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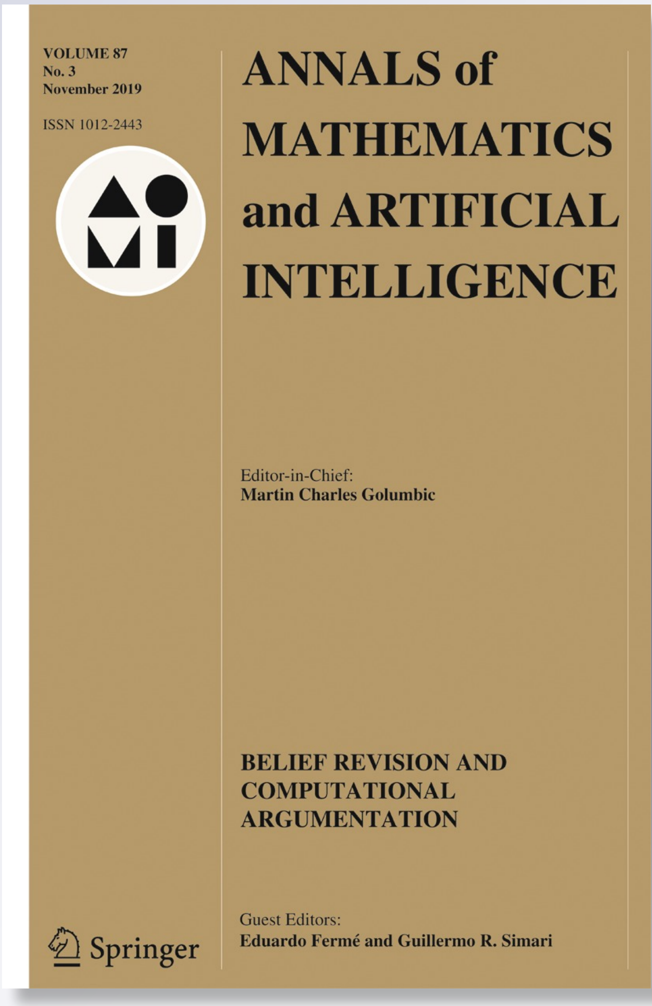

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A review of the relations between logical argumentation and reasoning with maximal consistency

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Abstract
This is a survey of some recent results relating Dung-style semantics for different types of logical argumentation frameworks and several forms of reasoning with maximally consistent sets (MCS) of premises. The related formalsims are also examined with respect to some rationality postulates and are carried on to corresponding proof systems for non-monotonic reasoning.

Keywords Logical argumentation · Structured argumentation · Reasoning with maximal consistency · Defeasible reasoning · Extension-based semantics · Dynamic proof systems

Mathematics Subject Classification (2010) 68-02 · 68T27 · 68T37

1 Introduction

Structured argumentation is concerned with the modeling of argumentation-based inferences, where arguments have a concrete structure. Logical argumentation (also called deductive argumentation) is a primary approach in this context, where the basic ingredients of the arguments (namely, their premises and claims) are expressed in terms of a formal language, and the relationship between these ingredients are justified by some underlying entailment relation. A primary goal of structured argumentation (and so of logical argumentation) is to prove a general and intuitive semantics for making consequences

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by corresponding argumentation-based logics in particular and non-monotonic logics in general.[1]

In this paper we review the relations of logical argumentation to one of the fundamental principles of non-monotonic reasoning and inconsistency handling, according to which the main information of a given set of assertions is carried by its consistent subsets and that such subsets should be as large as possible in order not to lose data. This principle, first introduced in the seminal paper of Rescher and Manor [66], has given a boost to a vast amount of research of consistency maintenance in different AI-related areas, including integration [16, 55] and knowledge-base systems [23], consistency operators for belief revision [45, 54], computational linguistics [56], and so forth. In this respect, this work is not only a retrospective on the way logical argumentation maintains consistency, but also a note on its potential application in related AI-based paradigms.

The study of the relations between logical argumentation theory and reasoning with maximal consistency may be traced back at least to Cayrol [36], with many follow-up or related papers, based either on classical logic (e.g., [5, 6, 43, 47, 72, 73]) or on arbitrary (Tarskian) propositional logic (e.g., [4, 7, 8, 29, 48]). In what follows we shall recall some of the latest results concerning this research area. To keep this paper self-contained we first review, in the next section, some basic notions from logical argumentation and set up our framework. Then we show the relations between logical argumentation and reasoning with maximally consistent premise sets (MCS) from several perspectives:

- In Section 3 we consider three basic forms of reasoning with MCS: by the transitive closure of their intersection, by the intersection of their transitive closures, and by taking into account consistent subsets that are not necessarily maximal. Each one of these forms of reasoning is then related to Dung's semantics for logical argumentation (see, respectively, Theorems 1–3).
- In Section 4 we consider the prioritized case, i.e., where formulas are assigned quantitative measurements to reflect their relative strength (or reliability, or any other preference criterion). In this case as well we show the relations between these kinds of logical argumentation and MCS-based reasoning (Theorem 4).
- In Section 5 we consider more general forms of arguments, in which the premises of the arguments may be of two different natures: some of them may be strict (non-attackable) assumptions, while others may be defeasible (i.e., retractable) ones. The correspondence to MCS-based reasoning is shown for different forms of this kind of setting, including assumption-based frameworks (Theorem 5) and dialectical frameworks (Theorem 6).
- In Section 6 we consider two further generalizations of the setting. One is concerned with extended forms of arguments, called hyperarguments, and the other one relaxes the notion of consistency to coherence. Again, in both cases we provide corresponding characterization results (see, respectively, Theorems 7 and 8).

In the last part of this paper, we discuss several classes of postulates for these formalisms (Section 7), and show that the correspondence between logical argumentation and reasoning with maximal consistency goes beyond the representation level, to the operational level (Section 8). For the latter, we introduce a proof-theoretic approach, imitating

[1] We refer to [24, 27, 62] for recent surveys on the subject. Some reviews on particular approaches to structured argumentation can be found, e.g., in [25, 42, 44, 47, 61, 71].

argumentation-based dialogues, by which it is possible to compute the above-mentioned MCS-based entailments (see Theorems 9 and 10).[2]

2 Preliminaries

As indicated previously, to define arguments in logical argumentation one needs a formal language for expressing assertions and a consequence relation for representing logical entailments. In what follows we shall concentrate on propositional languages, since most of the literature and the main problems involved in developing logical argumentation systems arise already on the propositional level. Accordingly, we denote by $\mathcal{L}$ an arbitrary propositional language. Atomic formulas in $\mathcal{L}$ are denoted by p, q, compound formulas are denoted by $\gamma, \delta, \psi, \phi, \sigma$, and sets of formulas are denoted by $\mathcal{S}, \mathcal{T}$. Now, a (Tarskian) *consequence relation* for a language $\mathcal{L}$ is a binary relation between sets of formulas and formulas in $\mathcal{L}$, satisfying the following conditions:

- *Reflexivity*: if $\psi \in \mathcal{S}$ then $\mathcal{S} \vdash \psi$.
- *Monotonicity*: if $\mathcal{S} \vdash \psi$ and $\mathcal{S} \subseteq \mathcal{S}'$ then $\mathcal{S}' \vdash \psi$.
- *Transitivity*: if $\mathcal{S} \vdash \psi$ and $\mathcal{S}', \psi \vdash \phi$ then $\mathcal{S}, \mathcal{S}' \vdash \phi$.

It is usual to assume that a consequence relation satisfies some further standard conditions:

- *Structurality*: for every $\mathcal{L}$-substitution θ and every $\mathcal{S}$ and ψ: if $\mathcal{S} \vdash \psi$ then $\theta(\mathcal{S}) \vdash \theta(\psi)$.
- *Non-Triviality*: $p \nvdash q$ for every two distinct atomic formulas p and q.
- *Finitariness*: for every $\mathcal{S}$ and ψ such that $\mathcal{S} \vdash \psi$, there is a *finite* set $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{T} \vdash \psi$.

Structurality means closure under substitutions of formulas. Non-triviality is convenient for excluding trivial logics, and finitariness is often essential for practical reasoning, where a conclusion is derived from a finite set of premises.

Together, a language and a consequence relation form a *logic*, which is a cornerstone of the whole setting.

Definition 1 (**logic**) A (propositional) *logic* is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural, non-trivial, and finitary consequence relation for $\mathcal{L}$.

We shall assume that the language $\mathcal{L}$ contains at least the following connectives and constant:

a $\vdash$-*negation* $\neg$, satisfying: $p \nvdash \neg p$ and $\neg p \nvdash p$ (for every atomic p),
a $\vdash$-*conjunction* $\wedge$, satisfying: $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$,
a $\vdash$-*disjunction* $\vee$, satisfying: $\mathcal{S}, \phi \vee \psi \vdash \sigma$ iff $\mathcal{S}, \phi \vdash \sigma$ and $\mathcal{S}, \psi \vdash \sigma$,
a $\vdash$-*implication* $\supset$, satisfying: $\mathcal{S}, \phi \vdash \psi$ iff $\mathcal{S} \vdash \phi \supset \psi$,
a $\vdash$-*falsity* F, satisfying: $\mathsf{F} \vdash \psi$ for every formula ψ.

[2]Since this is a survey of already known results it does not contain proofs but rather references to the relevant papers.

In what follows, we shall abbreviate $(\phi \supset \psi) \wedge (\psi \supset \phi)$ by $\phi \leftrightarrow \psi$. For a finite set of formulas $\mathcal{S}$ we denote by $\bigwedge\mathcal{S}$ (respectively, by $\bigvee\mathcal{S}$) the conjunction (respectively, the disjunction) of all the formulas in $\mathcal{S}$. We shall say that $\mathcal{S}$ is $\vdash$-*consistent* if $\mathcal{S} \nvdash \mathsf{F}$.

Given a logic, an *argument* (in that logic) consists of a premise set (of formulas) and a conclusion (a formula), such that the conclusion logically follows from the set of premises. Formally:

Definition 2 (**argument**) Given a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, an $\mathfrak{L}$-argument (an *argument* for short) is a pair $A = \langle \mathcal{S}, \psi \rangle$, where $\mathcal{S}$ is a set of $\mathcal{L}$-formulas and ψ is an $\mathcal{L}$-formula, such that $\mathcal{S} \vdash \psi$.

Given an argument $A = \langle \mathcal{S}, \psi \rangle$, we shall sometimes call $\mathcal{S}$ the *support set* (or the *premise set*) of A, and ψ the *conclusion* (or the *claim*) of A, denoting them by $\mathsf{Sup}(A)$ and $\mathsf{Conc}(A)$, respectively. For a set S of arguments, we denote: $\mathsf{Sup}(\mathsf{S}) = \bigcup_{A \in \mathsf{S}} \mathsf{Sup}(A)$ and $\mathsf{Conc}(\mathsf{S}) = \bigcup_{A \in \mathsf{S}} \mathsf{Conc}(A)$.

So far, the most studied type of arguments is what we call *classical arguments*. It is based on classical logic $\mathsf{CL} = \langle \mathcal{L}_{\mathsf{CL}}, \vdash_{\mathsf{CL}} \rangle$ (with a standard propositional language and the usual interpretations of the basic connectives $\vee, \wedge, \supset, \neg$, together with the falsity constant F) and regards only arguments with subset-minimal, classically consistent support sets.

Definition 3 (**classical argument**) A *classical argument* is a CL-argument $A = \langle \mathcal{S}, \psi \rangle$, where $\mathcal{S}$ is a CL-consistent and $\subseteq$-minimal support set for ψ, that is: (1) $\mathcal{S} \nvdash_{\mathsf{CL}} \mathsf{F}$, (2) $\mathcal{S} \vdash_{\mathsf{CL}} \psi$, and (3) there is no $\mathcal{S}' \subsetneq \mathcal{S}$ such that $\mathcal{S}' \vdash_{\mathsf{CL}} \psi$.

Intuitively, disagreements between arguments are described in terms of counter-arguments. It is often said that a counter-argument *attacks* (or *defeats*) the argument that it challenges (see [26, 47, 60, 63, 67, 72]). Attacks between arguments are usually described in terms of *attack rules* (with respect to the underlying logic). Table 1 lists some of them. For other attack rules (in terms of sequents, [46]) we refer to [11]. Attack rules incorporating modalities are introduced in [68].

Rules like those specified in Table 1 form attack schemes that are applied to particular arguments according to the underlying logic. For instance, when classical logic is the

Table 1 Some attack rules

Rule Name	Acronym	Attacking Argument	Attacked Argument	Attack Conditions
Defeat	Def	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\vdash \psi_1 \supset \neg\bigwedge\mathcal{S}_2$
Direct Defeat	DirDef	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\gamma_2\} \cup \mathcal{S}_2', \psi_2 \rangle$	$\vdash \psi_1 \supset \neg\gamma_2$
Undercut	Ucut	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	$\vdash \psi_1 \leftrightarrow \neg\bigwedge\mathcal{S}_2$
Direct Undercut	DirUcut	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\gamma_2\} \cup \mathcal{S}_2', \psi_2 \rangle$	$\vdash \psi_1 \leftrightarrow \neg\gamma_2$
Consistency Undercut	ConUcut	$\langle \emptyset, \neg\bigwedge\mathcal{S}_2 \rangle$	$\langle \mathcal{S}_2 \cup \mathcal{S}_2', \psi_2 \rangle$	
Rebuttal	Reb	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\vdash \psi_1 \leftrightarrow \neg\psi_2$
Defeating Rebuttal	DefReb	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \mathcal{S}_2, \psi_2 \rangle$	$\vdash \psi_1 \supset \neg\psi_2$
Big Argument Attack	BigArgAt	$\langle \mathcal{S}_1, \psi_1 \rangle$	$\langle \{\gamma_2\} \cup \mathcal{S}_2', \psi_2 \rangle$	$\vdash \bigwedge\mathcal{S}_1 \supset \neg\gamma_2$

The support sets of the attacked arguments are assumed to be nonempty (to avoid attacks on tautologies)

Fig. 1 Relations between attack relations from Table 1 when classical logic is the deductive base

$$\begin{tikzcd}
\text{DirUcut} \arrow[r] \arrow[dr] & \text{DirDef} \arrow[r] \arrow[dr] & \text{BigArgAt} \\
\text{ConUcut} \arrow[r] & \text{Ucut} \arrow[r] & \text{Def} \\
\text{Reb} \arrow[r] & \text{DefReb} \arrow[ur] &
\end{tikzcd}$$

underlying formalism, the attacks of $\langle p, p\rangle$ on $\langle\neg p, \neg p\rangle$ and of $\langle\neg p, \neg p\rangle$ on $\langle p \wedge q, p\rangle$[3] are obtained by applications of the Defeat rule. When an attack rule $\mathcal{R}$ is applied we shall sometimes say that its attacking argument $\mathcal{R}$*-attacks* the attacked argument.

Note 1 Clearly, the rules in Table 1 are related. The relations among some of the rules for classical arguments are considered in [47]. Figure 1 shows that when classical logic is the base logic these relations (together with other relations for ConUcut and BigArgAt) hold also for the more general definition of argument (Definition 2). In this figure, an arrow from $\mathcal{R}_1$ to $\mathcal{R}_2$ means that $\mathcal{R}_1 \subseteq \mathcal{R}_2$.

A logical argumentation formalism may be represented as an argumentation framework in the style of Dung [41]. This is defined next.

Definition 4 (logical argumentation framework) Let $\mathfrak{L} = \langle\mathcal{L}, \vdash\rangle$ be a logic and $\mathcal{A}$ a set of attack rules with respect to $\mathfrak{L}$. Let also $\mathcal{S}$ be a set of $\mathcal{L}$-formulas. The *(logical) argumentation framework* for $\mathcal{S}$, induced by $\mathfrak{L}$ and $\mathcal{A}$, is the pair $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}\rangle$, where $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set of the $\mathfrak{L}$-arguments whose supports are subsets of $\mathcal{S}$, and *Attack* is a relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, defined by $(A_1, A_2) \in \mathit{Attack}$ iff there is some $\mathcal{R} \in \mathcal{A}$ such that A_1 $\mathcal{R}$-attacks A_2.

Argumentation frameworks that are induced by classical logic (and some attack rules), and whose arguments are classical (Definition 3), are called *classical (logical) argumentation frameworks*.

In what follows, somewhat abusing the notations, we shall sometimes identify *Attack* with $\mathcal{A}$. To simplify the notations, we shall also frequently omit the subscripts $\mathfrak{L}$ and $\mathcal{A}$ in $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$, and just write $\mathcal{AF}(\mathcal{S})$.

Example 1 Let $\mathcal{S}_1 = \{r, p, q, \neg p \vee \neg q\}$ and let $\mathcal{AF}(\mathcal{S}_1)$ be the logical argumentation framework for $\mathcal{S}_1$, induced by classical logic CL and Undercut as the only attack rule. Some of the arguments in this case are the following:

$$\begin{array}{lll}
A_0 = \langle r, r\rangle & A_1 = \langle p, p\rangle & A_2 = \langle q, q\rangle \\
A_3 = \langle \neg p \vee \neg q, \neg p \vee \neg q\rangle & A_4 = \langle p, \neg((\neg p \vee \neg q) \wedge q)\rangle & A_5 = \langle q, \neg((\neg p \vee \neg q) \wedge p)\rangle \\
A_6 = \langle \{p, q\}, p \wedge q\rangle & A_7 = \langle \{\neg p \vee \neg q, q\}, \neg p\rangle & A_8 = \langle \{\neg p \vee \neg q, p\}, \neg q\rangle \\
A_\perp = \langle \{p, q, \neg p \vee \neg q\}, \neg r\rangle & A_\top = \langle \emptyset, \neg(p \wedge q \wedge (\neg p \vee \neg q))\rangle &
\end{array}$$

A graphical representation of these arguments and the attacks between them is given in Fig. 2, where nodes represent arguments and edges are directed from attacking arguments to attacked arguments.

[3] Here and in what follows we omit the set signs when the support of the arguments are singletons.

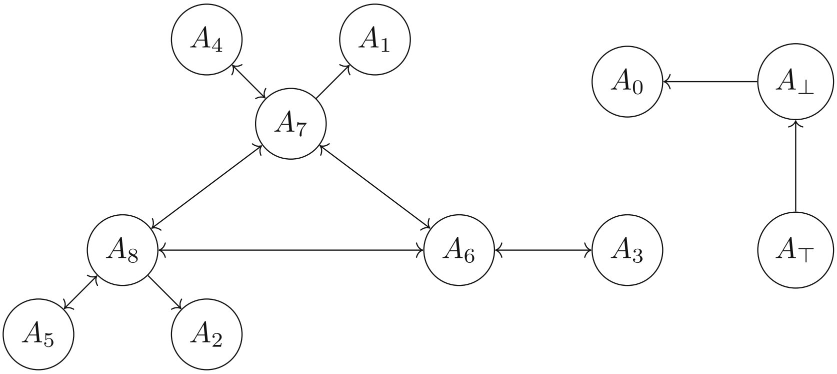

Fig. 2 Part of the logical argumentation framework for $\mathcal{S}_1 = \{r, p, q, \neg p \vee \neg q\}$ (Example 1)

Given an argumentation framework, a key issue in its understanding is the question what combinations of arguments (called *extensions*) can collectively be accepted from this framework. According to Dung [41], this is determined as follows:

Definition 5 (extension-based semantics) Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack} \rangle$ be a logical argumentation framework, and let $\mathcal{E} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Below, maximality and minimality are taken with respect to the subset relation.

