TMA4135 Matematikk 4D
Exercise 7

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1a. First we start by calculating the cardinal functions.

$\ell_0(x) = \frac{(x+\frac{1}{2})(x-\frac{1}{2})(x-1)}{(-1+\frac{1}{2})(-1-\frac{1}{2})(-1-1)} = \frac{x^3-x^2-\frac{1}{4}x+\frac{1}{4}}{-\frac{3}{2}}$

$\ell_1(x) = \frac{(x+1)(x-\frac{1}{2})(x-1)}{(-\frac{1}{2}+1)(-\frac{1}{2}-\frac{1}{2})(-\frac{1}{2}-1)} = \frac{x^3-\frac{1}{2}x^2-x+\frac{1}{2}}{\frac{3}{4}}$

$\ell_2(x) = \frac{(x+1)(x+\frac{1}{2})(x-1)}{(\frac{1}{2}+1)(\frac{1}{2}+\frac{1}{2})(\frac{1}{2}-1)} = \frac{x^3+\frac{1}{2}x^2-x-\frac{1}{2}}{-\frac{3}{4}}$

$\ell_3(x) = \frac{(x+1)(x+\frac{1}{2})(x-1\frac{1}{2})}{(1+1)(1+\frac{1}{2})(1-\frac{1}{2})} = \frac{x^3+x^2-\frac{1}{4}x-\frac{1}{4}}{\frac{3}{2}}$

Now, by definition a Lagrange interpolation is calculated with $p_n(x) = \Sigma_{i=0}^{n}\ell_i(x)f(x_i)$, which gives us:

$p_3(x) = -\frac{1}{2}\cdot\frac{x^3-x^2-\frac{1}{4}x+\frac{1}{4}}{-\frac{3}{2}} - \frac{5}{4}\cdot\frac{x^3-\frac{1}{2}x^2-x+\frac{1}{2}}{\frac{3}{4}} + \frac{1}{4}\cdot\frac{x^3+\frac{1}{2}x^2-x-\frac{1}{2}}{-\frac{3}{4}} + \frac{5}{2}\cdot\frac{x^3+x^2-\frac{1}{4}x-\frac{1}{4}}{\frac{3}{2}}$

$= \frac{6x^2+\frac{9}{2}x-3}{3} = \underline{\underline{2x^2+\frac{3}{2}x-1}}$

And thus $p_3(0) = -1$.

1b. See figure 1. Confirmed as asked in task 1c.

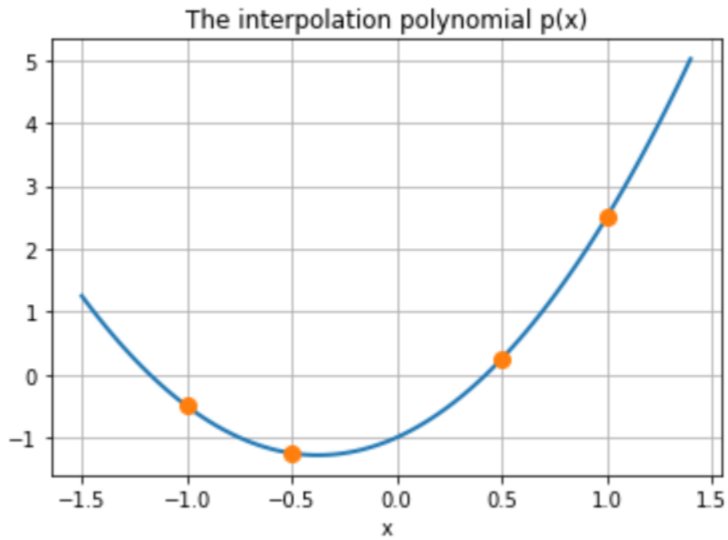

Figure 1: Lagrange interpolation in task 1b

2. See figure 2. The population growth from 2008 to 2013 is higher than from 2013 to 2018, therefore the interpolated polynomial uses this as a turning point, and interpolates a further descent from there on out. Since the polynomial is of maximum degree 5 (6 points are given), it will at one point or another have a steep descent.

