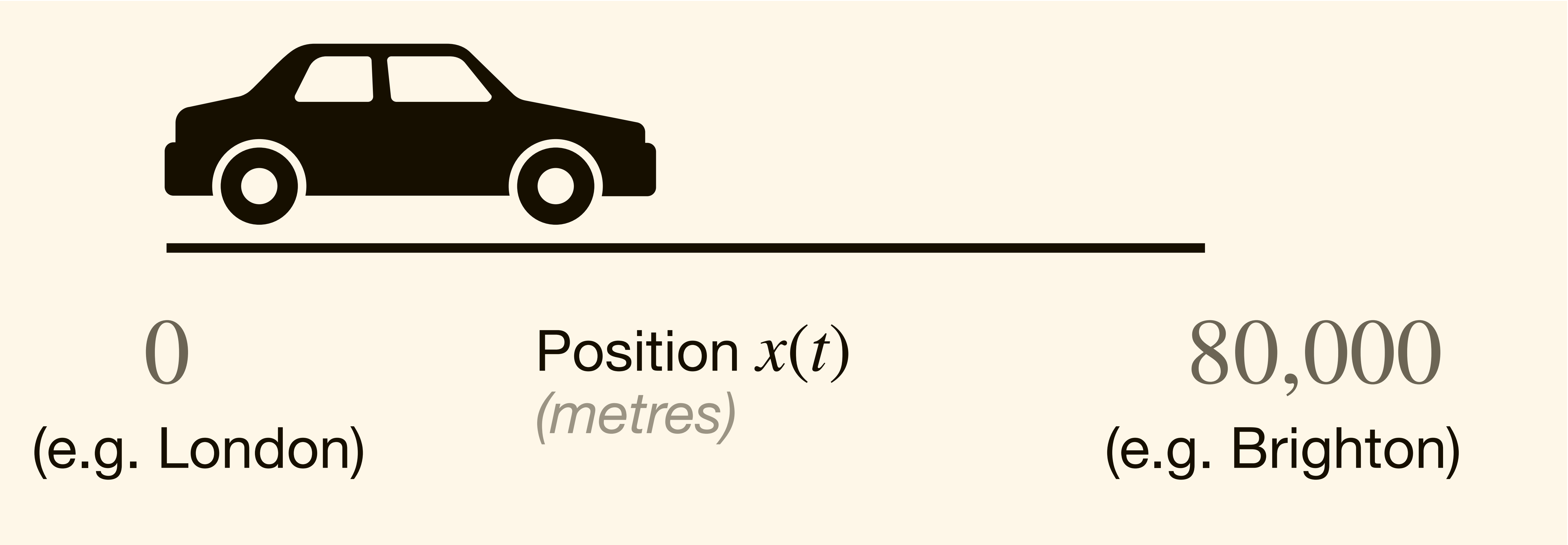
Dynamical systems

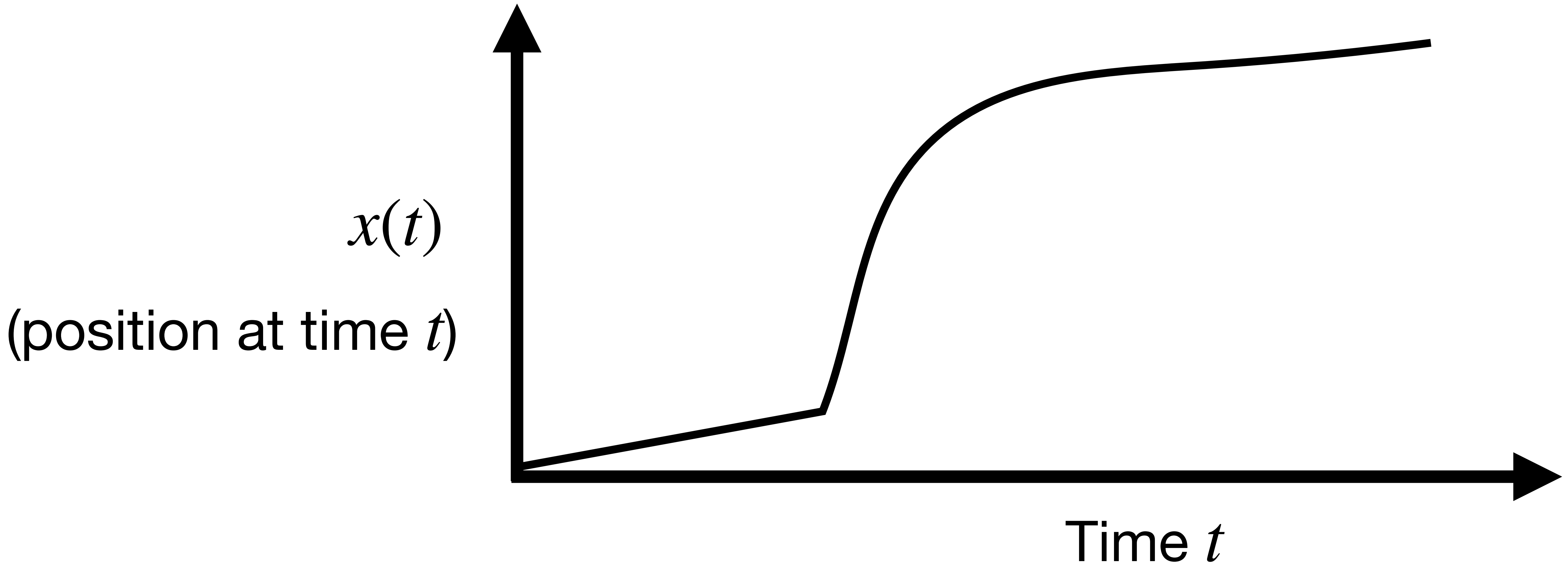
Optimisation

Complexity

Mid module feedback

Position of car as a function of time

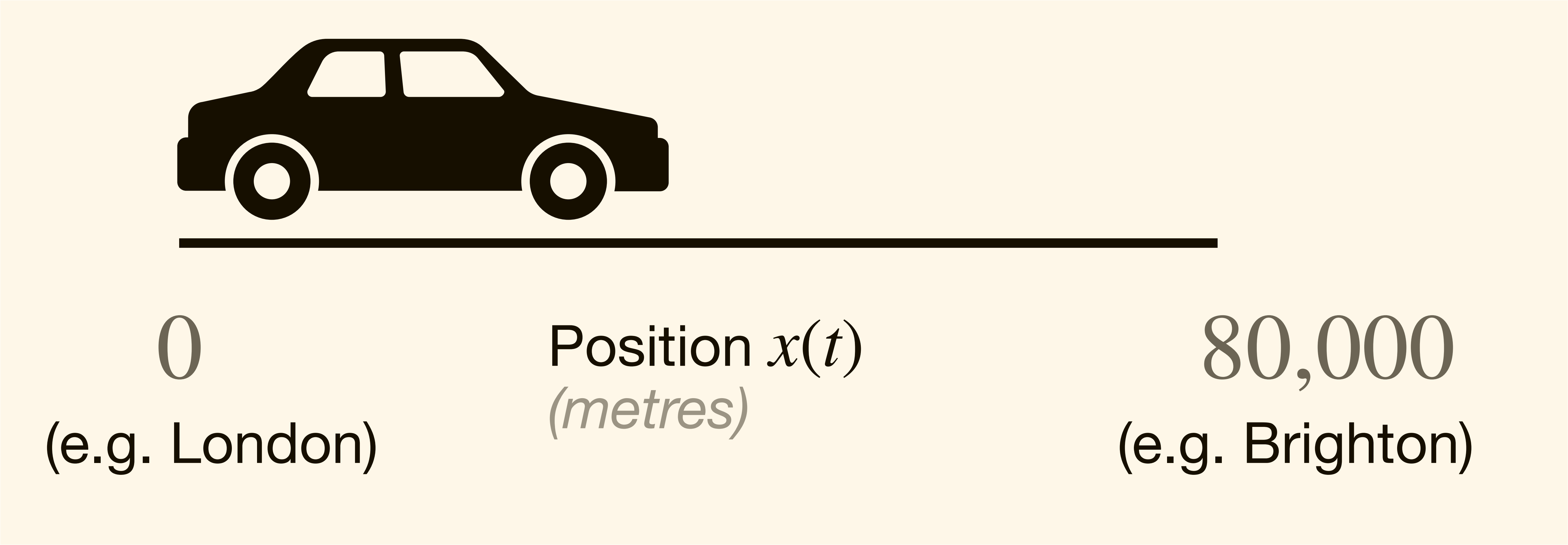




Velocity of car as a function of time

What does this even mean?

N.B. Velocity has a direction, so can be negative
Speed is the magnitude of velocity

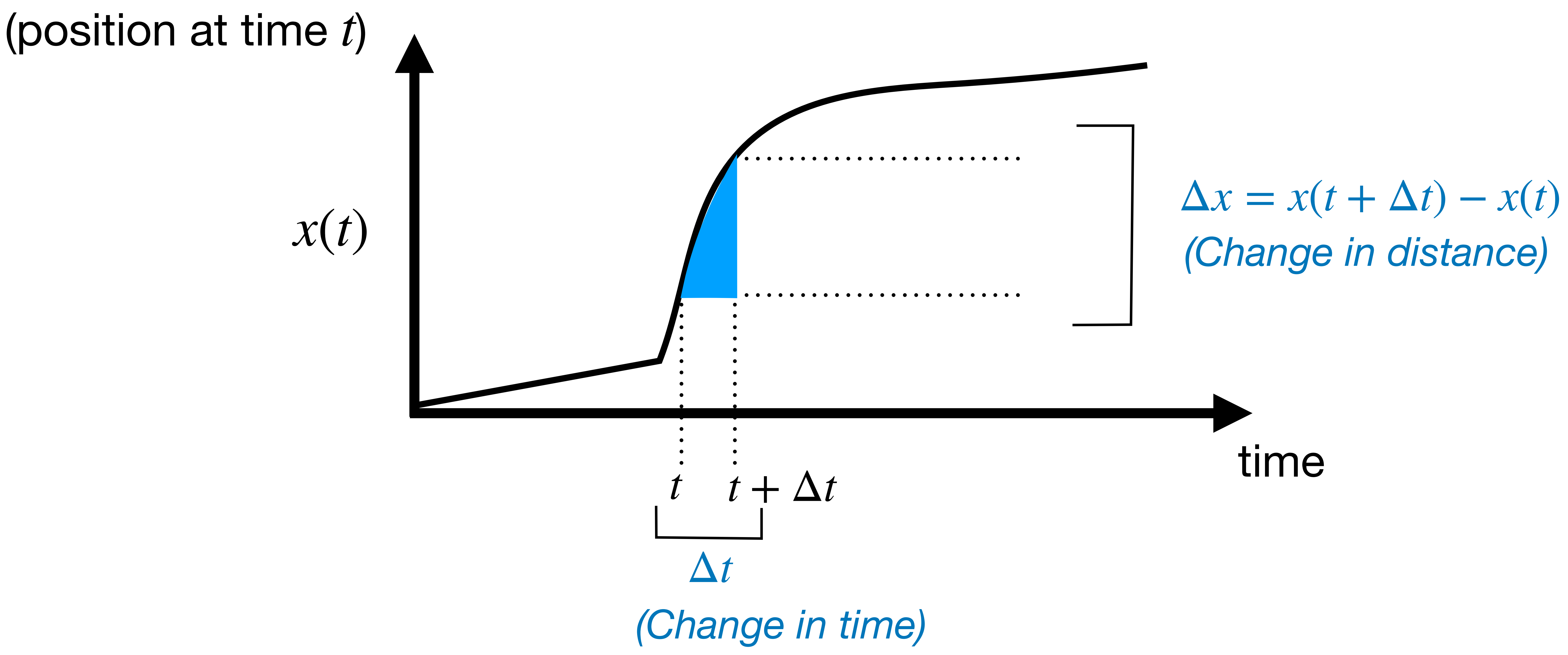


Differential quantity:

Velocity requires comparison of position at two points in time!

Average speed over a time interval

N.B. Δ often represents *'change in'*

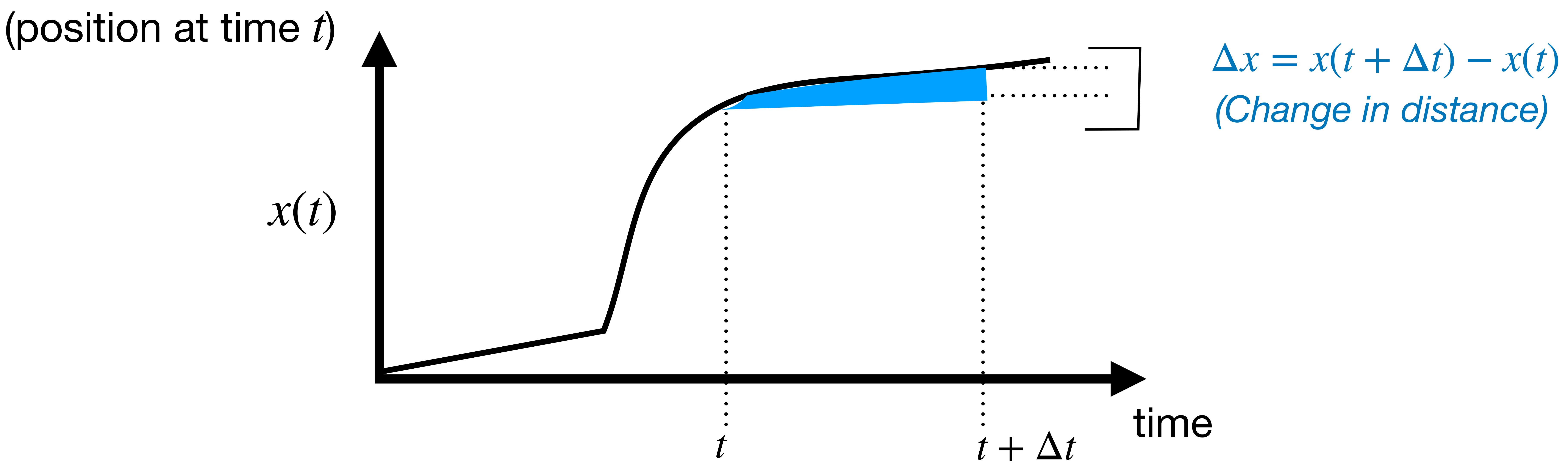


Average velocity between times t and $t + \Delta t$:

$$\frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Average speed over a time interval

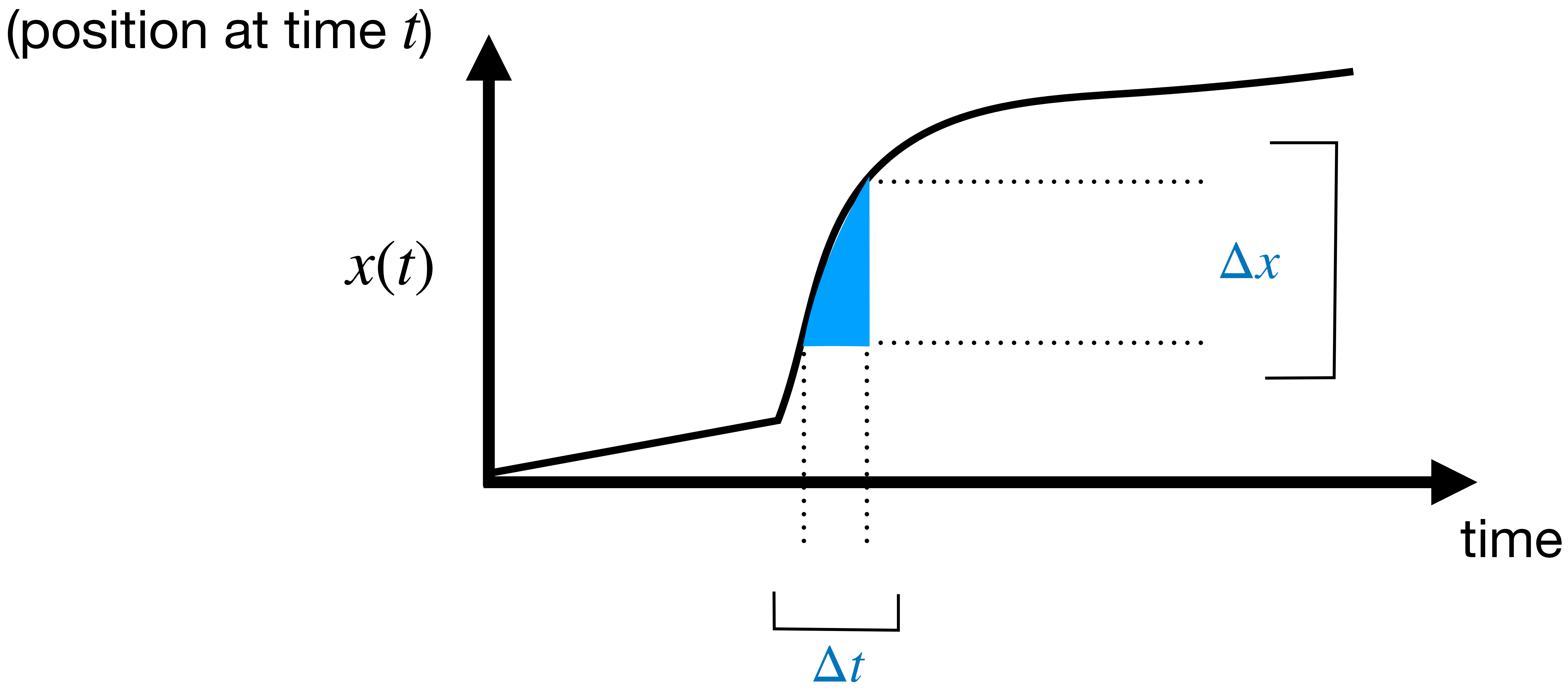
N.B. Δ often represents *'change in'*



Average velocity between times t and $t + \Delta t$:

$$\frac{x(t + \Delta t) - x(t)}{\Delta t}$$

Velocity at a point in time

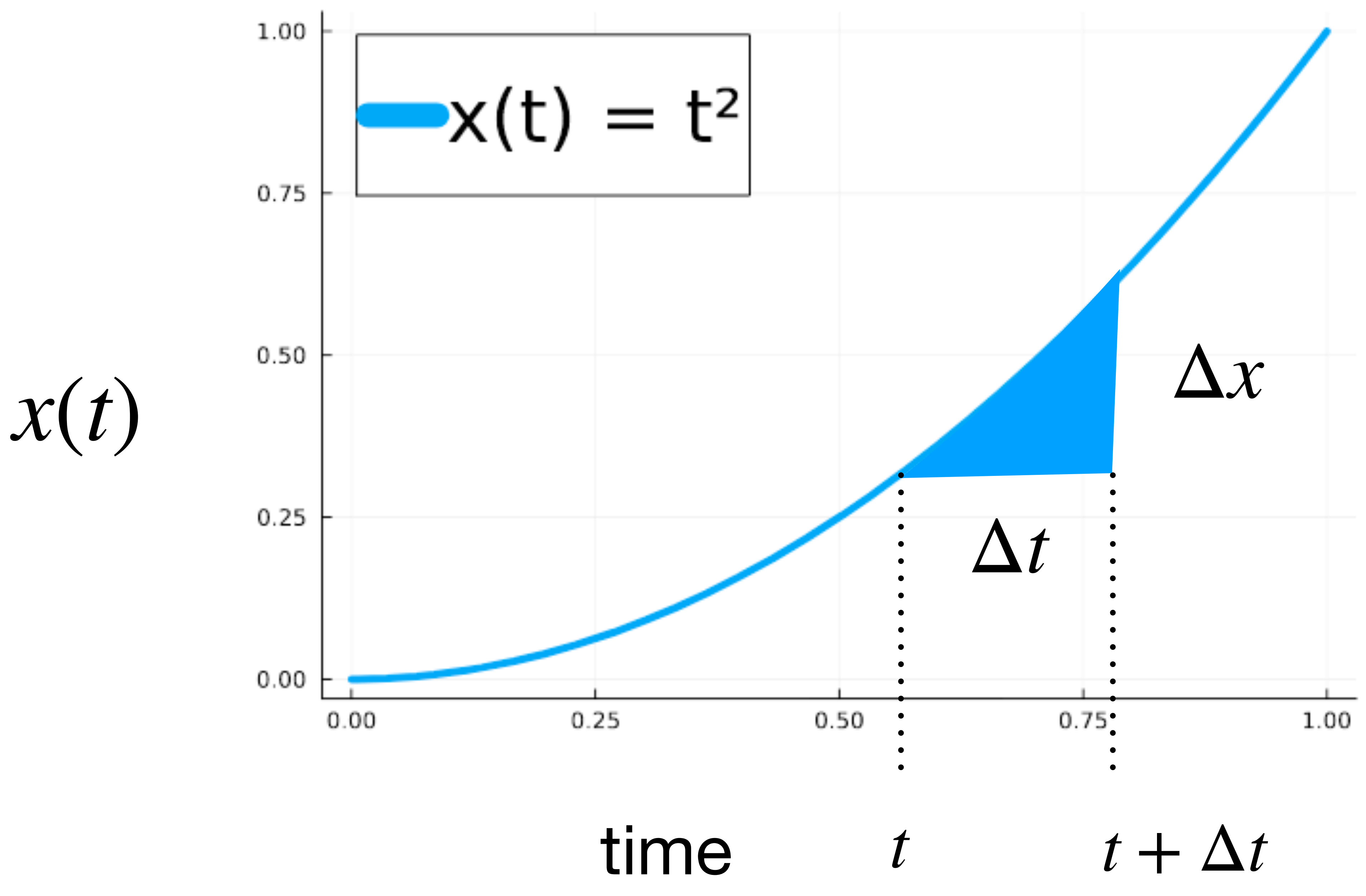


Velocity at time t : $$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

$$\left(= \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} \right)$$

An example

Suppose $x(t) = t^2$

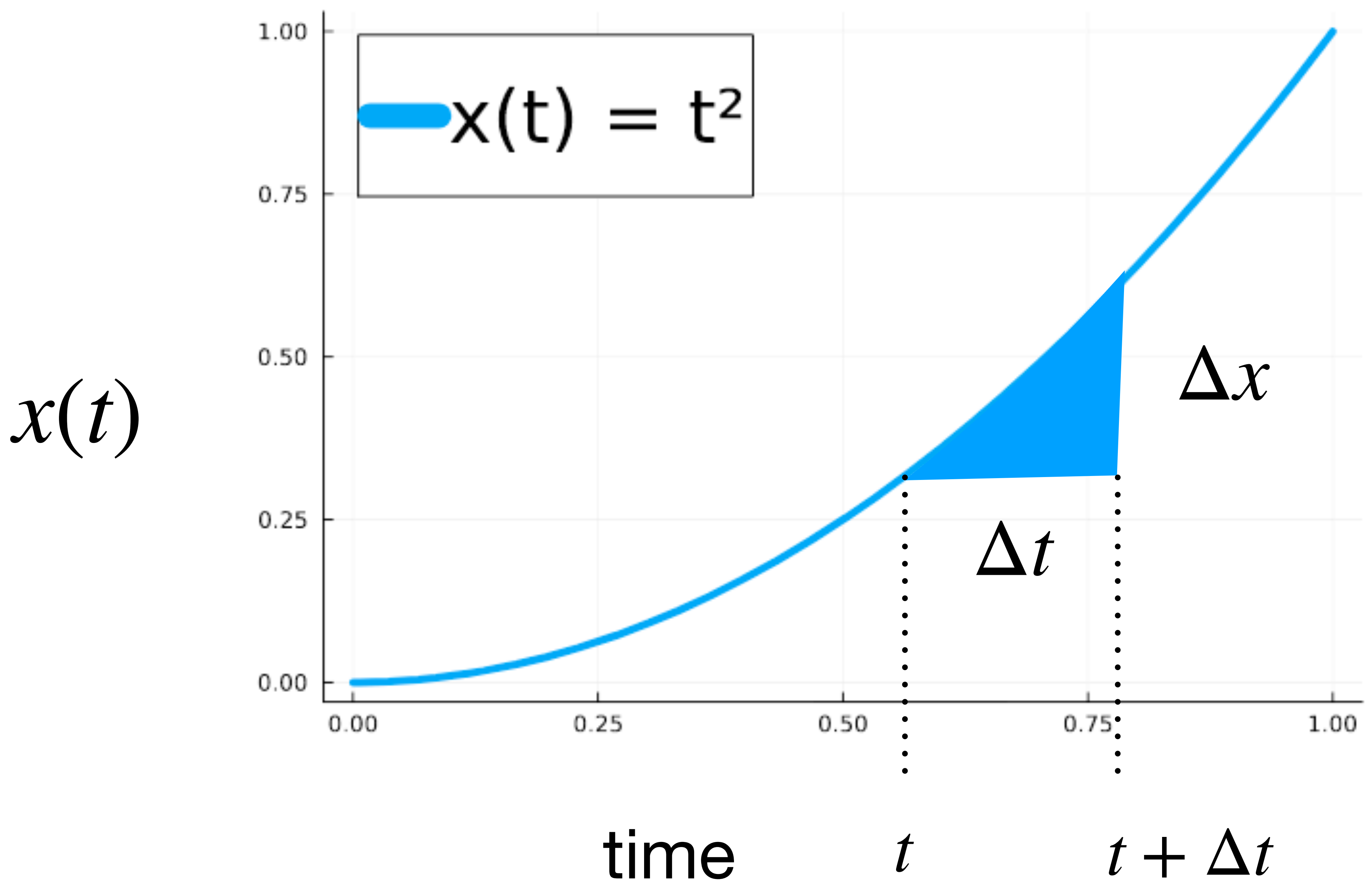


Velocity at time t : $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$

An example

Suppose $\quad x(t) = t^2$

Then $\quad \dfrac{x(t+\Delta t) - x(t)}{\Delta t} = \dfrac{(t+\Delta t)^2 - t^2}{\Delta t}$



