Stochastic Modelling of Spatial Data

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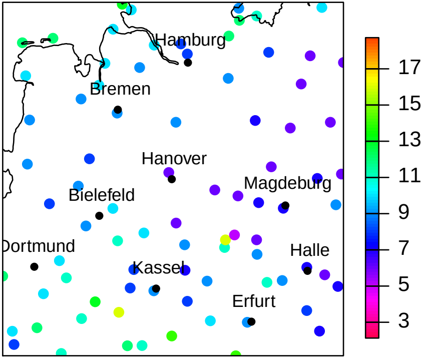
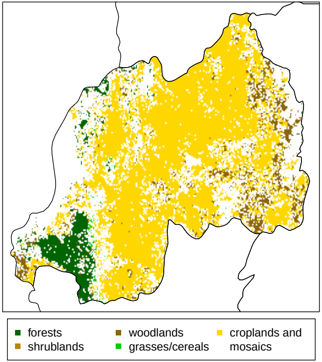
Spatial Data in Environmental Science

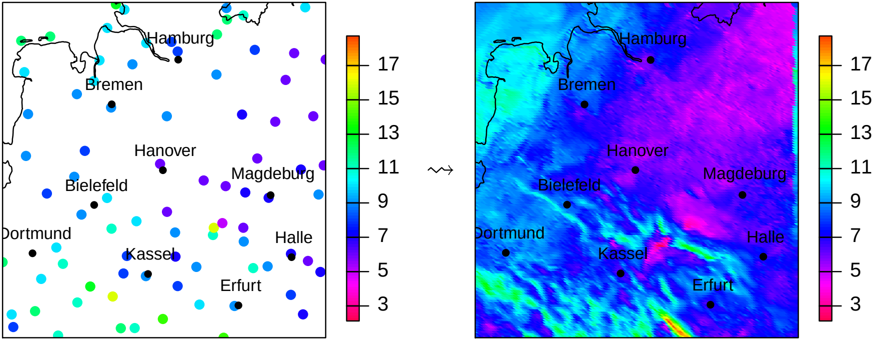




Two Important Types:

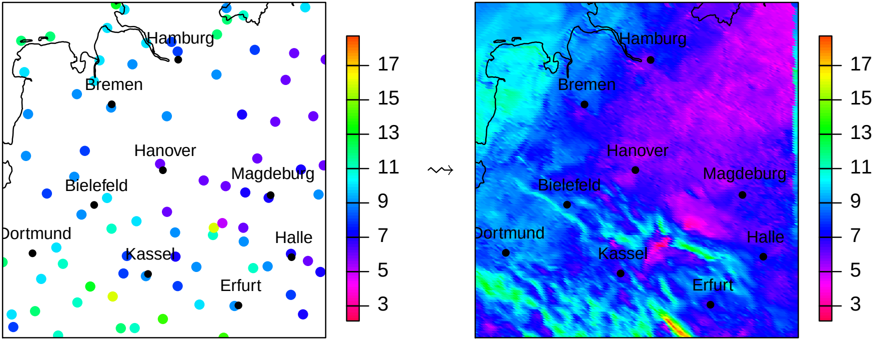
- data at scattered sites (in-situ measurements)
- gridded data (satellite data, output of numerical models)

Spatial Modelling: Aims



1. prediction at ungauged sites, higher resolution (interpolation, "downscaling")
2. simulation of "artificial data", unobserved scenarios

Spatial Modelling: Aims



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$\rightsquigarrow$ probabilistic models

Building Blocks of Probabilistic Models

1. probability distribution at a single site

   Assumption: same type of distribution in whole region of interest; exact distribution might depend on covariates (LON, LAT, ALT, ...)

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2. model for spatial dependence structure

   Tobler's First Law of Geography

   Everything is related to everything else, but near things are more related than distant things.

   Assumption: strength of dependence between data at two sites is function of distance and direction

Outline

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1. "Classical" Gaussian Modelling
   - Gaussian Modelling at a Single Site
   - Gaussian Modelling in Space
   - Simulation & Interpolation

2. Modelling of Extreme Events
   - Modelling an Extreme Event at a Single Site
   - Modelling an Extreme Event in Space
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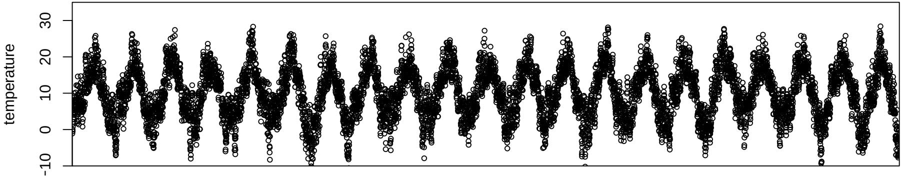
Gaussian Data

Aim: model environmental variable X at site s (Notation: $X(s)$)

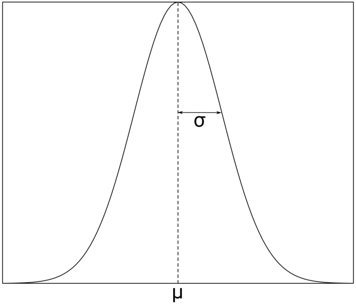
Example: Daily temperature data at Münster Airport 1990–2010



For continuous variables (e.g. temperature, pressure, air pollution): $X(s)$ can often be assumed to follow Gaussian distribution

$$X(s) \sim \mathcal{N}(\mu, \sigma^2)$$

Gaussian Distribution



$$X(s) \sim \mathcal{N}(\mu, \sigma^2)$$

μ: mean (location parameter)

σ: standard deviation (scale parameter)

Gaussian PDF

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\left(\frac{x-\mu}{\sigma} \right)^2 \right\}, \quad x \in \mathbb{R}$$

Gaussian Distribution (cont'd)

$$X(s) \sim \mathcal{N}(\mu \quad , \sigma^2 \quad)$$

Gaussian Distribution (cont'd)

mean and variance may differ from site to site due to conditions . . .

