10.4 Skip Lists

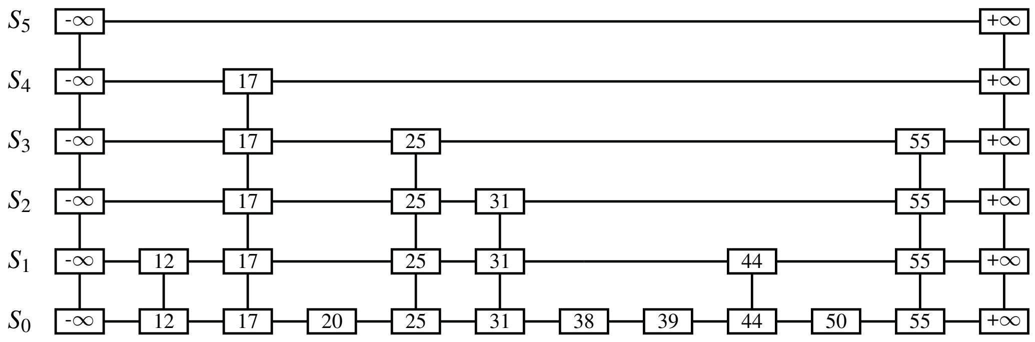
In Section 10.3.1, we saw that a sorted table will allow $O(\log n)$-time searches via the binary search algorithm. Unfortunately, update operations on a sorted table have $O(n)$ worst-case running time because of the need to shift elements. In Chapter 7 we demonstrated that linked lists support very efficient update operations, as long as the position within the list is identified. Unfortunately, we cannot perform fast searches on a standard linked list; for example, the binary search algorithm requires an efficient means for direct accessing an element of a sequence by index.

An interesting data structure for efficiently realizing the sorted map ADT is the ***skip list***. Skip lists provide a clever compromise to efficiently support search and update operations; they are implemented as the java.util.ConcurrentSkipListMap class. A ***skip list*** S for a map M consists of a series of lists $\{S_0, S_1, \ldots, S_h\}$. Each list S_i stores a subset of the entries of M sorted by increasing keys, plus entries with two sentinel keys denoted $-\infty$ and $+\infty$, where $-\infty$ is smaller than every possible key that can be inserted in M and $+\infty$ is larger than every possible key that can be inserted in M. In addition, the lists in S satisfy the following:

- List S_0 contains every entry of the map M (plus sentinels $-\infty$ and $+\infty$).
- For $i = 1, \ldots, h-1$, list S_i contains (in addition to $-\infty$ and $+\infty$) a randomly generated subset of the entries in list S_{i-1}.
- List S_h contains only $-\infty$ and $+\infty$.

An example of a skip list is shown in Figure 10.10. It is customary to visualize a skip list S with list S_0 at the bottom and lists $S_1, \ldots, S_h$ above it. Also, we refer to h as the ***height*** of skip list S.

Intuitively, the lists are set up so that S_{i+1} contains roughly alternate entries of S_i. However, the halving of the number of entries from one list to the next is not enforced as an explicit property of skip lists; instead, randomization is used. As



S_5: -∞, +∞
S_4: -∞, 17, +∞
S_3: -∞, 17, 25, 55, +∞
S_2: -∞, 17, 25, 31, 55, +∞
S_1: -∞, 12, 17, 25, 31, 44, 55, +∞
S_0: -∞, 12, 17, 20, 25, 31, 38, 39, 44, 50, 55, +∞

Figure 10.10: Example of a skip list storing 10 entries. For simplicity, we show only the entries' keys, not their associated values.

we shall see in the details of the insertion method, the entries in S_{i+1} are chosen at random from the entries in S_i by picking each entry from S_i to also be in S_{i+1} with probability $1/2$. That is, in essence, we "flip a coin" for each entry in S_i and place that entry in S_{i+1} if the coin comes up "heads." Thus, we expect S_1 to have about $n/2$ entries, S_2 to have about $n/4$ entries, and, in general, S_i to have about $n/2^i$ entries. As a consequence, we expect the height h of S to be about $\log n$.