Velocity at time t : $\quad \lim_{\Delta t \to 0} \dfrac{\Delta x}{\Delta t}$

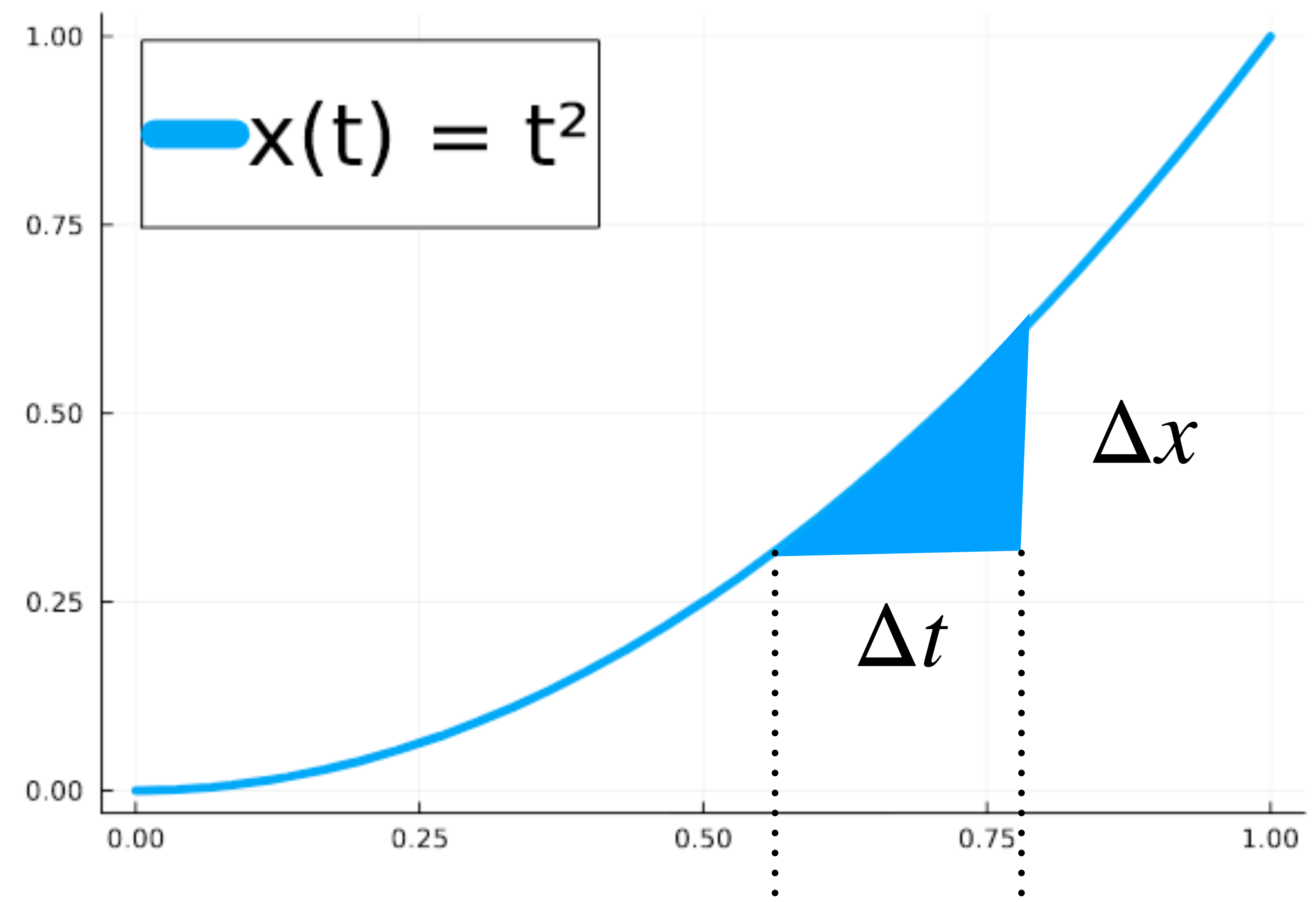
An example

Suppose $x(t) = t^2$

Then $$\frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{(t+\Delta t)^2 - t^2}{\Delta t}$$

$$= \frac{t^2 + 2t\Delta t + (\Delta t)^2 - t^2}{\Delta t}$$

$x(t)$



time t $t + \Delta t$

Velocity at time t : $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$

An example

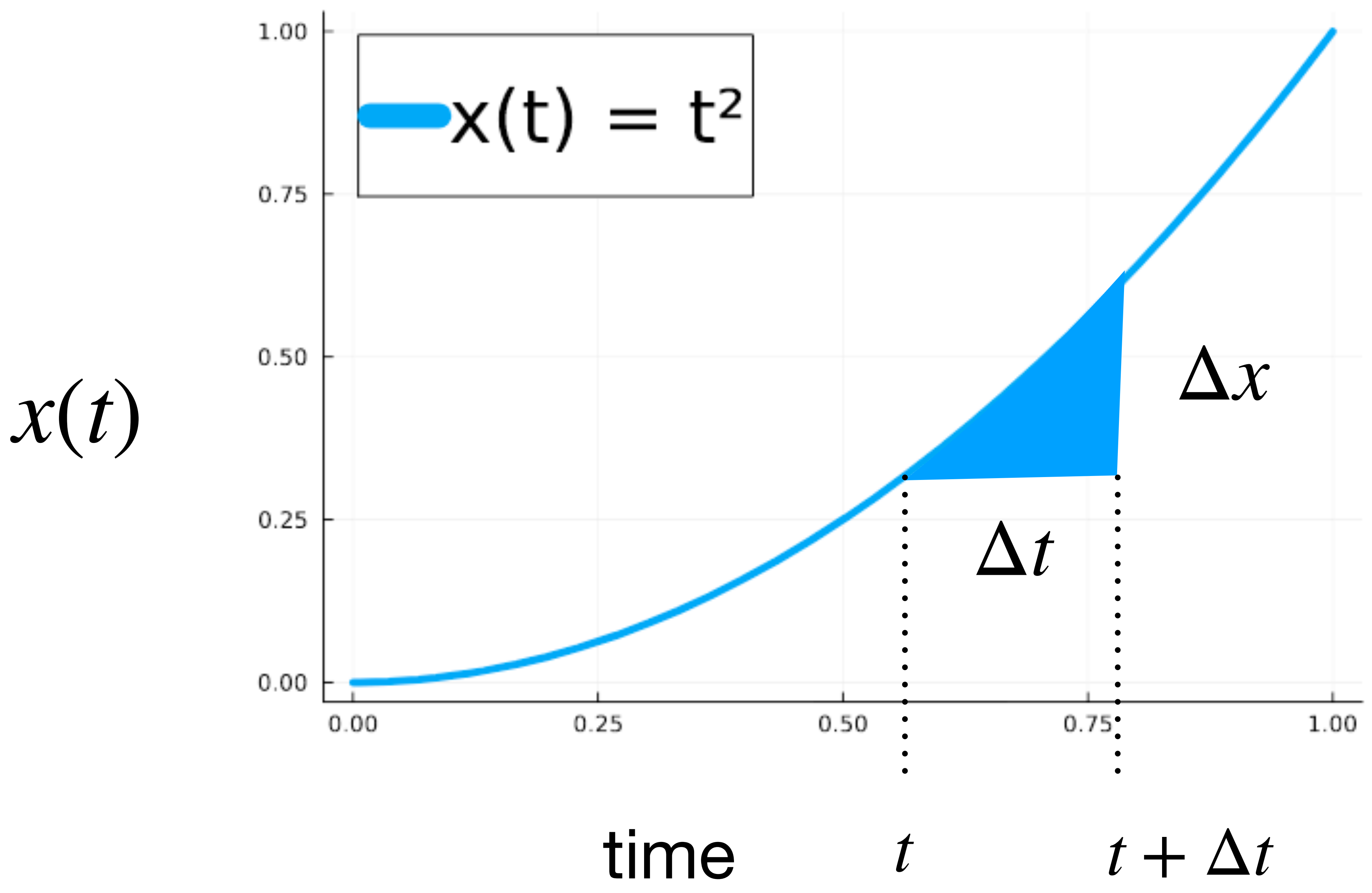
Suppose $x(t) = t^2$

Then
$$\frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{(t+\Delta t)^2 - t^2}{\Delta t}$$

$$= \frac{t^2 + 2t\Delta t + (\Delta t)^2 - t^2}{\Delta t}$$

$$= 2t + \Delta t$$

$$= 2t$$ (As $\Delta t \to 0$)



Velocity at time t : $\displaystyle\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$

An example

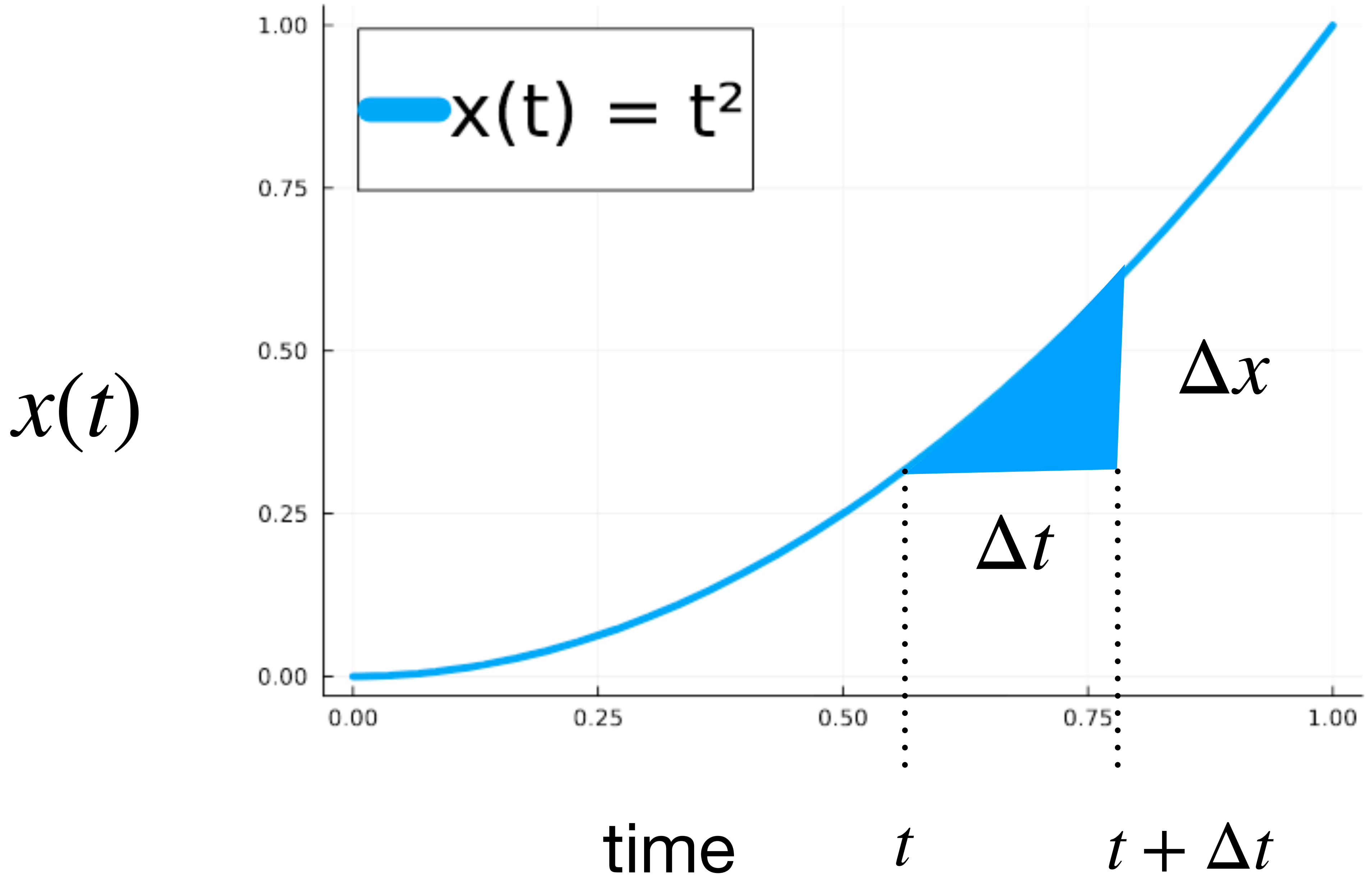
Suppose $x(t) = t^2$

Then $$\frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{(t+\Delta t)^2 - t^2}{\Delta t}$$

$$= \frac{t^2 + 2t\Delta t + (\Delta t)^2 - t^2}{\Delta t}$$

$$= 2t + \Delta t$$

$$= 2t \quad \text{(As } \Delta t \to 0\text{)}$$

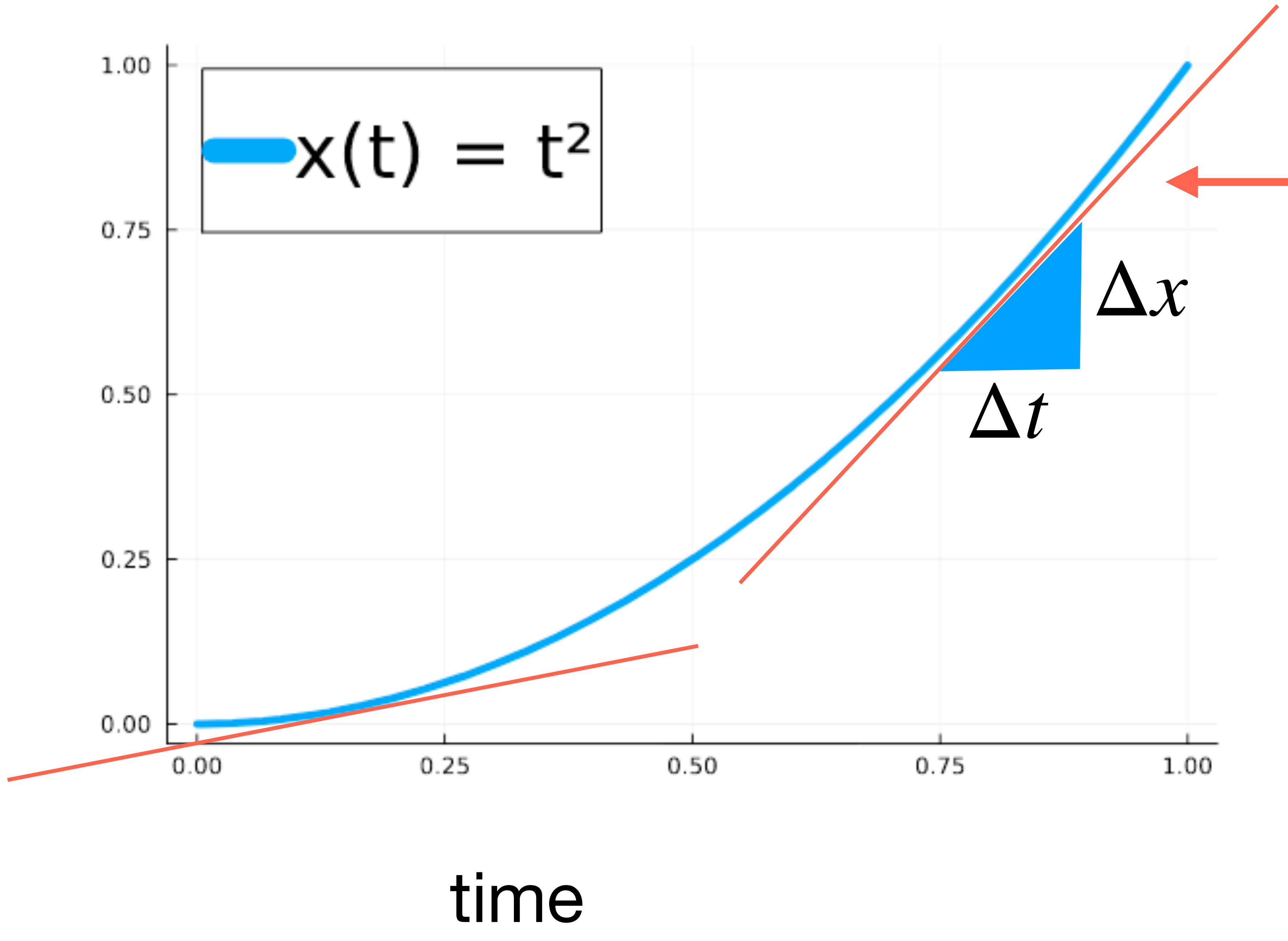


Velocity at time t : $\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$

Exercise

Did we assume Δt is positive?

Tangents



Tangent line

parallel where it touches

Velocity at time t is the steepness of the tangent

Velocity at time t : $$\lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

(i.e. as triangle width goes to zero)

Velocity is a differential quantity

Position function

$$x(t)$$

Time $\longrightarrow$ Position

Input of function:
"independent variable"

Output of function:
"dependent variable"

Velocity is a differential quantity

Position function

$$\text{Time} \xrightarrow{x(t)} \text{Position}$$

Input of function:
“independent variable”

Output of function:
“dependent variable”

Velocity function

$$\textit{Small change in } \text{Time} \xrightarrow{\frac{\Delta x}{\Delta t}(t)} \textit{Small change in } \text{Position}$$

Velocity is a differential quantity

“Velocity is the rate of change of position with respect to time”

Independent variable $x(t)$

Dependent variable t

Velocity is a differential quantity

“Velocity is the rate of change of position with respect to time”

Independent variable $x(t)$

Dependent variable t

“Velocity is the derivative of position with respect to time”

Velocity is a differential quantity

Notation: the differential

$\mathrm{d}r$:Infinitesimal change in variable r

Velocity is $\dfrac{\mathrm{d}x(t)}{\mathrm{d}t}$

↑

"Velocity is the derivative of position with respect to time"

Velocity is a differential quantity

Notation: the differential

$\mathrm{d}r$:Infinitesimal change in variable r

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t}$$

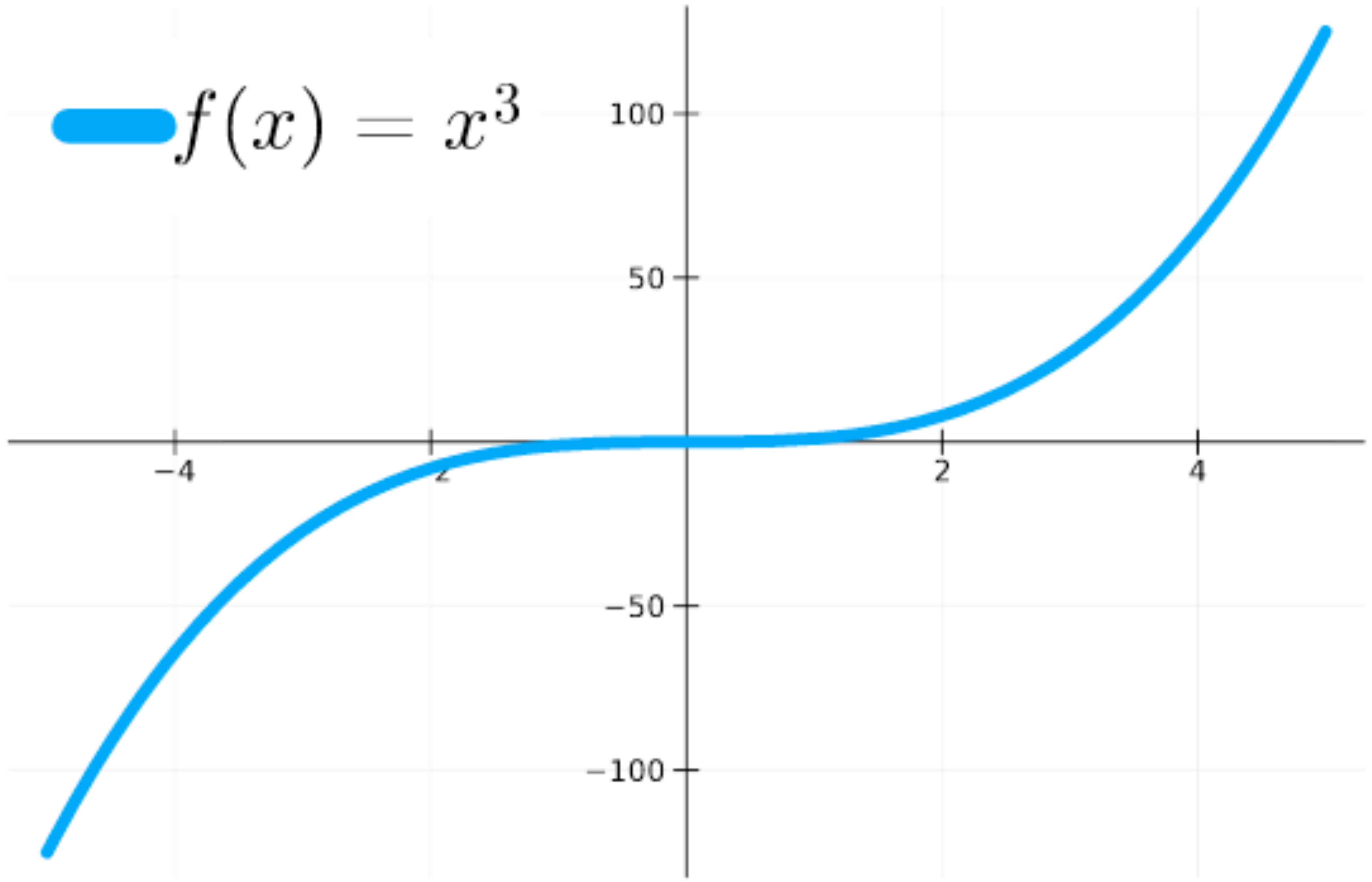
$\Downarrow$

Tip: read from the bottom

Infinitesimal change in denominator =>
How much change in numerator?

