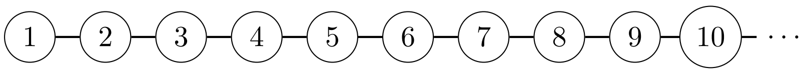
CITS2211 Assignment 2

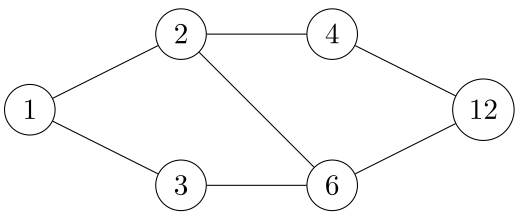
Name: Baasil Siddiqui
Student Id: 23895849

Question 1

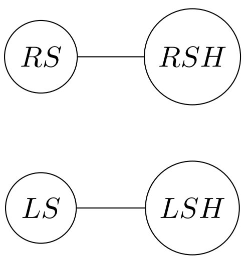
a)



b)



c)



Question 2

a)

the relation is reflexive if $\forall x \in \mathbb{R}((x,x) \in R)$
subtrating any real number by itself results in zero which is an integer, Hence R is reflexive.

The relation is symmetric if for all $(x,y) \in R$, $(y,x) \in R$ if $x \neq y$ we need to show that if x - y is an integer, y - x is an integer as well

$$\text{let } x - y = k \text{ where } x \in \mathbb{Z}$$
$$y - x = -k \text{ (arithmetic)}$$

k is an integer so -k is an integer as
hence, y - x is an integer
Therefore, we have proved that is $(x, y) \in R$ then $(y, x) \in R$ and relation R is symmetric
The relation R is transitive if for all $x, y, z \in \mathbb{R}$ if xRy and yRz then xRz holds
so we need to prove that if $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ then $x - z \in \mathbb{Z}$

$$\begin{aligned}
\text{let } x - y &= k_1 \text{ where } k_1 \text{ is an integer} \\
\text{let } y - z &= k_2 \text{ where } k_2 \text{ is an integer} \\
x - z &= (x - y) + (y - z) \text{ (adding and substracting y)} \\
&= k_1 + k_2
\end{aligned}$$

both k_1 and k_2 are integers so $k_1 + k_2$ is integer as well, so x - z is an integer and $(x, z) \in R$ and the relation R is transitive

Q.E.D.

b)

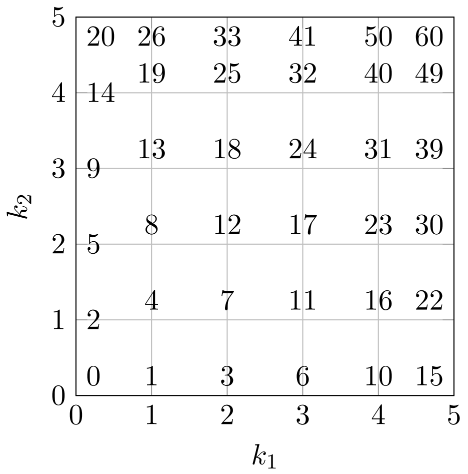
i.
the equivalence class of any real number x is given by
$[x] = \{y \in \mathbb{R} \mid y - x \in \mathbb{Z}\}$
let y - x be an integer k then y = x + k
Therefore, the equivalence class of any real number x consists of all numbers produced by adding an integer to x since the numbers integers is infinite, each equivalence class is infinite

ii.
as proved in part **i** every real number has an equivalence classes therefore, the total number of equivalence classes are infinite.

Question 4

a)

Values of $\pi(k_1, k_2)$



b)

The function π is bijection between $\mathbb{N}^2$ and $\mathbb{N}$. We need to prove that there exists a bijection between $\mathbb{N}^n$ and $\mathbb{N}$
We will prove this using induction
let $P(k)$ be $|\mathbb{N}^k| = |\mathbb{N}|$

<u>**Base case:**</u> The base case is n = 1
$\mathbb{N}^1 = \mathbb{N}$
$|\mathbb{N}| = |\mathbb{N}|$ is trivially true

<u>**Inductive Case:**</u>
We need to prove that $P(k) \to P(k+1)$ for some arbitary $k \geq 1$

Inductive hypothesis:
We can assume that $P(k)$ holds for an arbitary $k \geq 1$

Inductive step:
now we need to show that $P(k+1)$ holds given $P(k)$ is true
we know that there is a bijection f from $\mathbb{N}^n$ to $\mathbb{N}$ from the inductive hypothesis
So, $\mathbb{N}^n$ can be mapped to a single natural number $\mathbb{N}^{n+1}$ can be written as $\mathbb{N}^n \times \mathbb{N}$
$\mathbb{N}^n \times \mathbb{N}$ can further be mapped using the function π
since we know that both π and f function are bijections, they can be combined to a single bijective function from $\mathbb{N}^{n+1}$ to $\mathbb{N}$
Therefore, $|\mathbb{N}^{n+1}| = |\mathbb{N}|$
Q.E.D.

