TMA4135 Matematikk 4D
Exercise 4

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1a. The inner product is defined as $(f,g) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx$. Applying this to (sin nx,1) gives us the following:

$(1, sin(nx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} sin(nx)dx = -\frac{cos(nx)}{n}|_{-\pi}^{\pi} =$

$\frac{-cos(n\pi)+cos(n(-\pi))}{n} = \frac{-cos(n\pi)+cos(n\pi)}{n} = 0 \; \square$

Now, for (sin(mx),cos(nx)) we have:

$(sin(mx), cos(nx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} sin(mx)cos(nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{imx}-e^{-imx}}{2i}\cdot\frac{e^{inx}+e^{-inx}}{2}dx =$

$\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i(m+n)x}-e^{-i(m+n)x}}{2i} + \frac{e^{i(m-n)x}-e^{-i(m-n)x}}{2i}dx = \frac{1}{4\pi}(\int_{-\pi}^{\pi} sin((m+n)x)dx +$

$\int_{-\pi}^{\pi} sin((m-n)x)dx) = \frac{1}{2}((1, sin((m+n)x) + (1, sin((m-n)x)) = 0 \; \square$

1b. We start by looking at (sin(mx),sin(nx)) when $n \neq m$:

$(sin(mx), sin(nx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} sin(mx)sin(nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{imx}-e^{-imx}}{2i}\cdot\frac{e^{inx}-e^{-inx}}{2i}dx =$

$\frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{e^{i(m-n)x}+e^{-i(m-n)x}}{2} - \frac{e^{i(m+n)x}+e^{-i(m+n)x}}{2}dx = \frac{1}{4\pi}(\int_{-\pi}^{\pi} cos((m-n)x)dx -$

$\int_{-\pi}^{\pi} cos((m+n)x)dx) = \frac{1}{2}((1, cos((m-n)x)) - (1, cos((m+n)x))) = 0$

Now, we will be looking at (sin(mx),sin(nx)) when $n = m$:

$(sin(nx), sin(nx)) = \frac{1}{2\pi}\int_{-\pi}^{\pi} sin^2(nx)dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1-cos(2nx)}{2}dx =$

$\frac{1}{4\pi}(\int_{-\pi}^{\pi} dx - \int_{-\pi}^{\pi} cos(2nx)dx) = \frac{1}{4\pi}(x|_{-\pi}^{\pi} - (1, cos(2nx))) = \frac{1}{4\pi}\cdot 2\pi = \frac{1}{2}$

Thus we have that

$$(sin(mx), sin(nx)) = \begin{cases} 0 & m \neq n = 0, 1, 2, ... \\ \frac{1}{2} & m = n = 1, 2, 3, ... \end{cases} \tag{1}$$

Now, we will look at (cos(mx),cos(nx)) for $n \neq m$:

$(cos(mx), cos(nx)) = \frac{1}{2\pi} \int_{-\pi}^{\pi} cos(mx)cos(nx)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{imx}+e^{-imx}}{2} \cdot \frac{e^{inx}+e^{-inx}}{2} dx =$

$\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{e^{i(m+n)x}+e^{-i(m+n)x}}{2} + \frac{e^{(i(m-n)x}+e^{-i(m-n)x}}{2} dx = \frac{1}{4\pi}(\int_{-\pi}^{\pi} cos((m+n)x)dx +$

$\int_{-\pi}^{\pi} cos((m-n)x)dx) = \frac{1}{2}((1, cos((m+n)x)) + (1, cos((m-n)x))) = 0$

At last, looking at (cos(mx),cos(nx)) when $n = m$, we have:

$(cos(nx), cox(nx)) = \frac{1}{2\pi} \int_{\pi}^{\pi} cos^2(nx)dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1+cos(2nx)}{2} dx = \frac{1}{4\pi}(\int_{-\pi}^{\pi} dx +$

$\int_{-\pi}^{\pi} cos(2nx)dx = \frac{1}{4\pi}(x|_{-\pi}^{\pi} + (1, cos(2nx))) = \frac{1}{4\pi} \cdot 2\pi = \frac{1}{2}$

Which means that we have:

$$(cos(mx), cos(nx)) = (sin(mx), sin(nx)) = \begin{cases} 0 & m \neq n = 0, 1, 2, ... \\ \frac{1}{2} & m = n = 1, 2, 3, ... \end{cases} \square \tag{2}$$

1c. First we begin be multiplying by 1 an integrating on both sides of the function from $-\pi$ to π.