Functions that generate random-like numbers are built into most modern computers, because they are used extensively in computer games, cryptography, and computer simulations. Some functions, called ***pseudorandom number generators***, generate such numbers, starting with an initial ***seed***. (See discussion of the java.util.Random class in Section 3.1.3.) Other methods use hardware devices to extract "true" random numbers from nature. In any case, we will assume that our computer has access to numbers that are sufficiently random for our analysis.

An advantage of using ***randomization*** in data structure and algorithm design is that the structures and methods that result can be simple and efficient. The skip list has the same logarithmic time bounds for searching as is achieved by the binary search algorithm, yet it extends that performance to update methods when inserting or deleting entries. Nevertheless, the bounds are ***expected*** for the skip list, while binary search of a sorted table has a ***worst-case*** bound.

A skip list makes random choices in arranging its structure in such a way that search and update times are $O(\log n)$ ***on average***, where n is the number of entries in the map. Interestingly, the notion of average time complexity used here does not depend on the probability distribution of the keys in the input. Instead, it depends on the use of a random-number generator in the implementation of the insertions to help decide where to place the new entry. The running time is averaged over all possible outcomes of the random numbers used when inserting entries.

As with the position abstraction used for lists and trees, we view a skip list as a two-dimensional collection of positions arranged horizontally into ***levels*** and vertically into ***towers***. Each level is a list S_i and each tower contains positions storing the same entry across consecutive lists. The positions in a skip list can be traversed using the following operations:

next(p):	Returns the position following p on the same level.
prev(p):	Returns the position preceding p on the same level.
above(p):	Returns the position above p in the same tower.
below(p):	Returns the position below p in the same tower.

We conventionally assume that these operations return null if the position requested does not exist. Without going into the details, we note that we can easily implement a skip list by means of a linked structure such that the individual traversal methods each take $O(1)$ time, given a skip-list position p. Such a linked structure is essentially a collection of h doubly linked lists aligned at towers, which are also doubly linked lists.

10.4.1 Search and Update Operations in a Skip List

The skip-list structure affords simple map search and update algorithms. In fact, all of the skip-list search and update algorithms are based on an elegant SkipSearch method that takes a key k and finds the position p of the entry in list S_0 that has the largest key less than or equal to k (which is possibly $-\infty$).

Searching in a Skip List

Suppose we are given a search key k. We begin the SkipSearch method by setting a position variable p to the topmost, left position in the skip list S, called the ***start position*** of S. That is, the start position is the position of S_h storing the special entry with key $-\infty$. We then perform the following steps (see Figure 10.11), where $\mathsf{key}(p)$ denotes the key of the entry at position p:

1. If $S.\mathsf{below}(p)$ is null, then the search terminates—we are ***at the bottom*** and have located the entry in S with the largest key less than or equal to the search key k. Otherwise, we ***drop down*** to the next lower level in the present tower by setting $p = S.\mathsf{below}(p)$.
2. Starting at position p, we move p forward until it is at the rightmost position on the present level such that $\mathsf{key}(p) \leq k$. We call this the ***scan forward*** step. Note that such a position always exists, since each level contains the keys $+\infty$ and $-\infty$. It may be that p remains where it started after we perform such a forward scan for this level.
3. Return to step 1.

