

CITS2211 Assignment 1

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Question 1

Definitions:

A **Tautology** is a compound proposition which is true under all possible assignments of truth values of its identifiers

A **contradiction** is a compound proposition which is false under all possible assignments of truth values to its identifiers

A **contingent** proposition is one which is neither a tautology nor a contradiction.

a) $P \vee (Q \vee \neg P)$

The statement is a tautology because if P is true then so is $P \vee (Q \vee \neg P)$

if P is false then, $Q \vee \neg P$ is true

Therefore, $P \vee (Q \vee \neg P)$ is true

The proposition is a tautology because it's true regardless of the truth values of P or Q

b) $(P \wedge \neg P) \vee \neg Q$

$P \wedge \neg P = F$ (by contradiction)

$(P \wedge \neg P) \vee \neg Q = \neg Q$ (by absorption)

The proposition is a contingent because it depends on the truth value of Q

c) $Q \rightarrow (P \wedge \neg Q)$

if Q is true then, $(P \wedge \neg Q)$ is false, making the proposition false,

and if Q is false the proposition is true

The proposition is a contingent because it depends on the truth value of Q

Question 2

$$P \vee \neg(P \vee \neg Q) \equiv P \vee \neg Q$$

1. $P \vee \neg(P \vee \neg Q)$	Premise
2. $P \vee (\neg P \wedge Q)$	1, Demorgan's laws
3. $(P \vee \neg P) \wedge (P \vee Q)$	2, distributivity
4. $T \wedge (P \vee Q)$	3, excluded middle
5. $P \vee Q$	4, absorption

5. $P \vee Q$
Q.E.D.

Question 3

$$\forall n \exists x (n \leq x \leq n+5 \wedge (\exists a \exists b (a \neq x) \wedge (b \neq x) \wedge (a \times b = x)))$$

Question 4

let $N(x, y)$ be the predicate "x is a neighbour of y"

a) **Anna has no neighbours**

$$\neg(\exists x. N(x, a))$$

b) **Ben has two neighbours**

$$\exists x \exists y (N(x, b) \wedge N(y, b) \wedge x \neq y \wedge \forall z (N(z, b) \rightarrow (z = x \vee z = y)))$$

c) **If somebody is a neighbour of Ben, Ben is also a neighbour of that person**

$$\forall x (N(x, b) \rightarrow N(b, x))$$

d) **Except for Anna, everyone is the neighbour of someone**

$$\forall x (x \neq a \rightarrow \exists y (N(x, y)))$$

Question 5

An inference rule is sound iff assignment of truth values that makes all the antecedents of the rule true must also make the consequent true.

a) $\frac{P}{P}$

$P \equiv P$ by rule of identity

Therefore, the inference rule is sound

b) $\frac{P}{P \leftrightarrow Q}$

Counter example: if Q is false, the axiom is true but the conclusion is false

Therefore, the inference rule is unsound

$$c) \frac{P \leftrightarrow Q}{P}$$

Counter example: if P and Q are false, the axiom is true but the conclusion is false
Therefore, the inference rule is unsound

$$d) \frac{P \quad Q}{P \vee Q}$$

From the premise P we can infer $P \vee Q$ by disjunction introduction rule
Therefore, the inference rule is sound

$$e) \frac{P \rightarrow Q \quad \neg\neg P}{Q}$$

1. $\neg\neg P$	Premise
2. $P \rightarrow Q$	Premise
3. P	1, double negation
4. Q	3, 2, Demorgan's laws

Therefore, the inference rule is sound

Question 6

$$\forall x.(\neg Q(x) \vee P(x)) \vee \exists x(Q(x) \vee (P(x) \wedge R(x))) \rightarrow \exists x.R(x)$$

1. $\forall x(\neg Q(x) \wedge P(x))$	premise
2. $\exists x(Q(x) \vee (P(x) \wedge R(x)))$	premise
3. $Q(a) \vee (P(a) \wedge R(a))$	2, Exist elimination
4. $\neg Q(a) \wedge P(a)$	1, Forall elimination
5. $\neg Q(a)$	4, conjunction elimination
6. $\neg Q(a) \rightarrow (P(a) \wedge R(a))$	3, implication law
7. $P(a) \wedge R(a)$	5, 6, Modus ponens
8. R(a)	7, conjunction elimination
9. $\exists x.R(x)$	8, Exists introduction
Q.E.D.	

