Similarity-Based Inconsistency-Tolerant Logics

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Abstract. Many logics for AI applications that are defined by denotational semantics are trivialized in the presence of inconsistency. It is therefore often desirable, and practically useful, to refine such logics in a way that inconsistency does not cause the derivation of any formula, and, at the same time, inferences with respect to consistent premises are not affected. In this paper, we introduce a general method of doing so by incorporating preference relations defined in terms of similarities. We exemplify our method for three of the most common denotational semantics (standard many-valued matrices, their non-deterministic generalization, and possible worlds semantics), and demonstrate their usefulness for reasoning with inconsistency.

1 Introduction

Logics based on denotational semantics have many attractive properties for AI applications. However, most of the standard logics that are defined this way, including classical logic, intuitionistic logic, and some modal logics, are not inconsistency-tolerant, in the sense that they are trivialized for inconsistent theories: whenever the set of premises is not satisfiable, anything follows from it.[1] This renders such logics practically useless for reasoning with inconsistency.

In this paper, we introduce a general framework for adding inconsistency-maintenance capabilities to a wide range of logics that are defined by denotational semantics, without affecting their inferences with respect to consistent premises. More specifically, an *inconsistency-tolerant* variant of a logic L is a logic that is faithful to L with respect to consistent theories, but does not "explode" in the presence of inconsistency. For this, we incorporate the well-known preferential semantics of Shoham [13], in which for drawing conclusions from a set of premises, one takes into account its "most preferred" (or "plausible") valuations (rather than all of its models, none of which exists in case of contradictions).

Preferential semantics yields non-monotonic logics that often tolerate inconsistency in a proper, non-trivial way. However, in general this method does not guarantee faithfulness to the original logic (L) with respect to consistent theories. To achieve this, we consider a particular kind of preference criteria that are

[1] For languages with a negation $\neg$, this usually means that the underlying logic is not *paraconsistent* [6]: any formula ϕ follows from $\{\psi, \neg\psi\}$.

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based on the quantitative notion of *similarity*. Intuitively, similarities measure to what extent each valuation is "similar" to some model of a given theory, or how "close" each valuation is to satisfying the theory. This notion, which is more general than the notion of a distance, allows us to generalize many revision and merging operators considered in the literature for handling contradictory data.

We exemplify our similarity-based method on three of the most common types of denotational semantics, and demonstrate their usefulness by some concrete examples of reasoning with inconsistency.

2 Preliminaries

2.1 Denotational Semantics

In the sequel, $\mathcal{L}$ denotes a propositional language with a set Atoms of atomic formulas and a set $\mathcal{F}_\mathcal{L}$ of well-formed formulas. We denote the elements of Atoms by p, q, r, and the elements of $\mathcal{F}_\mathcal{L}$ by ψ, ϕ, σ. A theory Γ is a finite set of formulas in $\mathcal{F}_\mathcal{L}$. The atoms appearing in the formulas of Γ and the subformulas of Γ are denoted, respectively, $\mathsf{Atoms}(\Gamma)$ and $\mathsf{SF}(\Gamma)$. The set of all theories of $\mathcal{L}$ is $\mathcal{T}_\mathcal{L}$.

Definition 1. Given a language $\mathcal{L}$, a *propositional logic* for $\mathcal{L}$ is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, i.e., a binary relation satisfying the following conditions:
Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$.

A common (model-theoretical) way of defining consequence relations for $\mathcal{L}$ is based on *denotational semantics*:

Definition 2. A *denotational semantics* for a language $\mathcal{L}$ is a pair $\mathsf{S} = \langle S, \models_S \rangle$, where S is a nonempty set (of 'interpretations'), and $\models_S$ (the 'satisfiability relation' of S) is a binary relation on $S \times \mathcal{F}_\mathcal{L}$.

Let $\nu \in S$ and $\psi \in \mathcal{F}_\mathcal{L}$. If $\nu \models_S \psi$, we say that ν *satisfies* ψ and call ν an S-*model* of ψ. The set of the S-models of ψ is denoted by $mod_\mathsf{S}(\psi)$. If ν satisfies every formulas ψ in a theory Γ, it is called an S-model of Γ. The set of the S-models of Γ is denoted by $mod_\mathsf{S}(\Gamma)$. If $mod_\mathsf{S}(\Gamma) \neq \emptyset$ we say that Γ is S-*consistent*, otherwise Γ is S-inconsistent.

Below, we shall usually omit the prefix S of the above notions.

A denotational semantics S induces the following relation on $\mathcal{T}_\mathcal{L} \times \mathcal{F}_\mathcal{L}$:

Definition 3. We denote by $\Gamma \vdash_\mathsf{S} \psi$ that $mod_\mathsf{S}(\Gamma) \subseteq mod_\mathsf{S}(\psi)$.

Proposition 1. *Let* $\mathsf{S} = \langle S, \models_S \rangle$ *be a denotational semantics for* $\mathcal{L}$. *Then* $\langle \mathcal{L}, \vdash_\mathsf{S} \rangle$ *is a propositional logic for* $\mathcal{L}$.[2]

Next, we recall some common cases of denotational semantics and their corresponding logics.

[2] Proposition 1 is well-known and can be easily verified. Due to short of space, in what follows proofs are considerably reduced or omitted altogether.

2.2 Many-Valued Matrices

Definition 4. A *(multi-valued) matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and $\mathcal{O}$ contains an interpretation $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$ for every n-ary connective of $\mathcal{L}$.

Given a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we shall assume that $\mathcal{V}$ includes at least the two classical values t and f, and that only the former belongs to the set $\mathcal{D}$ of the *designated elements* in $\mathcal{V}$ (those that represent 'true assertions'). The set $\mathcal{O}$ contains the interpretations (the 'truth tables') of each connective in $\mathcal{L}$. The associated semantical notions are now defined as usual.

Definition 5. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$. An $\mathcal{M}$*-valuation* is a function $\nu : \mathcal{F}_{\mathcal{L}} \to \mathcal{V}$ so that, for every connective $\diamond$ in $\mathcal{L}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We shall sometimes denote by $\nu = \{p_1 : x_1, p_2 : x_2, \ldots\}$ the assignments $\nu(p_i) = x_i$, for $i = 1, 2 \ldots$. The set of all $\mathcal{M}$-valuations is denoted by $\Lambda_{\mathcal{M}}$. We say that $\nu \in \Lambda_{\mathcal{M}}$ is a *model* of ψ, denoted $\nu \models_{\mathcal{M}} \psi$, if $\nu(\psi) \in \mathcal{D}$.

