CITS2211 Assignment 1

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Question 1

Definitions:

A **Tautology** is a compound proposition which is true under all possible assignments of truth values of it's identifiers

A **contradiction** is a compound proposition which is false under all possible assignments of truth values to its identifiers

A **contingent** proposition is one which is neither a tautology nor a contradiction.

a) P ∨ (Q ∨ ¬P)

The statement is a tautology because if P is true then so is $P \lor (Q \lor \neg P)$
if P is false then, $Q \lor \neg P$ is true
Therefore, $P \lor (Q \lor \neg P)$ is true

The proposition is a tautology because it's true regardless of the truth values of P or Q

b) (P ∧ ¬P) ∨ ¬Q

P ∧ ¬P = F (by contradiction)
$(P \land \neg P) \lor \neg Q = \neg Q$ (by absorption)

The proposition is a contingent because it depends on the truth value of Q

c) Q → (P ∧ ¬Q)

if Q is true then, $(P \land \neg Q)$ is false, making the proposition false,
and if Q is false the proposition is true

The proposition is a contingent because it depends on the truth value of Q

Question 2

$P \vee \neg(P \vee \neg Q) \equiv P \vee \neg Q$

1. $P \vee \neg (P \vee \neg Q)$	Premise
2. $P \vee (\neg P \wedge Q)$	1, Demorgan's laws
3. $(P \vee \neg P) \wedge (P \vee Q)$	2, distributivity
4. $T \wedge (P \vee Q)$	3, excluded middle
5. $P \vee Q$	4, absorption

5. $P \vee Q$
Q.E.D.

Question 3

$\forall n \, \exists x \, (n \leq x \leq n+5 \wedge (\exists a \, \exists b \, (a \neq x) \wedge (b \neq x) \wedge (a \times b = x))$

Question 4

let N(x, y) be the predicate "x is a neighbour of y"

a) Anna has no neighbours

$\neg(\exists x . N(x, a))$

b) Ben has two neighbours

$\exists x \, \exists y \, (N(x, b) \wedge N(y, b) \wedge x \neq y \wedge \forall z \, (N(z, b) \rightarrow (z = x \vee z = y)))$

c) If somebody is a neighbour of Ben, Ben is also a neighbour of that person

$\forall x \, (N(x, b) \rightarrow N(b, x))$

d) Except for Anna, everyone is the neighbour of someone

$\forall x \, (x \neq a \rightarrow \exists y (N(x, y)))$

Question 5

An inference rule is sound iff assignment of truth values that makes all the antecedents of the rule true must also make the consequent true.

a) $\dfrac{\boldsymbol{P}}{\boldsymbol{P}}$

$P \equiv P$ by rule of identity
hence, the inference rule is sound

b) $\dfrac{\boldsymbol{P}}{\boldsymbol{P \leftrightarrow Q}}$

Counter example: if Q if false, the axiom is true but the conclusion is false
Hence, the inference rule is unsound

c) $\dfrac{\boldsymbol{P \leftrightarrow Q}}{\boldsymbol{P}}$

Counter example: if P and Q are false, the axiom is true but the conclusion is false
Hence, the inference rule is unsound

d) $\dfrac{\boldsymbol{P\ Q}}{\boldsymbol{P \vee Q}}$

From the premise P we can infer P $\vee$ Q by disjunction introduction rule
Hence, the inference rule is sound

e) $\dfrac{\boldsymbol{P \rightarrow Q\ \neg\neg P}}{\boldsymbol{Q}}$

1. $\neg\neg$P	Premise
2. P $\rightarrow$ Q	Premise
3. P	1, double negation
4. Q	3, 2, Demorgan's laws

Hence the inference rule is sound

Question 6

$\forall$x.($\neg$ Q(x) $\vee$ P(x)) $\vee$ $\exists$ x(Q(x) $\vee$ (P(x) $\wedge$ R(x))) $\rightarrow$ $\exists$x.R(x)

1. $\forall x(\neg Q(x) \wedge P(x))$	premise
2. $\exists x(Q(x) \vee (P(x) \wedge R(x)))$	premise
3. $Q(a) \vee (P(a) \wedge R(a))$	2, Exist elimination
4. $\neg Q(a) \wedge P(a)$	1, Forall elimination
5. $\neg Q(a)$	4, conjunction elimination
6. $\neg Q(a) \rightarrow (P(a) \wedge R(a))$	3, implication law
7. $P(a) \wedge R(a)$	5, 6, Modus ponens
8. R(a)	7, conjunction elimination
9. $\exists x.R(x)$	8, Exists introduction
Q.E.D.	

