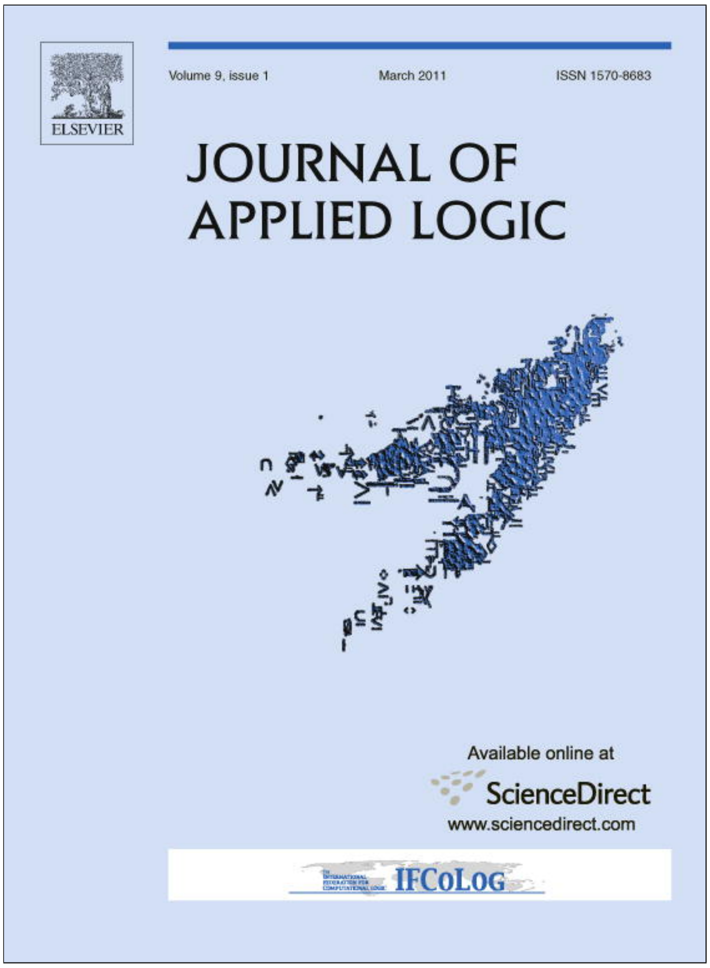
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Simplified forms of computerized reasoning with distance semantics

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ABSTRACT

The semantics of formalisms for representing dynamically evolving and possibly contradictory information is often determined by distance-based considerations. This implies that the reasoning process in such contexts, and the corresponding entailment relations, are induced, in one way or another, from an underlying metric space, in which distances are associated with measurements of plausibility. In this paper we show that in many cases such distance-based entailments can be computerized in a general and modular way. For this, we consider two different approaches for reasoning with distance semantics, one is based on set computations and the other one is defined by rule-based systems. These methods are then applied to some common cases of distance semantics, and the outcome is a specification of some simple and natural algorithms for reasoning in those frameworks. It is shown that what is obtained has some strong ties to well-known SAT-related problems.

1. Introduction

Distance semantics has a prominent role in reflecting the rationality behind the principle of minimal change. This is a primary motif in different areas, such as belief revision [8,19,24,37,46], decision making in the context of social choice theory [34,35,45], and integration of constraint data-sources [31–33,39] or database systems [1,5,16,42]. While there is no consensus about the exact nature of this semantics and the properties that it should satisfy, some particular distance-based approaches have been extensively used in those areas and are more common in practice. As shown in [2,3], many of these semantics have similar representations in terms of entailment relations, so it is not surprising that similar computational forms may be used for providing useful reasoning platforms in those cases.

The goal of this paper is to consider some of the *computational aspects* behind these approaches, that is: to identify some general principles for distance computations and apply them to some specific, nevertheless common test cases. The benefit of this is rather diverse, as distance-based reasoning provides, among others,

– a plausible approach to compromise among several (incomplete and often contradicting) information sources,
– a concise way of expressing operators (strategies) for belief revision and corresponding algorithms for belief update, and
– succinct and expressive representations of preferences for social choice methods, and decision algorithms for group decision making.

In the sequel, we consider the following two reasoning paradigms:

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– *Deductive systems*. This traditional approach for automated reasoning should be taken with care in our context, as many classically valid rules do not hold when distance semantics is involved. To see this, consider the following example.

Example 1. Suppose that one defines an inference system that is based on majority vote considerations (that is, ψ follows from Γ if there are more formulas in Γ implying ψ than those implying $\neg\psi$). As we shall see below, such a system can be represented in terms of distance semantics. Yet, it is evident that this entailment is neither reflexive nor monotonic (p follows from $\{p\}$, but it does not follow from $\{p,\neg p\}$). Moreover, it is also clear that such an entailment is not even closed under logical equivalence as, e.g., $\{p,\neg p\}$ and $\{p,p,\neg p\}$ have different conclusions. This also shows that contraction is not sound in this case either, so 'standard' structural rules may fail as well in the general case.

The discussion above indicates that even sound systems for distance semantics, based on deductive rules, are hard to get. In the sequel, we shall define a sound and complete system for one case (minmax reasoning with drastic distances), and consider some useful sound systems for the other cases.

– *Set computations*. The other approach for reasoning with distance semantics is based on computations of a minimal non-empty intersection of sets of valuations. The elements of each set are equally distant from the formulas that they evaluate, and the minimal non-empty intersection of those sets determines the valuations that are 'closest' to the premises. This idea is based on Grove's well-known spheres system [24], which is often used for defining revisions in terms of set intersections (see, e.g., [17,19,23]). We introduce iterative processes for computing those intersections for different distance-based settings, and show the correspondence between this problem and similar problems in the context of constraint programming.

In the sequel, we consider in greater details three common cases of reasoning with distance semantics: minmax reasoning, reasoning by voting, and reasoning by summation of distances. Each one of these reasoning strategies is augmented with different distances (metrics), and algorithms for computing entailments induced by these settings are provided. It is shown that in some cases distance semantics is reducible to other well-known problems (e.g., a variation of max-SAT), and so off-the-shelf solvers for those problems may be useful for distance-based reasoning as well.

We show also that our methods are easily extendable to the prioritized case, that is, when the premises are augmented with numeric information that represents the relative priority of each assertion. Interestingly, not only the entailment relations are defined in a way which is analogous to the non-prioritized case, but also the computation of their conclusions are derived from (the computations for) the non-prioritized case. For instance, we show that for reasoning with prioritized theories and summation of drastic distances one has to trade max-SAT solvers (as in the non-prioritized case) by solvers for the *weighted* version of max-SAT.

It should be noted that, as the purpose of this paper is to provide a step forward towards a unification and a systematization of computational approaches to distance-based reasoning, some techniques as well as several concrete cases covered in this paper have already been studied in other works. For instance, recently and independently, Gorogiannis and Hunter [23] have proposed some distance-based algorithms for implementing four merging operators using binary decision diagrams (BDDs). We believe, however, that the new results of this paper help to unify and clarify the studied cases. This includes, among others, the use of deductive methods in the context of distance-based reasoning, a computerized approach for reasoning with a variety of distances that can be represented in an inductive way (discussed in Sections 3.1 and 5.1), a comparison showing that the Hamming metric and the discrete metric are similar for distance-based reasoning with clauses (Corollary 4), decidability results for infinite languages (Section 6), incorporation of priorities, and algorithms for handling prioritized information (Section 7).

This paper is a revised and extended version of [6]; the rest of it is organized as follows: In the next section we recall the basic definitions of distance-based semantics by entailment relations as defined in [2,3], and consider the computational complexity of the general entailment problem. Then we consider the computational aspects of reasoning with minmax strategy (Section 3), majority votes (Section 4), and summations of distances (Section 5). In Section 6 we handle infinite languages and show that decidability is preserved in this case, and in Section 7 we investigate the corresponding cases for prioritized formulas and prioritized theories. Conclusions and future work are discussed in Section 8.

2. Distance-based semantics

2.1. Preliminaries

We fix a propositional language $\mathcal{L}$ with a finite set $\mathsf{Atoms} = \{p_1,\dots,p_m\}$ of atomic formulas. A finite multiset of formulas in $\mathcal{L}$ is called a *theory*.[1] For a theory Γ, we denote by $\mathsf{Atoms}(\Gamma)$ the set of atomic formulas that occur in Γ. The set of

[1] As theories may store several instances of the same assertion (coming, e.g., from different sources — see for instance Example 1), they are in the form of multisets rather than sets.

valuations for $\mathcal{L}$ is $\Lambda = \{\langle p_1 : a_1, \ldots, p_m : a_m\rangle \mid a_1, \ldots, a_m \in \{t, f\}\}$. The set of models of a formula ψ is a subset of Λ, defined as follows:

$$\mathsf{mod}(p_i) = \bigl\{\langle p_1 : a_1, \ldots, p_i : t, \ldots, p_m : a_m\rangle \bigm| a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m \in \{t, f\}\bigr\},$$

$$\mathsf{mod}(\neg\psi) = \Lambda \setminus \mathsf{mod}(\psi),$$

$$\mathsf{mod}(\psi \wedge \varphi) = \mathsf{mod}(\psi) \cap \mathsf{mod}(\varphi),$$

$$\mathsf{mod}(\psi \vee \varphi) = \mathsf{mod}(\psi) \cup \mathsf{mod}(\varphi).$$

Given a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, we define $\mathsf{mod}(\Gamma) = \mathsf{mod}(\psi_1) \cap \cdots \cap \mathsf{mod}(\psi_n)$. We say that Γ is *consistent* if $\mathsf{mod}(\Gamma)$ is not empty.

Definition 1 *(Distance functions).* A *pseudo-distance* on U is a total non-negative function $d : U \times U \to \mathbb{N}$,[2] satisfying the following conditions:

- *symmetry*: for all $\nu, \mu \in U$, $d(\nu, \mu) = d(\mu, \nu)$,
- *identity preservation*: for all $\nu, \mu \in U$, $d(\nu, \mu) = 0$ iff $\nu = \mu$.

A pseudo-distance d is a *distance* function on U if it has the following property:

- *triangular inequality*: for all $\nu, \mu, \sigma \in U$, $d(\nu, \sigma) \leqslant d(\nu, \mu) + d(\mu, \sigma)$.

Example 2. It is easy to verify that the following two functions are distances on the space Λ of two-valued valuations on Atoms:

- *The drastic distance*[3]: $d_U(\nu, \mu) = 0$ if $\nu = \mu$ and $d_U(\nu, \mu) = 1$ otherwise.
- *The Hamming distance*[4] [25]: $d_H(\nu, \mu) = |\{p \in \mathsf{Atoms} \mid \nu(p) \neq \mu(p)\}|$.

Definition 2 *(Aggregation functions).* A *numeric aggregation function* is a total function f whose argument is a multiset of real numbers and whose values are real numbers, such that:

(i) f is non-decreasing in the value of its argument,
(ii) $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = x_2 = \cdots = x_n = 0$, and
(iii) $f(\{x\}) = x$ for every $x \in \mathbb{R}$.

In what follows we shall aggregate (non-negative) distance values, and so we shall use functions such as summation, average, maximum, and so forth.

For defining distance-based entailments we recall the definitions in [2,3].

Definition 3. Given a finite set S of elements in U and a (pseudo-) distance d on U, denote: $\max_d S = \max\{d(s_1, s_2) \mid s_1, s_2 \in S\}$.

Definition 4. Given a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, a valuation $\nu \in \Lambda$, a pseudo-distance d, and an aggregation function f, define:

- $d(\nu, \psi_i) = \begin{cases} \min\{d(\nu, \mu) \mid \mu \in \mathsf{mod}(\psi_i)\} & \text{if } \mathsf{mod}(\psi_i) \neq \emptyset, \\ 1 + \max_d \Lambda & \text{otherwise.} \end{cases}$
- $\delta_{d,f}(\nu, \Gamma) = f(\{d(\nu, \psi_1), \ldots, d(\nu, \psi_n)\})$.

Note 1. Intuitively, the function d in the definition above represents ‘distances’ between valuations and formulas, and $\delta_{d,f}$ represents ‘distances’ between valuations and theories. In the two extreme degenerate cases, when the formula ψ is either a tautology or a contradiction, all the valuations are equally distant from ψ. In the other cases, the valuations that are closest to ψ are its models and their distance to ψ is zero. This also implies that $\delta_{d,f}(\nu, \Gamma) = 0$ iff $\nu \in \mathsf{mod}(\Gamma)$ (see [3]).

[2] Usually, distance values are real numbers. Here we restrict ourselves to the natural numbers in order to simplify the algorithms in what follows (these algorithms can be generalized to real numbers, though).

[3] Also known as the *discrete metric*.

[4] Also known as Dalal distance [18] or the *symmetric difference*.

The next definition captures the intuition behind distance semantics that the most relevant valuations for a theory Γ are those that are $\delta_{d,f}$-closest to Γ. Intuitively, then, such valuations model the 'most conceivable states of affairs' with respect to Γ (see also [3,4,31,33]).

Definition 5 *(Most plausible valuations).* The *most plausible* valuations of Γ, with respect to a pseudo-distance d and an aggregation function f on Λ, are defined as follows:

$$\Delta_{d,f}(\Gamma) = \begin{cases} \{\nu \in \Lambda \mid \forall \mu \in \Lambda,\ \delta_{d,f}(\nu,\Gamma) \leqslant \delta_{d,f}(\mu,\Gamma)\} & \text{if} \Gamma \neq \emptyset, \\ \Lambda & \text{otherwise.} \end{cases}$$

Note 2. The distance-like function $\delta_{d,f}$ in Definition 4 is not invariant with respect to the notion of logical equivalence. Yet, we note the following:

(a) According to Definition 5, consistent theories that are logically equivalent share *the same* most plausible valuations (see also Proposition 2 below). Inconsistent theories, on the other hand, are all logically equivalent. Thus, any definition of most plausible valuations that makes a distinction among inconsistent theories cannot preserve logical equivalence, but it should employ some other considerations. To see this consider, for instance, the theories $\Gamma_1 = \{p \wedge q, \neg p \vee \neg q\}$ and $\Gamma_2 = \{p, q, \neg p \vee \neg q\}$. In the context of mediator systems, Γ_1 and Γ_2 may reflect the results of a poll among two and three different sources, respectively (each source contributes one formula). According to this view, in the second case two out of three experts support (logically entail) the conclusion $p \vee q$, while in the first case there is no majority among the experts for this conclusion. This consideration shows that in certain circumstances the distance-based criterion may not be syntactically independent, so in such cases standard rewriting rules should *not* be employed (recall also Example 1). Clearly, this has a major effect on the algorithms for automated reasoning, developed for particular distance semantics (see also [3]).

