Rates, Ratios, and Percentages

Ratios are powerful mathematical tools that help you compare two values, and use their relationship to solve a problem. They take several common forms, which can be converted back and forth with careful multiplication, division, and knowledge of fractions.

Ratio, 1:2	Fraction, $\frac{1}{2}$	Decimal, 0.50	Percentage, 50%

- A "**Unit Ratio**" is a ratio where one of the numbers is 1.
- A "**Percentage**" is a ratio where the second number is implied to be 100.
- A "**Proportion**" happens when you put an equals sign between two ratios.
- A "**Rate**" is when the numbers have different units attached, like "33 miles per 1 hour".

CONVERSIONS:

From	To	How to convert	Examples
Ratio	**Fraction**	Put the first number on top, and the second number on bottom. To remember this more easily, remember that we read from the TOP+LEFT to the BOTTOM+RIGHT. ***however*, *see first sample.***	$1:2=\frac{1}{2}$ $2:1=\frac{2}{1}=2$
Ratio / Fraction	**Decimal**	Take the first/top number and divide by the second/bottom number. Calculators speed this up immensely.	$\frac{1}{2}=1\div 2=0.5$ $\frac{2}{1}=2\div 1=2$
Decimal	**Fraction**	Put the decimal on the top of the fraction, but erase the decimal point (i.e. : 0.5 -> 5). Count the number of digits that were to the right of the decimal point. On the bottom, put a 1, followed by that many zeroes. Simplify the fraction you get as much as you are asked to.	$0.5=\frac{5}{10}=\frac{1}{2}$ $1.5=\frac{15}{10}=\frac{3}{2}$ $0.05=\frac{5}{100}=\frac{1}{20}$
Decimal	**Percent**	Take the decimal between 0 and 1, and multiply that decimal by 100. Note: If you have something like $\frac{12.5}{50}$, or 2.5 : 10 those are **ratios** and **fractions**, even with a decimal in there. Use the other methods.	$0.05x100=5\%$ $0.5x100=50\%$ $5.0x100=500\%$
Percent	**Fraction**	Put the percent above 100, erase the % sign, and simplify.	$50\%=\frac{50}{100}=\frac{1}{2}$
Fraction	**Percent**	There are two methods. 1. Use a proportion. Set the fraction equal to $\frac{x}{100}$ and solve for x 2. Convert fraction → decimal → percent. IE, take the top, divide by the bottom, then multiply by 100. I sincerely recommend method 2 any time you have a calculator.	$1\div 2=0.5$ $0.5x100=50\%$ ——— $1\div 4=0.25$ $0.25x100=25\%$

The examples that follow show common tricks and problems to be aware of, especially #1.

#1. The importance of Reading Carefully.
There is a class containing 2 girls and 3 boys. There are several ratios you can produce from this information, and a test question might ask you to use any one of them! Read very, very carefully to be certain what they are looking for.

The ratio of **GIRLS to BOYS** is 2 girls to 3 boys, or **2:3**. The ratio of **BOYS to GIRLS** is 3 boys to 2 girls, or **3:2**. The ratio of **GIRLS in the CLASS** (or GIRLS to STUDENTS), is 2 girls : (total number of students). Boys and girls are both students, so 2+3 = 5 students. The ratio of girls in the class is **2:*5***. Likewise, the ratio of **BOYS in the CLASS is 3:5.**

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#2: Unit Conversion, and Unit Rates

2: There are 12 inches in one foot, so the ***unit rate*** of inches to feet is 12:1 and the rate of feet to inches is 1:12. Let's say we have four pieces of string of different sizes -- 24 inches long, 6 inches long, 3 feet long, and 1.5 feet long -- and I want to sort them in order from shortest to longest. How can we use the ***conversion factor*** of 12 inches to 1 foot to find the equivalent measures in the other unit?

The first, less-reliable method is using common sense to decide if you need to ***multiply*** or ***divide***. Feet are longer than inches, so if you measure a particular length, it will be many inches, but fewer feet. Multiplying whole numbers makes them bigger, and dividing makes them smaller. So, starting from a string that is 3 feet long, if I want its length in inches, I need to make the "3" bigger, and I know my conversion factor is 12:1. So I will multiply by 12. 3x12 = 36 inches. My string that is 1.5 feet long is 1.5 x 12 = 18 inches. My string that is 24 inches, I want to convert into feet, and feet are smaller than inches -- so I will divide by 12. 24 inches ÷ 12 = 2 feet. Likewise, 6 inches divided by 12 = ½, which makes sense because 6 inches is half of a foot, and 6 is half of 12. In order, my strings are 6 inches (½ ft), 18 inches (1.5 ft), 24 inches (2 ft), and 36 inches (3 ft).

