You may adopt the following steps:

1. Obtain the force-free (FF) electric field inside the neutron star
2. Derive the electric potential in the interior of the NS
3. Derive the electric potential exterior to the NS
4. Obtain the electric field exterior to the NS
5. Determine the electric field along the magnetic field lines
6. Compare electrical forces to the gravitational binding forces on particles

Step 1: Obtain the force-free (FF) electric field inside the neutron star

Assumptions:

- The magnetic axis is aligned with the rotational axis
- The magnetic field in the interior of the NS is either
  - uniform $\vec{B} = B_0\,\hat{e}_z$, or
  - a star-centered dipole $\vec{B} = \frac{B_0 R^3}{2r^3}\,(2\cos\theta\,\hat{e}_r + \sin\theta\,\hat{e}_\theta)$
- The NS interior has infinite conductivity

At any position $\vec{r}(r,\theta,\phi)$, inside the star, a Lorentz force will act on a charged particle of charge q, given by: [Write expression below]

$$\vec{F} = q\left(\frac{\vec{v}}{c}\times\vec{B}\right) \tag{1}$$

where, $\vec{v} = \vec{\Omega}\times\vec{r}$ and c is the speed of light.

The charged particles can move freely and hence they assume a configuration such that the resulting electric field ($\vec{E}$) nulls the original Lorentz force (Equation 1) inside the neutron star.

$$\cancelto{0}{\vec{F}} = q\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right)$$

$$\implies \vec{E} = -\frac{\vec{v}}{c}\times\vec{B} \tag{2}$$

Obtain the expression for the electric field assuming uniform rotation $\vec{\Omega} = \Omega\,\hat{e}_z$[1], and an uniform magnetic field along the z-direction $\vec{B} = B_0\,\hat{e}_z$

$$\begin{aligned}\vec{E} &= \frac{1}{c}\left(\vec{\Omega}\times\vec{r}\times\vec{B}\right)\\ &= \frac{1}{c}\left(\Omega\,\hat{e}_z\times r\,\hat{e}_r\times B_0\,(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)\right)\end{aligned}$$

$$\boxed{\vec{E}_{\text{in}}(r,\theta) = -\frac{\Omega\,r\,B_0}{c}\sin\theta\,(\sin\theta\,\hat{e}_r + \cos\theta\,\hat{e}_\theta)} \tag{3}$$

[1] $\hat{e}_z = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$

Step 2: Derive the electric potential in the interior of the NS

The FF solution has all forces inside the star equal to zero and hence no charge currents. The electric field can be derived from the gradient[2] of an electric potential (Φ)

$$\vec{\nabla} \times \vec{E} = 0 \implies \vec{E} = -\vec{\nabla}\Phi$$

Matching the vector components,

$$\begin{aligned} \frac{\partial}{\partial r}\Phi &= \frac{\Omega\, r\, B_0}{c}\sin^2\theta \\ \frac{1}{r}\frac{\partial}{\partial \theta}\Phi &= \frac{\Omega\, r\, B_0}{c}\sin\theta\cos\theta \end{aligned}$$

Integrating and equating the expressions for the electric potentials,

$$\begin{aligned} \Phi = \frac{\Omega B_0}{c}\sin^2\theta \int r\, dr &= \frac{\Omega B_0}{c} r^2 \int \sin\theta\cos\theta\, d\theta \\ \frac{\Omega B_0 r^2}{2c}\sin^2\theta + C_\theta &= \frac{\Omega B_0 r^2}{2c}\sin^2\theta + C_r \end{aligned} \tag{4}$$

where, $C_\theta \,\&\, C_r$ are constants of integration which are in general dependent on θ and r, respectively. But due to the equality in equation 4, they can only be a constant (say Φ_0), independent of θ and r. The electric potential inside the star has the form,

$$\boxed{\Phi_{\text{in}}(r,\theta) = \frac{\Omega\, r^2\, B_0}{2c}\sin^2\theta + \Phi_0} \tag{5}$$

[2] $\vec{\nabla} = \frac{\partial}{\partial r}\,\hat{e}_r + \frac{1}{r}\frac{\partial}{\partial \theta}\,\hat{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\,\hat{e}_\phi$

Step 3: Derive the electric potential exterior to the NS
Initially, there is no charge distribution outside the star. Hence,

$$\vec{\nabla} \cdot \vec{E}_{\text{out}} = 0$$
$$\implies \nabla^2 \Phi_{\text{out}} = 0$$

The Poisson equation has the general solution of the form:

$$\Phi_{\text{out}}(r, \theta) = \sum_{l=1}^{\infty} \frac{a_l}{r^{l+1}} P_l(\cos\theta) \tag{6}$$

where, a_l are constants and $P_l(x)$ are the Legendre polynomials[3].

