1.2.3 Exact Solution

Problem Statement:

Show that the function

$$u(t, x, y) = e^{i(k_x x + k_y y - \omega t)}$$

satisfies the two-dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u,$$

where $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ and $i = \sqrt{-1}$.

Solution:

To verify that $u(t, x, y)$ satisfies the wave equation, we'll compute the second partial derivatives and substitute them into the equation.

1. Compute $\frac{\partial u}{\partial t}$:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t} e^{i(k_x x + k_y y - \omega t)} = -i\omega e^{i(k_x x + k_y y - \omega t)} = -i\omega u.$$

2. Compute $\frac{\partial^2 u}{\partial t^2}$:

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}(-i\omega u) = -i\omega \frac{\partial u}{\partial t} = -i\omega(-i\omega u) = -(-1)\omega^2 u = -\omega^2 u.$$

3. Compute $\frac{\partial u}{\partial x}$ and $\frac{\partial^2 u}{\partial x^2}$:

$$\frac{\partial u}{\partial x} = ik_x u, \quad \frac{\partial^2 u}{\partial x^2} = ik_x \frac{\partial u}{\partial x} = ik_x(ik_x u) = -k_x^2 u.$$

4. Compute $\frac{\partial u}{\partial y}$ and $\frac{\partial^2 u}{\partial y^2}$:

$$\frac{\partial u}{\partial y} = ik_y u, \quad \frac{\partial^2 u}{\partial y^2} = ik_y \frac{\partial u}{\partial y} = ik_y(ik_y u) = -k_y^2 u.$$

5. Compute the Laplacian $\nabla^2 u$:

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -k_x^2 u - k_y^2 u = -\left(k_x^2 + k_y^2\right) u.$$

6. Substitute into the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = -\omega^2 u, \quad c^2 \nabla^2 u = c^2 \left(-\left(k_x^2 + k_y^2\right) u\right) = -c^2 \left(k_x^2 + k_y^2\right) u.$$

Therefore,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad \Longleftrightarrow \quad -\omega^2 u = -c^2 \left(k_x^2 + k_y^2\right) u.$$

7. Simplify and verify the dispersion relation:

$$\omega^2 = c^2 \left(k_x^2 + k_y^2\right).$$

1.2.4 Dispersion Coefficient

Problem Statement:

Assume $m_x = m_y$ such that $k_x = k_y = k$. A discrete version of the solution is:

$$u_{i,j}^n = e^{i(kh(i+j) - \tilde{\omega} n \Delta t)},$$

where $\tilde{\omega}$ is the numerical dispersion coefficient. Insert $u_{i,j}^n$ into the discretized wave equation and show that for the CFL number $C = \frac{1}{\sqrt{2}}$, we get $\tilde{\omega} = \omega$.

Solution:

1. Discretized Wave Equation:

$$\frac{u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1}}{\Delta t^2} = c^2 \left(\frac{u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n}{h^2} + \frac{u_{i,j+1}^n - 2u_{i,j}^n + u_{i,j-1}^n}{h^2} \right).$$

2. Substitute $u_{i,j}^n$ into the Left-Hand Side (LHS): - Compute the numerator:

$$u_{i,j}^{n+1} - 2u_{i,j}^n + u_{i,j}^{n-1} = e^{i(kh(i+j) - \tilde{\omega}(n+1)\Delta t)} - 2e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} + e^{i(kh(i+j) - \tilde{\omega}(n-1)\Delta t)}.$$

- Factor out common terms:

$$e^{i(kh(i+j) - \tilde{\omega} n \Delta t)} \left[e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t} \right].$$

- Simplify using trigonometric identities:

$$e^{-i\tilde{\omega}\Delta t} + e^{i\tilde{\omega}\Delta t} = 2\cos(\tilde{\omega}\Delta t), \quad e^{-i\tilde{\omega}\Delta t} - e^{i\tilde{\omega}\Delta t} = -2i\sin(\tilde{\omega}\Delta t).$$