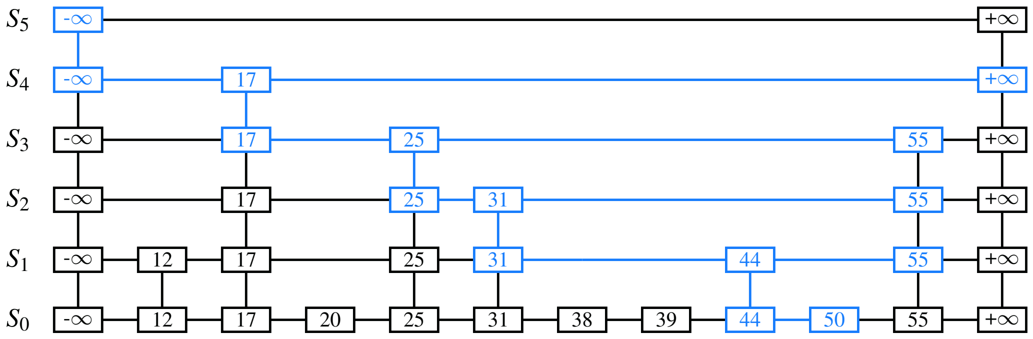


Figure 10.11: Example of a search in a skip list. The positions examined when searching for key 50 are highlighted.

We give a pseudocode description of the skip-list search algorithm, SkipSearch, in Code Fragment 10.14. Given this method, we perform the map operation $\mathsf{get}(k)$ by computing $p = \mathsf{SkipSearch}(k)$ and testing whether or not $\mathsf{key}(p) = k$. If these two keys are equal, we return the associated value; otherwise, we return null.

```
Algorithm SkipSearch($k$):
  Input: A search key $k$
  Output: Position $p$ in the bottom list $S_0$ with the largest key having key$(p) \leq k$
   $p = s$                                        {begin at start position}
   while below$(p) \neq$ null do
     $p =$ below$(p)$                             {drop down}
     while $k \geq$ key(next$(p)$) do
       $p =$ next$(p)$                            {scan forward}
   return $p$
```

Code Fragment 10.14: Algorithm to search a skip list S for key k. Variable s holds the start position of S.

As it turns out, the expected running time of algorithm SkipSearch on a skip list with n entries is $O(\log n)$. We postpone the justification of this fact, however, until after we discuss the implementation of the update methods for skip lists. Navigation starting at the position identified by SkipSearch(k) can be easily used to provide the additional forms of searches in the sorted map ADT (e.g., ceilingEntry, subMap).

Insertion in a Skip List

The execution of the map operation put(k, v) begins with a call to SkipSearch(k). This gives us the position p of the bottom-level entry with the largest key less than or equal to k (note that p may hold the special entry with key $-\infty$). If key$(p) = k$, the associated value is overwritten with v. Otherwise, we need to create a new tower for entry (k, v). We insert (k, v) immediately after position p within S_0. After inserting the new entry at the bottom level, we use randomization to decide the height of the tower for the new entry. We "flip" a coin, and if the flip comes up tails, then we stop here. Else (the flip comes up heads), we backtrack to the previous (next higher) level and insert (k, v) in this level at the appropriate position. We again flip a coin; if it comes up heads, we go to the next higher level and repeat. Thus, we continue to insert the new entry (k, v) in lists until we finally get a flip that comes up tails. We link together all the references to the new entry (k, v) created in this process to create its tower. A fair coin flip can be simulated with Java's built-in pseudorandom number generator java.util.Random by calling nextBoolean( ), which returns true or false, each with probability $1/2$.

We give the insertion algorithm for a skip list S in Code Fragment 10.15 and we illustrate it in Figure 10.12. The algorithm uses an insertAfterAbove$(p, q, (k, v))$ method that inserts a position storing the entry (k, v) after position p (on the same level as p) and above position q, returning the new position r (and setting internal references so that next, prev, above, and below methods will work correctly for p, q, and r). The expected running time of the insertion algorithm on a skip list with n entries is $O(\log n)$, as we show in Section 10.4.2.

```
Algorithm SkipInsert($k$, $v$):
  Input: Key $k$ and value $v$
  Output: Topmost position of the entry inserted in the skip list
  $p$ = SkipSearch($k$)      {position in bottom list with largest key less than $k$}
  $q$ = null                         {current node of new entry's tower}
  $i = -1$                           {current height of new entry's tower}
  repeat
    $i = i + 1$                      {increase height of new entry's tower}
    if $i \geq h$ then
      $h = h + 1$                    {add a new level to the skip list}
      $t$ = next($s$)
      $s$ = insertAfterAbove(null, $s$, $(-\infty, \text{null})$)   {grow leftmost tower}
      insertAfterAbove($s$, $t$, $(+\infty, \text{null})$)          {grow rightmost tower}
    $q$ = insertAfterAbove($p$, $q$, $(k, v)$)      {add node to new entry's tower}
    while above($p$) == null do
      $p$ = prev($p$)                {scan backward}
    $p$ = above($p$)                 {jump up to higher level}
  until coinFlip( ) == tails
  $n = n + 1$
  return $q$                         {top node of new entry's tower}
```

