CITS2211 Assignment 1

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Question 1

a) $\mathbf{P \lor (Q \lor \neg P)}$

The statement is a tautology because if P is true then so is $P \lor (Q \lor \neg P)$
if P is false $Q \lor \neg P$ is true and thus, $P \lor (Q \lor \neg P)$ is true

b) $\mathbf{(P \land \neg P) \lor \neg Q}$

P $\land$ $\neg$P = F (by contradiction)
$(P \land \neg P) \lor \neg Q = \neg Q$ (by absorption)

The proposition is a contingent because it depends on the truth value of Q

c) $\mathbf{Q \rightarrow (P \land \neg Q)}$

if Q is true then, $(P \land \neg Q)$ is false, making the proposition false,
and if Q is false the proposition is true
The proposition is a contingent because it depends on the truth value of Q

Question 2

$\mathbf{P \lor \neg(P \lor \neg Q) \equiv P \lor \neg Q}$

1. P $\lor$ $\neg$ (P $\lor$ $\neg$ Q)	Premise
2. P $\lor$ ($\neg$ P $\land$ Q)	1, Demorgan's laws
3. (P $\lor$ $\neg$P) $\land$ (P $\lor$ Q)	2, distributivity
4. T $\land$ (P $\lor$ Q)	3, excluded middle
5. P $\lor$ Q	4, absorption

5. P $\lor$ Q
Q.E.D.

Question 3

$\forall$x $\exists$n (x $\leq$ n $\leq$ x+5 $\land$ ($\exists$a $\exists$b (a $\neq$ n) $\land$ (a $\neq$ 1) $\land$ (b $\neq$ n) $\land$ (b $\neq$ 1) $\land$ (a $\times$ b = n))

Question 4

let N(x, y) be ”x is a neighbour of y”

a) Anna has no neighbours

$\neg(\exists\, x.N(x, a))$

b) Ben has two neighbours

$\exists\, x\, \exists\, y\, (N(x, b) \land N(y, b) \land x \neq y \land \forall\, z\, (N(z, b) \rightarrow (z = x \lor z = y)))$

c) If somebody is a neighbour of Ben, Ben is also a neighbour of that person

$\forall\, x\, (N(x, b) \rightarrow N(b, x))$

d) Except for Anna, everyone is the neighbour of someone

$\forall\, x\, (x \neq a \rightarrow \exists y(N(x, y)))$

Question 5

a) $\dfrac{\boldsymbol{P}}{\boldsymbol{P}}$

P ≡ P by rule of identity
hence, the inference rule is sound

b) $\dfrac{\boldsymbol{P}}{\boldsymbol{P \leftrightarrow Q}}$

if Q if false, the axiom is true but the conclusion is false
Hence, the inference rule is unsound

c) $\dfrac{\boldsymbol{P \leftrightarrow Q}}{\boldsymbol{P}}$

if P and Q are false, the axiom is true but the conclusion is false
Hence, the inference rule is unsound

d) $\dfrac{\boldsymbol{P \ Q}}{\boldsymbol{P \lor Q}}$

From the premise P we can infer P ∨ Q by disjunction introduction rule
Hence, the inference rule is sound

e) $\dfrac{\boldsymbol{P \rightarrow Q \ \neg\neg P}}{\boldsymbol{Q}}$

1. ¬¬P	Premise
2. P → Q	Premise
3. P	1, double negation
4. Q	3, 2, Demorgan's laws

Hence the rule is sound

Question 6

∀x.(¬ Q(x) ∨ P(x)) ∨ ∃ x(Q(x) ∨ (P(x) ∧ R(x))) → ∃x.R(x)

1. $\forall x(\neg Q(x) \wedge P(x))$	premise
2. $\exists x(Q(x) \vee (P(x) \wedge R(x)))$	premise
3. $Q(a) \vee (P(a) \wedge R(a))$	2, Exist elimination
4. $\neg Q(a) \wedge P(a)$	1, Forall elimination
5. $\neg Q(a)$	4, conjunction elimination
6. $\neg Q(a) \rightarrow (P(a) \wedge R(a))$	3, implication law
7. $P(a) \wedge R(a)$	5, 6, Modus ponens
8. R(a)	7, conjunction elimination
9. $\exists x.R(x)$	8, Exists introduction

Question 7

Assume that the difference beetween a squidgy and a non-squidgy number multiplied by 2 produces a squidgy number

let k be a non-squidgy number and l be a squidgy number
If l is a squidgy number there exists integers p and q such that
$l = \frac{p}{q}$

$$(\frac{p}{q} - k) * 2 = \frac{p\prime}{q\prime}$$

where $p\prime$ and $q\prime$ are integers (by our assumption that the result is a squidgy number)

$$k = \frac{2pq\prime - p\prime q}{2q\prime q}$$

By arithmetic

2p$q\prime$ - $p\prime$q and 2$q\prime$q are integers

k can be represented in the form of $\frac{a}{b}$ where a and b are integers,
therefore, we have a contradiction as k is a non-squidgy number yet
can be represented as a fraction with integers as numerator and denominator.
Our assumption that the result is a squidgy number must be false.
Therefore, the result of multiplying the difference betweeen a squidgy and a non-squidgy number by 2 is a non-squidgy number
Q.E.D.