Note that the pair $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ is a denotational semantics in the sense of Definition 2. By Proposition 1 we have, then, that:

Proposition 2. *The relation $\vdash_{\mathcal{M}}$, induced from a matrix $\mathcal{M}$ by Definition 3, is a Tarskian consequence relation.*

Example 1. The most common matrix-based entailments are induced from two-valued matrices. Thus, for instance, when $\mathcal{L}$ is the standard propositional language, $\mathcal{V} = \{\mathsf{t}, \mathsf{f}\}$, $\mathcal{D} = \{\mathsf{t}\}$, and $\mathcal{O}$ consists of the standard interpretations of the connectives in $\mathcal{L}$, $\langle \mathcal{L}, \models_{\mathcal{M}} \rangle$ is the classical propositional logic.

Three-valued logics are obtained by adding to $\mathcal{V}$ a third element. For instance, Kleene's logic [9] and McCarthy's logic [11] are obtained, respectively, from the matrices $\mathcal{M}_K^{3_\perp} = \langle \{\mathsf{t}, \mathsf{f}, \perp\}, \{\mathsf{t}\}, \mathcal{O}_K \rangle$ and $\mathcal{M}_M^{3_\perp} = \langle \{\mathsf{t}, \mathsf{f}, \perp\}, \{\mathsf{t}\}, \mathcal{O}_M \rangle$, in which the disjunction and conjunction are interpreted differently:

	$\tilde{\neg}$
f	t
$\perp$	$\perp$
t	f

(Kleene)

$\tilde{\wedge}$	f	$\perp$	t
f	f	f	f
$\perp$	f	$\perp$	$\perp$
t	f	$\perp$	t

$\tilde{\vee}$	f	$\perp$	t
f	f	$\perp$	t
$\perp$	$\perp$	$\perp$	t
t	t	t	t

(McCarthy)

$\tilde{\wedge}$	f	$\perp$	t
f	f	f	f
$\perp$	$\perp$	$\perp$	$\perp$
t	f	$\perp$	t

$\tilde{\vee}$	f	$\perp$	t
f	f	$\perp$	t
$\perp$	$\perp$	$\perp$	$\perp$
t	t	t	t

Priest's logic LP [12] is similar to Kleene's logic, but the third element is designated, so we denote it by $\top$ rather than $\perp$. This logic is induced by $\mathcal{M}_P^{3_\top} = \langle \{\mathsf{t}, \mathsf{f}, \top\}, \{\mathsf{t}, \top\}, \mathcal{O}_P \rangle$, where $\mathcal{O}_P$ is obtained from $\mathcal{O}_K$ by replacing $\perp$ by $\top$.

2.3 Non-deterministic Matrices

Matrix-based semantics is truth-functional in the sense that the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas. Such a semantics cannot be useful in capturing non-deterministic phenomena. This leads to the idea of non-deterministic matrices, introduced in [4], which allows non-deterministic evaluation of formulas:

Definition 6. A *non-deterministic matrix* (*Nmatrix*) for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and $\mathcal{O}$ contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective of $\mathcal{L}$.

An $\mathcal{M}$-*valuation* is a function $\nu : \mathcal{F}_{\mathcal{L}} \to \mathcal{V}$ such that for every connective $\diamond$ in $\mathcal{L}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. The set of all $\mathcal{M}$-valuations is denoted by $\Lambda_{\mathcal{M}}$. Again, $\nu \in \Lambda_{\mathcal{M}}$ is a *model* of ψ in $\mathcal{M}$ ($\nu \models_{\mathcal{M}} \psi$), if $\nu(\psi) \in \mathcal{D}$.

Ordinary matrices can be thought of as Nmatrices, the interpretations of which return singletons of truth-values. Henceforth, we shall identify deterministic Nmatrices and the corresponding ordinary matrices. Again, for an Nmatrix $\mathcal{M}$, the pair $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ is a denotational semantics and it induces a Tarskian consequence relation $\vdash_{\mathcal{M}}$.

Example 2. Consider an interaction with remote computers, where each computation may be either serial or parallel. This can be captured by non-deterministic interpretations, combining Kleene's and McCarthy's logics (Example 1):

	$\tilde{\neg}$
f	{t}
⊥	{⊥}
t	{f}

$\tilde{\wedge}$	f	⊥	t
f	{f}	{f}	{f}
⊥	{f, ⊥}	{⊥}	{⊥}
t	{f}	{⊥}	{t}

$\tilde{\vee}$	f	⊥	t
f	{f}	{⊥}	{t}
⊥	{⊥}	{⊥}	{t, ⊥}
t	{t}	{t}	{t}

Nmatrices have important applications in reasoning under uncertainty, proof theory, etc. We refer to [5] for a detailed discussion on Nmatrices.

2.4 Possible-Worlds Semantics

The last type of denotational semantics considered here is based on a many-valued extension of standard Kripke semantics (see [7]), where the logical connectives can be interpreted by a matrix $\mathcal{M}$,[3] and qualifications of the truth of a judgement is expressed by the necessitation operator "$\Box$". In case of the classical two-valued matrix we have the usual Kripke-style semantics.

Definition 7. Let $\mathcal{L}$ be a propositional language.

– A *frame* for $\mathcal{L}$ is a triple $\mathcal{F} = \langle W, R, \mathcal{M} \rangle$, where W is a non-empty set (of "worlds"), R (the "accessibility relation") is a binary relation on W, and $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a matrix for $\mathcal{L}$. We say that a frame is finite if so is W.
– Let $\mathcal{F} = \langle W, R, \mathcal{M} \rangle$ be a frame for $\mathcal{L}$. An $\mathcal{F}$-*valuation* is a function $\nu : W \times F_{\mathcal{L}} \to \mathcal{V}$ that assigns truth values to the $\mathcal{L}$-formulas at each world in W according to the following conditions: For every connective $\diamond$ in the language $\mathcal{L}$ (except for $\Box$),
  - $\nu(w, \diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}_{\mathcal{M}}(\nu(w, \psi_1), \ldots, \nu(w, \psi_n))$,
  - $\nu(w, \Box\psi) \in \mathcal{D}$ iff $\nu(w', \psi) \in \mathcal{D}$ for all w' such that $R(w, w')$.

[3] This framework can be extended to Nmatrices as well, but for simplicity we stick to deterministic matrices.

The set of $\mathcal{F}$-valuations is denoted by $\Lambda_{\mathcal{F}}$. The set of $\mathcal{F}$-valuations that satisfy a formula ψ in a world $w \in W$ is $mod^w_{\mathcal{F}}(\psi) = \{\nu \in \Lambda_{\mathcal{F}} \mid \nu(w, \psi) \in \mathcal{D}\}$.

