CITS2211 Assignment 2

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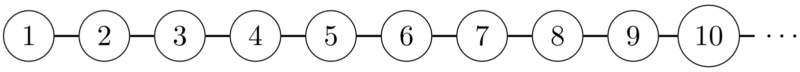
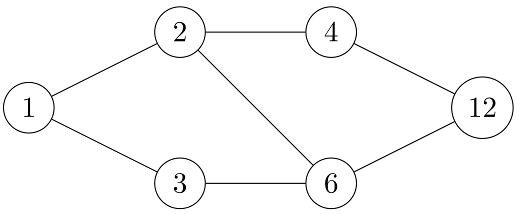
Question 1

a)

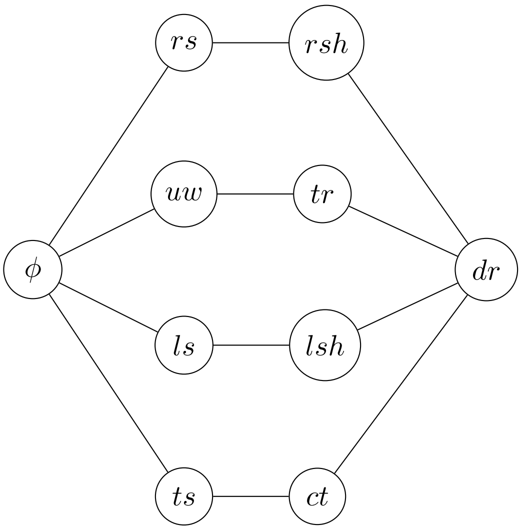


b)



c)

rs = right sock, ls = left sock, lsh = left shoe, uw = underwear, tr = trousers, rsh = right shoe, ts = t-shirt, ct = coat, dr = dressed



Question 2

a)

the relation is reflexive if $\forall x \in \mathbb{R}, ((x, x) \in R)$
subtrating any real number by itself results in zero which is an integer, Hence R is reflexive.

The relation is symmetric if for all $(x, y) \in R$, $(y, x) \in R$ if $x \neq y$
we need to show that if x - y is an integer, y - x is an integer as well

$$\text{let } x - y = k \text{ where } x \in \mathbb{Z}$$
$$y - x = -k \text{ (arithmetic)}$$

if k is an integer, -k is an integer as well
hence, y - x is an integer
Therefore, we have proved that if $(x, y) \in R$ then $(y, x) \in R$ and relation R is symmetric

The relation R is transitive if for all $x, y, z \in \mathbb{R}$ if xRy and yRz then xRz holds
so we need to prove that if $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ then $x - z \in \mathbb{Z}$

$$\begin{aligned} \text{let } x - y &= k_1 \text{ where } k_1 \text{ is an integer} \\ \text{let } y - z &= k_2 \text{ where } k_2 \text{ is an integer} \\ x - z &= (x - y) + (y - z) \text{ (adding and substracting y)} \\ &= k_1 + k_2 \end{aligned}$$

both k_1 and k_2 are integers so $k_1 + k_2$ is integer as well, so x - z is an integer and $(x, z) \in R$ and the relation R is transitive

We have proved that R is reflexive, transitive and symmetric Therefore R is an equivalence relation.

b)

i.
the equivalence class of any real number x is given by
$[x] = \{y \in \mathbb{R} \mid y - x \in \mathbb{Z}\}$
let y - x be an integer k then y = x + k
Therefore, the equivalence class of any real number x consists of all numbers produced by adding an integer to x since the numbers of integers is infinite, each equivalence class is infinite

ii.
as proved in part **i** every real number has an equivalence classes therefore, the total number of equivalence classes are infinite.

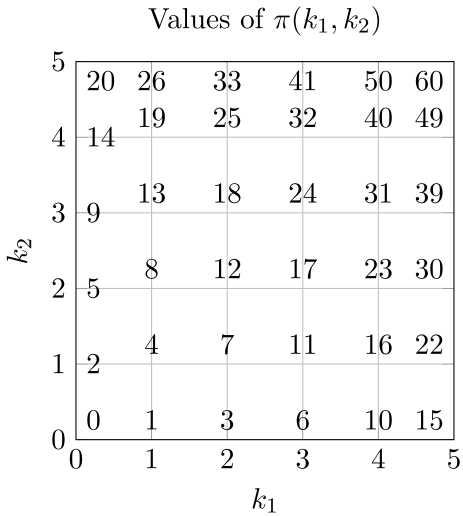
Question 3

Assume a new generation starts every 50 years
We have 40 generations in 2000 years so the person should have 2^{40} ancestors 2000 years ago
the current poopulation is less than 8 billion which is $2^3 \times 10^9 = 2^{12} \times 5^9$ which is less than 2^{39}
the population has increased exponentially in 2000 years.