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Gaussian Distribution (cont'd)

mean and variance may differ from site to site due to conditions ...

$$X(s) \sim \mathcal{N}(\mu(s), \sigma^2(s))$$

First Approach: estimate μ and σ^2 from data separately for each site, e.g. by empirical counterparts

$$\widehat{\mu}(s) = \frac{1}{n} \sum_{i=1}^{n} X_i(s)$$

$$\widehat{\sigma}^2(s) = \frac{1}{n-1} \sum_{i=1}^{n} (X_i(s) - \widehat{\mu}(s))^2$$

Gaussian Distribution (cont'd)

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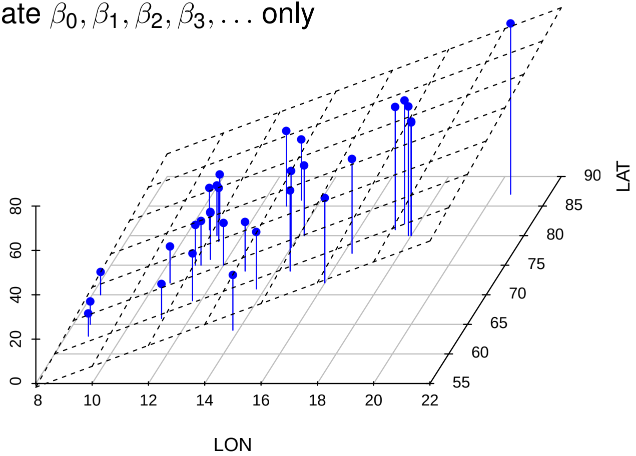
Problem: how to model variable at ungauged sites?

Gaussian Distribution (cont'd)

Second Approach: assume further spatial structure of μ and σ^2, e.g.

$$\mu(s) = \beta_0 + \beta_1 \cdot \mathrm{LON}(s) + \beta_2 \cdot \mathrm{LAT}(s) + \beta_3 \cdot \mathrm{ALT}(s) + \ldots$$

and estimate $\beta_0, \beta_1, \beta_2, \beta_3, \ldots$ only



$\rightsquigarrow$ model for any site within region

How to estimate the β's?

1. Least-Squares-Fit of the Raw Estimates

Minimize

$$\sum_s (\widehat{\mu}(s) - \beta_0 - \beta_1 \cdot \mathrm{LON}(s) - \beta_2 \cdot \mathrm{LAT}(s) - \beta_3 \cdot \mathrm{ALT}(s) - \ldots)^2$$

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2. Fit of Linear Regression Model for the Observations

   If the variance $\sigma^2(s) \equiv \sigma^2$ is nearly constant and the data are assumed to be independent (also across sites), we can write

   $$X_i(s) = \beta_0 + \beta_1 \cdot \mathrm{LON}(s) + \beta_2 \cdot \mathrm{LAT}(s) + \beta_3 \cdot \mathrm{ALT}(s) + \ldots + \varepsilon_i(s)$$

   where $\varepsilon_i(s) \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$.

How to estimate the β's?

1. Least-Squares-Fit of the Raw Estimates

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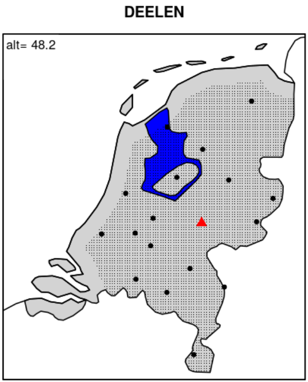
   where $\varepsilon_i(s) \sim_{i.i.d.} \mathcal{N}(0, \sigma^2)$.

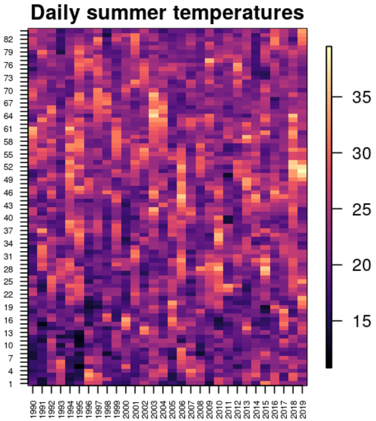
   Then, minimizing

   $$\sum_s \sum_{i=1}^n (X_i(s) - \beta_0 - \beta_1 \cdot \mathrm{LON}(s) - \beta_2 \cdot \mathrm{LAT}(s) - \beta_3 \cdot \mathrm{ALT}(s) - \ldots)^2$$

   is equivalent to the (independent) maximum likelihood estimator.

Data Example in Julia: Dutch Summer Temperatures





daily mean temperature data for. . .

- 18 Dutch weather stations
- 29 summers (JJA 1991–2019)

$\Big\} \Longrightarrow$ 14-day block means (6 per year)

Julia Packages

```julia
using Statistics
using DataFrames
using GLM
using Plots
using GaussianProcesses
using GaussianRandomFields
using Extremes
using Optim
using JLD2
```

Outline

Gaussian Processes

Main Assumption: Data belong to a Gaussian process $\{X(s)\}_{s \in S}$!

Ingredients:

- mean $\mu(s) = \mathbb{E}\{X(s)\}$ ✓
- variance $\sigma^2(s) = \mathrm{Var}\{X(s)\}$ ✓

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Ingredients:

- mean $\mu(s) = \mathbb{E}\{X(s)\}$ ✓
- variance $\sigma^2(s) = \mathrm{Var}\{X(s)\}$ ✓
- correlation $\rho(s_1, s_2) = \mathrm{Corr}\{X(s_1), X(s_2)\}$?

Tobler's First Law of Geography

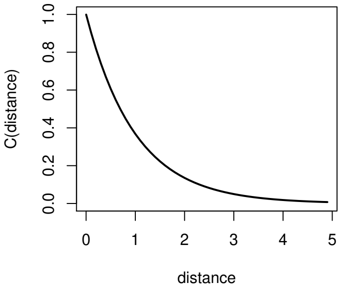
Everything is related to everything else, but near things are more related than distant things.