Question 7

Assume that the difference between a squidgy and a non-squidgy number multiplied by 2 produces a squidgy number

let k be a non-squidgy number and l be a squidgy number

If l is a squidgy number there exists integers p and q such that

$$l = \frac{p}{q}$$

$$\left(\frac{p}{q} - k\right) * 2 = \frac{p'}{q'}$$

where p' and q' are integers (by our assumption that the result is a squidgy number)

$$k = \frac{2pq' - p'q}{2q'q} \text{ (By arithmetic)}$$

$2pq' - p'q$ and $2q'q$ are integers

k can be represented in the form of $\frac{a}{b}$ where a and b are integers, therefore, we have a contradiction as k is a non-squidgy number yet, can be represented as a fraction of integers

Our assumption that the result is a squidgy number must be false.

Therefore, the result of multiplying the difference between a squidgy and a non-squidgy number by 2 is a non-squidgy number

Q.E.D.

Question 8

Definition: even numbers are divisible by 2 while odd numbers are not.

this is a bidirection proof so firstly, we prove that if x is odd then $5x-1$ is even, and then secondly, prove that if $5x-1$ is even then x is odd.

first direction: if x is odd then $5x-1$ is even

let x be an arbitrary odd integer then x is of the form $2k-1$ where k is an integer

$$x = 2k - 1$$

$$5x = 10k - 5 \text{ (multiplying both sides by 5)}$$

$$5x - 1 = 10k - 6 \text{ (subtracting 1 from both sides)}$$

$$= 2(5k - 3) \text{ (factoring out 2)}$$

$$= 2n \text{ (where } n = 5k-3)$$

$5x-1$ is of the form $2n$ which is an even number.

therefore, for any odd integer x , $5x-1$ is an even integer

second direction: if $5x-1$ is an even integer then x is an odd integer

The contrapositive of the statement is -

if x is not an odd integer then $5x-1$ is not an even integer

We need to prove that if x is an even integer then $5x-1$ is an odd integer

let x be an arbitrary even integer then x is of the form $2n$ where n is an integer

$$x = 2n$$

$$5x = 10n \text{ (multiplying both sides by 5)}$$

$$5x - 1 = 10n - 1 \text{ (subtracting 1 from both sides)}$$

$$= 2(5n) - 1 \text{ (factoring out 2)}$$

$$= 2k - 1 \text{ (where } k \text{ is } 5n)$$

$5x - 1$ is of the form $2k - 1$ which is an odd number

therefore, for any even integer x , $5x - 1$ is an odd integer

Conclusion

therefore, we have proved that if x is odd, then $5x-1$ is even

and if $5x-1$ is even then x is odd.

It follows that x is odd if and only if $5x-1$ is even

Question 9

let $P(n)$ be "there exists an ordering of players p_1, p_2, \dots, p_n such that p_i defeats p_{i+1} for all $i \in 1, 2, \dots, n-1$ "

Base case:

when $n = 1$, there is no valid value of i so there is no condition to satisfy

the 1 player can be arranged as " p_1 "

Therefore, $P(1)$ is true.

Inductive case:

We want to show that $P(K) \rightarrow P(K+1)$ for some arbitrary $k \geq 1$

Inductive hypothesis:

We can Assume that $P(K)$ holds for an arbitrary $K \geq 1$

Inductive step:

Now, we need to show that $P(K+1)$ holds given the inductive hypothesis

We can arrange the first K players as p_1, p_2, \dots, p_k such that p_i defeats p_{i+1} for all $i = 1, 2, \dots, k-1$.

Consider adding a new player p_{k+1} . Since p_{k+1}

plays a match against every other player, it either wins or loses each match.