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Predicted populations:
 2000: 4472371.58982     2010:  4848757.08742
 2025:  3829736.29382    2030:  -1174984.14138
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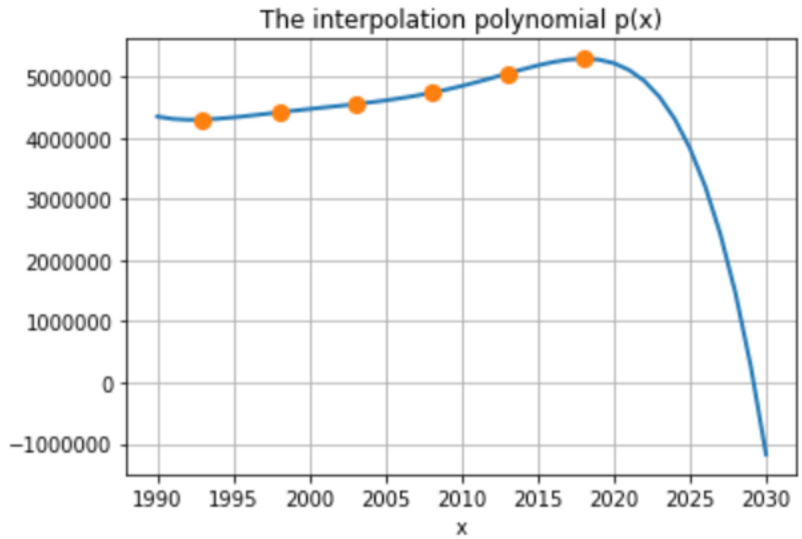

Figure 2: Lagrange interpolation in task 2 with predicted populations

3a. Part 1: Equal distribution.
Here we will use the points for x as [-1,0,1,2]. This gives the following values for $f(x_n)$:

x_n	-1	0	1	2
$f(x_n)$	0.540	0	0.540	-1.665

This gives us the following cardinals:

$\ell_0(x) = \frac{x(x-1)(x-2)}{(-1)(-1-1)(-1-2)} = \frac{x^3-3x^2+2x}{-6}$

$\ell_1(x) = \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)} = \frac{x^3-2x^2-x+2}{2}$

$\ell_2(x) = \frac{x(x+1)(x-2)}{(1+1)(1-0)(1-2)} = \frac{x^3-x^2-2x}{-2}$

$\ell_3(x) = \frac{x(x-1)(x-2)}{(2+1)(2-0)(2-1)} = \frac{x^3-x}{6}$

By applying Lagrange interpolation, we get the following:

$p_3(x) = 0.540 \cdot \frac{x^3-3x^2+2x}{-6} + 0 \cdot \frac{x^3-2x^2-x+2}{2} + 0.540 \cdot \frac{x^3-x^2-2x}{-2} - 1.665 \cdot \frac{x^3-x}{6} =$

$\underline{\underline{-0.6375x^3 + 0.54x^2 + 0.6375x}}$

Part 2: Chebyshev points.

Chebyshev points are defined by calculating the following for n partitions of the interval [a,b]:

$x_k = \frac{1}{2}(a+b) + \frac{1}{2}(b-a)cos(\frac{2k-1}{2n}\pi)$ for k=1,2,3,...,n.

Applying this to our situation, we get:

$x_1 = 1.886 \quad x_2 = 1.074 \quad x_3 = -0.074 \quad x_4 = -0.886$

This gives us the following values for $f(x_n)$:

x_n	1.886	1.074	-0.074	-0.886
$f(x_n)$	-1.103	0.55	0.005	0.497

This gives the following cardinals:

$$\ell_0(x) = \frac{(x-1.074)(x+0.074)(x+0.886)}{(1.886-1.074)(1.886+0.074)(1.886+0.886)} = \frac{x^3-0.114x^2-0.965x-0.07}{4.412}$$

$$\ell_1(x) = \frac{(x-1.886)(x+0.074)(x+0.886)}{(1.074-1.886)(1.074+0.074)(1.074+0.886)} = \frac{x^3-0.926x^2-1.745x-0.12}{-1.827}$$

$$\ell_2(x) = \frac{(x-1.886)(x-1.074)(x+0.886)}{(-0.074-1.886)(-0.074-1.074)(-0.074+0.886)} = \frac{x^3-2.074x^2-2.745x+1.795}{1.827}$$

$$\ell_3(x) = \frac{(x-1.886)(x-1.074)(x+0.074)}{(-0.886-1.886)(-0.886-1.074)(-0.886+0.074)} = \frac{x^3-2.886x^2+1.807x+0.149}{-4.412}$$