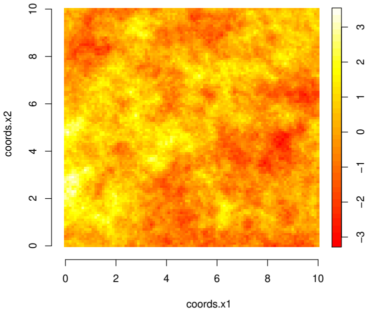
typical model for correlation:

- close to 1 (strong correlation) as s_1 and s_2 are close to each other
- decreasing as distance increases
- eventually tends to 0 (independence)

Example: Exponential Correlation

$$\rho(s_1, s_2) = \exp(-a\|s_1 - s_2\|), \quad a > 0$$

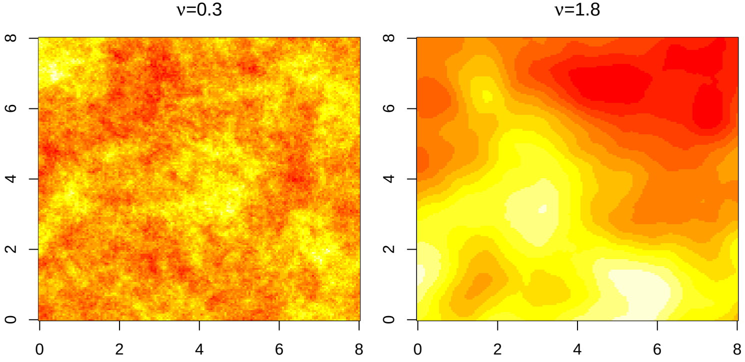




Note: Not every function is an admissible correlation function, only positive definite ones.

Example: Whittle–Matérn Model

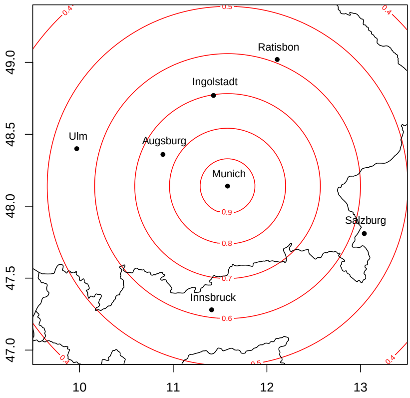
$$\rho(s_1, s_2) = \frac{2^{\nu-1}}{\Gamma(\nu)}(a\|s_1 - s_2\|)^{\nu} K_{\nu}(a\|s_1 - s_2\|), \quad a > 0, \nu > 0$$



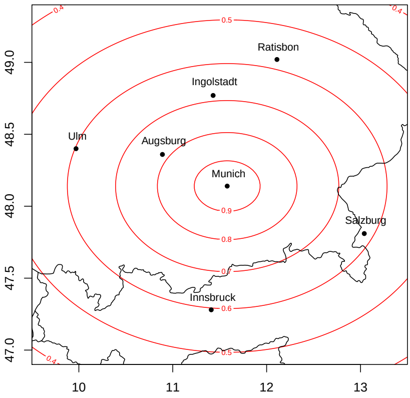
Note: parameter ν allows for different degrees of smoothness

Anisotropy: The direction matters

isotropic:



anisotropic:



Model Fit

1. build parametric models for
   - mean $\mu(s)$
   - variance $\sigma^2(s)$
   - correlation $\rho(s_1, s_2)$
2. estimate parameters from data
   - least squares fit to empirical estimates (see above)
   - joint maximum likelihood (including dependence structure)
   - ...

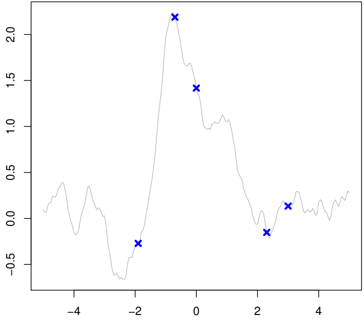
$\rightsquigarrow$ full probabilistic model for variable within whole region

Outline

Interpolation & Prediction

Given: data at sites $s_1, \ldots, s_n$ belonging to Gaussian process

Question: What does the process look like in between?



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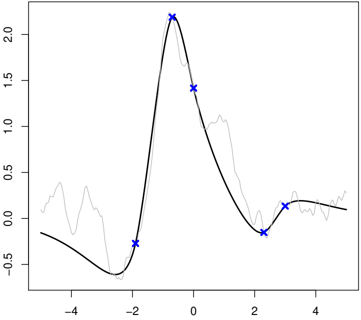
(At least) Two Options:

- pointwise prediction

  $\rightsquigarrow$ Kriging / Gaussian Process Regression

  pointwise "optimal" prediction, smooth interpolation by conditional mean



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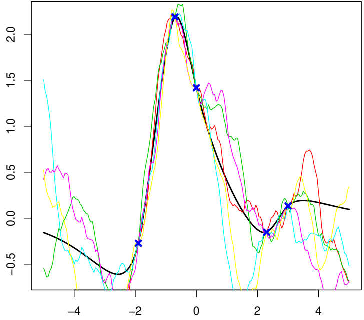
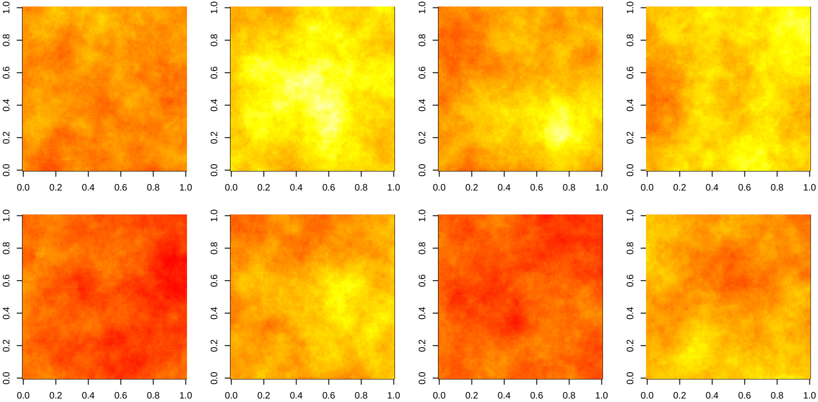
- structure of the sample path

  $\rightsquigarrow$ conditional simulation
  recovers all potential sample paths matching the data



Simulation of Gaussian Processes

Aim: generate realizations of process



Simulation of Gaussian Processes

How to simulate N-dimensional vector

$$\boldsymbol{Y} \sim \mathcal{N}(\mathbf{0}, \Sigma) \quad ?$$

1. Direct Simulation:
   - based on observation $\boldsymbol{X} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{I}) \quad \Rightarrow \boldsymbol{AX} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{AA}^\top)$
   - find decomposition (e.g. Cholesky) $\Sigma = \boldsymbol{AA}^\top$ and simulate $\boldsymbol{AX}$
   - computational costs typically of order $\mathcal{O}(N^3)$
2. Circulant Embedding:
   - based on FFT
   - works for GPs with isotropic covariance function on regular grid
   - computational costs of order $\mathcal{O}(N \log N)$

...

