Exercise 1   Odd André Owren

① We know that $e^{i\pi/2} = i$, which means that
$e^{ni\pi/2} = (e^{i\pi/2})^n = (i)^n$

By Eulers formula we have that $e^{i\theta} = \cos\theta + i\sin\theta$, s.t.
$e^{ni\pi/2} = \cos(n\pi/2) + i\sin(n\pi/2) = i^n$

This means that

(1) $\cos(n\pi/2) = i^n - i\sin(n\pi/2)$

and

(2) $\sin(n\pi/2) = \dfrac{i^n - \cos(n\pi/2)}{i}$

With whole numbers ($n \in \mathbb{Z}^+$) we will in this case only have to deal with odd and even numbers.

For $n = 2k$ (even):

3) $\cos(n\pi/2) = i^{2k} - i\sin(2k\pi/2) = i^{2k} - \underbrace{i\sin(k\pi)}_{0} = \underline{\underline{i^n}}$

and

4) $\sin(n\pi/2) = \dfrac{i^{2k} - \cos(n\pi/2)}{i} = \dfrac{i^n - i^n}{i} = \underline{\underline{0}}$

For $n = 2k+1$ (odd)

5) $\cos(n\pi/2) = i^{2k+1} - i\sin\left(\frac{(2k+1)\pi}{2}\right) = i^{2k} \cdot i - i\sin(k\pi + \frac{\pi}{2})$

From Euler's formula we have that

$\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \sin\theta_2\cos\theta_1$

Such that

$\sin(k\pi + \frac{\pi}{2}) = \underbrace{\sin(k\pi)\cos(\frac{\pi}{2})}_{0} + \sin(\frac{\pi}{2})\cos(k\pi) = \cos(k\pi) = (-1)^k$

Apply to 5):

$\cos(n\pi/2) = i^{2k} \cdot i - i \cdot (-1)^k = (-1)^k \cdot i - i(-1)^k = \underline{\underline{0}}$

At last we have

6) $\sin(n\pi/2) = \dfrac{i^{2k+1} - \cos(n\pi/2)}{i} = \dfrac{i^{2k+1}}{i} = i^{2k} = \underline{\underline{i^{n-1}}}$

(2) $\int_{-\pi}^{\pi} x e^{inx}\,dx$

For $n=0$:

$$\int_{-\pi}^{\pi} x\,dx = \frac{1}{2}x^2\Big|_{-\pi}^{\pi} = \frac{1}{2}\pi^2 - \frac{1}{2}(-\pi)^2 = \underline{\underline{0}}$$

For $n \neq 0$:

$$\int_{-\pi}^{\pi} x e^{inx}\,dx = x\frac{e^{inx}}{in}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{e^{inx}}{in}\,dx = x\frac{e^{inx}}{in} - \frac{e^{inx}}{(in)^2}\Big|_{-\pi}^{\pi}$$

Integration by parts:

$u = x \quad v' = e^{inx}$

$u' = 1 \quad v = \frac{e^{inx}}{in}$

$$= \frac{\pi e^{in\pi}}{in} - \frac{e^{in\pi}}{(in)^2} + \frac{\pi e^{in\pi}}{in} + \frac{e^{in\pi}}{(in)^2}$$

$$= \underline{\underline{\frac{2\pi}{in}e^{in\pi}}}$$

From Euler's formula we have that $e^{i\pi} = -1$, so $e^{in\pi} = (-1)^n$

Thus we have that for $n \neq 0$:

$$\int_{-\pi}^{\pi} x e^{inx}\,dx = \frac{2\pi}{in}e^{in\pi} = \frac{2\pi}{in}(-1)^n = \underline{\underline{\frac{2\pi}{n}i^{2n-1}}}$$

(3) a) $f(t) = \sinh(At)$

$$F(s) = \int_0^\infty e^{-st} \cdot \sinh(At)\, dt = \int_0^\infty e^{-st} \cdot \frac{e^{At} - e^{-At}}{2}\, dt$$

$$= \int_0^\infty \frac{e^{At-st} - e^{-At-st}}{2}\, dt = \int_0^\infty \frac{e^{(A-s)t} - e^{-(A+s)t}}{2}\, dt$$

$$\frac{1}{2}\left(\frac{e^{(A-s)t}}{A-s} + \frac{e^{-(A+s)t}}{A+s}\right)\Bigg|_0^\infty$$

Here we assume that $s > A$ such that $A - s < 0$ and that $A + s > 0$

This then gives:

$$\frac{1}{2}\left(\underbrace{\frac{e^{(A-s)\cdot\infty}}{A-s} + \frac{e^{-(A+s)\cdot\infty}}{A+s}}_{0} - \frac{e^{(A-s)\cdot 0}}{A-s} - \frac{e^{-(A+s)\cdot 0}}{A+s}\right)$$

$$= \frac{1}{2} \cdot \left(-\frac{1}{A-s} - \frac{1}{A+s}\right) = \frac{1}{2}\left(-\frac{A+s}{A^2-s^2} - \frac{A-s}{A^2-s^2}\right) = -\frac{A}{A^2-s^2} = \frac{A}{s^2-A^2}$$