– A *frame interpretation* is a pair $I = \langle \mathcal{F}, \nu \rangle$, in which $\mathcal{F} = \langle W, R, \mathcal{M} \rangle$ is a frame and ν is an $\mathcal{F}$-valuation. We say that I *satisfies* ψ (or that I is a model of ψ), if $\nu \in mod^w_{\mathcal{F}}(\psi)$ for every $w \in W$. We say that I satisfies Γ if it satisfies every $\psi \in \Gamma$.

Let $\mathcal{I}$ be a nonempty set of frame interpretations. Define a satisfaction relation $\models_{\mathcal{I}}$ on $\mathcal{I} \times F_{\mathcal{L}}$ by $I \models_{\mathcal{I}} \psi$ iff I satisfies ψ. Note that $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}} \rangle$ is a denotational semantics in the sense of Definition 2. By Proposition 1, then, the induced relation $\vdash_{\mathfrak{I}}$ is a Tarskian consequence relation for $\mathcal{L}$.

3 Inconsistency-Tolerant Logics

In the context of reasoning with uncertainty, a major drawback of a logic $\langle \mathcal{L}, \vdash_{\mathsf{S}} \rangle$, induced by a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$, is that it does not tolerate inconsistency properly. Indeed, if $mod_{\mathsf{S}}(\Gamma)$ is empty, then by Definition 3, $\Gamma \vdash_{\mathsf{S}} \psi$ for *every* formula $\psi \in \mathcal{F}_{\mathcal{L}}$. We therefore consider a 'refined' entailment relation, denoted $\mid\!\sim_{\mathsf{S}}$, that overcomes this explosive nature of $\vdash_{\mathsf{S}}$ but respects $\vdash_{\mathsf{S}}$ with respect to consistent theories. Formally, we require the following two properties:

I. Faithfulness: $\mid\!\sim_{\mathsf{S}}$ coincides with $\vdash_{\mathsf{S}}$ with respect to S-consistent theories, i.e., if $mod_{\mathsf{S}}(\Gamma) \neq \emptyset$ then for every $\psi \in \mathcal{F}_{\mathcal{L}}$, $\Gamma \mid\!\sim_{\mathsf{S}} \psi$ iff $\Gamma \vdash_{\mathsf{S}} \psi$.

II Non-Explosiveness: $\mid\!\sim_{\mathsf{S}}$ is not trivialized when the premises are not S-consistent, i.e., if $mod_{\mathsf{S}}(\Gamma) = \emptyset$ then there is $\psi \in \mathcal{F}_{\mathcal{L}}$ such that $\Gamma \not\mid\!\sim_{\mathsf{S}} \psi$.

We call $\mid\!\sim_{\mathsf{S}}$ an *inconsistency-tolerant* variant of $\vdash_{\mathsf{S}}$. When $\vdash_{\mathsf{S}}$ is clear from context, we shall just say that $\mid\!\sim_{\mathsf{S}}$ is inconsistency-tolerant.

Note 1. When $mod_{\mathsf{S}}(\Gamma) \neq \emptyset$ for every theory Γ (as in Priest's logic; see Example 1), $\vdash_{\mathsf{S}}$ itself is inconsistency-tolerant. In what follows we shall be interested in stronger logics (like classical logic) that do not tolerate inconsistency and so need to be refined. Moreover, being a consequence relation, Priest's logic is monotonic, but frequently commonsense reasoning is nonmonotonic, in particular in light of contradictions. Here, again, a refinement of the basic logic, adhering the two properties above, is called upon.

One way of achieving non-explosiveness is by incorporating Shoham's *preferential semantics* [13]: Given a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$ for $\mathcal{L}$, we define an S-*preferential operator* $\Delta_{\mathsf{S}} : \mathcal{F}_{\mathcal{L}} \to 2^S$ (where 2^S is the power-set of S), that relates a theory Γ to a set $\Delta_{\mathsf{S}}(\Gamma)$ of its 'most preferred' (or 'most plausible') elements in S. Then, the role of $mod_{\mathsf{S}}(\Gamma)$ in Definition 3 is taken now by $\Delta_{\mathsf{S}}(\Gamma)$:

Definition 8. Given a denotational semantics S and a S-preferential operator $\Delta_{\mathsf{S}} : \mathcal{F}_{\mathcal{L}} \to 2^S$, we denote by $\Gamma \mid\!\sim_{\Delta_{\mathsf{S}}} \psi$ that $\Delta_{\mathsf{S}}(\Gamma) \subseteq mod_{\mathsf{S}}(\psi)$.[4]

[4] In words: any conclusion should be satisfied by all the 'preferred' semantical objects (i.e., those elements in S describing the premises in the most plausible way).

Note 2. By faithfulness, every two S-consistent theories that are logically equivalent with respect to $\vdash_{\mathsf{S}}$ (that is, have the same S-models), must also share the same $\mid\!\sim_{\mathcal{S}}$-conclusions. On the other hand, while in any logic defined by denotational semantics (including classical logic) *all* inconsistent theories are logically equivalent, inconsistency-tolerant logics make a distinction between inconsistent theories, so they cannot preserve logical equivalence, and must employ other considerations. This is common to many methods for resolving inconsistencies, e.g., those that are based on information and inconsistency measures (see [8]).

Proposition 3. *Let* $\mathsf{S} = \langle S, \models_S \rangle$ *be a denotational semantics in which for every* $\nu \in S$ *there is some formula* $\psi \in \mathcal{F}_{\mathcal{L}}$*, such that* $\nu \not\models_S \psi$.[5] *Let* Δ_{S} *be a preferential operator for* S. *If (1)* $\Delta_{\mathsf{S}}(\Gamma)$ *is non-empty for every* Γ*, and (2)* $\Delta_{\mathsf{S}}(\Gamma) = mod_{\mathsf{S}}(\Gamma)$ *whenever* $mod_{\mathsf{S}}(\Gamma)$ *is not empty, then* $\mid\!\sim_{\Delta_{\mathsf{S}}}$ *is inconsistency-tolerant.*

Proof. Faithfulness follows from Condition (2); Non-explosiveness follows from the condition on S and from Condition (1). □

Proposition 3 shows that in many cases inconsistency-tolerant entailments can be obtained from a given denotational semantics S by a proper choice of a preferential operator Δ_{S}. Frequently, such an operator can be defined in terms of a preferential function P that maps every theory Γ to a strict partial order $<_\Gamma$ on S. In such cases,

$$\Delta^{\mathsf{P}}_{\mathsf{S}}(\Gamma) = \{\nu \in S \mid \neg\exists\mu \in S \text{ such that } \mu <_\Gamma \nu\}, \tag{1}$$

so, intuitively, $\Delta^{\mathsf{P}}_{\mathsf{S}}(\Gamma)$ consists of the 'best' elements in terms of $<_\Gamma$.