Outline

Extreme Events . . .

. . . have high impact

Examples of extreme events in environmental science:

- floods
- storms
- avalanches
- heat waves
- drought
- . . .

New Orleans after Hurricane Katrina (Photo: K. Niemi)

Extreme Events . . .

. . . are rare

Example: Dutch Delta Project
building dikes which are supposed to fail once per 10 000 years

⇝ **Challenge:** assess probability of unobserved events

New Orleans after Hurricane Katrina (Photo: K. Niemi)

Extreme Events . . .

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Example: Dutch Delta Project
building dikes which are supposed to fail once per 10 000 years

⇝ **Challenge:** assess probability of unobserved events

. . . often have a spatial extent

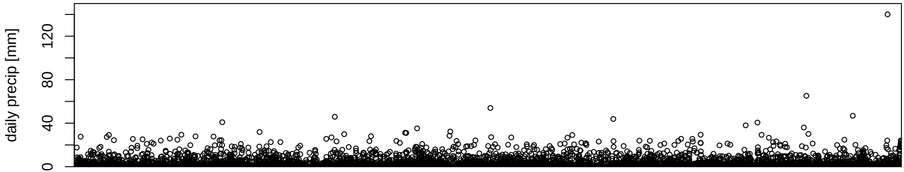
⇝ **Challenge:** accurate modelling of spatial dependence structure

New Orleans after Hurricane Katrina (Photo: K. Niemi)

Outline

Block Maxima Approach

Example: Daily precipitation at Münster Airport 1990–2010



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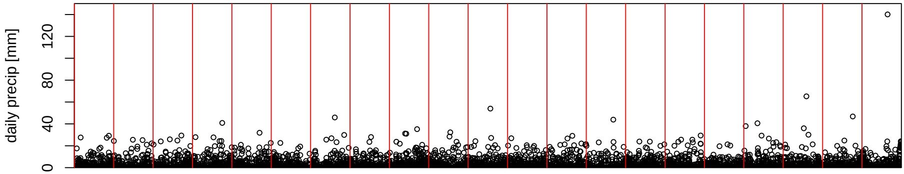

Idea: Divide time series into blocks of size 365 (or 366)

Block Maxima Approach

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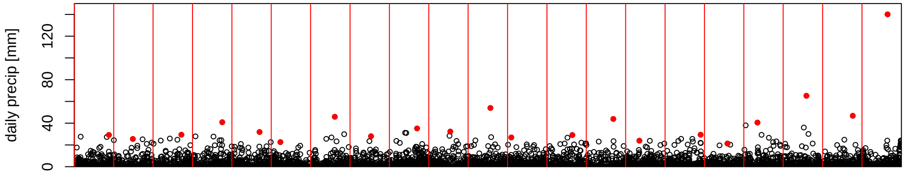

Idea: Divide time series into blocks of size 365 (or 366)
and take maximum within each block $\rightsquigarrow$ 21 annual maxima

Block Maxima Approach

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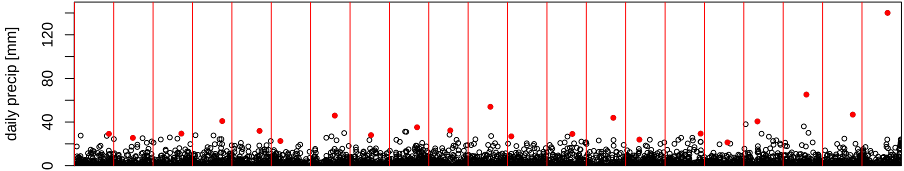

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Theoretical result: For large blocks, block maximum Z approximately follows Generalized Extreme Value (GEV) distribution.

$$\mathbb{P}(Z \leq z) \approx \exp\left(- \left(1 + \xi \frac{z - \mu}{\sigma} \right)^{-1/\xi} \right)$$

μ: location parameter
σ: scale parameter
ξ: shape parameter

Block Maxima Approach

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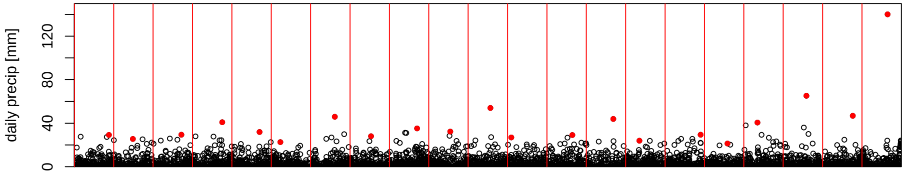

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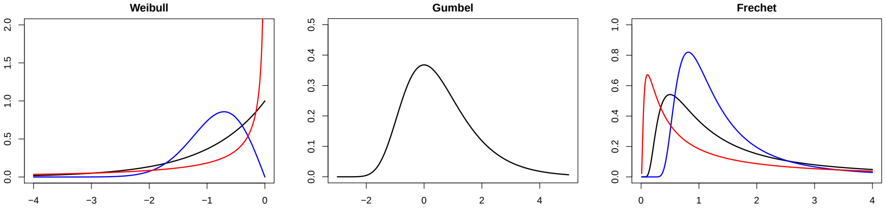
GEV: Flexible Modelling of Tails



$\xi < 0$
upper end point

$\xi = 0$
light tail

$\xi > 0$
heavy tail

GEV Distributions in Space

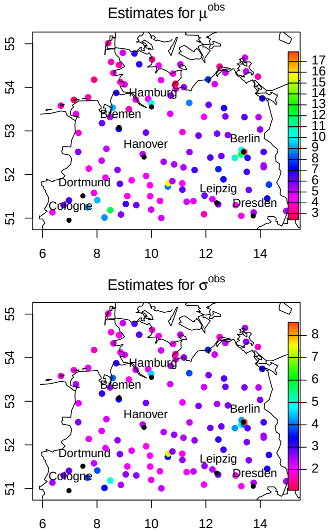
Block maximum Z at site s:

$$Z(s) \sim \mathrm{GEV}(\mu(s), \sigma(s), \xi(s))$$

First Approach:

Estimate μ, σ and ξ separately for each site, e.g. via individual maximum likelihood (derivative of c.d.f. can be calculated easily)

$\rightsquigarrow$ how to model at ungauged stations?