The electric potential is continuous across any boundry. Hence, matching the potential (equations 5 & 6) at the stellar surface,

$$\Phi_{\text{in}}(R, \theta) = \Phi_{\text{out}}(R, \theta)$$

$$\frac{\Omega R^2 B_0}{2c} \sin^2\theta + \Phi_0 = \sum_{l=1}^{\infty} \frac{a_l}{R^{l+1}} P_l(\cos\theta)$$

$$\frac{\Omega R^2 B_0}{2c} \sin^2\theta + \Phi_0 = \frac{\cancelto{0}{a_0}}{R} + \frac{\cancelto{0}{a_1}}{R^2}\cos\theta + \frac{a_2}{R^3}\frac{1}{2}(3\cos^2\theta - 1) + \sum_{l=3}^{\infty} \frac{\cancelto{0}{a_l}}{R^{l+1}} P_l(\cos\theta)$$

$$\frac{\Omega R^2 B_0}{2c} \sin^2\theta + \Phi_0 = -\frac{3a_2}{2R^3}\sin^2\theta + \frac{a_2}{R^3}$$

$$\implies a_2 = -\frac{\Omega R^5 B_0}{3c}; \quad \Phi_0 = -\frac{\Omega R^2 B_0}{3c}$$

$$\boxed{\Phi_{\text{out}}(r, \theta) = -\frac{\Omega R^5 B_0}{6cr^3}(3\cos^2\theta - 1) - \frac{\Omega R^2 B_0}{3c}} \tag{7}$$

[3] $P_l(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$

Step 4: Obtain the electric field exterior to the NS.
The electric field outside the star

$$\vec{E}_{\text{out}} = -\vec{\nabla}\Phi_{\text{out}}$$

$$\begin{aligned} -\vec{\nabla}\Phi_{\text{out}} &= \frac{\Omega R^5 B_0}{6c}(3\cos^2\theta - 1)\frac{\partial}{\partial r}\frac{1}{r^3}\,\hat{e}_r + \frac{\Omega R^5 B_0}{6cr^3}\frac{\partial}{r\partial\theta}(3\cos^2\theta - 1)\,\hat{e}_\theta \\ &= -\frac{\Omega B_0 R^5}{2cr^4}(3\cos^2\theta - 1)\,\hat{e}_r - \frac{\Omega B_0 R^5}{2cr^4}\sin 2\theta\,\hat{e}_\theta \end{aligned}$$

$$\boxed{\vec{E}_{\text{out}}(r,\theta) = -\frac{\Omega B_0 R^5}{2cr^4}\left[(3\cos^2\theta - 1)\,\hat{e}_r + \sin 2\theta\,\hat{e}_\theta\right]} \tag{8}$$

