CITS2211: Assignment One 2024

This assignment has 12 questions with a total value of 60 marks. Follow the instructions on LMS for submission.

1. Use a *truth table* to prove that following proposition is a contradiction.

$$\neg P \vee (\neg Q \wedge P) \leftrightarrow P \wedge Q$$

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Solution:

P	Q	$\neg P$	$\neg Q$	$\neg Q \wedge P$	LHS	RHS	$LHS \leftrightarrow RHS$
T	T	F	F	F	F	T	F
T	F	F	T	T	T	F	F
F	T	T	F	F	T	F	F
F	F	T	T	F	T	F	F

There are only F values in the last column so this proposition is a contradiction (false for any values of P and Q).

Hint: This question specifically asks for a truth table. No marks for answering the question using other methods.

Marking rubric:

- 1 mark for getting the final 4 columns ($\neg Q \wedge P$, LHS, RHS, $LHS \leftrightarrow RHS$) correct
- 1 mark for completing the proof, by stating "It is a contradiction because the final column is all false" (or similar)
- 1 mark penalty for minor errors, 2 mark penalty for major

2. Use the logical equivalence laws from lectures to show that

$$(\neg Q \vee P) \rightarrow \neg(\neg P \wedge Q)$$

is a tautology. In each step name the equivalence law you are using.

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SOLUTION: The aim is to simplify the expression to true.

You can use the following strategy:

(a) Use De Morgan's law to change $\neg(\neg P \wedge Q)$ to $P \vee \neg Q$.

(b) Use commutativity to change $P \vee \neg Q$ to $\neg Q \vee P$.

(c) Let R be $\neg Q \vee P$, and notice that we now have $R \rightarrow R$.

(d) $R \rightarrow R$ is a tautology.

So one solution is:

	$(\neg Q \vee P) \rightarrow \neg(\neg P \wedge Q)$	
$\equiv$	$(\neg Q \vee P) \rightarrow (P \vee \neg Q)$	(De Morgan)
$\equiv$	$(\neg Q \vee P) \rightarrow (\neg Q \vee P)$	(Commutativity)
$\equiv$	$R \rightarrow R$	(Define R as $\neg Q \vee P$)
$\equiv$	$R \vee \neg R$	(Implication)
$\equiv$	T	(Excluded middle)

Marking rubric:

- -1 mark for minor error, -2 for major
- Proof doesn't give names of steps being performed: -2 marks

3. State whether each of the following propositions is a **tautology** or a **contradiction** or **contingent** (i.e. neither). You must give a (brief) reason to justify each of your answers.

   (a) $P \to \neg P$

   (b) $P \to (Q \to P)$

   (c) $Q \wedge (P \wedge \neg Q)$

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SOLUTION:

Recall that a **tautology** is a proposition that is TRUE for all possible values of its prime propositions, a **contradiction** is one that is FALSE for all possible values and a **contingency** is TRUE for some and FALSE for others (that is, neither a tautology nor a contradiction). Suggestion: add these terms to your notes sheet if you have not already done so.

(a) $P \to \neg P$ is **contingent** because its value depends on the truth or falsity of P. If P is true the proposition is false, and if P is false then the proposition is true.

(b) $P \to (Q \to P)$ is a **tautology** because if P is true then so is $Q \to P$.

(c) $Q \wedge (P \wedge \neg Q)$ is a **contradiction**: If Q is true then the $P \wedge \neg Q$ is false, and if Q is false then the conjunction is false, so the proposition is false.

Marking rubric:

- 2 marks for (a), 1 mark for (b), 2 marks for (c)
- One mark off per minor error, two per major (down to a minimum of zero for each question part)
- Not giving any reason is a major error.

Tips from the marker:

How to approach this question: Making a full truth table takes a long time so practice checking for possible contradictions and simplification using reasoning by cases or the axioms of propositional logic. Only solve these questions by truth table as a last resort.

