

# CITS2211 Assignment 1

Baasil Siddiqui

## Question 1

a)  $P \vee (Q \vee \neg P)$

The statement is a tautology because if  $P$  is true then so is  $P \vee (Q \vee \neg P)$   
if  $P$  is false  $Q \vee \neg P$  is true and thus,  $P \vee (Q \vee \neg P)$  is true

b)  $(P \wedge \neg P) \vee \neg Q$

$P \wedge \neg P = F$  (by contradiction)  
 $(P \wedge \neg P) \vee \neg Q = \neg Q$  (by absorption)

The proposition is a contingent because it depends on the truth value of  $Q$

c)  $Q \rightarrow (P \wedge \neg Q)$

if  $Q$  is true then,  $(P \wedge \neg Q)$  is false, making the proposition false,  
and if  $Q$  is false the proposition is true  
The proposition is a contingent because it depends on the truth value of  $Q$

## Question 2

$$P \vee \neg(P \vee \neg Q) \equiv P \vee \neg Q$$

1. $P \vee \neg(P \vee \neg Q)$	Premise
2. $P \vee (\neg P \wedge Q)$	1, Demorgan's laws
3. $(P \vee \neg P) \wedge (P \vee Q)$	2, distributivity
4. $T \wedge (P \vee Q)$	3, excluded middle
5. $P \vee Q$	4, absorption

5.  $P \vee Q$   
*Q.E.D.*

## Question 3

$$\forall x \exists n (x \leq n \leq x+5 \wedge (\exists a \exists b (a \neq n) \wedge (a \neq 1) \wedge (b \neq n) \wedge (b \neq 1) \wedge (a \times b = n)))$$

## Question 4

let  $N(x, y)$  be "x is a neighbour of y"

a) **Anna has no neighbours**

$$\neg(\exists x.N(x, a))$$

b) **Ben has two neighbours**

$$\exists x \exists y (N(x, b) \wedge N(y, b) \wedge x \neq y \wedge \forall z (N(z, b) \rightarrow (z = x \vee z = y)))$$

c) **If somebody is a neighbour of Ben, Ben is also a neighbour of that person**

$$\forall x (N(x, b) \rightarrow N(b, x))$$

d) **Except for Anna, everyone is the neighbour of someone**

$$\forall x (x \neq a \rightarrow \exists y (N(x, y)))$$

## Question 5

a)  $\frac{P}{P}$

$P \equiv P$  by rule of identity

hence, the inference rule is sound

b)  $\frac{P}{P \leftrightarrow Q}$

if Q is false, the axiom is true but the conclusion is false

Hence, the inference rule is unsound

c)  $\frac{P \leftrightarrow Q}{P}$

if P and Q are false, the axiom is true but the conclusion is false

Hence, the inference rule is unsound

d)  $\frac{P \quad Q}{P \vee Q}$

From the premise P we can infer  $P \vee Q$  by disjunction introduction rule

Hence, the inference rule is sound

$$\text{e) } \frac{P \rightarrow Q \quad \neg\neg P}{Q}$$

1. $\neg\neg P$	Premise
2. $P \rightarrow Q$	Premise
3. $P$	1, double negation
4. $Q$	3, 2, Demorgan's laws

Hence the rule is sound

## Question 6

$$\forall x.(\neg Q(x) \vee P(x)) \vee \exists x(Q(x) \vee (P(x) \wedge R(x))) \rightarrow \exists x.R(x)$$

1. $\forall x(\neg Q(x) \wedge P(x))$	premise
2. $\exists x(Q(x) \vee (P(x) \wedge R(x)))$	premise
3. $Q(a) \vee (P(a) \wedge R(a))$	2, Exist elimination
4. $\neg Q(a) \wedge P(a)$	1, Forall elimination
5. $\neg Q(a)$	4, conjunction elimination
6. $\neg Q(a) \rightarrow (P(a) \wedge R(a))$	3, implication law
7. $P(a) \wedge R(a)$	5, 6, Modus ponens
8. $R(a)$	7, conjunction elimination
9. $\exists x.R(x)$	8, Exists introduction

## Question 7

Assume that the difference between a squidgy and a non-squidgy number multiplied by 2 produces a squidgy number

let k be a non-squidgy number and l be a squidgy number

If l is a squidgy number there exists integers p and q such that

$$l = \frac{p}{q}$$

$$\left(\frac{p}{q} - k\right) * 2 = \frac{p'}{q'}$$

where  $p'$  and  $q'$  are integers (by our assumption that the result is a squidgy number)

$$k = \frac{2pq' - p'q}{2q'q}$$

By arithmetic

$2pq' - p'q$  and  $2q'q$  are integers

k can be represented in the form of  $\frac{a}{b}$  where a and b are integers,

therefore, we have a contradiction as k is a non-squidgy number yet

can be represented as a fraction with integers as numerator and denominator.

Our assumption that the result is a squidgy number must be false.

Therefore, the result of multiplying the difference between a squidgy and a non-squidgy number by 2 is a non-squidgy number

Q.E.D.