- We say that $\mathcal{E}$ *attacks* an argument A, if there is an argument $B \in \mathcal{E}$ that attacks A (that is, $(B, A) \in \mathit{Attack}$). The set of arguments that are attacked by $\mathcal{E}$ is denoted $\mathcal{E}^+$.
- We say that $\mathcal{E}$ *defends* A, if $\mathcal{E}$ attacks every argument that attacks A.
- The set $\mathcal{E}$ is called *conflict-free* with respect to $\mathcal{AF}(\mathcal{S})$, if it does not attack any of its elements (i.e., $\mathcal{E}^+ \cap \mathcal{E} = \emptyset$). A set that is maximally conflict-free with respect to $\mathcal{AF}(\mathcal{S})$ is called a *naive extension* of $\mathcal{AF}(\mathcal{S})$.
- An *admissible extension* of $\mathcal{AF}(\mathcal{S})$ is a subset of $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ that is conflict-free with respect to $\mathcal{AF}(\mathcal{S})$ and defends all of its elements. A *complete extension* of $\mathcal{AF}(\mathcal{S})$ is an admissible extension of $\mathcal{AF}(\mathcal{S})$ that contains all the arguments that it defends.
- The minimal complete extension of $\mathcal{AF}(\mathcal{S})$ is called the *grounded extension* of $\mathcal{AF}(\mathcal{S})$,[4] and a maximal complete extension of $\mathcal{AF}(\mathcal{S})$ is called a *preferred extension* of $\mathcal{AF}(\mathcal{S})$. A complete extension $\mathcal{E}$ of $\mathcal{AF}(\mathcal{S})$ is called a *stable extension* of $\mathcal{AF}(\mathcal{S})$ if $\mathcal{E} \cup \mathcal{E}^+ = \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$.
- We write $\mathsf{Naive}(\mathcal{AF}(\mathcal{S}))$ [respectively: $\mathsf{Adm}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Cmp}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Prf}(\mathcal{AF}(\mathcal{S}))$, $\mathsf{Stb}(\mathcal{AF}(\mathcal{S}))$] for the set of all the naive [respectively: admissible, complete, preferred, stable] extensions of $\mathcal{AF}(\mathcal{S})$ and $\mathsf{Grd}(\mathcal{AF}(\mathcal{S}))$ for the unique grounded extension of $\mathcal{AF}(\mathcal{S})$.

Properties of the extensions defined above can be found in [41]. Further types of extensions are considered, e.g., in [17–19].

[4] As is shown in [41, Theorem 25], the grounded extension of $\mathcal{AF}(\mathcal{S})$ is unique.

Skeptical and credulous approaches for making inferences from the above-mentioned extensions are now defined as follows:

Definition 6 **(extension-based entailments)** Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}\rangle$ be a logical argumentation framework, and let $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. We denote:

- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\mathsf{Grd}} \psi$ if there is an argument $\langle \mathcal{T}, \psi\rangle \in \mathsf{Grd}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$,[5]
- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\cap\mathsf{Sem}} \psi$ if there is an argument $\langle \mathcal{T}, \psi\rangle \in \bigcap \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$,
- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\cup\mathsf{Sem}} \psi$ if there is an argument $\langle \mathcal{T}, \psi\rangle \in \bigcup \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$.

Example 2 Consider the argumentation framework of Fig. 2. In this figure, the grounded extension consists only of A_0 and $A_\top$, and the naive/preferred/stable extensions are the following:

- $\mathcal{E}_1 = \{A_\top, A_0, A_1, A_2, A_4, A_5, A_6\}$,
- $\mathcal{E}_2 = \{A_\top, A_0, A_1, A_3, A_4, A_8\}$,
- $\mathcal{E}_3 = \{A_\top, A_0, A_2, A_3, A_5, A_7\}$,
- $\mathcal{E}_4 = \{A_\top, A_0, A_1, A_2, A_3, A_4, A_5\}$.

Similarly, the grounded extension of the full framework $\mathcal{AF}(\mathcal{S}_1)$ defined in Example 1 for $\mathcal{S}_1 = \{r, p, q, \neg p \vee \neg q\}$ contains the arguments A_0 and $A_\top$ and the four naive/preferred/stable extensions of $\mathcal{AF}(\mathcal{S}_1)$ are supersets of $\mathcal{E}_i$ $(i = 1, 2, 3, 4)$. It follows that for every entailment $\mid\!\sim$ considered in Definition 6 we have that $\mathcal{S}_1 \mid\!\sim r$. The other formulas in $\mathcal{S}_1$ can only be credulously inferred: for every $\psi \in \mathcal{S}_1 - \{r\}$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$ we have that $\mathcal{S}_1 \mid\!\sim_{\cup\mathsf{Sem}} \psi$ but $\mathcal{S}_1 \not\mid\!\sim_{\cap\mathsf{Sem}} \psi$ and $\mathcal{S}_1 \not\mid\!\sim_{\mathsf{Grd}} \psi$.

Note 2 It is interesting to note that in the last example, $\mathsf{Conc}(\mathcal{E}_4) \vdash \mathsf{F}$, that is, the set of conclusions of the arguments in $\mathcal{E}_4$ is not consistent. The fact that not all argumentation frameworks always result in consistent extensions was pointed out in [36] and later extensively discussed in the literature. In Section 7 we will discuss consistency of extensions in more detail. We just note here that in what follows we provide two possible solutions to this phenomenon:

1. Replace Undercut by Direct Undercut as the sole attack rule. In this case $\mathcal{E}_4$ is no longer an admissible extension, yet the inferred conclusions discussed in Example 2 remain the same for the revised framework.
2. Incorporate hyperarguments instead of just arguments (see Section 6.1 and Example 17).

3 Patterns of reasoning with maximal consistency

In this section we consider several different patterns of reasoning with the maximally consistent subsets (MCS) of the premises, and show how these patterns can be represented by argumentation-based entailment relations. Unless otherwise stated, in this section we

[5] Recall that by the definition of $\mathsf{Grd}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ it holds that $\mathcal{T} \subseteq \mathcal{S}$. The same note holds for the other items in this definition.

shall use classical logic and a propositional language with the standard interpretations of its connectives.

3.1 Basic entailments

We start with a basic form of reasoning with maximal consistent sets, as explained in the next example.

Example 3 Consider again the set $\mathcal{S}_1 = \{r, p, q, \neg p \vee \neg q\}$ from Examples 1 and 2. This set has three maximally consistent subsets, each one contains r and two out of the three formulas in $\mathcal{S}_1' = \{p, q, \neg p \vee \neg q\}$. Intuitively, then, when *all* of these maximally consistent subsets are taken into account, the only formula in $\mathcal{S}_1$ that can be safely inferred from $\mathcal{S}_1$ is r, and when *some* of them are considered, any formula in $\mathcal{S}_1$ can be inferred from $\mathcal{S}_1$. This is formalized in the next definition.

Definition 7 ($\mid\!\sim^{\mathfrak{L}}_{\cap \mathsf{mcs}}, \mid\!\sim^{\mathfrak{L}}_{\cup \mathsf{mcs}}$) Let $\mathcal{S}$ be a set of formulas. We denote by $\mathsf{Cn}_{\mathfrak{L}}(\mathcal{S})$ the transitive closure of $\mathcal{S}$ with respect to the logic $\mathfrak{L}$ and by $\mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})$ the set of all the maximally $\mathfrak{L}$-consistent subsets of $\mathcal{S}$ (i.e., the set of the subsets $\mathcal{T} \subseteq \mathcal{S}$ such that $\mathcal{T} \nvdash_{\mathfrak{L}} \mathsf{F}$ and $\mathcal{T}' \vdash_{\mathfrak{L}} \mathsf{F}$ for any $\mathcal{T} \subsetneq \mathcal{T}' \subseteq \mathcal{S}$). We denote:

- $\mathcal{S} \mid\!\sim^{\mathfrak{L}}_{\cap \mathsf{mcs}} \psi$ iff $\psi \in \mathsf{Cn}_{\mathfrak{L}}(\bigcap \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S}))$.
- $\mathcal{S} \mid\!\sim^{\mathfrak{L}}_{\cup \mathsf{mcs}} \psi$ iff $\psi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})} \mathsf{Cn}_{\mathfrak{L}}(\mathcal{T})$.

The entailments $\mid\!\sim^{\mathfrak{L}}_{\cap \mathsf{mcs}}$ and $\mid\!\sim^{\mathfrak{L}}_{\cup \mathsf{mcs}}$ are sometimes called "free" and "existential", respectively (see [21, 23, 66][6]). In what follows, when $\mathfrak{L}$ is clear from the context, its notation will be omitted.

Example 3 (**continued**) Since $\mathsf{MCS}(\mathcal{S}_1) = \{\{r, p, q\}, \{r, p, \neg p \vee \neg q\}, \{r, q, \neg p \vee \neg q\}\}$, we have that $\mathcal{S}_1 \mid\!\sim_{\cap \mathsf{mcs}} r$ while $\mathcal{S}_1 \not\mid\!\sim_{\cap \mathsf{mcs}} \psi$ for every $\psi \in \mathcal{S}_1 - \{r\}$. Also, $\mathcal{S}_1 \mid\!\sim_{\cup \mathsf{mcs}} \psi$ for every $\psi \in \mathcal{S}_1$.

As the next theorem shows, the fact that in Examples 2 and 3 we reach the same conclusions from the same set of assertions is not a coincidence.[7]

Theorem 1 *Let $\mathcal{AF}(\mathcal{S})$ be a logical argumentation framework for $\mathcal{S}$, based on classical logic and Undercut as the sole attack rule. Then:*

1. $\mathcal{S} \mid\!\sim_{\mathsf{Grd}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cap \mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cap \mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cap \mathsf{mcs}} \psi$.
2. $\mathcal{S} \mid\!\sim_{\cup \mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cup \mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cup \mathsf{mcs}} \psi$.

Note 3 In [8] it is shown that under the conditions of the last theorem, for every $\mathcal{T} \in \mathsf{MCS}(\mathcal{S})$ there is a preferred or stable extension $\mathcal{E}$ of $\mathcal{AF}(\mathcal{S})$ such that $\mathcal{E} = \mathsf{Arg}_{\mathsf{CL}}(\mathcal{T})$. Yet, the converse of this result (i.e., that the stable and preferred extensions of $\mathcal{AF}(\mathcal{S})$ are of the form $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{T})$ for some $\mathcal{T} \in \mathsf{MCS}(\mathcal{S})$) is *not* true. The extension $\mathcal{E}_4$ of Example 2 provides a counter-example.

[6] All of these works refer to the particular case where $\mathfrak{L}$ is classical logic, CL.

[7] The results without references in Section 3 are taken from [8].

3.2 Inevitable entailments

Next, we refine the entailment relations that are considered in Definition 7. This type of relations was introduced in [66] as the *inevitable consequence*.[8] A motivation for this is given in the next example.

Example 4 Let $\mathcal{S}_2 = \{p \wedge q, \neg p \wedge q\}$. Here, $\bigcap \mathsf{MCS}(\mathcal{S}_2) = \emptyset$, and so only tautological formulas follow according to $\mid\!\sim_{\cap\mathsf{mcs}}$ from $\mathcal{S}_2$. Yet, one may argue that in this case formulas in $\mathsf{Cn}(\{q\})$ should also follow from $\mathcal{S}_2$, since they follow according to classical logic from every set in $\mathsf{MCS}(\mathcal{S}_2)$.

The last example gives rise to the following variation of $\mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}}$.

Definition 8 ($\mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{mcs}}$) We denote: $\mathcal{S} \mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$ iff $\psi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S})} \mathsf{Cn}_{\mathfrak{L}}(\mathcal{T})$.

Again, when $\mathfrak{L}$ is clear from the context or when its identity does not make a difference, it will be omitted from the notations.[9]

It is easy to see that if $\mathcal{S} \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$ then $\mathcal{S} \mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$. However, as follows from Example 4, the converse does not hold (indeed, $\mathcal{S}_2 \mid\!\sim_{\Cap\mathsf{mcs}} q$ while $\mathcal{S}_2 \not\mid\!\sim_{\cap\mathsf{mcs}} q$). For another example, let $\mathcal{S}_3 = \{p, q, p \supset \neg q\}$. Then $\mathcal{S}_3 \mid\!\sim_{\Cap\mathsf{mcs}} p \vee q$ but $\mathcal{S}_3 \not\mid\!\sim_{\cap\mathsf{mcs}} p \vee q$.

A natural counterpart of Definition 6 for dealing with the entailment of Definition 8 is the following:

Definition 9 (**extension-based entailments II**) Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}\rangle$ be a logical argumentation framework and let $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. We denote $\mathcal{S} \mid\!\sim^{\mathfrak{L},\mathcal{A}}_{\Cap\mathsf{Sem}} \psi$ if for every $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$ there is an argument $\langle \mathcal{S}_{\mathcal{E}}, \psi\rangle \in \mathcal{E}$ for some $\mathcal{S}_{\mathcal{E}} \subseteq \mathcal{S}$.

Indeed, for argumentation frameworks with DirUcut as the (single) attack rule, we have the following counterpart of Theorem 1.

Theorem 2 *Let $\mathcal{AF}(\mathcal{S})$ be a logical argumentation framework for $\mathcal{S}$, based on classical logic and Direct Undercut as the sole attack rule. Then $\mathcal{S} \mid\!\sim_{\Cap\mathsf{Prf}} \psi$ iff $\mathcal{S} \mid\!\sim_{\Cap\mathsf{Stb}} \psi$ iff $\mathcal{S} \mid\!\sim_{\Cap\mathsf{mcs}} \psi$.*

Note 4

1. It is interesting to note that in this case (unlike, e.g., the basic entailments; see Theorem 1), the grounded semantics does not coincide with the stable or the preferred semantics (this is the reason for the absence of $\mid\!\sim_{\mathsf{Grd}}$ from Theorem 2). Indeed, consider again the set $\mathcal{S}_2 = \{p \wedge q, \neg p \wedge q\}$ from Example 4. It holds that $\mathcal{S}_2 \mid\!\sim_{\Cap\mathsf{Stb}} q$ while $\mathcal{S}_2 \not\mid\!\sim_{\mathsf{Grd}} q$ (here, $\mathcal{S}_2 \mid\!\sim_{\mathsf{Grd}} \psi$ only if ψ is a tautology).
2. As shown in [8], another way of computing $\mid\!\sim_{\Cap\mathsf{mcs}}$ when the set of premises is finite, is by trading the argumentation framework of Theorem 2 by an argumentation framework with Undercut as the sole attack rule, and in which $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is replaced by $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}^{*})$, where $\mathcal{S}^{\star} = \{\phi_1 \vee \ldots \vee \phi_n \mid \phi_1, \ldots, \phi_n \in \mathcal{S}^{\wedge}\}$, and $\mathcal{S}^{\wedge} = \{\bigwedge \mathcal{S} \mid \mathcal{S}$ is a finite subset of $\mathcal{S}\}$.

[8] In [8] it is called "moderated entailment".

[9] As noted before, in this section unless $\mathfrak{L}$ is explicit, we refer to CL.

3.3 Lifting subset maximality

As we show next, there may be situations in which one would like to further refine the entailment relations considered in the last subsections.

Example 5 Consider the set $\mathcal{S}_4 = \{p \wedge q, \neg p\}$. According to $\mathrel{|\!\sim}_{\cap\mathsf{mcs}}$ and $\mathrel{|\!\sim}_{\Cap\mathsf{mcs}}$ only tautologies follow from $\mathcal{S}_4$, while in this case one would probably like to infer q (and everything in its transitive closure). This is possible by the following entailment relations, introduced by Benferhat, Dubois and Prade in [22, 23].

Definition 10 ($\mathrel{||\!\sim}^{\mathfrak{L}}_{\mathsf{mcs}}$) [22, 23] Given a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, a set $\mathcal{S}$ of $\mathcal{L}$-formulas, and an $\mathcal{L}$-formula ψ, we denote by $\mathcal{S} \mathrel{||\!\sim}^{\mathfrak{L}}_{\mathsf{mcs}} \psi$ that the following two conditions are satisfied:

1. It holds that $\mathcal{T} \vdash \psi$ for some $\mathfrak{L}$-consistent subset $\mathcal{T}$ of $\mathcal{S}$.
2. There is no $\mathfrak{L}$-consistent subset $\mathcal{T}'$ of $\mathcal{S}$ such that $\mathcal{T}' \vdash \neg\psi$.

Again, in what follows we shall frequently assume that $\mathfrak{L} = \mathsf{CL}$ (this is also what is assumed in [22] and [23]), in which case the superscript in the notations of the last definition will be omitted.

Note 5 If $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \psi$ or $\mathcal{S} \mathrel{|\!\sim}_{\Cap\mathsf{mcs}} \psi$, there is no consistent subset $\mathcal{T}$ of $\mathcal{S}$ such that $\mathcal{T} \vdash_{\mathsf{CL}} \neg\psi$. Thus, $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \psi$ and $\mathcal{S} \mathrel{|\!\sim}_{\Cap\mathsf{mcs}} \psi$ imply that $\mathcal{S} \mathrel{||\!\sim}_{\mathsf{mcs}} \psi$. The next example shows that the converse does not hold.

Example 5 (**continued**) We have that $\mathcal{S}_4 \mathrel{||\!\sim}_{\mathsf{mcs}} q$, while (as we have noted above) according to $\mathrel{|\!\sim}_{\cap\mathsf{mcs}}$ and $\mathrel{|\!\sim}_{\Cap\mathsf{mcs}}$ only tautologies follow from $\mathcal{S}_4$.

Again, logical argumentation provides a method of computing the entailment relations of Definition 10.

Theorem 3 *Let $\mathcal{AF}(\mathcal{S})$ be a logical argumentation framework for $\mathcal{S}$, based on classical logic and the attack rules Consistency Undercut and Defeating Rebuttal (ConUcut and DefReb; see* Table 1*). Then: $\mathcal{S} \mathrel{|\!\sim}_{\mathsf{Grd}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{Prf}} \psi$ iff $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{Stb}} \psi$ iff $\mathcal{S} \mathrel{||\!\sim}_{\mathsf{mcs}} \psi$.*

To gain some intuition on why Consistency Undercut and Defeating Rebuttal are useful for computing $\mathrel{||\!\sim}_{\mathsf{mcs}}$, we revisit Example 5.

Example 5 (**continued**) Consider again the set $\mathcal{S}_4 = \{p \wedge q, \neg p\}$. The arguments $\langle p \wedge q, p\rangle$ and $\langle \neg p, \neg p\rangle$ DefReb-attack each other. On the other hand, the only DefReb-attackers from $\mathsf{Arg}(\mathcal{S}_4)$ of arguments in $\mathsf{Arg}(\mathcal{S}_4)$ whose conclusion is q are those whose premise set is $\mathcal{S}_4$ itself. As $\mathcal{S}_4$ is not classically consistent, such attackers are counter attacked using ConUcut. It follows that arguments like $\langle p \wedge q, q\rangle$ are in the grounded extension of the logical argumentation framework of Theorem 3, and so in our case indeed $\mathcal{S}_4 \mathrel{|\!\sim}_{\mathsf{Grd}} q$.

4 Adding priorities

The use of priorities among the assertions, e.g. to model preferences among formulas, is an important and useful tool for representing knowledge and for non-monotonic reasoning. A wealth of research has been conducted on the formalization of reasoning with prioritized data (see, e.g., [20, 52, 59] for some recent overviews). Here we concentrate on extending the reasoning patterns of Section 3 to the prioritized case.[10]

4.1 Priority functions and prioritized MCS

A common method to make precedence among the assertions at hand is by assigning them quantitative measures, as defined next.