4. Express the following colloquial English statements using predicate logic, where the domain of discourse is all people.
Hint: First identify the predicates in each statement.
Use the constants a = "Anna" and b = "Ben".

   (a) Somebody is Ben's neighbour.
   (b) Anna is not anyone's neighbour.
   (c) Some people are neighbours.

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SOLUTION:

One possible set of translations is:

(a) $\exists x.N(x, b))$ where N(x,y) means "x is a neighbour of y".
(b) $\neg\exists x.N(a, x))$
(c) $\exists x.\exists y.\ N(x, y)$

It is also possible to use a 1-ary "is a neighbour" predicate as well.

Hints for answering these types of questions:

First identify the predicates.
Here there is one predicate over pairs of people: "is a neighbour of" which takes two arguments (the people who is are neighbours).
Then, correctly interpret "all" as $\forall$, and "somebody" and "anyone" as $\exists$. Finally, construct predicate statements using the correct scope for any quantified variables as per the sample solutions.

Marking rubric:

- 2 marks for identifying an appropriate predicate
- 1 mark each for correct translations of (a), (b) and (c).

5. Translate each of the assertions (a)-(e) below into predicate logic formulas, with the set of Perthians as the domain of discourse.

   The only **predicates** you may use are

   - $D(x, y)$ meaning that "x is a doctor of y", and
   - equality and inequality.

   However, you may define whatever variables and constants you wish, and you are also allowed to use all the symbols from the alphabet of predicate logic, i.e. all connectives, quantifiers, brackets etc.

   If you need to make any assumptions, state what they are.

   (a) Adrian is a doctor of Rachel.
   (b) Mary is not Rachel.
   (c) No Perthian is a doctor of Mary.
   (d) Mary is no Perthian's doctor.
   (e) Rachel is the only Perthian who is their own doctor.

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SOLUTION:

Let a be Adrian, let r be Rachel, let m be Mary and let t be Thor.

Adrian is a doctor of Rachel.	$D(a, r)$
Mary is not Rachel.	$m \neq r$.
No Perthian is a doctor of Mary.	$\neg\exists x.\ (D(x, m))$
Mary is no Perthian's doctor..	$\neg\exists x.\ D(m, x)$
Rachel is the only Perthian who is their own doctor.	$D(r, r) \wedge \forall x.(D(x, x) \implies x = r)$

Marking rubric:

- 1 mark per correct answer

6. Fill in the blanks in the following proof that

$$\exists x.(P(x) \lor (R(x) \land Q(x))) \land \forall x.(\neg P(x) \land R(x)) \to \exists x.Q(x)$$

1. $\exists x.(P(x) \lor (R(x) \land Q(x)))$	premise
2. $\forall x.(\neg P(x) \land R(x))$	premise
3.	1, exist elimination
4. $\neg P(a) \land R(a)$	
5. $\neg P(a)$	4, conjunction elimination
6.	3, implication law
7. $R(a) \land Q(a)$	5,6 modus ponens
8.	7, conjunction elimination
9. $\exists x.Q(x)$	

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SOLUTION:

1 mark per correctly added item

We aim to prove that:

$$\exists x.(P(x) \lor (R(x) \land Q(x))) \land \forall x.(\neg P(x) \land R(x)) \to \exists x.Q(x)$$

1. $\exists x.(P(x) \vee (R(x) \wedge Q(x)))$	premise
2. $\forall x.(\neg P(x) \wedge R(x))$	premise
3. $\mathbf{P}(\mathbf{a}) \vee (\mathbf{R}(\mathbf{a}) \wedge \mathbf{Q}(\mathbf{a}))$	1, exist elimination
4. $\neg P(a) \wedge R(a)$	**2, forall elimination**
5. $\neg P(a)$	4, conjunction elimination
6. $\neg \mathbf{P}(\mathbf{a}) \rightarrow (\mathbf{R}(\mathbf{a}) \wedge \mathbf{Q}(\mathbf{a}))$	3, implication law
7. $R(a) \wedge Q(a)$	5,6 modus ponens
8. $\mathbf{Q}(\mathbf{a})$	7, conjunction elimination
8. $\exists x.Q(x)$	**8, exist introduction**

7. A number is *squidgy* if it can be represented as a fraction $\frac{p}{q}$ where both p and q are integers. Using a proof by contradiction show that the sum of a squidgy and a non-squidgy number is a not a squidgy number.