This means that

$$\underline{\underline{\mathcal{L}(\sinh(At)) = \frac{A}{s^2-A^2}}} \quad \text{with } s > A$$

b) $f(t) = \cosh(At)$

$$F(s) = \int_0^\infty e^{-st} \cdot \cosh(At)\, dt = \int_0^\infty e^{-st} \cdot \frac{e^{At} + e^{-At}}{2}\, dt$$

$$= \int_0^\infty \frac{e^{(A-s)t} + e^{-(A+s)t}}{2}\, dt = \frac{1}{2}\left(\frac{e^{(A-s)t}}{A-s} - \frac{e^{-(A+s)t}}{A+s}\right)\Bigg|_0^\infty$$

Again we assume that $A<S$ Such that $A-S<0$ and $A+S>0$

We then have:

$$\frac{1}{2}\Big(\underbrace{\frac{e^{(A-S)\cdot\infty}}{A-S} - \frac{e^{-(A+S)\cdot\infty}}{A+S}}_{0} - \frac{e^{(A-S)\cdot 0}}{A-S} + \frac{e^{-(A+S)\cdot 0}}{A+S}\Big)$$

$$= \frac{1}{2}\left(\frac{1}{A+S} - \frac{1}{A-S}\right) = \frac{1}{2}\left(\frac{A-S}{A^2-S^2} - \frac{A+S}{A^2-S^2}\right) = -\frac{S}{A^2-S^2} = \frac{S}{S^2-A^2}$$

This means that

$$\underline{\mathcal{L}(\cosh(At)) = \frac{S}{S^2-A^2} \quad \text{Where } S>A}$$

c) $f(t) = \begin{cases} 0 & \text{if } 0 < t < \pi \\ 1 & \text{if } t \geq \pi \end{cases}$

$$F(s) = \underbrace{\int_0^{\pi} e^{-st} \cdot 0 \, dt}_{0} + \int_{\pi}^{\infty} e^{-st} \cdot 1 \, dt$$

$$= \int_{\pi}^{\infty} e^{-st} \, dt = -\frac{e^{-st}}{s} \Big|_{\pi}^{\infty} = \underbrace{-\frac{e^{-s \cdot \infty}}{s}}_{0} + \frac{e^{-s\pi}}{s} = \underline{\underline{\frac{e^{-s\pi}}{s}}}$$

d) $f(t) = \begin{cases} 0 & \text{if } 0 < t < \pi \\ \cos t & \text{if } t \geq \pi \end{cases}$

$$F(s) = \underbrace{\int_0^{\pi} e^{-st} \cdot 0 \, dt}_{0} + \int_{\pi}^{\infty} e^{-st} \cos t \, dt$$

$$= \int_{\pi}^{\infty} e^{-st} \cos t \, dt = e^{-st} \sin t - s e^{-st} \cos t - \int_{\pi}^{\infty} s^2 e^{-st} \cos t \, dt$$

↑

Partial integration:

	D	I
+	e^{-st}	$\cos t$
-	$-se^{-st}$	$\sin t$
+	$s^2 e^{-st}$	$-\cos t$

This leads to:

$$s^2 \int_{\pi}^{\infty} e^{-st} \cos t \, dt + \int_{\pi}^{\infty} e^{-st} \cos t = e^{-st} \sin t - s e^{-st} \cos t$$

$\Rightarrow$

$$\frac{(s^2+1)\int_{\pi}^{\infty} e^{-st}\cos t\,dt}{(s^2+1)} = \frac{e^{-st}\sin t - se^{-st}\cos t}{s^2+1}$$

$$\Rightarrow$$

$$\int_{\pi}^{\infty} e^{-st}\cos t\,dt = \frac{e^{-st}\sin t - se^{-st}\cos t}{s^2+1}\Bigg|_{\pi}^{\infty}$$

$$= \frac{\overbrace{e^{-s\cdot\infty}\sin(\infty) - se^{-s\cdot\infty}\cos(\infty)}^{0} - \overbrace{e^{-s\pi}\sin\pi}^{0} + se^{-s\pi}\cos\pi}{s^2+1}$$

$$= \frac{se^{-s\pi}\cos\pi}{s^2+1} = -\frac{se^{-s\pi}}{s^2+1}$$

e) $f(t) = t^2e^t$

$$F(s) = \int_0^{\infty} t^2e^t \cdot e^{-st}\,dt = \int_0^{\infty} t^2e^{(1-s)t}\,dt$$