Proposition 4. *Let* S *be a denotational semantics as in Proposition 3 and let* P *be a preferential function, mapping every theory* Γ *to a strict partial order* $<_\Gamma$ *on* S. *If (1) for every theory* Γ*,* $<_\Gamma$ *is well-founded, and (2) for every* S*-consistent* Γ*,* $\min_{<_\Gamma}(S)$ $[= \Delta^{\mathsf{P}}_{\mathsf{S}}(\Gamma)] = mod_{\mathsf{S}}(\Gamma)$*, then* $\mid\!\sim_{\Delta^{\mathsf{P}}_{\mathsf{S}}}$ *is inconsistency-tolerant.*

Proof. Clearly, the two conditions of this proposition imply, respectively, the two conditions of Proposition 3, and so $\mid\!\sim_{\Delta^{\mathsf{P}}_{\mathsf{S}}}$ is inconsistency-tolerant. □

A preferential function P as in Proposition 4 represents *preference by satisfiability*, that is: the models of the underlying theory (if such elements exist) are preferred over the other elements in S.

Proposition 4 specifies natural conditions under which a strict pre-order $<_\Gamma$ induces an inconsistency-tolerant entailment. However, this proposition does not give a method for defining such an order. Next, we consider a simple and intuitive way of doing so by introducing the notion of *similarity*. In what follows, we demonstrate similarity-based reasoning for the three types of denotational semantics discussed previously.

[5] This holds, e.g., when there is a contradictory formula $\bot_{\mathsf{S}}$, for which $mod_{\mathsf{S}}(\bot_{\mathsf{S}}) = \emptyset$.

4 Similarity-Based Reasoning

4.1 Inconsistency Tolerance by Matrix Semantics

Given a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we fix the corresponding denotational semantics $\mathsf{S} = \langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$. For simplicity, we shall identify S with $\mathcal{M}$. Now, the criterion of preference by satisfiability, considered previously for general denotational semantics, can be described in the case of (many-valued) valuations by the aspiration of being 'as similar as possible' to valuations that satisfy the set of premises, Γ. This is depicted in what follows by corresponding quantitative indications.

Definition 9. A *(numeric) aggregation function* is a total function f, such that: (1) for every multiset of real numbers, the value of f is a real number, (2) the value of f does not decrease when a number in its multiset increases, (3) $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = x_2 = \ldots x_n = 0$, and (4) $\forall x \in \mathbb{R}\ f(\{x\}) = x$.

Summation, average, and maximum, are all aggregation functions.

To keep the set of the "preferred valuations" computable, we restrict the comparison of valuations to relevant contexts:

Definition 10. A *context* is a finite set of formulas. A *context generator* is a function $\mathcal{G} : \mathcal{T}_{\mathcal{L}} \to \mathcal{T}_{\mathcal{L}}$, producing a context for every theory.

Simple examples for context generators are, e.g., the functions $\mathcal{G}^{\mathsf{At}}$, $\mathcal{G}^{\mathsf{SF}}$, $\mathcal{G}^{\mathsf{ID}}$, defined for every Γ by $\mathcal{G}^{\mathsf{At}}(\Gamma) = \mathsf{Atoms}(\Gamma)$, $\mathcal{G}^{\mathsf{SF}}(\Gamma) = \mathsf{SF}(\Gamma)$, and $\mathcal{G}^{\mathsf{ID}}(\Gamma) = \Gamma$.

Definition 11. Let $\mathcal{M}$ be a matrix, C a context, and $\mathcal{G}$ a context generator.

- An $\mathcal{M}$*-similarity with respect to* C is a symmetric function $\mathsf{s} : \Lambda_{\mathcal{M}} \times \Lambda_{\mathcal{M}} \to \mathbb{N}^+$, such that $\mathsf{s}(\nu, \mu) = 0$ iff $\nu(\phi) = \mu(\phi)$ for all $\phi \in \mathsf{C}$.
- Given an $\mathcal{M}$-similarity s with respect to C, we define:
$$\mathsf{m}^{\mathsf{s}}(\nu, \psi) = \begin{cases} min\{\mathsf{s}(\nu, \mu) \mid \mu \in mod_{\mathcal{M}}(\psi)\} & mod_{\mathcal{M}}(\psi) \neq \emptyset, \\ 1 + max\{\mathsf{s}(\nu, \mu) \mid \nu, \mu \in \Lambda_{\mathcal{M}}\} & \text{otherwise.} \end{cases}$$
- Given an aggregation function f, we define:
$$\mathsf{m}^{\mathsf{s}}_f(\nu, \Gamma) = f\big(\{\mathsf{m}^{\mathsf{s}}(\nu, \psi_1), \ldots, \mathsf{m}^{\mathsf{s}}(\nu, \psi_n)\}\big),$$
where $\Gamma = \{\psi_1, \ldots, \psi_n\}$ and m^{s} is defined as in the previous item by a similarity s with respect to $\mathcal{G}(\Gamma)$.

Note that *lower* values of similarities indicate *higher* correspondence between valuations. However, this correspondence is limited to the relevant contexts: if s is a similarity with respect to C, then $\mathsf{s}(\nu, \mu) = 0$ indicates that ν and μ agree on the formulas of C, but this does not necessarily mean that $\nu = \mu$.

Intuitively, $\mathsf{m}^{\mathsf{s}}(\nu, \psi)$ indices how 'close' ν is to be a model of ψ. The function $\mathsf{m}^{\mathsf{s}}_f$ extends m^{s} to theories: $\mathsf{m}^{\mathsf{s}}_f(\nu, \Gamma)$ indicates how 'close' is the valuation ν to satisfy Γ. Note that if ψ is *not* $\mathcal{M}$-satisfiable, then, as expected, all the valuations $\nu \in \Lambda_{\mathcal{M}}$ are equally close to ψ: $\mathsf{m}^{\mathsf{s}}(\nu, \psi) = 1 + max\{\mathsf{s}(\nu, \mu) \mid \nu, \mu \in \Lambda_{\mathcal{M}}\}$. By Proposition 5 below, when ψ is $\mathcal{M}$-satisfiable, the valuations ν for which $\mathsf{m}^{\mathsf{s}}(\nu, \psi)$ is minimal, are the models of ψ.

Example 3. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a finite Euclidean space, i.e., $\mathcal{V} = \mathbb{R}^m = \{\langle x_1, \ldots, x_m \rangle \mid \forall\, 1 \leq i \leq m \; x_i \in \mathbb{R}\}$.[6] The following are distances on $\mathbb{R}^m$:

- *the discrete distance*: $d_U(\overline{x}, \overline{x}) = 0$ and $d_U(\overline{x}, \overline{y}) = 1$ if $\overline{x} \neq \overline{y}$,
- *distance by average*: $d_{\frac{\Sigma}{m}}(\overline{x}, \overline{y}) = \frac{1}{m}\big(\sum_{i=1}^{m} |x_i - y_i|\big)$,
- *the k-norm distance* ($k \geq 1$): $||\overline{x}, \overline{y}||_k = \big(\sum_{i=1}^{m} |x_i - y_i|^k\big)^{\frac{1}{k}}$,
- *the infinity-norm distance*: $\lim_{k \to \infty} ||\overline{x}, \overline{y}||_k = \max(|x_1 - y_1|, \ldots, |x_n - y_n|)$.