Code Fragment 10.15: Insertion in a skip list of entry (k, v) We assume the skip list does not have an entry with key k. Method coinFlip( ) returns "heads" or "tails", each with probability 1/2. Instance variables n, h, and s respectively hold the number of entries, the height, and the start node of the skip list.

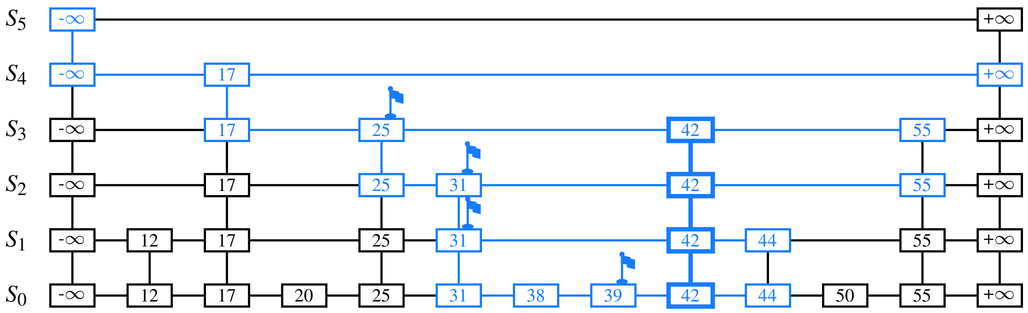


Figure 10.12: Insertion of an entry with key 42 into the skip list of Figure 10.10 using method SkipInsert (Code Fragment 10.15). We assume that the random "coin flips" for the new entry came up heads three times in a row, followed by tails. The positions visited are highlighted in blue. The positions of the tower of the new entry (variable q) are drawn with thick lines, and the positions preceding them (variable p) are flagged.

Removal in a Skip List

Like the search and insertion algorithms, the removal algorithm for a skip list is quite simple. In fact, it is even easier than the insertion algorithm. To perform the map operation remove(k), we will begin by executing method SkipSearch(k). If the returned position p stores an entry with key different from k, we return null. Otherwise, we remove p and all the positions above p, which are easily accessed by using above operations to climb up the tower of this entry in S starting at position p. While removing levels of the tower, we reestablish links between the horizontal neighbors of each removed position. The removal algorithm is illustrated in Figure 10.13 and a detailed description of it is left as an exercise (R-10.24). As we show in the next subsection, the remove operation in a skip list with n entries has $O(\log n)$ expected running time.

Before we give this analysis, however, there are some minor improvements to the skip-list data structure we would like to discuss. First, we do not actually need to store references to values at the levels of the skip list above the bottom level, because all that is needed at these levels are references to keys. In fact, we can more efficiently represent a tower as a single object, storing the key-value pair, and maintaining j previous references and j next references if the tower reaches level S_j. Second, for the horizontal axes, it is possible to keep the list singly linked, storing only the next references. We can perform insertions and removals in strictly a top-down, scan-forward fashion. We explore the details of this optimization in Exercise C-10.55. Neither of these optimizations improve the asymptotic performance of skip lists by more than a constant factor, but these improvements can, nevertheless, be meaningful in practice. In fact, experimental evidence suggests that optimized skip lists are faster in practice than AVL trees and other balanced search trees, which are discussed in Chapter 11.