By partial integration we have

D	I
$+\ t^2$	$e^{(1-s)t}$
$-\ 2t$	$\frac{e^{(1-s)t}}{1-s}$
$+\ 2$	$\frac{e^{(1-s)t}}{(1-s)^2}$

Such that

$$F(s) = t^2\frac{e^{(1-s)t}}{1-s} - 2t\frac{e^{(1-s)t}}{(1-s)^2} + \int_0^{\infty} 2\frac{e^{(1-s)t}}{(1-s)^2}\,dt =$$

$$t^2\frac{e^{(1-s)t}}{1-s} - 2t\frac{e^{(1-s)t}}{(1-s)^2} + 2\frac{e^{(1-s)t}}{(1-s)^3}\Bigg|_0^{\infty} = -\frac{2}{(1-s)^3} = \frac{2}{(s-1)^3}$$

Here we assume that $s > 1$

f) $f(t) = e^{t}\cos t$

$$F(s) = \int_0^\infty e^{-(s-1)t}\cos t \, dt$$

By partial integration as in 3d) we get:

D	I
$+\ e^{(1-s)t}$	$\cos t$
$-\ (1-s)e^{(1-s)t}$	$\sin t$
$+\ (1-s)^2 e^{(1-s)t}$	$-\cos t$

This gives:

$$F(s) = \left. \frac{e^{(1-s)t}\sin t + (1-s)e^{(1-s)t}\cos t}{s^2-2s+2} \right|_0^\infty$$

Here we assume that $s>1$, then $e^{(1-s)\cdot\infty} = 0$

$$F(s) = \frac{-\left(e^{(1-s)\cdot 0}\sin(0) + (1-s)e^{(1-s)\cdot 0}\cos(0)\right)}{s^2-2s+2} = \underline{\underline{\frac{s-1}{s^2-2s+2}}}$$

g) $f(t) = e^{t}\sin(t)$

$$F(s) = \int_0^\infty e^{(1-s)t}\sin t \, dt$$

Repeat the process from f):

$$F(s) = \left. -\frac{e^{(1-s)t}\cdot\cos t + (1-s)e^{(1-s)t}\cdot\sin t}{s^2-2s+2} \right|_0^\infty$$

Again we assume $s>1$ such that $e^{(1-s)\cdot\infty} = 0$

$$F(s) = \frac{(1-s)e^{(1-s)\cdot 0}\sin 0 - e^{(1-s)\cdot 0}\cos(0)}{s^2-2s+2} = \underline{\underline{-\frac{1}{s^2-2s+2}}}$$

4) a) $y'' - 2y' + 2y = 6e^{-t}$, $y(0) = 0$, $y'(0) = 1$

Apply Laplace to get:

(1) $s^2 Y(s) - s\,y(0) - y'(0) - 2(sY(s) - y(0)) + 2Y(s) = \mathcal{L}(6e^{-t})$

$$\mathcal{L}(6e^{-t}) = \int_0^\infty 6e^{-(s+1)t}\,dt = -\frac{6e^{-(s+1)t}}{s+1}\Big|_0^\infty$$

Here we assume that $s > -1$ such that $e^{-(s+1)\infty} = 0$

$$\mathcal{L}(6e^{-t}) = \frac{6e^{-(s+1)\cdot 0}}{s+1} = \frac{6}{s+1}$$

insert $y(0) = 0$ and $y'(0) = 1$ to (1):

$$s^2 Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{6}{s+1}$$

$$(s^2 - 2s + 2)\,Y(s) = \frac{6}{s+1} + 1$$

$$Y(s) = \frac{\frac{s+7}{s+1}}{(s^2 - 2s + 2)} = \frac{s+7}{(s+1)\cdot(s^2 - 2s + 2)}$$

$$y = \mathcal{L}^{-1}\left(\frac{s+7}{(s+1)(s^2 - 2s + 2)}\right)$$

b) $y'' + y = f(t)$, $y(0) = y'(0) = 0$ and $f(t) = \begin{cases} 0 & \text{if } 0 < t < \pi \\ 1 & \text{if } t \geq \pi \end{cases}$

Apply Laplace such that:

(1) $s^2 Y(s) - sy(0) - y'(0) + Y(s) = \mathcal{L}(f(t))$

From 3c) we have that

$$\mathcal{L}(f(t)) = \frac{e^{-s\pi}}{s}$$

Insert $y(0)$ and $y'(0) = 0$ in (1):

$$s^2 Y(s) + Y(s) = \frac{e^{-s\pi}}{s}$$

$$\underline{Y(s) = \frac{e^{-s\pi}}{s^3 + s}}$$

$$y = \mathcal{L}^{-1}\left(\frac{e^{-s\pi}}{s^3 + s}\right)$$