Question 8

this is a bidirection proof so firstly, we prove that if x is odd then 5x-1 is even,
and then secondly, prove that if 5x-1 is even then x is odd.

first direction: if x is odd then 5x-1 is even

let x be an arbitary odd integer then x is of the form 2k-1 where k is an integer

$$\begin{aligned} x &= 2k - 1 \\ 5x &= 10k - 5 \text{ (multiplying both sides by 5)} \\ 5x - 1 &= 10k - 6 \text{ (substracting 1 from both sides)} \\ &= 2(5k - 3) \text{ (factoring out 2)} \\ &= 2n \text{(where n = 5k-3)} \end{aligned}$$

which is an odd number.

therefore, for any odd integer x, 5x -1 is an even integer

second direction: if 5x-1 is an even integer then x is an odd integer

The contrapositive of the statement is -
if x is not an odd integer then 5x-1 is not an even integer
We need to prove that if x is an even integer then 5x-1 is an odd integer

let x be an arbitary even integer then x is of the form 2n where n is an integer

$$\begin{aligned} x &= 2n \\ 5x &= 10n(\text{multiplying both sides by 5}) \\ 5x - 1 &= 10n - 1(\text{substracting 1 from both sides}) \\ &= 2(5n) - 1(\text{factoring out 2}) \\ &= 2k - 1(\text{where k is 5n}) \end{aligned}$$

Hence, 5x - 1 is of the form 2k - 1 which is an odd number
therefore, for any even integer x, 5x - 1 is an odd integer

Conclusion
therefore, we have proved that if x is odd, then 5x-1 is even
and if 5x-1 is even then x is odd.
It follows that x is odd if and only if 5x-1 is even

Question 9

Skill issue

Question 10

Base cases:
in the formula a, A(ϕ) = 1 and B(ϕ) = 0
in the formula baa, A(ϕ) = 2 and B(ϕ) = 1

hence, A(ϕ) $\geq$ 2B(ϕ) holds for the base cases

Inductive case ψaϕ:
by the inductive Hypothesis, we can assume that $A(\psi) \geq 2B(\psi)$ and $A(\phi) \geq 2B(\phi)$

$$\begin{aligned} A(\psi a \phi) &= 1 + A(\psi) + A(\phi) \\ &\geq 1 + 2B(\psi) + 2B(\phi) \\ &= 1 + 2B(\psi\phi) \\ &= 1 + 2B(\psi a \phi) \\ &\geq 2B(\psi a \phi) \end{aligned}$$

Inductive case abaψ:
by the inductive Hypothesis, we can assume that $A(\psi) \geq 2B(\psi)$

$$
\begin{aligned}
A(aba\psi) &= 2 + A(\psi) \\
&\geq 2 + 2B(\psi) \\
&= 2B(aba) + 2B(\psi) \text{ (since 2B(aba) = 2)} \\
&= 2B(aba\psi)
\end{aligned}
$$

Question 11

(i) (A - B - C) ∪ (B ∩ C)

(ii) (A ∪ B ∪ C) - ((A ∩ B) ∪ (A ∩ C) ∪ (B ∩ C))

Question 12

(a) Definition: R is reflexive iff $\forall x \in X, R(x, x)$

I will disprove the given proposition with a counter-example:
let X = {1, 2, 3, 4}
Suppose R = {(1, 2)(2, 3)(3, 4)(4, 4)} and S = {(2, 1)(3, 2)(4, 3)(4, 4)}
then T = {(1, 1)(2, 2)(3, 3)(4, 4)(3, 4)(4, 3)}
Here, R and S are not reflexive since (1, 1) ∉ R and (1, 1) ∉ S
but T is reflexive

(b) Definition: R is reflexive iff $\forall x \in X, R(x, x)$

For any arbitary element x in set X,
(x, x) ∈ R and (x, x) ∈ S (by defination of reflexive relations)
(x, x) belongs to T as well (by definition of T)
we have proved that for any arbitary element x in X, (x, x) ∈ T
Therefore, if R and S are reflexive then so is T.
Q.E.D.

(c) Definition: $\forall a, b \in A.((a, b) \in R \to (b, a) \in R$

I will disprove the given proposition with a counter-example:
Suppose R = {(1, 2)(2, 1)(3, 4)(4, 3)(1, 3)(3, 1)} and S = {(5, 6)(6, 5)(4, 3)(3, 4)}
Then T = {(3, 3)(4, 4)(1, 4)}
here, R and S are symmetric
but T is not symmetric because (1, 4) ∈ T but (4, 1) ∉ T