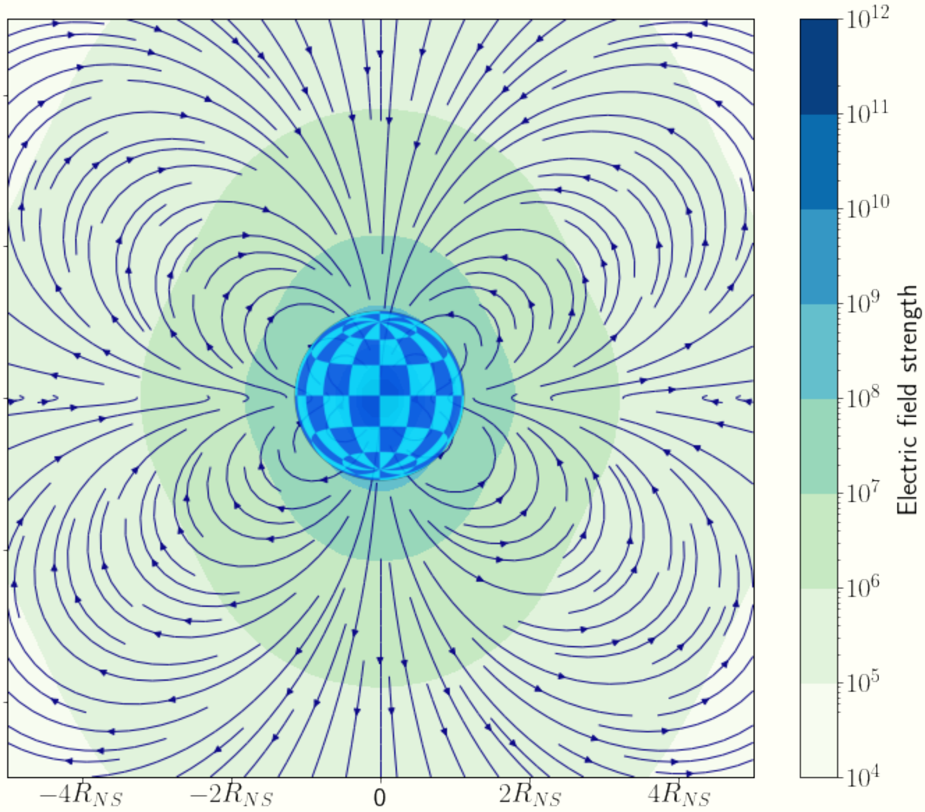
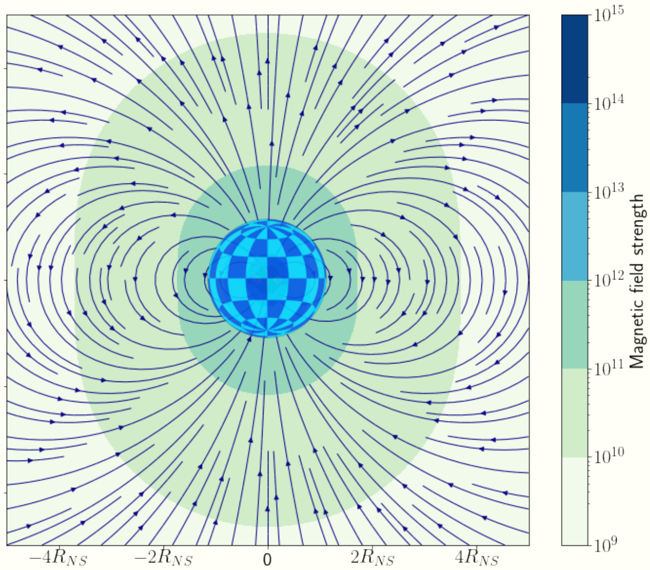


Figure 1: Plot showing quadrupolar electric field lines outside the star on any of the constant ϕ coordinate planes (i.e., $r-\theta$ plane). A color density plot in the background shows the field strength.

Step 5: Determine the electric field along the magnetic field lines.
The magnetic field outside the star is a dipole

$$\vec{B} = \frac{1}{2r^3}[3(\vec{m}\cdot\hat{e}_r)\hat{e}_r - \vec{m}]$$
$$\vec{m} = B_0 R^3\,\hat{e}_z$$

$$\boxed{\vec{B}_{\text{out}}(r,\theta) = \frac{B_0 R^3}{r^3}\left(\cos\theta\,\hat{e}_r + \frac{1}{2}\sin\theta\,\hat{e}_\theta\right)} \tag{9}$$