Source: O., Schlather & Friederichs 2017

GEV Distributions in Space (cont'd)

Block maximum Z at site s:

$$Z(s) \sim \mathrm{GEV}(\mu(s), \sigma(s), \xi(s))$$

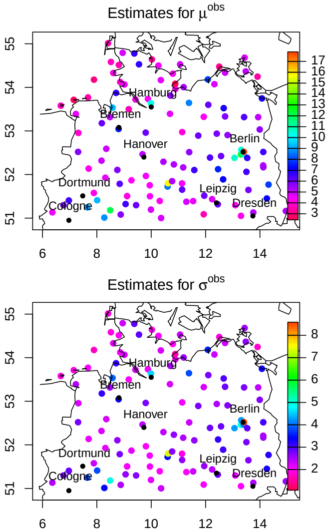
Second Approach:

Assume some spatial structure of μ, σ and ξ, e.g.

$$\mu(s) = \beta_0 + \beta_1 \cdot \mathrm{LON} + \beta_2 \cdot \mathrm{LAT} + \beta_3 \cdot \mathrm{ALT}$$

and estimate coefficients $\beta_0, \beta_1, \beta_2, \beta_3$ (e.g. by joint maximum likelihood assuming independence across sites)

$\rightsquigarrow$ model for any site within region



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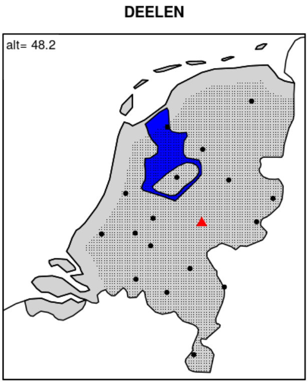

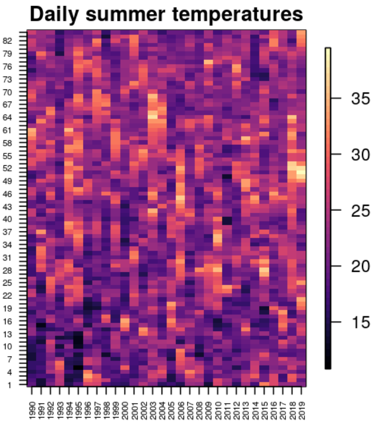

daily maxima temperature data for...

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$\Longrightarrow$ 14-day block maxima (6 per year)

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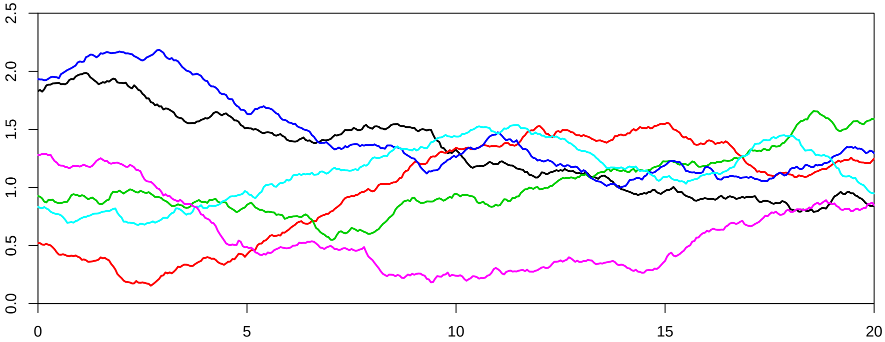
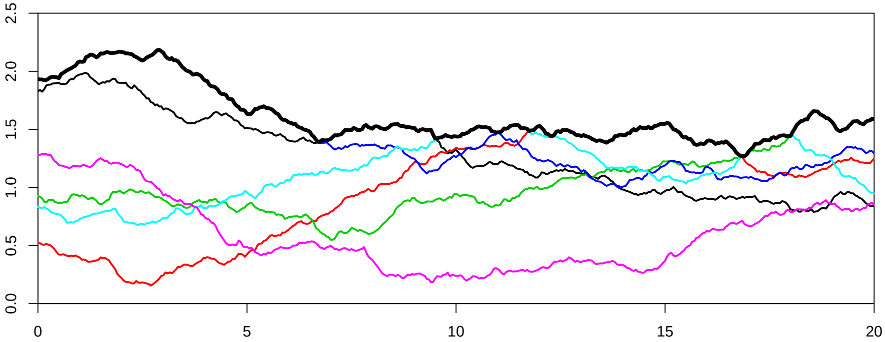
Block Maxima in Space

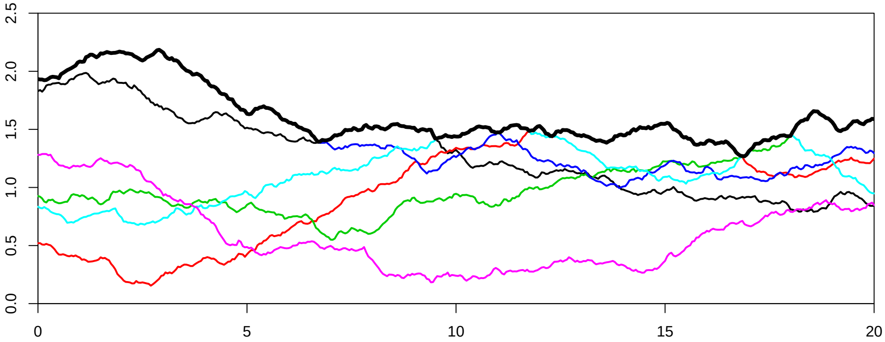


Block Maxima in Space



Take pointwise maxima!

