Inconsistency-Tolerance in Knowledge-Based Systems by Dissimilarities

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Abstract. Distance-based reasoning is a well-known approach for defining non-monotonic and paraconsistent formalisms, which so far has been mainly used in the context of standard two-valued semantics. In this paper, we extend this approach to *arbitrary denotational semantics* by considering *dissimilarities*, a generalization of distances. Dissimilarity-based reasoning is then applied for handling inconsistency in knowledge-based systems using various non-classical logics. This includes logics defined by multi-valued semantics, non-deterministic semantics, and possible-worlds (Kripke-style) semantics. In particular, we show that our approach allows to define a variety of inconsistency-tolerant entailment relations, and that it extends many well-studied forms of reasoning in the context of belief revision and database integration.

1 Introduction

A common, model-theoretic way, of defining a consequence relation for a logical system S, is to require that every model of the premises would also be a model of the conclusion. Symbolically, this can be represented as follows:

$$\Gamma \vdash_{\mathsf{S}} \psi \text{ if } mod_{\mathsf{S}}(\Gamma) \subseteq mod_{\mathsf{S}}(\psi). \qquad (1)$$

Logics that are based on this approach (including, e.g., classical logic, intuitionistic logic, and many forms of modal logics) face difficulties in handling inconsistent information, since, by (1), if Γ has no model it entails *any* conclusion. This problem has long been identified, and different solutions have been proposed to it. However, many of those solutions depend on the nature of the semantics at hand, and therefore they cannot be easily adapted to other contexts.

One way of properly maintaining inconsistency, and still being faithful to (1), is to substitute $mod_{\mathsf{S}}(\Gamma)$ in (1) by a nonempty set $\Delta_{\mathsf{S}}(\Gamma)$ that coincides with $mod_{\mathsf{S}}(\Gamma)$ whenever the latter is non-empty. Intuitively, $\Delta_{\mathsf{S}}(\Gamma)$ consists of the semantic elements that are 'as close as possible' to satisfying Γ. This intuition motivates the dissimilarity-based approach for defining $\Delta_{\mathsf{S}}(\Gamma)$, introduced in [6]. The main goal of this paper is to generalize this approach to *any* denotational semantics. To this end, we define in precise terms what a dissimilarity between

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semantic objects in a given denotational semantics is, and what properties it should satisfy in order to induce natural and useful entailments. This allows us to apply these abstract definitions, in a uniform way, on different kinds of denotational semantics, and not define them from scratch for each new type of semantics (as is done in [6]).

Next, given a logic L that is based on a denotational semantics, we provide a general way of constructing a logic L' that is an *inconsistency-tolerant* variant of L, in the sense that L' coincides with L with respect to consistent premises, and is non-trivial with respect to inconsistent premises. A major advantage of this approach is its uniformity: in order to construct an inconsistency-tolerant variant of one's favorite logic, one only needs to define a dissimilarity relation in this logic, and this induces a corresponding inconsistency-tolerant entailment. This approach may be useful, for instance, for applying distance-based strategies for revising or merging knowledge-bases, the semantics of which is not the standard classical one.

The rest of this paper is organized as follows: In the next section we recall some basic definitions and facts about denotational semantics and the logics induced by them. Then, in Section 3, we explain what we mean by 'inconsistency-tolerant' logics. In Section 4 we introduce dissimilarity-based semantic settings and show how they can be used for turning different types of logics, induced by denotational semantics, to inconsistency-tolerant ones. In Section 5 we examine some common properties of the entailment relations defined in Section 4, and in Section 6 we apply our framework to particular cases of denotational semantics. In Section 7 we conclude and consider some directions for future work.

2 Denotational Semantics and Their Logics

In the sequel, $\mathcal{L}$ denotes a propositional language with a countable set $\mathsf{Atoms} = \{p, q, r \ldots\}$ of atomic formulas and a (countable) set $\mathcal{F}_{\mathcal{L}} = \{\psi, \phi, \sigma, \ldots\}$ of well-formed formulas. A theory Γ is a finite set of formulas in $\mathcal{F}_{\mathcal{L}}$. The atoms appearing in the formulas of Γ and the subformulas of Γ are denoted, respectively, by $\mathsf{Atoms}(\Gamma)$ and $\mathsf{SF}(\Gamma)$. The set of all theories of $\mathcal{L}$ is denoted by $\mathcal{T}_{\mathcal{L}}$.

Definition 1. Given a language $\mathcal{L}$, a *propositional logic* for $\mathcal{L}$ is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, i.e., a binary relation between sets of formulas in $\mathcal{F}_{\mathcal{L}}$ and formulas in $\mathcal{F}_{\mathcal{L}}$, satisfying the following conditions:
Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$.

A common (model-theoretical) way of defining logics for $\mathcal{L}$ is based on the notion of *denotational semantics*:

Definition 2. A *denotational semantics* for a language $\mathcal{L}$ is a pair $\mathsf{S} = \langle S, \models_S \rangle$, where S is a nonempty set (of 'interpretations'), and $\models_S$ (the 'satisfiability relation' of S) is a binary relation on $S \times \mathcal{F}_{\mathcal{L}}$.

Example 1. The most common case of denotational semantics is classical logic, in which the elements of S are (two-valued) *valuations*, i.e., functions from the well-formed formulas of a standard propositional language to the set $\{t, f\}$ of the classical truth values, and $\models_S$ is the ordinary satisfaction relation, defined by $\nu \models_S \Gamma$ iff $\nu(\psi) = t$ for every $\psi \in \Gamma$.