Did dividing 6 by 12 to get a fraction feel vaguely wrong? Did any of that feel uncomfortable, or like you weren't sure where decisions were coming from? If so, you might prefer to use the technical and more detailed method. The advantage to knowing more than one way to solve a problem, is that you can use one method to check the answer you got from the other. It is important that the answer you get makes sense to you, that you believe it is correct, and that you reliably get the same answer.

The second, detailed method is to set up a proportion and solve it for your answer.

Proportions always look like: $$\frac{\text{something}}{\text{something else}} = \frac{\text{another thing}}{\text{another other thing}}$$

Each side of the proportion is a ratio (or percentage, or rate), in the form of a fraction.

If 12 inches is equivalent to 1 foot, then 24 inches is equivalent to how many feet?

$$\frac{12\text{ inch}}{1\text{ foot}} = \frac{24\text{ inch}}{?\text{ foot}}$$

Set up the proportion. One unit on top, the other on the bottom.

$$\frac{12\text{ inch} \times \cancel{1\text{ foot}}}{\cancel{1\text{ foot}}} = \frac{24\text{ inch} \times 1\text{ foot}}{?\text{ foot}}$$

$$\frac{12\text{ inch} \times ?\text{ foot}}{} = \frac{24\text{ inch} \times 1\text{ foot} \times \cancel{?\text{ foot}}}{\cancel{?\text{ foot}}}$$

$$12\text{ inch} \times ?\text{ foot} = 24\text{ inch} \times 1\text{ foot}$$

Get rid of fractions by multiplying both sides of the equals sign by the bottoms of the fractions, and then canceling out

Shortcut: *Notice that our result is the top of the original fraction, multiplied by the bottom of the other side? You can get this same answer by "**Cross multiply**"ing, where you multiply diagonally across the equals sign. Top left x bottom right = top right x bottom left. Always.*

$$?\text{ foot} \times \cancel{12\text{ inch}} \div \cancel{12\text{ inch}} = 24\text{ inch} \times 1\text{ foot} \div 12\text{ inch}$$

$$?\text{ foot} = 24\text{ inch} \times 1\text{ foot} \div 12\text{ inch}$$

$$?\text{ foot} = 24 \times 1 \div 12$$

$$?\text{ foot} = 24 \div 12$$

$$?\text{ foot} = 2$$

so 24 inches = 2 feet

Get the part you don't know alone on one side. The opposite of multiplying by 12 is dividing by 12, so they cancel out.

So, we want to know how many feet six inches converts to. So, we say that 6 inches : ? feet = 12 inches : 1 foot. We write that out as fractions, inches on top and feet on the bottom. We ***cross multiply***, and get 6 inches x 1 foot = ? feet x 12 inches. We divide both sides by 12 inches, and get 6 inches x 1 foot ÷ 12 inches = ? feet. 6x1÷12 = ½, so 6 inches = ½ foot.

$$\frac{6\text{ inches}}{?\text{ foot}} = \frac{12\text{ inches}}{1\text{ foot}}$$

6 inches x 1 foot = ? foot x 12 inches
6 inches x 1 foot ÷ 12 inches = ? foot
6 inches ÷ 12 inches = ½ foot
6 inches = ½ foot

How many inches is 3 feet? It works the same way, this time our answer will just come out in inches. Being careful to keep one unit on top and the other on the bottom, 3 feet : ? inches = 1 foot : 12 inches. Then 3 feet x 12 inches = 1 foot x ? inches, and 3 feet x 12 inches ÷ 1 foot = ? inches. 3x12=36 inches.

$$\frac{3\text{ feet}}{?\text{ inches}} = \frac{1\text{ foot}}{12\text{ inches}}$$

3 feet x 12 inches = 1 foot x ? inches
3 x 12 ÷ 1 = ? inches
3 feet = 36 inches

So how many inches is 1.5 feet? The same method keeps working. 1.5 feet : ? inches = 1 foot : 12 inches. Then 1.5 ft x 12 inches = 1 foot x ? inches, 1.5 x 12 = 18, so 1.5 feet = 18 inches.

$$\frac{1.5\text{ feet}}{?\text{ inches}} = \frac{1\text{ foot}}{12\text{ inches}}$$

1.5 feet x 12 inches = 1 foot x ? inches
1.5 x 12 ÷ 1 = ? inches
1.5 feet = 18 inches

You might notice that in all of these cases, we end up with a final step that is taking our initial number, and then either multiplying or dividing it by 12 inches -- exactly as in the common sense paragraph. Likewise, you might notice that when converting *from feet* to inches, with a ratio of 1 foot to 12 inches, you always multiply by 12 inches. When converting from inches *to feet*, with a ratio of 12 inches *to 1 foot*, you always end up dividing by 12. Put another way, ...