(b) Despite of the above, we note that distance-based reasoning *is* invariant with respect to more restricted notions of logical equivalence, even when the underlying theories are not consistent. To see this, consider the following definition.

Definition 6 *(Bi-equivalence).* (See [30].) A theory Γ is *bijection-equivalent* to a theory Γ' (bi-equivalent, for short), if there is a bijection $\sigma : \Gamma \to \Gamma'$ such that for all $\psi \in \Gamma$, $\mathsf{mod}(\psi) = \mathsf{mod}(\sigma(\psi))$.[5]

For instance, the theory Γ_1, considered in item (a) of this note, is not bi-equivalent to Γ_2, but it is bi-equivalent to $\Gamma_3 = \{\neg(\neg p \vee \neg q), \neg p \vee \neg q\}$.

Proposition 1. *Let Γ and Γ' be bi-equivalent theories. Then for every pseudo-distance d and aggregation function f, $\Delta_{d,f}(\Gamma) = \Delta_{d,f}(\Gamma')$.*

Proof. Given a pseudo-distance d and an aggregation function f. Let Γ and Γ' be bi-equivalent theories, and let σ be a corresponding bijection between them. Then for every $\psi \in \Gamma$, $\mathsf{mod}(\psi) = \mathsf{mod}(\sigma(\psi))$, and so, for every $\nu \in \Lambda$, $d(\nu,\psi) = d(\nu,\sigma(\psi))$. Now, since the number of occurrences of ψ in Γ is the same as the number of occurrences of $\sigma(\psi)$ in Γ', we have that for every $\nu \in \Lambda$, $\delta_{d,f}(\nu,\Gamma) = \delta_{d,f}(\nu,\Gamma')$. Thus, $\nu \in \Delta_{d,f}(\Gamma)$ iff $\nu \in \Delta_{d,f}(\Gamma')$. $\square$

Distance-based entailments are now defined as follows:

Definition 7 *(Distance-based entailments).* For a pseudo-distance d and an aggregation function f, define $\Gamma \models_{d,f} \psi$ if $\Delta_{d,f}(\Gamma) \subseteq \mathsf{mod}(\psi)$.

According to our distance-based entailments, then, conclusions should follow from *all* of the most plausible valuations of the premises.

Example 3. Let $\Gamma = \{p, \neg p, q\}$. According to classical logic, this theory is not consistent, and so everything follows from it. Yet, as q is not related to the contradiction in Γ, conclusions such as $\neg q$ look counterintuitive in this case.

This anomaly is lifted in our framework. To see this consider, for instance, the drastic distance d_U and the summation function Σ. We have that

$$\delta_{d_U,\Sigma}\big(\langle p:t, q:t\rangle, \Gamma\big) = 1, \qquad \delta_{d_U,\Sigma}\big(\langle p:f, q:t\rangle, \Gamma\big) = 1,$$

[5] Recall that Γ and Γ' are multisets, so a bijection between them is an injection and a surjection between their root sets (i.e., a bijection between the corresponding sets, obtained by reducing multiple occurrences of each element to a single occurrence of the same element), and such that the number of occurrences in Γ and in Γ' of elements that are matched by the bijection, is the same (see [11]).

Table 1
δ-Distances between the elements of Λ and Γ (Example 4).

	p	q	r	$d_H(\nu_i, \neg p \vee \neg q)$	$d_H(\nu_i, p \wedge q \wedge r)$	$d_H(\nu_i, \neg p \vee \neg r)$	$\delta_{d_H,\max}(\nu_i, \Gamma)$	$\delta_{d_H,\Sigma}(\nu_i, \Gamma)$
ν_1	t	t	t	1	0	1	**1**	2
ν_2	t	t	f	1	1	0	**1**	2
ν_3	t	f	t	0	1	1	**1**	2
ν_4	t	f	f	0	2	0	2	2
ν_5	f	t	t	0	1	0	**1**	**1**
ν_6	f	t	f	0	2	0	2	2
ν_7	f	f	t	0	2	0	2	2
ν_8	f	f	f	0	3	0	3	3

while

$$\delta_{d_U,\Sigma}\big(\langle p:t, q:f\rangle, \Gamma\big) = 2, \qquad \delta_{d_U,\Sigma}\big(\langle p:f, q:f\rangle, \Gamma\big) = 2,$$

thus valuations in which q is assigned f are more distant from Γ than valuations in which q is assigned t. It follows that

$$\Gamma \models_{d_U,\Sigma} q, \qquad \Gamma \not\models_{d_U,\Sigma} \neg q, \qquad \Gamma \not\models_{d_U,\Sigma} p, \qquad \Gamma \not\models_{d_U,\Sigma} \neg p,$$

as intuitively expected. Similar results are also obtained, e.g., for $\models_{d_H,\Sigma}$.

Example 4. Consider the theory $\Gamma = \{\neg p \vee \neg q, p \wedge q \wedge r, \neg p \vee \neg r\}$. Clearly, Γ is not consistent, and again classical logic is useless in this case. Table 1 lists the distances between the elements of Λ and Γ in two common cases where Hamming distance is incorporated. It follows that $\Delta_{d_H,\max}(\Gamma) = \{\nu_1, \nu_2, \nu_3, \nu_5\}$ while $\Delta_{d_H,\Sigma}(\Gamma) = \{\nu_5\}$. Thus, for instance, $\Gamma \models_{d_H,\Sigma} \neg p$, while $\Gamma \not\models_{d_H,\max} \neg p$.

2.2. Computing distance-based entailments

The following proposition states two important observations about reasoning with distance semantics.[6]

Proposition 2. *Denote by $\models$ the classical (two-valued) entailment. For every pseudo-distance d and aggregation function f,*

(a) *if Γ is satisfiable, then $\Gamma \models_{d,f} \psi$ iff $\Gamma \models \psi$,*
(b) *for every Γ there is a formula ψ such that $\Gamma \not\models_{d,f} \psi$.*

Proof. [outline] The first item follows from the fact that for every formula ψ and a set of formulas Γ we have that $d(\nu, \psi) = 0$ iff $\nu \in \mathsf{mod}(\psi)$ and $\delta_{d,f}(\nu, \Gamma) = 0$ iff $\nu \in \mathsf{mod}(\Gamma)$, thus Γ is satisfiable iff $\Delta_{d,f}(\Gamma) = \mathsf{mod}(\Gamma)$ (see also [3]). The second part follows from the fact that for every Γ, $\Delta_{d,f}(\Gamma) \neq \emptyset$ (indeed, since Λ is finite, there always exist valuations that are minimally $\delta_{d,f}$-distant from Γ). $\Box$

Taken together, the two items of Proposition 2 imply that every entailment relation induced by our framework coincides with the classical entailment with respect to consistent premises, while (unlike the classical case) distance-based reasoning is not trivial with respect to inconsistent theories.

Note 3. It is interesting to compare this result with the approach taken in belief revision in general and in the context of constraint merging operators [33] in particular. The latter assumes that belief sets are consistent and associates a belief set with the conjunction of its elements, while we do not do so: Proposition 2 indicates that distance-based entailments are faithful to classical logic only for consistent theories and are paraconsistent otherwise, thus inconsistency is not eliminated but is rather tolerated. This is also the reason why in our framework one cannot just take the conjunction of the premises (otherwise, for instance, one would not be able to distinguish between $\{p, p, \neg p\}$ and $\{p, \neg p, \neg p\}$, which have *different* consequences in, e.g., a majority-vote semantics; see Section 4). Moreover, merging operators regard the integrity constraints as preferable to the formulas of the belief set, while in our case all the formulas in a theory have the same importance. Priorities among formulas may, nevertheless, be simulated in our framework as well. This subject is considered in Section 7.

Proposition 2 has also some important consequences regarding computational aspects and the complexity of distance-based reasoning. First, it implies that one cannot hope for better complexity results than those for the classical propositional logic, as for consistent premises the entailment problem is coNP-Complete. On the other hand, it is clear from Definition 7

[6] For some other basic properties of entailments induced by distance semantics see, e.g., [3].

and the fact that Λ is finite, that for polynomially (or exponentially) bounded computable distances and aggregation functions, distance-based reasoning for finite propositional languages is in EXP, i.e., it is decidable and has (at most) exponential complexity.[7]

The purpose of this work is, therefore, to consider some useful distance-based settings for which there are some *practical* ways of computing entailments. In particular, we consider some cases in which distance-based reasoning is reducible to the question of satisfiability, and so off-the-shelf SAT-solvers may be incorporated for automated computations of distance-based consequences. For this, we first need the following definitions:

Definition 8. Let d be a pseudo-distance. The function $\mathcal{R}_d : \mathcal{L} \times \mathbb{N} \to 2^{\Lambda}$ is defined, for every formula ψ in $\mathcal{L}$ and every $i \in \mathbb{N}$, by

$$\mathcal{R}_d(\psi, i) = \{\nu \in \Lambda \mid \exists \mu \in \mathsf{mod}(\psi)\ d(\nu, \mu) \leqslant i\}.$$

Intuitively, $\mathcal{R}_d(\psi, i)$ is the 'sphere' of the valuations whose d-distance from ψ is bounded by i. Accordingly, the i-th 'buttonhole' of ψ with respect to d is defined for every $i \in \mathbb{N}^+$ as follows:

- $\mathcal{R}_d^0(\psi) = \mathcal{R}_d(\psi, 0)$,
- $\mathcal{R}_d^i(\psi) = \mathcal{R}_d(\psi, i) \setminus \mathcal{R}_d(\psi, i-1)$.

Note 4. Clearly, we have that $\mathcal{R}_d(\psi, i) = \{\nu \in \Lambda \mid d(\nu, \psi) \leqslant i\}$. In terms of set inclusion, for every formula ψ, the sequence $\mathcal{R}_d(\psi, r)$ is non-decreasing in the 'radius' i. Also, for every satisfiable ψ and $i \in \mathbb{N}$ it holds that $\nu \in \mathcal{R}_d^i(\psi)$ iff $d(\nu, \psi) = i$. Thus, for a satisfiable ψ, we have that $\mathcal{R}_d^0(\psi) = \mathsf{mod}(\psi) = \Delta_{d,f}(\psi)$, and $\mathcal{R}_d(\psi, k) = \Lambda$ for $k = \mathsf{max}_d \Lambda$. If ψ is not satisfiable, then $\mathcal{R}_d(\psi, i) = \mathcal{R}_d^i(\psi) = \emptyset$ for all i.

The notion of spheres may be traced back to Lewis's analysis of counterfactual conditionals in terms of the theory of possible worlds [38]. This notion is also a primary concept behind Grove's system [24], which is extensively studied in the context of belief revision (see also [19] and [23]). In [12], a similar concept is called *dilation*, and it is used for defining a variety of revision and merging operators. That concept also induces a dual notion, called *erosion* (see [12]), which can be defined in our terms as follows:

$$\mathcal{E}_d(\psi, i) = \Lambda - \mathcal{R}_d(\neg\psi, i) = \{\nu \in \Lambda \mid d(\nu, \neg\psi) > i\}.$$

In [34, Section 4] spheres are used for preference representation and group decision making, and as such they serve as a formal tool for bridging between distances-based representations and weighted formulas.

Spheres are also closely related to *supermodels* [22]. Given two integers i, j, a valuation $\nu \in \Lambda$ is an (i, j)-supermodel of a formula ψ, if for every consistent conjunction ϕ of at most i literals, there exists a valuation $\mu \in \mathsf{mod}(\{\psi, \phi\})$ and $d(\nu, \mu) \leqslant j$. In this sense, $\mathcal{R}_d(\psi, n)$ consists of the $(0, n)$-supermodels of ψ.

The following lemma and definition will be useful in what follows:

Lemma 1. *If every formula in $\Gamma = \{\psi_1, \ldots, \psi_n\}$ is satisfiable, then there is some $k \geqslant 0$, such that $\bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, k)$ is not empty.*

Proof. By Note 4, for every $1 \leqslant i \leqslant n$ there is a $k_i \leqslant \mathsf{max}_d \Lambda$ such that for every $j \geqslant k_i$, $\mathcal{R}_d(\psi_i, j) = \Lambda$. Let $k = \mathsf{max}\{k_i \mid 1 \leqslant i \leqslant n\}$. Then $\mathcal{R}_d(\psi, k) = \Lambda$ for all $1 \leqslant i \leqslant n$, and so $\bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, j) = \Lambda$. $\square$

Definition 9 *(Inductively representable distances).* A pseudo-distance d is called *inductively representable*, if there is a computable function $\mathsf{G} : 2^{\Lambda} \to 2^{\Lambda}$ such that for every formula ψ and every $i \in \mathbb{N}$, $\mathcal{R}_d(\psi, i) = \mathsf{G}(\mathcal{R}_d(\psi, i-1))$. G is called an *inductive representation* of d.

As Propositions 8 and 11 below show, both d_U and d_H are inductively representable.

3. MinMax reasoning

In this section we study distance-based reasoning by min-max methods, that is: minimization of maximal distances. This kind of reasoning may be viewed as a skeptical approach, as it takes into account the best options (minimal values) among the worst cases (maximal distances). Distance-entailments that correspond to this type of reasoning are induced by the max aggregation function.

[7] See [31,32] for some more detailed results on the complexity of computing distance-based operators.

```
MPV(G, {ψ_1, ..., ψ_n})
  /* Most Plausible Valuations of {ψ_1, ..., ψ_n} w.r.t. d and max */
  /* G – an inductive representation of d */
    for i ∈ {1, ..., n}, let X_i ← mod(ψ_i);
    if X_j is empty for some j ∈ {1, ..., n}, return Λ;
    while (X_1 ∩ ··· ∩ X_n) is empty, do:
        for i ∈ {1, ..., n}, let X_i ← G(X_i);
    return (X_1 ∩ ··· ∩ X_n);
```

Fig. 1. Computing the most plausible valuations of $\{\psi_1, \ldots, \psi_n\}$ w.r.t. d and max.

3.1. Inductively representable distances

Min-max distance-based reasoning is characterized as follows:

Proposition 3. *For any pseudo-distance d and theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$,*

(a) *If there is a non-satisfiable element in Γ, then $\Delta_{d,\max}(\Gamma) = \Lambda$.*
(b) *If all the elements in Γ are satisfiable, then $\Delta_{d,\max}(\Gamma) = \bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, m)$, where m is the minimal number such that $\bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, m)$ is not empty.*[8]

Proof. For item (a), let ψ_i be a non-satisfiable formula in Γ. Then, for every $\mu \in \Lambda$, $d(\mu, \psi_i) = 1 + \max_d \Lambda$. Thus, for all $\mu_1, \mu_2 \in \Lambda$ it holds that $\delta_{d,\max}(\mu_1, \Gamma) = \delta_{d,\max}(\mu_2, \Gamma)$, and so $\Delta_{d,\max}(\Gamma) = \Lambda$.