Definition 11 (**priority function**) A *priority function* for a language $\mathcal{L}$ is a function $\pi : \mathcal{L} \mapsto \mathbb{Q}^+$. Given a set of $\mathcal{L}$-formulas $\mathcal{S} = \{\phi_1, \ldots, \phi_n\}$, the π-value of $\mathcal{S}$ is defined as a set by $\pi(\mathcal{S}) = \{\pi(\phi) \mid \phi \in \mathcal{S}\}$; as a multiset by $\ddot{\pi}(\mathcal{S}) = [\pi(\phi_1), \ldots, \pi(\phi_n)]$; and as a π-vector by $\overrightarrow{\pi}(\mathcal{S})$, which is the tuple of the π-values of the formulas in $\mathcal{S}$, sorted in increasing order.

Intuitively, in what follows smaller values of π will represent higher priorities. The priorities may be provided by the reasoner, reflecting subjective preferences, or may be determined by some logical properties of the premises, as demonstrated in the next example.

Example 6 In the context of classical logic, Konieczny, Marquis, and Vesic [53] assign to a premise formula a score, representing the number of maximally consistent subsets it occurs in, namely: $\mathsf{score}_{\mathcal{S}}(\phi) = |\{\mathcal{T} \in \mathsf{MCS}(\mathcal{S}) \mid \phi \in \mathcal{T}\}|$. To get higher precedence (i.e., lower values) to formulas with higher scores, one may define:

$$\pi(\phi) = \begin{cases} \frac{1}{|\mathsf{MCS}(\mathcal{S})|+1} & \text{if } \psi \text{ is a (classical)tautology,} \\ \frac{1}{\mathsf{score}_{\mathcal{S}}(\phi)} & \text{otherwise, if } \mathsf{score}_{\mathcal{S}}(\phi) > 0, \\ 2 & \text{otherwise, if } \mathsf{score}_{\mathcal{S}}(\phi) = 0. \end{cases}$$

For instance, let $\mathcal{S}_5 = \{p, q, \neg q, \neg p \wedge \neg q, p \supset \neg q\}$. Then $\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}_5) = \{\{p, q\}, \{p, \neg q, p \supset \neg q\}, \{\neg q, \neg p \wedge \neg q, p \supset \neg q\}\}$, and so $\pi(p) = \pi(\neg q) = \pi(p \supset \neg q) = \frac{1}{2}$ and $\pi(q) = \pi(\neg p \wedge \neg q) = 1$.

A priority function π for $\mathcal{L}$ may be used for defining preference orders $\preceq_\pi$ on sets of $\mathcal{L}$-formulas. Some examples are considered next.

Example 7 Let π be a priority function for $\mathcal{L}$ and $\mathcal{S}_1, \mathcal{S}_2$ sets of $\mathcal{L}$-formulas. The following are possible conditions for letting $\mathcal{S}_1 \preceq_\pi \mathcal{S}_2$: [11]

1. $\mathcal{S}_1 \preceq_{\mathsf{mm}} \mathcal{S}_2$ if $\min(\pi(\mathcal{S}_1)) \leq \min(\pi(\mathcal{S}_2))$. In this case only the most preferred formulas in the sets are compared.
2. $\mathcal{S}_1 \preceq_{\mathsf{MM}} \mathcal{S}_2$ if $\max(\pi(\mathcal{S}_1)) \leq \max(\pi(\mathcal{S}_2))$. Here, the least preferred formulas in the sets are compared, thus every formula in $\mathcal{S}_1$ is preferred over at least one formula in $\mathcal{S}_2$.
3. $\mathcal{S}_1 \preceq_{\mathsf{Mm}} \mathcal{S}_2$ if $\max(\pi(\mathcal{S}_1)) \leq \min(\pi(\mathcal{S}_2))$. In this case all the formulas in $\mathcal{S}_1$ are at least as preferred as every formula in $\mathcal{S}_2$.
4. $\mathcal{S}_1 \preceq_{\mathsf{mmDif}} \mathcal{S}_2$ if $\min(\pi(\mathcal{S}_1 \setminus \mathcal{S}_2)) \leq \min(\pi(\mathcal{S}_2 \setminus \mathcal{S}_1))$. Like $\preceq_{\mathsf{mm}}$, the most preferred formulas are compared, but now with respect to the set difference operator.
5. $\mathcal{S}_1 \preceq_f \mathcal{S}_2$ if $f(\pi_{\mathcal{S}_1}) \leq f(\pi_{\mathcal{S}_2})$, where, for $i = 1, 2$, $\pi_{\mathcal{S}_i} \in \{\pi(\mathcal{S}_i), \ddot{\pi}(\mathcal{S}_i), \overrightarrow{\pi}(\mathcal{S}_i)\}$ (the exact form of $\pi_{\mathcal{S}_i}$ depends on the nature of f), and f is a numeric aggregation function (like the average, median, summation, or the max/min functions, as in previous items).

[10] Unless otherwise states, the material in this section is taken from [7].

[11] In this example, we let $\min(\emptyset) = \max(\emptyset) = f(\emptyset) = 0$.

6. $\mathcal{S}_1 \preceq_s \mathcal{S}_2$ if either $\mathcal{S}_1 = \emptyset$, or $\mathcal{S}_1 = \mathcal{S}_2$, or there is an $i \in \mathbb{Q}$, such that:
   - $\{\psi \in \mathcal{S}_1 \mid \pi(\psi) = i\} \supsetneq \{\psi \in \mathcal{S}_2 \mid \pi(\psi) = i\}$,
   - $\{\psi \in \mathcal{S}_1 \mid \pi(\psi) = j\} = \{\psi \in \mathcal{S}_2 \mid \pi(\psi) = j\}$ for every $j < i$.
7. $\mathcal{S}_1 \preceq_c \mathcal{S}_2$ if either $\mathcal{S}_1 = \emptyset$, or there is an $i \in \mathbb{Q}$ such that:
   - $|\{\psi \in \mathcal{S}_1 \mid \pi(\psi) = i\}| > |\{\psi \in \mathcal{S}_2 \mid \pi(\psi) = i\}|$,
   - $|\{\psi \in \mathcal{S}_1 \mid \pi(\psi) = j\}| = |\{\psi \in \mathcal{S}_2 \mid \pi(\psi) = j\}|$ for every $j < i$,

   or for every $i \in \mathbb{Q}$:

   $|\{\psi \in \mathcal{S}_1 \mid \pi(\psi) = i\}| = |\{\psi \in \mathcal{S}_2 \mid \pi(\psi) = i\}|$.
8. $\mathcal{S}_1 \preceq_{\mathsf{lex}} \mathcal{S}_2$ if $\vec{\pi}(\mathcal{S}_1)$ is lexicographically smaller than or equal to $\vec{\pi}(\mathcal{S}_2)$.

The order relations in Example 7 are used in different contexts. Items 1–7 are considered for prioritized logical argumentation in [7], the orderings $\preceq_{\mathsf{MM}}$ and $\preceq_s$ are applied in [39] (see Section 5.2 for more details), and $\preceq_{\mathsf{lex}}$ is used in [53]. The order relations in Items 6 and 7 are inspired by Brewka's approach to reasoning with preferred theories [33]. This approach is adjusted to our case by viewing the arguments' support sets as stratified theories, where each stratification consists of the formulas with the same π-value. Accordingly, $\preceq_s$ is a subset-inclusion comparison, and $\preceq_c$ is a comparison by cardinality. General considerations on which order to use for particular cases are beyond the scope of this paper. Such considerations (for abstract argumentation frameworks) are given, e.g., in [20, 52].

Note 6 If $\mathcal{S}_1 \preceq_\pi \mathcal{S}_2$ we shall sometimes write $\mathcal{S}_1 =_\pi \mathcal{S}_2$ and $\mathcal{S}_1 \prec_\pi \mathcal{S}_2$ to indicate, respectively, that $\mathcal{S}_2 \preceq_\pi \mathcal{S}_1$ and that $\mathcal{S}_2 \not\preceq_\pi \mathcal{S}_1$. Now,

1. Items 1, 2, 4, 6, 7 and 8 of Example 7 describe *pre-orders*, that is: $\preceq_\pi$ is reflexive ($\mathcal{S} \preceq_\pi \mathcal{S}$) and transitive (if $\mathcal{S}_1 \preceq_\pi \mathcal{S}_2$ and $\mathcal{S}_2 \preceq_\pi \mathcal{S}_3$ then $\mathcal{S}_1 \preceq_\pi \mathcal{S}_3$). Whether the relation in Item 5 is a pre-order depends on the function f.
2. The orders in Items 1, 4, 6, 7 and 8 and their strict counterparts are also *left monotonic*: if $\mathcal{S}_1 \preceq_\pi \mathcal{T}$ (respectively, if $\mathcal{S}_1 \prec_\pi \mathcal{T}$) and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\mathcal{S}_2 \preceq_\pi \mathcal{T}$ (respectively, then $\mathcal{S}_2 \prec_\pi \mathcal{T}$).

Given a priority function π for $\mathcal{L}$ and a corresponding preference order $\preceq_\pi$ on sets of $\mathcal{L}$-formulas (like those in Example 7) we lift $\preceq_\pi$ to arguments as follows:

$$A_1 \leq_\pi A_2 \text{ iff } \mathsf{Sup}(A_1) \preceq_\pi \mathsf{Sup}(A_2). \tag{1}$$

Intuitively, $A_1 \leq_\pi A_2$ indicates that the argument A_1 is at least as preferred as the argument A_2. In the sequel, we shall sometimes remove the subscript π when the priority function is known or fixed.

Example 8 In Example 1, let $\pi(r) = 1$, $\pi(p) = 2$, $\pi(q) = 3$ and $\pi(\neg p \vee \neg q) = 4$. Then, according to the instances for $\preceq_\pi$ from Example 7, we have that:

1. When the most preferred supports are compared we have that $A_0 <_{\mathsf{mm}} A_1 <_{\mathsf{mm}} A_2 <_{\mathsf{mm}} A_3$, $A_1 <_{\mathsf{mm}} A_7$, $A_8 <_{\mathsf{mm}} A_2$, $A_6 <_{\mathsf{mm}} A_3$, and $A_6 =_{\mathsf{mm}} A_8$.
2. When the least preferred supports are compared we still have $A_0 <_{\mathsf{MM}} A_1 <_{\mathsf{MM}} A_2 <_{\mathsf{MM}} A_3$, $A_1 <_{\mathsf{MM}} A_7$ and $A_6 <_{\mathsf{MM}} A_3$, but now $A_2 <_{\mathsf{MM}} A_8$ and only $A_6 <_{\mathsf{MM}} A_8$.
3. The max-min-comparison yields again $A_0 <_{\mathsf{Mm}} A_1 <_{\mathsf{Mm}} A_2 <_{\mathsf{Mm}} A_3$, $A_1 <_{\mathsf{Mm}} A_7$ and $A_6 <_{\mathsf{Mm}} A_3$, but this time A_2 and A_6 are $\leq_{\mathsf{Mm}}$-incomparable with A_8.

4. When the comparison takes place on the difference of the support sets, we have that $A_0 <_{\mathsf{mmDif}} A_1 <_{\mathsf{mmDif}} A_2 <_{\mathsf{mmDif}} A_3$, $A_1 <_{\mathsf{mmDif}} A_7$, $A_8 <_{\mathsf{mmDif}} A_2$ and $A_6 <_{\mathsf{mmDif}} A_3$, since the restriction on the support set has no effect here, and so the comparison is the same as the first item. However, $A_6 <_{\mathsf{mmDif}} A_8$, since $\pi(p) < \pi(\neg p \vee \neg q)$,
5. If $f(\mathcal{S}) = \frac{1}{|\mathcal{S}|} \sum_{\phi \in \mathcal{S}} \pi(\phi)$, then $A_2 =_f A_8$ and $A_6 <_f A_8$.
6. According to $\preceq_s$, we have that $A_8 <_s A_2$ and $A_6 <_s A_8$.
7. Similarly, according to $\preceq_c$, we have: $A_8 <_c A_2$ and $A_6 <_c A_8$.
8. The same holds for the lexicographic order: $A_8 <_{\mathsf{lex}} A_2$ and $A_6 <_{\mathsf{lex}} A_8$.

Since $A_\top$ has an empty support set, we have that $A_\top < A_i$ for every $1 \leq i \leq 8$ and every $\preceq_\pi$.

For reasoning with maximally consistent subsets in the prioritized case we now have to take into account also the priority function π. The set of the $\preceq_\pi$-most preferred maximally consistent subsets of $\mathcal{S}$ is defined as follows:

Definition 12 (**prioritized MCS**) Given a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, a set $\mathcal{S}$ of $\mathcal{L}$-formulas, and a preference order $\preceq$ (induced from a priority function π), we define:

$$\mathsf{MCS}_{\mathfrak{L}}^{\preceq}(\mathcal{S}) = \{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S}) \mid \text{there is no } \mathcal{T}' \in \mathsf{MCS}_{\mathfrak{L}}(\mathcal{S}) \text{ such that } \mathcal{T}' \prec \mathcal{T}\}.$$

Example 9 As noted in Example 3, $\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}_1) = \{\{r, p, q\}, \{r, p, \neg p \vee \neg q\}, \{r, q, \neg p \vee \neg q\}\}$. Thus, using the priority assignment π from Example 8 on $\mathcal{S}_1$, we have that according to the preference orders of Items 2, 4, 5 (where f is the average function), 6, 7 and 8 of Example 7, $\mathsf{MCS}_{\mathsf{CL}}^{\preceq_\pi}(\mathcal{S}_1) = \{\{r, p, q\}\}$, while according to Item 1 of the same example, $\mathsf{MCS}_{\mathsf{CL}}^{\preceq_\pi}(\mathcal{S}_1) = \{\{r, p, q\}, \{r, p, \neg p \vee \neg q\}\}$.

Example 10 Recall the priority assignment π on $\mathcal{S}_5$ from Example 6, and consider the preference order $\preceq_{\mathsf{lex}}$ from Item 8 of Example 7. As noted in Example 6, $\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}_5)$ consists of three sets: $\mathcal{T}_1 = \{p, q\}$, $\mathcal{T}_2 = \{p, \neg q, p \supset \neg q\}$, and $\mathcal{T}_3 = \{\neg q, \neg p \wedge \neg q, p \supset \neg q\}$. Their corresponding π-vectors are $\vec{\pi}(\mathcal{T}_1) = (\frac{1}{2}, 1)$, $\vec{\pi}(\mathcal{T}_2) = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\vec{\pi}(\mathcal{T}_3) = (\frac{1}{2}, \frac{1}{2}, 1)$. Thus, $\mathsf{MCS}_{\mathsf{CL}}^{\preceq_{\mathsf{lex}}}(\mathcal{S}_5) = \{\mathcal{T}_2\}$. This is the same set of preferred MCSs, obtained in [53] for $\mathcal{S}_5$.

Now we can consider the prioritized versions of the entailment relations from Definitions 7 and 8.

Definition 13 ($\mid\!\sim^{\mathfrak{L},\preceq_\pi}_{\cap\mathsf{mcs}}$, $\mid\!\sim^{\mathfrak{L},\preceq_\pi}_{\cup\mathsf{mcs}}$, $\mid\!\sim^{\mathfrak{L},\preceq_\pi}_{\Cap\mathsf{mcs}}$) Let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas, π a priority function on $\mathcal{L}$, and $\preceq_\pi$ a corresponding preference order. We denote:

- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\preceq_\pi}_{\cap\mathsf{mcs}} \psi$ iff $\psi \in \mathsf{Cn}_{\mathfrak{L}}(\bigcap \mathsf{MCS}_{\mathfrak{L}}^{\preceq_\pi}(\mathcal{S}))$.
- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\preceq_\pi}_{\cup\mathsf{mcs}} \psi$ iff $\psi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}^{\preceq_\pi}(\mathcal{S})} \mathsf{Cn}_{\mathfrak{L}}(\mathcal{T})$.
- $\mathcal{S} \mid\!\sim^{\mathfrak{L},\preceq_\pi}_{\Cap\mathsf{mcs}} \psi$ iff $\psi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}_{\mathfrak{L}}^{\preceq_\pi}(\mathcal{S})} \mathsf{Cn}_{\mathfrak{L}}(\mathcal{T})$.

Again, whenever possible we shall omit the superscript $\mathfrak{L}$ from the notations of the entailments.

Example 11 In Example 9 we have that $\mathcal{S}_1 \mid\!\sim_{\cup\mathsf{mcs}} \psi$ for every $\psi \in \mathcal{S}_1$, but $\mathcal{S}_1 \mid\!\sim_\star \phi$ when $\star \in \{\cap\mathsf{mcs}, \Cap\mathsf{mcs}\}$ only if $\phi \in \mathsf{CN}_{\mathsf{CL}}(\{r\})$ (since $\bigcap \mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}_1) = \{r\}$). In the prioritized case, when $\preceq$ is defined as in Items 2, 4, 5 (for the average function), 6, 7 and 8 of Example

7, we have that $\mathcal{S}_1 \mid\!\sim^{\preceq}_{\star} \phi$ for every $\star \in \{\cap\mathsf{mcs}, \cup\mathsf{mcs}, \Cap\mathsf{mcs}\}$ and $\phi \in \{p, q, r\}$. If $\leq_\pi$ is as in Item 1 of Example 7, then $\mathcal{S}_1 \mid\!\sim^{\preceq}_{\star} p$ and $\mathcal{S}_1 \mid\!\sim^{\preceq}_{\star} r$ for $\star \in \{\cap\mathsf{mcs}, \Cap\mathsf{mcs}\}$ and $\mathcal{S}_1 \mid\!\sim^{\preceq}_{\cup\mathsf{mcs}} \phi$ for every $\phi \in \mathcal{S}_1$.

In Example 10, for every $\star \in \{\cap\mathsf{mcs}, \cup\mathsf{mcs}, \Cap\mathsf{mcs}\}$ we have that $\mathcal{S}_5 \mid\!\sim^{\preceq_{\mathsf{lex}}}_{\star\mathsf{mcs}} \phi$ iff $\phi \in \mathsf{Cn}_{\mathsf{CL}}(\mathcal{T}_2)$.

In the next section we show how reasoning with the entailments of Definition 13 can be represented by (Dung semantics for) logical argumentation.

4.2 Prioritized argumentation frameworks

The definition of a logical argumentation framework, now with a priority function, is very similar to the one given in Definition 4, except that now the attacking arguments must be at least as preferred as the attacked arguments:

Definition 14 (**prioritized argumentation framework**) Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, $\mathcal{A}$ a set of attack rules with respect to $\mathfrak{L}$, π a priority function on $\mathcal{L}$, and $\leq_\pi$ a corresponding preference order on $\mathfrak{L}$-arguments (defined according to (1) above). Let also $\mathcal{S}$ be a set of $\mathcal{L}$-formulas. The *prioritized (logical) argumentation framework* for $\mathcal{S}$, induced by $\mathfrak{L}$, $\mathcal{A}$, and $\leq_\pi$, is the triple $\mathcal{AF}^{\leq_\pi}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}, \leq_\pi \rangle$, where $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set of the $\mathfrak{L}$-arguments whose supports are subsets of $\mathcal{S}$, and *Attack* is a relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, defined by $(A_1, A_2) \in \mathit{Attack}$ iff $A_1 \leq_\pi A_2$ and there is some $\mathcal{R} \in \mathcal{A}$ such that A_1 $\mathcal{R}$-attacks A_2 (in which case we shall say that A_1 $\mathcal{R}_{\leq_\pi}$-attacks A_2).