The population 2000 years ago was certainly way less than 2^{40}
The population is less than the number of ancestors
Therefore, by pigeonhole principle at least two ancestors of the person were the same person
Q.E.D.

Question 4

a)



b)

If the function π is bijection between $\mathbb{N}^2$ and $\mathbb{N}$. We need to prove that there exists a bijection between $\mathbb{N}^n$ and $\mathbb{N}$
We will prove this using induction
let $P(k)$ be $|\mathbb{N}^k| = |\mathbb{N}|$

Base case: The base case is n = 1
$\mathbb{N}^1 = \mathbb{N}$
$|\mathbb{N}| = |\mathbb{N}|$ is trivially true

Inductive Case:
We need to prove that $P(k) \rightarrow P(k+1)$ for some arbitary $k \geq 1$

Inductive hypothesis:
We can assume that $P(k)$ holds for an arbitary $k \geq 1$

Inductive step:
now we need to show that $P(k+1)$ holds given $P(k)$ is true
we know that there is a bijection f from $\mathbb{N}^k$ to $\mathbb{N}$ from the inductive hypothesis
So, $\mathbb{N}^k$ can be mapped to a single natural number

$\mathbb{N}^{k+1}$ can be written as $\mathbb{N}^k \times \mathbb{N}$
$\mathbb{N}^k \times \mathbb{N}$ can further be mapped to $\mathbb{N}$ using the function π

Construct a new function $f\prime$ from N^{k+1} to N such that
$f\prime = \pi(f(a_1, a_2 \ldots a_k), a_{k+1})$ where $a \in \mathbb{N}$
since both π and f are biections $f\prime$ is a bijection as well

Hence, The function $f\prime$ is a bijection from N^{k+1} to N
Therefore, $|\mathbb{N}^{k+1}| = |\mathbb{N}|$
Q.E.D.

Question 5

We need to prove that a bijection from B to A^B doesn't exist
assume a bijection f from B to A^B exists
for every $x \in B$, $f(x) \in A^B$. f(x) is a function from B to A.

Now, we can define new function f' such that $f'(x) \neq (f(x))(x)$
since, $|A| \geq 2$ we can define this function by choosing a different value from set A (using the diagonal argument)
the function f is a bijection so there must be some value of $x \in B$ such that $f(x) = f'$

This contradicts with the definition of f' function since this would imply that $f(x)(x) = f'(x)$ our assumption that f is a bijection must be false Therefore, a bijection from set B to A^B doesn't exist and $|A^B| \neq |B|$.
Q.E.D.

Question 6

a)

States: $Q = \{q_1, q_2, q_3\}$
Start state: $q_0 = q_1$
Alphabet: $\Sigma = \{0, 1\}$
Accepting states: $F = \{q_1, q_3\}$
State transition: $\delta : Q \times \Sigma - > Q$

b)

FSM recognises the symbols 1 and 0

- q_1 is the starting state and it stays there if it reads 0 and moves to q_2 if it reads 1
- in state q_2 it doesn't move if it reads 0 and moves to q_3 if it reads 1
- in state q_3 it only accepts the symbol 0 and stays in q_3

states q_1 and q_3 are the accepting states
Thus, the FSM accepts strings that contain exactly two 1s and any number of 0s including none anywhere

c)

0*10*10*

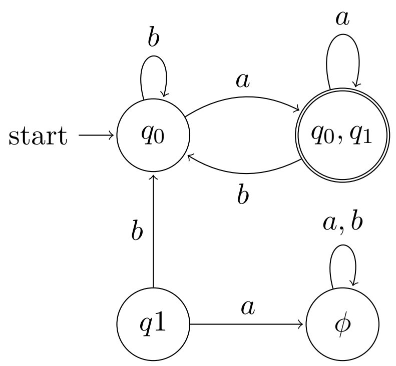
Question 7

States:
set of States of the DFSM is the powerset of the states of the original NFSM
$Q = \{\phi, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}$

Initial state:
The initial state of the DFSM is a singleton set containing the initial state of the NFSM
$\{q_0\}$ is the initial state

Accepting states:
accepting states of the DFSM are the states containing any accepting states of NFSM
$\{q_1\}$ and $\{q_0, q_1\}$ are the accepting states

The Transition Function:
The transition function, δ , returns the union of all states that are reachable via the original transition function δ' , by consuming the input from any of the NFSM states in the current DFSM state, i.e., $\delta(s, x) = \bigcup\{\delta'(q, x) \mid q \in s\}$



The Result could be simplified by removing the q_1 and null state since they are unreachable.