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SOLUTION: Let p be a squidgy number and q be a non-squidgy number.

Assume that $p + q$ is a rational number.

As p is rational there exists a and b such that $p = \frac{a}{b}$.

As $p + q$ is rational there exists c and d such that $p + q = \frac{c}{d}$.

Therefore we have that $\frac{a}{b} + q = \frac{c}{d}$.

Rearranging this gives us $q = \frac{c}{d} - \frac{a}{b} = \frac{cb-da}{db}$.

Therefore we have a contradiction as q is irrational yet can be expressed as a fraction where the top and bottom are both integers.

Hence $p + q$ must be irrational.

Marking rubric:

- 1 mark for correct assumptions
- 1 mark for unfolding definition of squidgy for the assumptions
- 1 mark for the manipulation into the correct form
- 1 mark for demonstrating a valid contradiction
- 1 mark for the final conclusion (must have the previous mark)

8. Consider the following (incomplete) proof that $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$, for all $n \geq 1$ — i.e., that

$$2^1 + 2^2 + \ldots + 2^n = 2^{n+1} - 2$$

PROOF

Let $P(n)$ be the predicate: $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$. We will prove that this equality holds for all $n \geq 1$.

Base case

When $n = 1$, the left hand side in the equality is $\sum_{i=1}^{n} 2^i = 2^1 = 2$.

And the right-hand side is

(a)

which is equal to the left-hand side.

Therefore, ____ **(b)** for the base case.

Inductive step: prove that ____ **(c)**

Assume that ____ **(d)**.

We will show that the equality $P(k+1)$ holds, given this assumption.

Our predicates are:

$$P(k): \quad \sum_{i=1}^{k} 2^i = 2^1 + 2^2 + \ldots + 2^k \qquad = 2^{k+1} - 2$$

$$P(k+1): \quad \sum_{i=1}^{k+1} 2^i = 2^1 + 2^2 + \ldots + 2^k + 2^{k+1} \quad = 2^{(k+1)+1} - 2 = 2^{k+2} - 2$$

Therefore, the left-hand side of $P(k+1)$ is

$$\begin{aligned} & (2^1 + 2^2 + \ldots + 2^k) \ + 2^{k+1} \\ \equiv & (2^{k+1} - 2) \quad + 2^{k+1} \\ \equiv & 2^{k+1} - 2 + 2^{k+1} \\ \equiv & \underline{\quad \textbf{(e)} \quad} - 2 \\ \equiv & 2^{k+2} - 2 \end{aligned}$$

which equals the right-hand side of $P(k+1)$.

Therefore, we've proved that $P(k+1)$ holds, given the assumption that **(d)**. Since we've also proved $P(1)$ holds, then by the principle of induction, $P(n)$ holds for all $n \geq 1$. $\square$

Complete the **missing portions** of the proof that go in the empty boxes **(a)**, **(b)**, **(c)**, **(d)** and **(e)** in the above proof. You do **not** need to rewrite the whole proof.

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Solution:

PROOF

Let $P(n)$ be the predicate: $\sum_{i=1}^{n} 2^i = 2^{n+1} - 2$. We will prove that this equality holds for all $n \geq 1$.

Base case

When $n = 1$, the left hand side in the equality is $\sum_{i=1}^{n} 2^i = 2^1 = 2$.