As before, we will omit the subscripts $\mathfrak{L}$, $\mathcal{A}$ and/or π if they are known or arbitrary.

Example 12 Consider a prioritized framework for $\mathcal{S}_1 = \{r, p, q, \neg p \vee \neg q\}$, based on CL and Undercut as the sole attack rule. The flat case (i.e., when all the arguments have the same priority) is the same as the framework of Example 1 (see Fig. 2). Suppose now that the π-assignment from Example 8 and $\preceq_s$ (the preference order in Item 6 of Example 7) are used. Figure 3 depicts (part of) the corresponding prioritized framework. In this case, A_1 and A_2 are no longer attacked, and while A_4 and A_7 $\mathcal{R}$-attack each other in the original framework, in the prioritized setting A_4 $\mathcal{R}_{\leq_\pi}$-attacks A_7 but not vice versa. Indeed, $\{p\} = \{\psi \in \mathsf{Sup}(A_4) \mid \pi(\psi) = 2\} \supsetneq \{\psi \in \mathsf{Sup}(A_7) \mid \pi(\psi) = 2\} = \emptyset$.

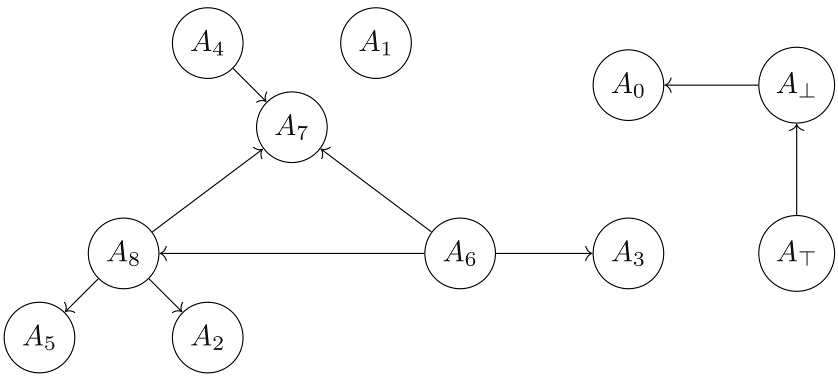

Fig. 3 Part of the prioritized argumentation framework for $\mathcal{S}_1 = \{r, p, q, \neg p \vee \neg q\}$ and $\preceq_s$ (Example 12).

The Dung-style semantics for the prioritized case is defined as in Definition 5, and so are the entailment relations, which are the following counterparts of those in Definitions 6 and 9.

Definition 15 (**entailments for prioritized AF**) Let $\mathcal{AF}^{\leq}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be a prioritized logical argumentation framework, $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, and $\star \in \{\cup, \cap, \Cap\}$. The entailment relations $\mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A},\leq}_{\star\mathsf{Sem}}$ are defined just as $\mathrel{|\!\sim}^{\mathfrak{L},\mathcal{A}}_{\star\mathsf{Sem}}$, but with respect to $\mathcal{AF}^{\leq}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ instead of $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$.

Example 13 Consider again a prioritized argumentation framework for $\mathcal{S}_1 = \{r, p, q, \neg p \vee \neg q\}$, based on CL and Undercut as the sole attack rule. When using the priority function π from Example 8, in any of the definitions for $\preceq_\pi$ from Example 7, A_0 and A_1 cannot be attacked. Thus, $\mathcal{S}_1 \mathrel{|\!\sim}^{\leq_\pi}_{\mathsf{Grd}} r$ and $\mathcal{S}_1 \mathrel{|\!\sim}^{\leq_\pi}_{\mathsf{Grd}} p$. Concerning q, the result depends on the choice of $\preceq_\pi$ (which determines the preference order $\leq_\pi$ on arguments by (1)):

- When using $\preceq_{\mathsf{mm}}$ (Item 1 of Example 7) $A_8 \leq_\pi A$, for any $A \in \{\langle \mathcal{S}, \psi\rangle \mid \emptyset \subsetneq \mathcal{S} \subseteq \mathcal{S}_1\}$. Moreover, since A_6 and A_8 attack each other, one can construct two different admissible sets, one in which A_6 defends A and one in which it does not. Therefore, $\mathcal{S}_1 \not\mathrel{|\!\sim}^{\leq_{\mathsf{mm}}}_{\mathsf{Grd}} q$.
- According to $\preceq_{\mathsf{mmDif}}$ and $\preceq_s$ (Items 4 and 6, respectively, of Example 7), A_8 does not attack A_6, thus A_6 is no longer attacked, and so it defends A_2. Hence, in this case, $\mathcal{S}_1 \mathrel{|\!\sim}^{\star}_{\mathsf{Grd}} q$ for $\star \in \{\leq_{\mathsf{mmDif}}, \leq_s\}$.

4.3 Relating the two approaches

We now show the correspondence between the MCS-based entailments for prioritized assertions (Definition 13) and entailments that are induced by prioritized argumentation frameworks (Definition 15). The two kinds of entailments coincide under the following conditions.

1. The base logic is *contrapositive*:
$\mathcal{T} \vdash \neg \bigwedge \mathcal{S}$ iff $\mathcal{T} \setminus \mathcal{T}', \mathcal{S}' \vdash \neg \bigwedge((\mathcal{S} \setminus \mathcal{S}') \cup \mathcal{T}')$ for every $\mathcal{T}' \subseteq \mathcal{T}$ and $\mathcal{S}' \subseteq \mathcal{S}$.
2. The preference order is *left monotonic*:
If $\mathcal{S}_1 \preceq_\pi \mathcal{T}$ and $\mathcal{S}_1 \subseteq \mathcal{S}_2$, then $\mathcal{S}_2 \preceq_\pi \mathcal{T}$.

Both of the conditions above are quite common. For instance, classical logic CL, intuitionistic logic (the central logic in the family of constructive logics), and normal modal logics, are all contrapositive. The preference orders in Items 1, 4, 6, 7 and 8 of Example 7 are left monotonic.

Theorem 4 *Let $\mathcal{AF}^{\leq}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be a prioritized argumentation framework for a finite $\mathcal{S}$, based on a contrapositive propositional logic $\mathfrak{L}$, the set $\mathcal{A}$ of the attack rules DirUcut and ConUcut (*Table 1*), and a preference order $\leq$ on the arguments that is induced by a left monotonic order $\preceq$ according to* (1). *Then:*

1. $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\cap\mathsf{Grd}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\cap\mathsf{mcs}} \psi$.
2. $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\cup\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\cup\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\cup\mathsf{mcs}} \psi$.
3. $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\Cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\Cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}^{\leq}_{\Cap\mathsf{mcs}} \psi$.

Note 7 The requirement that $\mathcal{S}$ is finite is indeed necessary for Theorem 4. To see this, let $\mathcal{S} = \{p_i \mid i \geq 1\} \cup \{q, \neg q\}$, and suppose that $\forall i \geq 1\ \pi(p_i) = 1$, $\pi(q) = 2$, and

$\pi(\neg q) = 3$. Consider now the prioritized argumentation framework $\mathcal{AF}^{\leq_s}_{\mathsf{CL},\mathcal{A}}(\mathcal{S})$, where the preference relation $\leq_s$ is defined by (1) and the order $\preceq_s$ of Example 7, and where $\mathcal{A}$ consists of the attack rules DirUcut and ConUcut. Denote: $\mathcal{S}^1 = \{p_i \mid i \geq 1\} \cup \{q\}$ and $\mathcal{S}^2 = \{p_i \mid i \geq 1\} \cup \{\neg q\}$. Then it is easy to see that $\mathsf{MCS}(\mathcal{S}) = \{\mathcal{S}^1, \mathcal{S}^2\}$ and $\mathsf{MCS}^{\preceq_s}\{\mathcal{S}^1\}$. Thus, for instance, $\mathcal{S} \mathrel{|\!\sim}^{\preceq_s}_{\cap\mathsf{mcs}} q$. Yet, we have that $\mathcal{S} \not\mathrel{|\!\sim}^{\leq_s}_{\cap\mathsf{Stb}} q$, since $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}^2) \in \mathsf{Stb}(\mathcal{S})$. Indeed, every argument $\langle \mathcal{T} \cup \{q\}, \{\psi\}\rangle \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S} \setminus \mathcal{S}^2)$ where $\mathcal{T} \subseteq \mathcal{S}$ is attacked by $\langle \mathcal{T} \cup \{p_k, \neg q\}, \{\neg q\}\rangle \in \mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}^2)$ where $p_k \notin \mathcal{T}$.[12]

Note 8 Consider the degenerate priority and preference orders π, $\preceq_\pi$, and $\leq_\pi$, in which all the formulas, sets of formulas, and argument (respectively) have the same priority/preference. As $\preceq_\pi$ is in particular left-monotonic, we get as a particular case of Theorem 4 the following corollary for the flat case and any contrapositive logic:

Corollary 1 *Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be a logical argumentation framework for a finite $\mathcal{S}$, based on a contrapositive propositional logic $\mathfrak{L}$ and a set $\mathcal{A}$ of the attack rules DirUcut and ConUcut. Then:*

1. $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{Grd}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \psi$.
2. $\mathcal{S} \mathrel{|\!\sim}_{\cup\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}_{\cup\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}_{\cup\mathsf{mcs}} \psi$.
3. $\mathcal{S} \mathrel{|\!\sim}_{\Cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}_{\Cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mathrel{|\!\sim}_{\Cap\mathsf{mcs}} \psi$.

Note 9 The contraposition requirement on the underlying logic is essential for the last corollary. To see this, consider Asenjo-Priest's 3-valued logic LP [14, 64, 65]. This logic consists of the truth values t, f and $\top$, where intuitively t represents truth, f represents falsity, and $\top$ represents conflicting (i.e., both true and false) situations. In LP, both t and $\top$ are designated (i.e., formulas having these values are considered valid) and f is non-designated. The language of LP consists of the connectives $\{\neg, \vee, \wedge\}$, where it holds that $\neg t = f$, $\neg f = t$, $\neg\top = \top$, and the interpretations of $\vee$ and $\wedge$ are given, respectively, by the maximum and the minimum function on the order $f < \top < t$. An interpretation ν is a model of a formula ψ if $\nu(\psi)$ is designated (i.e., if $\nu(\psi) = t$ or $\nu(\psi) = \top$).

Now, LP is *not* contrapositive. Indeed, $p, q \vdash_{\mathsf{LP}} p \wedge q$, but $p, \neg(p \wedge q) \not\vdash_{\mathsf{LP}} \neg q$ (a counter-model assigns $\top$ to p and t to q), and so Corollary 1 cannot be applied to it. Consider for instance the set $\mathcal{S} = \{p, \neg p\}$. In LP this set (as well as any other set) *is* consistent (and even satisfiable, as the valuation ν that assigns $\top$ to p is a model of $\mathcal{S}$), thus $\mathsf{MCS}(\mathcal{S}) = \{\mathcal{S}\}$. It follows (by the reflexivity of $\vdash_{\mathsf{LP}}$) that $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} p$ and $\mathcal{S} \mathrel{|\!\sim}_{\cap\mathsf{mcs}} \neg p$, yet $\mathcal{S} \not\mathrel{|\!\sim}_{\mathsf{Grd}} p$ and $\mathcal{S} \not\mathrel{|\!\sim}_{\mathsf{Grd}} \neg p$, since $\mathsf{Grd}(\mathcal{S})$ consists only of tautological arguments.

5 Using strict and defeasible assertions

A common practice in argumentation theory is to distinguish between strict and defeasible assertions. The former are taken for granted and so cannot be challenged (i.e, attacked), while the latter are debatable and so are attackable. According to the reasoning methods considered in Section 3, all the arguments are defeasible. One way of making a distinction between strict and defeasible arguments is to use the methods considered in Section 4, giving precedence to strict arguments over defeasible ones. Yet, these methods involve extra

[12] By reflexivity, the latter is obviously an argument in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}^2)$. Moreover, it attacks $\langle \mathcal{T} \cup \{q\}, \{\psi\}\rangle$, since $\mathcal{T} \cup \{p_k, \neg q\} \leq_s \mathcal{T} \cup \{q\}$ for every $p_k \notin \mathcal{T}$.

machinery already in the representation level, and do not necessarily avoid situations in which there are attacks among strict arguments. In this section we consider approaches in which strict and defeasible assumptions are totally separated, implemented by three instances of logical argumentation: assumption-based argumentation (ABA) frameworks, dialectical argumentation, and ASPIC systems. We show that these settings also have tight links to reasoning with maximal consistency.

5.1 Assumption-based frameworks

First, we examine the relations between entailments induced by assumption-based frameworks and MCS-based reasoning.[13] The next definition is a generalization of the definition from [28].

Definition 16 (assumption-based framework) An *assumption-based framework* (ABF, for short) is a tuple $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{S}, Ab, \sim \rangle$ where:

- $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is a propositional logic,
- $\mathcal{S}$ (the *strict assumptions*) and Ab (the *candidate/defeasible assumptions*) are distinct (countable) sets of $\mathcal{L}$-formulas, where the former is assumed to be $\vdash$-consistent and the latter is assumed to be nonempty,
- $\sim : Ab \to \wp(\mathcal{L})$ is a contrariness operator, assigning a finite set of $\mathcal{L}$-formulas to every defeasible assumption in Ab, such that for every $\psi \in Ab$ for which $\psi \not\vdash \mathsf{F}$, it holds that $\psi \not\vdash \bigwedge \sim \psi$ and $\bigwedge \sim \psi \not\vdash \psi$.

Note 10 Unlike the setting of [28], an ABF according to the definition above may be based on *any* propositional logic $\mathfrak{L}$. Also, the strict as well as the candidate assumptions are formulas that may not be just atomic. Concerning the contrariness operator, note that it is not a connective of $\mathcal{L}$, as it is restricted only to the candidate assumptions.

Defeasible assertions in an ABF may be attacked in the presence of counter defeasible information. This is described in the next definition.

Definition 17 (attacks in ABFs) Let $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{S}, Ab, \sim \rangle$ be an assumption-based framework, $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$. We say that Δ *attacks* ψ iff $\mathcal{S}, \Delta \vdash \phi$ for some $\phi \in \sim \psi$. Accordingly, Δ attacks Θ if Δ attacks some $\psi \in \Theta$.

The semantics of ABFs is similar to the one in Definition 5, reflecting the fact that only the elements of Ab are attackable.

Definition 18 (semantics for ABFs) ([28], see also [37, 70]) Let $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{S}, Ab, \sim \rangle$ be an assumption-based framework, and let $\Delta \subseteq Ab$. Below, maximum and minimum are taken with respect to set inclusion.

- Δ is *closed* (with respect to $\mathcal{ABF}$), iff $\Delta = Ab \cap \mathsf{Cn}_{\mathfrak{L}}(\mathcal{S} \cup \Delta)$.
- Δ is *conflict-free* (with respect to $\mathcal{ABF}$), iff there is no $\Delta' \subseteq \Delta$ that attacks some $\psi \in \Delta$.
- Δ is *naive* (with respect to $\mathcal{ABF}$), iff it is closed and maximally conflict-free.

[13] Unless otherwise states, the material in this section is taken from [48].

- Δ *defends* (with respect to $\mathcal{ABF}$) a set $\Delta' \subseteq Ab$, iff for every closed set Θ that attacks Δ' there is a set $\Delta'' \subseteq \Delta$ that attacks Θ.
- Δ is *admissible* (with respect to $\mathcal{ABF}$), iff it is closed, conflict-free, and defends every $\Delta' \subseteq \Delta$.
- Δ is *complete* (with respect to $\mathcal{ABF}$), iff it is admissible and contains every $\Delta' \subseteq Ab$ that it defends.
- Δ is *grounded* (with respect to $\mathcal{ABF}$), iff it is minimally complete.
- Δ is *preferred* (with respect to $\mathcal{ABF}$), iff it is maximally admissible.
- Δ is *stable* (with respect to $\mathcal{ABF}$), iff it is conflict-free, and attacks every $\psi \in Ab \setminus \Delta$.

As before, we denote by $\mathsf{Naive}(\mathcal{ABF})$ [respectively: $\mathsf{Adm}(\mathcal{ABF})$, $\mathsf{Cmp}(\mathcal{ABF})$, $\mathsf{Grd}(\mathcal{ABF})$, $\mathsf{Prf}(\mathcal{ABF})$ and $\mathsf{Stb}(\mathcal{ABF})$] the set of all the naive [respectively: admissible, complete, grounded, preferred and stable] extensions of $\mathcal{ABF}$. The induced entailments are defined as in Definition 6. For an assumption-based framework $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{S}, Ab, \sim \rangle$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$ we denote:

- $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{Sem}} \psi$ if $\mathcal{S}, \Delta \vdash \psi$ for every $\Delta \in \bigcap \mathsf{Sem}(\mathcal{ABF})$.
- $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{Sem}} \psi$ if $\mathcal{S}, \Delta \vdash \psi$ for every $\Delta \in \mathsf{Sem}(\mathcal{ABF})$.
- $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cup\mathsf{Sem}} \psi$ if $\mathcal{S}, \Delta \vdash \psi$ for some $\Delta \in \mathsf{Sem}(\mathcal{ABF})$.

Note that in the definition above, unlike the previous definitions of the entailment relations, we use the symbol $\uplus$ instead of the comma, to distinguish between the strict and the defeasible assumptions. As before, to shorten a bit the notations, we shall sometimes omit the symbol for the base logics, which will be clear from the context.

In what follows we shall concentrate on *simple contrapositive* ABFs. These are assumption-based frameworks of the following form:

1. The base logic $\mathfrak{L}$ is contrapositive (recall Item 1 in Section 4.3) and explosive (for every $\mathcal{L}$-formulas ψ, ϕ it holds that $\psi, \neg\psi \vdash \phi$),[14]
2. The contrariness operator is defined by the negation connective: for every $\psi \in Ab$, $\sim\psi = \{\neg\psi\}$.

As before, we shall now show the correspondence between Dung's semantics and MCS-based reasoning, this time for simple contrapositive ABFs. Given an assumption-based framework $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{S}, Ab, \sim \rangle$, the set of all the maximally $\mathfrak{L}$-consistent subsets of Ab with respect to $\mathcal{S}$ is defined by

$$\mathsf{MCS}(\mathcal{ABF}) = \{\mathcal{S} \cup \mathcal{T} \mid \mathcal{T} \subseteq Ab \text{ and } \mathcal{S} \cup \mathcal{T} \nvdash \mathsf{F} \text{ and } \mathcal{S} \cup \mathcal{T}' \vdash \mathsf{F} \text{ for every } \mathcal{T} \subsetneq \mathcal{T}' \subseteq Ab\}.$$

The induced entailment relations are defined by:

- $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$ iff $\psi \in \mathsf{Cn}_{\mathfrak{L}}(\bigcap \mathsf{MCS}(\mathcal{ABF}))$ (that is, $\bigcap \mathsf{MCS}(\mathcal{ABF}) \vdash \psi$).
- $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$ iff $\psi \in \bigcap_{\mathcal{T} \in \mathsf{MCS}(\mathcal{ABF})} \mathsf{Cn}_{\mathfrak{L}}(\mathcal{T})$ (that is, $\Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\mathcal{ABF})$).
- $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cup\mathsf{mcs}} \psi$ iff $\psi \in \bigcup_{\mathcal{T} \in \mathsf{MCS}(\mathcal{ABF})} \mathsf{Cn}_{\mathfrak{L}}(\mathcal{T})$ (that is, $\Delta \vdash \psi$ for some $\Delta \in \mathsf{MCS}(\mathcal{ABF})$).