If t changed infinitesimally, how much would $x(t)$ (infinitesimally) change?

Lots of* functions have derivatives

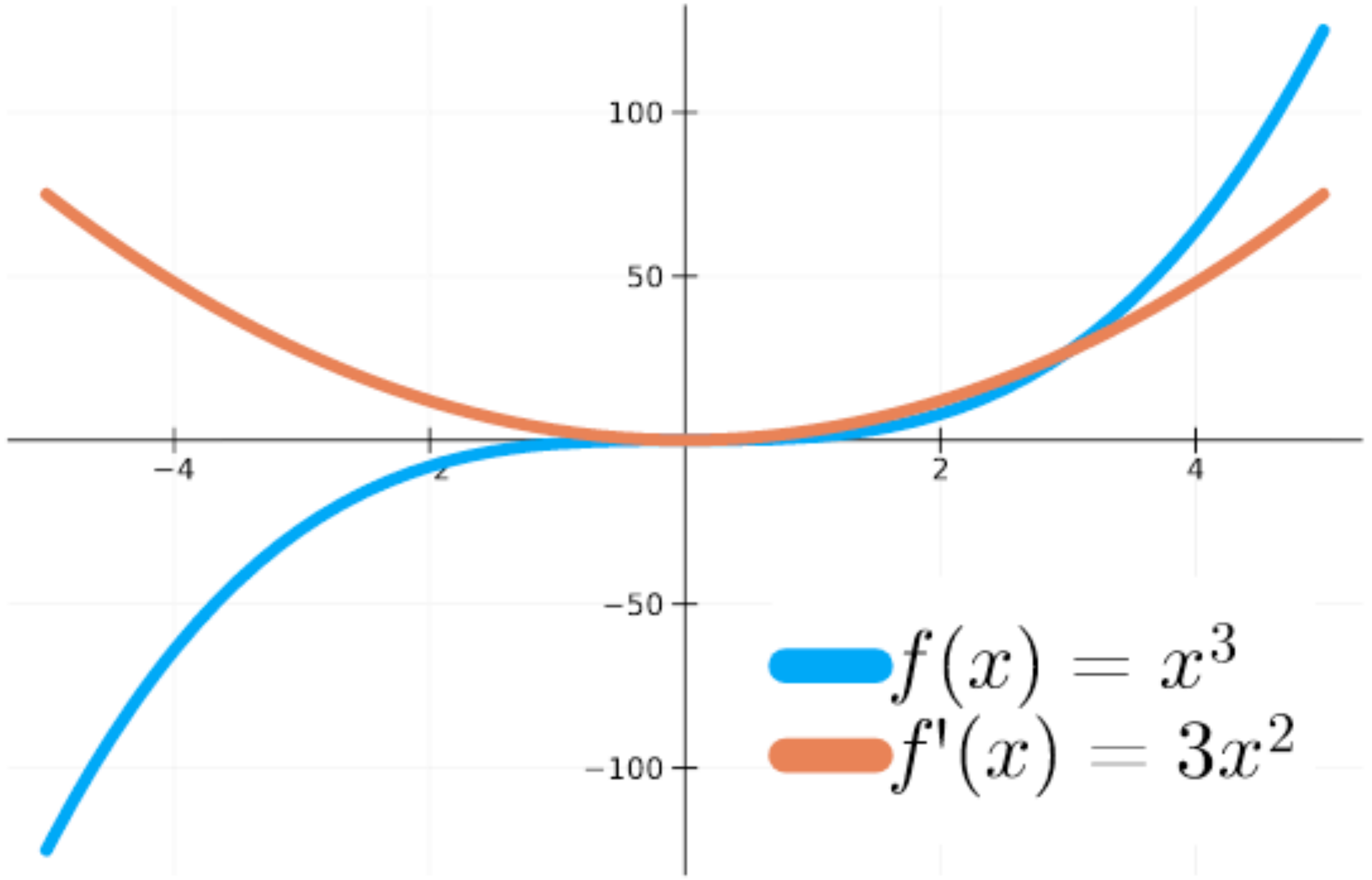


Mathematical notation for derivative?

Sketch the derivative from intuition

Terms and conditions apply

Lots of* functions have derivatives



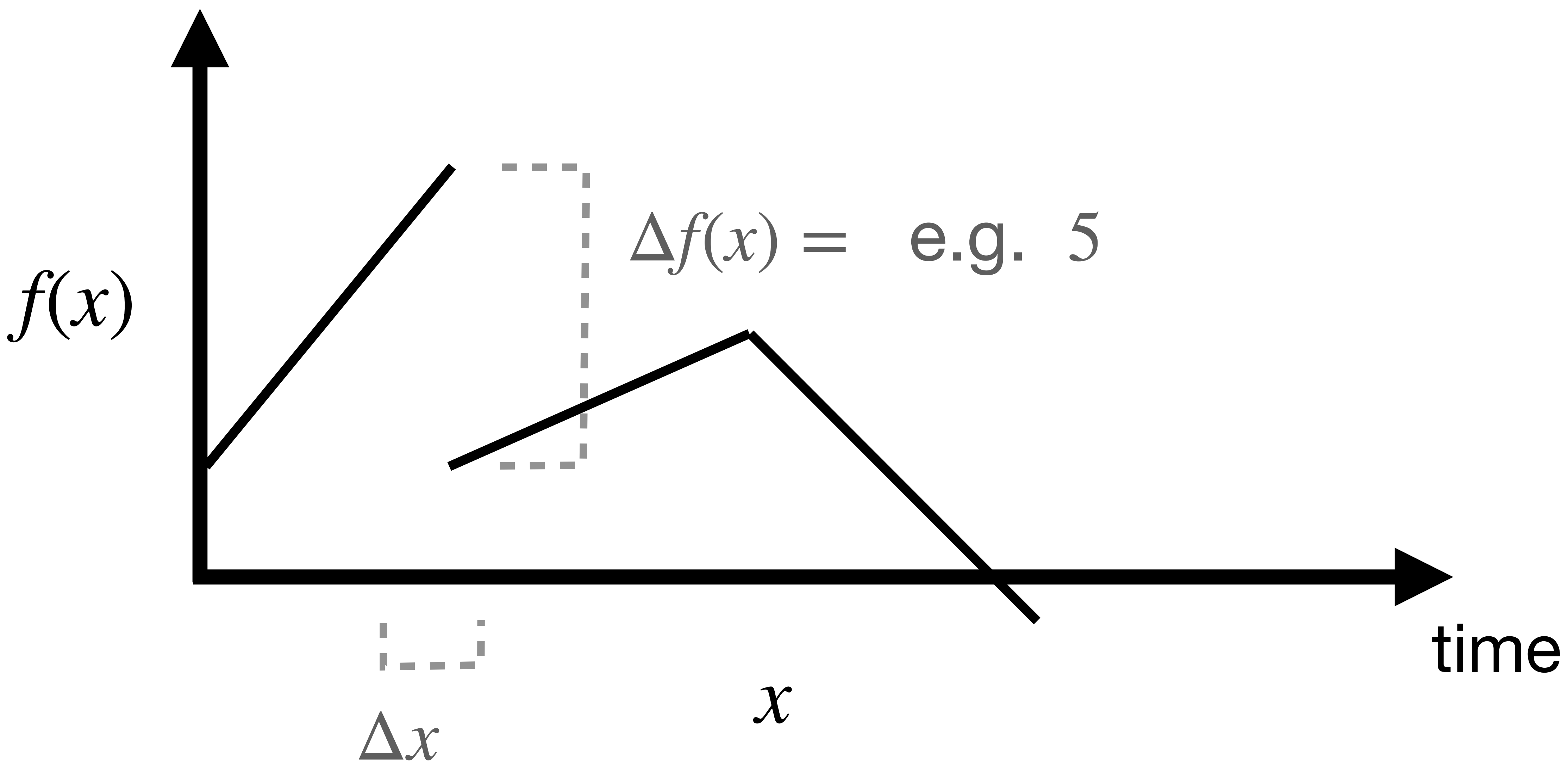
Common notations:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \qquad \frac{\mathrm{d}f}{\mathrm{d}x}(x)$$

$$f'(x) \qquad \dot{f}(x)$$

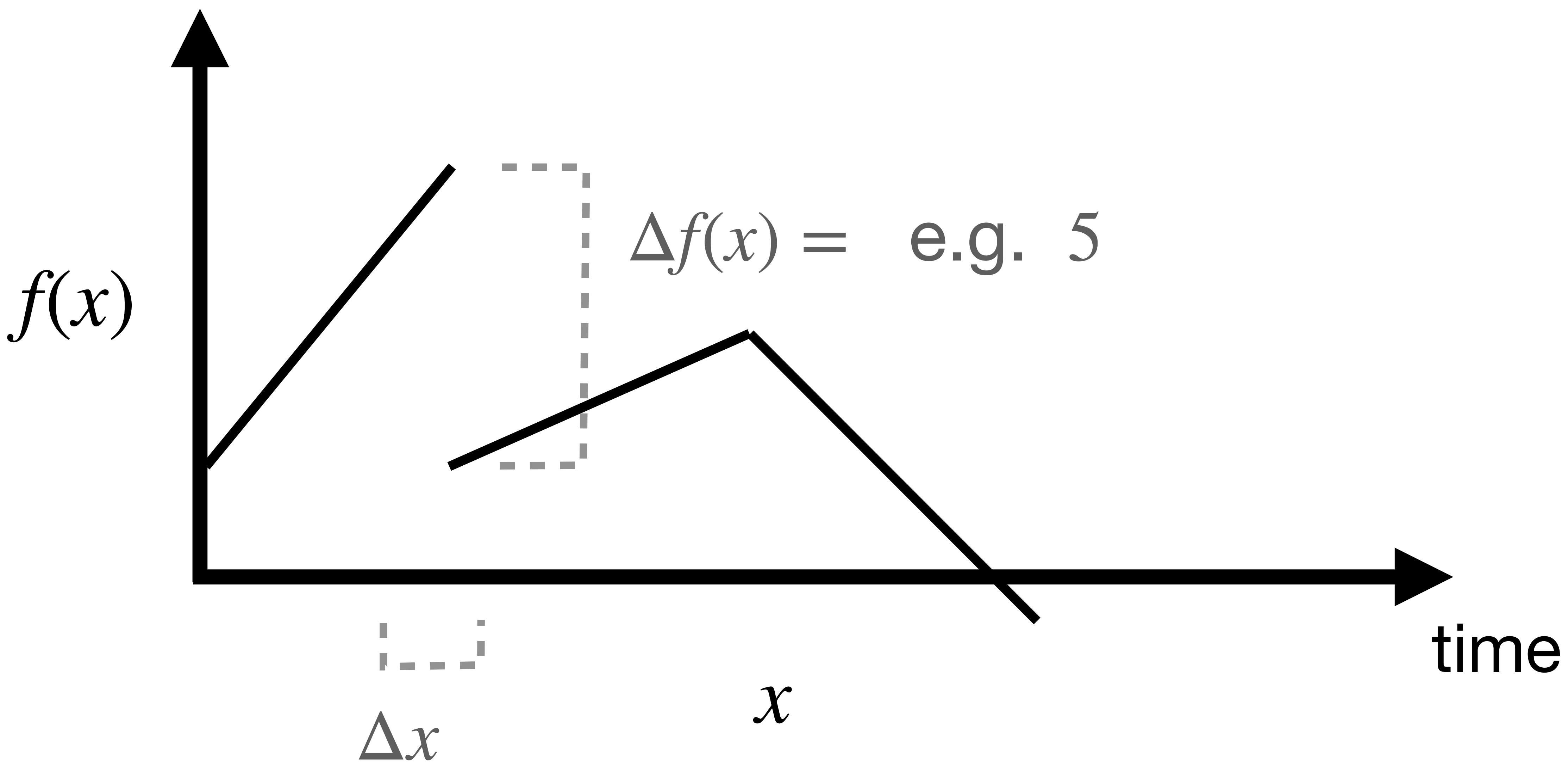
**Terms and conditions apply*

Derivatives are undefined at corners and jumps



$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x}?$$

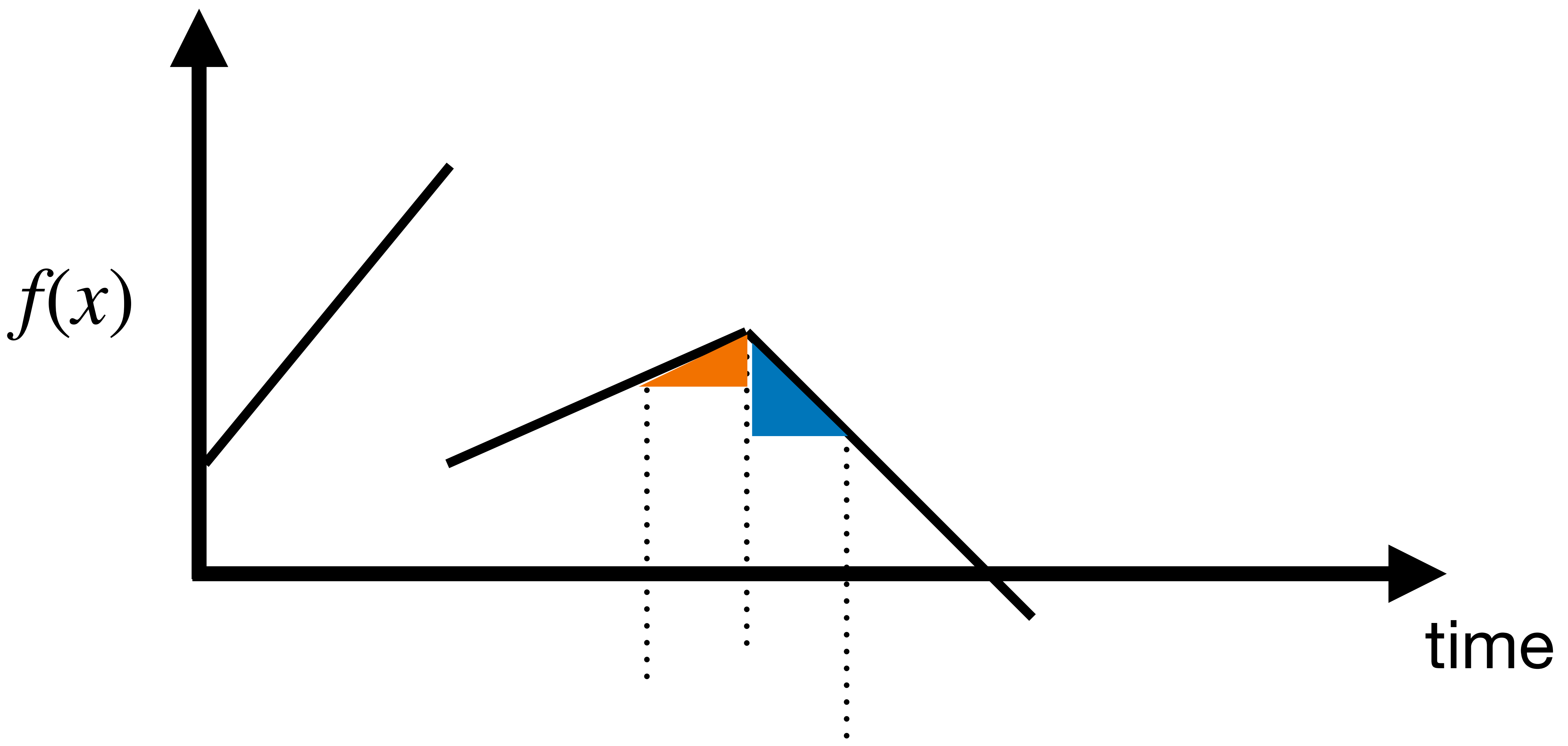
Derivatives are undefined at corners and jumps



$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{5}{\Delta x}$$

Derivatives are undefined at sharp corners and jumps

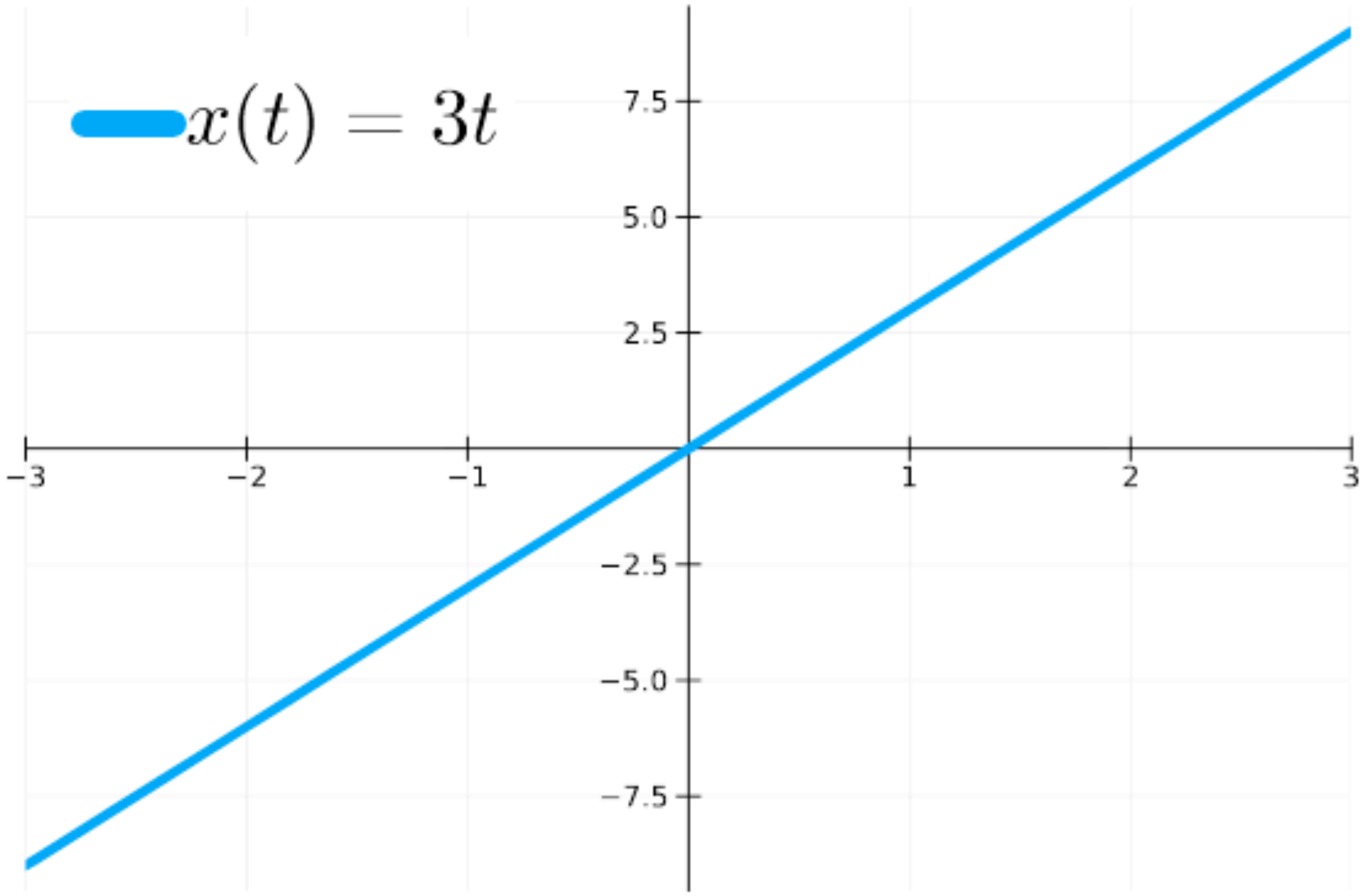
$$\lim_{\Delta x \to 0, \Delta x < 0} \frac{\Delta f(x)}{\Delta x} \neq \lim_{\Delta x \to 0, \Delta x > 0} \frac{\Delta f(x)}{\Delta x}$$



$$\lim_{\Delta x \to 0} \frac{\Delta f(x)}{\Delta x} :$$

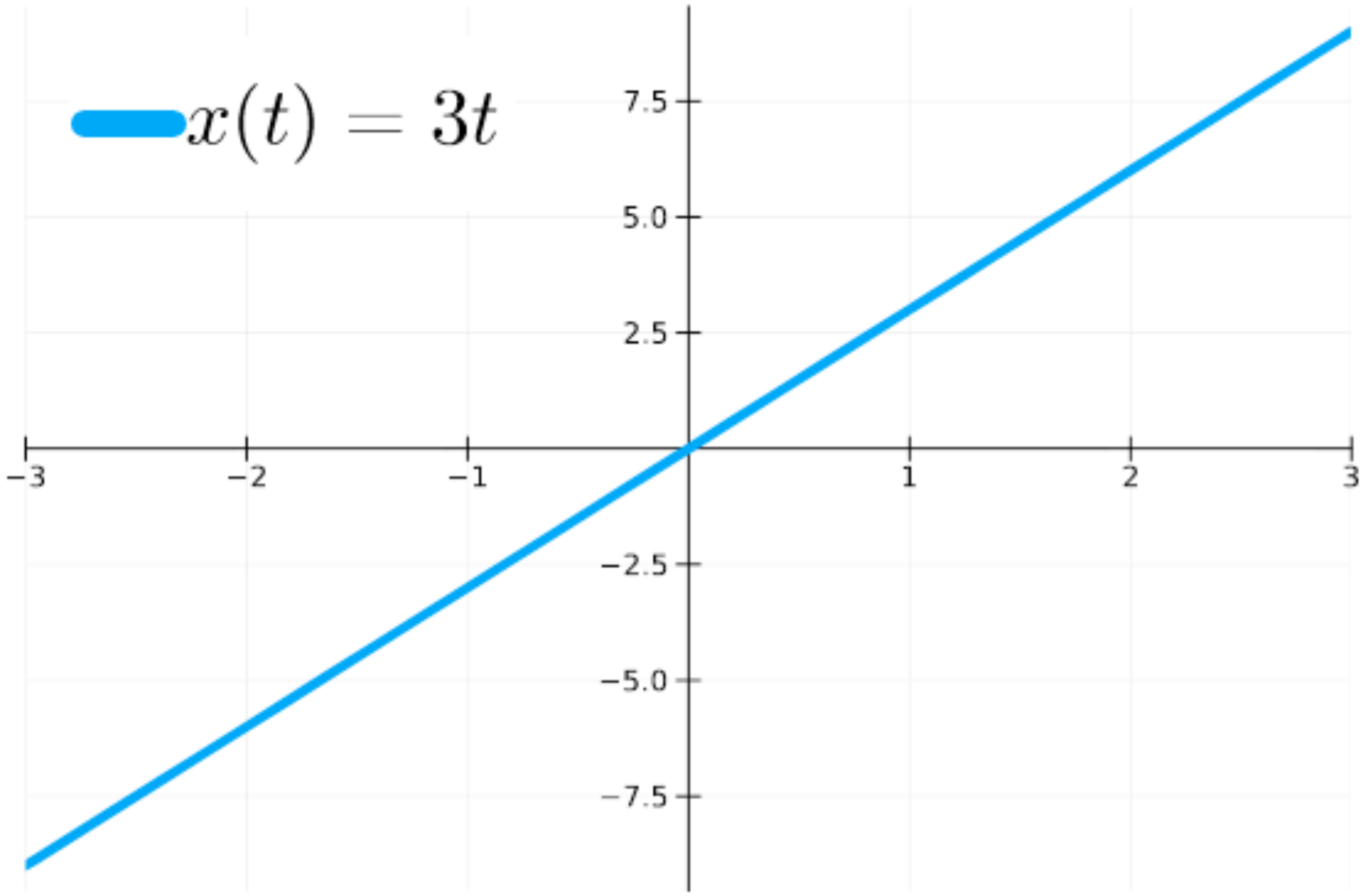
Δx can be negative or positive.
Limit should be the same!

