

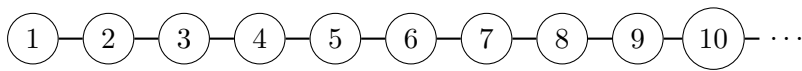
# CITS2211 Assignment 2

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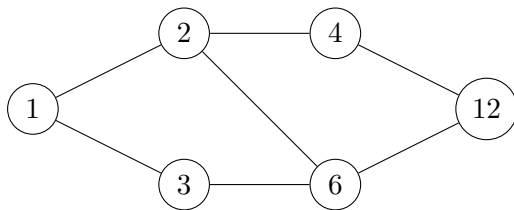
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## Question 1

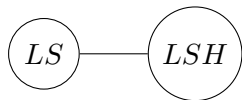
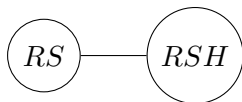
a)



b)



c)



## Question 2

a)

the relation is reflexive if  $\forall x \in \mathbb{R} ((x, x) \in R)$

subtracting any real number by itself results in zero which is an integer, Hence R is reflexive.

The relation is symmetric if for all  $(x, y) \in R$ ,  $(y, x) \in R$  if  $x \neq y$  we need to show that if  $x - y$  is an integer,  $y - x$  is an integer as well

let  $x - y = k$  where  $x \in \mathbb{Z}$

$y - x = -k$  (arithmetic)

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k is an integer so -k is an integer as

hence,  $y - x$  is an integer

Therefore, we have proved that is  $(x, y) \in R$  then  $(y, x) \in R$  and relation R is symmetric

The relation R is transitive if for all  $x, y, z \in \mathbb{R}$  if  $xRy$  and  $yRz$  then  $xRz$  holds

so we need to prove that if  $x - y \in \mathbb{Z}$  and  $y - z \in \mathbb{Z}$  then  $x - z \in \mathbb{Z}$

let  $x - y = k_1$  where  $k_1$  is an integer

let  $y - z = k_2$  where  $k_2$  is an integer

$$\begin{aligned}x - z &= (x - y) + (y - z) \text{ (adding and subtracting } y\text{)} \\&= k_1 + k_2\end{aligned}$$

both  $k_1$  and  $k_2$  are integers so  $k_1 + k_2$  is integer as well, so  $x - z$  is an integer and  $(x, z) \in R$  and the relation R is transitive

Q.E.D.

b)

i.

the equivalence class of any real number x is given by

$$[x] = \{y \in \mathbb{R} \mid y - x \in \mathbb{Z}\}$$

let  $y - x$  be an integer k then  $y = x + k$

Therefore, the equivalence class of any real number x consists of all numbers produced by adding an integer to x since the numbers integers is infinite, each equivalence class is infinite

ii.

as proved in part i every real number has an equivalence classes therefore, the total number of equivalence classes are infinite.

### Question 3

Assume a new generation starts every 50 years

We have 40 generations in 2000 years so the person should have  $2^{40}$  ancestors 2000 years ago the current population is less than 8 billion which is  $2^3 \times 10^9 = 2^{12} \times 5^9$  which is less than  $2^{39}$  the population has increased exponentially in 2000 years.

2000 years ago was certainly way less than  $2^{40}$

by pigeonhole principle at least two ancestors of the person were the same person

Q.E.D.

## Question 4

a)

Values of  $\pi(k_1, k_2)$

5	20	26	33	41	50	60
4	14	19	25	32	40	49
3	9	13	18	24	31	39
2	5	8	12	17	23	30
1	2	4	7	11	16	22
0	0	1	3	6	10	15
	0	1	2	3	4	5

$k_1$

b)

The function  $\pi$  is bijection between  $\mathbb{N}^2$  and  $\mathbb{N}$ . We need to prove that there exists a bijection between  $\mathbb{N}^n$  and  $\mathbb{N}$