[14] Again, classical logic, intuitionistic logic and standard modal logics are notable examples for logics that are both contrapositive and explosive.

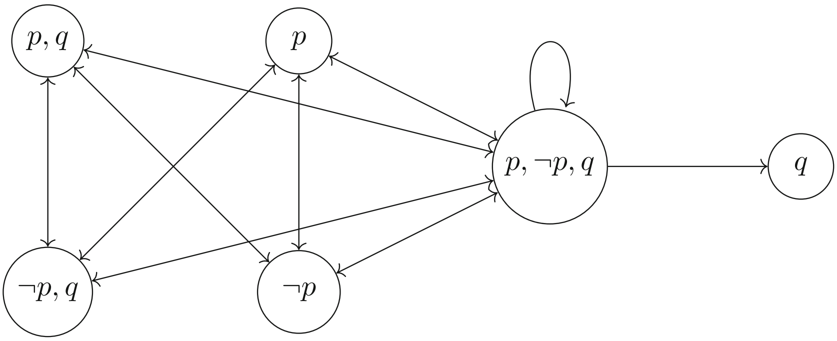

Fig. 4 Attack diagram for $\mathcal{S} = \emptyset$ and $Ab = \{p, \neg p, q\}$ (Example 14)

Theorem 5 *Let $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{S}, Ab, \sim \rangle$ be a simple contrapositive assumption-based framework. Then:*

1. $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\Cap\mathsf{mcs}} \psi$.
2. $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cup\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cup\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cup\mathsf{mcs}} \psi$.
3. *If* $\mathsf{F} \in Ab$ *then* $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{Grd}} \psi$ *iff* $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$.

Note 11 Item 3 in Theorem 5 follows from the fact that under the conditions of the theorem it holds that $\mathsf{Grd}(\mathcal{ABF}) = \bigcap \mathsf{MCS}(\mathcal{ABF})$ (see [48]). To see that the condition $\mathsf{F} \in Ab$ is indeed necessary for this, consider the following example (taken from [48]):

Example 14 Let $\mathcal{ABF} = \langle \mathfrak{L}, \mathcal{S}, Ab, \sim \rangle$ be a simple contrapositive assumption-based framework for $\mathfrak{L} = \mathsf{CL}$, $\mathcal{S} = \emptyset$, and $Ab = \{p, \neg p, q\}$. A corresponding attack diagram is shown in Fig. 4.

Note that the grounded set of assumptions in this case is the empty set, since there are no unattacked arguments. However, $\bigcap \mathsf{MCS}(\mathcal{ABF}) = \{q\}$. This may be intuitively explained by the fact that the inconsistent set $\{p, \neg p, q\}$ contaminates the argumentation framework, thus keeping q out of the grounded set of assumptions.

Consider now the same ABF, except that F is added to Ab. Note that $\{p, \neg p\} \vdash \mathsf{F}$ and consequently $\{p, \neg p\}$ is not closed, whereas $\{p, \neg p, q, \mathsf{F}\}$ is closed. Furthermore, $\emptyset \vdash \neg\mathsf{F}$ and so the grounded set of assumptions is $\{q\}$, as depicted by the attack diagram, shown in Fig. 5.

Note 12 The condition in Item 3 of Theorem 5 that $\mathsf{F} \in Ab$, may be lifted under the generalized attacks considered in [49]: a set $\Delta \subseteq Ab$ *disjunctively attacks* a set $\Theta \subseteq Ab$, if there is a finite subset $\Theta' \subseteq \Theta$ such that $\mathcal{S}, \Delta \vdash \bigvee \neg \Theta'$ (cf. Definition 17). Indeed, it is shown in [49] that for every simple contrapositive ABF with disjunctive attacks, and where the base logic satisfied de-Morgan's laws $\bigvee \neg \Delta \vdash \neg \bigwedge \Delta$ and $\neg \bigwedge \Delta \vdash \bigvee \neg \Delta$, a similar result as that of Theorem 5 still holds even when $\mathsf{F} \notin Ab$.

Note 13 Items 1 and 2 of Theorem 5 were independently shown in [29], where in addition it is also proved that for a contrapositive $\mathfrak{L}$ it holds that $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{Prf}} \psi$ iff $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{Stb}} \psi$ iff $\mathcal{S} \uplus Ab \mid\!\sim^{\mathfrak{L}}_{\cap\mathsf{mcs}} \psi$. The idea there is to consider logical argumentation frameworks that consist of (extended) arguments of the form $\langle \mathcal{S} \uplus Ab, \psi \rangle$ (cf. Definition 2), whose support sets consist of non-attackable (strict) formulas ($\mathcal{S}$), and attackable (defeasible) formulas (Ab).

5.2 Dialectical argumentation

Another setting in which arguments' supports consist of two different sets is considered in [38, 39], where D'Agostino and Modgil present a different way of lifting the set minimality and consistency restrictions of classical arguments (Definition 3). The splitting of the supports of the arguments to two disjoint sets is intuitively understood as follows: "An argument entails a conclusion from assumptions regarded as premises assumed to be true, and assumptions that are supposed true for the 'sake of argument' (i.e., those premises that an interlocutor commits to)" [39]. In the structures that are obtained in this way, called *dialectical argumentation frameworks*, Brewka's order on preferred subtheories [33] (recall Example 7, Item 7) can be represented.[15]

Definition 19 (**dialectical argument**) A *dialectical argument* is a triple $A = \langle \mathcal{S}, \mathcal{T}, \psi \rangle$, where $\mathcal{S}$ and $\mathcal{T}$ are disjoint sets of CL-formulas and ψ is a CL-formula, such that $\mathcal{S} \cup \mathcal{T} \vdash \psi$. We denote: $\mathsf{Sup}(A) = \mathcal{S}$, $\mathsf{Asm}(A) = \mathcal{T}$ and $\mathsf{Conc}(A) = \psi$.[16]

The dialectical arguments that are based on a set $\mathcal{S}$ of CL-formulas (i.e., the arguments A for which $\mathsf{Sup}(A) \cup \mathsf{Asm}(A) \subseteq \mathcal{S}$) are denoted by $\mathsf{Arg}_{\mathsf{dial}}(\mathcal{S})$. Also, in what follows we shall sometimes write $\phi = -\psi$ to indicate that $\phi = \neg\psi$ or $\psi = \neg\phi$.

Example 15 Let $\mathcal{S}_6 = \{p, \neg p, q\}$. The following dialectical arguments are some elements in $\mathsf{Arg}_{\mathsf{dial}}(\mathcal{S}_6)$:[17] $A_1 = \langle p, \emptyset, p \rangle$, $A_2 = \langle \neg p, \emptyset, \neg p \rangle$, $A_3 = \langle \{p, \neg p\}, \emptyset, \mathsf{F} \rangle$, $A_4 = \langle \emptyset, \{p, \neg p\}, \mathsf{F} \rangle$, $A_5 = \langle q, \emptyset, q \rangle$.

Given a priority function π over CL-formulas and a corresponding preference orders $\preceq_\pi$ on sets of CL-formulas (like those in Example 7), we lift $\preceq_\pi$ to dialectical arguments as follows (cf. (1) above): [18]

$$A_1 \leq_\pi A_2 \text{ iff } \mathsf{Sup}(A_1) \cup \mathsf{Asm}(A_1) \preceq_\pi \mathsf{Sup}(A_2) \cup \mathsf{Asm}(A_2). \tag{2}$$

The notion of dialectical defeat (attack) is defined accordingly as follows:

Definition 20 (**attack and defeat**) Let $\mathsf{Arg}_{\mathsf{dial}}(\mathcal{S})$ be a set of dialectical arguments based on $\mathcal{S}$, π a priority function, and $\leq_\pi$ a corresponding preference order on $\mathsf{Arg}_{\mathsf{dial}}(\mathcal{S})$ (defined according to (2) above). Let $A = \langle \mathcal{S}, \mathcal{T}, \psi \rangle$ and $B = \langle \mathcal{S}', \mathcal{T}', \psi' \rangle$ be dialectical arguments, and suppose that $\{A, B\} \cup \mathcal{E} \subseteq \mathsf{Arg}_{\mathsf{dial}}(\mathcal{S})$. Then:

- *A attacks B*, if:
  1. $\psi = \mathsf{F}$, or
  2. $\psi = -\phi$ for some $\phi \in \mathcal{S}'$.
- *A defeats B*, if:

[15] The notions and results in this section are taken from [39]; Notations are adjusted to this paper.

[16] In [38] the second component of a dialectical argument is called supposition. Here we denote it by Asm (for assumptions), to avoid confusion with Sup.

[17] Here and in what follows we omit the set signs when the assumptions or support of the arguments are singletons.

[18] In [39], instantiations are allowed with any preference relation $\preceq$ that satisfies the following requirement: if $\mathcal{S}_1 \cup \mathcal{T}_1 = \mathcal{S}_2 \cup \mathcal{T}_2$ and $\mathcal{S}_3 \cup \mathcal{T}_3 = \mathcal{S}_4 \cup \mathcal{T}_4$ then $\langle \mathcal{S}_1, \mathcal{T}_1, \phi_1 \rangle \preceq \langle \mathcal{S}_3, \mathcal{T}_3, \phi_3 \rangle$ iff $\langle \mathcal{S}_2, \mathcal{T}_2, \phi_2 \rangle \preceq \langle \mathcal{S}_4, \mathcal{T}_4, \phi_4 \rangle$ (i.e., the preference ordering is invariant under the distinction between assumptions and support). Clearly Condition (2), formulated to retain uniformity and simplicity, satisfies this requirement.

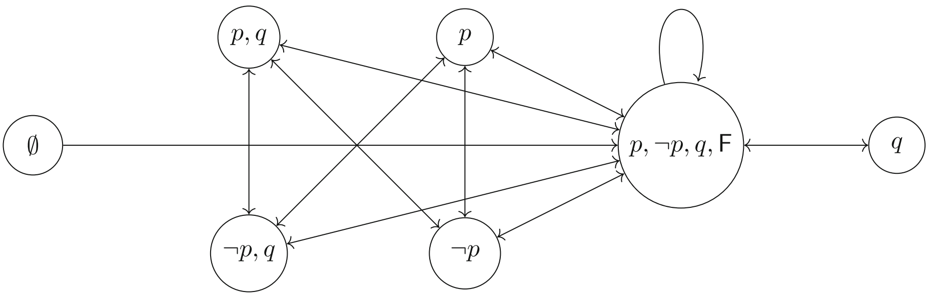

Fig. 5 Attack diagram for $\mathcal{S} = \emptyset$ and $Ab = \{p, \neg p, q, \mathsf{F}\}$ (Example 14).

1. $\psi = \mathsf{F}$ and $\mathcal{S} = \emptyset$, or
2. $\psi = -\phi$ for some $\phi \in \mathcal{S}'$ and $A \not>_\pi \langle\{\phi\}, \emptyset, \phi\rangle$.[19]

- *A defeats B with respect to* $\mathcal{E}$, if A defeats B, and $\mathsf{Asm}(A) \subseteq \mathsf{Sup}(B) \cup \mathsf{Sup}(\mathcal{E})$.

Definition 21 (**dialectical argumentation framework**) Let π a priority function on $\mathcal{L}$, and $\leq_\pi$ a corresponding preference order on dialectical arguments (defined according to (2) above). Let also $\mathcal{S}$ be a set of CL-formulas. The *dialectical (logical) argumentation framework* for $\mathcal{S}$, induced by $\leq_\pi$, is the triple $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathsf{dial}}(\mathcal{S}), Attack, \leq_\pi\rangle$, where *Attack* is the dialectical defeat relation on $\mathsf{Arg}_{\mathsf{dial}}(\mathcal{S}) \times \mathsf{Arg}_{\mathsf{dial}}(\mathcal{S})$ as defined in Definition 20.

Definition 22 (**dialectical argumentation semantics**) Let $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathsf{dial}}(\mathcal{S}), Attack, \leq_\pi\rangle$ be a dialectical argumentation framework (for $\mathcal{S}$, induced by $\leq_\pi$) and $\mathcal{E} \subseteq \mathsf{Arg}_{\mathsf{dial}}(\mathcal{S})$. Then:

- $\mathcal{E}$ is *dialectical conflict-free*, iff there are no $A, B \in \mathcal{E}$ such that A defeats B with respect to $\mathcal{E}$.
- B is *acceptable with respect to* $\mathcal{E}$, if for every A that defeats B with respect to $\mathcal{E}$, there is a $C \in \mathcal{E}$ that defeats A with respect to $\{A\}$.[20]
- $\mathcal{E}$ is *dialectically admissible* (with respect to $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S})$), iff every $A \in \mathcal{E}$ is dialectically acceptable with respect to $\mathcal{E}$.
- $\mathcal{E}$ is *dialectically complete* (with respect to $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S})$), iff it is dialectically admissible and for every $A \in \mathsf{Arg}_{\mathsf{dial}}(\mathcal{S})$, if A is dialectically acceptable with respect to $\mathcal{E}$, then $A \in \mathcal{E}$.
- A maximal complete extension of $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S})$ is called a *dialectically preferred extension* of $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S})$.
- A dialectically conflict-free extension $\mathcal{E}$ of $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S})$ is called a *dialectically stable extension* of $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S})$, iff $\mathcal{E} \cup \mathcal{E}^+ = \mathsf{Arg}_{\mathsf{dial}}(\mathcal{S})$.

As before, we denote the set of preferred (respectively, stable) extensions of $\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S})$ by $\mathsf{Prf}(\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S}))$ (respectively, $\mathsf{Stb}(\mathcal{AF}^{\leq_\pi}_{\mathsf{dial}}(\mathcal{S}))$).

[19] We note that in [38, 39] the preference order is reversed (meaning that higher values correspond to higher priorities), and so is the second condition in this item. We reversed the original order to keep the intuitive meaning of preference orders the same as those in Section 4.

[20] We note that C defeats A with respect to $\{A\}$ iff it defeats A with respect to $\emptyset$.

Now we can consider the dialectical versions of the entailment relations from Definition 7. Note that entailment is defined by taking into account arguments with the empty set of assumptions only.

Definition 23 (**extension-based entailments**) Let $\mathcal{AF}^{\leq\pi}_{\mathsf{dial}}(\mathcal{S})$ be a dialectical argumentation framework, and let $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$. We denote:

- $\mathcal{S} \mathrel{|\!\sim}^{\mathsf{dial},\leq\pi}_{\cap\mathsf{Sem}} \psi$ if there is a dialectical argument $\langle \mathcal{T}, \emptyset, \psi\rangle \in \bigcap \mathsf{Sem}(\mathcal{AF}^{\leq\pi}_{\mathsf{dial}}(\mathcal{S}))$,
- $\mathcal{S} \mathrel{|\!\sim}^{\mathsf{dial},\leq\pi}_{\cup\mathsf{Sem}} \psi$ if there is an argument $\langle \mathcal{T}, \emptyset, \psi\rangle \in \bigcup \mathsf{Sem}(\mathcal{AF}^{\leq\pi}_{\mathsf{dial}}(\mathcal{S}))$.

Example 16 Consider again Example 15 and suppose that all the arguments have the same priority. Below are some observations concerning the induced dialectical framework (for the non-prioritized case):

1. Since A_4 has the empty set as a support and its conclusion is F, it defeats A_1, A_2, A_3 and A_5.
2. A_4 defeats A_3 with respect to the empty set of arguments.
3. A_4 defeats A_1 with respect to $\{A_2\}$, since $\mathsf{Asm}(A_4) \subseteq \mathsf{Sup}(A_1) \cup \mathsf{Sup}(A_2)$ (note that A_4 does *not* defeat A_1 with respect to the empty set, since $\mathsf{Asm}(A_4) \not\subseteq \mathsf{Sup}(A_1)$).
4. Similarly, A_4 defeats A_2 with respect to $\{A_1\}$.
5. A_3 defeats itself with respect to the empty set of arguments since its conclusion is F and $\mathsf{Asm}(A_3) \subseteq \mathsf{Sup}(A_3)$.
6. Since A_3 defeat itself, any set containing A_3 cannot be dialectically conflict-free.
7. Since A_4 has the empty set as a support, it cannot be defeated. This means that A_4 is always acceptable. In view of this and Item 2, A_3 cannot be part of any dialectically admissible set.
8. A_1 defeats A_2 with respect to the empty set. Indeed, A_1 concludes p, thus it attacks A_2, and since all arguments have the same priority, this attack results in a defeat. This defeat is with respect to the empty set, since $\mathsf{Asm}(A_1) = \emptyset$.
9. Analogously, A_2 defeats A_1 with respect to the empty set.
10. A_5 is not defeated (with respect to the empty set), and so it is always acceptable. Note that the only defeat of A_5 is by A_4, but this defeat does not hold with respect to the empty set but only with respect to a set of arguments that includes p and $\neg p$ as assumptions.

The observations above imply that there are two dialectically preferred sets in this case: one contains A_1, A_4 and A_5, and the other one contains A_2, A_4 and A_5. Notice that these sets are dialectically stable as well. It follows that for a uniform π and for $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$, we have that $p, \neg p, q \mathrel{|\!\not\sim}^{\mathsf{dial},\leq\pi}_{\cap\mathsf{Sem}} p$ and $p, \neg p, q \mathrel{|\!\sim}^{\mathsf{dial},\leq\pi}_{\cup\mathsf{Sem}} p$. The same holds when $\neg p$ is inferred. On the other hand, $p, \neg p, q \mathrel{|\!\sim}^{\mathsf{dial},\leq\pi}_{\cap\mathsf{Sem}} q$.

Note 14 The example above indicates that in dialectical frameworks it is not necessary to use an attack like consistency undercut (ConUcut; see Table 1) to filter out inconsistent arguments, since an argument like A_4 filters out any inconsistent argument (like A_3).

We are now ready to show the correspondence between reasoning with preferred subtheories over the most consistent subsets of the premises, and dialectical frameworks using the $<_{\mathsf{MM}}$ ordering (obtained by (2) from Item 2 of Example 7) and assuming a total order over the premises.

Theorem 6 *Let $\mathcal{AF}^{<_{\mathsf{MM}}}_{\mathsf{dial}}(\mathcal{S})$ be a dialectical framework for a finite $\mathcal{S}$, where $<_{\mathsf{MM}}$ is the preference order, obtained by* (2) *and $\preceq_{\mathsf{MM}}$ (Item 2 of Example 7), using a total priority order π on the formulas in $\mathcal{S}$. Then:*

1. $\mathcal{S} \mid\!\sim^{\mathsf{dial},<_{\mathsf{MM}}}_{\cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{dial},<_{\mathsf{MM}}}_{\cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\preceq_s}_{\cap\mathsf{mcs}} \psi$.
2. $\mathcal{S} \mid\!\sim^{\mathsf{dial},<_{\mathsf{MM}}}_{\cup\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{dial},<_{\mathsf{MM}}}_{\cup\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\preceq_s}_{\cup\mathsf{mcs}} \psi$.

Note 15 The grounded semantics in this case is yet to be investigated.