Derivative?



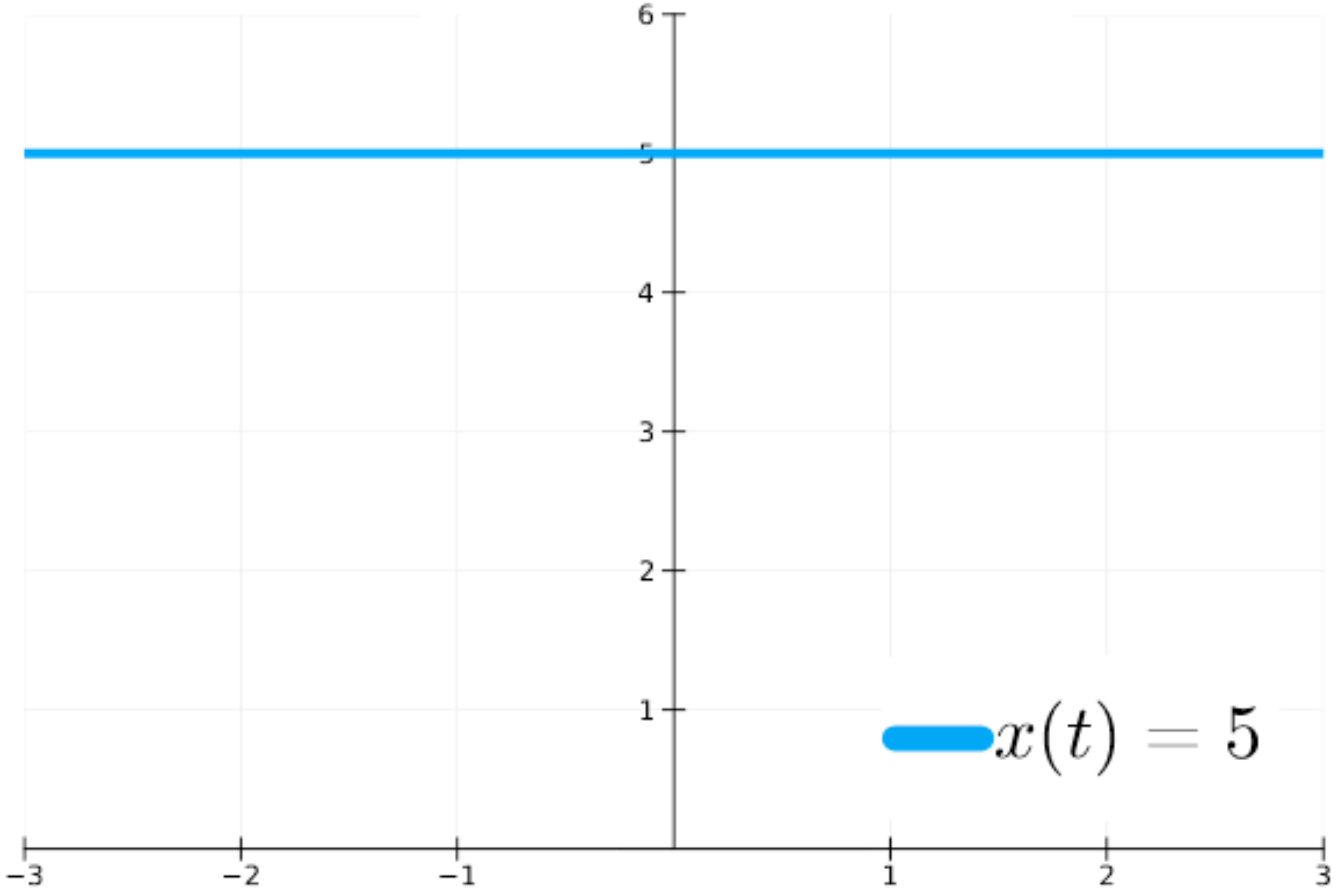
Derivative?

Linear functions have constant derivatives

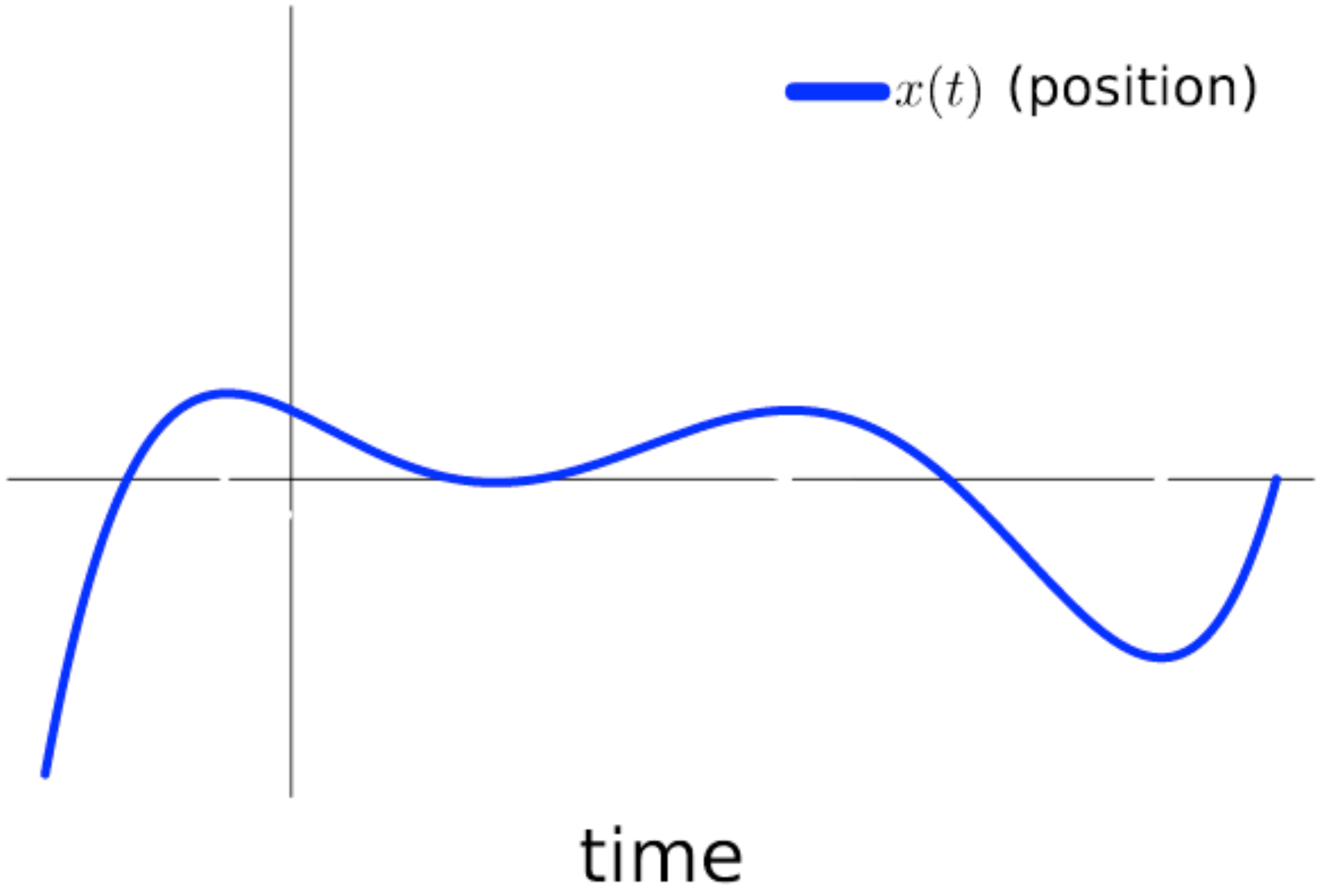


Derivative?

Constant functions
have zero derivatives

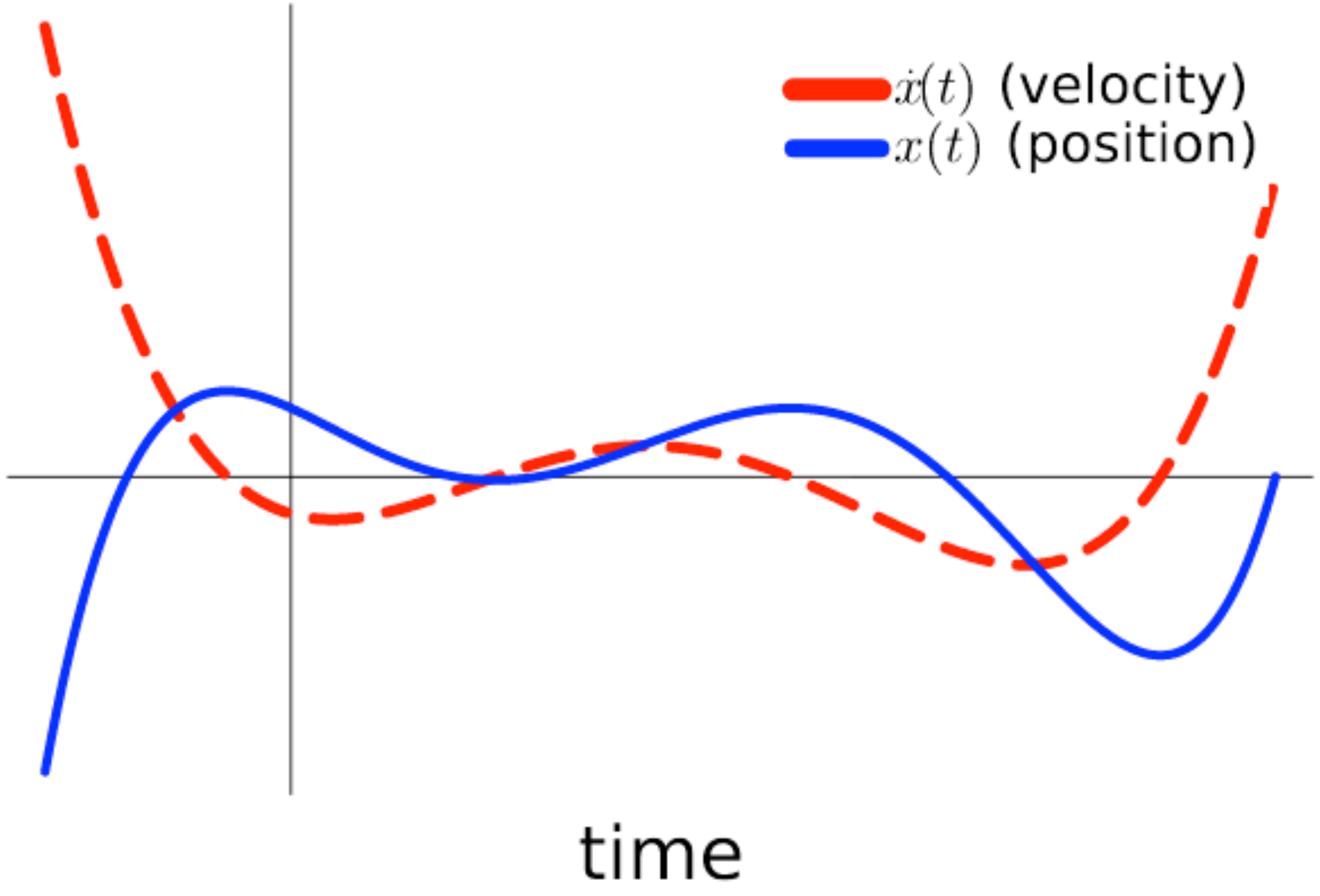


Acceleration



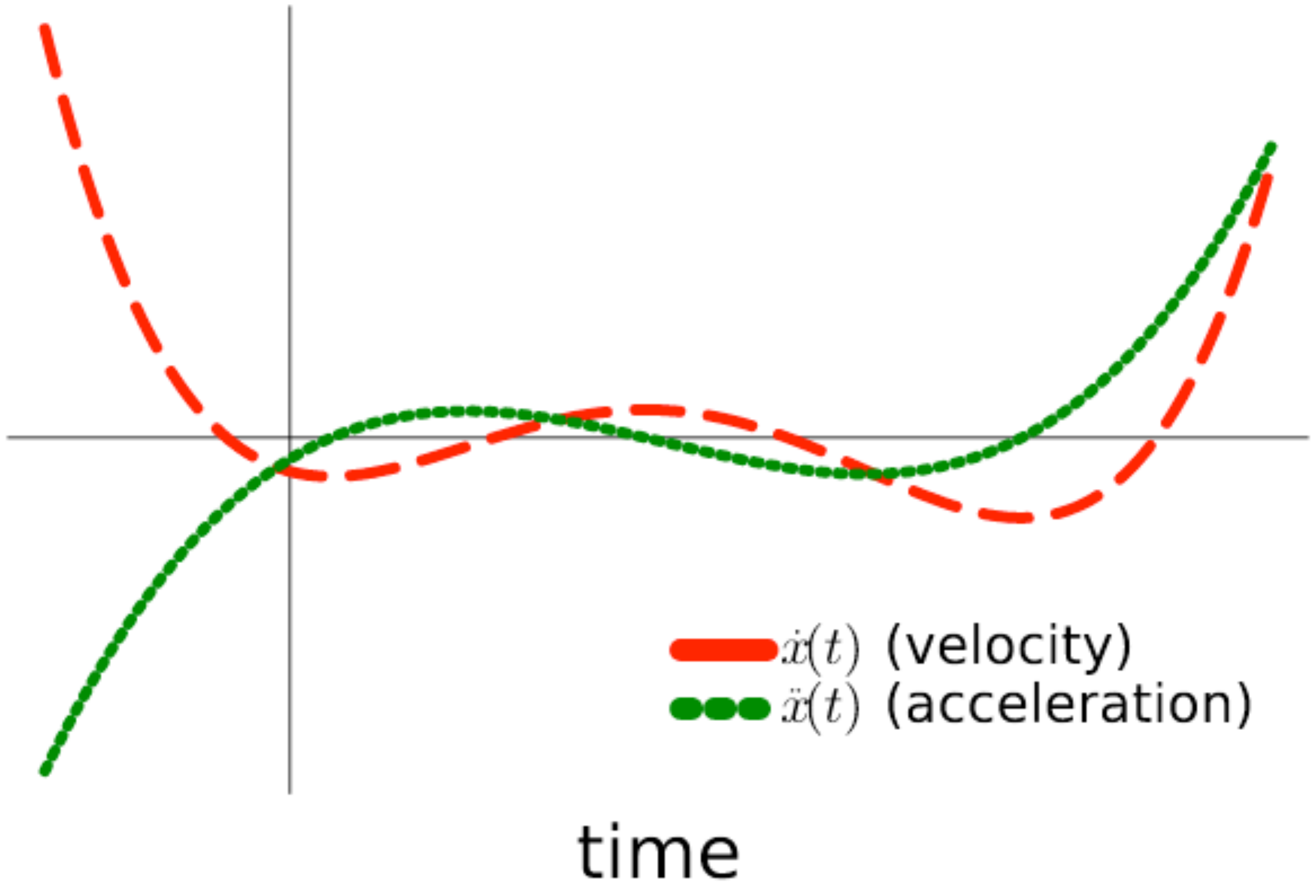
Geometrical analogue?

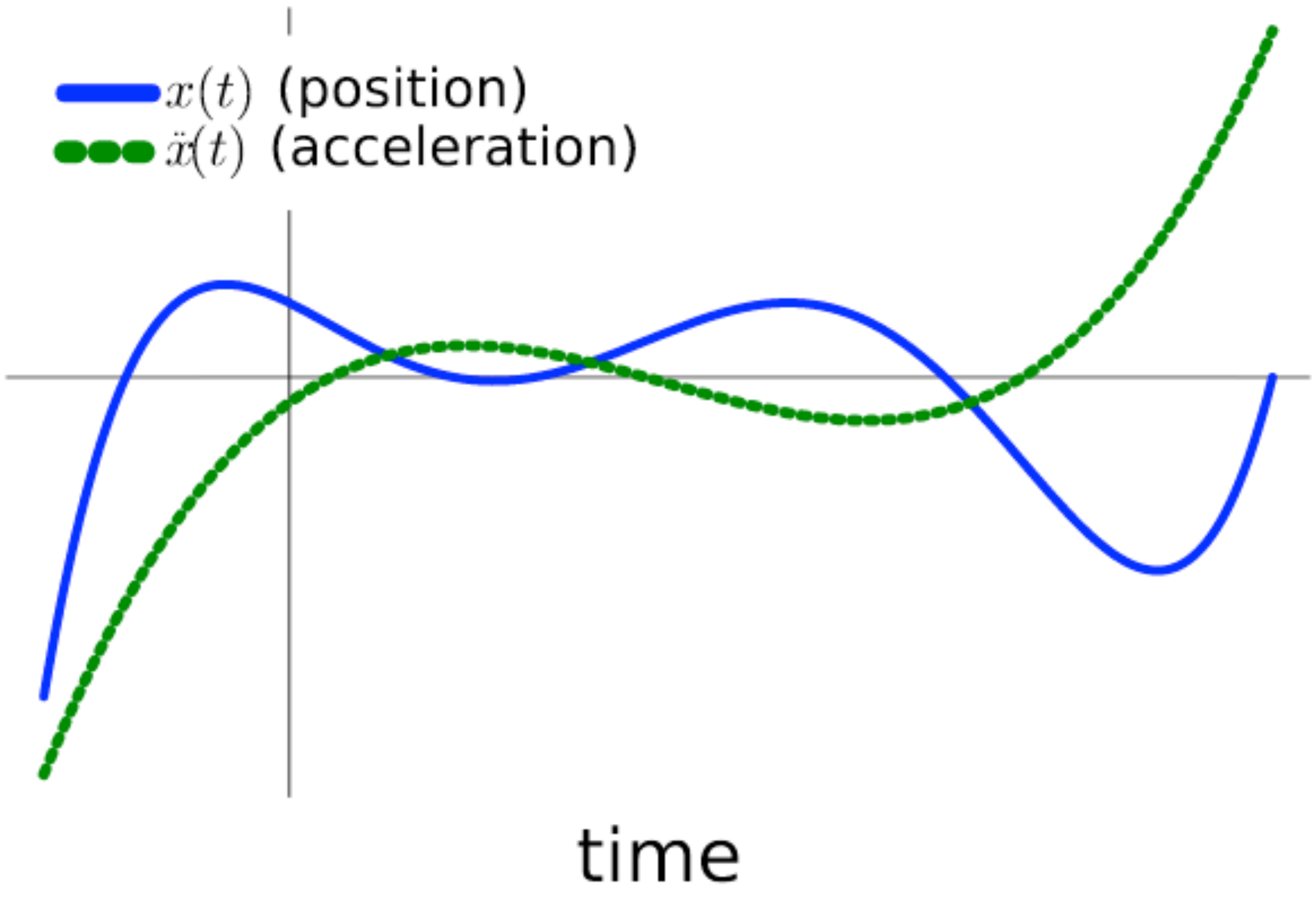
Acceleration



Geometrical analogue?