Consider the context $\mathsf{C} = \mathsf{Atoms}(\Gamma)$ for some theory Γ. A function s_g, defined for every $\nu, \mu \in \Lambda_{\mathcal{M}}$ by

$$\mathsf{s}_g(\nu, \mu) = g(\{d(\nu(\psi), \mu(\psi)) \mid \psi \in \mathsf{C}\}), \tag{2}$$

where d is one of the distances above and g is an aggregation function, is a similarity function with respect to C. For instance, in the two-valued case, we get the uniform distance with respect to C when $g = max$, and the Hamming distance with respect to C when $g = \Sigma$.

Definition 12. A (semantical) *setting* for $\mathcal{L}$ is a quadruple $\mathcal{K} = \langle \mathcal{M}, \mathcal{G}, \mathcal{S}, f \rangle$, where $\mathcal{M}$ is a matrix, $\mathcal{G}$ is a context generator, f is an aggregation function, and $\mathcal{S}$ is a *similarity generator*, i.e., a function so that for all Γ $\mathcal{S}(\Gamma)$ is an $\mathcal{M}$-similarity with respect to $\mathcal{G}(\Gamma)$.

Preference by similarities is now defined as follows:

Definition 13. Given a setting $\mathcal{K} = \langle \mathcal{M}, \mathcal{G}, \mathcal{S}, f \rangle$, the *most plausible valuations* with respect to $\mathcal{K}$ of a (nonempty) theory Γ, are the elements of the set

$$\Delta_{\mathcal{K}}(\Gamma) = \big\{\nu \in \Lambda_{\mathcal{M}} \mid \forall \mu \in \Lambda_{\mathcal{M}} \; \mathsf{m}_f^{\mathcal{S}(\Gamma)}(\nu, \Gamma) \leq \mathsf{m}_f^{\mathcal{S}(\Gamma)}(\mu, \Gamma)\big\}.$$

In case that Γ is empty, we define $\Delta_{\mathcal{K}}(\emptyset) = \Lambda_{\mathcal{M}}$.

Note that $\Delta_{\mathcal{K}}$ can be represented in the form of (1), where $<_\Gamma$ is defined by $\nu <_\Gamma \mu$ iff $\nu \in \Delta_{\mathcal{K}}(\Gamma)$ and $\mu \notin \Delta_{\mathcal{K}}(\Gamma)$. Now, similarity-based entailments are defined as in Definition 8:

$$\Gamma \mid\!\sim_{\Delta_{\mathcal{K}}} \psi \text{ iff } \Delta_{\mathcal{K}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi). \tag{3}$$

Example 4. Let $\mathcal{K} = \langle \mathcal{M}_K^{3_\perp}, \mathcal{G}^{\mathsf{At}}, \mathcal{S}, \Sigma \rangle$, where $\mathcal{M}_K^{3_\perp}$ is Kleene's three-valued matrix (Example 1), $\mathcal{G}^{\mathsf{At}}$ is the atom-based context generator (see below Definition 10), Σ is a summation function, and $\mathcal{S}$ is a similarity generator that for

[6] This includes, among others, linearly ordered values (as in the three-valued logics considered above, or the elements of the unit interval), that are represented by a one-dimensional space; partial orders in which there are at most $i-1$ different x_j's such that $\mathsf{f} < x_1 < \ldots < x_{i-1} < \mathsf{t}$, that may be represented by pairs of numbers in $\{0, \ldots, i-1\}$ (see [2] for this kind of representation for Belnap's four-valued logic); the elements of an Nmatrix for mbC that can be represented by triples (see [5]), etc.

each Γ produces a similarity s_Σ in the form of (2), i.e., $\mathcal{S}(\Gamma)(\nu,\mu) = \mathsf{s}_\Sigma(\nu,\mu) = \Sigma\,\{d_\Sigma(\nu(p),\mu(p)) \mid p \in \mathsf{Atoms}(\Gamma)\}$. Here, d_Σ is a distance on $\{\mathsf{t},\mathsf{f},\bot\}$, in which $d_\Sigma(\mathsf{t},\mathsf{f}) = 1$ and $d_\Sigma(\mathsf{t},\bot) = d_\Sigma(\mathsf{f},\bot) = \frac{1}{2}$ (see Example 3).

Now, let $\Gamma = \{\neg p, \neg q, p \vee q\}$. Clearly, Γ is not $\mathcal{M}_K^{3_\bot}$-satisfiable. We compute its most plausible models w.r.t. $\mathcal{K}$:

	p	q	$\neg p$	$\neg q$	$p \vee q$	1	2	3	$\mathsf{m}_\Sigma(\nu_i,\Gamma)$
ν_1	t	t	f	f	t	1	1	0	2
ν_2	t	f	f	t	t	1	0	0	**1**
ν_3	t	$\bot$	f	$\bot$	t	1	0.5	0	1.5
ν_4	f	t	t	f	t	0	1	0	**1**
ν_5	f	f	t	t	f	0	0	1	**1**
ν_6	f	$\bot$	t	$\bot$	$\bot$	0	0.5	0.5	**1**
ν_7	$\bot$	t	$\bot$	f	t	0.5	1	0	1.5
ν_8	$\bot$	f	$\bot$	t	$\bot$	0.5	0	0.5	**1**
ν_9	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	0.5	0.5	0.5	1.5

Legend. $1 = \mathsf{m}^{\mathsf{s}}(\nu_i,\neg p)$, $2 = \mathsf{m}^{\mathsf{s}}(\nu_i,\neg q)$, $3 = \mathsf{m}^{s}(\nu_i, p \vee q)$.

Hence, $\Delta_\mathcal{K}(\Gamma) = \{\nu_2,\nu_4,\nu_5,\nu_6,\nu_8\}$, and so, for instance, $\Gamma \mid\!\sim_{\Delta_\mathcal{K}} \neg p \vee \neg q$ (even though $\Gamma \not\mid\!\sim_{\Delta_\mathcal{K}} \neg p$ and $\Gamma \not\mid\!\sim_{\Delta_\mathcal{K}} \neg q$).

In what follows, we shall abbreviate $\mid\!\sim_{\Delta_\mathcal{K}}$ by $\mid\!\sim_\mathcal{K}$. Next, we show that entailments of this type are inconsistency tolerant.

