CITS2211 Assignment 2

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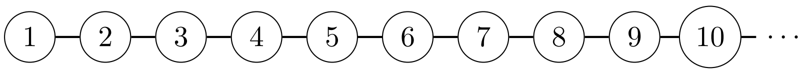
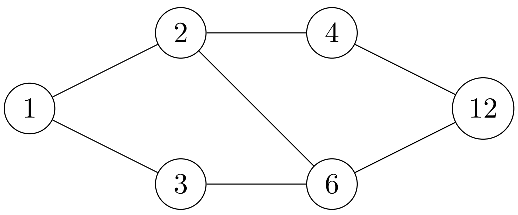
Question 1

a)

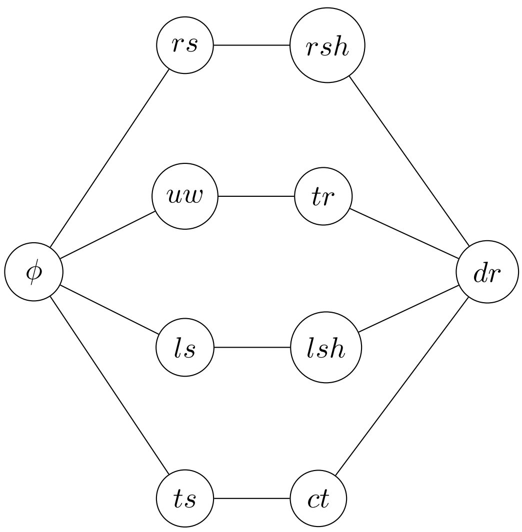


b)



c)

rs = right sock, ls = left sock, lsh = left shoe, uw = underwear, tr = trousers, rsh = right shoe, ts = t-shirt, ct = coat, dr = dressed



Question 2

a)

the relation is reflexive if $\forall x \in \mathbb{R}, ((x, x) \in R)$
subtrating any real number by itself results in zero which is an integer, Hence R is reflexive.

The relation is symmetric if for all $(x, y) \in R$, $(y, x) \in R$ if $x \neq y$
we need to show that if x - y is an integer, y - x is an integer as well

$$\begin{aligned} \text{let } x - y &= k \text{ where } x \in \mathbb{Z} \\ y - x &= -k \text{ (arithmetic)} \end{aligned}$$

if k is an integer, -k is an integer as well
hence, y - x is an integer
Therefore, we have proved that if $(x, y) \in R$ then $(y, x) \in R$ and relation R is symmetric

The relation R is transitive if for all $x, y, z \in \mathbb{R}$ if xRy and yRz then xRz holds
so we need to prove that if $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$ then $x - z \in \mathbb{Z}$

$$\begin{aligned} \text{let } x - y &= k_1 \text{ where } k_1 \text{ is an integer} \\ \text{let } y - z &= k_2 \text{ where } k_2 \text{ is an integer} \\ x - z &= (x - y) + (y - z) \text{ (adding and substracting y)} \\ &= k_1 + k_2 \end{aligned}$$

both k_1 and k_2 are integers so $k_1 + k_2$ is integer as well, so x - z is an integer and $(x, z) \in R$ and the relation R is transitive

We have proved that R is reflexive, transitive and symmetric Therefore R is an equivalence relation.

b)

i.
the equivalence class of any real number x is given by
$[x] = \{y \in \mathbb{R} \mid y - x \in \mathbb{Z}\}$
let y - x be an integer k then y = x + k
Therefore, the equivalence class of any real number x consists of all numbers produced by adding an integer to x since the numbers of integers is infinite, each equivalence class is infinite

ii.
as proved in part **i** every real number has an equivalence classes therefore, the total number of equivalence classes are infinite.

Question 3

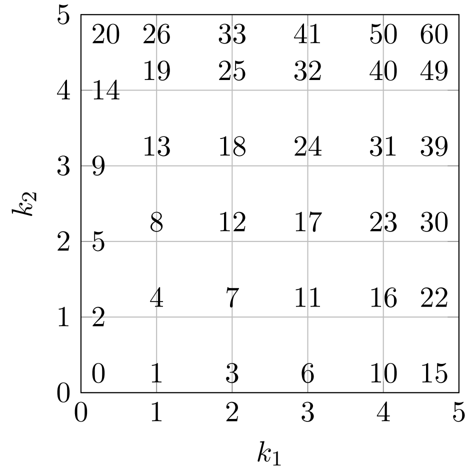
Assume a new generation starts every 50 years
We have 40 generations in 2000 years so the person should have 2^{40} ancestors 2000 years ago the current poopulation is less than 8 billion which is $2^3 \times 10^9 = 2^{12} \times 5^9$ which is less than 2^{39} the population has increased exponentially in 2000 years.