Acceleration



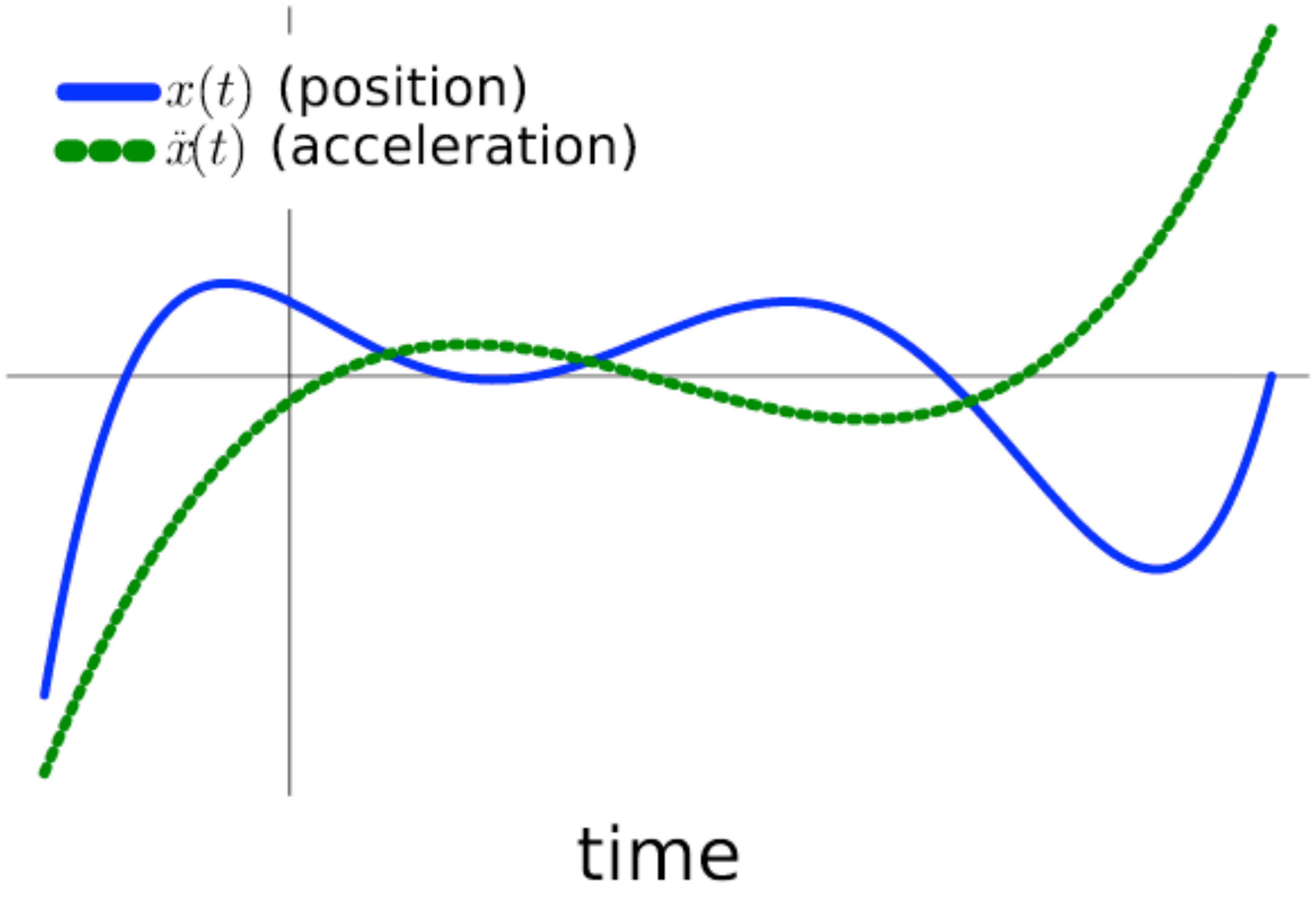


Geometrical analogue:

Derivative of velocity (= steepness)

Double derivative of position (= curviness)

Acceleration is a differential quantity of a differential quantity



Common notations:

$$\frac{\mathrm{d}^2 f(x)}{\mathrm{d}x^2} \qquad \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)$$

$$f''(x) \qquad \ddot{f}(x)$$

Geometrical analogue:

Derivative of velocity (= steepness)

Double derivative of position (= curviness)

Higher-order derivatives

Common notations:

$$\frac{\mathrm{d}^n f(x)}{\mathrm{d}x^n} \qquad \frac{\mathrm{d}^n f}{\mathrm{d}x^n}(x)$$

(Fourth order) $\longrightarrow$ $f''''(x) \qquad \ddot{\ddot{f}}(x)$

A bog-standard function
Mapping between values

$$f : \mathbb{R} \to \mathbb{R}$$

$$y = f(x)$$

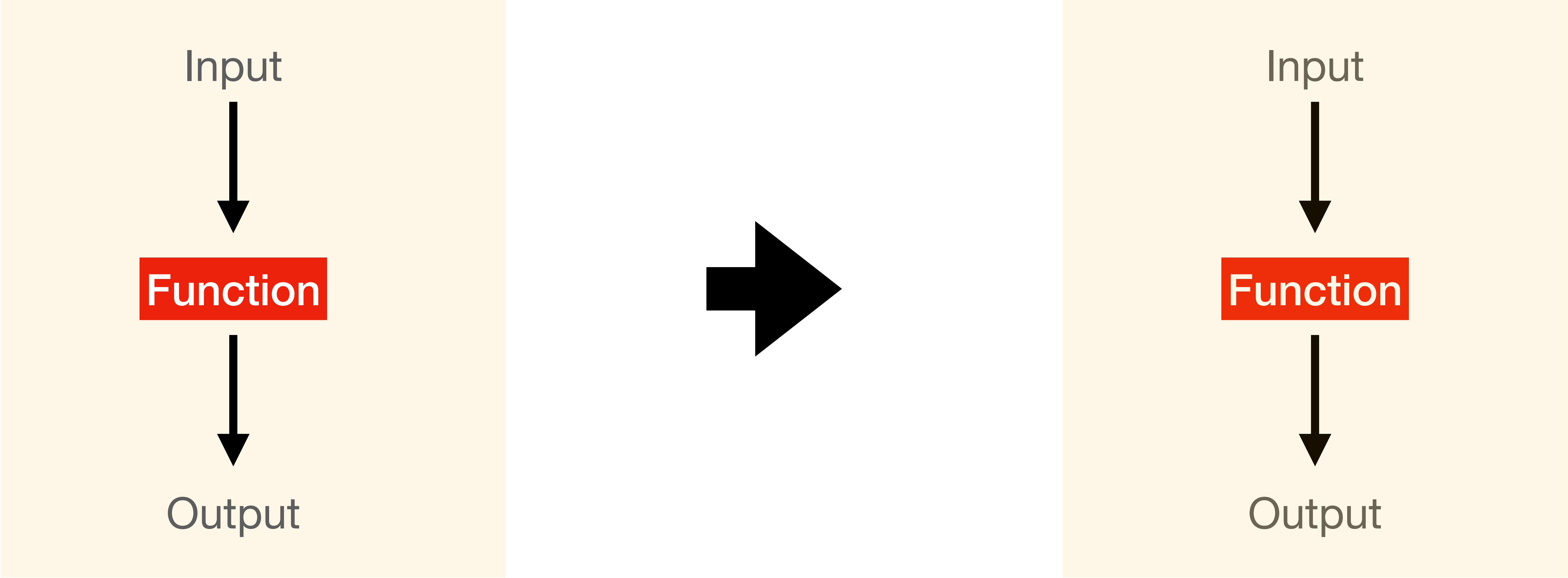
y is the output .
f is the mapping

They are not the same

$$x \xrightarrow{\quad f \quad} y$$

Differentiation is an operator

Mapping between functions

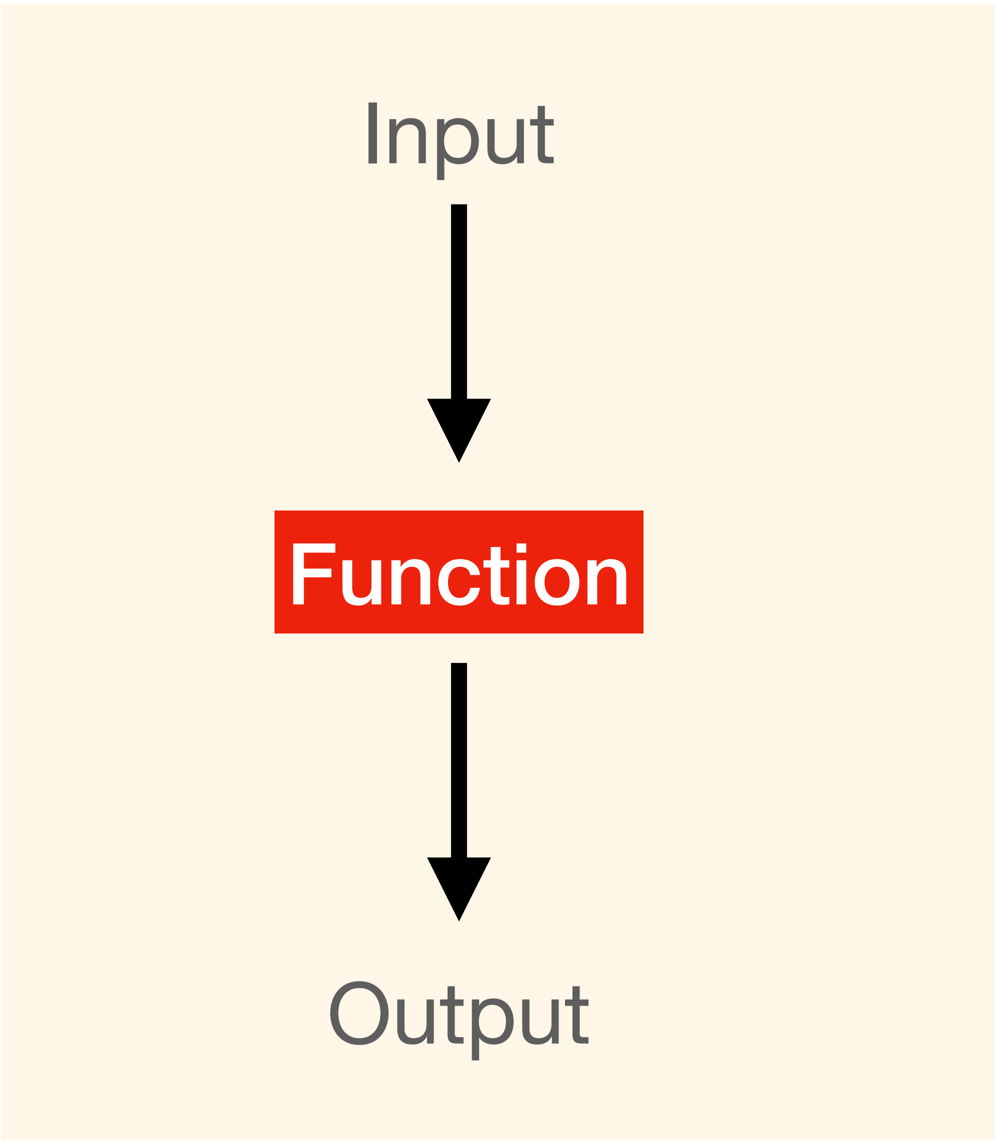


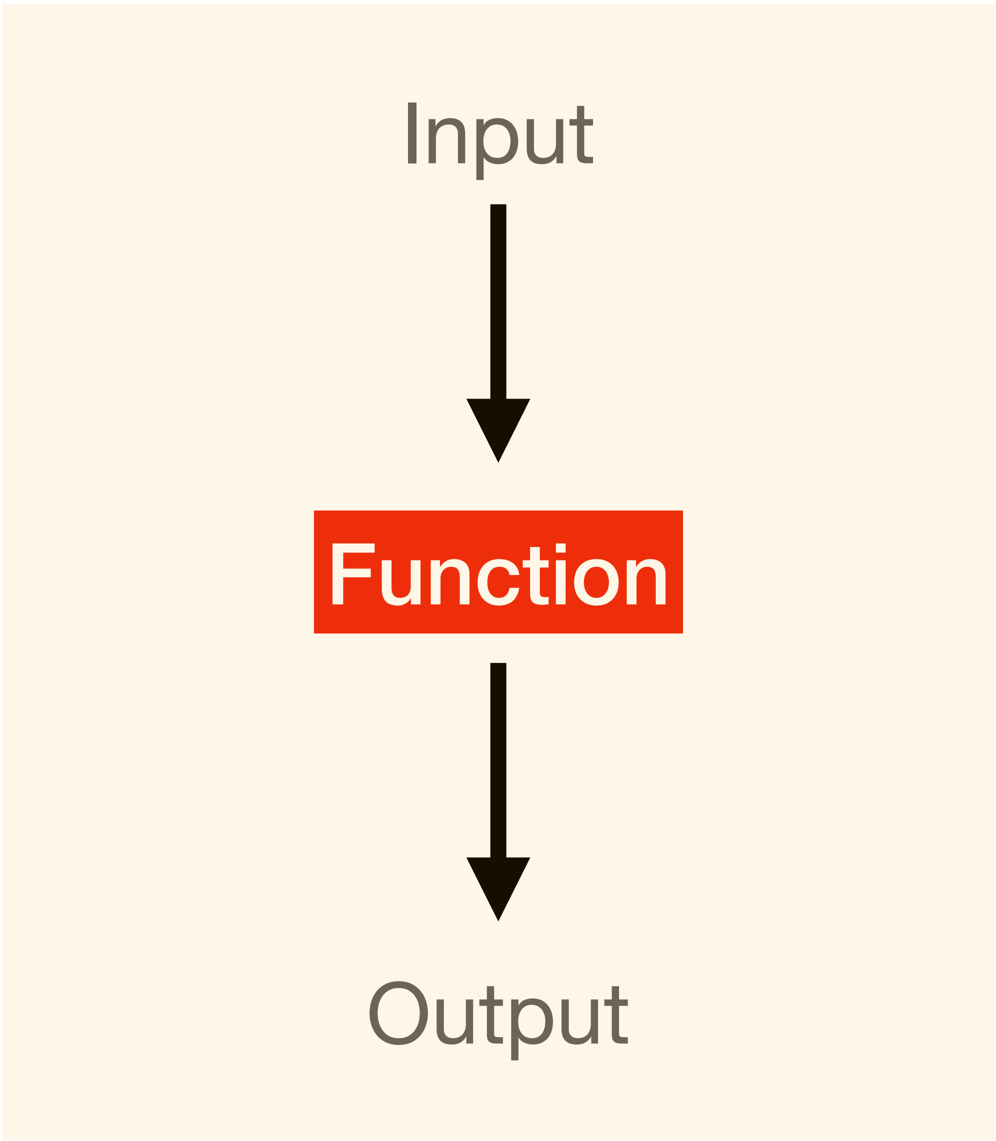
$$D_x(f)$$

$$f(x) \rightarrow \frac{\mathrm{d}f}{\mathrm{d}x}(x)$$

Differentiation is an operator

Mapping between functions





$\frac{\mathrm{d}f}{\mathrm{d}x}(x)$ is the output .

D_x is the mapping

f is the input

They are not the same

$$f(x) \xrightarrow{D_x(f)} \frac{\mathrm{d}f}{\mathrm{d}x}(x)$$

Differentiation is an operator
Mapping between functions

$$D_x(f)(x) = \frac{\mathrm{d}f}{\mathrm{d}x}(x)$$ (=value at x)

Common notations:

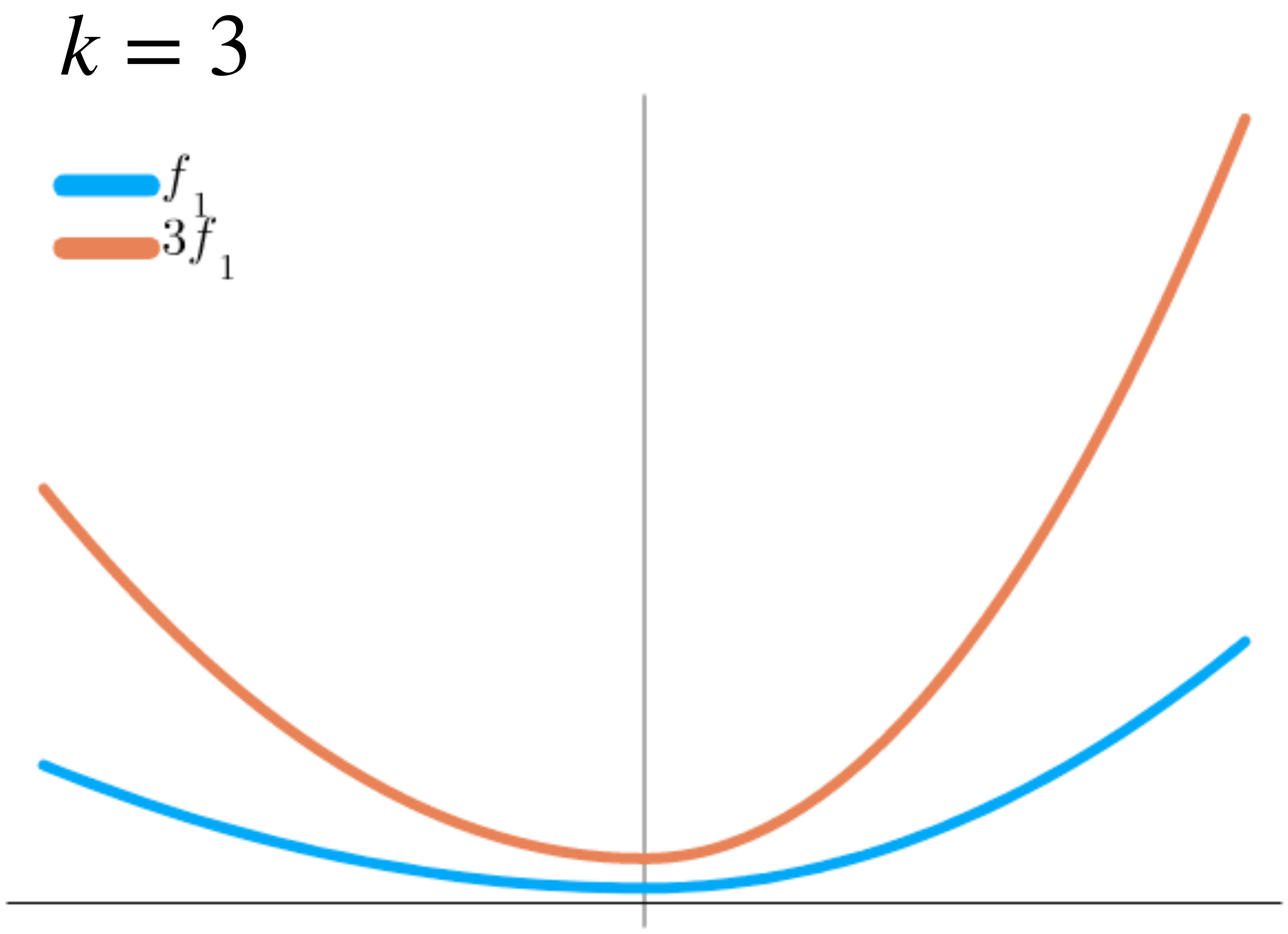
$D_x(f)$: differential operator with respect to x

$$f(x) \xrightarrow{D_x(f)} \frac{\mathrm{d}f}{\mathrm{d}x}(x) \xrightarrow{D_x \circ D_x(f)} \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)$$

Differentiation is a linear operator

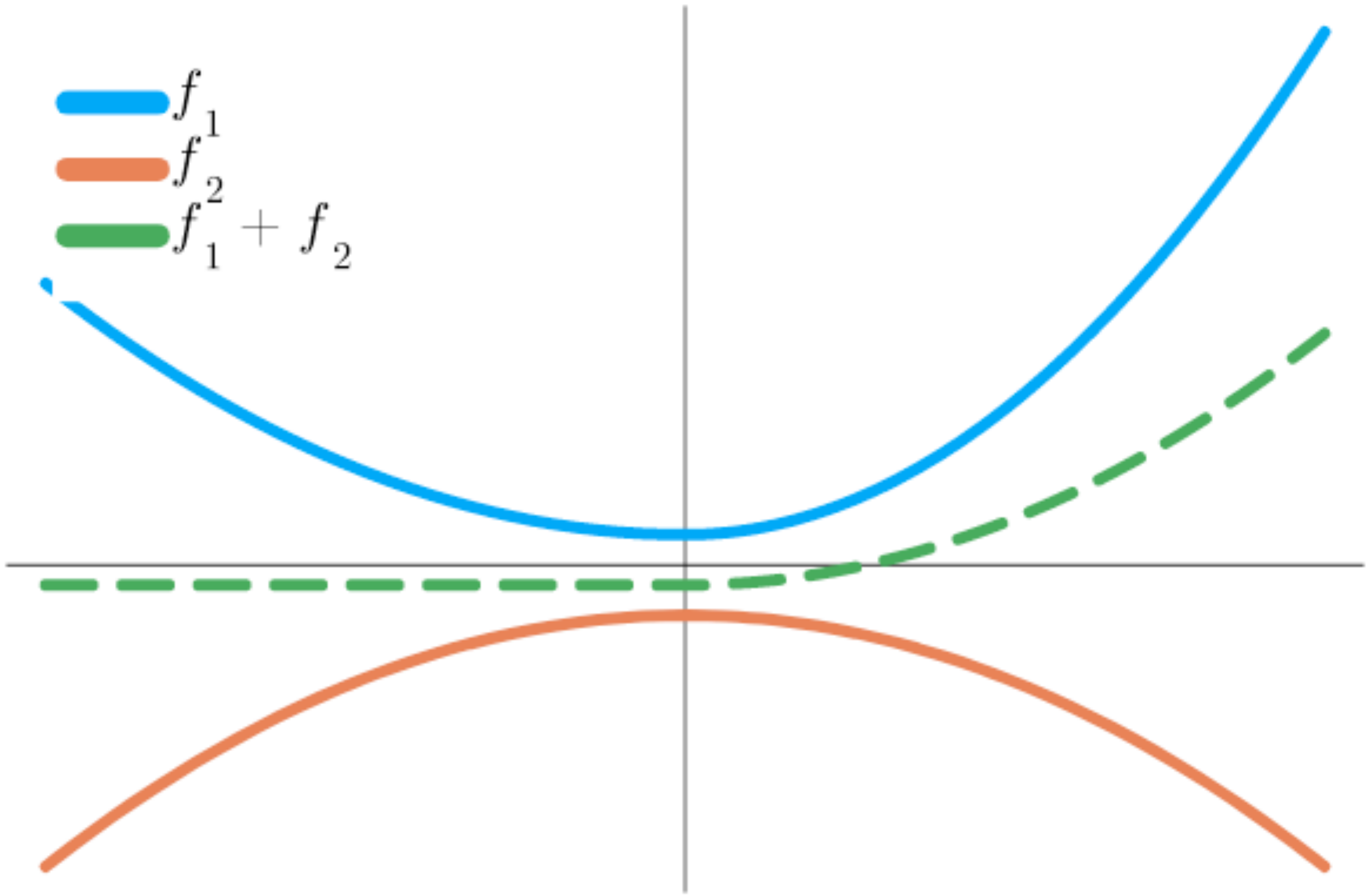
$$kD_x(f) = D_x(kf)$$

Where $k \in \mathbb{R}$ and
f is a function



Differentiation is a linear operator

$$D_x(f_1) + D_x(f_2) = D_x(f_1 + f_2)$$



Differentiation is a linear operator

Overall:

$$\boxed{D_x\left(\sum_{i=1}^{n} a_i f_i\right) = \sum_{i=1}^{n} a_i D_x(f_i)}$$

for any $a_i \in \mathbb{R}, \quad n \in \mathbb{N}$

Differentiation is a linear operator

$$f(x) = x^2 + 3x - 4$$

$$f = f_1 + 3f_2 - 4f_3$$

$$f_1(x) = x^2$$

$$f_2(x) = x$$

$$f_3(x) = 1$$

Differentiation is a linear operator

$$f(x) = x^2 + 3x - 4$$

$$f = f_1 + 3f_2 - 4f_3$$

$$f_1(x) = x^2$$

$$f_2(x) = x$$

$$f_3(x) = 1$$

$$D_x(f) = \frac{\mathrm{d}f(x)}{\mathrm{d}x} = D_x(f_1) + 3D_x(f_2) - 4D_x(f_3)$$

Standard derivatives

Function	Derivative
$y = x^n$	$y' = nx^{n-1}$
$y = \sin(x)$	$y' = \cos(x)$
$y = \cos(x)$	$y' = -\sin(x)$
$y = \ln(x)$	$y' = 1/x$
$y = e^x$	$y' = e^x$

Standard derivatives

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$y = \ln(x)$	$y' = 1/x$
$y = e^x$	$y' = e^x$

$$f(x) = 2x^3 + 3x^4 + 5\ln(x)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = ?$$

Standard derivatives

Function	Derivative
$y = x^n$	$y' = nx^{n-1}$
$y = \sin(x)$	$y' = \cos(x)$
$y = \cos(x)$	$y' = -\sin(x)$
$y = \ln(x)$	$y' = 1/x$
$y = e^x$	$y' = e^x$

$$f(x) = 2x^3 + 3x^4 + 5\ln(x)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x) = 2 * (3x^2) + 3 * (4x^3) + \frac{5}{x}$$

$$= 6x^2 + 12x^3 + \frac{5}{x}$$

Computing derivatives

Drag to move cell

```julia
begin
    using Symbolics
    @variables t # independent variable
    Dₜ = Differential(t)
    y = t^2 + 4sin(t) + log(t)
end
```