For item (b), note that by Lemma 1 the number m defined in the proposition indeed exists. Now, let $\mu \in \bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, m)$. As $\mu \in \mathcal{R}_d(\psi_i, m)$ for every $1 \leqslant i \leqslant n$, we have that $d(\mu, \psi_i) \leqslant m$ for all $\psi_i \in \Gamma$, and so $\delta_{d,\max}(\mu, \Gamma) \leqslant m$. If $\mu \notin \Delta_{d,\max}(\Gamma)$, then there is a valuation $\nu \in \Lambda$ such that $\delta_{d,\max}(\nu, \Gamma) < \delta_{d,\max}(\mu, \Gamma) \leqslant m$. Thus, $\max\{d(\nu, \psi_1), \ldots, d(\nu, \psi_n)\} < m$, and so $d(\nu, \psi_i) = k_i < m$ for every $1 \leqslant i \leqslant n$. Now, let $k = \max\{k_1, \ldots, k_n\}$. Then $\nu \in \mathcal{R}_d(\psi_i, k)$ for every $1 \leqslant i \leqslant n$, and so $\bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, k)$ is non-empty for some $k < m$, in contradiction to the minimality of m.

For the converse, let $\mu \in \Delta_{d,\max}(\Gamma)$. As $d(\mu, \psi_i) \leqslant \delta_{d,\max}(\mu, \Gamma)$ for all $1 \leqslant i \leqslant n$, necessarily $\mu \in \mathcal{R}_d(\psi_i, \delta_{d,\max}(\mu, \Gamma))$ and so $\bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, \delta_{d,\max}(\mu, \Gamma))$ is non-empty. Suppose for a contradiction that $m = \delta_{d,\max}(\mu, \Gamma)$ is not the minimal number, for which $\bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, m)$ is non-empty. Then there is some $k < m$ such that $\bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, k)$ is non-empty, and let $\nu \in \bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, k)$. Then $\nu \in \mathcal{R}_d(\psi_i, k)$ for every $1 \leqslant i \leqslant n$, and so $d(\nu, \psi_i) \leqslant k$ for every $\psi_i \in \Gamma$. Thus, $\delta_{d,\max}(\nu, \Gamma) \leqslant k < m = \delta_{d,\max}(\mu, \Gamma)$, a contradiction to our assumption that $\mu \in \Delta_{d,\max}(\Gamma)$. □

By the proof of part (b) of the last proposition, we have the following corollary, which emphasizes the fact that this kind of distance reasoning applies a 'min-max' approach:

Corollary 1. *For every pseudo-distance d and a theory Γ of satisfiable formulas it holds that $\Delta_{d,\max}(\Gamma) = \bigcap_{1\leqslant i\leqslant n} \mathcal{R}_d(\psi_i, m)$, where $m = \min\{\delta_{d,\max}(\nu, \Gamma) \mid \nu \in \Lambda\}$.*

In case that d is inductively representable by G, the results above induce the iterative procedure in Fig. 1, for computing $\Delta_{d,\max}(\Gamma)$.

Proposition 4. *If d is inductively representable by* G*, then for every theory Γ, the procedure* **MPV**(G, Γ) *of Fig. 1 terminates and computes $\Delta_{d,\max}(\Gamma)$.*

Proof. It is easy to see that in the i-th iteration it holds that $X_j = \mathcal{R}_d(\psi_j, i)$ for every $1 \leqslant j \leqslant n$. Hence, by Lemma 1, the condition in the loop is satisfied after a finite number of iterations and so the procedure always terminates. Also, by Proposition 3, the procedure returns $\Delta_{d,\max}(\Gamma)$. □

Note 5. If there are tautologies among the premises, they can be discarded in the computations above, as the following proposition shows that in the min-max reasoning tautologies have no effect on the consequences.

Proposition 5. *If $\varphi \in \Gamma$ is a tautology, then $\Delta_{d,\max}(\Gamma) = \Delta_{d,\max}(\Gamma \setminus \{\varphi\})$.*

Proof. Let $\Gamma = \{\psi_1, \ldots, \psi_n, \varphi\}$. If $n = 0$, then as all the valuations are equally distant from a tautology, we have that $\Delta_{d,\max}(\Gamma) = \Lambda = \Delta_{d,\max}(\emptyset)$, and so the claim holds. If $n \geqslant 1$, then for every valuation $\mu \in \Lambda$, $\delta_{d,\max}(\mu, \Gamma) =$

[8] Part (b) of Proposition 3 in also noted (without a proof) in [12], for the particular case where $d = d_H$.

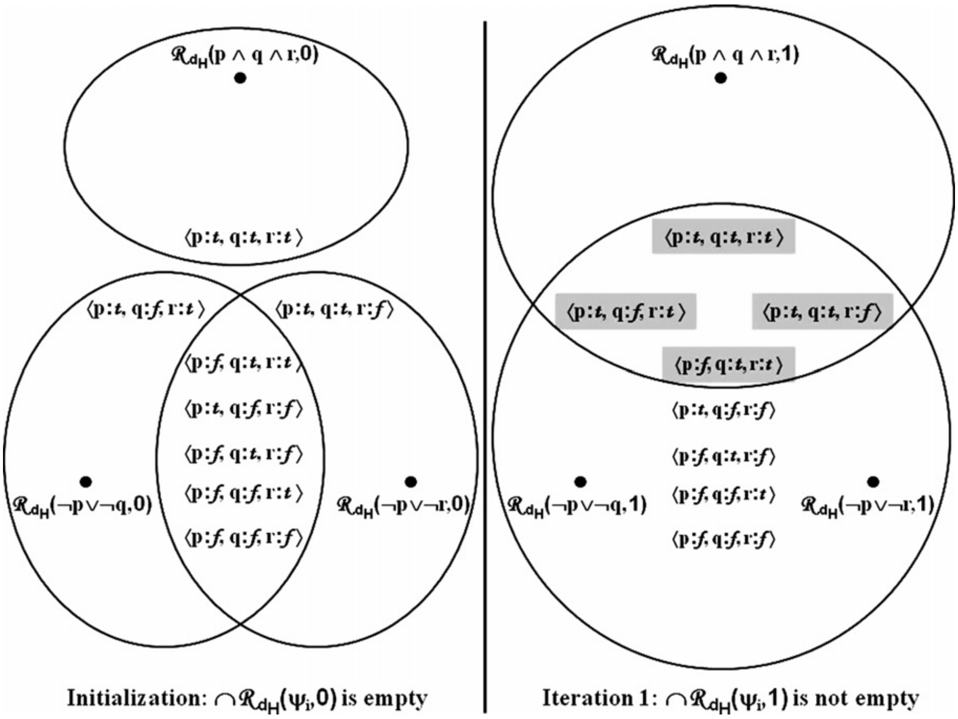

Fig. 2. Iterative computations of $\Delta_{d_H,\max}(\Gamma)$ (Example 5) by intersection of spheres.

$\max\{d(\mu,\psi_1),\dots,d(\mu,\psi_n),d(\mu,\varphi)\} = \max\{d(\mu,\psi_1),\dots,d(\mu,\psi_n),0\} = \max\{d(\mu,\psi_1),\dots,d(\mu,\psi_n)\} = \delta_{d,\max}(\mu,\Gamma\setminus\{\varphi\})$. Thus, $\Delta_{d,\max}(\Gamma) = \Delta_{d,\max}(\Gamma\setminus\{\varphi\})$. $\square$

Example 5. Consider again the theory Γ of Example 4. Fig. 2 shows a graphical representation of the computations of **MPV** in case of minmax reasoning with Hamming distances. After one iteration, the intersections of the spheres becomes non-empty, and so, by Proposition 3, its elements (the shadowed valuations in Fig. 2) are the most plausible valuations of Γ (cf. Example 4).

3.2. Drastic distances

In this section we concentrate on minmax reasoning with drastic distances (see Example 2).

Proposition 6. *For every theory $\Gamma = \{\psi_1,\dots,\psi_n\}$,*

$$\Delta_{d_U,\max}(\Gamma) = \begin{cases} \mathsf{mod}(\Gamma) & \text{if } \mathsf{mod}(\Gamma) \neq \emptyset, \\ \Lambda & \text{otherwise}. \end{cases}$$

Proof. If $\mathsf{mod}(\Gamma) \neq \emptyset$, then for *every* distance d and aggregation function f, $\Delta_{d,f}(\Gamma) = \mathsf{mod}(\Gamma)$ (see [3]). Otherwise, there is at least one element in Γ that is not satisfiable, and so, for every valuation $\mu \in \Lambda$, $\delta_{d_U,\max}(\mu,\Gamma) = \max\{d_U(\mu,\psi_1),\dots,d_U(\mu,\psi_n)\} = 1$. It follows, then, that in this case $\Delta_{d_U,\max}(\Gamma) = \Lambda$. $\square$

Proposition 6 implies, in particular, that only tautological formulas follow from inconsistent theories:

Corollary 2. *Let Γ be an inconsistent theory. Then $\Gamma \models_{d_U,\max} \psi$ iff ψ is a tautology.*

Proof. Suppose that Γ is not consistent. By Proposition 6, $\Delta_{d_U,\max}(\Gamma) = \Lambda$, and so $\Gamma \models_{d_U,\max} \psi$ iff $\Delta_{d_U,\max}(\Gamma) \subseteq \mathsf{mod}(\psi)$, iff $\mathsf{mod}(\psi) = \Lambda$, iff ψ is a tautology. $\square$

While reasoning with the drastic distance and the max aggregation function are well studied in the literature of belief revision, merging operators and data integration, Proposition 2 and Corollary 2 indicate that reasoning with the drastic distance under the minmax strategy has a somewhat 'crude nature' and is, ultimately, a classical one: either the set of premises is classically consistent, in which case the set of conclusions coincides with that of the classical entailment, or, in case of contradictory premises, only tautologies are entailed. It follows that in this case questions of satisfiability and logical

– Axioms:

$$\emptyset : \Lambda \qquad (A_0)$$

$$\{\psi\} : \mathsf{mod}(\psi) \quad \text{if } \mathsf{mod}(\psi) \neq \emptyset \quad (A_1)$$

$$\{\psi\} : \Lambda \quad \text{if } \mathsf{mod}(\psi) = \emptyset \quad (A_2)$$

– Inference rules:

$$\frac{\Gamma_1 : V_1 \quad \Gamma_2 : V_2}{\Gamma_1 \cup \Gamma_2 : V_1 \cap V_2} \quad \text{if } \mathsf{mod}(\Gamma_1 \cup \Gamma_2) \neq \emptyset \quad (I_1)$$

$$\frac{\Gamma_1 : V_1 \quad \Gamma_2 : V_2}{\Gamma_1 \cup \Gamma_2 : \Lambda} \quad \text{if } \mathsf{mod}(\Gamma_1 \cup \Gamma_2) = \emptyset \quad (I_2)$$

Fig. 3. The system $\mathbf{S}^{\mathbf{u}}_{\max}$.

entailment are reducible to the similar problems in standard propositional logic, and distance considerations do not add extra computational complications.

Next, we provide two methods for computerized reasoning in this case. One is based on the procedure **MPV** defined in the previous section, and the other is based on deduction systems.

By Proposition 6 we have the following result:

Proposition 7. *If ψ is satisfiable, then $\mathcal{R}_{d_U}(\psi, 0) = \mathsf{mod}(\psi)$ and $\mathcal{R}_{d_U}(\psi, i) = \Lambda$ for every $i > 0$.*

Proposition 8. *The function $\mathsf{G}_U : 2^\Lambda \to 2^\Lambda$ defined by $\mathsf{G}_U(V) = \Lambda$ for all $V \subseteq \Lambda$, is an inductive representation of d_U.*

Proof. Immediate from Proposition 7. $\Box$

By Proposition 8, d_U is inductively representable, and so, by Proposition 4, for every theory Γ, **MPV**(G_U, Γ) terminates and returns $\Delta_{d_U,\max}(\Gamma)$.

Another way of computing consequences of the entailment relation $\models_{d_U,\max}$ is by the deduction system $\mathbf{S}^{\mathbf{u}}_{\max}$, defined in Fig. 3. This system manipulates expressions of the form $\Gamma : V$, where Γ is a theory and $V \subseteq \Lambda$.[9]

Definition 10. For a theory Γ and a set $V \subseteq \Lambda$, denote by $\vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \Gamma : V$ that $\Gamma : V$ is provable in $\mathbf{S}^{\mathbf{u}}_{\max}$, and by $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$ that $\vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \Gamma : V$ for some $V \subseteq \mathsf{mod}(\psi)$.

Example 6. Let $\Gamma = \{p, q, \neg p \wedge \neg q\}$. We show that $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} p \vee \neg p$. Indeed,

$$\frac{\neg p \wedge \neg q : \{\langle p : f, q : f\rangle\} \qquad \dfrac{p : \{\langle p : t, q : t\rangle, \langle p : t, q : f\rangle\} \quad q : \{\langle p : t, q : t\rangle, \langle p : f, q : t\rangle\}}{p, q : \{\langle p : t, q : t\rangle\}}}{p, q, \neg p \wedge \neg q : \Lambda}.$$

Thus, $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$ iff $\mathsf{mod}(\psi) = \Lambda$. In particular, as $\mathsf{mod}(p \vee \neg p) = \mathsf{mod}(p) \cup \mathsf{mod}(\neg p) = \Lambda$, we have that $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} p \vee \neg p$.

Proposition 9 *(Soundness and completeness). $\Gamma \models_{d_U,\max} \psi$ iff $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$.*

For the proof of Proposition 9 we need the following lemma:

Lemma 2. *For every theory Γ we have that $\vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \Gamma : V$ iff $V = \Delta_{d_U,\max}(\Gamma)$.*

Proof. Suppose first that $\vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \Gamma : V$. We show by induction on the length of the proof that $V = \Delta_{d_U,\max}(\Gamma)$. Indeed, the claim clearly holds if $\Gamma : V$ is an axiom. If $\Gamma : V$ is obtained by (I_1), then $\Gamma = \Gamma_1 \cup \Gamma_2$, and as $\mathsf{mod}(\Gamma_1 \cup \Gamma_2) \neq \emptyset$, also $\mathsf{mod}(\Gamma_1) \neq \emptyset$ and $\mathsf{mod}(\Gamma_2) \neq \emptyset$. Hence, by induction hypothesis, $\Delta_{d_U,f}(\Gamma_1) = V_1 = \mathsf{mod}(\Gamma_1)$ and $\Delta_{d_U,f}(\Gamma_2) = V_2 = \mathsf{mod}(\Gamma_2)$. Thus, by Proposition 6, $V = V_1 \cap V_2 = \mathsf{mod}(\Gamma_1) \cap \mathsf{mod}(\Gamma_2) = \mathsf{mod}(\Gamma) = \Delta_{d_U,\max}(\Gamma)$. If $\Gamma : V$ is obtained by (I_2), then $\Gamma = \Gamma_1 \cup \Gamma_2$ and $\mathsf{mod}(\Gamma_1 \cup \Gamma_2) = \emptyset$. By Proposition 6 again, $V = \Lambda = \Delta_{d_U,\max}(\Gamma)$.