When using a unit ratio (like 1 foot : 12 inches, or 1 week : 7 days) to convert between units,

1. You will always either multiply or divide by the number that is not 1.
2. You *multiply* when you are converting *from* the unit that is 1 to the unit that is a different number.
3. You *divide* when you are converting from the other unit, to the unit that is one.

When using a non-unit ratio (like 10 minutes : 5 miles), you will multiply your unknown value by the number on top, and divide by the number on bottom. Figuring out which number belongs on top and which belongs on bottom is part of setting up the problem. Typically, life is easier when your unknown is on top.

Patterns like this can be useful to learn, because they take less steps than the long way, they are faster than the long way, and they are probably easier to remember than the long way. However, there is a different pattern for every type of problem, and the long way (setting up a proportion) is a single method that works for every problem. This gives you as a student a choice, and neither answer is wrong: Would you rather memorize a lot of small math tricks, or one very complicated one?

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#3: Unit Conversion, The Metric System and Scientific Units
When you multiply by ten, you move the decimal point one to the left. When you divide by ten, you move the decimal point one to the right. All metric units are related to eachother by multiplying or dividing by ten a certain number of times. For example, 10x10x10 is 1000.

To convert between kilometers and meters, the ratio is 1000:1. Kilograms:grams is also 1000:1. Kiloliters:liters is also 1000:1. The same ratio is true for meters:millimeters, grams:milligrams, and liters:milliliters. To convert between those units, you will move the decimal point 3 places in the appropriate direction. 2 kilograms = 2,000 grams = 2,000,000 milligrams.

#4: The Multi-Part Gas Mileage Problem
You have a car that can go 30 miles ***per*** gallon, and holds 12 gallons of gas.

A. How many gallons will you need for a 120 mile roadtrip?

$$\frac{120 \text{ miles}}{? \text{ gallons}} = \frac{30 \text{ miles}}{1 \text{ gallon}}$$

$$\begin{aligned} 120 \text{ miles} \times 1 \text{ gallon} &= 30 \text{ miles} \times ? \text{ gallons} \\ 120 \times 1 \div 30 &= ? \text{ gallons} \cancel{\times 30 \div 30} \\ 120 \div 30 &= ? \text{ gallons} \\ &= 4 \text{ gallons} \end{aligned}$$

B. How many miles can a full tank go?

$$\frac{? \text{ miles}}{12 \text{ gal.}} = \frac{30 \text{ miles}}{1 \text{ gallon}}$$

$$\begin{aligned} ? \text{ miles} \times 1 \text{ gallon} &= 30 \text{ miles} \times 12 \text{ gallons} \\ ? \text{ miles} &= 30 \times 12 \div 1 \text{ gallons} \\ &= 360 \text{ gallons} \end{aligned}$$

C. If you drive at 60 miles *per* hour, how many gallons per hour is that?

In one hour, the car will drive 60 miles. So how many gallons of gas does it take to go 60 miles? The car gets 30 miles per gallon, so divide 60 by 30 to get 2. When you drive 60 miles per hour, the car will use 2 gallons per hour.

D. If you drive at 60 miles an hour, how many hours can you last on a full tank?

From part C, we know it uses two gallons an hour when driving 60 miles an hour. 12 ÷ 2 = 6. You can drive for six hours on a full tank.

#5: Tipping at restaurants

<u>What is 20% of $23.90?</u>

20% means 20 out of 100. So, to find 20% of a number, we set up a proportion like this:

$$\frac{?}{\$23.90} = \frac{20}{100}$$

$$\begin{aligned} ? \times 100 &= \$23.90 \times 20 \\ ? \times \cancel{100 \div 100} &= \$23.90 \times 20 \div 100 \\ ? &= \$23.90 \times 20 \div 100 \\ ? &= \$4.78 \end{aligned}$$

<u>SHORTCUT 1:</u> *Remember that you can find 10% of any decimal number by moving the decimal point one place to the left. By this logic, 10% of 23.90 is 2.39. 20% is twice as much as 10%, and 2 x 2.39 = 4.78, so $4.78 is a 20% tip.*

If you round results to the nearest 50 cents as you go along, you should be able to calculate reasonably accurate tips in your head! Using this example, 10% of 23.90 is 2.39, which is almost $2.50. 2 x 2.50 = $5. $5 is pretty close to $4.78.

<u>SHORTCUT 2:</u> To find a percentage, multiply the number ($23.90) by the percent you want (20) and then divide by 100.

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#6: Price Changes

There are two snacks you like, that used to cost $2.50 and $4.50. Thanks to a new tax, each one costs $0.25 more. What is the increase in price on each snack, as a percentage?