We will prove this using induction

let  $P(k)$  be  $|\mathbb{N}^k| = |\mathbb{N}|$

**Base case:** The base case is  $n = 1$

$\mathbb{N}^1 = \mathbb{N}$

$|\mathbb{N}| = |\mathbb{N}|$  is trivially true

**Inductive Case:**

We need to prove that  $P(k) \rightarrow P(k+1)$  for some arbitrary  $k \geq 1$

**Inductive hypothesis:**

We can assume that  $P(k)$  holds for an arbitrary  $k \geq 1$

**Inductive step:**

now we need to show that  $P(k+1)$  holds given  $P(k)$  is true

we know that there is a bijection  $f$  from  $\mathbb{N}^n$  to  $\mathbb{N}$  from the inductive hypothesis

So,  $\mathbb{N}^n$  can be mapped to a single natural number  $\mathbb{N}^{n+1}$  can be written as  $\mathbb{N}^n \times \mathbb{N}$

$\mathbb{N}^n \times \mathbb{N}$  can further be mapped using the function  $\pi$

since we know that both  $\pi$  and  $f$  function are bijections, they can be combined to a single bijective function from  $\mathbb{N}^{n+1}$  to  $\mathbb{N}$

Therefore,  $|\mathbb{N}^{n+1}| = |\mathbb{N}|$

Q.E.D.

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## Question 5

We need to prove that a bijection from  $B$  to  $A^B$  doesn't exist

assume a bijection  $f$  from  $B$  to  $A^B$  exists

for every  $x \in B$ ,  $f(x) \in A^B$ .  $f(x)$  is a function from  $B$  to  $A$ .

Now, we can define new function  $f'$  such that  $f'(x) \neq (f(x))(x)$

since,  $|A| \geq 2$  we can define this function by choosing a different value from set  $A$  for every value of  $x \in B$  (using the diagonal argument)

the function  $f$  is a bijection so there must be some value of  $x \in B$  such that  $f(x) = f'$

This contradicts with the definition of  $f'$  function since this would imply that  $f(x)(y) = f'(y)$  for all  $y \in B$

our assumption that  $f$  is a bijection must be false Therefore, a bijection from set  $B$  to  $A^B$  doesn't exist and  $|A^B| \neq |B|$ .

Q.E.D.

## Question 6

a)

**States:**  $Q = \{q_1, q_2, q_3\}$

**Start state:**  $q_0 = q_1$

**Alphabet:**  $\Sigma = \{0, 1\}$

**Accepting states:**  $F = \{q_1, q_3\}$

**State transition:**  $\delta : Q \times \Sigma \rightarrow \mathcal{P}(Q)$

b)

FSM recognises the symbols 1 and 0

- $q_1$  is the starting state and it stays there if it reads 0 and moves to  $q_2$  if it reads 1
- in state  $q_2$  it doesn't move if it reads 0 and moves to  $q_3$  if it reads 1
- in state  $q_3$  it only accepts the symbol 0 and stays in  $q_3$

states  $q_1$  and  $q_3$  are the accepting states

Thus, the FSM accepts strings that contain exactly two 1s and any number of 0s including none anywhere

c)

$0^*10^*10^*$

## Question 7

**States:**

set of States of the DFSM is the powerset of the states of the original NFSM

$$Q = \{\phi, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}$$

**Initial state:**

The initial state of the DFSM is a singleton set containing the initial state of the NFSM  
 $\{q_0\}$  is the initial state

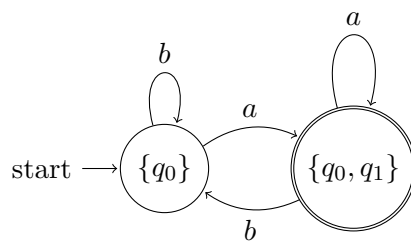
**Accepting states:**

accepting states of the DFSM are the states containing any accepting states of NFSM  
 $\{q_1\}$  and  $\{q_0, q_1\}$  are the accepting states