Question 5

We need to prove that a bijection from B to A^B doesn't exist
assume a bijection f from B to A^B exists
for every $x \in B$, $f(x) \in A^B$. f(x) is a function from B to A.
Now, we can define new function f' such that $f'(x) \neq (f(x))(x)$
since, $|A| \geq 2$ we can define this function by choosing a different value from set A for every value of $x \in B$ (using the diagonal argument)
the function f is a bijection so there must be some value of $x \in B$ such that $f(x) = f'$

This contradicts with the definition of f' function since this would imply that $f(x)(y) = f'(y)$ for all $y \in B$
our assumption that f is a bijection must be false Therefore, a bijection from set B to A^B doesn't exist and $|A^B| \neq |B|$.
Q.E.D.

Question 6

a)

States: $Q = \{q_1, q_2, q_3\}$
Start state: $q_0 = q_1$

Alphabet: $\Sigma = \{0, 1\}$
Accepting states: $F = \{q_1, q_3\}$
State transition: $\delta : Q \times \Sigma- > \mathcal{P}(Q)$

b)

FSM recognises the symbols 1 and 0

- q_1 is the starting state and it stays there if it reads 0 and moves to q_2 if it reads 1
- in state q_2 it doesn't move if it reads 0 and moves to q_3 if it reads 1
- in state q_3 it only accepts the symbol 0 and stays in q_3

states q_1 and q_3 are the accepting states
Thus, the FSM accepts strings that contain exactly two 1s and any number of 0s including none anywhere

c)

0*10*10*

Question 7

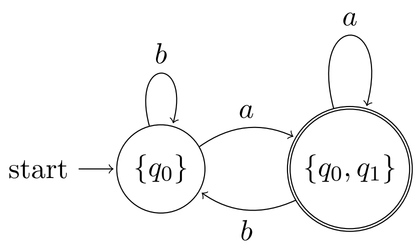
States:
set of States of the DFSM is the powerset of the states of the original NFSM
$Q = \{\phi, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}$

Initial state:
The initial state of the DFSM is a singleton set containing the initial state of the NFSM
$\{q_0\}$ is the initial state

Accepting states:
accepting states of the DFSM are the states containing any accepting states of NFSM
$\{q_1\}$ and $\{q_0, q_1\}$ are the accepting states

The Transition Function:
The transition function, δ , returns the union of all states that are reachable via the original transition function δ' , by consuming the input from any of the NFSM states in the current DFSM state, i.e., $\delta(s, x) = \bigcup\{\delta'(q, x) \mid q \in s\}$

The Result could be simplified by removing the state $\{q_1\}$ since there is no way to reach the state and it's part of the combined state $\{q_0, q_1\}$



Question 8

a)

b*ab*aab*aaab*

b)

- ((11)*)1*)* can be simplified to 1* since (11)* means even number of 1s and the 1* provides additional 1s which means an empty string of any string made of 1s is accepted
- (11 + 1)* can be simplified to 1* because 11* would be even number of 1s and 1* would make any number of 1s including none.
- (0 + ϵ)* can be simplified to 0* since ϵ* can be covered by no zeroes included in 0*
- the expression can be further simplified by combining 1* + 1* to 1*

final expression is 1* + 0*

Question 9

The claim is similar to the pumping lemma but the pumping length needs to be different for every language the condition $|w| \geq 1$ is not sufficient for all languages

Counter example:
Consider the language (ab)* let's take w as ab, $|w| \geq 1$ so we satisfy the condition
if y = a, and i = 2, the resulting string aab is not in the language (ab)*
similarly if y = b and i = 2 the resulting string abb is not in the language
those are the only two possibilities of y since cardinality of y needs to be greater than 1

Question 10

a) $\{a^n b^m a^n \mid m, n \in \mathbb{N}\}$

The automata reads at least one 'a' and pushes each 'a' read onto the stack.
It requires at least one b and doesn't change the stack.
Then it requires at least one a and pops a from the stack for every a read. The PDA accepts if the stack is empty after reading the input.

a) $\{a^n b^m \mid n \leq m \leq 2n \wedge m, n \in \mathbb{N}\}$

The PDA reads at least one 'a' and pushed each 'a' read onto the stack
When it starts reading 'b's it non deterministically chooses whether to pop 'a' from the stack for each b read or pop 'a' on reading two b's. The automata accepts if the stack is empty after reading the input. This ensures that for valid strings there is atleast one path to acceptance and the numebr of b's read is in the range (n, 2n).

Question 11

a)

$A \to 0A \mid 1A \mid \epsilon$
$S \to 0A1 \mid 1A0$

if the string starts at 0 then it must end with 1 and vice versa. within the 1 and 0, A accepts any combination of 1s and 0s

b)

$A \to 01A \mid 10A \mid \epsilon$
$S \to AS \mid SA \mid 1$

A has equal number of ones and zeroes and S adds the additional ones.