And the right-hand side is

$$\boxed{\begin{aligned} &2^{n+1} - 2 = \\ &2^{1+1} - 2 = \\ &2^2 - 2 = \\ &4 - 2 = \\ &2 \end{aligned}} \quad \textbf{(a)}$$

which is equal to the left-hand side.

Therefore, $\boxed{P(n) \text{ holds}}$ **(b)** for the base case.

Inductive step: prove that $\boxed{P(k) \to P(k+1)}$ **(c)**

Assume that $\boxed{P(k) \text{ holds}}$ **(d)**.

We will show that the equality $P(k+1)$ holds, given this assumption.

Our predicates are:

$$\begin{aligned} P(k):&\quad \sum_{i=1}^{k} 2^i = 2^1 + 2^2 + \ldots + 2^k &&= 2^{k+1} - 2 \\ P(k+1):&\quad \sum_{i=1}^{k+1} 2^i = 2^1 + 2^2 + \ldots + 2^k + 2^{k+1} &&= 2^{(k+1)+1} - 2 = 2^{k+2} - 2 \end{aligned}$$

Therefore, the left-hand side of $P(k+1)$ is

$$\begin{aligned} &(2^1 + 2^2 + \ldots + 2^k) \ + 2^{k+1} \\ \equiv&(2^{k+1} - 2) \quad + 2^{k+1} \\ \equiv&2^{k+1} - 2 + 2^{k+1} \\ \equiv&\boxed{2(2^{k+1})}\text{(e)} - 2 \\ \equiv&2^{k+2} - 2 \end{aligned}$$

which equals the right-hand side of $P(k+1)$.

Therefore, we've proved that $P(k+1)$ holds, given the assumption that **(d)**. Since we've also proved $P(1)$ holds, then by the principle of induction, $P(n)$ holds for all $n \geq 1$. $\square$

Marks: 1 mark each for boxes (b), (c), (d) and (e), and two marks for box (a).

9. Consider the following formal language (no semantics provided):

- alphabet: a, b, c, $\oplus$ and $\ominus$
- syntax: a, b and c are all formulas. If ψ, ϕ and δ are formulas then so are $\psi \oplus \phi$ and $\psi \ominus \phi \oplus \delta$.

Let $P(\psi)$ be the number of $\oplus$ symbols in formula ψ, and let $M(\psi)$ be the number of $\ominus$ symbols.

Let the domain of discourse is the set of formulas in the language. Use structural induction to prove that:

$$\forall \phi . P(\phi) \geq M(\phi)$$

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Solution:

Proof by induction.

Base cases: in the atomic formulae a, b and c, $P(\psi) = M(\psi) = 0$ and hence the required result $P(\psi) \geq M(\psi)$ holds.

Inductive case $\psi \oplus \phi$:

Assume $P(\psi) \geq M(\psi)$ and $P(\phi) \geq M(\phi)$.

Then:

$$\begin{aligned} P(\psi \oplus \phi) &= 1 + P(\psi) + P(\phi) \\ &\geq 1 + M(\psi) + M(\phi) \\ &= 1 + M(\psi \oplus \phi) \\ &\geq M(\psi \oplus \phi) \end{aligned}$$

Inductive case $\psi \ominus \phi \oplus \delta$:

Assume $P(\psi) \geq M(\psi)$, $P(\phi) \geq M(\phi)$, $P(\delta) \geq M(\delta)$.

Then:

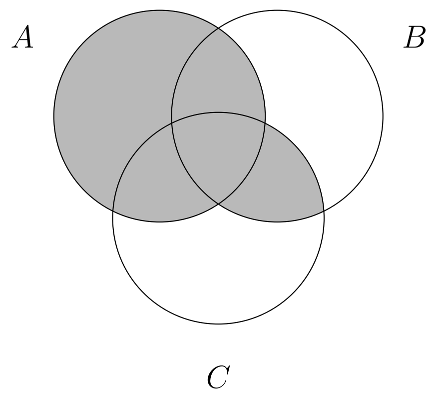
$$\begin{aligned} P(\psi \ominus \phi \oplus \delta) &= 1 + P(\psi) + P(\phi) + P(\delta) \\ &\geq 1 + M(\psi) + M(\phi) + M(\delta) \\ &= M(\psi \ominus \phi \oplus \delta) \end{aligned}$$