**The Transition Function:**

The transition function,  $\delta$ , returns the union of all states that are reachable via the original transition function  $\delta'$ , by consuming the input from any of the NFSM states in the current DFSM state, i.e.,  $\delta(s, x) = \bigcup \{\delta'(q, x) \mid q \in s\}$

The Result could be simplified by removing the state  $\{q_1\}$  since there is no way to reach the state and it's part of the combined state  $\{q_0, q_1\}$



## Question 8

a)

- i.  $b^*ab^*$
- ii.  $(ab + b)^*(aa)(ba + b)^*$
- iii.  $(ab + aab + b)^*(aaa)(ba + baa + b)^*$

b)

- $((11)^*1^*)^*$  can be simplified to  $1^*$  since  $(11)^*$  means even number of 1s and the  $1^*$  provides additional 1s which means an empty string of any string made of 1s is accepted
- $(11 + 1)^*$  can be simplified to  $1^*$  because  $11^*$  would be even number of 1s and  $1^*$  would make any number of 1s including none.
- $(0 + \epsilon)^*$  can be simplified to  $0^*$  since  $\epsilon^*$  can be covered by no zeroes included in  $0^*$
- the expression can be further simplified by combining  $1^* + 1^*$  to  $1^*$

final expression is  $1^* + 0^*$

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## Question 9

The claim is similar to the pumping lemma but the pumping length needs to be different for every language the condition  $|w| \geq 1$  is not sufficient for all languages

### Counter example:

Consider the language  $(ab)^*$  let's take  $w$  as  $ab$ ,  $|w| \geq 1$  so we satisfy the condition if  $y = a$ , and  $i = 2$ , the resulting string  $aab$  is not in the language  $(ab)^*$  similarly if  $y = b$  and  $i = 2$  the resulting string  $abb$  is not in the language those are the only two possibilities of  $y$  since cardinality of  $y$  needs to be greater than 1

## Question 10

a)  $\{a^n b^m a^n \mid m, n \in \mathbb{N}\}$

The automata reads at least one 'a' and pushes each 'a' read onto the stack.

It requires at least one b and doesn't change the stack.

Then it requires at least one a and pops a from the stack for every a read. The PDA accepts if the stack is empty after reading the input.

a)  $\{a^n b^m \mid n \leq m \leq 2n \wedge m, n \in \mathbb{N}\}$

The PDA reads at least one 'a' and pushed each 'a' read onto the stack

When it starts reading 'b's it non deterministically chooses whether to pop 'a' from the stack for each b read or pop 'a' on reading two b's. The automata accepts if the stack is empty after reading the input. This ensures that for valid strings there is atleast one path to acceptance and the numebr of b's read is in the range  $(n, 2n)$ .

## Question 11

a)

$$A \rightarrow 0A \mid 1A \mid \epsilon$$

$$S \rightarrow 0A1 \mid 1A0$$

if the string starts at 0 then it must end with 1 and vice versa. within the 1 and 0, A accepts any combination of 1s and 0s

b)

$$A \rightarrow 01A \mid 10A \mid \epsilon$$

$$S \rightarrow ASA \mid 1$$

A has equal number of ones and zeroes and S adds the additional ones.

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## Question 12

Assume a turing machine  $X$  exists such that it returns 1 if  $M$  accepts  $x$  and returns 0 otherwise

Construct a new turing machine  $Y$  that behaves as follows -

- Constructs a new machine  $Z$  which takes input  $x$ , runs  $M$  on  $x$  then accepts  $x$
- executes  $X(Z)$

if  $M$  halts on  $x$  then then,  $Z$  accepts the input  $x$  and  $X$  returns 1

$$Y(M, x) = X(Z) = 1$$

if  $M$  does not halt on  $x$  then  $Z$  does not halt and does not accept  $x$

$$Y(M, a) = X(Z) = 0$$

hence,  $X$  decides the halting problem

this is a contradiction since the halting problem is undecidable.

Our assumption that such a turing machine  $X$  must be false.