Applying Lagrange interpolation then gives us:

$$p_3(x) = -1.103 \cdot \frac{x^3-0.114x^2-0.965x-0.07}{4.412} + 0.55 \cdot \frac{x^3-0.926x^2-1.745x-0.12}{-1.827} +$$

$$0.005 \cdot \frac{x^3-2.074x^2-2.745x+1.795}{1.827} + 0.497 \cdot \frac{x^3-2.886x^2+1.807x+0.149}{-4.412}$$

$$= \underline{\underline{-0.661x^3 + 0.627x^2 + 0.556x + 0.042}}$$

3b. The formula for error for a polynomial interpolation is given by $f(x) - p_n(x) = \frac{f^{n+1}(\xi)}{(n+1)!}\prod_{i=0}^{n}(x-x_i)$ where n+1 is the number of points that is interpolated. To get the maximum error bound we need to find the absolute and maximize the error. This is done by using the formula above with some modifications; $max|f(x)-p_n(x)| \leq \frac{max|f^{n+1}(\xi)|}{(n+1)!}max|\prod_{i=0}^{n}(x-x_i)|$.

Part 1: Equal distribution.

From the hint given in the task, we have that

$$f^{(4)}(x) = (-1)^2(x^2cos(x) + 8xsin(x) - 12cos(x))$$

Now, looking at the next derivative, $f^{(5)}$, we can see that x=0 gives us an extremum, which is the bottom of the function. Looking at the endpoints of the interval [-1,2] we get that $f^{(4)}$ has its max when x=2. In this point, $f^{(4)}(2) = 4cos(2) + 16sin(2) - 12cos(2) = \underline{17.878}$.

Now, we need to maximize the second part of the error-function.

$g(x) = \prod_{i=0}^{4}(x - x_i) = x(x+1)(x-1)(x-2) = x^4 - 2x^3 - x^2 + 2x$

$g'(x) = 4x^3 - 6x^2 - 2x + 2 = 0 \Rightarrow x_0 = \frac{1}{2} \quad x_1 = \frac{1-\sqrt{5}}{2} \quad x_2 = \frac{1+\sqrt{5}}{2}$

$\Rightarrow g(x_0) = (\frac{1}{2})^4 - 2 \cdot (\frac{1}{2})^3 - (\frac{1}{2})^2 + 2 \cdot (\frac{1}{2}) = 0.563$

$g(x_1) = (\frac{1-\sqrt{5}}{2})^4 - 2 \cdot (\frac{1-\sqrt{5}}{2})^3 - (\frac{1-\sqrt{5}}{2})^2 + 2 \cdot (\frac{1-\sqrt{5}}{2}) = -1$

$g(x_2) = (\frac{1+\sqrt{5}}{2})^4 - 2 \cdot (\frac{1+\sqrt{5}}{2})^3 - (\frac{1+\sqrt{5}}{2})^2 + 2 \cdot (\frac{1+\sqrt{5}}{2}) = -1$

Here we see that g(x) is at it's max when $x = \frac{1+\sqrt{5}}{2}$, where its absolute value is 1. Putting this back into the maximized error-function, we have

$max|f(x) - p_n(x)| \leq \frac{17.878}{4!} = \underline{\underline{0.745}}$

<u>Part 2: Chebyshev points.</u>
Here we will repeat the process from the previous part, but since we already have the maximized value of $f^{(4)}$ we will not have to calculate this again.

$g(x) = \prod_{i=0}^{4}(x - x_i) = (x - 1.886)(x - 1.074)(x + 0.074)(x + 0.886) = x^4 - 2x^3 - 0.75x^2 + 1.75x + 0.132$

$g'(x) = 4x^3 - 6x^2 - 1.5x + 1.75 = 0 \Rightarrow x_0 = \frac{1}{2} \quad x_1 = -0.561 \quad x_2 = 1.561$

$\Rightarrow g(x_0) = (\frac{1}{2})^4 - 2 \cdot (\frac{1}{2})^3 - 0.75 \cdot (\frac{1}{2})^2 + 1.75 \cdot (\frac{1}{2}) + 0.132 = 0.563$