For the converse, it is easy to see by induction on the size of a non-empty set Γ, that there is some non-empty set $V \subseteq \Lambda$, such that $\vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \Gamma : V$. Thus, for every theory Γ, it holds that $\Gamma : V$ is provable in $\mathbf{S}^{\mathbf{u}}_{\max}$ for some $V \neq \emptyset$. On the other hand, we have already shown that if $\vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \Gamma : V$, then $V = \Delta_{d_U,\max}(\Gamma)$. It follows, then, that $\Gamma : \Delta_{d_U,\max}(\Gamma)$ is provable in $\mathbf{S}^{\mathbf{u}}_{\max}$. $\Box$

[9] Here, $\Gamma : V$ is a kind of a 'labeled expression', in which V is a string associated with a set of valuations.

Proof of Proposition 9. Suppose that $\Gamma \models_{d_U,\max} \psi$. Then $\Delta_{d_U,\max}(\Gamma) \subseteq \mathsf{mod}(\psi)$. By Lemma 2, $\Gamma : \Delta_{d_U,\max}(\Gamma)$ is provable in $\mathbf{S}^{\mathbf{u}}_{\max}$, and so $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$. Conversely, if $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$, then there is some $V \subseteq \mathsf{mod}(\psi)$ such that $\Gamma : V$ is provable in $\mathbf{S}^{\mathbf{u}}_{\max}$. By Lemma 2 again, $V = \Delta_{d_U,\max}(\Gamma)$. Thus, $\Delta_{d_U,\max}(\Gamma) \subseteq \mathsf{mod}(\psi)$, and so $\Gamma \models_{d_U,\max} \psi$. $\square$

Proposition 9 shows that, eventually, reasoning with the drastic distance and the maximum function boils down to satisfiability checking. Apart of this observation, one could use Proposition 9 for showing more concrete properties of the computations in this case. For instance, by Proposition 9 it is evident that for checking entailments in this case, one has to consider only those premises that are 'relevant' for the conclusion. Such a situation is described below:

Corollary 3. *Suppose that a theory Γ can be partitioned to two independent subtheories. I.e., there are theories Γ' and Γ'' so that $\Gamma = \Gamma' \cup \Gamma''$ and $\mathsf{Atoms}(\Gamma') \cap \mathsf{Atoms}(\Gamma'') = \emptyset$. Then for every formula ψ such that $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\Gamma')$, we have that $\Gamma \models_{d_U,\max} \psi$ iff $\Gamma' \models_{d_U,\max} \psi$.*

Proof. By the conditions of the corollary it is evident that if $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$, there is a derivation process for ψ that does not involve any formula from Γ'' (this follows directly from the definition of the rules of $\mathbf{S}^{\mathbf{u}}_{\max}$), and so $\Gamma' \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$ as well. Thus, by Proposition 9, $\Gamma \models_{d_U,\max} \psi$ iff $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$, iff $\Gamma' \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$, iff $\Gamma' \models_{d_U,\max} \psi$. $\square$

3.3. Hamming distances

In this section we investigate reasoning with the min-max approach, together with Hamming distances (denoted d_H; see Example 2). For this, we need the following (technical) definitions:

Definition 11. Denote: $\mathbf{K}^n_i = \sum_{j=1}^{i} \binom{n}{j}$.

Definition 12 *(i-Validated formulas).* A formula ψ is called *i-validated*, if among the $2^{|\mathsf{Atoms}(\psi)|}$ valuations on $\mathsf{Atoms}(\psi)$, at most $\mathbf{K}^{|\mathsf{Atoms}(\psi)|}_i$ valuations do not satisfy ψ.

Note that an i-validated formula ψ is also j-validated, for all $1 \leqslant i \leqslant j \leqslant |\mathsf{Atoms}(\psi)|$.

Example 7. Clearly, any tautology is 1-validated (and so it is i-validated for any i). Also, a literal (i.e., an atomic formula or its negation) is 1-validated. This implies the following result:

Lemma 3. *Every clause is* 1*-validated.*

Proof. Follows from the fact that a disjunction of literals is either a tautology or is falsified only by the valuation that assigns f to all its literals. In both cases the formulas are 1-validated. $\square$

Checking the i-validity of a formula is more cumbersome when conjunctions are introduced. For instance, $\psi = p \wedge q$ is 2-validated but not 1-validated, as $|\mathsf{Atoms}(\psi)| = 2$, $\mathbf{K}^2_1 = 2$, $\mathbf{K}^2_2 = 3$, and there are three valuations on $\{p, q\}$ that falsify ψ.

Now we can identify some important cases in which reasoning with Hamming distances coincides with reasoning with the drastic distance.

Proposition 10. *If Γ consists of* 1*-validated formulas, then for every aggregation function f and every formula ψ, $\Gamma \models_{d_H,f} \psi$ iff $\Gamma \models_{d_U,f} \psi$.*

The main observation for Proposition 10 is the following:

Lemma 4. *If a formula ψ is i-validated, then for every $\nu \in \Lambda$, $d_H(\nu, \psi) \leqslant i$.*

Proof. Suppose that ψ is i-validated and let $n = |\mathsf{Atoms}(\psi)|$. If ν is a model of ψ, then $d_H(\nu, \psi) = 0$ and we are done. Suppose then that $\nu(\psi) = f$. As ψ is i-validated, it is in particular satisfiable, so we need to show that there exists a model μ of ψ, such that $d_H(\nu, \mu) \leqslant i$. Indeed, denote by $\sigma^{\downarrow \mathsf{Atoms}(\psi)}$ the restriction of a valuation σ to $\mathsf{Atoms}(\psi)$. Since ψ is i-validated, there are at most $\mathbf{K}^n_i$ partial valuations on $\mathsf{Atoms}(\psi)$ that do not satisfy ψ, and one of them is of course $\nu^{\downarrow \mathsf{Atoms}(\psi)}$. Denote this set of partial valuations by S. Then $|S| \leqslant \mathbf{K}^n_i$. Now, let T be the set of all partial valuations on $\mathsf{Atoms}(\psi)$ that differ in at most i atoms from $\nu^{\downarrow \mathsf{Atoms}(\psi)}$. As for every $j \geqslant 1$ the number of partial valuations on $\mathsf{Atoms}(\psi)$ that differ in *exactly* j atoms from $\nu^{\downarrow \mathsf{Atoms}(\psi)}$ is equal to n over j, and since $\nu^{\downarrow \mathsf{Atoms}(\psi)} \in T$, we have that $|T| = \mathbf{K}^n_i + 1$. By the pigeon-hole principle, then, it follows that there must be some partial valuation σ on $\mathsf{Atoms}(\psi)$ such that $\sigma \in T \setminus S$. Now, let μ be an extension to Atoms of σ, defined as follows:

$$\mu(p) = \begin{cases} \sigma(p) & \text{if } p \in \mathsf{Atoms}(\psi), \\ \nu(p) & \text{if } p \in \mathsf{Atoms} \setminus \mathsf{Atoms}(\psi). \end{cases}$$

As $\sigma \notin S$, $\sigma(\psi) = t$, and so $\mu(\psi) = t$ as well. Also, $d_H(\nu, \mu) \leqslant i$, thus $d_H(\nu, \psi) \leqslant i$. □

Proof of Proposition 10. Let ϕ be a 1-validated formula and let $\nu \in \Lambda$. We have that $\nu \in \mathsf{mod}(\phi)$ iff $d_H(\nu, \phi) = d_U(\nu, \phi) = 0$. By Lemma 4, $\nu \notin \mathsf{mod}(\phi)$ iff $d_H(\nu, \phi) = d_U(\nu, \phi) = 1$. In any case, then, it holds that $d_H(\nu, \phi) = d_U(\nu, \phi)$. This implies that for every aggregation function f and every set Γ of 1-validated formulas, $\Delta_{d_H,f}(\Gamma) = \Delta_{d_U,f}(\Gamma)$, and so Proposition 10 is obtained. □

Corollary 4. *Let Γ be a set of clauses. Then*:

1. $\Gamma \models_{d_H,f} \psi$ *iff* $\Gamma \models_{d_U,f} \psi$, *for every aggregation function f and every formula ψ.*
2. $\Gamma \models_{d_H,\max} \psi$ *iff* $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\max}} \psi$.

Proof. The first item follows from Lemma 3 and Proposition 10, the second item follows from the first item and Proposition 9. □

Note 6. Item 1 of Corollary 4 also indicates that, under distance semantics, syntactic transformations of inconsistent premises will often change the intended meaning of the underlying theory. Indeed, reasoning based on Hamming distance and the drastic distance are, in general, different.

For 1-validated premises we therefore have a sound and complete proof system. Next, we consider the other cases.

Definition 13. For $\nu \in \Lambda$, denote by $\mathsf{Diff}(\nu, i)$ the set of valuations differing from ν in exactly i atoms.

The following proposition is the analogue, for Hamming distances, of Proposition 8.

Proposition 11. *The function $\mathsf{G}_H : 2^\Lambda \to 2^\Lambda$, defined for every $V \subseteq \Lambda$ by*

$$\mathsf{G}_H(V) = V \cup \bigcup_{\mu \in V} \mathsf{Diff}(\mu, 1),$$

is an inductive representation of d_H.

Proof. Straightforward from the definition of $\mathcal{R}_{d_H}$. □

Proposition 12. $\mathbf{MPV}(\mathsf{G}_H, \Gamma)$ *terminates after no more than $\max_{d_H} \Lambda$ iterations and returns $\Delta_{d_H,\max}(\Gamma)$. If Γ consists of i-validated formulas,* $\mathbf{MPV}(\mathsf{G}_H, \Gamma)$ *terminates after at most i iterations.*

Proof. The first part follows from Propositions 4 and 11. The second part follows from the fact that by Lemma 4, $\mathcal{R}_{d_H}(\psi, i) = \Lambda$ for every $\psi \in \Gamma$. It follows that, in the notations of Fig. 1, after i iterations $X_1 \cap \cdots \cap X_n = \Lambda$, so **MPV** must terminate by the i-th iteration. □

4. Reasoning by voting

A common way of reasoning in the presence of contradictions is to draw conclusions that are supported by “sufficiently many” evidences. Prominent examples for this method are operators for database merging by majority votes [15,21,40] and various voting methods that are used in social choice theory [7,41,45]. Below, we define a corresponding distance-based setting and show how the most plausible valuations may be computed in this case. Logical properties of voting operators and their computational complexity are considered in [21].

Definition 14. Given a multiset $D = \{d_1, \ldots, d_n\}$, denote the number of zeros in D by $\mathsf{Zero}(D)$. A $\frac{k}{m}$*-voting function* $\mathsf{vote}_{\frac{k}{m}}$, where $1 \leqslant k \leqslant m \in \mathbb{N}$, is defined as follows:

$$\mathsf{vote}_{\frac{k}{m}}(D) = \begin{cases} 0 & \text{if } \mathsf{Zero}(D) = n, \\ \frac{1}{2} & \text{if } \lceil \frac{k}{m} n \rceil \leqslant \mathsf{Zero}(D) < n, \\ 1 & \text{otherwise.} \end{cases}$$

In what follows, we shall assume that the argument of $\mathsf{vote}_{\frac{k}{m}}$ is a multiset of elements in $\{0,1\}$ (e.g., a multiset of drastic distances). In this case, it is easy to verify that $\mathsf{vote}_{\frac{k}{m}}$ is an aggregation function. Intuitively, $\mathsf{vote}_{\frac{k}{m}}$ simulates a poll and requires a quorum of at least $\lceil \frac{k}{m} \rceil$ of the 'votes' to determine implications of inconsistent theories. Indeed, if Γ is not consistent and there are valuations that satisfy at least $\lceil \frac{k}{m} \rceil$ of the elements of Γ, then $\Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma)$ contains all such valuations. Otherwise, $\Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma) = \Lambda$. It follows that for $\frac{k}{m} = \frac{1}{2}$, $\mathsf{vote}_{\frac{k}{m}}$ acts as a *majority-vote function*, so we denote

$$\mathsf{majority}(D) = \mathsf{vote}_{\frac{1}{2}}(D).$$

Definition 15. Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$. Let $\mathsf{Sub}_{\frac{k}{m}}(\Gamma)$ be the set of all sub-multisets of Γ of size $\lceil \frac{k}{m} n \rceil$, and denote $\mathsf{mod}_{\frac{k}{m}}(\Gamma) = \bigcup_{H \in \mathsf{Sub}_{\frac{k}{m}}(\Gamma)} \mathsf{mod}(H)$.

Proposition 13. *For every theory Γ,*

$$\Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma) = \begin{cases} \mathsf{mod}(\Gamma) & \text{if } \mathsf{mod}(\Gamma) \neq \emptyset, \\ \mathsf{mod}_{\frac{k}{m}}(\Gamma) & \text{otherwise, if } \mathsf{mod}_{\frac{k}{m}}(\Gamma) \neq \emptyset, \\ \Lambda & \text{otherwise}. \end{cases}$$

Proof. For the proof we need the following lemma:

Lemma 5. *For all $\mu \in \Lambda$, $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\mu, \Gamma) < 1$ iff $\mu \in \mathsf{mod}(H)$ for some $H \in \mathsf{Sub}_{\frac{k}{m}}(\Gamma)$.*

Proof. It holds that $\mu \in \mathsf{mod}(H)$ for some $H \in \mathsf{Sub}_{\frac{k}{m}}(\Gamma)$ iff μ satisfies at least $\lceil \frac{k}{m} \cdot n \rceil$ formulas in Γ, iff $\mathsf{Zero}(\{d_U(\mu, \psi_1), \ldots, d_U(\mu, \psi_n)\}) \geqslant \lceil \frac{k}{m} n \rceil$, iff $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\mu, \Gamma) < 1$. $\square$

Back to the proof of Proposition 13. If $\mathsf{mod}(\Gamma) \neq \emptyset$ then Γ is consistent and so $\Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma) = \mathsf{mod}(\Gamma)$.