The population 2000 years ago was certainly way less than 2^{40}
The population is less than the number of ancestors
Therefore, by pigeonhole principle at least two ancestors of the person were the same person
Q.E.D.

Question 4

a)

Values of $\pi(k_1, k_2)$



b)

If the function π is bijection between $\mathbb{N}^2$ and $\mathbb{N}$. We need to prove that there exists a bijection between $\mathbb{N}^n$ and $\mathbb{N}$
We will prove this using induction
let $P(k)$ be $|\mathbb{N}^k| = |\mathbb{N}|$

<u>Base case:</u> The base case is n = 1
$\mathbb{N}^1 = \mathbb{N}$
$|\mathbb{N}| = |\mathbb{N}|$ is trivially true

<u>Inductive Case:</u>
We need to prove that $P(k) \to P(k+1)$ for some arbitary $k \geq 1$

Inductive hypothesis:
We can assume that $P(k)$ holds for an arbitary $k \geq 1$

Inductive step:
now we need to show that $P(k+1)$ holds given $P(k)$ is true
we know that there is a bijection f from $\mathbb{N}^k$ to $\mathbb{N}$ from the inductive hypothesis
So, $\mathbb{N}^k$ can be mapped to a single natural number

$\mathbb{N}^{k+1}$ can be written as $\mathbb{N}^k \times \mathbb{N}$
$\mathbb{N}^k \times \mathbb{N}$ can further be mapped to $\mathbb{N}$ using the function π

Construct a new function $f\prime$ from N^{k+1} to N such that
$f\prime = \pi(f(a_1, a_2 \ldots a_k), a_{k+1})$ where $a \in \mathbb{N}$
since both π and f are biections $f\prime$ is a bijection as well

Hence, The function $f\prime$ is a bijection from N^{k+1} to N
Therefore, $|\mathbb{N}^{k+1}| = |\mathbb{N}|$
Q.E.D.

Question 5

We need to prove that a bijection from B to A^B doesn't exist
assume a bijection f from B to A^B exists
for every $x \in B$, $f(x) \in A^B$. f(x) is a function from B to A.

Now, we can define new function f' such that $f'(x) \neq (f(x))(x)$
since, $|A| \geq 2$ we can define this function by choosing a different value from set A (using the diagonal argument)
the function f is a bijection so there must be some value of $x \in B$ such that $f(x) = f'$

This contradicts with the definition of f' function since this would imply that $f(x)(x) = f'(x)$
our assumption that f is a bijection must be false Therefore, a bijection from set B to A^B doesn't exist and $|A^B| \neq |B|$.
Q.E.D.

Question 6

a)

States: $Q = \{q_1, q_2, q_3\}$
Start state: $q_0 = q_1$
Alphabet: $\Sigma = \{0, 1\}$
Accepting states: $F = \{q_1, q_3\}$
State transition: $\delta : Q \times \Sigma -> Q$

b)

FSM recognises the symbols 1 and 0

- q_1 is the starting state and it stays there if it reads 0 and moves to q_2 if it reads 1
- in state q_2 it doesn't move if it reads 0 and moves to q_3 if it reads 1
- in state q_3 it only accepts the symbol 0 and stays in q_3

states q_1 and q_3 are the accepting states
Thus, the FSM accepts strings that contain exactly two 1s and any number of 0s including none anywhere

c)

0*10*10*

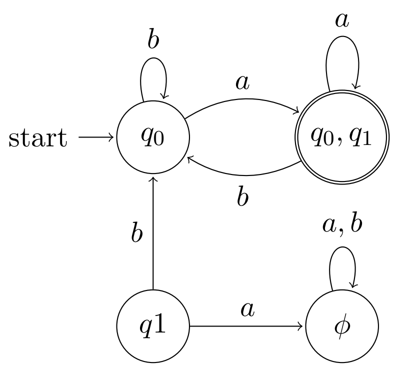
Question 7

States:
set of States of the DFSM is the powerset of the states of the original NFSM
$Q = \{\phi, \{q_0\}, \{q_1\}, \{q_0, q_1\}\}$

Initial state:
The initial state of the DFSM is a singleton set containing the initial state of the NFSM
$\{q_0\}$ is the initial state

Accepting states:
accepting states of the DFSM are the states containing any accepting states of NFSM
$\{q_1\}$ and $\{q_0, q_1\}$ are the accepting states

The Transition Function:
The transition function, δ , returns the union of all states that are reachable via the original transition function δ' , by consuming the input from any of the NFSM states in the current DFSM state, i.e., $\delta(s, x) = \bigcup\{\delta'(q, x) \mid q \in s\}$



The Result could be simplified by removing the q_1 and null state since they are unreachable.