$g(x_1) = (-0.561)^4 - 2 \cdot (-0.561)^3 - 0.75 \cdot (-0.561)^2 + 1.75 \cdot (-0.561) + 0.132 = -0.634$

$g(x_2) = (1.561)^4 - 2 \cdot (1.561)^3 - 0.75 \cdot (1.561)^2 + 1.75 \cdot (1.561) + 0.132 = -0.634$

Here we see that g(x) is at its max when $x = -0.561$, where its absolute value is 0.634. Putting this back into the maximized error-function, we have

$max|f(x) - p_n(x)| \leq \frac{17.878 \cdot 0.634}{4!} = \underline{\underline{0.472}}$

3c. See figure 3 for equal distribution and figure 4 for Chebyshev points. Error is marked in orange, and as we can see, the maximum error found in 3b is never surpassed.

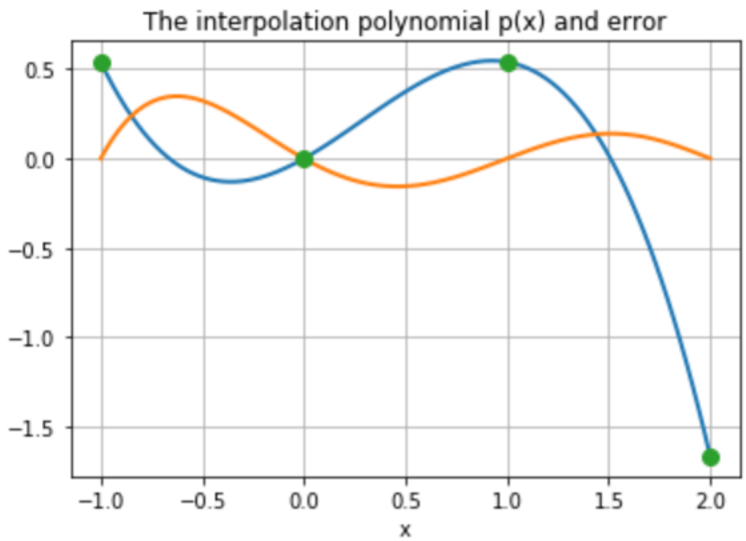

Figure 3: Equal distribution interpolation with error marked in orange

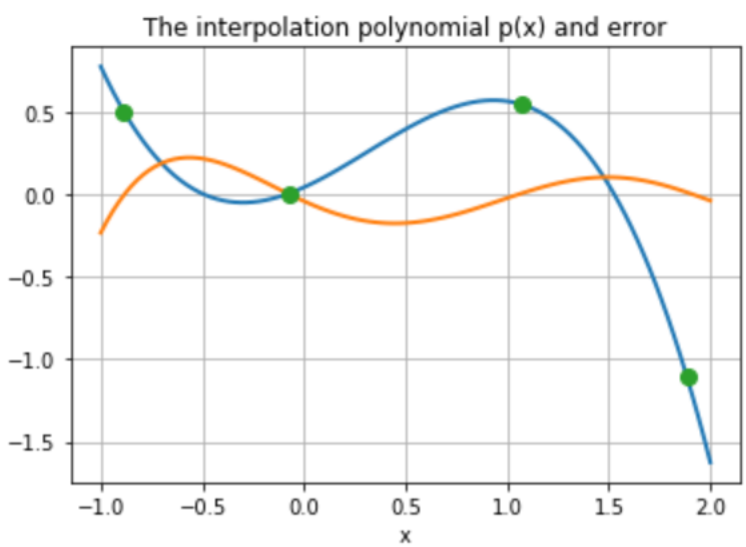

Figure 4: Chebyshev points interpolation with error marked in orange

3d. Figures 5 and 6 shows respectively equal distribution and chebyshev points with 5,10,15 and 20 interpolation points and respective error in orange. I was not able to calculate the error bounds numerically, so I only included the maximum measured error.