For the second case, suppose that $\mathsf{mod}(\Gamma) = \emptyset$ but $\mathsf{mod}_{\frac{k}{m}}(\Gamma) \neq \emptyset$. Let $\mu \in \mathsf{mod}_{\frac{k}{m}}(\Gamma)$. If $\mu \notin \Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma)$, then there is some $\nu \in \Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma)$ such that $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\nu, \Gamma) < \delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\mu, \Gamma)$. But as $\mu \in \mathsf{mod}_{\frac{k}{m}}(\Gamma)$, it is a model of some subset $H \in \mathsf{Sub}_{\frac{k}{m}}(\Gamma)$, and so, by Lemma 5, $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\mu, \Gamma) < 1$. It follows that necessarily $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\mu, \Gamma) = \frac{1}{2}$ and $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\nu, \Gamma) = 0$, which means that ν satisfies all the formulas in Γ, in contradiction to the assumption that $\mathsf{mod}(\Gamma) = \emptyset$. We conclude, therefore, that $\mu \in \Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma)$. For the converse, let $\mu \in \Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma)$. If $\mu \notin \mathsf{mod}_{\frac{k}{m}}(\Gamma)$, then $\mu \notin \mathsf{mod}(H)$ for every subset $H \in \mathsf{Sub}_{\frac{k}{m}}(\Gamma)$. Thus, by Lemma 5, $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\mu, \Gamma) = 1$. But as $\mathsf{mod}_{\frac{k}{m}}(\Gamma) \neq \emptyset$, there is some valuation ν that satisfies an element $H \in \mathsf{Sub}_{\frac{k}{m}}(\Gamma)$, so by Lemma 5, $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\nu, \Gamma) < 1$. It follows that $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\nu, \Gamma) < \delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\mu, \Gamma)$, but this is a contradiction to the assumption that $\mu \in \Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma)$.

For the last case, suppose that $\mathsf{mod}(\Gamma) = \emptyset$ and $\mathsf{mod}_{\frac{k}{m}}(\Gamma) = \emptyset$. Then there is no valuation that satisfies some $H \in \mathsf{Sub}_{\frac{k}{m}}(\Gamma)$, and so by Lemma 5 again, for all $\mu \in \Lambda$, $\delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\mu, \Gamma) = 1$. Thus, $\Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma) = \Lambda$. $\square$

The following conclusions easily follow from Proposition 13.

Corollary 5. *If $\Gamma \models_{d_U,\mathsf{vote}_{\frac{k}{m}}} \psi$ then $\Gamma \models_{d_U,\mathsf{vote}_{\frac{l}{m}}} \psi$ for every $l \geqslant k$.*

Proof. By Proposition 13, $\Delta_{d_U,\mathsf{vote}_{\frac{l}{m}}}(\Gamma) \subseteq \Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma)$. Thus, if $\Delta_{d_U,\mathsf{vote}_{\frac{k}{m}}}(\Gamma) \subseteq \mathsf{mod}(\Gamma)$, so $\Delta_{d_U,\mathsf{vote}_{\frac{l}{m}}}(\Gamma) \subseteq \mathsf{mod}(\Gamma)$, which implies that if $\Gamma \models_{d_U,\mathsf{vote}_{\frac{k}{m}}} \psi$ then $\Gamma \models_{d_U,\mathsf{vote}_{\frac{l}{m}}} \psi$. $\square$

Corollary 6. *Let Γ be an inconsistent theory. Then $\Gamma \models_{d_U,\mathsf{vote}_1} \psi$ iff ψ is a tautology.*

Proof. When $k = m$, $\mathsf{mod}_{\frac{k}{m}}(\Gamma) = \mathsf{mod}(\Gamma)$, and so, by Proposition 13, when Γ is not consistent $\Delta_{d_U,\mathsf{vote}_1}(\Gamma) = \Lambda$. Thus, $\Gamma \models_{d_U,\mathsf{vote}_1} \psi$ iff $\Lambda \subseteq \mathsf{mod}(\psi)$ iff ψ is a tautology. $\square$

Taken together, Proposition 2 and Corollary 6 mean that if a 'consensus' is required for making decisions, then consistent premises entail exactly their standard logical consequences, and inconsistent premises entail just tautological assertions, as intuitively expected (cf. Corollary 2).

Vote$_{\frac{k}{m}}$ $(\{\psi_1, \ldots, \psi_n\})$
/* Most plausible valuations of $\{\psi_1, \ldots, \psi_n\}$ w.r.t. d_U and vote$_{\frac{k}{m}}$ */
for $i \in \{1, \ldots, n\}$: $X_i \leftarrow \mathsf{mod}(\psi_i)$;
$Y \leftarrow \emptyset$;
if $(X_1 \cap \cdots \cap X_n)$ is non-empty, return $(X_1 \cap \cdots \cap X_n)$;
for every subset $\mathbf{I}$ of $\{1, \ldots, n\}$ of size $\lceil \frac{k}{m} n \rceil$: $Y \leftarrow Y \cup \bigcap_{j \in \mathbf{I}} X_j$;
if Y is non-empty return Y else return Λ;

Fig. 4. Computing the most plausible valuations of $\{\psi_1, \ldots, \psi_n\}$ w.r.t. d_U and vote$_{\frac{k}{m}}$.

– Axioms:

$$\emptyset : \Lambda \qquad (A_0)$$

$$\{\psi\} : \mathsf{mod}(\psi) \quad \text{if } \mathsf{mod}(\psi) \neq \emptyset \qquad (A_1)$$

$$\{\psi\} : \Lambda \quad \text{if } \mathsf{mod}(\psi) = \emptyset \qquad (A_2)$$

• Inference rule:

$$\frac{\Gamma_1 : V_1 \quad \Gamma_2 : V_2}{\Gamma_1 \cup \Gamma_2 : V_1 \cap V_2} \quad \text{if } V_1 \cap V_2 \neq \emptyset \qquad (J_1)$$

Fig. 5. The system $\mathbf{S}_{\boldsymbol{\Sigma}}$.

Another consequence of Proposition 13 is the algorithm in Fig. 4 for computing $\Delta_{d_U, \mathsf{vote}_{\frac{k}{m}}}(\Gamma)$.

Proposition 14. *The procedure* **Vote**$_{\frac{k}{m}}(\Gamma)$ *in Fig. 4 always terminates and returns* $\Delta_{d_U, \mathsf{vote}_{\frac{k}{m}}}(\Gamma)$.

Proof. Immediately follows from Proposition 13. $\square$

5. Summation of distances

Summation of distances is probably the most common semantics for distance-based reasoning. In this section we consider this kind of reasoning. Again, we first examine the general case and then concentrate on more specific distances.

5.1. Arbitrary pseudo-distances

Consider the system $\mathbf{S}_{\boldsymbol{\Sigma}}$ in Fig. 5. Again, $\mathbf{S}_{\boldsymbol{\Sigma}}$ manipulates expressions of the form $\Gamma : V$, where Γ is a theory and $V \subseteq \Lambda$. We denote by $\Gamma \vdash_{\mathbf{S}_{\boldsymbol{\Sigma}}} \psi$ that $\Gamma : V$ is provable in $\mathbf{S}_{\boldsymbol{\Sigma}}$ (i.e., that $\vdash_{\mathbf{S}_{\boldsymbol{\Sigma}}} \Gamma : V$) for some $V \subseteq \mathsf{mod}(\psi)$.

Proposition 15 *(Soundness). For every pseudo-distance d, if* $\Gamma \vdash_{\mathbf{S}_{\boldsymbol{\Sigma}}} \psi$ *then* $\Gamma \models_{d,\Sigma} \psi$.

The proof of Proposition 15 follows from the next lemma:

Lemma 6. *For every pseudo-distance d, if* $\vdash_{\mathbf{S}_{\boldsymbol{\Sigma}}} \Gamma : V$ *then* $V = \Delta_{d,\Sigma}(\Gamma)$.

Proof. By induction on the length of the proof of $\Gamma : V$ in $\mathbf{S}_{\boldsymbol{\Sigma}}$. The claim clearly holds for the axioms. Assume now that $\Gamma : V$ is obtained by (J_1). By induction hypothesis, we have to show that in case that $\Delta_{d,\Sigma}(\Gamma_1) \cap \Delta_{d,\Sigma}(\Gamma_2) \neq \emptyset$ (that is, the side condition of (J_1) is satisfied), then $\Delta_{d,\Sigma}(\Gamma_1 \cup \Gamma_2) = \Delta_{d,\Sigma}(\Gamma_1) \cap \Delta_{d,\Sigma}(\Gamma_2)$. Indeed, let $\nu_1 \in \Delta_{d,\Sigma}(\Gamma_1)$ and $\nu_2 \in \Delta_{d,\Sigma}(\Gamma_2)$. Denote $k_1 = \delta_{d,\Sigma}(\nu_1, \Gamma_1)$ and $k_2 = \delta_{d,\Sigma}(\nu_2, \Gamma_2)$. Then for every valuation $\mu \in \Lambda$, $\delta_{d,\Sigma}(\mu, \Gamma_1) \geqslant k_1$ and $\delta_{d,\Sigma}(\mu, \Gamma_2) \geqslant k_2$. Suppose for a contradiction that there is some $\mu \in \Delta_{d,\Sigma}(\Gamma_1 \cup \Gamma_2)$ such that $\mu \notin \Delta_{d,\Sigma}(\Gamma_1)$. Then $\delta_{d,\Sigma}(\mu, \Gamma_1) > k_1$ and $\delta_{d,\Sigma}(\mu, \Gamma_2) \geqslant k_2$. Thus, $\delta_{d,\Sigma}(\mu, \Gamma_1 \cup \Gamma_2) > k_1 + k_2$. But since $\Delta_{d,\Sigma}(\Gamma_1) \cap \Delta_{d,\Sigma}(\Gamma_2)$ is not empty, there is some ν_0, for which $\delta_{d,\Sigma}(\nu_0, \Gamma_1 \cup \Gamma_2) = k_1 + k_2$, in contradiction to our assumption that $\mu \in \Delta_{d,\Sigma}(\Gamma_1 \cup \Gamma_2)$. Thus $\mu \in \Delta_{d,\Sigma}(\Gamma_1)$. The proof for $\mu \in \Delta_{d,\Sigma}(\Gamma_2)$ is symmetric. For the converse, suppose that $\mu \in \Delta_{d,\Sigma}(\Gamma_1) \cap \Delta_{d,\Sigma}(\Gamma_2)$ and assume for a contradiction that $\mu \notin \Delta_{d,\Sigma}(\Gamma_1 \cup \Gamma_2)$. Then there is some $\nu \in \Delta_{d,\Sigma}(\Gamma_1 \cup \Gamma_2)$, such that $\delta_{d,\Sigma}(\nu, \Gamma_1 \cup \Gamma_2) < \delta_{d,\Sigma}(\mu, \Gamma_1 \cup \Gamma_2)$. Denote: $k_1 = \delta_{d,\Sigma}(\nu, \Gamma_1)$ and $k_2 = \delta_{d,\Sigma}(\nu, \Gamma_2)$. Since $\delta_{d,\Sigma}(\nu, \Gamma_1 \cup \Gamma_2) = k_1 + k_2$, it must be the case that $\delta_{d,\Sigma}(\mu, \Gamma_1 \cup \Gamma_2) > k_1 + k_2$. But this means that either $\delta_{d,\Sigma}(\mu, \Gamma_1) > k_1$ or $\delta_{d,\Sigma}(\mu, \Gamma_2) > k_2$, thus either $\mu \notin \Delta_{d,\Sigma}(\Gamma_1)$ or $\mu \notin \Delta_{d,\Sigma}(\Gamma_2)$, in contradiction to our assumption that $\mu \in \Delta_{d,\Sigma}(\Gamma_1) \cap \Delta_{d,\Sigma}(\Gamma_2)$. $\square$

Proof of Proposition 15. Suppose that $\Gamma \vdash_{\mathbf{S}_{\boldsymbol{\Sigma}}} \psi$. Then $\vdash_{\mathbf{S}_{\boldsymbol{\Sigma}}} \Gamma : V$ for some $V \subseteq \mathsf{mod}(\psi)$. By Lemma 6, $V = \Delta_{d,\Sigma}(\Gamma)$, and so $\Delta_{d,\Sigma}(\Gamma) \subseteq \mathsf{mod}(\psi)$. Thus, $\Gamma \models_{d,\Sigma} \psi$. $\square$

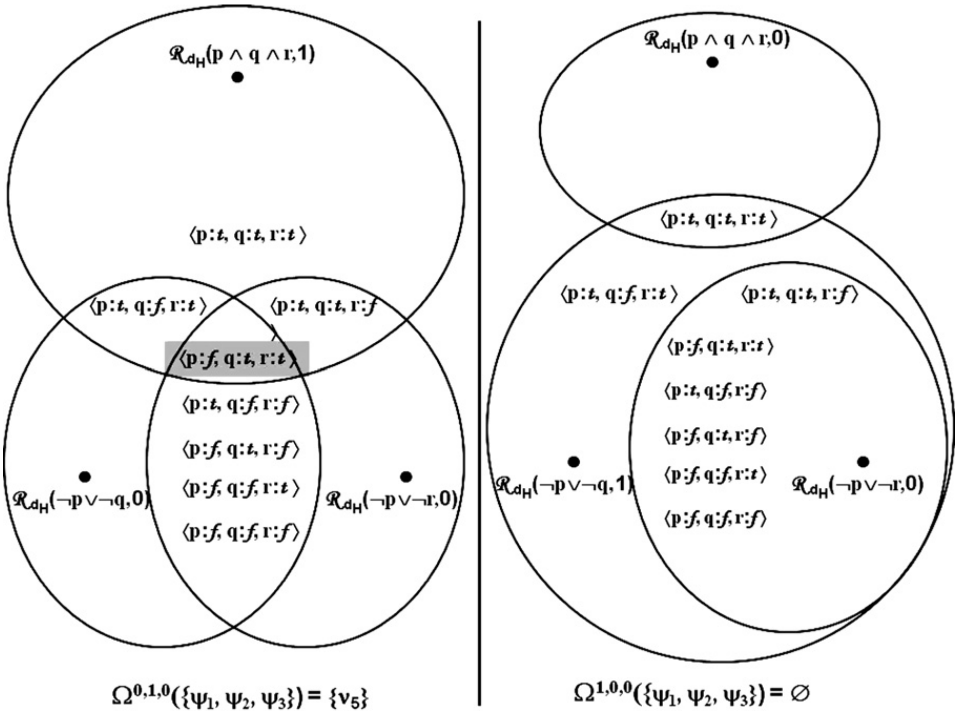

Fig. 6. Computations of $\Delta_{d_H,\Sigma}(\Gamma)$ (Examples 4 and 8) by intersection of spheres.

For another way of reasoning with summation of distances we note that, as in the case of max and the voting function, it is possible to characterize distance-summation conclusions by a set-theoretical condition.