Definition 14. Let $\mathcal{M} = \langle \mathcal{V},\mathcal{D},\mathcal{O}\rangle$ be a matrix for $\mathcal{L}$. A context C is *proper for* ψ (in $\mathcal{M}$), if for every $\nu,\mu \in \Lambda_\mathcal{M}$, if $\nu(\phi) = \mu(\phi)$ for all $\phi \in \mathsf{C}$, then $\nu(\psi) = \mu(\psi)$ as well. C is a proper context for a theory Γ, if it is proper for every $\psi \in \Gamma$.

In what follows we consider only *proper settings*, that is: settings $\mathcal{K} = \langle \mathcal{M},\mathcal{G},\mathcal{S},f\rangle$ in which for every theory Γ, $\mathcal{G}(\Gamma)$ is a proper context for Γ (in $\mathcal{M}$). Note that for all the context generators considered above ($\mathcal{G}^{\mathsf{At}}$, $\mathcal{G}^{\mathsf{SF}}$, and $\mathcal{G}^{\mathsf{ID}}$), the corresponding setting is proper.

Proposition 5. *Let* s *be a similarity with respect to a context* C *that is proper for* ψ*. Then* $\mathsf{m}^{\mathsf{s}}(\nu,\psi) = 0$ *iff* $\nu \in mod_\mathcal{M}(\psi)$*.*

Corollary 1. *Let* s *be a similarity with respect to a context* C *that is proper for* Γ*. Then* $\mathsf{m}^{\mathsf{s}}_f(\nu,\Gamma) = 0$ *iff* $\nu \in mod_\mathcal{M}(\Gamma)$*.*

Proof. By Proposition 5 and since f is an aggregation function. □

Definition 15. Γ_1 and Γ_2 are *independent*, if $\mathsf{Atoms}(\Gamma_1) \cap \mathsf{Atoms}(\Gamma_2) = \emptyset$.

Proposition 6. *Let* $\mathcal{K} = \langle \mathcal{M},\mathcal{G},\mathcal{S},f\rangle$ *be a semantic setting. If* $\mathcal{G}(\Gamma)$ *and* $\{\psi\}$ *are independent, then* $\Gamma \mid\!\sim_\mathcal{K} \psi$ *iff* ψ *is an* $\mathcal{M}$*-tautology.*

Corollary 2. *Let* $\mathcal{K} = \langle \mathcal{M},\mathcal{G},\mathcal{S},f\rangle$ *be a semantic setting. For every* Γ *there is a formula* ψ *such that* $\Gamma \not\mid\!\sim_\mathcal{K} \psi$*.*

Proof. Given Γ, let $p \in \mathsf{Atoms} \setminus \mathcal{G}(\Gamma)$ (such a p exists, since Atoms is infinite and $\mathcal{G}(\Gamma)$ is not). As $\mathcal{G}(\Gamma)$ and $\{p\}$ are independent, by Proposition 6, $\Gamma \not\mid\!\sim_{\mathcal{K}} p$. □

Proposition 7. *For every setting* $\mathcal{K} = \langle \mathcal{M}, \mathcal{G}, \mathcal{S}, f \rangle$, $\mid\!\sim_{\mathcal{K}}$ *is an inconsistency-tolerant variant of* $\vdash_{\mathcal{M}}$.

Proof. Faithfulness to $\vdash_{\mathcal{M}}$ follows from Corollary 1; Non-explosiveness follows from Corollary 2. □

4.2 Inconsistency Tolerance by Nmatrices

Similarity-based entailments can be defined in the non-deterministic case just as in the deterministic case. Given an Nmatrix $\mathcal{M}$, similarities and satisfiability measures are defined according to Definition 11. This induces the operator $\Delta_{\mathcal{K}}$ and the entailment $\mid\!\sim_{\Delta_{\mathcal{K}}}$, as in Definition 13 and in (3), respectively.

The results in the previous section also carry on to non-deterministic semantics. It is important to note, though, that in the non-deterministic case the context generator $\mathcal{G}^{\mathsf{At}}$ does *not* produce proper contexts. This is explained by the fact that, unlike deterministic valuations, non-deterministic valuations are not truth functional, so they can agree on atomic formulas, but make different non-deterministic choices on complex formulas. Yet, as the next proposition shows, the other two context generators do provide proper contexts:

Proposition 8. *For every Nmatrix* $\mathcal{M}$*, similarity generator* $\mathcal{S}$*, and aggregation* f*, both* $\langle \mathcal{M}, \mathcal{G}^{\mathsf{SF}}, \mathcal{S}, f \rangle$ *and* $\langle \mathcal{M}, \mathcal{G}^{\mathsf{ID}}, \mathcal{S}, f \rangle$ *are proper.*

Proof. By the fact that if $\psi \in \mathsf{C}$ then C is proper in $\mathcal{M}$ for ψ. □

Example 5. Let $\mathcal{K} = \langle \mathcal{M}^{3_\perp}_{KM}, \mathcal{G}^{\mathsf{ID}}, \mathcal{S}, \Sigma \rangle$ be a setting in which $\mathcal{M}^{3_\perp}_{KM}$ is the Nmatrix of Example 2, combining Kleene's and McCarthy's three-valued logics, $\mathcal{G}^{\mathsf{ID}}$ is the context generator by identity, Σ is a summation function, and $\mathcal{S}$ a similarity generator defined for every Γ by $\mathcal{S}(\Gamma)\,(\nu, \mu) = \Sigma\,\{d_\Sigma(\nu(\psi), \mu(\psi)) \mid \psi \in \Gamma\}$. Again, here d_Σ is the distance on $\{\mathsf{t}, \mathsf{f}, \perp\}$ defined in Example 3.

As in Example 4, we let $\Gamma = \{\neg p, \neg q, p \vee q\}$. Clearly, Γ is not $\mathcal{M}^{3_\perp}_{KM}$-satisfiable. Note that in addition to the nine valuations in Example 4 we also have $\nu_{10} = \{p\!:\!\perp,\, q\!:\!\mathsf{t},\, \neg p\!:\!\perp,\, \neg q\!:\!\mathsf{f},\, p\vee q\!:\!\perp\}$. It can be verified that this time $\Gamma \not\mid\!\sim_{\Delta_{\mathcal{K}}} \neg p \vee \neg q$ (cf. Example 4).

4.3 Inconsistency Tolerance by Possible Worlds

We now extend similarities to the context of finite frames.

Definition 16. Let $\mathcal{F} = \langle W, R, \mathcal{M} \rangle$ be a *finite* frame, C a context, and $\mathcal{G}$ a context generator.