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Max measured error for  5 points: 0.110778540819
Max measured error for  10 points: 1.51984892479e-05
Max measured error for  15 points: 8.17321765822e-11
Max measured error for  20 points: 2.61679566904e-13
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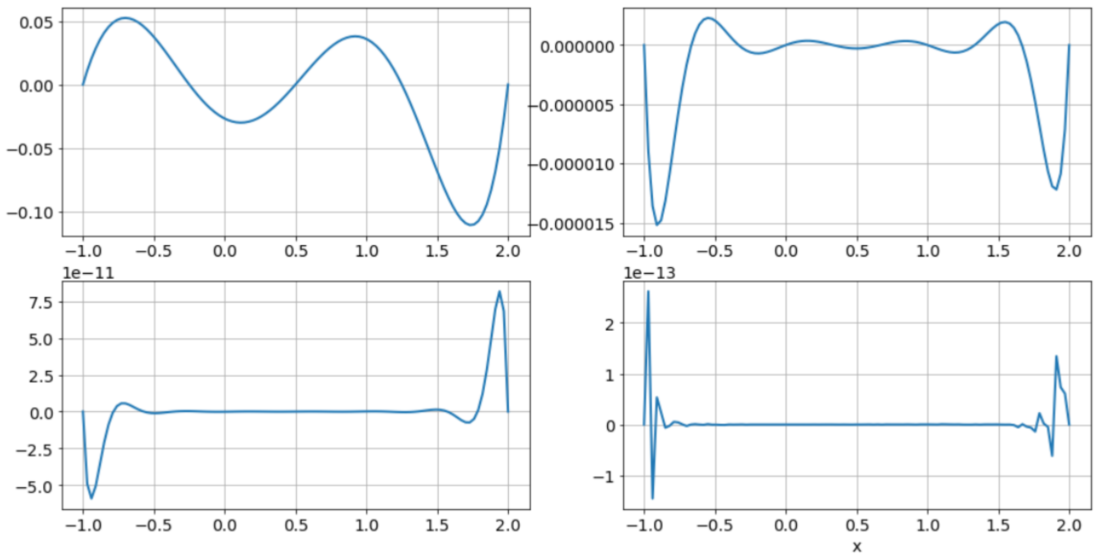

Figure 5: Errors in equal distribution

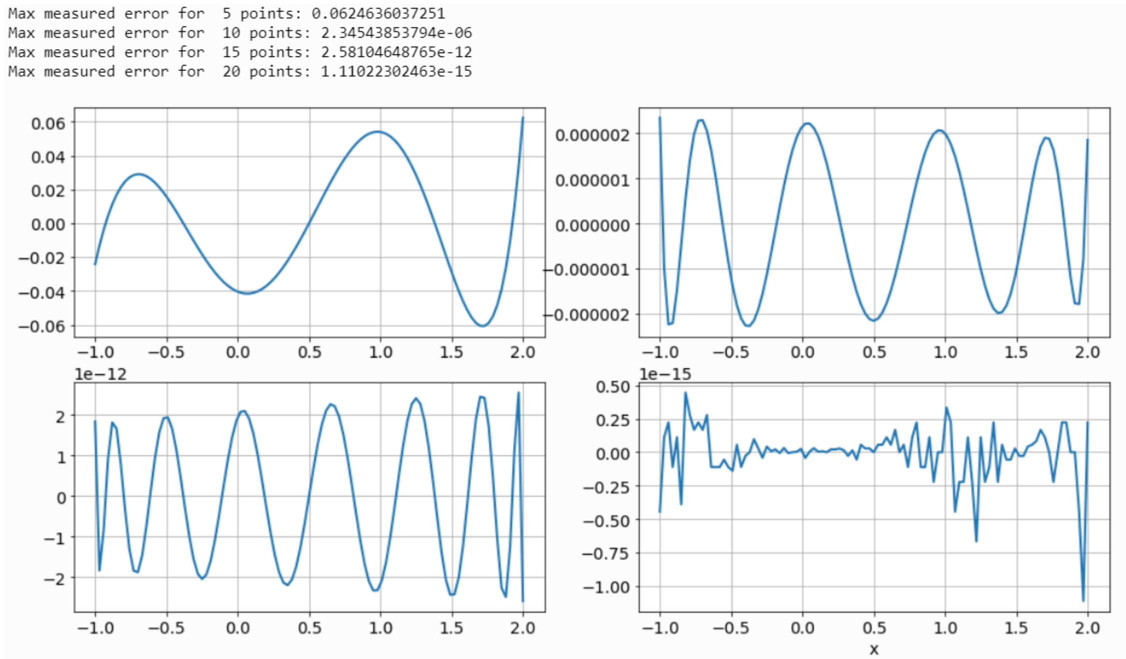

Figure 6: Errors in Chebyshev points