Question 8

a)

i. b*ab*
ii. $((ab) + b)^*(aa)((ba) + b)^*$
iii. $((ab) + (aab) + b)^*(aaa)((ba) + (baa) + b)^*$

b)

- ((11)*)1*)* can be simplified to 1* since (11)* means even number of 1s and the 1* provides additional 1s which means an empty string of any string made of 1s is accepted

- (11 + 1)* can be simplified to 1* because 11* would be even number of 1s and 1* would make any number of 1s including none.
- (0 + ϵ)* can be simplified to 0* since ϵ* can be covered by no zeroes included in 0*
- the expression can be further simplified by combining 1* + 1* to 1*

final expression is 1* + 0*

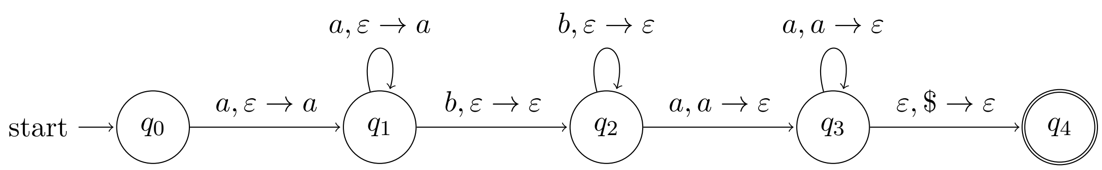
Question 9

The claim is similar to the pumping lemma but the pumping length needs to be different for every language the condition $|w| \geq 1$ is not sufficient for all languages

Counter example:
Consider the language (ab)* let's take w as ab, $|w| \geq 1$ so we satisfy the condition
if y = a, and i = 2, the resulting string aab is not in the language (ab)*
similarly if y = b and i = 2 the resulting string abb is not in the language
those are the only two possibilities of y since cardinality of y needs to be greater than 1
Therefore such an expression does not exist in this language.

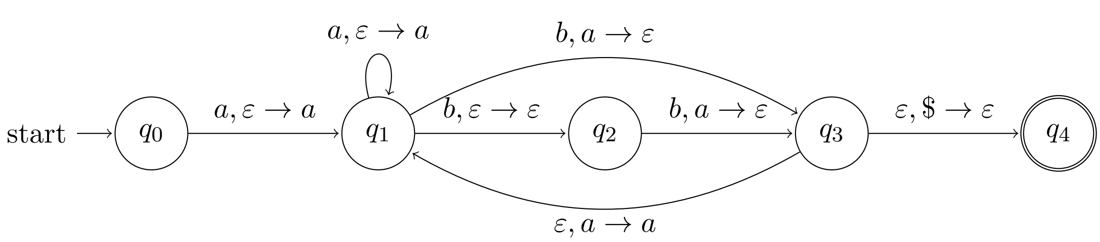
Question 10

a) $\{a^n b^m a^n \mid m, n \in \mathbb{N}\}$



The pda reads at least one 'a' and pushes each 'a' read onto the stack.
It requires at least one b and doesn't change the stack.
Then it requires at least one a and pops a from the stack for every a read. The PDA accepts if the stack is empty after reading the input.

b) $\{a^n b^m \mid n \leq m \leq 2n \wedge m, n \in \mathbb{N}\}$



The PDA reads at least one 'a' and pushes each 'a' read onto the stack
When it reads 'b's, it non-deterministically chooses whether to pop 'a' from the stack for one b read or pop 'a' on reading two b's.
The automata accepts if the stack is empty after reading the input.

If one a is popped for every two bs read the number of bs would be 2n and if an a is popped for each b read then the number of bs would be n.
Since the choice is made non-deterministically every time. The number of bs read would be in the range (n, 2n)

Question 11

a)

$A \rightarrow 0A \mid 1A \mid \epsilon$
$S \rightarrow 0A1 \mid 1A0$

S is the start point
if the string starts at 0 then it must end with 1 and vice versa. within the 1 and 0, A accepts any combination of 1s and 0s

b)

$A \rightarrow 01A \mid 10A \mid \epsilon$
$S \rightarrow ASA \mid 1$

S is the start point
A has equal number of ones and zeroes and S adds the additional ones.

Question 12

Assume a turing machine X exists such that it returns 1 if M accepts x and returns 0 otherwise

Construct a new turing machine Y that behaves as follows -

- Constructs a new machine Z which takes input x, runs M on x then accepts x
- executes X(Z)

if M halts on x then then, Z accepts the input x and X returns 1
Y(M, x) = X(Z) = 1
if M does not halt on x then Z does not halt and does not accept x
Y(M, a) = X(Z) = 0

hence, X decides the halting problem
this is a contradiction since the halting problem is undecidable.
Our assumption that such a turing machine X must be false.