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\log\left(t\right) + 4\sin\left(t\right) + t^{2}\right)$$

```julia
Dₜ(y)
```

$$2t + \frac{1}{t} + 4\cos\left(t\right)$$

```julia
expand_derivatives(Dₜ(y))
```

Alternatives

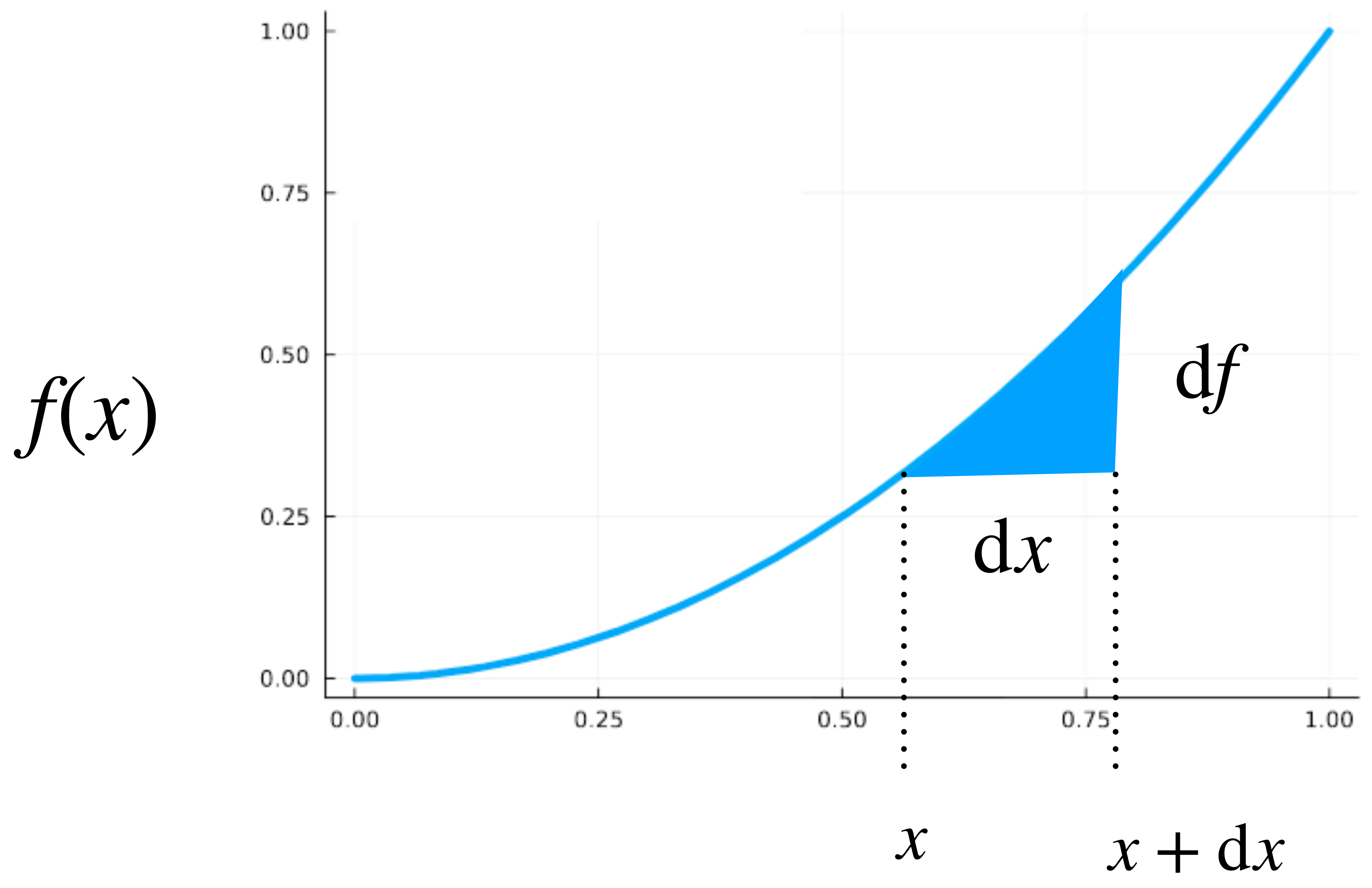
SymPy in Python

Wolfram alpha online

Not ChatGPT

Product and chain rules

Differentiate product and composition of functions

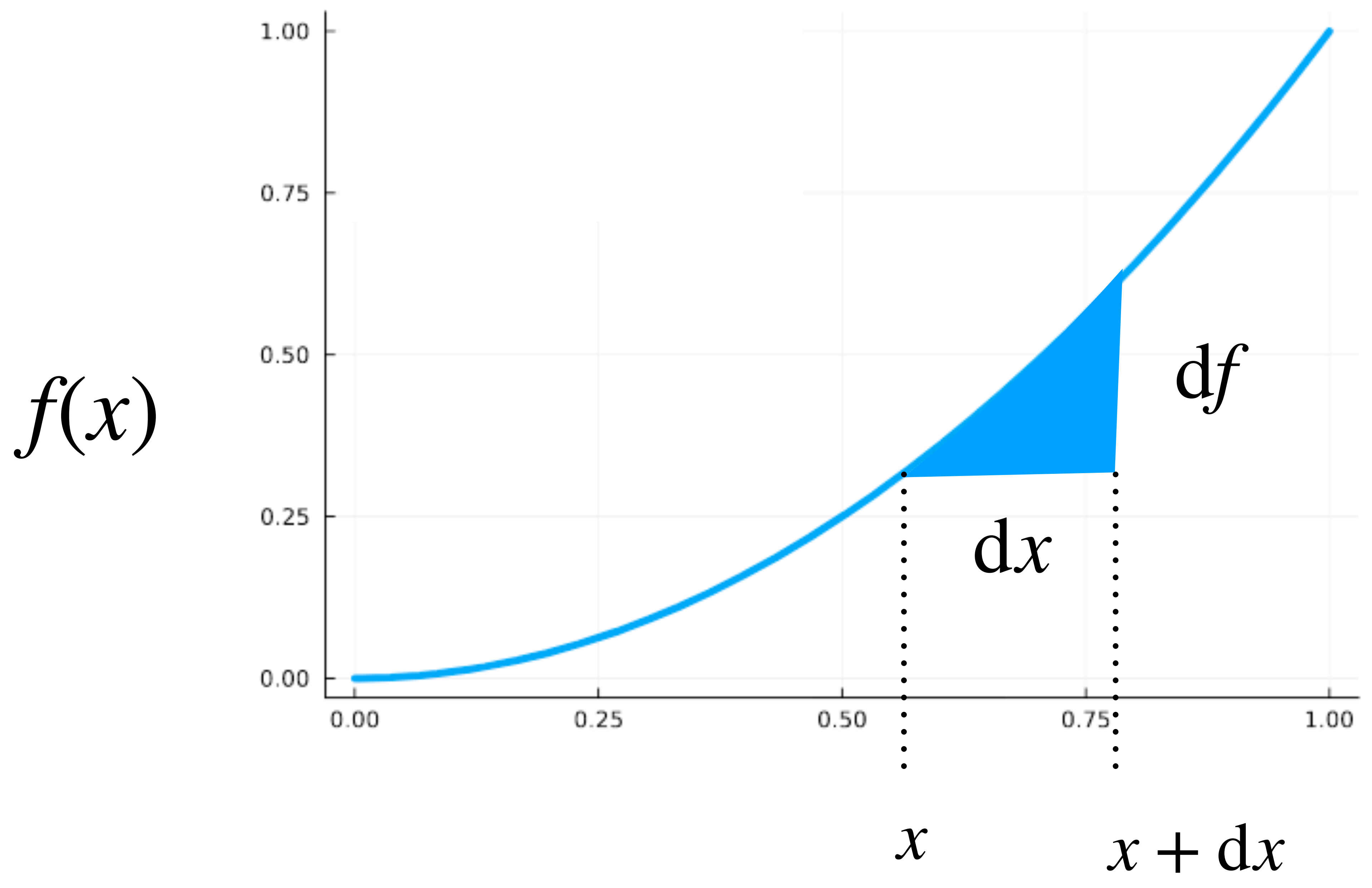


$$\mathrm{d}f = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x$$

Product rule

$$f(x) = g(x)h(x)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x)?$$



$$\mathrm{d}f = \left(\frac{\mathrm{d}f}{\mathrm{d}x}\right)\mathrm{d}x$$

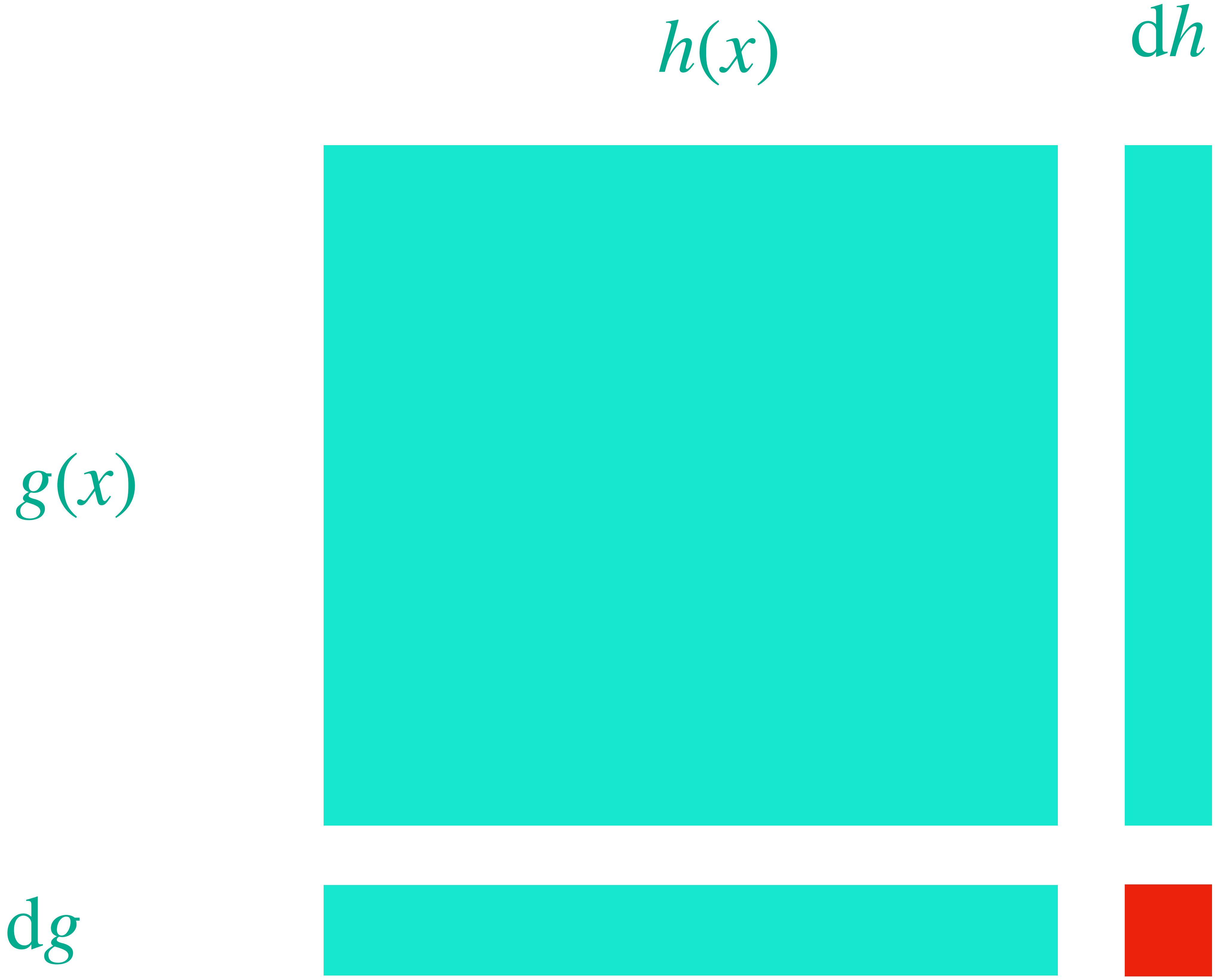
Product rule

$$f(x) = g(x)h(x)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x)?$$

$$x \to x + \mathrm{d}x$$

$$f(x) \to f(x) + \mathrm{d}f$$



Product rule

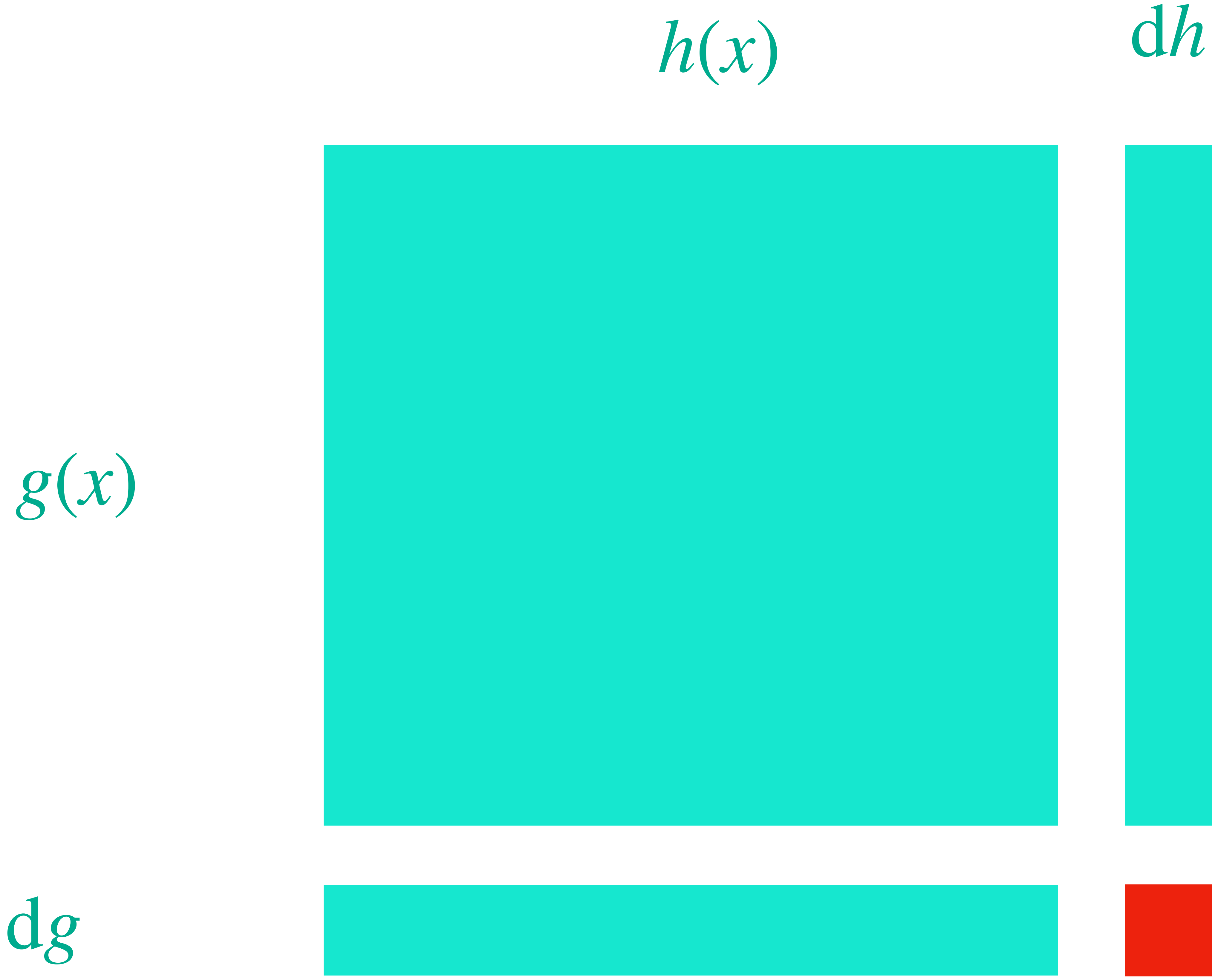
$$x \to x + \mathrm{d}x$$

$$f(x) \to f(x) + \mathrm{d}f$$

$$f(x) = g(x)h(x)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x)?$$

$$\mathrm{d}f = h(x)\mathrm{d}g + g(x)\mathrm{d}h + \mathrm{d}g\mathrm{d}h$$



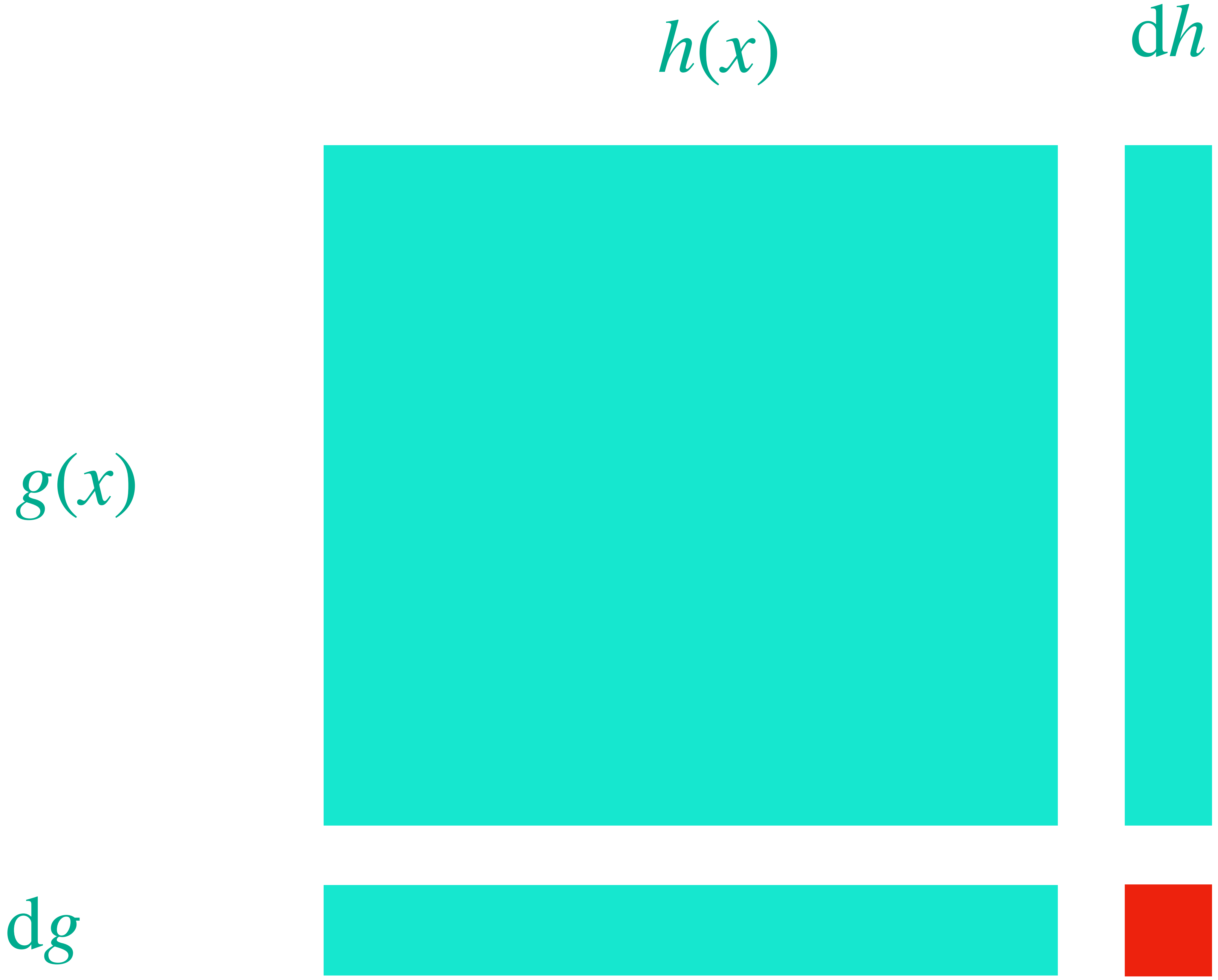
Product rule

$$x \to x + \mathrm{d}x$$

$$f(x) \to f(x) + \mathrm{d}f$$

$$f(x) = g(x)h(x)$$

$$\frac{\mathrm{d}f}{\mathrm{d}x}(x)\,?$$



$$\mathrm{d}f = h(x)\mathrm{d}g + g(x)\mathrm{d}h + \mathrm{d}g\mathrm{d}h$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = h(x)\frac{\mathrm{d}g}{\mathrm{d}x} + g(x)\frac{\mathrm{d}h}{\mathrm{d}x} + \cancel{\frac{\mathrm{d}g\mathrm{d}h}{\mathrm{d}x}} \to 0$$

Product rule

$$\frac{\mathrm{d}g\mathrm{d}h}{\mathrm{d}x} \to 0 \quad ?$$

There must be some $k \in \mathbb{R}$ such that:

$$|\mathrm{d}g| \leq k|\mathrm{d}x|$$

$$|\mathrm{d}h| \leq k|\mathrm{d}x|$$

e.g. twice the max steepness of $g(x)$ and $h(x)$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = h(x)\frac{\mathrm{d}g}{\mathrm{d}x} + g(x)\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{d}g\mathrm{d}h}{\mathrm{d}x}$$

Product rule

$$\left|\frac{\mathrm{d}g\mathrm{d}h}{\mathrm{d}x}\right| \leq \left|\frac{k^2\mathrm{d}x^2}{\mathrm{d}x}\right| = k|\mathrm{d}x|$$

$$= 0 \text{ as } \mathrm{d}x \to 0$$

There must be some $k \in \mathbb{R}$ such that:

$$|\mathrm{d}g| \leq k|\mathrm{d}x|$$

$$|\mathrm{d}h| \leq k|\mathrm{d}x|$$

e.g. twice the max steepness of $g(x)$ and $h(x)$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = h(x)\frac{\mathrm{d}g}{\mathrm{d}x} + g(x)\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{d}g\mathrm{d}h}{\mathrm{d}x}$$

Product rule

$$\boxed{D_x\left(fg\right) = gD_x(f) + fD_x(g)}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} = h(x)\frac{\mathrm{d}g}{\mathrm{d}x} + g(x)\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{d}g\mathrm{d}h}{\mathrm{d}x}$$

Product rule

Example

$f(x) = x^3 \cos(x)$

$$f(x) = g(x)h(x)$$

$g(x) = x^3$

$h(x) = \cos(x)$

$$\boxed{D_x(fg) = gD_x(f) + fD_x(g)}$$

Product rule

Example

$$f(x) = x^3 \cos(x)$$

$$\boxed{D_x\big(fg\big) = gD_x(f) + fD_x(g)}$$

$$f(x) = g(x)h(x)$$

$$g(x) = x^3$$
$$\mathrm{d}g = 3x^2\mathrm{d}x$$

$$h(x) = \cos(x)$$
$$\mathrm{d}h = -\sin(x)\mathrm{d}x$$

$$\mathrm{d}f = \cos(x)(3x^2\mathrm{d}x) - x^3(\sin(x)\mathrm{d}x)$$

Chain rule

$\cancel{f(x) = g(x)h(x)}$ (Product rule)

