Odd André Ovren TMA4135 Ex 2

1a) $F(s) = \frac{1}{s^2(s^2+1)}$

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \underline{\underline{t - \sin t}}$$

b) $F(s) = \frac{s}{s^2+2s+1}$

$$f(t) = \mathcal{L}^{-1}\left(\frac{s}{s^2+2s+1}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t} - te^{-t}$$

$$= (1-t)e^{-t}$$

$$= \underline{\underline{-(t-1)e^{-t}}}$$

c)

Here we can see that $F(s) = \frac{2s}{(s^2+1)^2} = -\left(\frac{1}{s^2+1}\right)'$

$$f(t) = \mathcal{L}^{-1}(F(s)) = \mathcal{L}^{-1}\left(-\frac{1}{s^2+1} \cdot \frac{d}{ds}\right) = t\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right)$$

$$= \underline{\underline{t \sin t}}$$

d) $F(s) = (s-3)^{-5} = \frac{1}{(s-3)^5}$

We recall that $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ and that $\mathcal{L}(e^{at}) = \frac{1}{s-a}$.

This gives us that $\mathcal{L}(t^4 e^{3t}) = \frac{24}{(s-3)^5}$, which means that

$$f(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-3)^5}\right) = \underline{\underline{\frac{1}{24}e^{3t}t^4}}$$

2a) $f(t) = (u(t) - u(t-\pi))\cos t = u(t)\cdot\cos t - u(t-\pi)\cos t.$

$F(s) = \mathcal{L}(f(t)) = \mathcal{L}(u(t)\cos t) - \mathcal{L}(u(t-\pi)\cos t)$

$= \mathcal{L}(u(t)\cos t) - \mathcal{L}(-u(t-\pi)\cos(t-\pi))$ | $\cos(t) = -\cos(t-\pi)$

$= \mathcal{L}(u(t)\cos t) + \mathcal{L}(u(t-\pi)\cos(t-\pi))$ | $\mathcal{L}(u(t-a)f(t-a)) = e^{-sa}\mathcal{L}(f(t))$

$= \mathcal{L}(\cos t)\cdot(1+e^{-\pi s}) = \underline{\underline{\frac{s}{s^2+1}\cdot(1+e^{-\pi s})}}$

b) $f(t) = u(t-3)t^4$

If we define $V(t) = (t+3)^4$, s.t $V(t-3) = t^4$, then we have

$f(t) = u(t-3)V(t-3)$

this gives us

$F(s) = \mathcal{L}(u(t-3)V(t-3)) = e^{-3s}\mathcal{L}(V(t)) = e^{-3s}\mathcal{L}((t+3)^4)$

$\mathcal{L}((t+3)^4) = \int_0^\infty e^{-st}(t+3)^4\,dt$

Integration by parts gives us

D	I
$+\ (t+3)^4$	e^{-st}
$-\ 4(t+3)^3$	$-\frac{e^{-st}}{s}$
$+\ 12(t+3)^2$	$\frac{e^{-st}}{s^2}$
$-\ 24(t+3)$	$-\frac{e^{-st}}{s^3}$
$+\ 24$	$\frac{e^{-st}}{s^4}$

$\rightarrow \int_0^\infty 24\frac{e^{-st}}{s^4}\,dt = 24\frac{e^{-st}}{s^5}\Big|_0^\infty$

$\mathcal{L}((t+3)^4) = -e^{-st}\left(\frac{(t+3)^4}{s} + \frac{4(t+3)^3}{s^2} + \frac{12(t+3)^2}{s^3} + \frac{24(t+3)}{s^4} + \frac{24}{s^5}\right)\Big|_0^\infty$ $s>0$

$= \frac{24}{s^5} + \frac{72}{s^4} + \frac{108}{s^3} + \frac{108}{s^2} + \frac{81}{s}$

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this means that

$$\mathcal{L}(u(t-3)t^4) = e^{-3s}\left(\frac{24}{s^5} + \frac{72}{s^4} + \frac{108}{s^3} + \frac{108}{s^2} + \frac{81}{s}\right)$$

3) $y'' + 2y = \delta(t-1)$, $y(0) = y'(0) = 0$

Applying Laplace gives us

$$s^2Y - \underbrace{sy'(0) - y(0)}_{0} + 2Y = \mathcal{L}(\delta(t-1))$$

$$(s^2+2)Y = \mathcal{L}(\delta(t-1))$$

$$Y = \mathcal{L}^{-1}\left(e^{-s}\cdot\frac{1}{s^2+2}\right)$$

We know that $e^{-s}\mathcal{L}(f(t)) = \mathcal{L}(u(t-1)f(t-1))$

We also know that $\mathcal{L}(\sin at) = \frac{a}{s^2+a^2}$ so $\mathcal{L}(\sin\sqrt{2}t) = \frac{\sqrt{2}}{s^2+2}$

and $\frac{1}{\sqrt{2}}\mathcal{L}(\sin\sqrt{2}t) = \frac{1}{s^2+2}$

If we now define $v(t) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}(t))$, then we have $v(t-1)\frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-1))$. This means that

$$Y = \mathcal{L}^{-1}\left(e^{-s}\cdot\frac{1}{s^2+2}\right) = \mathcal{L}^{-1}\left(e^{-s}\mathcal{L}(v(t))\right) = u(t-1)\frac{1}{\sqrt{2}}\sin(\sqrt{2}(t-1))$$

4a) $\int_{nT}^{(n+1)T} e^{-st} f(t)\, dt$

If we replace t with $t+nT$ we get

$\int_0^T e^{-s(t+nT)} f(t+nT)\, dt$

Since f is periodic we have that $f(t+nT) = f(t)$

$\int_0^T e^{-s(t+nT)} f(t)\, dt = e^{-snT} \int_0^T e^{-st} f(t)\, dt$ $\square$

b) $\mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt + \int_T^{2T} e^{-st} f(t)\, dt + \ldots$

$= \sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt$

From a) we have that $\int_{nT}^{(n+1)T} e^{-st} f(t)\, dt = \int_0^T e^{-st} f(t)\, dt \cdot e^{-snT}$

Thus, we have

$\sum_{n=0}^{\infty} \int_{nT}^{(n+1)T} e^{-st} f(t)\, dt = \sum_{n=0}^{\infty} e^{-snT} \int_0^T e^{-st} f(t)\, dt$ $\square$

c) If we assume s and $T > 0$, such that $0 < e^{-sT} < 1$, we have that

$\sum_{n=0}^{\infty} (e^{-sT})^n \int_0^T e^{-st} f(t)\, dt = \frac{1}{1-e^{-sT}} \int_0^T e^{-sT} f(t)\, dt$ $\square$