Block Maxima in Space



Take pointwise maxima! $\rightsquigarrow$ max-stable processes

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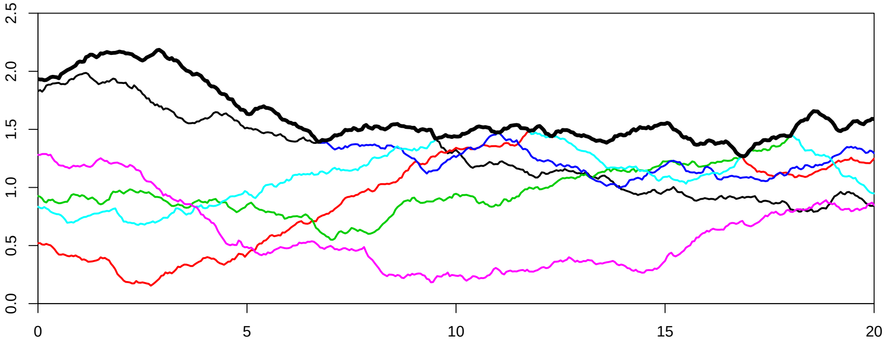


Take pointwise maxima! $\rightsquigarrow$ max-stable processes

Example: Mixed Moving Maxima Processes (Smith, 1990)

$\rightsquigarrow$ interpretation as models for storms

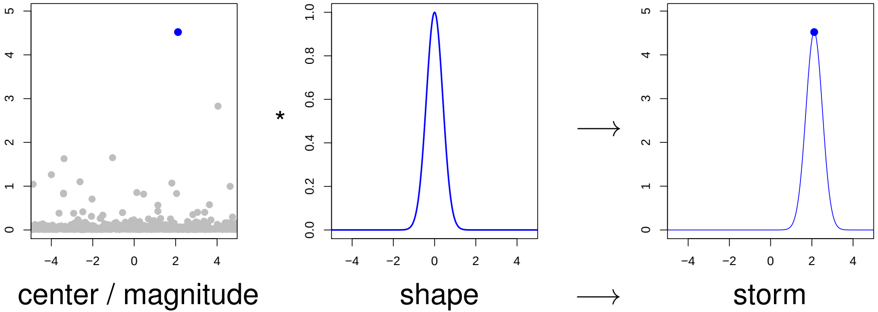
Components:

- magnitude of storm
- storm center
- shape of storm

Construction of Mixed Moving Maxima Processes

- U_i: magnitudes
- S_i: centers
- F_i: shapes

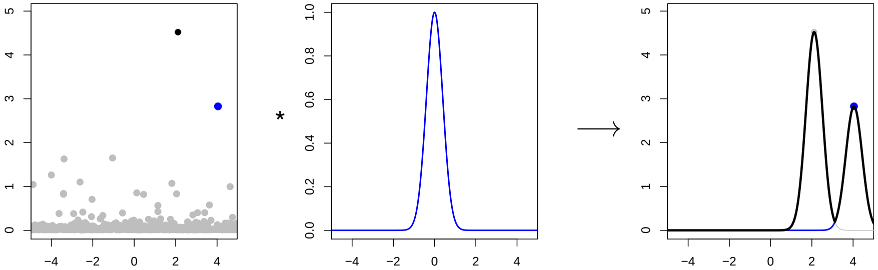
$$U_i \cdot F_i(x - S_i)$$



Construction of Mixed Moving Maxima Processes

- U_i: magnitudes
- S_i: centers
- F_i: shapes

$$Z(x) = \max_{i=1,2,\ldots} U_i \cdot F_i(x - S_i)$$



center / magnitude   shape   ⟶   storm

Construction of Mixed Moving Maxima Processes

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- S_i: centers
- F_i: shapes

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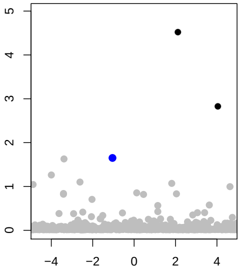

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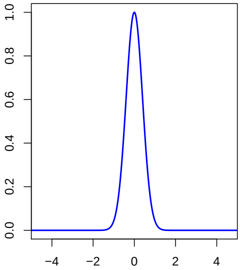
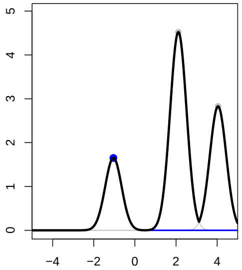


⟶



center / magnitude   shape   ⟶   storm

Construction of Mixed Moving Maxima Processes

- U_i: magnitudes
- S_i: centers
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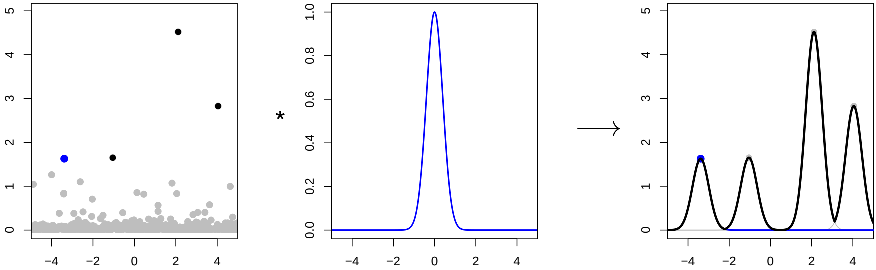

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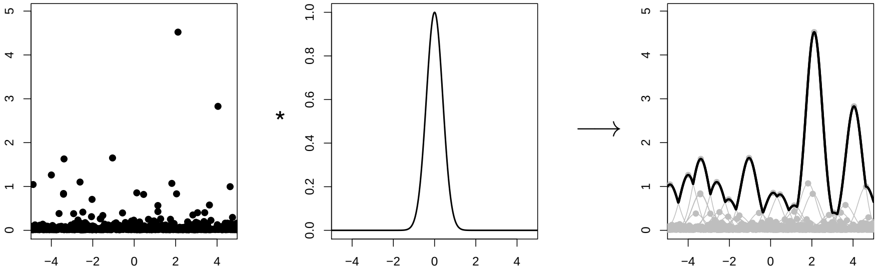

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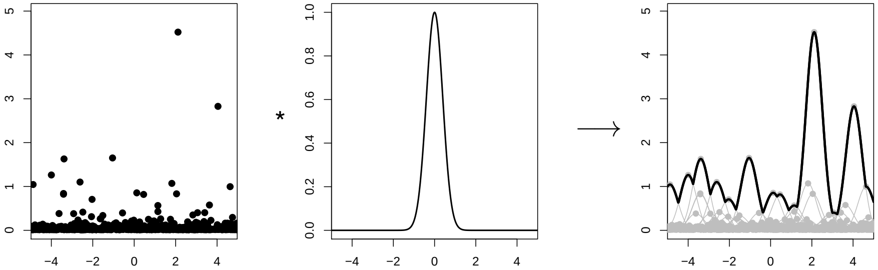

center / magnitude   shape $\longrightarrow$ storm

$\rightsquigarrow$ storm shape determines the spatial dependence structure!