A. The $2.50 snack

$$\frac{\$2.50+0.25}{\$2.50} = \frac{?\%}{100\%} \qquad 100\% \times (\$2.50 + \$0.25) = \$2.50 \times ?\%$$

$$100\% \times \$2.75 \div \$2.50 = ?\%$$

$$110\% = ?\%$$

B. The $4.50 snack

$$\frac{\$4.50+0.25}{\$4.50} = \frac{?\%}{100\%} \qquad 100\% \times (\$4.50 + \$0.25) = \$4.50 \times ?\%$$

$$100\% \times \$4.75 \div \$4.50 = ?\%$$

$$105\% = ?\%$$

The expensive snack is now 105% of its original price, and the cheaper one is 110% of its original price. Their prices have increased by 5% and 10%, respectively.

<u>Shortcut Zone:</u> You may notice that we found these values by taking the new price, dividing by the old price, and then multiplying by 100 to get a percentage. We then subtract 100 from that percentage to get just the change. Another way to calculate this is to take the tax ($0.25), and divide it by each price ($4.50 or $2.50), then multiply by 100 ($0.25 ÷ $2.50 = 0.1, 0.1 x 100 = 10, the tax is 10% of the cheap price. Likewise, $0.25 ÷ 4.50 x 100 = a quarter is around 5% of the expensive price). ***We take a part, and divide it by the total, then multiply by 100 to find the percentage.***

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#7A: Diluting Medicine

You need to give your pet some medicated eyedrops, but the only bottle you can find comes in a concentrated 50% solution, and your pet needs a 10% solution. The instructions say you can use distilled water to dilute it. If you use 1 ml of 50% solution, how much water do you add to get a 10% solution?

We're going to set this problem up a little differently. We know that we will be adding one ml of a solution that is 50% medicine to an unknown quantity distilled water (which is 0% medicine). Our goal is to get an amount (1 ml + ? ml) of 10% solution. We can write this as:

Amount x (decimal percentage) + Amount x (Decimal Percentage) = Amount x (decimal percentage)

$$1 \text{ ml} \times (0.50) + ? \text{ ml} \times (0.00) = (1 \text{ ml} + ? \text{ ml}) \times (0.10)$$
$$0.50 + 0 = (1 \text{ ml} + ? \text{ ml}) \times (0.10)$$
$$0.50 \div 0.10 = (1 \text{ ml} + ? \text{ ml})$$
$$5 = 1 \text{ ml} + ? \text{ ml}$$
$$5-1 = ? \text{ ml}$$
$$4 = ? \text{ ml}$$

If we add 4 ml of distilled water to 1 ml of 50% solution, we will get a total of 5 ml of 10% solution.

#7B: Mixing Solutions

You have 1 ounce of a solution that is 50% ink, and 50% water. You have 3 ounces of a 25% ink solution. If you mix them together, how many ounces of solution do you have, and what percentage of the mixture is ink?

First, the easy part. One ounce of solution plus 3 ounces of solution is four ounces. So, whatever percentage it is, we will have four ounces of it.

Amount x (decimal percentage) + Amount x (Decimal Percentage) = Amount x (decimal percentage)

$$1 \times 0.50 + 3 \times 0.25 = (1 + 3) \times ?$$
$$0.50 + 0.75 = 4 \times ?$$
$$1.25 = 4 \times ?$$
$$0.31 = ?$$

So, our mixed solution will be 31% ink, and there will be 4 ounces of it.

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Percentages are when one of the rates is out of 100. It gives us a universal way to compare things, and is a convenient denominator for a lot of purposes.
> (Some examples involving dilution or pill dosages.)
--> Ordinarily the medicine is given in x% solution. A: An adult with bodyweight blah needs x grams of the thing, how many ml do you administer? B: OK, if that grams:bodyweight ratio is the same, how many grams does a small child with bodyweight Bluh need? C: How many ml is that? D: If the child is very sick and requires a gentler (smaller, not-evenly-divisible percentage) solution to not upset their stomach, how much water do you add to the previous amount?

> "30% of a number is 12", "100% of 20 is 20, so 50% of 20 is .5 * 20 = 10"
> "write each increase as a percent"
> "part to part" vs "part to whole" ratios?
> "write two more ratios equivalent to this ratio" "write this ratio in simplest form"
> "write these ratios as unit ratios"
> "the ratio of boys to girls is b:g, there are this many students, how many are boys and girls"
> Which is bigger, a/b or c/d?
> "There is a total of 600 blue, yellow, and red balls in a machine. The ratio of blue balls to the total number of balls is 1:4. The ratio of yellow balls to blue balls is 7:3. The ratio of blue balls to red balls is 3:2. Which colour of balls is most common?"
> "x:5 = 2:1, what is x?"