Definition 16. Let d be an inductively representable pseudo-distance. Denote:

$$\Omega_d^{i_1,\dots,i_n}\big(\{\psi_1,\dots,\psi_n\}\big) = \bigcap_{k=1}^{n} \mathcal{R}_d^{i_k}(\psi_i).$$

Proposition 16. *For an inductively representable pseudo-distance d and a theory $\Gamma = \{\psi_1,\dots,\psi_n\}$, let m be the minimal number s.t. $\Omega_d^{i_1,\dots,i_n}(\{\psi_1,\dots,\psi_n\})$ is not empty for a sequence $i_1,\dots,i_n$, in which $\sum_{k=1}^n i_k = m$. Then $\Delta_{d,\Sigma}(\Gamma) = \bigcup_{j_1+\dots+j_n=m} \Omega_d^{j_1,\dots,j_n}(\Gamma)$.*

Example 8. Consider again the theory Γ of Example 4. As is shown in Fig. 6,

$$\mathcal{R}^0_{d_H}(\neg p \vee \neg q) \cap \mathcal{R}^1_{d_H}(p \wedge q \wedge r) \cap \mathcal{R}^0_{d_H}(\neg p \vee \neg r) \neq \emptyset,$$

therefore, in terms of Proposition 16, $\Omega^{0,1,0}_{d_H}(\Gamma) \neq \emptyset$ and $m = 1$. Now, since $\Omega^{1,0,0}_{d_H}(\Gamma) = \Omega^{0,0,1}_{d_H}(\Gamma) = \emptyset$ (see again Fig. 6), by Proposition 16 it holds that $\Delta_{d_H,\Sigma}(\Gamma) = \Omega^{0,1,0}_{d_H}(\Gamma)$, i.e., in the notations of Table 1, $\Delta_{d_H,\Sigma}(\Gamma) = \{\nu_5\}$. Indeed, this set is the one that is obtained in Examples 4 for the same theory and semantic setting.

Proof of Proposition 16. Let $\mu \in \Delta_{d,\Sigma}(\Gamma)$, and suppose that $d(\mu,\psi_i) = k_i$ for $i = 1,\dots,n$. Then for every $1 \leqslant i \leqslant n$, $\mu \in \mathcal{R}_d^{k_i}(\psi_i)$. Now, if there is no sequence $i_1,\dots,i_n$, such that $\sum_{k=1}^n i_k = m$, and for which $\mu \in \Omega_d^{i_1,\dots,i_n}(\Gamma)$, then, by the minimality of m, it is not the case that $k_1 + \dots + k_n < m$. Thus, since $k_1 + \dots + k_n \neq m$, necessarily $k_1 + \dots + k_n > m$. By the assumption of the proposition, there is a sequence $i_1,\dots,i_n$ such that $\sum_{k=1}^n i_k = m$ and for which $\Omega_d^{i_1,\dots,i_n}(\{\psi_1,\dots,\psi_n\})$ is not empty. So let $\nu \in \Omega_d^{i_1,\dots,i_n}(\{\psi_1,\dots,\psi_n\})$. Then $\nu \in \bigcap_{k=1}^n \mathcal{R}_d^{i_k}(\psi_i)$, and so $\delta_{d,\Sigma}(\nu,\Gamma) = m < \sum_{i=1}^n k_i = \delta_{d,\Sigma}(\mu,\Gamma)$, in contradiction to our assumption that $\mu \in \Delta_{d,\Sigma}(\Gamma)$.

For the converse, suppose that $\mu \in \bigcup_{j_1+\dots+j_n=m} \Omega_d^{j_1,\dots,j_n}(\Gamma)$. Then $d(\mu,\psi_i) = j_i$ for some sequence $j_1,\dots,j_n$, such that $j_1 + \dots + j_n = m$. If $\mu \notin \Delta_{d,\Sigma}(\Gamma)$ there is some $\nu \in \Delta_{d,\Sigma}$ such that $\delta_{d,\Sigma}(\nu,\Gamma) < \delta_{d,\Sigma}(\mu,\Gamma)$. But then $d(\nu,\psi_1) + \dots + d(\nu,\psi_n) < m$ and so there is a sequence $d(\nu,\psi_1),\dots,d(\nu,\psi_n)$, such that $\Omega_d^{d(\nu,\psi_1),\dots,d(\nu,\psi_n)}(\Gamma)$ is not empty, in contradiction to the minimality of m. $\square$

As a matter of fact, Proposition 16 indicates that reasoning with summation of distances is a constraint programming problem: for a given theory $\Gamma = \{\psi_1,\dots,\psi_n\}$, the goal is to minimize the value of $\sum_{j=i}^n i_j$ for which the intersection

$\mathcal{R}_d^{i_1}(\psi_1) \cap \cdots \cap R_d^{i_n}(\psi_n)$ is not empty. This problem may be implemented, e.g., by off-the-shelf constraint logic programming (CLP) solvers, such as SICStus Prolog CLP(FD) [14].

We note, finally, that for summation of distances, tautologies and contradictions have a degenerate role (cf. Proposition 5).

Proposition 17. *For any pseudo-distance d, a tautology or a contradiction φ, and a theory Γ, it holds that $\Delta_{d,\Sigma}(\Gamma) = \Delta_{d,\Sigma}(\Gamma \setminus \{\varphi\})$.*

Proof. Similar to that of Proposition 5, replacing max by Σ, and using the fact that for every $\mu_1, \mu_2 \in \Lambda$, $\delta_{d,\Sigma}(\mu_1, \varphi) = \delta_{d,\Sigma}(\mu_2, \varphi)$. $\square$

5.2. Drastic distances

As we show below, summation of drastic distances is closely related to the all-max-SAT problem, in which, given a theory, the valuations that satisfy subtheories with maximal cardinality should be computed.[10]

Definition 17. Given a theory Γ. Denote:

- $\mathsf{SAT}(\Gamma)$ is the set of all the *satisfiable multisets* in Γ,
- $\mathsf{mSAT}(\Gamma)$ is the set of the *cardinality maximally satisfiable* elements in $\mathsf{SAT}(\Gamma)$,[11]
- $\mathsf{mod}(\mathsf{mSAT}(\Gamma)) = \{\mu \in \Lambda \mid \mu \in \mathsf{mod}(\Upsilon) \text{ for some } \Upsilon \in \mathsf{mSAT}(\Gamma)\}$.

Note 7. Clearly, $\mathsf{mSAT}(\Gamma)$ is not empty for any Γ that contains a satisfiable element. Also, all the elements in $\mathsf{mSAT}(\Gamma)$ have the same size.

Proposition 18. *For every theory Γ,*

$$\Delta_{d_U,\Sigma}(\Gamma) = \begin{cases} \mathsf{mod}(\mathsf{mSAT}(\Gamma)) & \textit{if } \mathsf{mSAT}(\Gamma) \neq \emptyset, \\ \Lambda & \textit{otherwise}. \end{cases}$$

Proof. Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$. First assume that $\mathsf{mSAT}(\Gamma) \neq \emptyset$. Then for every $\psi \in \Gamma$ and every $\nu \in \Lambda$, $d_U(\nu, \psi) = 0$ if $\nu \in \mathsf{mod}(\psi)$ and otherwise $d_U(\nu, \psi) = 1$. Now,

$$\delta_{d_U,\Sigma}(\nu, \Gamma) = d_U(\nu, \psi_1) + \cdots + d_U(\nu, \psi_n) = \left|\left\{\psi \in \Gamma \;\middle|\; \nu \notin \mathsf{mod}(\psi)\right\}\right|.$$

Thus, $\nu \in \Delta_{d_U,\Sigma}(\Gamma)$ iff the set $\{\psi \in \Gamma \mid \nu \notin \mathsf{mod}(\psi)\}$ is minimal in its size, iff the set $\{\psi \in \Gamma \mid \nu \in \mathsf{mod}(\psi)\}$ is maximal in its size, iff this set belongs to $\mathsf{mSAT}(\Gamma)$.

If $\mathsf{mSAT}(\Gamma) = \emptyset$ then every $\psi \in \Gamma$ is not satisfiable. Thus,

$$\delta_{d_U,\Sigma}(\nu, \Gamma) = d_U(\nu, \psi_1) + \cdots + d_U(\nu, \psi_n) = n \cdot (1 + \max_{d_U} \Lambda) = 2n,$$

and so each valuation is equally distant from Γ. $\square$

By Proposition 18 we conclude that reasoning with $\models_{d_U,\Sigma}$ is reducible to the all-max-SAT problem. Consequence with respect to $\models_{d_U,\Sigma}$ may thus be computed by using max-SAT solving techniques and by incorporating off-the-shelf max-SAT solvers (see, e.g., [13,26,28,47]) for computing the valuations that satisfy a maximally consistent multiset in Γ.

The system $\mathbf{S}_{\boldsymbol{\Sigma}}$ defined above is not complete for $\models_{d_U,\Sigma}$, as its inference rule does not cover all the inter-relations among the premises. Proposition 18 implies that for a complete system one has to add the following rule:

$$\frac{\Gamma_1 : V_1 \quad \Gamma_2 : V_2}{\Gamma_1 \cup \Gamma_2 : \mathsf{mod}(\mathsf{mSAT}(\Gamma_1 \cup \Gamma_2))} \quad \text{if } V_1 \cap V_2 = \emptyset \quad (J_2)$$

Obviously, (J_2) is not an inference rule in the usual sense, as its conclusion is not affected by V_1 and V_2. As such, this rule is not very useful. Yet, the combination of (J_1) and (J_2) is helpful, e.g., in the context of belief revision, as:

(a) if the condition of (J_1) is satisfied, the most plausible valuations of the revised theory should not be recomputed, and
(b) if the condition of (J_1) is not satisfied, (J_2) indicates what is the auxiliary source of computations, namely: revision should be determined by maxSAT calculations.

[10] Note that the original formulation of max-SAT is about finding a valuation that satisfies a maximally satisfiable set of clauses in a set Γ (see, e.g., [44] for some complexity results and [29] for related approximation methods). By the all-max-SAT problem in our context we mean a harder version of this problem, according to which one has to find *all* the valuations that satisfy a maximally satisfiable multiset of *formulas* of a multiset Γ.

[11] That is, $\mathsf{mSAT}(\Gamma)$ consists of all $\Upsilon \in \mathsf{SAT}(\Gamma)$ such that $|\Upsilon'| \leqslant |\Upsilon|$ for every $\Upsilon' \in \mathsf{SAT}(\Gamma)$.

Definition 18. Denote by $\mathbf{S}^{\mathbf{u}}_{\boldsymbol{\Sigma}}$ the system $\mathbf{S}_{\boldsymbol{\Sigma}}$ together with (J_2).

Proposition 19 *(Soundness and completeness). $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\boldsymbol{\Sigma}}} \psi$ iff $\Gamma \models_{d_U,\Sigma} \psi$.*

5.3. Hamming distances

Summation of Hamming distances is a common approach in the context of belief revision and database integration. Yet, the deductive systems developed so far for automated reasoning with this kind of semantics are limited to a very narrow fragment of propositional languages. This is the case, for instance, with the logic *MF* defined in [15], which is suitable only for theories that consist of sets of literals. In this case, reasoning with $\models_{d_H,\Sigma}$ reduces to 'counting' majority votes:

Note 8. Let Γ be a multiset of literals. Then $\Gamma \models_{d_H,\Sigma} \psi$ iff ψ belongs to the transitive closure of $\mathsf{Maj}(\Gamma)$, where $\mathsf{Maj}(\Gamma)$ is the set of literals in Γ whose number of appearances in Γ is strictly bigger than the number of appearances in Γ of their negations.

Following this, Cholvy and Garion [15] developed a modal logic for reasoning with summation of Hamming distances when the information sources are sets of literals. The idea is to gather these sources in a multiset, and represent the fact that a literal l appears i times in Γ by the modal operator B^i_Γ. Then, l follows from Γ (l is believed; $B_\Gamma l$) if it holds that $B^i_\Gamma l \wedge B^j_\Gamma \neg l$ for some $i > j \geqslant 0$.

The following result suggests an alternative way for automated reasoning with summation of Hamming distances, in a more general context:

Proposition 20. *For every theory Γ and formula ψ,*

- *if $\Gamma \vdash_{\mathbf{S}_{\boldsymbol{\Sigma}}} \psi$ then $\Gamma \models_{d_H,\Sigma} \psi$,*
- *if Γ is a set of clauses, then $\Gamma \models_{d_H,\Sigma} \psi$ iff $\Gamma \vdash_{\mathbf{S}^{\mathbf{u}}_{\boldsymbol{\Sigma}}} \psi$.*

Proof. The first part is a particular case of Proposition 15; the second part follows from Corollary 4 and Proposition 19. $\square$

When the premises are in a clause form, we also have the following result:

Proposition 21. *Let Γ be a multiset of clauses. Then*:

$$\Delta_{d_H,\Sigma}(\Gamma) = \begin{cases} \mathsf{mod}(\mathsf{mSAT}(\Gamma)) & \textit{if } \mathsf{mSAT}(\Gamma) \neq \emptyset, \\ \Lambda & \textit{otherwise}. \end{cases}$$

Proof. By the first item of Corollary 4 and by Proposition 18. $\square$

Proposition 21 resembles a similar result by Lin and Mendelzon [39] for merging sets of atomic formulas (database instances) under integrity constraints. As shown in [39], for merging distributed database instances by summation of Hamming distances, one has to compute the multiset Γ obtained by combining all the databases, and then to take into account the maximal (with respect to cardinality) sub-multisets that are consistent with respect to a certain (consistent) set of constraints. In our terms, this is the analogue of $\mathsf{mSAT}(\Gamma)$. It follows, then, that the computational framework that is described in [39] is extended in our context to (multi)sets of clauses. Indeed,

Proposition 22. *Let $\Gamma_1, \ldots, \Gamma_n$ be n sets of clauses in $\mathcal{L}$ (the distributed sources) and $\mathcal{IC}$ a consistent set of formulas in $\mathcal{L}$ (the integrity constraints of the merging). Define*:

$$\mathsf{Merge}\big(\{\Gamma_1, \ldots, \Gamma_n\}, \mathcal{IC}\big) = \left\{ \nu \in \mathsf{mod}(\mathcal{IC}) \;\middle|\; \forall \mu \in \Lambda,\, \delta_{d_H,\Sigma}\left(\nu, \bigcup_{i=1}^{n} \Gamma_i\right) \leqslant \delta_{d_H,\Sigma}\left(\mu, \bigcup_{i=1}^{n} \Gamma_i\right) \right\}.$$

Then:

$$\mathsf{Merge}\big(\{\Gamma_1, \ldots, \Gamma_n\}, \mathcal{IC}\big) = \begin{cases} \mathsf{mod}(\mathcal{IC}) \cap \mathsf{mod}(\mathsf{mSAT}(\Gamma)) & \textit{if } \mathsf{mSAT}(\Gamma) \neq \emptyset, \\ \mathsf{mod}(\mathcal{IC}) & \textit{otherwise}. \end{cases}$$

Again, the last propositions suggest that max-SAT solvers may be incorporated for reasoning with (or for merging of) multisets of clauses by summation of Hamming distances.

