TMA4135 Matematikk 4D
Exercise 3

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1a. $\mathcal{L}(f(t)) = \int_0^a te^{-st}dt + \int_a^\infty 0dt$

Since the integral of 0 is 0, we only need to calculate the first part. Using partial integration we get

$\int_0^a te^{-st}dt = -t\frac{e^{-st}}{s} - \int_0^a -\frac{e^{-st}}{s}dt = [-t\frac{e^{-st}}{s} - \frac{e^{-st}}{s^2}]_0^a = \frac{1}{s^2} - \frac{e^{-as}}{s^2} - a\frac{e^{-as}}{s}$ $\square$

1b. Here we start by defining $v(t) = sin(t+\pi)$ such that $v(t-\pi) = sin(t)$. This gives us (by the t-shifting theorem) that

(1) $\mathcal{L}(f(t)) = \mathcal{L}(u(t-\pi)sin(t)) = \mathcal{L}(u(t-\pi)v(t-\pi)) = e^{-\pi s}\mathcal{L}(v(t))$

Since $sin(t+\pi) = -sin(t)$ we have that $\mathcal{L}(v(t)) = -\mathcal{L}(sin(t)) = -\frac{1}{s^2+1}$
Now, inserting this into (1) we get

$\mathcal{L}(f(t)) = e^{-\pi s} \cdot -\frac{1}{s^2+1} = -\frac{e^{-\pi s}}{s^2+1}$ $\square$

1c. $i'(t) + 2i(t) + \int_0^t i(\tau)d\tau = \delta(t-1), i(0) = 0$

Here it is worth noting that $\mathcal{L}(\int_0^t i(\tau)d\tau) = \frac{\mathcal{L}(i(t))}{s}$, and by applying Laplace to the equation above, we get

$s\mathcal{L}(i(t)) - i(0) + 2\mathcal{L}(i(t)) + \frac{1}{s}\mathcal{L}(i(s)) = \mathcal{L}(\delta(t-1))$
$\Rightarrow (s+2+\frac{1}{s})\mathcal{L}(i(t)) = e^{-s}$
$\Rightarrow \mathcal{L}(i(t)) = \frac{e^{-s}}{s+2+\frac{1}{s}}$

Now, e^{-s} reminds us of Laplace transform of the Heaviside-function for t-1, such that we can expect a solution of the form $i(t) = u(t-1)f(t-1)$. This means that we are now looking for f(t). We pull e^{-s} outside of the inverse Laplace-transform, so that we can find f(t).

$f(t) = \mathcal{L}^{-1}(\frac{1}{s+2+\frac{1}{s}}) = \mathcal{L}^{-1}(\frac{s}{(s+1)^2}) = \mathcal{L}^{-1}(\frac{1}{s+1} - \frac{1}{(s+1)^2}) = e^{-t} - te^{-t}$

By substituting in f(t-1) into i(t), we get the solution for the equation:

$i(t) = u(t-1)f(t-1) = u(t-1)(e^{-(t-1)} - e^{-(t-1)}(t-1)) \square$

2. $y - y \star t = t$

By applying Laplace to the equation we get

$\mathcal{L}(y) - \mathcal{L}(y \star t) = \mathcal{L}(t)$
$\Rightarrow \mathcal{L}(y) - \mathcal{L}(y) \cdot \mathcal{L}(t) = \mathcal{L}(t)$

Since $\mathcal{L}(t) = \frac{1}{s^2}$ we replace that in the equation.

$(1 - \frac{1}{s^2})\mathcal{L}(y) = \frac{1}{s^2}$
$\Rightarrow y = \mathcal{L}^{-1}(\frac{\frac{1}{s^2}}{1-\frac{1}{s^2}}) = \mathcal{L}^{-1}(\frac{1}{s^2-1}) = \underline{\underline{sinh(t)}}$

3. (1) $x' = 2x - y$
(2) $y' = 3x - 2y$

Applying Laplace to equations (1) and (2), we get

(3) $\mathcal{L}(x') = 2\mathcal{L}(x) - \mathcal{L}(y)$
$\Rightarrow s\mathcal{L}(x) - x(0) = 2\mathcal{L}(x) - \mathcal{L}(y)$
$\Rightarrow s\mathcal{L}(x) = 2\mathcal{L}(x) - \mathcal{L}(y)$

(4) $\mathcal{L}(y') = 3\mathcal{L}(x) - 2\mathcal{L}(y)$
$\Rightarrow s\mathcal{L}(y) - y(0) = 3\mathcal{L}(x) - 2\mathcal{L}(y)$
$\Rightarrow s\mathcal{L}(y) - 1 = 3\mathcal{L}(x) - 2\mathcal{L}(y)$

Solving (3) with regard to x

(5) $(2-s)\mathcal{L}(x) = \mathcal{L}(y)$
$\Rightarrow x = \mathcal{L}^{-1}(\frac{\mathcal{L}(y)}{2-s})$