5.3 ASPIC$^+$

ASPIC is a family of well-known, widely used and conceptually rich formalisms for logical argumentation, which also make a distinction between strict and defeasible assumptions. These formalisms include, among others, ASPIC$^+$ [57, 58], ASPIC$^-$ [35], ASPIC$^\ominus$ [51] and ASPIC$^{\mathsf{END}}$ [40]. In the context of this survey, we restrict our attention to ASPIC$^+$, which allows for preferences over defeasible premises and makes use of attacks of the form of restricted rebut and undermining (see [57, 58]). In more details, in ASPIC$^+$ arguments are built by constructing inference trees with strict and/or defeasible premises as leaves, and consequents of strict rules as non-leaf nodes.[21] An argument *undermines* another argument (in ϕ) iff the attacker concludes $\sim\phi$ for some defeasible premise ϕ occurring as a leaf in the attacked argument. Furthermore, defeat is determined as in Definition 14: an argument A defeats another argument B iff A undermines B (in ϕ) and $B \not\prec \phi$ according to the weakest link lifting (see Item 2 in Example 7). In [57] Modgil and Prakken show that for such frameworks, the preferred and stable extensions coincide and correspond to the set of preferred sub-theories of the set of premises under consideration (see [33] and Item 7 in Example 7). It follows that they provide a result that can be linked to Theorem 4 (where $\preceq_s$ is the preference order) and Theorem 6. Moreover, in [50, 61] it has been shown that ASPIC$^+$ and ABA are equi-expressive (when not taking into account priorities). Thus, the results from [48], stated in Section 5, carry over to ASPIC$^+$. This includes as a special case ASPIC$^+$-knowledge-bases without any defeasible rules (but a possibly nonempty set of defeasible premises), which is exactly the setting of this survey.

6 Further generalizations

In this section we consider two further generalizations to the settings described in the previous sections: one extends the notion of an argument and the other one relaxes the notion of consistency.

6.1 Hyperarguments

The following notion, corresponding to the notion of a *hypersequent* in the context of proof theory (see [15]), is a natural extension of the notion of an argument (cf. Definition 2).

Definition 24 (hyperargument) Given a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, an $\mathfrak{L}$-*hyperargument* (a *hyperargument* for short) is an expression of the form $H = \langle \mathcal{S}_1, \psi_1 \mid \ldots \mid \mathcal{S}_n, \psi_n \rangle$, where $\mid$ is a

[21] That is, a non-leaf node A can be connected to leafs $A_1, \ldots, A_n$ iff $A_1, \ldots, A_n \to A$ is a strict rule.

Table 2 Attack rules for hyperarguments

Rule Name	Acronym	Attacker	Attacked	Attack Conditions
H-Defeat	H-Def	H_1	H_2	$\vdash \mathsf{Conc}(H_1) \supset \neg \bigwedge \mathcal{S}_i \ \ (\mathcal{S}_i \neq \emptyset)$
H-Undercut	H-Ucut	H_1	H_2	$\vdash \mathsf{Conc}(H_1) \leftrightarrow \neg \bigwedge \mathcal{S}'_i \ (\mathcal{S}_i \supseteq \mathcal{S}'_i \neq \emptyset)$
H-Direct Undercut	H-DirUcut	H_1	H_2	$\vdash \mathsf{Conc}(H_1) \leftrightarrow \neg\psi \ \ (\exists i \ \psi \in \mathcal{S}_i)$
H-Consistency Undercut	H-ConUcut		H_2	$\vdash \neg \bigwedge \bigcup_{i-=1}^{n} \mathcal{S}_i \ (\bigcup_{i=1}^{n} \mathcal{S}_i \neq \emptyset)$
H-Rebuttal	H-Reb	H_1	H_2	$\vdash \mathsf{Conc}(H_1) \leftrightarrow \neg\psi_i \ \ (\langle \mathcal{S}_i, \psi_i \rangle$ in $H_2)$

In the table, H_1 and H_2 denote hyperarguments, where $\mathsf{Sup}(H_2) = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$ and $1 \leq i \leq n$

new symbol, not appearing in $\mathcal{L}$, and $\mathcal{S}_1, \ldots, \mathcal{S}_n \vdash \psi_1 \vee \ldots \vee \psi_n$. A pair $\mathcal{S}_i, \psi_i$ $(1 \leq i \leq n)$ is called a *component* of H.[22]

The common, intuitive interpretation of the sign "|" is by disjunction. Note, in particular, that an $\mathfrak{L}$-argument is a special case of an $\mathfrak{L}$-hyperargument (when $n = 1$).

Given a hyperargument $H = \langle \mathcal{S}_1, \psi_1 \mid \ldots \mid \mathcal{S}_n, \psi_n \rangle$, the support set (or the premise set) of H is $\mathsf{Sup}(H) = \{\mathcal{S}_1, \ldots, \mathcal{S}_n\}$, and the conclusion of H is the formula $\mathsf{Conc}(H) = \psi_1 \vee \ldots \vee \psi_n$. Accordingly, attack rules may be extended to hyperarguments as demonstrated in Table 2 (cf. Table 1).

Hyperargument-based logical frameworks are now defined just as ordinary logical argumentation frameworks (Definition 4), where arguments are replaced by hyperarguments and attack rules for arguments are traded by attack rules for hyperarguments. Dung's semantics and the entailments relations induced by such frameworks are defined just as in Section 2.

Note 16 The use of hyperarguments provides some flexibility in maintaining their components. For instance, when CL is the underlying logic, the following rule allows to split a hyperargument component into two new components:

$$\frac{\langle \mathcal{S}_1, \psi_1 \mid \ldots \mid \mathcal{S}_i^1 \cup \mathcal{S}_i^2, \psi_i^1 \vee \psi_i^2 \mid \ldots \mid \mathcal{S}_n, \psi_n \rangle}{\langle \mathcal{S}_1, \psi_1 \mid \ldots \mid \mathcal{S}_i^1, \psi_i^1 \mid \mathcal{S}_i^2, \psi_i^2 \mid \ldots \mid \mathcal{S}_n, \psi_n \rangle}$$

Example 17 Consider again the argumentation framework $\mathcal{AF}(\mathcal{S}_1)$ from Example 1, now in a hyperargument setting, where $\mathcal{S}_1 = \{r, p, q, \neg p \vee \neg q\}$, classical logic is the base logic, and H-Undercut is the sole attack rule. In addition to the arguments $A_0 - A_8$ in the setting of Example 1, we now get, by the splitting rule of Note 16, hyperarguments such as the following:

$$\begin{aligned} H_9 &= \langle \neg p \vee \neg q, \neg p \mid q, \neg p \rangle, \\ H_{10} &= \langle \neg p \vee \neg q, \neg q \mid p, \neg q \rangle, \\ H_{11} &= \langle p, p \wedge q \mid q, p \wedge q \rangle. \end{aligned}$$

See Fig. 6 for the extension of the graph in Fig. 2 (the dashed graph) with the additional hyperarguments and attacks (the solid parts of the graph).

Recall that as indicated in Note 2, $\mathcal{AF}(\mathcal{S}_1)$ has inconsistent extensions. This problem may be avoided by using a hyperargument-based framework as in the current

[22]In practice, the components are usually produced by a calculus for the core logic. For more details on this see [32].

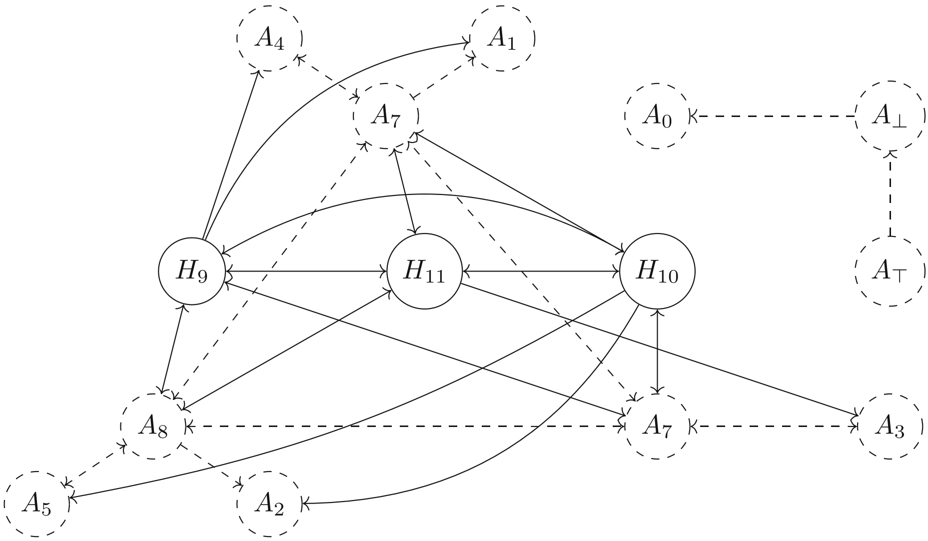

Fig. 6 Part of the hyperargument framework for $\mathcal{S}_1 = \{p, q, \neg p \vee \neg q, r\}$, based on classical logic and the attack rule H-Undercut (Example 17). The dashed graph is the same as the one in Fig. 2, the solid nodes and arrows become available when generalizing to the hyperargument-based setting

example. Indeed, in the present setting, the following three sets of hyperarguments are parts of different complete extensions: $\mathcal{E}_1 = \{A_\top, A_0, A_1, A_2, A_4, A_5, A_6, H_{11}\}$, $\mathcal{E}_2 = \{A_\top, A_0, A_2, A_3, A_5, A_7, H_9\}$ and $\mathcal{E}_3 = \{A_\top, A_0, A_1, A_3, A_4, A_8, H_{10}\}$ (see Fig. 6). Now, the 'problematic set' $\mathcal{E}_4 = \{A_\top, A_0, A_1, A_2, A_3, A_4, A_5\}$ discussed in Note 2 is no longer a complete extension, since, for instance, A_1 is attacked by H_9. In order to defend A_1, $\mathcal{E}_4$ must be extended with a hyperargument like A_6, A_8, or H_{10}, and so the new set of arguments is not conflict-free anymore. In Section 7 we show that the consistency of extensions of hyperargument-based frameworks, like the ones of this example, is guaranteed.

Argumentation frameworks that are based on hyperarguments are considered in detail in [32] (see also [30] and [31]). Apart of the benefits mentioned in the last example (see also Proposition 6 below), these frameworks are particularly useful for logics, such as the intermediate Gödel-Dummett logic LC and the modal logic S5, that lack 'good' (i.e., cut-free) proof systems for producing ordinary arguments. The correspondence to MCS-based reasoning of entailments induced by hyperargument-based frameworks for such logics is shown next.

Theorem 7 [32] *Let* $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ *be a hyperargument-based argumentation framework for* $\mathcal{S}$*, based on a logic* $\mathfrak{L} \in \{\mathsf{CL}, \mathsf{LC}, \mathsf{S5}\}$ *and the attack rules* $\mathcal{A} = \{\text{H-ConUcut}\} \cup \mathcal{A}'$*, where* $\emptyset \neq \mathcal{A}' \subseteq \{\text{H-Def}, \text{H-Ucut}\}$*. Then:*

1. $\mathcal{S} \mid\!\sim_{\mathsf{Grd}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$.
2. $\mathcal{S} \mid\!\sim_{\cup\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cup\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mid\!\sim_{\cup\mathsf{mcs}} \psi$.

Note 17 In [32] the theorem above is shown for a larger family of base logics (and a slightly different definition of hyperarguments). We refer to [32] for further details.

6.2 *g*-coherence

The second generalization to the settings of the previous sections is related to the notion of consistency. Here we trade it by the weaker notion of *coherence* (with respect to what we call below *reversing functions*). Accordingly, the set $\mathsf{MCS}(\mathcal{S})$ of the maximally consistent subsets of $\mathcal{S}$ is replaced by the set $\mathsf{MAX}_g(\mathcal{S})$ of the maximally g-coherent subsets of $\mathcal{S}$, where g is a function that allows to 'reverse' the roles of some premises and conclusions with respect to the consequence relation of the base logic. This is formalized below.[23]

Definition 25 (reversing function) Let $\rho(\mathcal{L})$ be the set of the finite sets of the formulas in $\mathcal{L}$. A function $g : \rho(\mathcal{L}) \to \mathcal{L}$ is called $\vdash_{\mathsf{CL}}$-*reversing*, if for every finite sets $\Gamma, \Delta, \mathcal{S}_1, \mathcal{S}_2$ of $\mathcal{L}$-formulas it holds that $\Gamma, \mathcal{S}_1 \vdash_{\mathsf{CL}} g(\mathcal{S}_2 \cup \Delta)$ iff$\Gamma, \mathcal{S}_2 \vdash_{\mathsf{CL}} g(\mathcal{S}_1 \cup \Delta)$.

Intuitively, $\vdash_{\mathsf{CL}}$-reversibility means that it is possible to "reverse" the roles of $\mathcal{S}_1$ and $\mathcal{S}_2$ in the two sides of $\vdash_{\mathsf{CL}}$.

Definition 26 (g-**coherence**) Let $g : \rho(\mathcal{L}) \to \mathcal{L}$.

- $\mathcal{S}_1, \mathcal{S}_2 \in \rho(\mathcal{L})$ are g-*reversible* (with respect to $\vdash_{\mathsf{CL}}$), if $\mathcal{S}_1 \vdash_{\mathsf{CL}} g(\mathcal{S}_2)$ or $\mathcal{S}_2 \vdash_{\mathsf{CL}} g(\mathcal{S}_1)$.
- $\mathcal{S}_1, \mathcal{S}_2 \in \rho(\mathcal{L})$ are g-*coherent* (with respect to $\vdash_{\mathsf{CL}}$), if there are no subsets $\mathcal{S}'_1$ and $\mathcal{S}'_2$ of $\mathcal{S}_1$ and $\mathcal{S}_2$ that are g-reversible.
- $\mathcal{S} \in \rho(\mathcal{L})$ is g-*coherent* (with respect to $\vdash_{\mathsf{CL}}$), if every $\mathcal{S}_1, \mathcal{S}_2 \subseteq \mathcal{S}$ are g-coherent.
- A g-coherent set $\mathcal{S}$ is *maximal*, if none of its proper supersets is g-coherent. We denote by $\mathsf{MAX}_g(\mathcal{S})$ the set of the maximally g-coherent subsets of $\mathcal{S}$.

Example 18 Here are some examples of $\vdash_{\mathsf{CL}}$-reversing functions:

1. For $\mathcal{S} \in \rho(\mathcal{L})$, let $g(\mathcal{S}) = \neg \bigwedge \mathcal{S}$. Then g is $\vdash_{\mathsf{CL}}$-reversing. It holds that $\mathcal{S}_1 \in \rho(\mathcal{L})$ and $\mathcal{S}_2 \in \rho(\mathcal{L})$ are g-coherent iff $\mathcal{S}_1 \cup \mathcal{S}_2$ is consistent, and $\mathcal{S} \in \rho(\mathcal{L})$ is g–coherent iff it is consistent. In this case, then, for a set $\mathcal{S}$ of formulas in $\mathcal{L}$, we have that $\mathsf{MAX}_g(\mathcal{S}) = \mathsf{MCS}(\mathcal{S})$.
2. For $\mathcal{S} \in \rho(\mathcal{L})$, let $g(\mathcal{S}) = \bigwedge \mathcal{S} \supset \phi$, where ϕ is a fixed formula. Intuitively, ϕ may represent a state of affairs that the reasoner wants to avoid. Again, g is $\vdash_{\mathsf{CL}}$-reversing, and this time $\mathcal{S}_1, \mathcal{S}_2 \in \rho(\mathcal{L})$ are g-coherent if their conjunction $\bigwedge(\mathcal{S}_1 \cup \mathcal{S}_2)$ does not imply ϕ. Hence, the elements in $\mathsf{MAX}_g(\mathsf{S})$ are the $\subseteq$-maximally consistent subsets of S that do not imply ϕ.

   Note that in case that a propositional constant F for representing falsity is available in $\mathcal{L}$, the function in this item generalizes the function in the previous item, since the previous function is obtained when $\phi = \mathsf{F}$.[24]

Reasoning with maximally g-coherent sets is now defined as in Definition 7.

Definition 27 ($\mid\!\sim^{\mathsf{CL}}_{\cap\mathsf{MAX}_g}, \mid\!\sim^{\mathsf{CL}}_{\cup\mathsf{MAX}_g}$) Let g be a $\vdash_{\mathsf{CL}}$-reversing function and let $\mathcal{S}$ a set of formulas.

- $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\cap\mathsf{MAX}_g} \psi$ iff $\psi \in \mathsf{Cn}_{\mathsf{CL}}(\bigcap \mathsf{MAX}_g(\mathcal{S}))$.

[23] The material in this section is taken from [8].

[24] Indeed, in CL the formulas $\neg \bigwedge \Gamma$ can equivalently be expressed by $\bigwedge \Gamma \supset \mathsf{F}$, thus the g-function of the previous item is the same as the function $g(\Gamma) = \bigwedge \Gamma \supset \mathsf{F}$.

- $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\cup\mathsf{MAX}_g} \psi$ iff $\psi \in \bigcup_{\mathcal{T} \in \mathsf{MAX}_g(\mathcal{S})} \mathsf{Cn}_{\mathsf{CL}}(\mathcal{T})$.

To show that the entailments of the last definition correspond to the entailments induced by logical argumentation frameworks, we need an attack relation that reflects g-reversibility.

Definition 28 (g-**Undercut**) Let g be $\vdash_{\mathsf{CL}}$-reversing. For $\mathcal{S}'_2 \neq \emptyset$ we define:

$$g\text{-Undercut} \quad (g\text{Ucut}): \langle \mathcal{S}_1, \psi_1 \rangle \text{ attacks } \langle \mathcal{S}_2 \cup \mathcal{S}'_2, \psi_2 \rangle \text{ iff } \vdash_{\mathsf{CL}} \psi_1 \leftrightarrow g(\mathcal{S}'_2).$$

The following theorem shows the correspondence between the entailment relations of Definition 27 and those that are induced by the corresponding argumentation framework.

Theorem 8 *Let g be a $\vdash_{\mathsf{CL}}$-reversing function and let $\mathcal{AF}_{\mathsf{CL},\{\mathsf{gUcut}\}}(\mathcal{S})$ be an argumentation framework for $\mathcal{S}$, based on classical logic and the attack rule gUcut. Let $\mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\mathsf{Grd}}$, $\mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\cap\mathsf{Prf}}$, $\mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\cap\mathsf{Stb}}$, $\mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\cup\mathsf{Prf}}$ and $\mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\cup\mathsf{Stb}}$ be the entailments induced by $\mathcal{AF}_{\mathsf{CL},\{\mathsf{gUcut}\}}(\mathcal{S})$ (see Definition 6). Then:*

1. $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\mathsf{Grd}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\cap\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\cap\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\cap\mathsf{MAX}_g} \psi$.
2. $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\cup\mathsf{Prf}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{CL},\{\mathsf{gUcut}\}}_{\cup\mathsf{Stb}} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\cup\mathsf{MAX}_g} \psi$.

Note 18 g-coherence is defined here with respect to classical logic. However, extending this notion to other logics is also possible. In fact, one of the main motivations of introducing g-coherence in [8] is to allow a weaker notion of consistency for logics like LP [14, 64, 65] (recall Note 9). This involves some extensions of the basic definitions. We refer to Section 7 of [8] for the details.

7 A postulate-based analysis

In this section we consider the settings of the previous sections from a postulate-oriented point of view. First, we concentrate on the attack relations and then check the frameworks and their semantic extensions.

7.1 Postulates concerning the attack relations

The results in the previous sections mainly answer the question whether some specific attack relations (in a context of some base logic) are able to represent reasoning with maximally consistent subsets. There is, however, a line of work that asks the reverse question: what are the conditions on the attack relations that guarantee the correspondence between argumentation-based entailment relations and reasoning with maximal consistent subsets. The gist of this research has been done for classical argumentation frameworks (see the paragraph below Definition 4). In this section, we summarize the main results of this research.[25]

We start by recalling the conditions on attack relations studied by [3, 72].