Question 7

Assume that the difference beetween a squidgy and a non-squidgy number multiplied by 2 produces a squidgy number

let k be a non-squidgy number and l be a squidgy number
If l is a squidgy number there exists integers p and q such that
l = $\frac{p}{q}$

$$(\frac{p}{q} - k) * 2 = \frac{p\prime}{q\prime}$$

where $p\prime$ and $q\prime$ are integers (by our assumption that the result is a squidgy number)

$$k = \frac{2pq\prime - p\prime q}{2q\prime q} \text{ (By arithmetic)}$$

2p$q\prime$ - $p\prime$q and 2$q\prime$q are integers

k can be represented in the form of $\frac{a}{b}$ where a and b are integers,
therefore, we have a contradiction as k is a non-squidgy number yet,
can be represented as a fraction of integers
Our assumption that the result is a squidgy number must be false.
Therefore, the result of multiplying the difference betweeen a squidgy and a non-squidgy number by 2 is a non-squidgy number
Q.E.D.

Question 8

Definition: even numbers are divisible by 2 while odd numbers are not.

this is a bidirection proof so firstly, we prove that if x is odd then 5x-1 is even,
and then secondly, prove that if 5x-1 is even then x is odd.

first direction: if x is odd then 5x-1 is even

let x be an arbitary odd integer then x is of the form 2k-1 where k is an integer

$$\begin{aligned} x &= 2k - 1 \\ 5x &= 10k - 5 \text{ (multiplying both sides by 5)} \\ 5x - 1 &= 10k - 6 \text{ (substracting 1 from both sides)} \\ &= 2(5k - 3) \text{ (factoring out 2)} \\ &= 2n \text{ (where n = 5k-3)} \end{aligned}$$

5x-1 is of the form 2n which is an even number.
therefore, for any odd integer x, 5x -1 is an even integer

second direction: if 5x-1 is an even integer then x is an odd integer

The contrapositive of the statement is -
if x is not an odd integer then 5x-1 is not an even integer
We need to prove that if x is an even integer then 5x-1 is an odd integer

let x be an arbitary even integer then x is of the form 2n where n is an integer

$$\begin{aligned} x &= 2n \\ 5x &= 10n \text{ (multiplying both sides by 5)} \\ 5x-1 &= 10n-1 \text{ (substracting 1 from both sides)} \\ &= 2(5n)-1 \text{ (factoring out 2)} \\ &= 2k-1 \text{ (where k is 5n)} \end{aligned}$$

5x - 1 is of the form 2k - 1 which is an odd number
therefore, for any even integer x, 5x - 1 is an odd integer

Conclusion
therefore, we have proved that if x is odd, then 5x-1 is even
and if 5x-1 is even then x is odd.
It follows that x is odd if and only if 5x-1 is even

Question 9

let P(n) be "there exists an ordering of players $p_1, p_2...p_n$ such that p_i defeats p_{i+1} for all $i \in 1, 2, ...n-1$"

Base case:
when k = 1, the 1 player can be arranged as "p_1" since no matches are played
Therefore, P(1) is true.

Inductive case:
We want to show that $P(K) \to P(K+1)$ for some arbitary $k \geq 1$

Inductive hypothesis:
We can Assume that P(K) holds for an arbitary $K \geq 1$

Inductive step:
Now, we need to show that P(K+1) holds given the inductive hypothesis

We can arrange the first K players as $p_1, p_2, \ldots, p_k$ such that p_i defeats p_{i+1} for all $i = 1, 2, \ldots, k-1$.