Question 8

a)

i. b*ab*
ii. $((ab) + b)^*(aa)((ba) + b)^*$
iii. $((ab) + (aab) + b)^*(aaa)((ba) + (baa) + b)^*$

b)

- ((11)*)1*)* can be simplified to 1* since (11)* means even number of 1s and the 1* provides additional 1s which means an empty string of any string made of 1s is accepted

- (11 + 1)* can be simplified to 1* because 11* would be even number of 1s and 1* would make any number of 1s including none.
- (0 + ϵ)* can be simplified to 0* since ϵ* can be covered by no zeroes included in 0*
- the expression can be further simplified by combining 1* + 1* to 1*

final expression is 1* + 0*

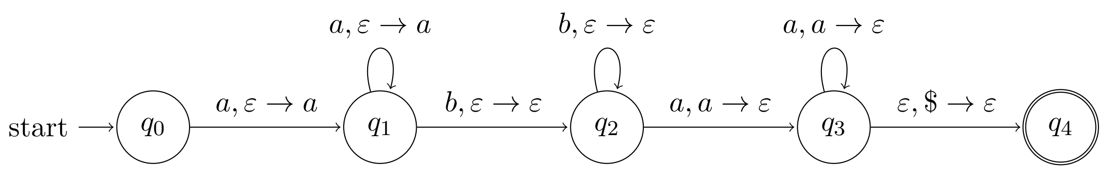
Question 9

The claim is similar to the pumping lemma but the pumping length needs to be different for every language the condition $|w| \geq 1$ is not sufficient for all languages

Counter example:
Consider the language (ab)* let's take w as ab, $|w| \geq 1$ so we satisfy the condition
if y = a, and i = 2, the resulting string aab is not in the language (ab)*
similarly if y = b and i = 2 the resulting string abb is not in the language
those are the only two possibilities of y since cardinality of y needs to be greater than 1
Therefore such an expression does not exist in this language.

Question 10

a) $\{a^n b^m a^n \mid m, n \in \mathbb{N}\}$



The pda reads at least one 'a' and pushes each 'a' read onto the stack.
It requires at least one b and doesn't change the stack.
Then it requires at least one a and pops a from the stack for every a read. The PDA accepts if the stack is empty after reading the input.

b) $\{a^n b^m \mid n \leq m \leq 2n \wedge m, n \in \mathbb{N}\}$

The PDA reads at least one 'a' and pushed each 'a' read onto the stack
When it starts reading 'b's it non deterministically chooses whether to pop 'a' from the stack for each b read or pop 'a' on reading two b's. The automata accepts if the stack is empty after reading the input. This ensures that for valid strings there is atleast one path to acceptance and the number of b's read is in the range (n, 2n).

Question 11

a)

$A \to 0A \mid 1A \mid \epsilon$

$S \rightarrow 0A1 \mid 1A0$

S is the start point
if the string starts at 0 then it must end with 1 and vice versa. within the 1 and 0, A accepts any combination of 1s and 0s

b)

$A \rightarrow 01A \mid 10A \mid \epsilon$
$S \rightarrow ASA \mid 1$

S is the start point
A has equal number of ones and zeroes and S adds the additional ones.

Question 12

Assume a turing machine X exists such that it returns 1 if M accepts x and returns 0 otherwise

Construct a new turing machine Y that behaves as follows -

- Constructs a new machine Z which takes input x, runs M on x then accepts x
- executes X(Z)

if M halts on x then then, Z accepts the input x and X returns 1
Y(M, x) = X(Z) = 1
if M does not halt on x then Z does not halt and does not accept x
Y(M, a) = X(Z) = 0

hence, X decides the halting problem
this is a contradiction since the halting problem is undecidable.
Our assumption that such a turing machine X must be false.