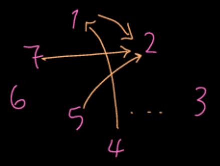
Linear Algebra 1

Monday, November 27, 2023   17:55

- linear Systems
  - Inconsistent : no solution → $\left[\begin{array}{ccccc|c} 0&0&0&0&0&b \end{array}\right] \to 0x+0y+\dots=b$, $b\neq 0$
  - consistent: +1 solution
    - Homogeneous $A\underline{x}=\underline{0}$
      - With unique solution → No free variable → $\left[\begin{array}{cccc|c} 1&&&&0\\ &1&&&0\\ &&1&&0\\ &&&1&0 \end{array}\right] \to$ $x_1=0$, $x_2=0$, $x_3=0$, $x_4=0$
      - without unique solution
        - → Trivial solution : Independent
        - → Non trivial solution : Dependent → +1 free variable
    - Non homogeneous $A\underline{x}=\underline{b}$
      - Trivial solution : Independent → b in span$\{\bar{a}_1, \bar{a}_2, a_3 \dots\}$ if $b_n=0$ and there would be no pivot in right most column $[\ \ \underset{\times \text{ pivot}}{|}\ ]$
      - Non trivial solution : Dependent

Systems of linear equations



$x_1 = x_4$

$x_2 = \frac{1}{3}x_1 + \frac{1}{2}x_5 + \frac{1}{3}x_7$

.....

$$A = \begin{bmatrix} 0&0&0&1&0&0&0 \\ \frac{1}{3}&0&0&0&\frac{1}{2}&0&\frac{1}{3} \\ 0&\frac{1}{2}&&&&& \\ \frac{1}{3}&0&&\ddots&&& \\ 0&0&&&&& \\ 0&\frac{1}{2}&&&&& \\ \frac{1}{3}&0&&&&& \end{bmatrix}$$

$A = (x_1, x_2, x_3, x_4, x_5, x_6, x_7)^T$ → eignvector $\lambda = 1$ → eignvalue

S = solution set eg. $(x_1, x_2, x_3 \dots)^{\text{values}}$ → equivalent → 2 linear System with same Solution Sets.

- → inconsistent : emty solution set
- → consistent : non inconsistent

$\begin{bmatrix} a_{11} & a_{12} & \dots \\ a_{21} & a_{22} & \dots \\ a_{31} & a_{32} & \dots \end{bmatrix}$ coefficient matrix

$\left[\begin{array}{ccc|c} a_{11} & a_{12} & \dots & b_1 \\ a_{21} & a_{22} & \dots & b_2 \\ a_{31} & a_{32} & \dots & b_3 \end{array}\right]$ augmented matrix → Basic Operations {Replace, Interchange, Scale (multiply)}

↳ Should be in echelon form {

REF $\begin{bmatrix} 1&0&3&4&0&3&3&4 \\ 0&1&0&2&1&2&2&6 \\ 0&0&0&1&0&0&7&8 \\ 0&0&0&0&1&0&9&3 \\ 0&0&0&0&0&1&2&2 \end{bmatrix}$

Reduced REF (RREF) ↘

$$\begin{array}{cccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ \boxed{1} & 0 & 0 & 0 & 0 & 0 & 3 & 4 \\ 0 & \boxed{1} & 0 & 0 & 0 & 0 & 2 & 6 \\ 0 & 0 & 0 & \boxed{1} & 0 & 0 & 7 & 8 \\ 0 & 0 & 0 & 0 & \boxed{1} & 0 & 9 & 3 \\ 0 & 0 & 0 & 0 & 0 & \boxed{1} & 2 & 2 \end{array}$$

☐ Pivots Position

↩ Pivot Columns

basic variables $(x_1, x_2, x_4, x_5, x_6)$

free variables (x_3, x_7, x_8)

Statements {
- Existance ★ Consistent linear system ⇄ $[\ \ \ {}_x|\ |\ ]$ right most column shouldn't be pivot
- Uniqueness ★ Consistent linear system with unique solution ⇄ No free variables

Vector Equation

$$\text{Span}\{\underline{v_1}, \underline{v_2}, \dots\} = c_1v_1 + c_2v_2 + \dots$$

span of a vectors set is all the combinationals of a set of vectors

and it exist when: $c_1v_1 + c_2v_2 \in \text{span}\{v_1, v_2\} \mid c_1, c_2 \in \mathbb{R}$