Therefore,

$$e^{-i\tilde{\omega}\Delta t} - 2 + e^{i\tilde{\omega}\Delta t} = 2\left(\cos(\tilde{\omega}\Delta t) - 1\right) = -4\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right).$$

- The LHS becomes:

$$\text{LHS} = \frac{-4u_{i,j}^n \sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right)}{\Delta t^2}.$$

3. Substitute $u_{i,j}^n$ into the Right-Hand Side (RHS): - Compute the spatial differences similarly:

$$u_{i+1,j}^n - 2u_{i,j}^n + u_{i-1,j}^n = e^{i(kh(i+1))} - 2e^{i(khi)} + e^{i(kh(i-1))} = e^{i(khi)}\left(e^{ikh} - 2 + e^{-ikh}\right).$$

- Simplify:

$$e^{ikh} - 2 + e^{-ikh} = 2\left(\cos(kh) - 1\right) = -4\sin^2\left(\frac{kh}{2}\right).$$

- The RHS becomes:

$$\text{RHS} = c^2 \left(\frac{-4u_{i,j}^n \sin^2\left(\frac{kh}{2}\right)}{h^2} + \frac{-4u_{i,j}^n \sin^2\left(\frac{kh}{2}\right)}{h^2} \right) = \frac{-8c^2 u_{i,j}^n \sin^2\left(\frac{kh}{2}\right)}{h^2}.$$

4. Equate LHS and RHS:

$$\frac{-4u_{i,j}^n \sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right)}{\Delta t^2} = \frac{-8c^2 u_{i,j}^n \sin^2\left(\frac{kh}{2}\right)}{h^2}.$$

- Simplify and cancel $u_{i,j}^n$ and negative signs:

$$\frac{\sin^2\left(\frac{\tilde{\omega}\Delta t}{2}\right)}{\Delta t^2} = 2c^2 \frac{\sin^2\left(\frac{kh}{2}\right)}{h^2}.$$

5. Express in Terms of the CFL Number C: - The CFL number is $C = \frac{c\Delta t}{h}$, so $\Delta t = \frac{Ch}{c}$. - Substitute Δt into the equation:

$$\frac{\sin^2\left(\frac{\tilde{\omega}Ch}{2c}\right)}{\left(\frac{Ch}{c}\right)^2} = 2c^2 \frac{\sin^2\left(\frac{kh}{2}\right)}{h^2}.$$

- Simplify:

$$\frac{c^2}{C^2 h^2} \sin^2\left(\frac{\tilde{\omega}Ch}{2c}\right) = 2c^2 \frac{\sin^2\left(\frac{kh}{2}\right)}{h^2}.$$

- Cancel c^2 and h^2:

$$\frac{1}{C^2} \sin^2\left(\frac{\tilde{\omega}Ch}{2c}\right) = 2\sin^2\left(\frac{kh}{2}\right).$$

6. Set $C = \frac{1}{\sqrt{2}}$ and Simplify: - Substitute $C = \frac{1}{\sqrt{2}}$:

$$2\sin^2\left(\frac{\tilde{\omega}h}{2\sqrt{2}c}\right) = 2\sin^2\left(\frac{kh}{2}\right).$$

- Cancel the factor of 2:

$$\sin^2\left(\frac{\tilde{\omega}h}{2\sqrt{2}c}\right) = \sin^2\left(\frac{kh}{2}\right).$$

- Therefore:

$$\frac{\tilde{\omega}h}{2\sqrt{2}c} = \frac{kh}{2} + n\pi, \quad n \in Z.$$

- Choose $n = 0$ for the principal value:

$$\frac{\tilde{\omega}h}{2\sqrt{2}c} = \frac{kh}{2}.$$

- Simplify:

$$\tilde{\omega} = k\sqrt{2}c.$$

7. Compare with the Exact Dispersion Relation: - The exact dispersion relation is $\omega = c\sqrt{k_x^2 + k_y^2}$. - Since $k_x = k_y = k$:

$$\omega = c\sqrt{k^2 + k^2} = k\sqrt{2}c.$$

- Therefore, $\tilde{\omega} = \omega$.