$f(x) = g \circ h(x)$ (Chain rule)

$$g(x) = x^2$$
$$h(x) = \sin(x)$$

$g \circ h?$ $h \circ g?$ $gh?$ $hg?$

Chain rule

$$\cancel{f(x) = g(x)h(x)}$$ (Product rule)

$$f(x) = g \circ h(x)$$ (Chain rule)

$$g(x) = x^2$$
$$h(x) = \sin(x)$$

$g \circ h?$	$h \circ g?$	$gh?$	$hg?$
$[sin(x)]^2$	$\sin(x^2)$	$x^2 \sin(x)$	$x^2 \sin(x)$

Chain rule

$$f(x) = g \circ h(x)$$

$$\boxed{D_x\left(g \circ h\right)?}$$

Step 1: Rewrite

$$f(x) = g(u)$$
$$u = h(x)$$

$$\begin{tikzcd}
 & \mathrm{d}u \arrow[dr] & \\
\mathrm{d}x \arrow[ur] \arrow[rr] & & \mathrm{d}f
\end{tikzcd}$$

Chain rule

$$f(x) = g \circ h(x)$$

Step 1: Rewrite

$$f(x) = g(u)$$

$$u = h(x)$$

Step 2:
Individual derivatives

$$\mathrm{d}u = \frac{\mathrm{d}h(x)}{\mathrm{d}x}\mathrm{d}x$$

$$\mathrm{d}f = \frac{\mathrm{d}g(u)}{\mathrm{d}u}\mathrm{d}u$$

$$\begin{tikzcd}
& \mathrm{d}u \arrow[dr] & \\
\mathrm{d}x \arrow[ur] \arrow[rr] & & \mathrm{d}f
\end{tikzcd}$$

Chain rule

$$f(x) = g \circ h(x)$$

Step 1: Rewrite

$$f(x) = g(u)$$
$$u = h(x)$$

Step 2:
Individual derivatives

$$\mathrm{d}u = \frac{\mathrm{d}h(x)}{\mathrm{d}x}\mathrm{d}x$$

$$\mathrm{d}f = \frac{\mathrm{d}g(u)}{\mathrm{d}u}\mathrm{d}u$$

Step 3:
Put together

$$\mathrm{d}f = \frac{\mathrm{d}g(u)}{\mathrm{d}u}\frac{\mathrm{d}h(x)}{\mathrm{d}x}\mathrm{d}x$$

$$\begin{tikzcd} & \mathrm{d}u \arrow[dr] & \\ \mathrm{d}x \arrow[ur] \arrow[rr] & & \mathrm{d}f \end{tikzcd}$$

Chain rule

$$f(x) = g \circ h(x)$$

Example

$$f(x) = \cos(x^3)$$

Chain rule

$$f(x) = g \circ h(x)$$

Step 1: break into pieces

$$g(u) = \cos(u)$$

$$u = h(x) = x^3$$

Example

$$f(x) = \cos(x^3)$$

Chain rule

$$f(x) = g \circ h(x)$$

Example

$$f(x) = \cos(x^3)$$

Step 1: break into pieces

$$g(u) = \cos(u)$$

$$u = h(x) = x^3$$

Step 2: calculate derivatives

$$\mathrm{d}g(u) = -\sin(u)\mathrm{d}u$$

$$\mathrm{d}u = 3x^2\mathrm{d}x$$

Chain rule

$$f(x) = g \circ h(x)$$

Example

$$f(x) = \cos(x^3)$$

Step 1: break into pieces

$$g(u) = \cos(u)$$

$$u = h(x) = x^3$$

Step 2: calculate derivatives

$$\mathrm{d}g(u) = -\sin(u)\mathrm{d}u$$

$$\mathrm{d}u = 3x^2\mathrm{d}x$$

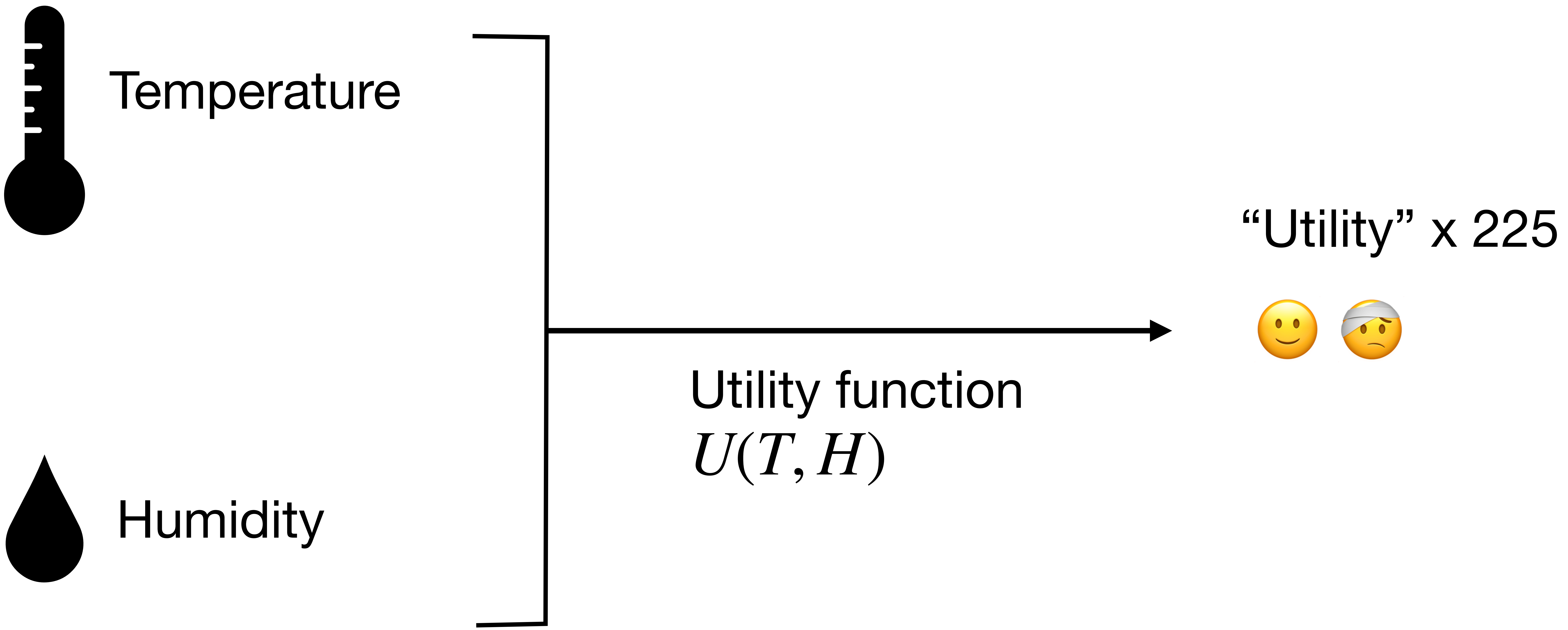
Step 3: Put together

$$\mathrm{d}f(x) = -\sin(x^3)3x^2\mathrm{d}x$$

Multidimensional derivatives

Question
Dimensionality of derivative?

$$U : \mathbb{R}^2 \to \mathbb{R}^{225}$$



Multidimensional derivatives

$$U : \mathbb{R}^2 \to \mathbb{R}^{225}$$

$$x = (T, H)$$

$$\frac{\mathrm{dU}}{\mathrm{d}x} = \begin{bmatrix} \frac{\mathrm{dU_1}}{\mathrm{d}x_1} & \frac{\mathrm{dU_1}}{\mathrm{d}x_2} \\ \frac{\mathrm{dU_2}}{\mathrm{d}x_1} & \frac{\mathrm{dU_2}}{\mathrm{d}x_2} \\ \vdots & \vdots \\ \frac{\mathrm{dU_n}}{\mathrm{d}x_1} & \frac{\mathrm{dU_n}}{\mathrm{d}x_2} \end{bmatrix}$$

Multidimensional derivatives

$f : \mathbb{R}^m \to \mathbb{R}^n$

Question

Dimensionality of derivative?

Multidimensional derivatives

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

Question
Dimensionality of derivative?

The Jacobian matrix

$$\frac{\mathrm{df}}{\mathrm{d}x} = \begin{array}{cc} \text{m columns} & \\ \begin{bmatrix} \frac{\mathrm{df}_1}{\mathrm{d}x_1} & \frac{\mathrm{df}_1}{\mathrm{d}x_2}, \ldots, \frac{\mathrm{df}_1}{\mathrm{d}x_m} \\ \frac{\mathrm{df}_2}{\mathrm{d}x_1} & \frac{\mathrm{df}_2}{\mathrm{d}x_2}, \ldots, \frac{\mathrm{df}_2}{\mathrm{d}x_m} \\ \vdots & \vdots \\ \frac{\mathrm{df}_\mathrm{n}}{\mathrm{d}x_1} & \frac{\mathrm{df}_\mathrm{n}}{\mathrm{d}x_2}, \ldots, \frac{\mathrm{df}_\mathrm{n}}{\mathrm{d}x_m} \end{bmatrix} & \text{n rows} \end{array}$$

Multidimensional derivatives

$$f : \mathbb{R}^m \rightarrow \mathbb{R}^n$$

Question

Dimensionality of derivative?

The Jacobian matrix

$$\begin{array}{cc} & m \textbf{ columns} \\ \dfrac{\mathrm{df}}{\mathrm{d}x} = & \begin{bmatrix} \dfrac{\mathrm{df}_1}{\mathrm{d}x_1} & \dfrac{\mathrm{df}_1}{\mathrm{d}x_2}, \ldots, \dfrac{\mathrm{df}_1}{\mathrm{d}x_m} \\ \dfrac{\mathrm{df}_2}{\mathrm{d}x_1} & \dfrac{\mathrm{df}_2}{\mathrm{d}x_2}, \ldots, \dfrac{\mathrm{df}_2}{\mathrm{d}x_m} \\ \vdots & \vdots \\ \dfrac{\mathrm{df}_\mathrm{n}}{\mathrm{d}x_1} & \dfrac{\mathrm{df}_\mathrm{n}}{\mathrm{d}x_2}, \ldots, \dfrac{\mathrm{df}_\mathrm{n}}{\mathrm{d}x_m} \end{bmatrix} & n \textbf{ rows} \end{array}$$

Notation

$$D_x(f) : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times n}$$

$$J_x(f) : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times n}$$

$$\nabla(f) : \mathbb{R}^m \rightarrow \mathbb{R}^{m \times n}$$

Practice

$$f(x) = [x_1^2 + 4x_3, 5x_2 + 6x_1^3]$$

$$f : \mathbb{R}^n \to \mathbb{R}^2$$

Function	Derivative
$y = x^n$	$y' = nx^{n-1}$
$y = \sin(x)$	$y' = \cos(x)$
$y = \cos(x)$	$y' = -\sin(x)$
$y = \ln(x)$	$y' = 1/x$
$y = e^x$	$y' = e^x$

Practice

$$f(x) = [x_1^2 + 4x_3, 5x_2 + 6x_1^3]$$

$$f : \mathbb{R}^n \to \mathbb{R}^2$$

$$D_x(f) : \mathbb{R}^n \to \mathbb{R}^{n\times 2}$$

$$\begin{bmatrix} \bullet & \bullet \\ \bullet & \bullet \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{bmatrix}$$

Function	Derivative
$y = x^n$	$y' = nx^{n-1}$
$y = \sin(x)$	$y' = \cos(x)$
$y = \cos(x)$	$y' = -\sin(x)$
$y = \ln(x)$	$y' = 1/x$
$y = e^x$	$y' = e^x$

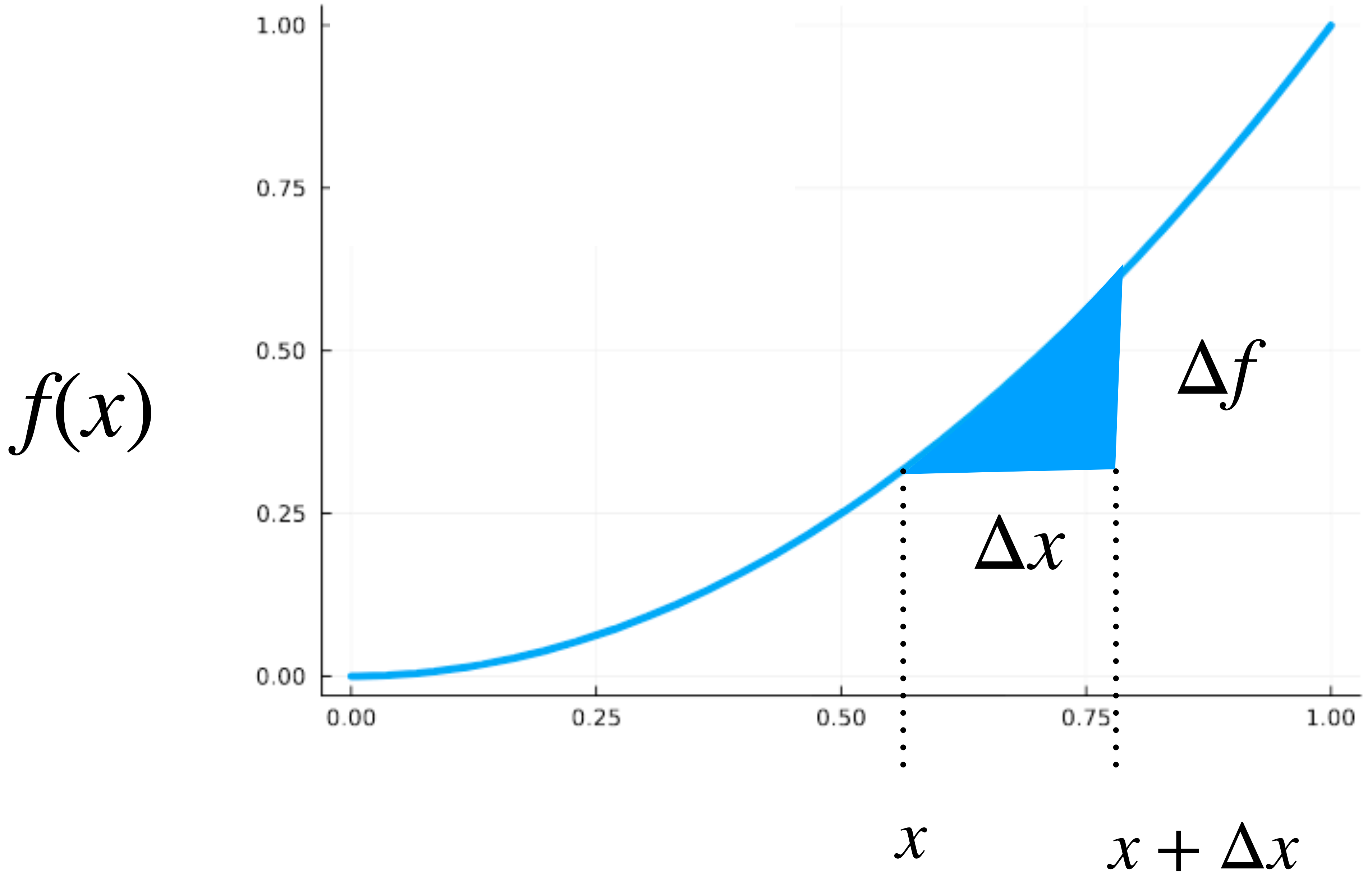
Finite difference approximation

Scalar functions

$$\frac{\mathrm{df}}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\frac{\mathrm{df}}{\mathrm{d}x} \approx \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

(Small Δx)

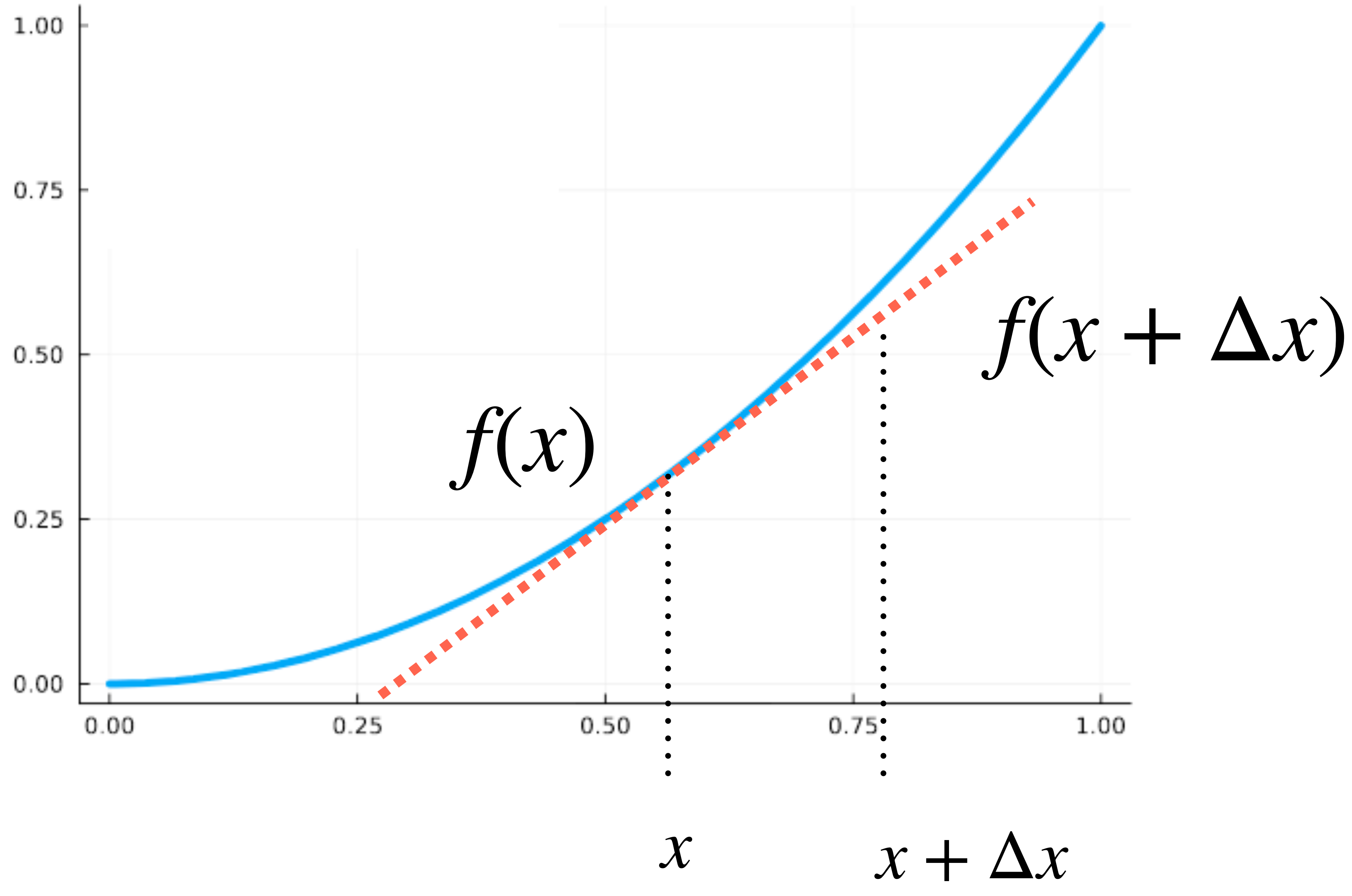


Don’t know derivative? Approximate!

Finite differences

Scalar functions

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$



$$\frac{\mathrm{d}f}{\mathrm{d}x} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(Small Δx)

Don't know function value? Approximate!

Extrapolation

$$f(x+\Delta x) \approx \frac{\mathrm{d}f}{\mathrm{d}x}\Delta x$$

Approximate this ← Know these

Finite differences

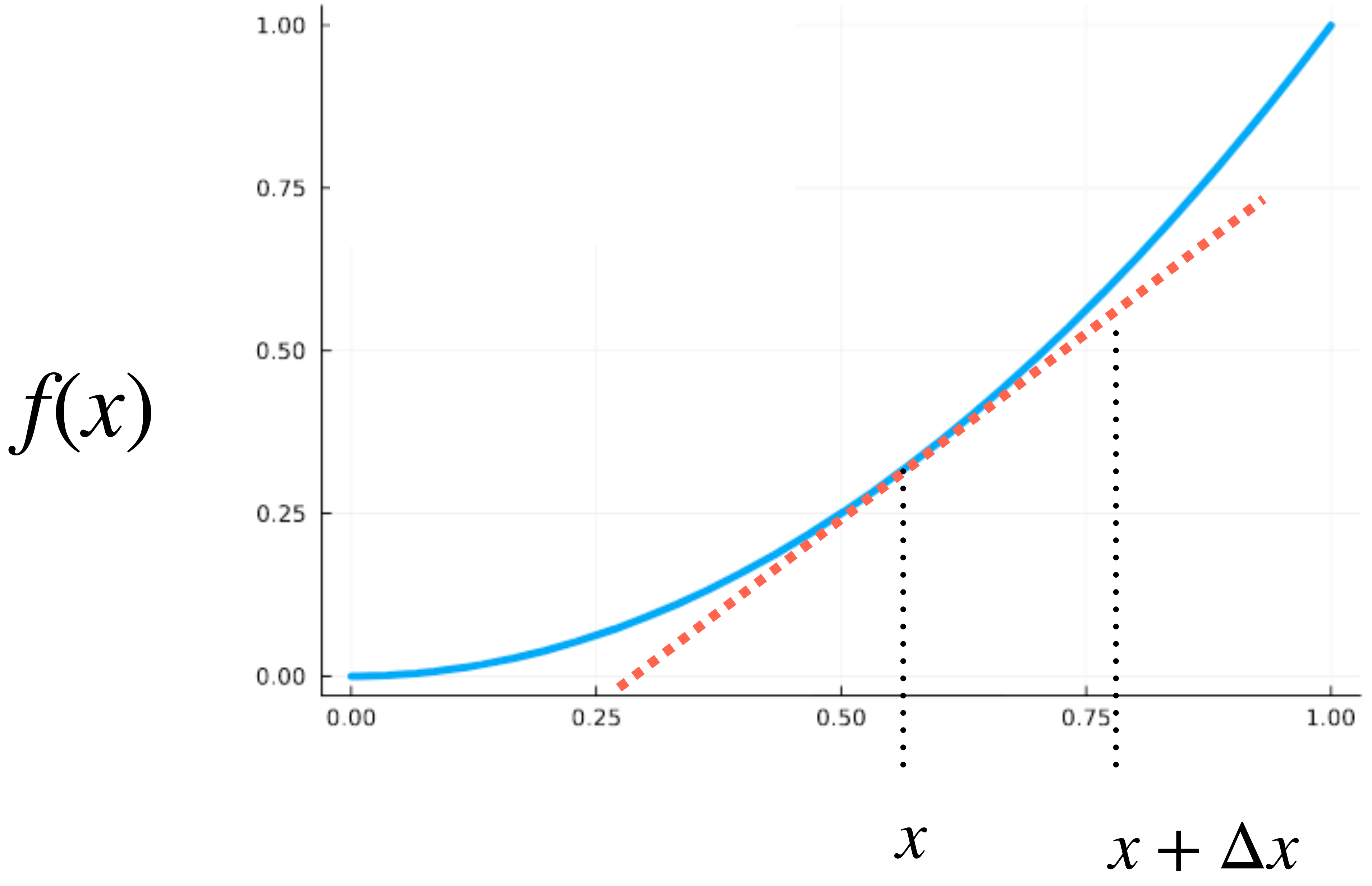
Approximation error

$$\frac{\mathrm{d}f}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$\frac{\mathrm{d}f}{\mathrm{d}x} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

(Small Δx)

Approximation error



Issues

How small?