[25] Unless otherwise stated, the results in this Section 7.1 are taken from [72].

Definition 29 (properties of attack relations) Let $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), \mathit{Attack}\rangle$ be a classical argumentation framework for $\mathcal{S}$. Then *Attack* is said to be:

- *Conflict-dependent* (with respect to $\mathcal{AF}(\mathcal{S})$), iff for every $(\langle \mathcal{S}_1, \psi_1\rangle, \langle \mathcal{S}_2, \psi_2\rangle) \in \mathit{Attack}$ it holds that $\mathcal{S}_1 \cup \mathcal{S}_2 \vdash \mathsf{F}$.
- *Conflict-sensitive* (with respect to $\mathcal{AF}(\mathcal{S})$), iff for every $\langle \mathcal{S}_1, \psi_1\rangle, \langle \mathcal{S}_2, \psi_2\rangle \in \mathsf{Arg}(\mathcal{S})$, if $\mathcal{S}_1 \cup \mathcal{S}_2 \vdash \mathsf{F}$ then $(\langle \mathcal{S}_1, \psi_1\rangle, \langle \mathcal{S}_2, \psi_2\rangle) \in \mathit{Attack}$.
- *Valid* (with respect to $\mathcal{AF}(\mathcal{S})$), iff for every $\mathcal{E} \subseteq \mathsf{Arg}(\mathcal{S})$, if $\mathcal{E}$ is conflict-free then $\mathsf{Sup}(\mathcal{E})$ is consistent.
- *Conflict-complete* (with respect to $\mathcal{AF}(\mathcal{S})$), iff for every minimally inconsistent set $\mathcal{T} \subseteq \mathcal{S}$, for every $\mathcal{T}_1 \cup \mathcal{T}_2 \subseteq \mathcal{T}$ such that $\mathcal{T}_1 \neq \emptyset$, $\mathcal{T}_2 \neq \emptyset$ and $\mathcal{T}_1 \cup \mathcal{T}_2 = \mathcal{T}$, and for every $\langle \mathcal{T}_1, \psi_1\rangle \in \mathsf{Arg}(\mathcal{S})$, there is an argument $\langle \mathcal{T}_2, \psi_2\rangle \in \mathsf{Arg}(\mathcal{S})$ such that $\langle \mathcal{T}_2, \psi_2\rangle$ attacks $\langle \mathcal{T}_1, \psi_1\rangle$.
- *Symmetric* (with respect to $\mathcal{AF}(\mathcal{S})$), iff when $(\langle \mathcal{S}_1, \psi_1\rangle, \langle \mathcal{S}_2, \psi_2\rangle) \in \mathit{Attack}$, we also have that $(\langle \mathcal{S}_2, \psi_2\rangle, \langle \mathcal{S}_1, \psi_1\rangle) \in \mathit{Attack}$.

Note 19 In [72] it is shown that there is no classical argumentation framework $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), \mathit{Attack}\rangle$ for which *Attack* is both valid and conflict-dependent with respect to $\mathcal{AF}(\mathcal{S})$.[26]

So far we have concentrated on the question whether an argumentative entailment relation coincides with entailments for reasoning with maximal consistency. In [72], the satisfaction of the following, stronger condition is investigated:

$$\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}) = \{\mathsf{Sup}(\mathcal{E}) \mid \mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))\}\,. \tag{3}$$

The next result shows that any classical argumentation framework that is based on some conflict-free semantics, and in which Condition (3) is satisfied, will have a conflict-dependent attack relation.

Proposition 1 *Let* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), \mathit{Attack}\rangle$ *be a classical argumentation framework and let* Sem *be a semantics such that every* $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}(\mathcal{S}))$ *is conflict-free.*[27] *If Condition* (3) *is met, then Attack is conflict-dependent.*

It follows that when looking for an attack relation that satisfies Condition (3), we can restrict our attention to conflict-dependent attack relations.

By the last proposition and Note 19, we get:

Corollary 2 *If* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), \mathit{Attack}\rangle$ *is a classical argumentation framework and* Sem *is a semantics consisting only of conflict-free extensions, then if Attack is valid with respect to* $\mathcal{AF}(\mathcal{S})$*, Condition (3) is not satisfied.*

The next propositions show that besides valid attack relations, also symmetric and conflict-complete attack relations do not necessarily give rise to a correspondence between extensions and maximally consistent subsets.

[26] This, in turn, shows that the conditions of some of the results in [2, 3], regarding the links between reasoning with maximal consistent subsets and argumentation-based inferences, are in fact vacuous.

[27] All the semantics in Definition 5 satisfy this condition.

Proposition 2 *Let* Sem $\in$ {Cmp, Prf, Stb}. *There is a classical framework* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), Attack\rangle$ *with respect to which Attack is symmetric, and in which Condition* (3) *is not satisfied.*

Proposition 3 *Let* Sem $\in$ {Cmp, Prf, Stb}. *There is a classical framework* $\mathcal{AF}(\mathcal{S}) = \langle \mathsf{Arg}(\mathcal{S}), Attack\rangle$ *with respect to which Attack is conflict-complete, and in which Condition* (3) *is not satisfied.*

Based on the propositions above and using the results in [3, 72], Table 3 summarizes, for classical frameworks and a given attack relation $\mathcal{R}$, what properties from Definition 29 are satisfied (in case that *Attack* $= \{\mathcal{R}\}$), and for a given semantics Sem $\in$ {Prf, Stb}, whether Condition (3) holds (✓) or not (×).

Some notes are in order here:

- ConUcut does not appear in Table 3 since it is not applicable in classical frameworks.
- The correspondence to $\bigcap \mathsf{MCS}_{\mathsf{CL}}(\mathcal{S})$ (as investigated in Theorem 1) and logical frameworks that are not classical are not considered in [2, 3, 72]. A postulate-based study of these cases is therefore a topic yet to be studied.
- As Table 3 indicates, validity and conflict-sensitivity are rather strict postulates (at least as far as single attack rules are concerned). However, one may think of more liberal versions of these postulates that are satisfied by some attack rules. For instance, each one of the rules Def, DirDef, Ucut and DirUcut, satisfies a weaker version of validity (with a stronger condition), where the conflict-freeness of $\mathcal{E}$ is traded by its completeness (in which case we get the support consistency postulate of [9]).
- It is important to note that Condition (3) provides a useful indication to the suitability of the underlying framework for reasoning with maximal consistency, but it is *not* a necessary condition for the latter. Indeed, Example 2 provides an example in which Condition (3) is violated, yet – as e.g. Theorem 1 shows – the underlying framework *is* capable of representing reasoning with the maximally consistent subsets of the premises.

Table 3 The satisfaction of the properties from Definition 29 for classical frameworks, different attack rules and the satisfaction of Condition (3) with respect to Sem $\in$ {Prf, Stb}

Attack rule	conflict-dependence	conflict-sensitivity	validity	conflict-completeness	symmetry	Eq. 3 w.r.t. Prf	Eq. 3 w.r.t. Stb
Def	✓	×	×	✓	×	×	×
DirDef	✓	×	×	×	×	✓	✓
Ucut	✓	×	×	✓	×	×	×
DirUcut	✓	×	×	×	×	✓	✓
Reb	✓	×	×	×	✓	×	×
DefReb	✓	×	×	×	✓	×	×
Reb + DirUcut	✓	×	×	×	×	×	×
BigArgAt	✓	×	×	×	×	✓	✓

7.2 Postulates concerning the frameworks and their extensions

We now turn to rationality postulates concerning the frameworks described in some of the previous sections.[28] In what follows we shall refer to the postulates proposed by Caminada and Amgoud in [1, 34]. First, we need the following terminology:

Definition 30 ($\mathsf{Free}(\mathcal{S})$) Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic and $\mathcal{S}$ a set of $\mathcal{L}$-formulas.

- We say that $\mathcal{S}$ is $\vdash$-*consistent* if there is no finite $\mathcal{T} \subseteq \mathcal{S}$ such that $\vdash \neg \bigwedge \mathcal{T}$.[29]
- A subset $\mathcal{T} \subseteq \mathcal{S}$ is an $\mathfrak{L}$-*minimal conflict* of $\mathcal{S}$, if it is not $\vdash$-consistent, but $\mathcal{T} \setminus \{\psi\}$ *is* $\vdash$-consistent for every $\psi \in \mathcal{T}$.
- We denote by $\mathsf{Free}(\mathcal{S})$ the set of formulas that are not part of any minimal conflict of $\mathcal{S}$.

Definition 31 (rationality postulates) Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack \rangle$ be a logical argumentation framework, $\mathcal{E} \in \mathsf{Sem}(\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S}))$, and $A, B \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Consider the following postulates:[30]

- *Closure of extensions* (with respect to Sem): $\mathsf{Conc}(\mathcal{E}) = \mathsf{Cn}_{\mathfrak{L}}(\mathsf{Conc}(\mathcal{E}))$.
- *Closure under sub-arguments* (with respect to Sem): if $A \in \mathcal{E}$ and $\mathsf{Sup}(B) \subseteq \mathsf{Sup}(A)$ then $B \in \mathcal{E}$.
- *Consistency* (with respect to Sem): $\mathsf{Conc}(\mathcal{E})$ is consistent.
- *Exhaustiveness* (with respect to Sem): if $\mathsf{Sup}(A) \cup \{\mathsf{Conc}(A)\} \subseteq \mathsf{Conc}(\mathcal{E})$ then $A \in \mathcal{E}$.
- *Weak exhaustiveness* (with respect to Sem): if $\mathsf{Sup}(A) \subseteq \mathsf{Sup}(\mathcal{E})$ then $A \in \mathcal{E}$.
- *Free precedence* (with respect to Sem): $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{Free}(\mathcal{S})) \subseteq \mathcal{E}$.

The following follows from what is shown in [7]:

Proposition 4 *Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be a logical argumentation framework, based on a contrapositive propositional logic $\mathfrak{L}$ and where DirUcut is the attack rule, and let* $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}\}$. *Then closure of extensions, closure under sub-arguments, consistency, and weak exhaustiveness are satisfied with respect to* Sem. *When ConUcut is also an attack rule, free precedence is satisfied as well with respect to* Sem.

In case of priorities among arguments, closure under sub-arguments is weakened to the following postulate, stated for Sem-extensions and with respect to a preference order $\leq$ among arguments:

- *Weak closure under sub-arguments*: if $A \in \mathcal{E}$, $\mathsf{Sup}(B) \subseteq \mathsf{Sup}(A)$, and $B \leq A$, then $B \in \mathcal{E}$.

Proposition 5 *Let $\mathcal{AF}^{\leq_\pi}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be a prioritized argumentation framework, based on a contrapositive propositional logic $\mathfrak{L}$, and where DirUcut is the attack rule. Suppose also that the preferential order $\preceq_\pi$ and its strict counterpart $\prec_\pi$ (from which $\leq_\pi$ is obtained by Condition* (1) *in* Section 4*) are reflexive, transitive, and left monotonic. Then closure of*

[28] For frameworks that are not mentioned in this section the satisfiability of the postulates is an open question.
[29] To keep the terminology of [1] and [34], we use in this section a different notion for $\vdash$-consistency than the one used previously in this paper (namely, that $\mathcal{S} \not\vdash \mathsf{F}$). Note, nevertheless, that for explosive logics (i.e., in which for every $\mathcal{L}$-formulas ψ, ϕ it holds that $\psi, \neg\psi \vdash \phi$) these two notions coincide.
[30] We refer to [1, 34] for a discussion and justifications of these postulates.

extensions, weak closure under sub-arguments, consistency, and weak exhaustiveness are satisfied with respect to any Sem ∈ {Cmp, Prf, Stb}. *When ConUcut is also an attack rule, free precedence is satisfied as well with respect to any* Sem ∈ {Cmp, Prf, Stb}.

Note 20 The following remarks and examples indicate that the conditions specified in Propositions 4 and 5 are indeed necessary:

1. When Ucut is the attack rule (instead of DirUcut), consistency is not necessarily satisfied. Indeed, the extension $\mathcal{E}_4$ in Example 2 is not consistent.[31]
2. Even though weak exhaustiveness is satisfied by the frameworks considered in Proposition 4 and 5, exhaustiveness is not always satisfied by the frameworks considered in these propositions:

Example 19 ([7]) Let $\mathcal{S}' = \{p \wedge q, q, s, \neg s, t \wedge (\neg s \vee \neg q), \neg t\}$ with $\pi(p \wedge q) = 1$, $\pi(s) = \pi(\neg s) = \pi(t \wedge (\neg s \vee \neg q) = \pi(\neg t) = 2$ and $\pi(q) = 3$, and let $\leq_\pi$ be any order of those in Items 1, 4, 5 or 6 of Example 7. Consider $A_1 = \langle s, t \wedge (\neg s \vee \neg q)\}, \neg q\rangle$ and $A_2 = \langle q, q\rangle$. Note that $\mathcal{E} = \{\langle p \wedge q, \phi\rangle \mid \{p \wedge q\} \vdash \phi\} \cup \{\langle \emptyset, \phi\rangle \mid \emptyset \vdash \phi\}$ is a complete extension when DirUcut is the attack rule. In this case it can be shown that $A_2 \notin \mathcal{E}$, since A_1 attacks A_2 and A_2 is unattacked by $\mathcal{E}$. Moreover, it can be easily checked that $\mathcal{E}$ does not defend any argument in $\mathsf{Arg}_{\mathsf{CL},\{\mathrm{DirUcut}\}}(\mathcal{S}') \setminus \mathcal{E}$. This means that this argumentation framework violates exhaustiveness, since there is an argument with conclusion q in $\mathcal{E}$ (namely $\langle p \wedge q, q\rangle$), yet $A_2 = \langle q, q\rangle \notin \mathcal{E}$.

3. When ConUcut is among the attack rules, sub-argument closure for complete extensions can be violated:

Example 20 ([7]) Let $\mathcal{S}'' = \{p \wedge q, s, r, r \supset (\neg p \wedge \neg s)\}$ and $\pi(p \wedge q) = 1$, $\pi(r) = \pi(r \supset (\neg p \wedge \neg s)) = 2$ and $\pi(s) = 3$ where $\leq_\pi$ is defined as in Example 7, Item 1. Consider the following arguments: $A_1 = \langle s, s\rangle$, $A_2 = \langle\{p \wedge q, s\}, s\rangle$, $A_3 = \langle\{p \wedge q, r, r \supset (\neg p \wedge \neg s), \neg s\rangle$, and $A_4 = \langle \emptyset, \neg((p \wedge q) \wedge r \wedge (r \supset (\neg p \wedge \neg s))\rangle$. Note that there is a complete extension containing A_2 yet not containing A_1, since A_2 has only one attacker, A_3, which is ConUcut-attacked by A_4 and thus cannot be defended. This amounts to a violation of sub-argument closure, since $\mathsf{Sup}(A_1) \subset \mathsf{Sup}(A_2)$.

Similar results to those of Propositions 4 and 5 are shown in [32] for frameworks that are based on hyperarguments. Here we recall one of them, concerning classical logic CL and intermediate Gödel-Dummett logic LC.

Proposition 6 *Let $\mathcal{AF}_{\mathfrak{L},\mathcal{A}}(\mathcal{S})$ be a hyperargument-based argumentation framework for $\mathcal{S}$, based on a logic $\mathfrak{L} \in$ {CL, LC} and the attack rules $\mathcal{A} = \{\mathrm{H}-ConUcut\} \cup \mathcal{A}'$, where $\emptyset \neq \mathcal{A}' \subseteq \{\mathrm{H}-Def, \mathrm{H}-Ucut\}$. Then closure of extensions, closure under sub-arguments, consistency, free precedence and exhaustiveness are satisfied with respect to any* Sem ∈ {Cmp, Prf, Stb}.

[31] See Example 17 in Section 6.1 for a discussion how the above problem is solved with hyperarguments. In [49, Proposition 3.18], it is shown how this example can be solved in the context of assumption-based argumentation as well.

Table 4 Arguments construction rules according to LK

Rule Name	Acronym	Rule's conditions	Rule's conclusion
Axiom			$\langle \psi, \psi \rangle$
Weakening		$\langle \mathcal{S}, \mathcal{T} \rangle$	$\langle \mathcal{S} \cup \mathcal{S}', \mathcal{T} \cup \mathcal{T}' \rangle$
Cut		$\langle \mathcal{S}_1, \mathcal{T}_1 \cup \{\psi\} \rangle$, $\langle \mathcal{S}_2 \cup \{\psi\}, \mathcal{T}_2 \rangle$	$\langle \mathcal{S}_1 \cup \mathcal{S}_2, \mathcal{T}_1 \cup \mathcal{T}_2 \rangle$
Left-$\wedge$	[$\wedge$L]	$\langle \mathcal{S} \cup \{\psi\} \cup \{\phi\}, \mathcal{T} \rangle$	$\langle \mathcal{S} \cup \{\psi \wedge \phi\}, \mathcal{T} \rangle$
Right-$\wedge$	[$\wedge$R]	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi\} \rangle$, $\langle \mathcal{S}, \mathcal{T} \cup \{\phi\} \rangle$	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi \wedge \phi\} \rangle$
Left-$\vee$	[$\vee$L]	$\langle \mathcal{S} \cup \{\psi\}, \mathcal{T} \rangle$, $\langle \mathcal{S} \cup \{\phi\}, \mathcal{T} \rangle$	$\langle \mathcal{S} \cup \{\psi \vee \phi\}, \mathcal{T} \rangle$
Right-$\vee$	[$\vee$R]	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi\} \cup \{\phi\} \rangle$	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi \vee \phi\} \rangle$
Left-$\supset$	[$\supset$L]	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi\} \rangle$, $\langle \mathcal{S} \cup \{\phi\}, \mathcal{T} \rangle$	$\langle \mathcal{S} \cup \{\psi \supset \phi\}, \mathcal{T} \rangle$
Right-$\supset$	[$\supset$R]	$\langle \mathcal{S} \cup \{\psi\}, \mathcal{T} \cup \{\phi\} \rangle$	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi \supset \phi\} \rangle$
Left-$\neg$	[$\neg$L]	$\langle \mathcal{S}, \mathcal{T} \cup \{\psi\} \rangle$	$\langle \mathcal{S} \cup \{\neg\psi\}, \mathcal{T} \rangle$
Right-$\neg$	[$\neg$R]	$\langle \mathcal{S} \cup \{\psi\}, \mathcal{T} \rangle$	$\langle \mathcal{S}, \mathcal{T} \cup \{\neg\psi\} \rangle$

8 Proof systems

Reasoning with (maximally) consistent subsets is computationally demanding, as already detection of [in]consistency in classical logic is a [co] NP-complete problem. Moreover, the number of maximally inconsistent subsets of a premise set $\mathcal{S}$ may grow exponentially in the size of $\mathcal{S}$, and as shown in [69], computing the size of $\mathsf{MCS}(\mathcal{S})$ is beyond the second level of the polynomial hierarchy. This calls upon finding effective methods for reasoning with MCSs. In this section we briefly present such a method, based on what is known as *dynamic proof systems* (or, *dynamic derivations*), introduced in [10, 12]. For a more detailed description of dynamic proof systems and their properties (together with some further examples), we refer to [12, 13].