Popular Max-Stable Models

Several Models use Gaussian processes as building blocks (storm shapes) . . .

- extremal Gaussian processes
  (Schlather 2002)
- Brown–Resnick processes
  (Kabluchko, Schlather & de Haan 2009)
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Some dependence properties inheritated by Gaussian process, e.g.

- smoothness of the process
- anisotropy

How to Measure Spatial Dependence in Extremes

Problem: Correlation not always defined!

Extremal Coefficient (Smith 1990, Schlather & Tawn 2003)

$$\mathbb{P}(Z(s_2) \text{ large} \mid Z(s_1) \text{ large})$$

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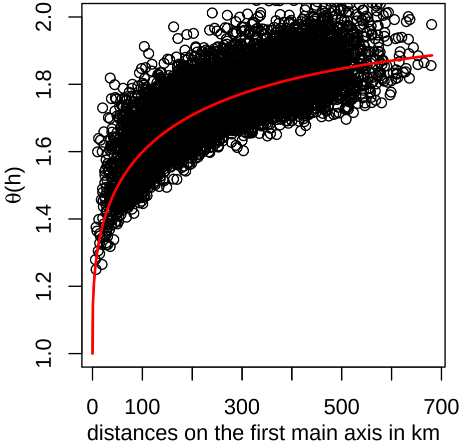
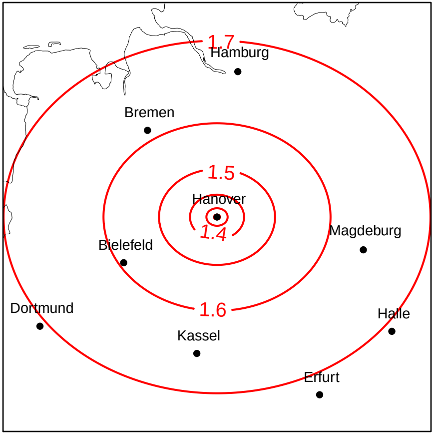
Properties:

- $1 \leq \theta(s_1, s_2) \leq 2$
- $\theta(s_1, s_2) = 1$: $Z(s_1)$ and $Z(s_2)$ fully dependent
- $\theta(s_1, s_2) = 2$: $Z(s_1)$ and $Z(s_2)$ (asymptotically) independent
- typically: $\theta(s_1, s_2)$ increases as distance $\|s_1 - s_2\|$ increases

Example: Extreme Wind Gusts in Northern Germany





Source: O., Schlather & Friederichs 2017

Model Fit

1. Parametric models for ...
   - marginal parameters $\mu(s)$, $\sigma(s)$ and $\xi(s)$
   - parameters of underlying Gaussian process

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   - marginal parameters $\mu(s)$, $\sigma(s)$ and $\xi(s)$
   - parameters of underlying Gaussian process
2. Estimation of parameters:
   - first estimation marginal parameters, then least squares fit to empirical extremal coefficients
   - maximum likelihood more complicated, but feasible, cf. Huser, Dombry, Ribatet & Genton 2019, Dombry, Engelke & O. 2017
   - ...

$\rightsquigarrow$ full probabilistic model for variable within whole region

Outline

Simulation of Spatial Extremes

Aim: create large amount of “artificial data” from fitted model (including rare/unobserved events) by simulations

$\rightsquigarrow$ assess risk of certain “extreme” scenarios

Remind: max-stable process $=$ maximum over infinitely many “storms”

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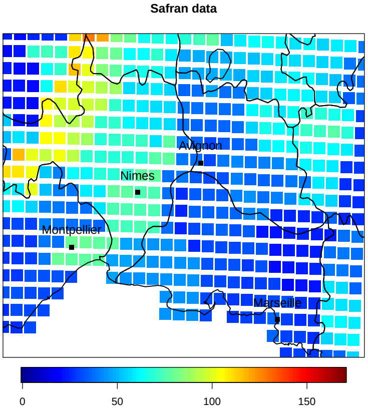
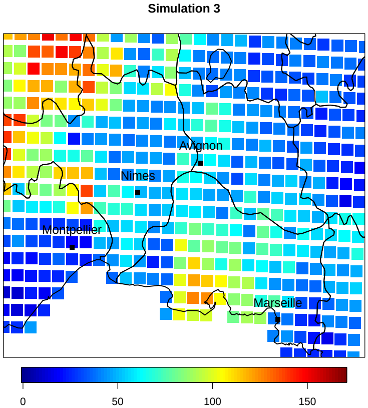
Algorithms for exact simulation:

- Dieker & Mikosch 2015
- Dombry, Engelke & O. 2016
- O., Schlather & Zhou 2018

Example: Autumnal Rainfall in Southern France





Source: O., Bel & Lantuéjoul 2018

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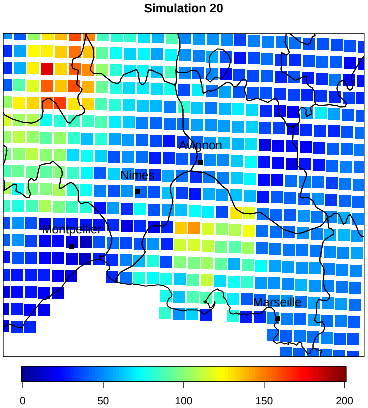
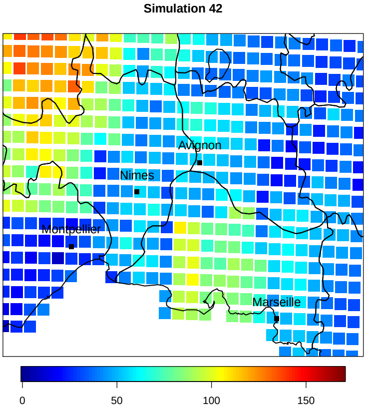
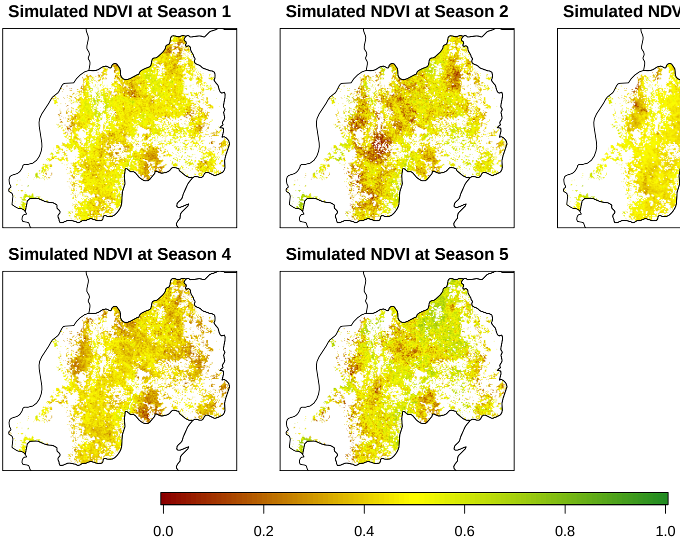