Mark scheme

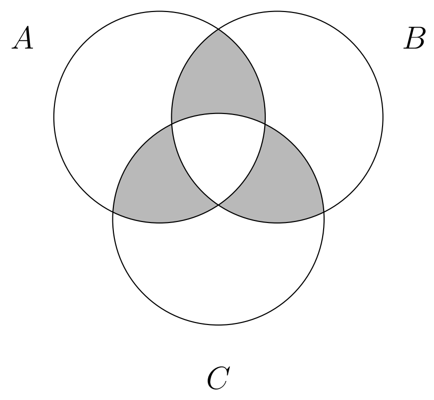
- 1 mark for base cases
- 1 mark for considering first inductive case
- 1 mark for first inductive case reasoning correct
- 1 mark for considering second inductive case
- 1 mark for second inductive case reasoning correct

10. For each of the Venn diagrams below, write down a set theory expression which equals the shaded area of the diagram.

(i)



(ii)



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SOLUTION:

(i) One possible expression is: $((A \cap B) \cup (A \cap C) \cup (B \cap C)) - (A \cap B \cap C)$

(ii) One possible expression is: $A \cup (B \cap C)$

Marking rubric:

(i) 2 marks
- 1 mark off for minor errors

(ii) 3 marks
- 1 mark off for minor errors
- 2 marks off for major errors

11. Consider two relations R, S on a set X (so that $R, S \subseteq X \times X$). Prove or disprove (by giving a counter example) the following statements:

(a) If R and S are transitive, then so is $R \cup S$.

(b) If R and S are transitive, then so is $R \cap S$.

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SOLUTION:

(a) Definition: R is transitive iff $aRb \wedge bRc \to aRc$

The proposition that "If R and S are transitive, then so is $R \cup S$" is FALSE.

Here is a counter-example. Suppose $R = \{(1,2)(2,3)(1,3)\}$ and $S = \{(3,4)(4,5)(3,5)\}$. Then R and S are transitive, but $R \cup S$ is not transitive because $\{(1,3),(3,4)\} \in R \cup S$ but $(1,4) \notin R \cup S$.

(b) The proposition "If R and S are transitive, then so is $R \cap S$" is TRUE.

Proof. Consider any $\{(a,b),(b,c)\} \in R \cap S$. By definition we also have those pairs in R and also in S since $R \cap S \subseteq R$ and $R \cap S \subseteq R$. But since R and S are both transitive we can conclude $(a,c) \in R$ and also in S and so in $R \cap S$. Therefore $R \cap S$ is also transitive. QED

12. Consider the two sets R and S, where

$$R = \{\ \varnothing,\ \{\varnothing, 2, \{5, 7\}\}\ \}$$

and

$$S = \{\ \{\varnothing, 2, \{5, 7\}\},\ \varnothing,\ \{1, 2, 5, 7\}\ \}$$

Answer the following questions about them, giving a brief explanation.

(i) Which of the sets contain the empty set as an element?
(ii) Which of the sets have the empty set as a subset?
(iii) Which set has the greater number of members?
(iv) Is S a subset of itself?

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SOLUTION:

(i) They both do – it is explicitly listed.
(ii) They both do, since every set has the empty set as a subset.
(iii) The second set (S) has the greater number of members – it has 3, whereas R has only 2.
(iv) S is a subset of itself, because every set is a subset of itself.

Marking rubric:

- 1 mark for each part (i) and (ii)
- 1.5 mark for each part (iii) and (iv)
- 0.5 marks penalty for a minor error
- 1-1.5 marks penalty for a major error
- Failing to give an explanation for a part results in 0 marks for that part.