Homogeneous systems: if $A\underline{x} = \underline{0}$

Trivial solution : if $A=1$ in Homogeneous linear System

nontrivial solution: if and only if the system has at least one free variable

Example:
$$\begin{aligned} x_1 + \quad -x_3 - \tfrac{1}{2}x_4 \quad &= 0 \\ 2x_1 + 6x_2 + 10x_3 + 13x_4 + 2x_5 &= 0 \\ -2x_1 + 2x_2 + 6x_3 + 7x_4 + 2x_5 &= 0 \end{aligned} \to \begin{bmatrix} 1&0&-1&\frac{-1}{2}&0 \\ 2&6&10&13&2 \\ -2&2&6&7&2 \end{bmatrix} \to \begin{bmatrix} 1&0&-1&0&\frac{1}{2} \\ 0&1&2&0&-2 \\ 0&0&0&1&1 \end{bmatrix}$$

$$\begin{aligned} x_1 - x_3 + \tfrac{1}{2}x_5 &= 0 \\ x_2 + 2x_3 - 2x_5 &= 0 \\ x_4 + x_5 &= 0 \end{aligned} \Rightarrow \underline{x} = \begin{bmatrix} x_1\\x_2\\x_3\\x_4\\x_5 \end{bmatrix} = \begin{bmatrix} x_3 - \frac{1}{2}x_5 \\ -2x_3 + 2x_5 \\ x_3 \\ -x_5 \\ x_5 \end{bmatrix} = x_3\begin{bmatrix} 1\\-2\\1\\0\\0 \end{bmatrix} + x_5\begin{bmatrix} \frac{-1}{2}\\2\\0\\-1\\1 \end{bmatrix}$$

↝ ★ We write $\underline{X}$ in form of using free variables

for $a_1 = \begin{bmatrix} \frac{1}{2}\\2\\-3 \end{bmatrix}$ $a_2 = \begin{bmatrix} -2\\-5\\6 \end{bmatrix}$ $a_3 = \begin{bmatrix} 0\\2\\-4 \end{bmatrix}$ $b = \begin{bmatrix} b_1\\b_2\\b_3 \end{bmatrix}$ → General vector equation : $\underline{b} = x_1\begin{bmatrix} \frac{1}{2}\\2\\-3 \end{bmatrix} + x_2\begin{bmatrix} -2\\-5\\6 \end{bmatrix} + x_3\begin{bmatrix} 0\\2\\-4 \end{bmatrix}$

b in span$\{a_1, a_2, a_3\}$ for: $\left[\begin{array}{ccc|c} \frac{1}{2}&-2&0&b_1 \\ 2&-5&2&b_2 \\ -3&6&-4&b_3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc} 1&-4&0&2b_1 \\ 0&3&2&b_2-4b_1 \\ 0&0&0&\underbrace{b_3+6b_1+2(b_2-4b_1)} \end{array}\right]$ ↪ shouldn't be pivot

if and only if $-2b_1 + 2b_2 + b_3 = 0$ → if $\neq 0$ → $\underline{b}$ can be in linear combination of $\underline{a_1}, \underline{a_2}, \underline{a_3}$

Nonhomogeneous Systems

★ $\{v_1, v_2 \dots v_k\}$ is linearly independant if $x_1v_1 + \dots + x_kv_k = 0$ has trivial solution → $\underline{x} = 0$
and linearly dependant if $c_1, c_2 \dots$ not all be zero

how to find linearly independency: 1) $x_1v_1 + \dots = \underline{0}$ ⇒ $\underbrace{[v_1\ v_2\ v_3]}_{A}\underline{x} = 0$

A → REF → Consistent (right most nonpivot) → uniqe solution (no free var)

★ $A\underline{x} = 0$ where $A = [\underline{v_1}\ \underline{v_2}\ \underline{v_3} \dots]$ {
- A columns are independent ⇔ $A\underline{x} = \underline{0}$ trivial solution
- A columns are dependent ⇒ $A\underline{x} = \underline{0}$ nontrivial solution

★ A way to prove having trivial solution without computation:

$\begin{bmatrix} 1&0&3&2 \\ 0&1&2&4 \\ 2&0&3&1 \end{bmatrix}$ → 3 row → at most 3 pivots ≥ ; 4 columns → There must be free variable ⇒ trivial solution

☆ A way to prove having trivial solution without computation:

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & 4 \\ 2 & 0 & 3 & 1 \end{bmatrix} \to \begin{matrix} 3\text{ row} \to \text{at most 3 pivots} \geq \\ 4\text{ columns} \to \text{There must be free variable} \end{matrix} \Rightarrow \text{trivial solution}$$

☆ Notation $A\underline{x} = b \to \underline{x} = \begin{bmatrix} \partial \\ \gamma \\ \alpha \end{bmatrix} \to P + x_i(\text{free var}) \begin{bmatrix} \beta \\ \lambda \\ \pi \end{bmatrix} \to u$ v, s …

Transformation

$T(\underline{x}) = A\underline{x}$ $\quad T: \underset{\text{domain}}{R^{\overset{\text{total row}}{n}}} \longrightarrow \underset{\text{codomain}}{R^m}$

☆ Notations {

1) Standard Basis: $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x_1 \underset{e_1}{\begin{bmatrix} 1 \\ \vdots \end{bmatrix}} + x_2 \underset{e_2}{\begin{bmatrix} \vdots \\ 1 \\ \vdots \end{bmatrix}} + x_n \underset{e_n}{\begin{bmatrix} \vdots \\ 1 \end{bmatrix}}$