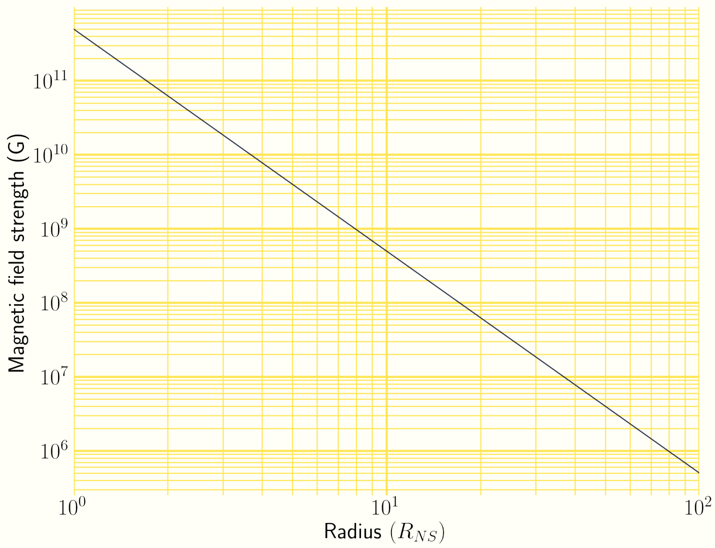


Figure 2: Left: Plot showing dipolar magnetic field lines outside the star in the constant ϕ coordinate planes (i.e., $r-\theta$ plane). A color density plot in the background shows the field strength. Right: Plot showing the measured field strength as a function of radial distance (in units of NS radius) from the centre of the star.

Therefore, the electric field along the B-field

$$\begin{aligned}
\text{Inside:}\qquad \vec{E}.\frac{\vec{B}}{|\vec{B}|} &= -\frac{\Omega\, r\, B_0}{c}\sin\theta(\sin\theta\,\hat{e}_r+\cos\theta\,\hat{e}_\theta)\cdot\frac{B_0\,\hat{e}_z}{B_0}\\
&= -\frac{\Omega\, r\, B_0}{c}\sin\theta(\sin\theta\,\cos\theta-\cos\theta\sin\theta)\\
\vec{E}.\frac{\vec{B}}{|\vec{B}|} &= 0\\
\text{Outside:}\qquad \vec{E}.\vec{B} &= -\frac{\Omega B_0 R^5}{2cr^4}\left[(3\cos^2\theta-1)\,\hat{e}_r+\sin 2\theta\,\hat{e}_\theta\right]\\
&\quad\cdot\frac{B_0 R^3}{r^3}(\cos\theta\,\hat{e}_r+\frac{1}{2}\sin\theta\,\hat{e}_\theta)\\
&= -\frac{\Omega B_0^2 R^8}{2cr^7}\left[(3\cos^2\theta-1)\cos\theta+\sin^2\theta\cos\theta\right]\\
&= -\frac{\Omega B_0^2 R^8}{cr^7}\cos^3\theta
\end{aligned}$$

$$\vec{E}.\frac{\vec{B}}{|\vec{B}|} = -\frac{2\Omega B_0 R^5}{cr^4}\frac{\cos^3\theta}{\sqrt{1+3\cos^2\theta}} \tag{10}$$

Step 6: Compare electrical forces to the gravitational forces on particles.

$$\frac{F_E}{F_g} = \frac{q}{m}\frac{\vec{E}.\hat{B}}{g_{NS}}$$
$$\frac{F_E}{F_g} = -\frac{q}{m}\frac{2\Omega B_0 R^5}{cr^4}\frac{\cos^3\theta}{\sqrt{1+3\cos^2\theta}}\frac{r^2}{G\,M_{NS}}$$

where, q/m is the particle's charge-to-mass ratio, $g_{NS} = G\,M_{NS}/r^2$ is gravitational acceleration, and $\vec{E}.\hat{B}$ is the expression obtained in Equation 10.

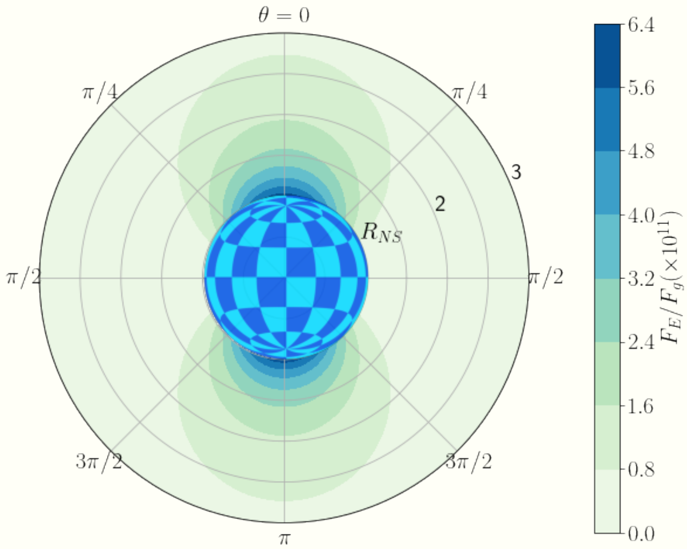


Figure 3: Color density plot: Ratio of electric force to the gravitational force around the NS for upto 3 times its radius.