– An $\mathcal{F}$*-similarity with respect to* C is a symmetric function $\mathsf{s} : \Lambda_{\mathcal{F}} \times \Lambda_{\mathcal{F}} \to \mathbb{N}^+$, such that $\mathsf{s}(\nu, \mu) = 0$ iff $\nu(w, \psi) = \mu(w, \psi)$ for every $w \in W$ and $\psi \in \mathsf{C}$.

– Given an $\mathcal{F}$-similarity s with respect to C, we define:

$$\mathsf{m}^{\mathsf{s}}(w,\nu,\psi) = \begin{cases} min\{\mathsf{s}(\nu,\mu) \mid \mu \in mod^{w}_{\mathcal{F}}(\psi)\} & mod^{w}_{\mathcal{F}}(\psi) \neq \emptyset, \\ 1 + max\{\mathsf{s}(\nu,\mu) \mid \nu,\mu \in \Lambda_{\mathcal{F}}\} & \text{otherwise.} \end{cases}$$

– For a frame interpretation $I = \langle \mathcal{F}, \nu \rangle$ and an aggregation function f, define: $\mathsf{m}^{\mathsf{s}}_{f}(I,\psi) = f\big(\{\mathsf{m}^{\mathsf{s}}(w,\nu,\psi) \mid w \in W\}\big)$, where m^{s} is defined as in the previous item by an $\mathcal{F}$-similarity s.

– For a frame interpretation $I = \langle \mathcal{F}, \nu \rangle$ and aggregation functions g, f, define: $\mathsf{m}^{\mathsf{s}}_{g,f}(I,\Gamma) = g\big(\{\mathsf{m}^{\mathsf{s}}_{f}(I,\psi_1),\ldots,\mathsf{m}^{\mathsf{s}}_{f}(I,\psi_n)\}\big)$, where $\Gamma = \{\psi_1,\ldots,\psi_n\}$ and $\mathsf{m}^{\mathsf{s}}_{f}$ is defined as in the item above by an $\mathcal{F}$-similarity s with respect to $\mathcal{G}(\Gamma)$.

Definition 17. A *setting* for $\mathcal{L}$ is a quintuple $\mathcal{K} = \langle \mathcal{I}, \mathcal{G}, \mathcal{S}, f, g \rangle$, where $\mathcal{I}$ is a set of finite frames, $\mathcal{G}$ is a context generator, f and g are aggregation functions, and $\mathcal{S}$ is a *similarity generator* for $\mathcal{G}$, i.e., for every $I = \langle \mathcal{F}, \nu \rangle \in \mathcal{I}$ and $\Gamma \in \mathcal{T}_{\mathcal{L}}$, $\mathcal{S}(I,\Gamma)$ is an $\mathcal{F}$-similarity with respect to $\mathcal{G}(\Gamma)$.

Example 6. Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix where $\mathcal{V} \subseteq \mathbb{R}^m$. A variety of similarity generators can be defined by letting d be one of the distances on $\mathbb{R}^m$ from Example 3: $\mathcal{S}(I,\Gamma)\,(\nu,\mu) = f_2\big(\big\{ f_1\big(\big\{ d(\nu(w,\psi),\mu(w,\psi)) \mid \psi \in \mathcal{G}(\Gamma)\big\}\big) \mid w \in W\big\}\big)$, where f_1, f_2 are some aggregation functions and $\mathcal{G}$ is a context generator.

Definition 18. Let $\mathcal{K} = \langle \mathcal{I}, \mathcal{G}, \mathcal{S}, f, g \rangle$ be a setting. The set of the *most plausible frame interpretations* of $\Gamma \neq \emptyset$ with respect to $\mathcal{K}$ is defined as follows:

$$\Delta_{\mathcal{K}}(\Gamma) = \big\{ I \in \mathcal{I} \mid \forall J \in \mathcal{I} \;\; \mathsf{m}^{\mathcal{S}(I,\Gamma)}_{g,f}(I,\Gamma) \leq \mathsf{m}^{\mathcal{S}(J,\Gamma)}_{g,f}(J,\Gamma) \big\}.$$

If $\Gamma = \emptyset$, we define $\Delta_{\mathcal{K}}(\emptyset) = \mathcal{I}$.

Again, $\Delta_{\mathcal{K}}$ is a particular case of (1). Now, for a (multi-valued) possible world semantics $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}} \rangle$ and a corresponding semantic setting $\mathcal{K} = \langle \mathcal{I}, \mathcal{G}, \mathcal{S}, f, g \rangle$ we define, like before, $\Gamma \mid\!\sim_{\mathcal{K}} \psi$ iff $\Delta_{\mathcal{K}}(\Gamma) \subseteq mod_{\mathfrak{I}}(\psi)$.

To show that $\mid\!\sim_{\mathcal{K}}$ is an inconsistency-tolerant variant of $\vdash_{\mathfrak{I}}$, we consider a natural extension to possible-world semantics of the notion of properness:

Definition 19. Let $\mathcal{F} = \langle W, R, \mathcal{M} \rangle$ be a frame for $\mathcal{L}$. A context C is *proper for* ψ (in $\mathcal{F}$), if for every $\nu, \mu \in \Lambda_{\mathcal{F}}$ and every $w \in W$, if $\nu(w,\phi) = \mu(w,\phi)$ for all $\phi \in \mathsf{C}$, then $\nu(w,\psi) = \mu(w,\psi)$ as well. We say that C is proper for Γ if it is proper for every $\psi \in \Gamma$.

Note that, as before, for all the context generators considered above ($\mathcal{G}^{\mathsf{At}}$, $\mathcal{G}^{\mathsf{SF}}$, and $\mathcal{G}^{\mathsf{ID}}$), the corresponding setting is proper.

Proposition 9. *Let $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}} \rangle$ be a multi-valued possible world semantics, and let $\mathcal{K} = \langle \mathcal{I}, \mathcal{G}, \mathcal{S}, f, g \rangle$ be a corresponding proper setting. Then $\mid\!\sim_{\mathcal{K}}$ is an inconsistency-tolerant variant of $\vdash_{\mathfrak{I}}$.*

Example 7. Consider two companies a and b and two investment houses, h_1 and h_2. An investment house h buys shares of a company if the latter is recommended by *all* the investment houses that h knows; otherwise h sells its shares. This can be modeled by a language $\mathcal{L} = \{\Box, \wedge, \neg\}$, and the classical two-valued matrix $\mathcal{M}_{cl}$ with the standard interpretations. We use two atoms in $\mathcal{L}$: $\mathsf{R_a}$ and $\mathsf{R_b}$ (where $\mathsf{R_x}$ intuitively means that 'company x is recommended') and denote by $\mathsf{Buy}(x)$ and by $\mathsf{Sell}(x)$ (for $x \in \{a,b\}$) the formulas $\Box\mathsf{R}_x$, and $\neg\Box\mathsf{R}_x$, respectively.