Source: O., Bel & Lantuéjoul 2018

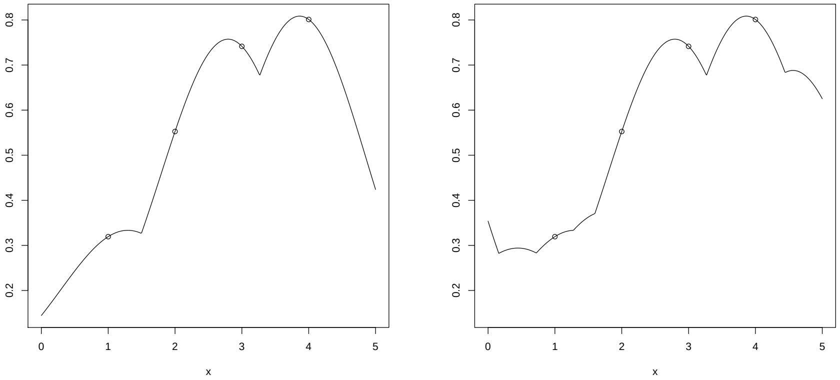
Example: Drought in Rwanda



Source: O. & Stein 2018

Prediction & Interpolation

- no analogue to Kriging for max-stable processes
- conditional simulations more difficult, but feasible for small number of conditions (Dombry, Eyi-Minko & Ribatet 2013)



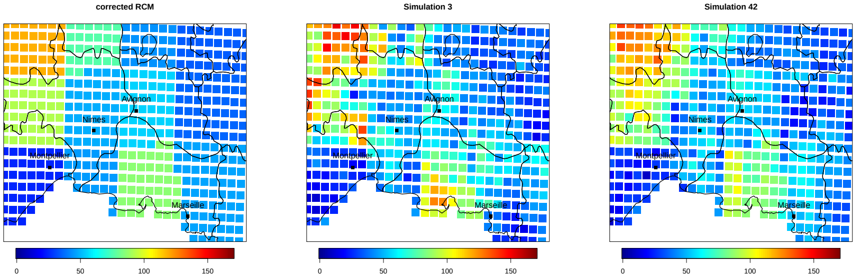
Source: Dombry, O. & Ribatet 2016

Conditional Simulation & Downscaling

also other conditions possible, e.g. average over large grid cell

$\rightsquigarrow$ "downscaling"

Example: Rainfall in Southern France



Source: O., Bel & Lantuéjoul 2018

Conclusion

- probabilistic spatial models include distributions for each site (with spatially varying parameters) and spatial dependence structure
- type of model depends on variable of interest and situation, e.g. Gaussian processes for “normal” data and max-stable processes for “extreme” data
- flexible models for a wide range of applications are available
- simulations enable us to go beyond the data
- conditional simulations allow for interpolation and downscaling

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Caution:

- Risk of model misspecification!
- High uncertainties in extreme events!

References I

A. B. Dieker and T. Mikosch.
Exact simulation of Brown-Resnick random fields at a finite number of locations.
Extremes, 18(2):301–314, 2015.

C. Dombry, S. Engelke, and M. Oesting.
Exact simulation of max-stable processes.
Biometrika, 103(2):303–317, 2016.

C. Dombry, S. Engelke, and M. Oesting.
Bayesian inference for multivariate extreme value distributions.
Electron. J. Stat., 11(2):4813–4844, 2017.

C. Dombry, F. Éyi-Minko, and M. Ribatet.
Conditional simulation of max-stable processes.
Biometrika, 100(1):111–124, 2013.

C. Dombry, M. Oesting, and M. Ribatet.
Conditional simulation of max-stable processes.
In D. K. Dey and J. Yan, editors, *Extreme Value Modeling and Risk Analysis*, pages 215–238. CRC Press, 2016.

R. Huser, C. Dombry, M. Ribatet, and M. G. Genton.
Full likelihood inference for max-stable data.
Stat, 8(1):e218, 2019.

Z. Kabluchko, M. Schlather, and L. de Haan.
Stationary max-stable fields associated to negative definite functions.
Ann. Probab., 37(5):2042–2065, 2009.

M. Oesting, L. Bel, and C. Lantuéjoul.
Sampling from a max-stable process conditional on a homogeneous functional with an appl. for downscaling climate data.
Scand. J. Stat., 45(2):382–404, 2018.

References II

M. Oesting, M. Schlather, and P. Friederichs.
Statistical post-processing of forecasts for extremes using bivariate Brown–Resnick processes with an appl. to wind gusts.
Extremes, 20(2):309–332, 2017.

M. Oesting, M. Schlather, and C. Zhou.
Exact and fast simulation of max-stable processes on a compact set using the normalized spectral representation.
Bernoulli, 24(2):1497–1530, 2018.

M. Oesting and A. Stein.
Spatial modeling of drought events using max-stable processes.
Stoch. Env. Res. Risk A., 32(1):63–81, 2018.

T. Opitz.
Extremal t processes: Elliptical domain of attraction and a spectral representation.
J. Multivar. Anal., 122:409–413, 2013.

M. Schlather.
Models for stationary max-stable random fields.
Extremes, 5(1):33–44, 2002.

M. Schlather and J. A. Tawn.
A dependence measure for multivariate and spatial extreme values: Properties and inference.
Biometrika, 90(1):139–156, 2003.

R. L. Smith.
Max–stable processes and spatial extremes.
Unpublished Manuscript, 1990.