2) $T(\underline{x}) \to \begin{matrix} T(e_1) = \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ T(e_2) = \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} \end{matrix} : T(\underline{x}) = T(x_1 e_1 + \dots) = x_1 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} + x_2 \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \beta & \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Onto: if columns of A span R^m and linear independent ($A\underline{x} = \underline{0}$ trivial) so Pivot in each Row

One to One: if $T(\underline{x}) = \underline{0}$ trivial solution also Pivot in every column

☆ A in $T: R^n \to R^m$ is $m \times n$

☆ $T: R^n \to R^m$ {

- Onto $n \geq m$
- One to One $n \leq m$
- Both $n = m$

Questions {

Does $x \in R^n$ exist whose image is $b \in R^m$?

$\Rightarrow$ Is $A\underline{x} = \underline{b}$ consistent.

Is $b \in R^m$ image of unique $x \in R^n$?

$\Rightarrow$ Does $A\underline{x} = \underline{b}$ admit unique solution.

Matrix Operations

a_{ii} in A_{mn} matrix is called Diagnoal Entries. like a_{11} or a_{22} $[\searrow]$

I matrix (Identity) → 1 in Diagonals & $\underline{0}$ on others

Transpose of Matrix: Columns of A will be rows of A^T in correct order

☆ $(AB)^T = B^T A^T$

Invertible matrix ~~if~~ $AC = CA = I_n \xrightarrow{\text{if not}}$ Singular Matrix

☆ if $|A| \neq 0 \to A$ invertible and $A^{-1} = \frac{1}{|A|} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

☆ $(AB)^{-1} = B^{-1} A^{-1}$

$(A^T)^{-1} = (A^{-1})^T$

Algorithm for A^{-1}

$A_n \to [A \mid I_n] \longrightarrow [\text{RREF} \mid A^{-1}] \to A^{-1}$

↳ Meanwhile if it got REF and we had pivots in each Row/Column → invertable

Determinant

☆ $A = \begin{bmatrix} \alpha & \boxed{\beta} & \gamma \\ \lambda & \psi & \pi \\ \phi & \varphi & \Omega \end{bmatrix} \to A_{12} = \begin{bmatrix} \lambda & \pi \\ \phi & \Omega \end{bmatrix}$

☆ cofactor $\to C_{ij} = (-1)^{ij} \det(A_{ij})$

$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \dots + a_{1n}C_{1n}$ → expantion of 1st row

☆ Det triangular matrix $= \prod_{i=1}^{n} a_{nn}$

Theorems {

A) $|A| = |A'| \to A' = A_{\text{row}\,i} + c\,A_{\text{row}\,!i}$

B) $|A| = |A'| \to A' = A_{\text{row}\,a} \overset{\text{interchanged}}{\rightleftarrows} A_{\text{row}\,b}$

C) $|A'| = k|A| \to A' = A$ with $k \times A_{\text{row}\,i}$

☆ $|A| = |A^T|$

☆ $|AB| = |A||B|$

Carmer and Transformation

Carmer Rule → if $[\text{Coefficient}]_{n \times n}$ invertible

$$\left.\begin{matrix} a_1 x + b_1 y = c_1 \\ a_2 x + b_2 y = c_2 \end{matrix}\right\} \quad x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \to \det A_1(b)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \to \det(A)} \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix} \to \det A_2(b)}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} \quad S = \begin{bmatrix} x \\ y \end{bmatrix}$$

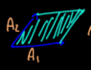
$A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ $\quad b = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

Inversion:

$A^{-1} = \frac{1}{|A|} \underline{\text{adj}(A)} \to$ adjugate matrix (transposed of C which is $C \begin{bmatrix} c_{11} & c_{12} & c_{\dots} \\ \vdots & \ddots & \end{bmatrix}$ where C_{ij} is cofactor of A)

If A

- 2x2 → $||A||$ = Area of parallelogram made by A vectors(columns)
- 3x3 → $||A||$ = Volume of parallelepiped



$A = [A_1 A_2]$

Transformed (Mapped) Area

$$\begin{matrix} e_1 \xrightarrow{T} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \\ e_2 \xrightarrow{T} \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} \end{matrix} \xrightarrow{\text{Transformation}} A = \begin{bmatrix} \alpha & \gamma \\ \beta & \lambda \end{bmatrix}$$

Transformed (Mapped) Area

$$e_1 \xrightarrow{T} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$e_2 \xrightarrow{T} \begin{bmatrix} \gamma \\ \lambda \end{bmatrix} \xrightarrow{\text{Transformation}} A = \begin{bmatrix} \alpha & \gamma \\ \beta & \lambda \end{bmatrix}$$

↳ Area of Transformed (A) Matrix(s)
is $\text{Area}(T(S)) = |A|.|S|$

★ System have unique solution if $|coef| \neq 0$