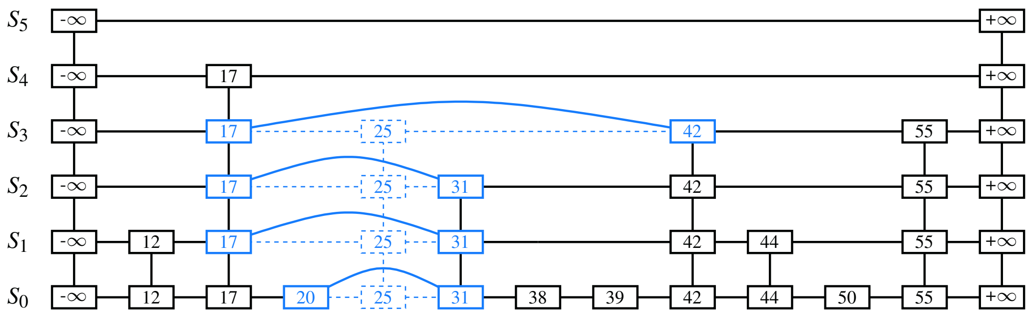


Figure 10.13: Removal of the entry with key 25 from the skip list of Figure 10.12. The positions visited after the search for the position of S_0 holding the entry are highlighted in blue. The positions removed are drawn with dashed lines.

Maintaining the Topmost Level

A skip list S must maintain a reference to the start position (the topmost, leftmost position in S) as an instance variable, and must have a policy for any insertion that wishes to continue growing the tower for a new entry past the top level of S. There are two possible courses of action we can take, both of which have their merits.

One possibility is to restrict the top level, h, to be kept at some fixed value that is a function of n, the number of entries currently in the map (from the analysis we will see that $h = \max\{10, 2\lceil \log n \rceil\}$ is a reasonable choice, and picking $h = 3\lceil \log n \rceil$ is even safer). Implementing this choice means that we must modify the insertion algorithm to stop inserting a new position once we reach the topmost level (unless $\lceil \log n \rceil < \lceil \log(n+1) \rceil$, in which case we can now go at least one more level, since the bound on the height is increasing).

The other possibility is to let an insertion continue growing a tower as long as heads keep getting returned from the random number generator. This is the approach taken by algorithm SkipInsert of Code Fragment 10.15. As we show in the analysis of skip lists, the probability that an insertion will go to a level that is more than $O(\log n)$ is very low, so this design choice should also work.

Either choice will still result in the expected $O(\log n)$ time to perform search, insertion, and removal, as we will show in the next section.

10.4.2 Probabilistic Analysis of Skip Lists ⋆

As we have shown above, skip lists provide a simple implementation of a sorted map. In terms of worst-case performance, however, skip lists are not a superior data structure. In fact, if we do not officially prevent an insertion from continuing significantly past the current highest level, then the insertion algorithm can go into what is almost an infinite loop (it is not actually an infinite loop, however, since the probability of having a fair coin repeatedly come up heads forever is 0). Moreover, we cannot infinitely add positions to a list without eventually running out of memory. In any case, if we terminate position insertion at the highest level h, then the ***worst-case*** running time for performing the get, put, and remove map operations in a skip list S with n entries and height h is $O(n+h)$. This worst-case performance occurs when the tower of every entry reaches level $h-1$, where h is the height of S. However, this event has very low probability. Judging from this worst case, we might conclude that the skip-list structure is strictly inferior to the other map implementations discussed earlier in this chapter. But this would not be a fair analysis, for this worst-case behavior is a gross overestimate.

⋆We use a star (⋆) to indicate sections containing material more advanced than the material in the rest of the chapter; this material can be considered optional in a first reading.

Bounding the Height of a Skip List

Because the insertion step involves randomization, a more accurate analysis of skip lists involves a bit of probability. At first, this might seem like a major undertaking, for a complete and thorough probabilistic analysis could require deep mathematics (and, indeed, there are several such deep analyses that have appeared in data structures research literature). Fortunately, such an analysis is not necessary to understand the expected asymptotic behavior of skip lists. The informal and intuitive probabilistic analysis we give below uses only basic concepts of probability theory.