Values of parameters used in the evaluation:
Fundamental Cosntants:

- Graviational constant $[G] = 6.6743 \times 10^{-11}$ m^3 kg^{-1} s^{-2} $= 6.6743 \times 10^{-8}$ cm^3 g^{-1} s^{-2}
- Speed of light $[c] = 2.998 \times 10^8$ m s^{-1} $= 2.998 \times 10^{10}$ cm s^{-1}
- Charge-to-mass ratio of electron $[e^-/m_e] = -175882001076$ C kg^{-1} $= -5.2728 \times 10^{17}$ statcoulomb g^{-1}
- Charge-to-mass ratio of proton $[p^+/m_e] = 95788332$ C kg^{-1} $= 2.8717 \times 10^{14}$ statcoulomb g^{-1}

Astrophysical values:

- Mass of the neutron star $[M_{ns}] = 2.784 \times 10^{33}$ g
- Radius of the neutron star $[R] = 1.0 \times 10^{6}$ cm
- Angular velocity of the neutron star $[\Omega] = 2\pi^{c}/\mathrm{s}$
- Surface magnetic field at the pole $[B_0] = 1.0 \times 10^{12}$ Gauss

At the pole ($\theta = 0$) on the NS surface, the ratio of the force $F_E/F_g = 6 \times 10^{11}$ for electrons and -3×10^{8} for protons[4].

[4] For anti-aligned rotation and magnetic axes, the signs are reversed.

Derivation 8 A surface charge density results in discontinuity in the normal component electric field at any surface. We obtain the surface charge density (σ) on the stellar surface using (what's the use of this quantity?)

$$4\pi\sigma = (\vec{E}_{\text{out}} - \vec{E}_{\text{in}}) \cdot \hat{e}_r \tag{11}$$

$$\sigma = \frac{1}{4\pi}\left(-\frac{\Omega B_0 R^5}{2cR^4}(3\cos^2\theta - 1) + \frac{\Omega\, R\, B_0}{c}\sin^2\theta\right) \tag{12}$$

$$= \frac{\Omega\, R\, B_0}{4\pi c}\left(-\frac{1}{2}(3\cos^2\theta - 1) + \sin^2\theta\right) \tag{13}$$

$$= -\frac{\Omega\, R\, B_0}{4\pi c}\left(\cos^2\theta - \frac{3}{2}\sin^2\theta\right) \tag{14}$$

$$= -\frac{\Omega\, R\, B_0}{4\pi c}\left(1 - \frac{5}{2}\sin^2\theta\right) \tag{15}$$

$$\boxed{\sigma_{\text{surf}}(\theta) = \frac{\Omega\, R\, B_0}{4\pi c}\left(\frac{5}{2}\sin^2\theta - 1\right)} \tag{16}$$

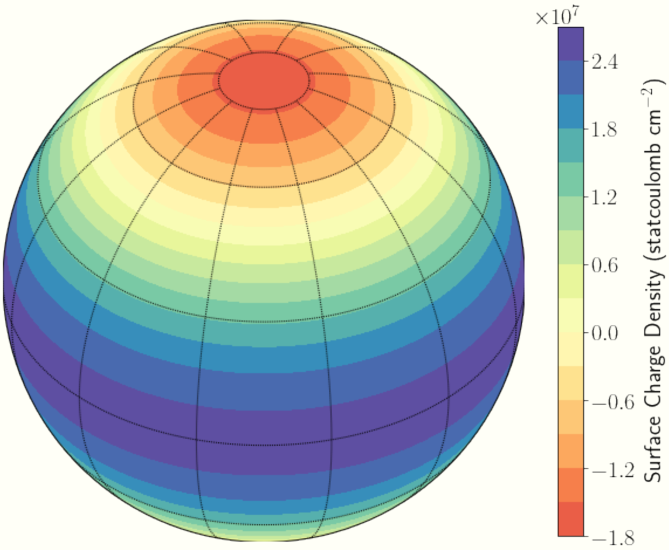


Figure 4: Surface charge density on the neutron star.