## Question 8

this is a bidirection proof so firstly, we prove that if x is odd then  $5x-1$  is even, and then secondly, prove that if  $5x-1$  is even then x is odd.

**first direction: if x is odd then  $5x-1$  is even**

let x be an arbitrary odd integer then x is of the form  $2k-1$  where k is an integer

$$x = 2k - 1$$

$$5x = 10k - 5 \text{ (multiplying both sides by 5)}$$

$$5x - 1 = 10k - 6 \text{ (subtracting 1 from both sides)}$$

$$= 2(5k - 3) \text{ (factoring out 2)}$$

$$= 2n \text{ (where } n = 5k-3 \text{)}$$

which is an odd number.

therefore, for any odd integer x,  $5x-1$  is an even integer

**second direction: if  $5x-1$  is an even integer then x is an odd integer**

The contrapositive of the statement is -

if x is not an odd integer then  $5x-1$  is not an even integer

We need to prove that if x is an even integer then  $5x-1$  is an odd integer

let  $x$  be an arbitrary even integer then  $x$  is of the form  $2n$  where  $n$  is an integer

$$\begin{aligned}x &= 2n \\5x &= 10n \text{ (multiplying both sides by 5)} \\5x - 1 &= 10n - 1 \text{ (subtracting 1 from both sides)} \\&= 2(5n) - 1 \text{ (factoring out 2)} \\&= 2k - 1 \text{ (where } k \text{ is } 5n)\end{aligned}$$

Hence,  $5x - 1$  is of the form  $2k - 1$  which is an odd number  
therefore, for any even integer  $x$ ,  $5x - 1$  is an odd integer

### Conclusion

therefore, we have proved that if  $x$  is odd, then  $5x-1$  is even  
and if  $5x-1$  is even then  $x$  is odd.  
It follows that  $x$  is odd if and only if  $5x-1$  is even

### Question 9

Skill issue

### Question 10

#### Base cases:

in the formula  $a$ ,  $A(\phi) = 1$  and  $B(\phi) = 0$   
and in the formula  $baa$ ,  $A(\phi) = 2$  and  $B(\phi) = 1$

hence,  $A(\phi) \geq 2B(\phi)$  holds for the base cases

#### Inductive case $\psi a\phi$ :

by the inductive Hypothesis, we can assume that  $A(\psi) \geq 2B(\psi)$  and  $A(\phi) \geq 2B(\phi)$

$$\begin{aligned}A(\psi a\phi) &= 1 + A(\psi) + A(\phi) \\&\geq 1 + 2B(\psi) + 2B(\phi) \\&= 1 + 2B(\psi\phi) \\&= 1 + 2B(\psi a\phi) \\&\geq 2B(\psi a\phi)\end{aligned}$$

#### Inductive case $aba\psi$ :

by the inductive Hypothesis, we can assume that  $A(\psi) \geq 2B(\psi)$

$$\begin{aligned}
A(aba\psi) &= 2 + A(\psi) \\
&\geq 2 + 2B(\psi) \\
&= 2B(aba) + 2B(\psi) \text{ (since } 2B(aba) = 2\text{)} \\
&= 2B(aba\psi)
\end{aligned}$$

### Question 11

- (i)  $(A - B - C) \cup (B \cap C)$
- (ii)  $(A \cup B \cup C) - ((A \cap B) \cup (A \cap C) \cup (B \cap C))$

### Question 12

- (a) Definition: R is reflexive iff  $\forall x \in X, R(x, x)$

I will disprove the given proposition with a counter-example:

let  $X = \{1, 2, 3, 4\}$

Suppose  $R = \{(1, 2)(2, 3)(3, 4)(4, 4)\}$  and  $S = \{(2, 1)(3, 2)(4, 3)(4, 4)\}$

then  $T = \{(1, 1)(2, 2)(3, 3)(4, 4)(3, 4)(4, 3)\}$

Here, R and S are not reflexive since  $(1, 1) \notin R$  and  $(1, 1) \notin S$   
but T is reflexive

- (b) Definition: R is reflexive iff  $\forall x \in X, R(x, x)$

For any arbitrary element x in set X,

$(x, x) \in R$  and  $(x, x) \in S$  (by definition of reflexive relations)

$(x, x)$  belongs to T as well (by definition of T)

we have proved that for any arbitrary element x in X,  $(x, x) \in T$

Therefore, if R and S are reflexive then so is T.

Q.E.D.

- (c) Definition:  $\forall a, b \in A. ((a, b) \in R \rightarrow (b, a) \in R$

I will disprove the given proposition with a counter-example:

Suppose  $R = \{(1, 2)(2, 1)(3, 4)(4, 3)(1, 3)(3, 1)\}$  and  $S = \{(5, 6)(6, 5)(4, 3)(3, 4)\}$

Then  $T = \{(3, 3)(4, 4)(1, 4)\}$

here, R and S are symmetric

but T is not symmetric because  $(1, 4) \in T$  but  $(4, 1) \notin T$