Our approach is a proof-theoretical one, using inference rules in Gentzen-style sequent calculi [46] for constructing arguments from simpler arguments, and elimination rules for renouncing arguments. Sequent calculi are particularly useful in our case, since an argument $\langle \mathcal{S}, \psi \rangle$ may be viewed as a sequent $\mathcal{S} \Rightarrow \psi$ (see, e.g. [11]). For instance, Table 4 presents a reformulation of Gentzen's well-known proof system LK for classical logic in terms of our representation of arguments. Note that here the conclusion of an argument may be a *set* of formulas (and not only a single formula), and that $\langle \mathcal{S}, \mathcal{T} \rangle$ is provable in LK[32] for finite sets $\mathcal{S}$ and $\mathcal{T}$ iff $\langle \bigwedge \mathcal{S}, \bigvee \mathcal{T} \rangle$ is a CL-argument in the sense of Definition 2.

In what follows we shall assume that the proof systems that we are using are sound and complete for the underlying logic, that is, given a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and a proof system $\mathfrak{C}$ for it, we have that $\mathcal{S} \vdash \psi$ iff $\langle \mathcal{S}, \psi \rangle$ is provable in $\mathfrak{C}$.

Now, (simple) *dynamic derivations* are sequences of applications of inference rules like those in Table 4 (for producing arguments), and attack rules like those in Table 1 (for renouncing, or discharging arguments). Such derivations may be viewed as representing debates, in which arguments are introduced or eliminated in the presence of counter-arguments. For defining dynamic proof systems we therefore need a *proof setting* $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathcal{A} \rangle$ consisting of a logic $\mathfrak{L}$, a corresponding sound and complete proof calculus $\mathfrak{C}$ for

[32] That is, there is a sequence of expressions of the form $\langle \mathcal{S}_i, \mathcal{T}_i \rangle$ $i = 1, \ldots, n$ such that $\langle \mathcal{S}_n, \mathcal{T}_n \rangle = \langle \mathcal{S}, \mathcal{T} \rangle$, and each expression is either an LK-axiom, or is obtained from previous expressions in the sequence by an application of an LK-rule.

producing $\mathfrak{L}$-arguments, and a set $\mathcal{A}$ of attack rules for renouncing (undefended) attacked arguments. An argument $\langle \mathcal{S}, \psi \rangle$ that is renounced (i.e., is attacked by an application of a rule in $\mathcal{A}$) will be denoted in what follows by $\cancel{\langle \mathcal{S}, \psi \rangle}$.

Given a setting $\mathfrak{S}$ and a set $\mathcal{S}$ of $\mathcal{L}$-formulas, an $\mathfrak{S}$-based *simple dynamic derivation* $\mathcal{D}$ for $\mathcal{S}$ is a finite sequence consisting of three kinds of subsequences:

1. derivations of $\mathcal{S}$-based arguments (i.e., elements in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$),
2. derivations of expressions of the form $\langle \mathcal{T}_1, \mathcal{T}_2 \rangle$,
3. eliminations of $\mathcal{S}$-based arguments.

Items 1 and 2 above are produced by the inference rules in $\mathfrak{C}$, and Item 3 is produced by the attack rules in $\mathcal{A}$. In an $\mathcal{S}$-based dynamic derivation, an argument in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ can be renounced only if it is attacked by another argument in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Items 1 and 3 are the 'heart' of the derivation; The purpose of Item 2 is to provide derivations of the arguments in Item 1, or to produce the conditions that are required for applying the attack rules in Item 3 (see the rightmost column in Table 1).[33]

Example 21 Consider the proof setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \{\text{Ucut}\} \rangle$ and the set of formulas $\mathcal{S}_6 = \{p, \neg p, q\}$. A simple $\mathfrak{S}$-derivation for $\mathcal{S}_6$ may look as follows (for simplicity, we omit the set notations $\{\cdot\}$ whenever possible and add numbers to refer to the elements in the derivation according to their order).

$$\begin{gathered} 1 : \langle p, p \rangle (\text{Axiom}), \quad 2 : \langle \neg p, \neg p \rangle (\text{Axiom}), \ \cdots, \\ k : \langle \emptyset, p \leftrightarrow \neg\neg p \rangle, \quad k+1 : \cancel{\langle \neg p, \neg p \rangle}\, (\text{Ucut of 1 on 2 by k}) \end{gathered}$$

The first two elements in the derivations are $\mathcal{S}_6$-based arguments (allowing to conclude p and $\neg p$, respectively). The k'th element in the sequence indicates that $p \leftrightarrow \neg\neg p$ is a tautology (follows from an empty support set), and since LK is sound and complete for CL, this implies that $\vdash_{\mathsf{CL}} p \leftrightarrow \neg\neg p$. Thus, the condition for the Ucut-attack of $\langle p, p \rangle$ (Argument 1 in the derivation) on $\langle \neg p, \neg p \rangle$ (Argument 2) is met (see Table 1), and so this attack can be applied on step k+1 of the derivation, which causes the renouncing of $\langle \neg p, \neg p \rangle$. Note that the sequence above may be extended by a derivation of $\langle \emptyset, \neg p \leftrightarrow \neg p \rangle$, which provides the condition for a counter Ucut-attack of $\langle \neg p, \neg p \rangle$ on $\langle p, p \rangle$, and as a result a renouncing of the latter.

The last example shows that the status of an argument may be changed as a derivation progresses: a derived argument may be renounced later on, and if an attacker A_1 of a renounced argument A_2 is itself attacked (and so A_1 is renounced), the attack on A_2 may become obsolete, in which case A_2 should be validated again. These considerations are captured by the evaluation process of Fig. 7.

Given a simple derivation $\mathcal{D}$, the iterative top-down algorithm in Fig. 7 computes the following three sets: $\mathsf{Elim}(\mathcal{D})$ – the renounced (eliminated) arguments whose attacker is not already eliminated, $\mathsf{Attack}(\mathcal{D})$ – the arguments that attack an argument in $\mathsf{Elim}(\mathcal{D})$, and $\mathsf{Accept}(\mathcal{D})$ – the derived arguments in $\mathcal{D}$ that are not in $\mathsf{Elim}(\mathcal{D})$.

Definition 32 (**(strongly) coherent derivation**) A simple derivation $\mathcal{D}$ is *coherent*, if there is no argument that eliminates another argument, and later is eliminated itself, that

[33] Note, in particular, that the expressions derived in Item 2 are not necessarily $\mathcal{S}$-based arguments, as in Item 1.

Input: a simple derivation $\mathcal{D}$.

Let $\mathsf{Attack} := \mathsf{Elim} := \mathsf{Derived} := \emptyset$;

Given a simple derivation $\mathcal{D}$, while ($\mathcal{D}$ is not empty) do {

if the last element in $\mathcal{D}$ is a derived argument A, then

add A to the set $\mathsf{Derived}$;

if the last element in $\mathcal{D}$ is an attack of $A_1 \notin \mathsf{Elim}$ on A_2, then

add A_1 to Attack and A_2 to Elim;

remove the last element from $\mathcal{D}$ }

Let $\mathsf{Accept} := \mathsf{Derived} - \mathsf{Elim}$;

Output: Attack, Elim, Accept.

Fig. 7 Evaluation of a simple derivation

is: $\mathsf{Attack}(\mathcal{D}) \cap \mathsf{Elim}(\mathcal{D}) = \emptyset$. We say that $\mathcal{D}$ is *strongly coherent*, if $\mathsf{Sup}(\mathsf{Attack}(\mathcal{D})) = \bigcup_{A \in \mathsf{Attack}(\mathcal{D})} \mathsf{Sup}(A)$ is consistent.[34]

Example 22 Consider the simple derivation $\mathcal{D}$ of Example 21. We have: $\mathsf{Accept} = \mathsf{Attack} = \{\langle p, p\rangle\}$ and $\mathsf{Elim} = \{\langle \neg p, \neg p\rangle\}$. In particular, $\mathcal{D}$ is both coherent and strongly coherent.

Now we can define what dynamic derivations are.

Definition 33 (dynamic derivation) Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathcal{A}\rangle$ be a proof setting and let $\mathcal{S}$ be a set of formulas in $\mathcal{L}$. A *(dynamic)* $\mathfrak{S}$*-derivation* for $\mathcal{S}$ is a simple derivation $\mathcal{D}$ of one of the following forms:

a) $\mathcal{D}$ is a singleton consisting of an axiom of $\mathfrak{C}$.
b) $\mathcal{D}$ is an extension of a dynamic derivation by a derivation in $\mathfrak{C}$, where all the derived arguments are not in $\mathsf{Elim}(\mathcal{D})$.
c) $\mathcal{D}$ is an extension of a dynamic derivation by a sequence of renounced arguments, the attackers of which are in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and are not attacked by arguments in $\mathsf{Accept}(\mathcal{D}) \cap \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, and the attacks are based on conditions that are proved in $\mathcal{D}$.[35]

One may think of a dynamic derivation as a proof that progresses over derivation steps. At each step the current derivation is extended by a 'block' of introduced arguments or renounced arguments. As a result, the statuses of the arguments in the derivation are updated accordingly. In particular, a derived argument may be renounced in light of new derived arguments, but also the other way around is possible: a renounced argument may be 'restored' if its attacking argument is counter-attacked. It follows that previously derived data may not be derived anymore (and vice-versa) until and unless new derived information revises the state of affairs.

[34] As shown in [8], in the proof setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \{\text{Ucut}\}\rangle$, strong coherence implies coherence (but not vice-versa).

[35] This condition assures that the attacks are 'sound': the attacking arguments are not counter-attacked by an accepted $\mathcal{S}$-based argument.

The next definition, of the outcomes of a dynamic derivation, indicates when it is 'safe' to conclude that a derived argument must hold under any circumstances.

Definition 34 (final derivability) Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathcal{A} \rangle$ be a proof setting and $\mathcal{S}$ a set of $\mathcal{L}$-formulas.

- A formula ψ is *finally derived* (or *safely derived*) in a dynamic $\mathfrak{S}$-derivation $\mathcal{D}$ for $\mathcal{S}$, if there is an argument $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \cap \mathsf{Accept}(\mathcal{D})$ such that $\psi = \mathsf{Conc}(A)$, and for every extension $\mathcal{D}'$ of $\mathcal{D}$, still $A \in \mathsf{Accept}(\mathcal{D}')$.
- A formula ψ is *sparsely finally derived* in a dynamic $\mathfrak{S}$-derivation $\mathcal{D}$ for $\mathcal{S}$, if there is an argument $A \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \cap \mathsf{Accept}(\mathcal{D})$ such that $\psi = \mathsf{Conc}(A)$, and for every extension $\mathcal{D}'$ of $\mathcal{D}$ there is an argument $A' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \cap \mathsf{Accept}(\mathcal{D}')$ such that $\psi = \mathsf{Conc}(A')$.

Example 23 Consider again the proof setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \{\text{Ucut}\}\rangle$ and the set of formulas $\mathcal{S}_6 = \{p, \neg p, q\}$. The following $\mathfrak{S}$-derivation for $\mathcal{S}_6$ is obviously (strongly) coherent (since nothing is renounced in it):

$$1 : \langle q, q\rangle,\ \cdots,\ i : \langle \emptyset, p \vee \neg p\rangle,\ \cdots,\ j : \langle \emptyset, (p \vee \neg p) \leftrightarrow \neg(p \wedge \neg p)\rangle$$

Moreover, q is finally derived (and so also sparsely finally derived) in this derivation. Indeed, the only arguments in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}_6)$ that can potentially Ucut-attack $\langle q, q\rangle$ are of the form $\langle \{p, \neg p\}, \psi\rangle$ or $\langle \{p, \neg p, q\}, \psi\rangle$, where ψ is logically equivalent to $\neg q$. However, such arguments are counter-attacked by the argument $\langle \emptyset, p \vee \neg p\rangle$, obtained in step i of the derivation above, based on the attack condition in step j of the same derivation. It follows, by the conditions in Item (c) of Definition 33, that no renounced tuple in which $\langle q, q\rangle$ is attacked can be derived in any extension of the derivation above, thus q is finally derived in this derivation.

Example 24 To see the need for sparse final derivability, let again $\mathfrak{S} = \langle \mathsf{CL}, LK, \{\text{Ucut}\}\rangle$ and consider the set $\mathcal{S}_2 = \{p \wedge q, \neg p \wedge q\}$ from Example 4. Note that both $A_1 = \langle p \wedge q, q\rangle$ and $A_2 = \langle \neg p \wedge q, q\rangle$ are LK-derivable in this case, but neither of them is finally derivable, since any $\mathfrak{S}$-derivation that includes them can be extended with derivations of $A_3 = \langle \neg p \wedge q, \neg(p \wedge q)\rangle$ and $A_4 = \langle p \wedge q, \neg(\neg p \wedge q)\rangle$ that respectively Ucut-attack A_1 and A_2. Note, however, that these attacks cannot be applied *simultaneously*, since the attackers A_3 and A_4 counter-attack each other. It follows that in each extension either A_1 or A_2 is accepted, and so q is sparsely finally derived from $\mathcal{S}_2$.

The entailment relations induced by the dynamic proof systems described above are defined next.

Definition 35 ($\mid\!\sim^{\mathfrak{S}}_{\cap}$, $\mid\!\sim^{\mathfrak{S}}_{\Cap}$) Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathcal{A} \rangle$ be a proof setting, $\mathcal{S}$ a set of $\mathcal{L}$-formulas, and ψ an $\mathcal{L}$-formula.

- $\mathcal{S} \mid\!\sim^{\mathfrak{S}}_{\cap} \psi$ iff there is a coherent $\mathfrak{S}$-derivation $\mathcal{D}$ for $\mathcal{S}$ in which ψ is finally derived.
- $\mathcal{S} \mid\!\sim^{\mathfrak{S}}_{\Cap} \psi$ iff there is a strongly coherent $\mathfrak{S}$-derivation $\mathcal{D}$ for $\mathcal{S}$ in which ψ is sparsely finally derived.

Example 25 Let $\mathfrak{S} = \langle \mathsf{CL}, LK, \{\text{Ucut}\}\rangle$.

- By Examples 21 and 23 we have that $\{p, \neg p, q\} \mid\!\sim^{\mathfrak{S}}_{\cap} q$ and $\{p, \neg p, q\} \mid\!\sim^{\mathfrak{S}}_{\Cap} q$, while $\{p, \neg p, q\} \not\mid\!\sim^{\mathfrak{S}}_{\star} p$ and $\{p, \neg p, q\} \not\mid\!\sim^{\mathfrak{S}}_{\star} \neg p$ for either $\star \in \{\cap, \Cap\}$.

- By Example 24 we have that $\{p \wedge q, \neg p \wedge q\} \mid\!\sim^{\mathfrak{S}}_{\Cap} q$ (and it is easy to verify that $\{p \wedge q, \neg p \wedge q\} \not\mid\!\sim^{\mathfrak{S}}_{\Cap} p$ and $\{p \wedge q, \neg p \wedge q\} \not\mid\!\sim^{\mathfrak{S}}_{\Cap} \neg p$).

The next theorem, introduced in [8], shows that dynamic proofs are faithful for computing the three patterns of reasoning with maximal consistency considered in Section 3.

Theorem 9 *Let $\mathfrak{S} = \langle \mathsf{CL}, LK, \{Ucut\}\rangle$ and $\mathfrak{S}' = \langle \mathsf{CL}, LK, \{ConUcut, DefReb\}\rangle$ be two proof settings. Then for every finite set $\mathcal{S}$ of formulas and formula ψ, it holds that:*

- $\mathcal{S} \mid\!\sim^{\mathfrak{S}}_{\cap} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\cap \mathsf{mcs}} \psi$
- $\mathcal{S} \mid\!\sim^{\mathfrak{S}}_{\Cap} \psi$ *iff* $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\Cap \mathsf{mcs}} \psi$
- $\mathcal{S} \mid\!\sim^{\mathfrak{S}'}_{\cap} \psi$ *iff* $\mathcal{S} \mid\!\mid\!\sim^{\mathsf{CL}}_{\mathsf{mcs}} \psi$

Example 26 The first item of Example 25 demonstrates the first two items of the last theorem for $\mathcal{S}_6 = \{p, \neg p, q\}$ (Examples 21 and 23), as $\mathsf{MCS}_{\mathsf{CL}}(\mathcal{S}_6) = \{\{q\}\}$. The second item of Example 25 exemplifies the second item of Theorem 9, where $\mathcal{S}_2$ (Examples 4 and 24) is the set of assertions.

For the extension to g-coherent subsets of premises considered in Section 6.2, the following is shown in [8]:

Theorem 10 *Let g be a $\vdash_{\mathsf{CL}}$-reversing function and let $\mathfrak{S}_g = \langle \mathsf{CL}, LK, \{gUcut\}\rangle$ be a proof settings. Then for every finite set $\mathcal{S}$ of formulas and a formula ψ it holds that $\mathcal{S} \mid\!\sim^{\mathfrak{S}_g}_{\cap} \psi$ iff $\mathcal{S} \mid\!\sim^{\mathsf{CL}}_{\cap \mathsf{Max}g} \psi$.*

9 Summary and conclusion

In this paper we have surveyed different settings for reasoning with maximal consistency in the context of logical argumentation. This includes:

- Rescher and Manor's basic MCS-based entailments [66] and their representations by argumentation frameworks based on classical logic and Undercut,
- Rescher and Manor's inevitable MCS-based entailments [66] and their representations by argumentation frameworks based on classical logic and Direct Undercut,
- Benferhat, Dubois and Prade's entailments based on consistent subsets [22, 23] and their representations by argumentation frameworks based on classical logic together with Consistency Undercut and Defeating Rebuttal,
- Prioritized MCS-based reasoning, like those considered by Amgoud, Cayrol [5], and Vesic [6], for incorporating preference-based approaches like Brewka's preferred subtheories [33], and Konieczny, Marquis, and Vesic's ranking methods [53], represented by prioritized argumentation frameworks based on contrapositive logics together with Direct Undercut and Consistency Undercut,
- Reasoning with maximally consistent subsets of premises that include both strict and defeasible assertions, represented by Bondarenko, Dung, Kowalski and Toni's assumption-based argumentation (ABA) frameworks [28], based on contrapositive logics, and whose contrariness operator is defined by a negation connective,

- Reasoning with preferred subtheories over the most consistent subsets of the premises by D'Agostino and Modgil's dialectical argumentation frameworks [38, 39],
- Relations to the reasoning patterns depicted by the ASPIC$^+$ system [58],
- Extensions to hyperarguments and the correspondence of the associated argumentation frameworks to MCS-based reasoning,
- Reasoning with maximally g-coherent subsets of premise and their correspondence to entailment relations that are induced by argumentation frameworks based on classical logic and g-Undercut.

The two disciplines for handling inconsistency in the various contexts considered above are to some extent complementary: the MCS-based reasoning is defined in a simple and intuitively appealing way, while logical argumentation provides a machinery for justifying the conclusions, including the presentation of arguments that support the conclusions, and the construction of a partial argumentation graph that contains the supporting arguments. From a knowledge representation point of view this is therefore a useful combination.

Another benefit of providing different approaches to MCS-based reasoning is that from a computational perspective the latter may be quite demanding (as indicated in the first paragraph of Section 8). Thus, proof-theoretic procedures like the dynamic derivations considered in Section 8 may serve as a successful platform for reaching conclusions based on the data depicted by the maximally consistent premise sets.

When it comes to first-order languages, MCS-based reasoning becomes even more computationally demanding. In such cases, the provision of equivalent reasoning methods is crucial for applications in which maximal consistency has a primary role. A natural step for further research is therefore an investigation of the relations between MCS-based reasoning and logical argumentation in the context of higher-order languages, implementations of systems for such cases, and verifications of their behavior with respect to large datasets.

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