Suppose now that a third party, call it h_3, wants to detect the trading intentions of the two investment houses. However, h_3 faces two problems. One is that h_3 gets contradictory rumors about these intentions: One rumor says that both houses are going to buy shares of a and b: $\mathsf{Buy(a,b)} = \mathsf{Buy(a)} \wedge \mathsf{Buy(b)}$, and the other rumor claims that they will sell the shares of a. The third party has, then, an inconsistent theory describing the situation $\Gamma = \{\mathsf{Buy(a,b)}, \mathsf{Sell(a)}\}$.

The other problem of h_3 is that it does not know whether h_1 and h_2 have access to each other (but it does know that accessibility must be symmetric and reflexive). This can be represented by two frames $\mathcal{F}_i = \langle W, R_i, \mathcal{M}_{cl}\rangle$ (for $i = 1,2$), in which $W = \{h_1, h_2\}$, $R_1 = \{\langle h_1,h_2\rangle, \langle h_2,h_1\rangle, \langle h_1,h_1\rangle, \langle h_2,h_2\rangle\}$, and $R_2 = \{\langle h_1,h_1\rangle, \langle h_2,h_2\rangle\}$. The corresponding possible world semantics is $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}}\rangle$ with $\mathcal{I} = \cup_{i=1,2}\{\langle \mathcal{F}_i, \nu\rangle \mid \nu \in \Lambda_{\mathcal{F}_i}\}$.

For making plausible decisions despite these uncertainties, h_3 uses $\mid\!\sim_{\mathcal{K}}$, the inconsistency-tolerant variant of $\vdash_{\mathfrak{I}}$, induced by the setting $\mathcal{K} = \langle \mathcal{I}, \mathcal{G}^{\mathsf{At}}, \mathcal{S}, \Sigma, \Sigma\rangle$, where $\mathcal{S}$ is defined by $\mathcal{S}(I,\Gamma)(\nu,\mu) = \Sigma_{w\in W}\, \Sigma_{\psi \in \mathsf{Atoms}(\Gamma)}\, d_U\big(\nu(w,\psi), \mu(w,\psi)\big)$. The relevant frame interpretations are represented in the table below:

I_i	1	2	3	4	5	6	7	8	$\mathsf{m^s}(I_i,\Gamma)$	I_i	1	2	3	4	5	6	7	8	$\mathsf{m^s}(I_i,\Gamma)$
I^1_1	f	f	f	f	0	0	4	4	8	I^2_1	f	f	f	f	0	0	2	2	4
I^1_2	f	f	f	t	0	0	3	3	6	I^2_2	f	f	f	t	0	0	2	1	3
I^1_3	f	f	t	f	0	0	3	3	6	I^2_3	f	f	t	f	0	1	2	1	4
I^1_4	f	f	t	t	0	0	2	2	4	I^2_4	f	f	t	t	0	1	2	0	3
I^1_5	f	t	f	f	0	0	3	3	6	I^2_5	f	t	f	f	0	0	1	2	3
I^1_6	f	t	f	t	0	0	2	2	4	I^2_6	f	t	f	t	0	0	1	1	**2**
I^1_7	f	t	t	f	0	0	2	2	4	I^2_7	f	t	t	f	0	1	1	1	3
I^1_8	f	t	t	t	0	0	1	1	**2**	I^2_8	f	t	t	t	0	1	1	0	**2**
I^1_9	t	f	f	f	0	0	3	3	6	I^2_9	t	f	f	f	1	0	1	2	4
I^1_{10}	t	f	f	t	0	0	2	2	4	I^2_{10}	t	f	f	t	1	0	1	1	3
I^1_{11}	t	f	t	f	1	1	2	2	6	I^2_{11}	t	f	t	f	1	1	1	1	4
I^1_{12}	t	f	t	t	1	1	1	1	4	I^2_{12}	t	f	t	t	1	1	1	0	3
I^1_{13}	t	t	f	f	0	0	2	2	4	I^2_{13}	t	t	f	f	1	0	0	2	3
I^1_{14}	t	t	f	t	0	0	1	1	**2**	I^2_{14}	t	t	f	t	1	0	0	1	**2**
I^1_{15}	t	t	t	f	1	1	1	1	4	I^2_{15}	t	t	t	f	1	1	0	1	3
I^1_{16}	t	t	t	t	1	1	0	0	**2**	I^2_{16}	t	t	t	t	1	1	0	0	**2**

Legend: $1 = \nu_i(h_1, \mathsf{R_a})$, $2 = \nu_i(h_1, \mathsf{R_b})$, $3 = \nu_i(h_2, \mathsf{R_a})$, $4 = \nu_i(h_2, \mathsf{R_b})$,
$5 = \mathsf{m^s}(h_1, \nu_i, \mathsf{Sell(a)})$, $6 = \mathsf{m^s}(h_2, \nu_i, \mathsf{Sell(a)})$, $7 = \mathsf{m^s}(h_1, \nu_i, \mathsf{Buy(a,b)})$,
$8 = \mathsf{m^s}(h_2, \nu_i, \mathsf{Buy(a,b)})$.

It follows that $\Delta_{\mathcal{K}}(\Gamma) = \{I^1_8, I^1_{14}, I^1_{16}, I^2_6, I^2_8, I^2_{14}, I^2_{16}\}$ and so $\Gamma \mid\!\sim_{\mathcal{K}} \mathsf{Buy}(\mathsf{b})$ while $\Gamma \not\mid\!\sim_{\mathcal{K}} \mathsf{Sell}(\mathsf{a})$. The third party anticipates, then, that the other houses will buy b, but it cannot infer that they will sell a.

5 Conclusion

We have introduced a general method of supplementing different logics, based on denotational semantics, with extra apparatus assuring a proper tolerance of inconsistency. This is also the main motivation of other works, such as [1] that introduced distance-based reasoning in deterministic matrices, and [3] that considers distance reasoning in two-valued non-deterministic matrices. This paper generalizes and extends those works in the following senses: First, the notion of similarities is a generalization of the notion of distances, allowing to incorporate a wider range of measures. This also admits the definition of some preferential logics that are not even cumulative (the weakest family of preferential logics considered in the well-known framework of Kraus-Lehmann-Magidor [10]), but which still have some merit for AI applications. Second, our framework captures some common properties shared by inconsistency-tolerant logics based on *any* kind of denotational semantics, whereas the other works handle only specific cases. In particular, new reasoning platforms are investigated, including applications within generalized Kripke-structures, and the extension of the similarity-based approach to many-valued matrices. The latter has not been investigated for Nmatrices, and yields some natural generalizations of well-studied distances.

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