Let us begin by determining the expected value of the height h of a skip list S with n entries (assuming that we do not terminate insertions early). The probability that a given entry has a tower of height $i \geq 1$ is equal to the probability of getting i consecutive heads when flipping a coin, that is, this probability is $1/2^i$. Hence, the probability P_i that level i has at least one position is at most

$$P_i \leq \frac{n}{2^i},$$

because the probability that any one of n different events occurs is at most the sum of the probabilities that each occurs.

The probability that the height h of S is larger than i is equal to the probability that level i has at least one position, that is, it is no more than P_i. This means that h is larger than, say, $3\log n$ with probability at most

$$\begin{aligned} P_{3\log n} &\leq \frac{n}{2^{3\log n}} \\ &= \frac{n}{n^3} = \frac{1}{n^2}. \end{aligned}$$

For example, if $n = 1000$, this probability is a one-in-a-million long shot. More generally, given a constant $c > 1$, h is larger than $c\log n$ with probability at most $1/n^{c-1}$. That is, the probability that h is smaller than $c\log n$ is at least $1 - 1/n^{c-1}$. Thus, with high probability, the height h of S is $O(\log n)$.

Analyzing Search Time in a Skip List

Next, consider the running time of a search in skip list S, and recall that such a search involves two nested **while** loops. The inner loop performs a scan forward on a level of S as long as the next key is no greater than the search key k, and the outer loop drops down to the next level and repeats the scan forward iteration. Since the height h of S is $O(\log n)$ with high probability, the number of drop-down steps is $O(\log n)$ with high probability.

So we have yet to bound the number of scan-forward steps we make. Let n_i be the number of keys examined while scanning forward at level i. Observe that, after the key at the starting position, each additional key examined in a scan-forward at level i cannot also belong to level $i+1$. If any of these keys were on the previous level, we would have encountered them in the previous scan-forward step. Thus, the probability that any key is counted in n_i is $1/2$. Therefore, the expected value of n_i is exactly equal to the expected number of times we must flip a fair coin before it comes up heads. This expected value is 2. Hence, the expected amount of time spent scanning forward at any level i is $O(1)$. Since S has $O(\log n)$ levels with high probability, a search in S takes expected time $O(\log n)$. By a similar analysis, we can show that the expected running time of an insertion or a removal is $O(\log n)$.

Space Usage in a Skip List

Finally, let us turn to the space requirement of a skip list S with n entries. As we observed above, the expected number of positions at level i is $n/2^i$, which means that the expected total number of positions in S is

$$\sum_{i=0}^{h} \frac{n}{2^i} = n \sum_{i=0}^{h} \frac{1}{2^i}.$$

Using Proposition 4.5 on geometric summations, we have

$$\sum_{i=0}^{h} \frac{1}{2^i} = \frac{\left(\frac{1}{2}\right)^{h+1} - 1}{\frac{1}{2} - 1} = 2 \cdot \left(1 - \frac{1}{2^{h+1}}\right) < 2 \quad \text{for all } h \geq 0.$$

Hence, the expected space requirement of S is $O(n)$.

Table 10.4 summarizes the performance of a sorted map realized by a skip list.

Method	Running Time
size, isEmpty	$O(1)$
get	$O(\log n)$ expected
put	$O(\log n)$ expected
remove	$O(\log n)$ expected
firstEntry, lastEntry	$O(1)$
ceilingEntry, floorEntry lowerEntry, higherEntry	$O(\log n)$ expected
subMap	$O(s + \log n)$ expected, with s entries reported
entrySet, keySet, values	$O(n)$

Table 10.4: Performance of a sorted map implemented with a skip list. We use n to denote the number of entries in the dictionary at the time the operation is performed. The expected space requirement is $O(n)$.