How accurate?

Numerical error: $\dfrac{\text{small}}{\text{small}}$

Multidimensional finite differences

$$f : \mathbb{R}^m \to \mathbb{R}^n$$

$$\frac{\mathrm{df_i}}{\mathrm{d}x_j} \approx \frac{f_i(x + \Delta x_j) - f_i(x)}{\|\Delta x_j\|_2}$$

$$\Delta x_j = [0,0,\ldots \underset{\substack{\uparrow \\ \text{Entry } j}}{\Delta},0]$$

$$x = [x_1, x_2, \ldots, x_n]$$

$$\begin{array}{cc} & m \textbf{ columns} \\ \dfrac{\mathrm{df}}{\mathrm{d}x} = & \begin{bmatrix} \dfrac{\mathrm{df_1}}{\mathrm{d}x_1} & \dfrac{\mathrm{df_1}}{\mathrm{d}x_2}, \ldots, \dfrac{\mathrm{df_1}}{\mathrm{d}x_m} \\ \dfrac{\mathrm{df_2}}{\mathrm{d}x_1} & \dfrac{\mathrm{df_2}}{\mathrm{d}x_2}, \ldots, \dfrac{\mathrm{df_2}}{\mathrm{d}x_m} \\ \vdots & \vdots \\ \dfrac{\mathrm{df_n}}{\mathrm{d}x_1} & \dfrac{\mathrm{df_n}}{\mathrm{d}x_2}, \ldots, \dfrac{\mathrm{df_n}}{\mathrm{d}x_m} \end{bmatrix} & n \textbf{ rows} \end{array}$$

Must take $n \times m$ finite differences

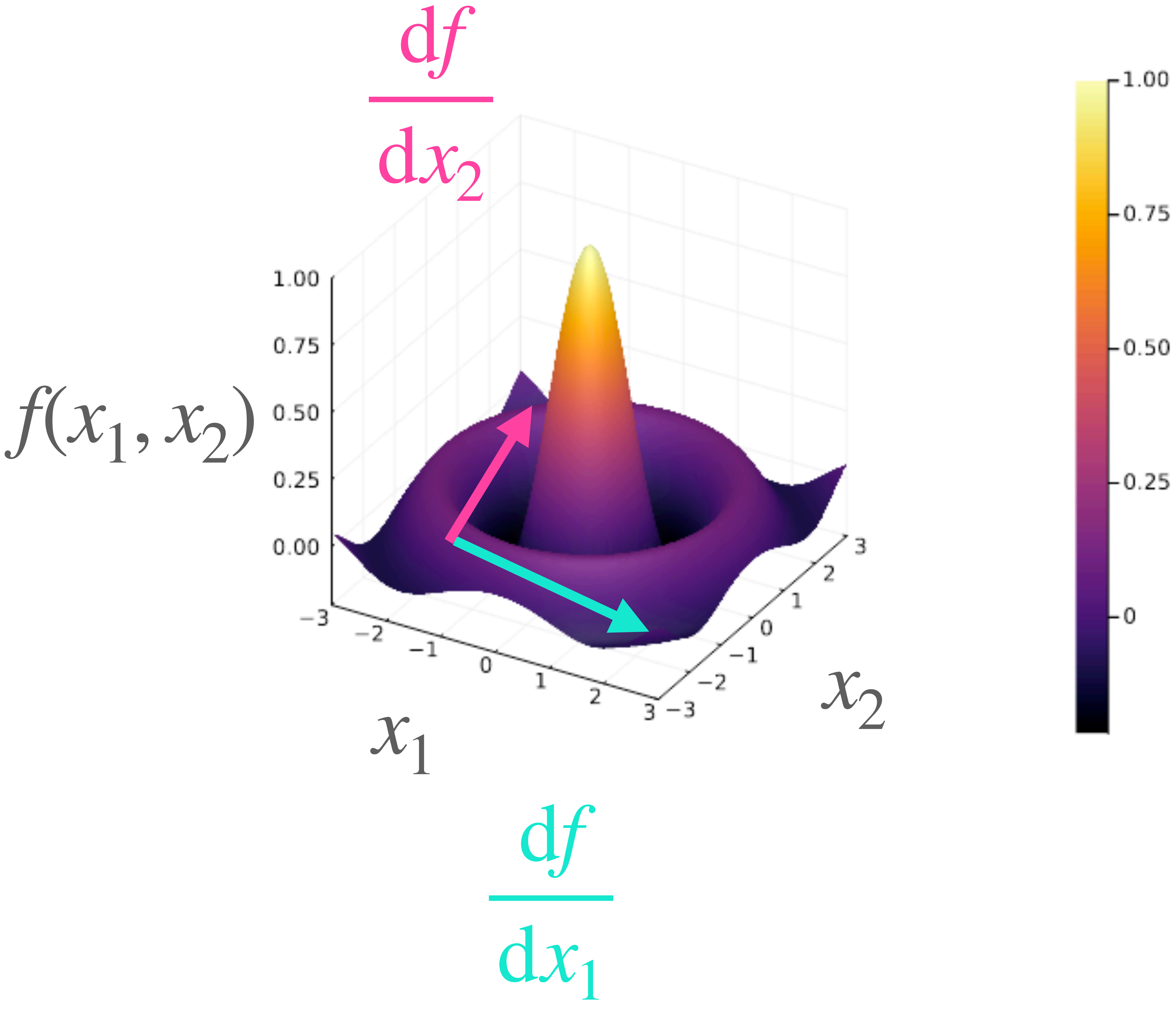
Example

$f : \mathbb{R}^2 \rightarrow \mathbb{R}$

$$\frac{\mathrm{df_i}}{\mathrm{d}x_j} \approx \frac{f_i(x + \Delta x_j) - f_i(x)}{\|\Delta x_j\|_2}$$

$\Delta x_1 = 0.01[1,0]$

$\Delta x_2 = 0.01[0,1]$



Example

$f : \mathbb{R}^2 \to \mathbb{R}$

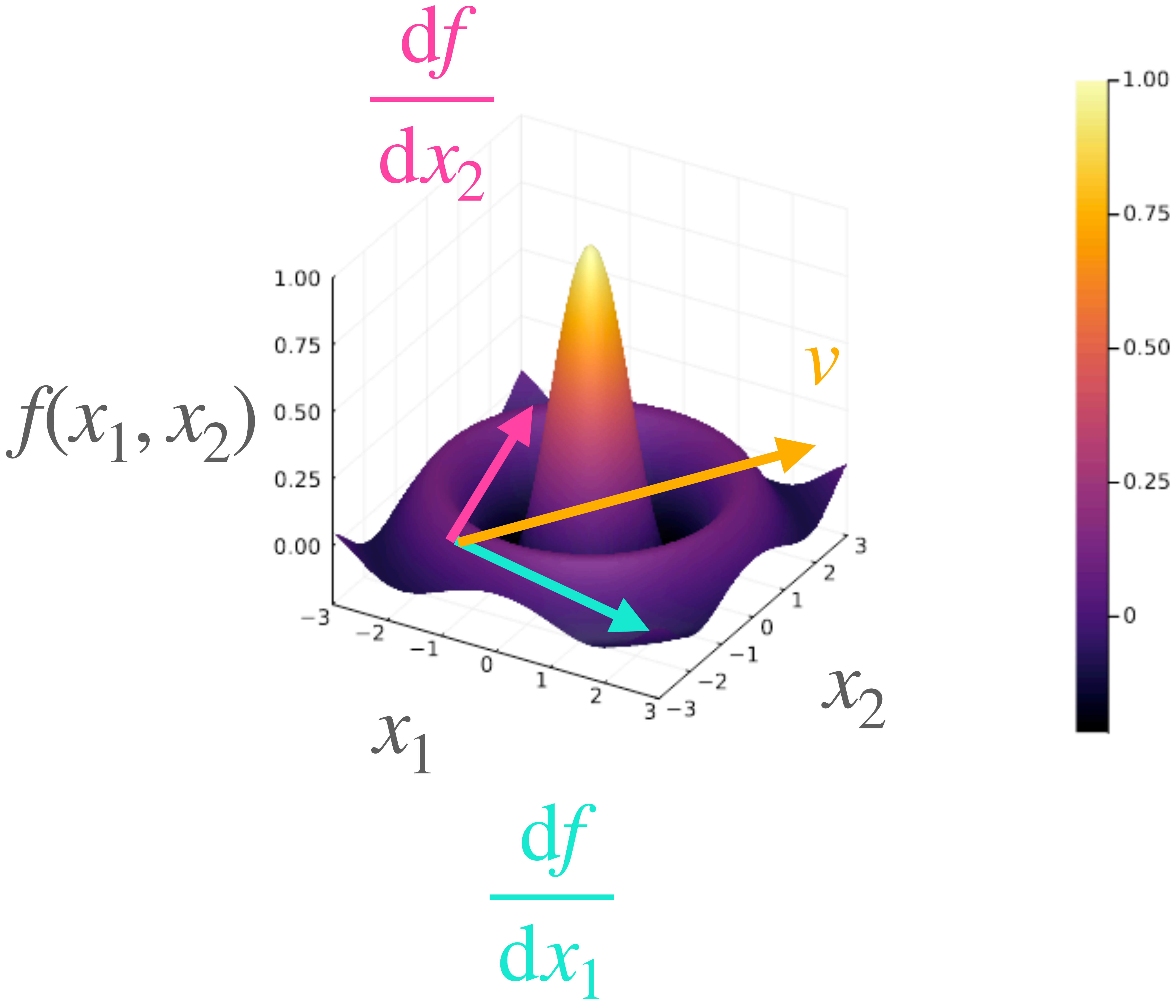
$$\frac{\mathrm{df_i}}{\mathrm{d}x_j} \approx \frac{f_i(x + \Delta x_j) - f_i(x)}{\|\Delta x_j\|_2}$$

$\Delta x_1 = 0.01[1,0]$

$\Delta x_2 = 0.01[0,1]$

$v = 3\Delta x_1 + 2\Delta x_2$

$f(x + v)$?



Example

$f : \mathbb{R}^2 \to \mathbb{R}$

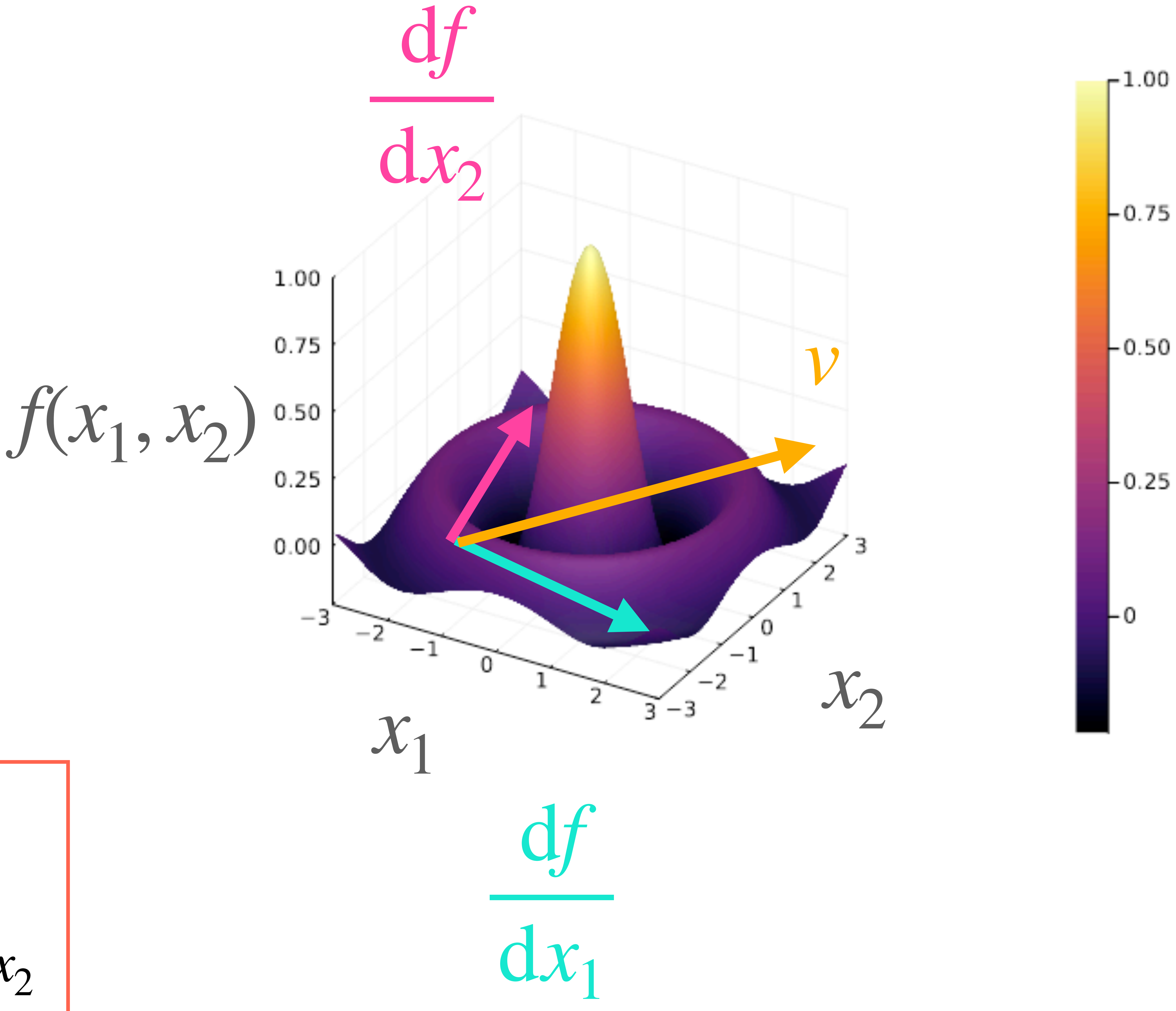
$$\frac{\mathrm{df_i}}{\mathrm{d}x_j} \approx \frac{f_i(x + \Delta x_j) - f_i(x)}{\|\Delta x_j\|_2}$$

$\Delta x_1 = 0.01[1,0]$

$\Delta x_2 = 0.01[0,1]$

$v = 3\Delta x_1 + 2\Delta x_2$

$$f(x + v) = f(x) + 3\frac{\mathrm{d}f}{\mathrm{d}x_1}\Delta x_1 + 2\frac{\mathrm{d}f}{\mathrm{d}x_2}\Delta x_2$$



Directional derivatives (more to come)

$f : \mathbb{R}^n \to \mathbb{R}$

$$v = [v_1, v_2, \ldots, v_n]$$

$$f(x + v)\,?$$

$$\frac{\mathrm{df}}{\mathrm{d}x} = [\,\bullet \quad \bullet \quad \ldots \quad \bullet\,]$$

$$\boxed{f(x + v) \approx f(x) + \sum_{i=1}^{n} v_i \frac{\mathrm{d}f}{\mathrm{d}x_i}}$$

Directional derivatives (more to come)

$f : \mathbb{R}^n \to \mathbb{R}$

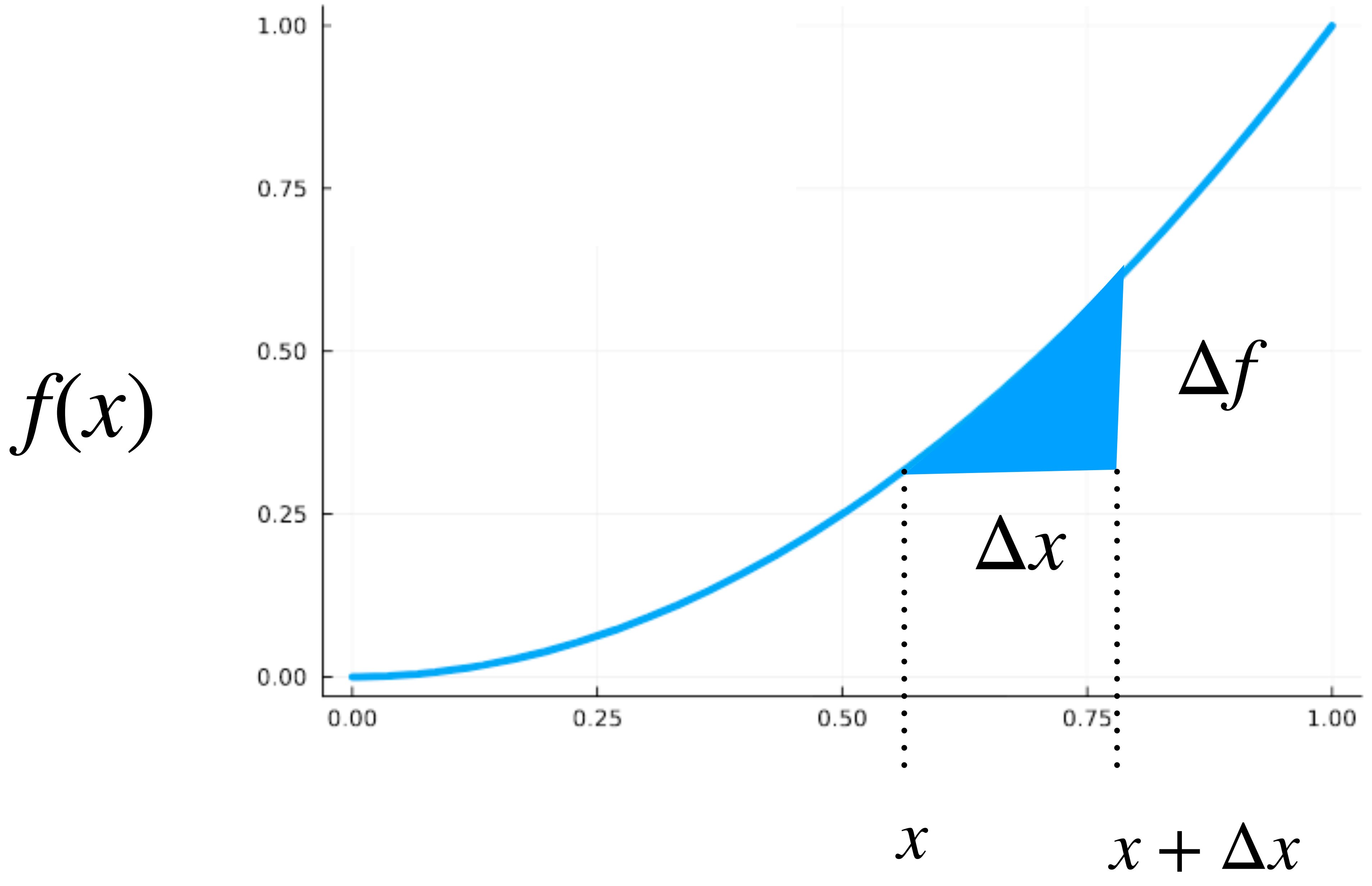
$$v = [v_1, v_2, \ldots, v_n]$$

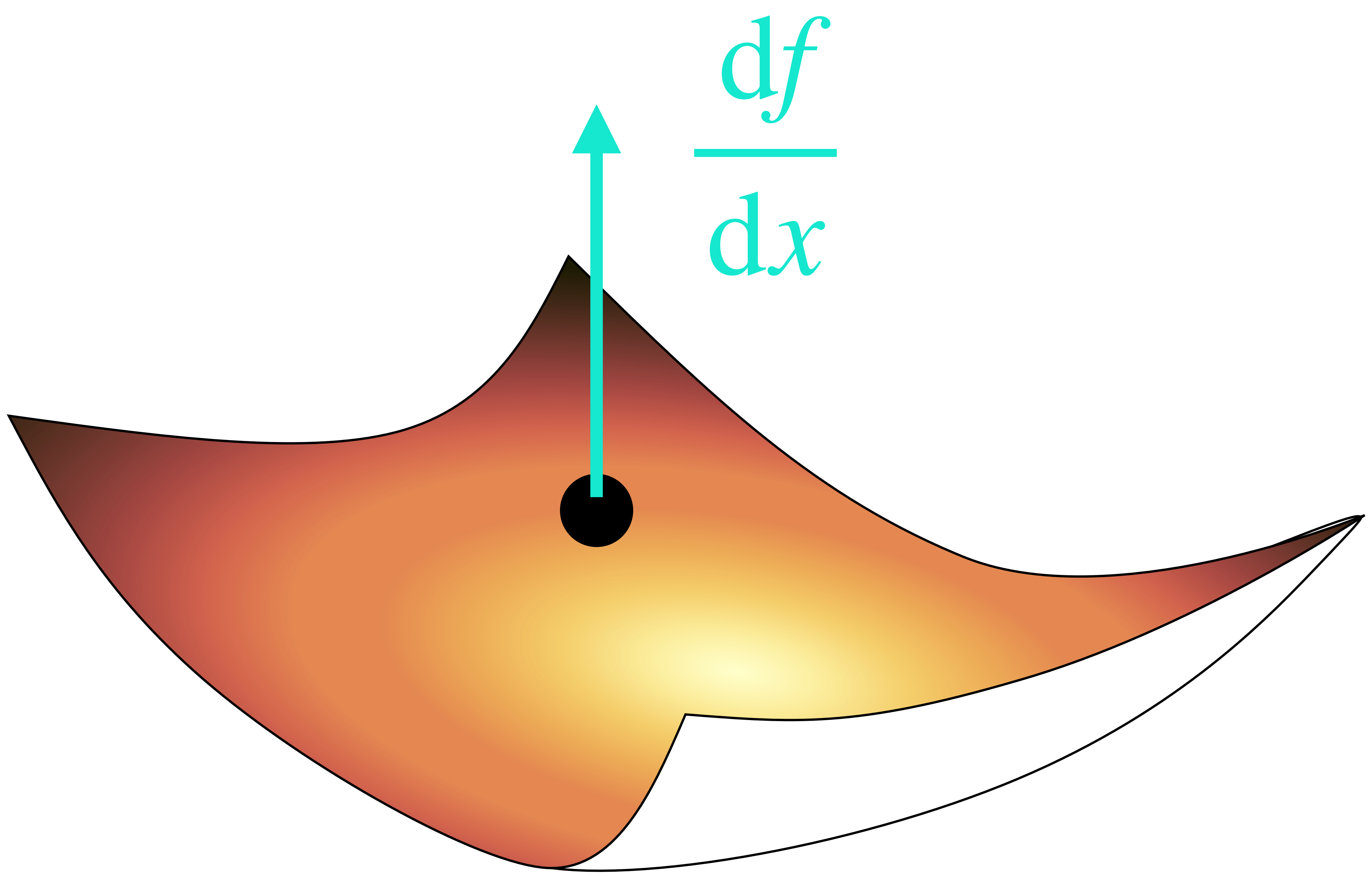
$$\frac{\mathrm{df}}{\mathrm{d}x} = [\, \bullet \quad \bullet \quad \ldots \quad \bullet \,]$$

$$\boxed{f(x+v) \approx f(x) + v^T \frac{\mathrm{d}f}{\mathrm{d}x}}$$

Dot product!

Visualising the multidimensional derivative





$$f(x+v) \approx f(x) + v^T \frac{\mathrm{d}f}{\mathrm{d}x}$$

$$v = \frac{\mathrm{d}f}{\mathrm{d}x}?$$