Consider adding a new player p_{k+1}. Since p_{k+1}
plays a match against every other player, it either wins or loses each match.

Find a player pj among the existing k players such that:
p_j defeats p_{k+1} and
p_{k+1} defeats p_{j+1}, or if $j = k$, p_{k+1} defeats no player after p_j.

Insert p_{k+1} between p_j and p_{j+1}, resulting in the new sequence:
$p1, p_2, \ldots, p_j, p_{k+1}, p_{j+1}, \ldots, p_k$.

In this new order:
p_i defeats p_{i+1} for all $i = 1, 2, \ldots, j-1$.
p_j defeats p_{k+1}.
p_{k+1} defeats p_{j+1}.
p_i defeats p_{i+1} for all $i = j+1, j+2, \ldots, k$.

Thus, we have arranged $k+1$ players in the required order.

Question 10

<u>**Base cases**</u>:
in the formula a, A(ϕ) = 1 and B(ϕ) = 0
in the formula baa, A(ϕ) = 2 and B(ϕ) = 1

hence, A(ϕ) $\geq$ 2B(ϕ) holds for the base cases

<u>**Inductive case ψaϕ**</u>:
by the inductive Hypothesis, we can assume that $A(\psi) \geq 2B(\psi)$ and $A(\phi) \geq 2B(\phi)$

$$\begin{aligned} A(\psi a\phi) &= 1 + A(\psi) + A(\phi) \\ &\geq 1 + 2B(\psi) + 2B(\phi) \\ &= 1 + 2B(\psi\phi) \\ &= 1 + 2B(\psi a\phi) \\ &\geq 2B(\psi a\phi) \end{aligned}$$

<u>**Inductive case abaψ**</u>:
by the inductive Hypothesis, we can assume that $A(\psi) \geq 2B(\psi)$

$$\begin{aligned} A(aba\psi) &= 2 + A(\psi) \\ &\geq 2 + 2B(\psi) \\ &= 2B(aba) + 2B(\psi) \text{ (since 2B(aba) = 2)} \\ &= 2B(aba\psi) \end{aligned}$$

Therefore, $\forall\phi. A(\phi) \geq 2B(\phi)$
Q.E.D

Question 11

<u>**Definitions**</u>:

Union: The union of sets A and B is a set containing all elements appearing in either A or B or both.

Intersection: The intersection of sets A and B is a set containing all elements appearing in both A and B
(i) $(A \cap \neg B \cap \neg C) \cup (B \cap C)$

(ii) $(A \cap \neg B \cap \neg C) \cup (B \cap \neg A \cap \neg C) \cup (C \cap \neg A \cap \neg B)$

Question 12

(a) Definition: R is reflexive iff $\forall x \in X, R(x, x)$

I will disprove the given proposition with a counter-example:
let X = {1, 2, 3, 4}
Suppose R = {(1, 2)(2, 3)(3, 4)(4, 4)} and S = {(2, 1)(3, 2)(4, 3)(4, 4)}
then T = {(1, 1)(2, 2)(3, 3)(4, 4)(3, 4)(4, 3)}
Here, R and S are not reflexive since (1, 1) ∉ R and (1, 1) ∉ S
but T is reflexive

(b) Definition: R is reflexive iff $\forall x \in X, R(x, x)$

For any arbitary element x in set X,
(x, x) ∈ R and (x, x) ∈ S (by defination of reflexive relations)
(x, x) belongs to T as well (by definition of T)
we have proved that for any arbitary element x in X, (x, x) ∈ T
Therefore, if R and S are reflexive then so is T.
Q.E.D.

(c) Definition: R is symmetric iff $\forall a, b \in X.((a, b) \in R \to (b, a) \in R)$

I will disprove the given proposition with a counter-example:
Suppose R = {(1, 2)(2, 1)(3, 4)(4, 3)(1, 3)(3, 1)} and S = {(5, 6)(6, 5)(4, 3)(3, 4)}
Then T = {(3, 3)(4, 4)(1, 4)}
here, R and S are symmetric
but T is not symmetric because (1, 4) ∈ T but (4, 1) ∉ T