6. Incorporation of infinite languages

Often, it is convenient to have an infinite repository of atomic formulas at our disposal. Since in such cases valuations are functions over an infinite domain, already at the representation level some modifications in the distance definitions are usually necessary (see, e.g., the definition of Hamming distance in Example 2, which is meaningless when Atoms is not finite). In this section we introduce a uniform way of handling these cases which, moreover, preserves the decidability of the inference process. As a consequence, the assumption about the finiteness of Atoms in Section 2.1 may be safely lifted.

We therefore assume that $\mathcal{L}$ is a propositional language with a possibly infinite set of atomic formulas Atoms. The idea is to limit the distance computations to a certain *context*, which is a finite set of formulas, closed under subformulas. In what follows we shall concentrate on contexts of the form $\mathcal{C}_\Gamma = \mathsf{Atoms}(\Gamma)$ for a theory Γ. Note that as Γ is finite, $\mathcal{C}_\Gamma$ is indeed a context.

Definition 19 *(Restrictions).* Let $\mathcal{C}$ be a context.

- The *restriction to* $\mathcal{C}$ of a valuation $\nu \in \Lambda$ is a valuation $\nu^{\downarrow\mathcal{C}}$ on $\mathcal{C}$, such that $\nu^{\downarrow\mathcal{C}}(\psi) = \nu(\psi)$ for every ψ in $\mathcal{C}$.
- The *restriction to* $\mathcal{C}$ of a set of valuations $S \subseteq \Lambda$ is the set $S^{\downarrow\mathcal{C}} = \{\nu^{\downarrow\mathcal{C}} \mid \nu \in S\}$. In particular, $\Lambda^{\downarrow\mathcal{C}}$ consists of all the valuations for $\mathcal{L}$ on $\mathcal{C}$.
- The *restriction to* $\mathcal{C}$ of a (pseudo) distance d is a function $d^{\downarrow\mathcal{C}}$ that applies d on valuations in $\Lambda^{\downarrow\mathcal{C}}$.[12]

Example 9. It is easy to verify that for every context $\mathcal{C}$ the following functions are distances on $\Lambda^{\downarrow\mathcal{C}}$:

- The drastic distance restricted to $\mathcal{C}$: $d_U^{\downarrow\mathcal{C}}(\nu,\mu) = 0$ if $\nu = \mu$, otherwise $d_U^{\downarrow\mathcal{C}}(\nu,\mu) = 1$.
- The Hamming distance restricted to $\mathcal{C}$: $d_H^{\downarrow\mathcal{C}}(\nu,\mu) = |\{p \in \mathsf{Atoms}(\mathcal{C}) \mid \nu(p) \neq \mu(p)\}|$.

Next, we show that distance-based entailments are not 'biased' by information that is not an actual part of the premises.

Proposition 23. *$\Gamma \models_{d,f} \psi$ iff $(\Delta_{d,f}(\Gamma))^{\downarrow\mathcal{C}_\Gamma} \subseteq (\mathsf{mod}(\psi))^{\downarrow\mathcal{C}_\Gamma}$.*

Proof. This is immediate from the fact that for every valuation $\nu \in \Lambda$, $\nu \in \Delta_{d,f}(\Gamma)$ iff $\nu^{\downarrow\mathcal{C}_\Gamma} \in (\Delta_{d,f}(\Gamma))^{\downarrow\mathcal{C}_\Gamma}$. □

Proposition 24. *Let d be a computable pseudo-distance[13] and f a computable aggregation function. Then the question whether $\Gamma \models_{d,f} \psi$ is decidable.*

Proof. First, we extend to *partial valuations* the notions of distance between valuations and formulas, and distance between valuations and theories. Given a theory Γ we define, for every $\nu \in \Lambda^{\downarrow\mathcal{C}_\Gamma}$ and every $\psi \in \Gamma$,

- $d^{\downarrow\mathcal{C}_\Gamma}(\nu,\psi) = \begin{cases} \min\{d^{\downarrow\mathcal{C}_\Gamma}(\nu^{\downarrow\mathcal{C}_\Gamma}, \mu^{\downarrow\mathcal{C}_\Gamma}) \mid \mu \in (\mathsf{mod}(\psi))^{\downarrow\mathcal{C}_\Gamma}\} & \text{if } (\mathsf{mod}(\psi))^{\downarrow\mathcal{C}_\Gamma} \neq \emptyset, \\ 1 + \max\{d^{\downarrow\mathcal{C}_\Gamma}(\mu_1^{\downarrow\mathcal{C}_\Gamma}, \mu_2^{\downarrow\mathcal{C}_\Gamma}) \mid \mu_1, \mu_2 \in \Lambda^{\downarrow\mathcal{C}_\Gamma}\} & \text{otherwise.} \end{cases}$
- $\delta_{d,f}^{\downarrow\mathcal{C}_\Gamma}(\nu,\Gamma) = f(\{d^{\downarrow\mathcal{C}_\Gamma}(\nu,\psi_1), \dots, d^{\downarrow\mathcal{C}_\Gamma}(\nu,\psi_n)\})$.

Note that since all the partial valuations involved in the definitions above are defined on (finite) contexts, there are finitely many such valuations to check, and so $d^{\downarrow\mathcal{C}_\Gamma}(\nu,\psi)$ and $\delta_{d,f}^{\downarrow\mathcal{C}_\Gamma}(\nu,\Gamma)$ are computable for every $\nu \in \Lambda^{\downarrow\mathcal{C}_\Gamma}$. Now, consider the following set of partial valuations on $\mathcal{C}_\Gamma$:

$$\Delta_{d,f}^{\downarrow\mathcal{C}_\Gamma}(\Gamma) = \begin{cases} \{\nu \in \Lambda^{\downarrow\mathcal{C}_\Gamma} \mid \forall \mu \in \Lambda^{\downarrow\mathcal{C}_\Gamma},\ \delta_{d,f}^{\downarrow\mathcal{C}_\Gamma}(\nu,\Gamma) \leqslant \delta_{d,f}^{\downarrow\mathcal{C}_\Gamma}(\mu,\Gamma)\} & \text{if } \Gamma \neq \emptyset, \\ \Lambda^{\downarrow\mathcal{C}_\Gamma} & \text{otherwise.} \end{cases}$$

It is easy to see that $\Delta_{d,f}^{\downarrow\mathcal{C}_\Gamma}(\Gamma) = (\Delta_{d,f}(\Gamma))^{\downarrow\mathcal{C}_\Gamma}$. Moreover, both of $\Delta_{d,f}^{\downarrow\mathcal{C}_\Gamma}(\Gamma)$ and $(\mathsf{mod}(\psi))^{\downarrow\mathcal{C}_\Gamma}$ are computable. Decidability now follows from the fact that $\Gamma \models_{d,f} \psi$ iff $\Delta_{d,f}(\Gamma) \subseteq \mathsf{mod}(\psi)$, iff $(\Delta_{d,f}(\Gamma))^{\downarrow\mathcal{C}_\Gamma} \subseteq (\mathsf{mod}(\psi))^{\downarrow\mathcal{C}_\Gamma}$ (by Proposition 23), iff $\Delta_{d,f}^{\downarrow\mathcal{C}_\Gamma}(\Gamma) \subseteq (\mathsf{mod}(\psi))^{\downarrow\mathcal{C}_\Gamma}$. The latter is decidable by the fact that both sets are computable. □

[12] Formally, one defines a function d on $\bigcup_{\{\mathcal{C}_\Gamma \mid \Gamma \in 2^{\mathcal{L}} \setminus \emptyset\}} \Lambda^{\downarrow\mathcal{C}_\Gamma} \times \Lambda^{\downarrow\mathcal{C}_\Gamma}$, and then $d^{\downarrow\mathcal{C}}$ is a function on $\Lambda^{\downarrow\mathcal{C}} \times \Lambda^{\downarrow\mathcal{C}}$, obtained by letting $d^{\downarrow\mathcal{C}}(\nu,\mu) = d(\nu,\mu)$ for every $\nu, \mu \in \Lambda^{\downarrow\mathcal{C}}$.

[13] That is, there is an algorithm that computes $d^{\downarrow\mathcal{C}}(\mu,\nu)$ for every context $\mathcal{C} = \mathsf{Atoms}(\Gamma)$ and valuations $\nu, \mu \in \Lambda^{\downarrow\mathcal{C}}$.

7. Reasoning with priorities

There are many situations in which different assertions have different importance or reliability. For instance, default assumptions are usually less accurate than actual information, in database systems integrity constraints are usually superior to arbitrary data about the underlying domain of discourse, in expert systems different sources of information may have different reputations, and so forth. This kind of situations may be captured by attaching a quantitative information to each formula, intuitively representing the formula's *priority* (or *weight*). As shown, e.g. in [4], distance-based semantics is a very useful framework for representing situations as those described above. In this section we show that the reasoning methods presented in the previous sections can be naturally extended to the prioritized case.

Definition 20 *(Prioritized theories).* A *prioritized formula* is an expression of the form $\psi : \mathsf{w}$, where ψ is a formula and $\mathsf{w} \in \mathbb{Q}^+ \cap (0,1]$. A *prioritized theory* is a multiset of prioritized formulas.

A prioritized formula is therefore a well-formed formula ψ augmented with a quantitative factor, w, representing the formula's weight. Intuitively, a formula with a higher weight has a higher priority. Note that different occurrences of the same formula in a theory may have different weights. This may happen, for instance, when a theory reflects several opinions of different experts on the same assertions.

Definition 21. Given a pseudo-distance d and an aggregation function f, define for every prioritized theory $\Gamma = \{\psi_1 : \mathsf{w}_1, \ldots, \psi_n : \mathsf{w}_n\}$ and valuation $\nu \in \Lambda$,

$$\delta_{d,f}(\nu, \Gamma) = f\big(\{\mathsf{w}_1 \cdot d(\nu, \psi_1), \ldots, \mathsf{w}_n \cdot d(\nu, \psi_n)\}\big),$$

where $d(\nu, \psi)$ is a 'distance' between ν and a formula ψ, just as in Definition 4.

The set $\Delta_{d,f}(\Gamma)$ of the most plausible valuations of a prioritized theory Γ, and the entailment relation $\models_{d,f}$ between prioritized theories and formulas, are defined just like in Definitions 5 and 7, respectively.

Note 9. When the weight of all the formulas in a prioritized theory is uniformly 1, the definition of $\delta_{d,f}$ in Definition 21 coincides with that of Definition 4. Thus, prioritized distance-based reasoning can be viewed as an extension of the non-prioritized case.

Below, we extend the distance-based reasoning strategies considered in the previous sections to the prioritized case.

7.1. MinMax reasoning

Definition 22. As weights are rational numbers in the unit interval, they are represented by $\mathsf{w} = \frac{s}{t}$ for some $t \geqslant s \in \mathbb{N}^+$. Given a prioritized theory $\Gamma = \{\psi_1 : \frac{s_1}{t_1}, \ldots, \psi_n : \frac{s_n}{t_n}\}$, we denote $\mathsf{M}_\Gamma = t_1 \cdot t_2 \cdot \ldots \cdot t_n$.

The following proposition is an analogue, for prioritized theories, of Proposition 3.

Proposition 25. *Let d be any pseudo-distance and $\Gamma = \{\psi_1 : \mathsf{w}_1, \ldots, \psi_n : \mathsf{w}_n\}$ a prioritized theory.*

(a) *If there is a non-satisfiable formula in Γ, then $\Delta_{d,\max}(\Gamma) = \Lambda$.*
(b) *If all the formulas in Γ are satisfiable, then*

$$\Delta_{d,\max}(\Gamma) = \bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d\left(\psi_i, \left\lfloor \frac{m}{\mathsf{w}_i \mathsf{M}_\Gamma} \right\rfloor\right),$$

where $m \in \mathbb{N}$ is the minimal number such that $\bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, \lfloor \frac{m}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor)$ is not empty.

Proof. The proof of the first part is the same as the proof of part (a) of Proposition 3. For the second part we need the following lemma, which is a straightforward consequence of Lemma 1:

Lemma 7. *Let d be a pseudo-distance. Suppose that $\Gamma = \{\psi_1 : \mathsf{w}_1, \ldots, \psi_n : \mathsf{w}_n\}$ contains only satisfiable formulas. Then there is an $m \in \mathbb{N}$, such that $\bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, \lfloor \frac{m}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor) \neq \emptyset$.*

Note that by the lemma above, the number m from Proposition 25 indeed exists. Now, let $\mu \in \bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, \lfloor \frac{m}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor)$. Thus, $d(\mu, \psi_i) \leqslant \frac{m}{\mathsf{w}_i \mathsf{M}_\Gamma}$ for all $\psi_i \in \Gamma$, and so $\mathsf{w}_i \cdot d(\mu, \psi_i) \leqslant \frac{m}{\mathsf{M}_\Gamma}$. Thus, $\delta_{d,\max}(\mu, \Gamma) = \max\{\mathsf{w}_1 d(\mu, \psi_1), \ldots, \mathsf{w}_n d(\mu, \psi_n)\} \leqslant$

$\frac{m}{\mathsf{M}_\Gamma}$. If $\mu \notin \Delta_{d,\max}(\Gamma)$, there is a valuation $\nu \in \Lambda$ such that $\delta_{d,\max}(\nu,\Gamma) < \delta_{d,\max}(\mu,\Gamma) \leqslant \frac{m}{\mathsf{M}_\Gamma}$. For $\mathsf{w}_j d(\nu,\psi_j) = \max\{\mathsf{w}_1 d(\nu,\psi_1),\ldots,\mathsf{w}_n d(\nu,\psi_n)\}$, $\mathsf{w}_j \cdot d(\nu,\psi_j) < \frac{m}{\mathsf{M}_\Gamma}$ holds, and so $\mathsf{w}_j \mathsf{M}_\Gamma \cdot d(\nu,\psi_j) < m$. But $d(\nu,\psi_j) \in \mathbb{N}$ and $\mathsf{w}_j \mathsf{M}_\Gamma \in \mathbb{N}$ (as $\mathsf{M}_\Gamma = t_1 \cdot t_2 \cdot \ldots \cdot t_n$ and $\mathsf{w}_j = \frac{s_j}{t_j}$), thus $\mathsf{w}_j \mathsf{M}_\Gamma \cdot d(\nu,\psi_j) \leqslant (m-1)$, i.e., $\mathsf{w}_j \cdot d(\nu,\psi_j) \leqslant \frac{m-1}{\mathsf{M}_\Gamma}$. It follows that for every $1 \leqslant i \leqslant n$, $\mathsf{w}_i \cdot d(\nu,\psi_i) \leqslant \mathsf{w}_j \cdot d(\nu,\psi_j) \leqslant \frac{m-1}{\mathsf{M}_\Gamma}$, and since $d(\nu,\psi_i) \in \mathbb{N}$, we conclude that $d(\nu,\psi_i) \leqslant \lfloor \frac{m-1}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor$. Thus $\nu \in \bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, \lfloor \frac{m-1}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor)$, in contradiction to the minimality of m.