Inserting (5) into (4) we get

(6) $s\mathcal{L}(y) - 3\frac{\mathcal{L}(y)}{2-s} + 2\mathcal{L}(y) = 1$
$\Rightarrow (2s - s^2 - 3 + 4 - 2s)\mathcal{L}(y) = 2 - s$
$\Rightarrow \mathcal{L}(y) = \frac{2-s}{1-s^2} = \frac{s-2}{s^2-1}$

By splitting the fraction, we get that $\frac{A}{s+1} + \frac{B}{s-1} = \frac{s-2}{s^2-1}$ such that $A = -\frac{1}{2}, B = \frac{3}{2}$, inserting this into (6) we get

$y = \mathcal{L}^{-1}(\frac{3}{2(s-1)} - \frac{1}{s(s+1)}) = \frac{3}{2}\mathcal{L}^{-1}(\frac{1}{s-1}) - \frac{1}{2}\mathcal{L}(\frac{1}{s+1}) = \underline{\frac{3e^{-t}-e^{t}}{2}}$

Inserting the result from (6) back into (5), we get

$x = \mathcal{L}^{-1}(\frac{\frac{2-s}{1-s^2}}{2-s}) = \mathcal{L}^{-1}(\frac{1}{1-s^2}) = \mathcal{L}^{-1}(\frac{1}{2(s+1)} - \frac{1}{2(s-1)}) = \frac{e^t - e^{-t}}{2} = -sinh(t)$

Thus, we have the solutions $\underline{\underline{x = \frac{e^t - e^{-t}}{2} = -sinh(t) \text{ and } y = \frac{3e^{-t} - e^t}{2}}}$

4a. First we need to find the complex Fourier coefficients, c_0 and c_n

$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x dx = 0$

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx = \frac{1}{2\pi}[-\frac{xe^{-inx}}{in} - \frac{e^{-inx}}{i^2n^2}]_{-\pi}^{\pi} = \frac{1}{\pi}(-\frac{\pi}{in}(\frac{e^{in\pi}+e^{-in\pi}}{2}) + \frac{1}{in^2}(\frac{e^{in\pi}-e^{-in\pi}}{2i})) = \frac{1}{\pi}(\frac{-\pi}{in}cos(n\pi) + \frac{1}{in^2}sin(n\pi))$

Considering that $sin(n\pi) = 0$ and $cos(n\pi) = (-1)^n$ we get

$c_n = \frac{1}{\pi} \cdot \frac{-\pi(-1)^n}{in} = \frac{i(-1)^n}{n}$

Inserting this into the complex Fourier identity gives us

$x = c_0 + \sum_{n\neq 0} c_n e^{in\pi} = \sum_{n\neq 0} \frac{i(-1)^n}{n} e^{in\pi} \square$

4b. Again, we need to find the complex Fourier coefficients, c_0 and c_n

$c_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\pi - x)dx = \frac{1}{2\pi}[\pi x^2 - \frac{x^3}{3}]_{-\pi}^{\pi} = -\frac{\pi^2}{3}$

$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(2\pi - x)e^{-inx}dx = \frac{1}{2\pi}(2\pi\int_{-\pi}^{\pi} xe^{e-inx}dx - \int_{-\pi}^{\pi} x^2 e^{-inx}dx)$

Here we can see that we solved the first part in the previous task, and know that $2\pi(\frac{1}{2\pi}\int_{-\pi}^{\pi} xe^{-inx}dx) = \frac{2\pi i(-1)^n}{n}$. Solving the second part gives us

$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}dx = \frac{1}{2\pi}[-x^2\frac{e^{-inx}}{in} - 2x\frac{e^{-inx}}{i^2n^2} + 2\frac{e^{-inx}}{i^3n^3}]_{-\pi}^{\pi} = \frac{1}{\pi}(\frac{\pi^2}{n}(\frac{e^{in\pi}-e^{-in\pi}}{2i}) - \frac{2\pi}{i^2n^2}(\frac{e^{in\pi}+e^{-in\pi}}{2}) + \frac{2}{i^2n^3}(\frac{e^{in\pi}-e^{-in\pi}}{2i})) = \frac{1}{\pi}(\frac{\pi^2}{n}sin(n\pi) - \frac{2\pi}{i^2n^2}cos(n\pi) + \frac{2}{i^2n^3}sin(n\pi))$

Since $sin(n\pi) = 0$, we get

$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx}dx = \frac{2}{n^2}cos(n\pi) = \frac{2(-1)^n}{n^2}$

Adding this to the original equation for c_n, we get

$c_n = \frac{2\pi i(-1)^n}{n} - \frac{2(-1)^n}{n^2} = \frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{n+1}}{n^2}$

This then gives us the complex Fourier expansion of $x(2\pi - x)$:

$x(2\pi - x) = c_0 + \sum_{n\neq 0} c_n e^{in\pi} = -\frac{\pi^2}{3} + \sum_{n\neq 0}(\frac{2\pi i(-1)^n}{n} + \frac{2(-1)^{(n+1)}}{n^2})e^{in\pi} \square$