Find a player p_j among the existing k players such that:

p_j defeats p_{k+1} and

p_{k+1} defeats p_{j+1} , or if $j = k$, p_{k+1} defeats no player after p_j .

Insert p_{k+1} between p_j and p_{j+1} , resulting in the new sequence:

$p_1, p_2, \dots, p_j, p_{k+1}, p_{j+1}, \dots, p_k.$

In this new order:

p_i defeats p_{i+1} for all $i = 1, 2, \dots, j - 1.$

p_j defeats $p_{k+1}.$

p_{k+1} defeats $p_{j+1}.$

p_i defeats p_{i+1} for all $i = j + 1, j + 2, \dots, k.$

Thus, we have arranged $k + 1$ players in the required order.

Question 10

Base cases:

in the formula a, $A(\phi) = 1$ and $B(\phi) = 0$

in the formula baa, $A(\phi) = 2$ and $B(\phi) = 1$

Thus, $A(\phi) \geq 2B(\phi)$ holds for the base cases

Inductive case $\psi a \phi$:

by the inductive Hypothesis, we can assume that $A(\psi) \geq 2B(\psi)$ and $A(\phi) \geq 2B(\phi)$

$$\begin{aligned} A(\psi a \phi) &= 1 + A(\psi) + A(\phi) \\ &\geq 1 + 2B(\psi) + 2B(\phi) \\ &= 1 + 2B(\psi \phi) \\ &= 1 + 2B(\psi a \phi) \\ &\geq 2B(\psi a \phi) \end{aligned}$$

Inductive case $aba\psi$:

by the inductive Hypothesis, we can assume that $A(\psi) \geq 2B(\psi)$

$$\begin{aligned} A(aba\psi) &= 2 + A(\psi) \\ &\geq 2 + 2B(\psi) \\ &= 2B(aba) + 2B(\psi) \text{ (since } 2B(aba) = 2) \\ &= 2B(aba\psi) \end{aligned}$$

Therefore, $\forall \phi. A(\phi) \geq 2B(\phi)$

Q.E.D

Question 11

Definitions:

Union: The union of sets A and B is a set containing all elements appearing in either A or B or both.

Intersection: The intersection of sets A and B is a set containing all elements appearing in both A and B

(i) $(A \cap \neg B \cap \neg C) \cup (B \cap C)$

(ii) $(A \cap \neg B \cap \neg C) \cup (B \cap \neg A \cap \neg C) \cup (C \cap \neg A \cap \neg B)$

Question 12

(a) Definition: R is reflexive iff $\forall x \in X, R(x, x)$

I will disprove the given proposition with a counter-example:

let $X = \{1, 2, 3, 4\}$

Suppose $R = \{(1, 2)(2, 3)(3, 4)(4, 4)\}$ and $S = \{(2, 1)(3, 2)(4, 3)(4, 4)\}$

then $T = \{(1, 1)(2, 2)(3, 3)(4, 4)(3, 4)(4, 3)\}$

Here, R and S are not reflexive since $(1, 1) \notin R$ and $(1, 1) \notin S$

but T is reflexive

(b) Definition: R is reflexive iff $\forall x \in X, R(x, x)$

For any arbitrary element x in set X,

$(x, x) \in R$ and $(x, x) \in S$ (by definition of reflexive relations)

(x, x) belongs to T as well (by definition of T)

we have proved that for any arbitrary element x in X, $(x, x) \in T$

Therefore, if R and S are reflexive then so is T.

Q.E.D.

(c) Definition: R is symmetric iff $\forall a, b \in X. ((a, b) \in R \rightarrow (b, a) \in R)$

I will disprove the given proposition with a counter-example:

Suppose $R = \{(1, 2)(2, 1)(3, 4)(4, 3)(1, 3)(3, 1)\}$ and $S = \{(5, 6)(6, 5)(4, 3)(3, 4)\}$

Then $T = \{(3, 3)(4, 4)(1, 4)\}$

here, R and S are symmetric

but T is not symmetric because $(1, 4) \in T$ but $(4, 1) \notin T$