For the converse, suppose that $\mu \in \Delta_{d,\max}(\Gamma)$ and let $m = \mathsf{M}_\Gamma \cdot \delta_{d,\max}(\mu,\Gamma)$. Then for every $1 \leqslant i \leqslant n$, $\mathsf{w}_i \cdot d(\mu,\psi_i) \leqslant \delta_{d,\max}(\mu,\Gamma) = \frac{m}{\mathsf{M}_\Gamma}$. Since $d(\mu,\psi_i) \in \mathbb{N}$, $d(\mu,\psi_i) \leqslant \lfloor \frac{m}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor$, and so $\mu \in \mathcal{R}_d(\psi_i, \lfloor \frac{m}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor)$. Suppose that there is some $r < m$ such that $\bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, \lfloor \frac{r}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor)$ is non-empty, and let $\nu \in \bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, \lfloor \frac{r}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor)$. Then $d(\nu,\psi_i) \leqslant \lfloor \frac{r}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor \leqslant \frac{r}{\mathsf{w}_i \mathsf{M}_\Gamma}$, and so $\mathsf{w}_i \cdot d(\nu,\psi_i) \leqslant \frac{r}{\mathsf{M}_\Gamma}$ for every $\psi_i \in \Gamma$. Thus, $\delta_{d,\max}(\nu,\Gamma) \leqslant \frac{r}{\mathsf{M}_\Gamma} < \frac{m}{\mathsf{M}_\Gamma} = \delta_{d,\max}(\mu,\Gamma)$, in contradiction to the assumption that $\mu \in \Delta_{d,\max}(\Gamma)$. $\square$

As in the non-prioritized case, the proof of part (b) of the last proposition, implies the following corollary:

Corollary 7. *For a pseudo-distance d and a prioritized theory Γ that contains only satisfiable formulas, it holds that $\Delta_{d,\max}(\Gamma) = \bigcap_{1 \leqslant i \leqslant n} \mathcal{R}_d(\psi_i, \lfloor \frac{m}{\mathsf{w}_i \mathsf{M}_\Gamma} \rfloor)$, where $m = \mathsf{M}_\Gamma \cdot \min\{\delta_{d,\max}(\nu,\Gamma) \mid \nu \in \Lambda\}$.*

The results above imply that when the underlying pseudo-distance d is inductively representable, the procedure **MPV** of Fig. 1 may be incorporated also in the prioritized case for computing $\Delta_{d,\max}(\Gamma)$.

7.2. Summation of distances

We now consider summation of weighted distances. We show that, interestingly, the prioritized case is closely related to weighted counterpart of the maxSAT-based approach, used in Section 5.2 for the non-prioritized case. We call it *weighted max-SAT*. According to this variant, one has to find a valuation that maximizes the sum of the weights of the satisfiable set of clauses in a theory Γ of weighted clauses (see, e.g. [27,28,47]). Again, in our context we consider an extension of the problem, *weighted all-max-SAT*, in which one has to find *all* the valuations that maximize the sum of weights of satisfiable sets of *formulas* in a theory Γ.

Definition 23. For a prioritized theory $\Gamma = \{\psi_1 : \mathsf{w}_1, \ldots, \psi_n : \mathsf{w}_n\}$, let $\overline{\Gamma} = \{\psi_1, \ldots, \psi_n\}$, and $\mathsf{w}(\Gamma) = \mathsf{w}_1 + \cdots + \mathsf{w}_n$.

Definition 24. Let Γ be a prioritized theory.

(a) The set of the *weighted maximally satisfiable subtheories of* Γ is defined by

$$\mathsf{wmSAT}(\Gamma) = \left\{\Upsilon \subseteq \Gamma \mid \overline{\Upsilon} \in \mathsf{SAT}(\overline{\Gamma}) \text{ and for all } \overline{\Upsilon'} \in \mathsf{SAT}(\overline{\Gamma}), \mathsf{w}\left(\Upsilon'\right) \leqslant \mathsf{w}(\Upsilon)\right\}.$$

(b) Define: $\mathsf{mod}(\mathsf{wmSAT}(\Gamma)) = \{\nu \in \Lambda \mid \nu \in \mathsf{mod}(\overline{\Upsilon}) \text{ for some } \Upsilon \in \mathsf{wmSAT}(\Gamma)\}$.[14]

Proposition 26. *For every prioritized theory Γ,*

$$\Delta_{d_U,\Sigma}(\Gamma) = \begin{cases} \mathsf{mod}(\mathsf{wmSAT}(\Gamma)) & \text{if } \mathsf{wmSAT}(\Gamma) \neq \emptyset, \\ \Lambda & \text{otherwise}. \end{cases}$$

Proof. Let $\Gamma = \{\psi_1 : \mathsf{w}_1, \ldots, \psi_n : \mathsf{w}_n\}$. Assume first that $\mathsf{wmSAT}(\Gamma) \neq \emptyset$. Then $d_U(\nu,\psi) = 0$ if $\nu \in \mathsf{mod}(\psi)$ and $d_U(\nu,\psi) = 1$ otherwise. Now,

$$\delta_{d_U,\Sigma}(\nu,\Gamma) = \mathsf{w}_1 \cdot d(\nu,\psi_1) + \cdots + \mathsf{w}_n \cdot d(\nu,\psi_n) = \mathsf{w}\left(\left\{\psi_i \in \Gamma \mid \nu \notin \mathsf{mod}(\psi_i)\right\}\right).$$

So $\nu \in \Delta_{d_U,\Sigma}(\Gamma)$ iff $\mathsf{w}(\{\psi \in \Gamma \mid \nu \notin \mathsf{mod}(\psi)\})$ is minimal, iff $\mathsf{w}(\{\psi \in \Gamma \mid \nu \in \mathsf{mod}(\psi)\})$ is maximal, iff $\{\psi \in \Gamma \mid \nu \in \mathsf{mod}(\psi)\} \in \mathsf{wmSAT}(\Gamma)$.

If $\mathsf{wmSAT}(\Gamma) = \emptyset$ then every $\psi \in \Gamma$ is not satisfiable. Thus,

$$\delta_{d_U,\Sigma}(\nu,\Gamma) = \mathsf{w}_1 \cdot d(\nu,\psi_1) + \cdots + \mathsf{w}_n \cdot d(\nu,\psi_n) = \mathsf{w}(\Gamma) \cdot (1 + \max_{d_U} \Lambda),$$

and so each valuation is equally distant from Γ. $\square$

[14] Again, this is a generalization of the non-prioritized case, as $\mathsf{wmSAT}(\Gamma) = \mathsf{mSAT}(\Gamma)$ when $\mathsf{w}_i = 1$ for all $1 \leqslant i \leqslant n$.

Note 10. Proposition 26 may be generalized as follows. For a prioritized theory $\Gamma = \{\psi_1 : \mathsf{w}_1, \ldots, \psi_n : \mathsf{w}_n\}$ and a numeric aggregation function f, denote: $\mathsf{w}_f(\Gamma) = f(\{\mathsf{w}_1, \ldots, \mathsf{w}_n\})$ and $\mathsf{wmSAT}_f(\Gamma) = \{\Upsilon \subseteq \Gamma \mid \overline{\Upsilon} \in \mathsf{SAT}(\overline{\Gamma})$ and for all $\overline{\Upsilon'} \in \mathsf{SAT}(\overline{\Gamma}), \mathsf{w}_f(\Upsilon') \leqslant \mathsf{w}_f(\Upsilon)\}$. Suppose that f is an aggregation function such that for every $x_1, \ldots, x_n$, and $k \in \mathbb{N}^+$,

$$f\big(\{k \cdot x_1, \ldots, k \cdot x_n\}\big) = k \cdot f\big(\{x_1, \ldots, x_n\}\big).$$

Then for every prioritized theory Γ,

$$\Delta_{d_U, f}(\Gamma) = \begin{cases} \mathsf{mod}(\mathsf{wmSAT}_f(\Gamma)) & \text{if } \mathsf{wmSAT}_f(\Gamma) \neq \emptyset, \\ \Lambda & \text{otherwise.} \end{cases}$$

7.3. Handling priorities among different theories

The algorithms above can be easily generalized for simultaneously processing several theories with different priorities. This is useful, for instance, when integrating distributed sources that are not equally reliable.

Suppose that there are m sources, where source i is of priority w_i. Let Γ_i be a theory representing the information provided by source i, and denote by $\overline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_m\}$ the overall information depicted by all the sources. The following methods may be applied for computing consequences of $\overline{\Gamma}$:

I In case that all the source theories are non-prioritized and that each theory Γ_i may be associated with the conjunction of its elements $\psi_i = \bigwedge_{\gamma \in \Gamma_i} \gamma$ (as, e.g., is the case in [21,31,32]), the prioritized theory $\Gamma = \{\psi_1 : \mathsf{w}_1, \ldots, \psi_m : \mathsf{w}_m\}$ faithfully represents the integrated data and its respective priority, so the algorithms considered previously in this section may be applied to this theory for drawing (distance-based) conclusions from $\overline{\Gamma}$. In other words, in this case a formula ϕ follows from $\overline{\Gamma}$ with respect to a pseudo-distance d and an aggregation function f, if $\Gamma \models_{d,f} \phi$.

II Suppose, without loss of generality, that the sources are indexed by their priorities: lower indices mean higher priorities (that is, $\mathsf{w}_1 > \cdots > \mathsf{w}_m$). In case that several sources have the same priority they may be considered as one source whose theory is the union of the original source theories. Drawing conclusions from $\overline{\Gamma}$ with respect to a pseudo-distance d and an aggregation function f may be done by an iterative process, induced by the following sequence of sets (see also [4]):

$$\nabla^1_{d,f}(\overline{\Gamma}) = \Delta_{d,f}(\Gamma_1),$$
$$\nabla^i_{d,f}(\overline{\Gamma}) = \big\{\nu \in \nabla^{i-1}_{d,f}(\overline{\Gamma}) \,\big|\, \forall \mu \in \nabla^{i-1}_{d,f}(\overline{\Gamma}),\ \delta_{d,f}(\nu, \Gamma_i) \leqslant \delta_{d,f}(\mu, \Gamma_i)\big\} \quad (1 < i \leqslant m).$$

Here, ϕ follows from $\overline{\Gamma}$ (with respect to d and f) if $\nabla^m_{d,f}(\overline{\Gamma}) \subseteq \mathsf{mod}(\phi)$. The sequence $\nabla^1_{d,f}(\overline{\Gamma}), \ldots, \nabla^m_{d,f}(\overline{\Gamma})$ is clearly non-increasing, as sets with higher indices are subsets of those with smaller indices. This reflects the intuitive idea that smaller-indexed sources are preferred over higher-indexed sources, thus the most plausible valuations of the (theories of the) latter are determined by the most plausible valuations of the former. Since the relevant valuations are derived by distance considerations, each set in the sequence above contains the valuations that are $\delta_{d,f}$-closest to the corresponding source theory among the elements of the preceding set in the sequence. Note that in this case the source theories themselves may also be prioritized and the computation of each $\nabla^i_{d,f}$ can be performed by the distance-based algorithms introduced previously.

Example 10. Given three sources, suppose that the theory of the most preferred one is $\Gamma_1 = \{p\}$, the theory of the intermediate one is $\Gamma_2 = \{\neg p \vee q\}$, and the theory of the least preferred source is $\Gamma_3 = \{\neg q, r\}$. Here, the combined theory of the two preferred sources classically entails q, while the least prioritized source indicates the opposite, so the indication of the third source should be ruled out. On the other hand, the assertion r of the same source should be accepted, as there is no counter information from more reliable sources. Indeed, according to the iterative method described above, each one of p, q and r follows from $\overline{\Gamma}$, while their complements are not deducible from $\overline{\Gamma}$. Note that the same conclusions remain the same even in case that Γ_i are prioritized theories and the assertion $\neg q$ has higher priority than any other formula. This follows from the facts that 'external' preferences (among sources) override 'internal' preferences (among formulas in the same theory), and that preferences of formulas are relative to other formulas in the same theory only.

The above iterative method of integrating ranked sources by distance considerations was introduced and investigated in [4], where is it shown that for $d = d_U$ and $f = \max$ this method corresponds to the possibilistic merging operator of Benferhat et al. [9,10] and for $d = d_U$ and $f = \Sigma$ we get the merging operator considered in [36] (see also [20] for a discussion in the context of iterated belief revision). When each source theory Γ_i is a singleton, this is a linear revision in the sense of Nebel [43]. All of these methods are therefore implementable by consecutive applications of the algorithms for computing most plausible valuations, considered in this paper.

8. Conclusion

The principle of minimal change has a major role in different areas of artificial intelligence, and it is often implicitly derived by distance considerations. Yet, inference systems for automated reasoning with distance-based entailments are still in their early stages. In this paper, we have considered two simple methods for distance-based computerized reasoning, applied them to several common forms of distance semantics, and demonstrated the (so far implicit) relations between distance-based reasoning and several variants of known SAT problems.

By the discussion in this paper, it is evident that automated reasoning with distance semantics is not an easy task. One of the main reasons for this difficulty is, as implied by Note 2, that standard syntactic transformations (such as idempotence, commutativity, associativity, distributivity and de-Morgan's laws) may harm the intended inferences in specific cases, so in general their application is not possible in a general distance-based framework (see also Note 6). This also implies that many structural rules like contraction (if $\Gamma, \psi, \psi \vdash \phi$ then $\Gamma, \psi \vdash \phi$) or expansion (if $\Gamma, \psi \vdash \phi$ then $\Gamma, \psi, \psi \vdash \phi$), and logical rules for introducing or eliminating connectives (e.g., if $\Gamma, \psi, \phi \vdash \sigma$ then $\Gamma, \psi \wedge \phi \vdash \sigma$) are not sound for many forms of distance-based reasoning. The 'best' we can achieve in general is what Proposition 2 suggests: as long as the set of premises is consistent, we are safe, and classically valid rules may be applied. Once the consistency of the premises is lost, extra cautiousness is required and the (paraconsistent) reasoning process is determined by the choice of the distance semantics at hand.

We conclude, then, that there is still a way to go before a full automation for distance-based logics is reached, and most probably the more successful systems will be adequate for very specific forms of distance-based reasoning. Introducing a comprehensive system for reasoning with summation of Hamming distances with respect to premises that are not necessarily in a clause form, and efficient implementations of such systems, are among the most challenging and practical open questions in this respect.

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