$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{\pi} a_0 dx + \Sigma_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n cos(nx)dx + \int_{-\pi}^{\pi} b_n sin(nx)dx = 2\pi a_0$

$\Rightarrow \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)dx = (f, 1) = a_0 \square$

Doing the same with cos(mx) and integrating on both sides gives us

$\int_{-\pi}^{\pi} f(x)cos(mx)dx = \int_{-\pi}^{\pi} a_0 cos(mx) + \Sigma_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n cos(nx)cos(mx)dx + \int_{-\pi}^{\pi} b_n sin(nx)cos(mx)dx =$

$\int_{-\pi}^{\pi} a_n cos(nx)cos(mx)dx = \int_{-\pi}^{\pi} a_m cos(mx)cos(mx) = \frac{2\pi}{2} a_m$

$\Rightarrow a_m = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x)cos(mx)dx$

$\Rightarrow a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)2cos(nx)dx = (f, 2cos(nx)) \square$

Repeating the process with sin(mx) instead of cos(mx), we get

$\int_{-\pi}^{\pi} f(x)sin(mx)dx = \int_{-\pi}^{\pi} a_0 sin(mx) + \Sigma_{n=1}^{\infty} \int_{-\pi}^{\pi} a_n cos(nx)sin(mx)dx + \int_{-\pi}^{\pi} b_n sin(nx)sin(mx)dx =$

$\int_{-\pi}^{\pi} b_n sin(nx)sin(mx)dx = \int_{-\pi}^{\pi} b_m sin(mx)sin(mx)dx = \frac{2\pi}{2} b_m$

$\Rightarrow b_m = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x)sin(mx)dx$

$\Rightarrow b_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)2sin(nx)dx = (f, 2sin(nx))$ $\square$

Now, for $||f||^2$, we have that

$f^2 = a_0^2 + 2a_0\Sigma_{n=1}^{\infty}a_n cos(nx) + b_n sin(nx) + \Sigma_{n=1}^{\infty}\Sigma_{m=1}^{\infty}a_n a_m cos(nx)cos(mx) +$

$a_n b_m cos(nx)sin(mx) + a_m b_n cos(mx)sin(nx) + b_n b_m sin(nx)sin(mx)$

Integrating both sides of the equation from $-\pi$ to π then gives us

$\int_{-\pi}^{\pi} f^2(x)dx = |a_0|^2\int_{-\pi}^{\pi} dx + a_0\int_{-\pi}^{\pi}\Sigma_{n=1}^{\infty}a_n cos(nx) + b_n sin(nx)dx + \int_{-\pi}^{\pi}\Sigma_{n=1}^{\infty}\Sigma_{m=1}^{\infty}a_n a_m cos(nx)cos(mx) +$

$a_n b_m cos(nx)sin(mx) + a_m b_n cos(mx)sin(nx) + b_n b_m sin(nx)sin(mx)dx = 2\pi|a_0|^2 +$

$\int_{-\pi}^{\pi}\Sigma_{n=1}^{\infty}\Sigma_{m=1}^{\infty}a_n a_m cos(nx)cos(mx) + b_n b_m sin(nx)sin(mx)dx =$

$2\pi|a_0|^2 + \Sigma_{n=1}^{\infty}|a_n|^2 \cdot \frac{2\pi}{2} + |b_n|^2 \cdot \frac{2\pi}{2}$

This gives us

$||f||^2 = (f, f) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f^2(x)dx = \frac{1}{2\pi}(2\pi|a_0|^2 + \Sigma_{n=1}^{\infty}|a_n|^2 \cdot \frac{2\pi}{2} + |b_n|^2 \cdot \frac{2\pi}{2}) =$

$|a_0|^2 + \frac{1}{2}\Sigma_{n=1}^{\infty}|a_n|^2 + |b_n|^2$ $\square$

2. The figure below shows the odd and even extensions of f(x).

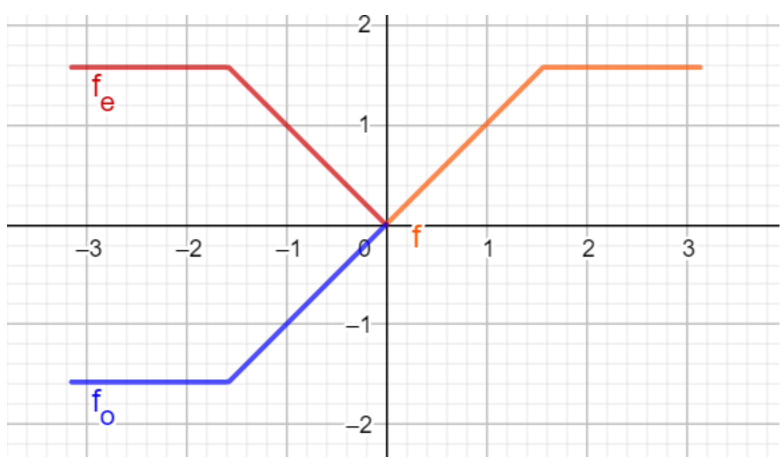

Figure 1: f_e is marked in red, f_o in blue

When computing the Fourier cosine series, we need to find a_0 and a_n:

$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{2}{\pi}\int_0^{\pi} f(x)dx = \frac{2}{\pi}(\int_0^{\frac{\pi}{2}} xdx + \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}dx) = \frac{2}{\pi}(\frac{x^2}{2}|_0^{\frac{\pi}{2}} + \frac{\pi}{2}x|_{\frac{\pi}{2}}^{\pi}) =$

$\frac{2}{\pi}(\frac{\pi^2}{8} + \frac{\pi^2}{4}) = \frac{6\pi}{8}$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)cos(nx)dx = \frac{2}{\pi}(\int_0^{\frac{\pi}{2}} xcos(nx)dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}cos(nx)dx) = \frac{2}{\pi}(\frac{xsin(nx)}{n} +$

$\frac{cos(nx)}{n^2}|_0^{\frac{\pi}{2}} + \frac{\pi}{2}\cdot\frac{sin(nx)}{n}|_{\frac{\pi}{2}}^{\pi}) = \frac{2}{\pi}\cdot\frac{cos(nx)}{n^2}|_0^{\frac{\pi}{2}}$

Now, depending on what n is, we have different values of $cos(n\frac{\pi}{2})$. When n is odd, we have that $cos(n\frac{\pi}{2}) = 0$. Else we have that $cos(n\frac{\pi}{2}) = (-1)^{\frac{n}{2}}$. In other words, we have that

$$a_n = \begin{cases} \frac{2}{\pi}(-\frac{1}{n^2}) & n = 2k+1 \\ \frac{2}{\pi}((-1)^{\frac{n}{2}} - \frac{1}{n^2}) & n = 2k \end{cases} \tag{3}$$

By putting this into the Fourier cosine series gives us

$f_e = \frac{a_0}{2} + \Sigma_{n=1}^{\infty} a_n cos(nx) = \frac{3\pi}{8} + \frac{2}{\pi}(-cosx - \frac{2cos(2x)}{2^2} - \frac{cos(3x)}{3^2} - \frac{cos(5x)}{5^2} - ...)$ $\square$

For the Fourier sine expansion of f, we have an odd function, such that $b_0 = 0$. Now, we need to find b_n:

$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)sin(nx)dx = \frac{2}{\pi}(\int_0^{\frac{\pi}{2}} xsin(nx)dx + \int_{\frac{\pi}{2}}^{\pi} \frac{\pi}{2}sin(nx)dx) = \frac{2}{\pi}(-\frac{xcos(nx)}{n} +$

$\frac{sin(nx)}{n^2}|_0^{\frac{\pi}{2}} - \frac{\pi}{2}\cdot\frac{cos(nx)}{n}|_{\frac{\pi}{2}}^{\pi}) = \frac{\pi}{2}(\frac{sin(\frac{\pi}{2}n)}{n^2} - \frac{\pi}{2}\cdot\frac{(-1)^n}{n})$

Again, $sin(n\frac{\pi}{2})$ depends on whether n is even or odd. If n is even, $sin(n\frac{\pi}{2}) = 0$, else, $sin(n\frac{\pi}{2}) = (-1)^{\frac{n-1}{2}}$. This gives us the following b_n:

$$b_n = \begin{cases} -\frac{1}{n} & n = 2k \\ \\ \frac{2\cdot(-1)^{\frac{n-1}{2}}}{\pi n^2} + \frac{1}{n} & n = 2k+1 \end{cases} \tag{4}$$

Inserting this into the definition of the Fourier sine series, we get:

$f_o = b_0 + \Sigma_{n=1}^{\infty} b_n sin(nx) = (\frac{2}{\pi} + 1)sinx - \frac{1}{2}sin(2x) + (-\frac{2}{3^2\pi} + \frac{1}{3})sin(3x) -$

$\frac{1}{4}sin(4x) + (\frac{2}{5^2\pi} + \frac{1}{5})sin(5x) + ...$ $\square$