Standard generalizations of classical logic to multiple-valued logics can also be described in terms of denotational semantics and so are, e.g., the various kinds of Kripke-structures for modal and for intuitionistic logics (some of which are described in greater detail in Section 6 below).

Let $\nu \in S$ be an interpretation and $\psi \in \mathcal{F}_\mathcal{L}$ a formula. If $\nu \models_S \psi$, we say that ν *satisfies* ψ and call ν an S-*model* of ψ. The set of the S-models of ψ is denoted by $mod_\mathsf{S}(\psi)$. If $mod_\mathsf{S}(\psi) = S$ then ψ is called an S-*tautology*, and if $mod_\mathsf{S}(\psi) = \emptyset$ then ψ is called an S-*contradiction*. If ν satisfies every formula ψ in a theory Γ, it is called an S-model of Γ. The set of the S-models of Γ is denoted by $mod_\mathsf{S}(\Gamma)$. If $mod_\mathsf{S}(\Gamma) \neq \emptyset$ we say that Γ is S-*consistent*, otherwise Γ is S-*inconsistent*. In what follows we shall sometimes omit the prefix S from the above notions.

Definition 3. A denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$ is *normal*, if for each $\nu \in S$ there is a formula ψ, such that $\nu \not\models_S \psi$.

Example 2. Any denotational semantics S for which there is an S-contradiction, is normal.

A denotational semantics S induces the following relation on $\mathcal{T}_\mathcal{L} \times \mathcal{F}_\mathcal{L}$:

Definition 4. $\Gamma \vdash_\mathsf{S} \psi$ if $mod_\mathsf{S}(\Gamma) \subseteq mod_\mathsf{S}(\psi)$.

The following proposition is easily verified.[1]

Proposition 1. *Let* $\mathsf{S} = \langle S, \models_S \rangle$ *be a denotational semantics for* $\mathcal{L}$*. Then* $\langle \mathcal{L}, \vdash_\mathsf{S} \rangle$ *is a propositional logic for* $\mathcal{L}$*.*

3 Inconsistency Tolerance

In the context of reasoning with uncertainty, a major drawback of a logic $\langle \mathcal{L}, \vdash_\mathsf{S} \rangle$, induced by a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$, is that it may not tolerate inconsistency properly. Indeed, if $mod_\mathsf{S}(\Gamma)$ is empty, then by Definition 4 it holds that $\Gamma \vdash_\mathsf{S} \psi$ for *every* formula ψ. We thus consider a 'refined' entailment relation, denoted $\mid\!\sim_\mathsf{S}$, that overcomes this explosive nature of $\vdash_\mathsf{S}$, but respects $\vdash_\mathsf{S}$ with respect to consistent theories. Formally, we require the following two properties:

I. Faithfulness: If $mod_\mathsf{S}(\Gamma) \neq \emptyset$ then for all $\psi \in \mathcal{F}_\mathcal{L}$, $\Gamma \mid\!\sim_\mathsf{S} \psi$ iff $\Gamma \vdash_\mathsf{S} \psi$.

II Non-Explosiveness: If $mod_\mathsf{S}(\Gamma) = \emptyset$ then there is a formula $\psi \in \mathcal{F}_\mathcal{L}$ such that $\Gamma \not\mid\!\sim_\mathsf{S} \psi$.

[1] Due to a lack of space, some proofs are omitted or shortened.

Faithfulness guarantees that $\mid\!\sim_{\mathsf{S}}$ coincides with $\vdash_{\mathsf{S}}$ with respect to S-consistent theories, and non-explosiveness assures that $\mid\!\sim_{\mathsf{S}}$ is not trivialized when the set of premises is not S-consistent. We call $\mid\!\sim_{\mathsf{S}}$ an *inconsistency-tolerant* variant of $\vdash_{\mathsf{S}}$. In what follows, when $\vdash_{\mathsf{S}}$ is clear from context, we just say that $\mid\!\sim_{\mathsf{S}}$ is inconsistency-tolerant.

Note 1. When $mod_{\mathsf{S}}(\Gamma)$ is not empty for every Γ, then $\vdash_{\mathsf{S}}$ itself is inconsistency-tolerant, but in such logics the notion of inconsistency is degenerated. In what follows we shall be interested in stronger logics (like classical logic) that do not tolerate inconsistency and so need to be refined. Moreover, by Proposition 1, $\vdash_{\mathsf{S}}$ is a consequence relation and so it is monotonic, but frequently commonsense reasoning is nonmonotonic, in particular in the presence of contradictions.

4 Dissimilarity-Based Entailments

For defining inconsistency-tolerant variants of consequence relations that are induced by a denotational semantics, we introduce the notion of *dissimilarity*. Intuitively, dissimilarities are quantitative indications about the distinction between interpretations.

Definition 5. Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics. An *S-dissimilarity* is a function $\mathsf{d} : S \times S \to \mathbb{R}^+$, satisfying the following properties for all $\nu, \mu \in S$:

Symmetry: $\mathsf{d}(\nu, \mu) = \mathsf{d}(\mu, \nu)$,
Reflexivity: $\mathsf{d}(\nu, \nu) = 0$,
Absorption: if $\mathsf{d}(\nu, \mu) = 0$ then $\mathsf{d}(\nu, \sigma) = \mathsf{d}(\mu, \sigma)$ for every $\sigma \in S$.

Example 3. The discrete (uniform) metric d^{u} on S, defined by $\mathsf{d}^{\mathsf{u}}(\nu, \mu) = 0$ if $\nu = \mu$ and $\mathsf{d}^{\mathsf{u}}(\nu, \mu) = 1$ otherwise, is an S-dissimilarity. The Hamming distance d^{h} [11], where $\mathsf{d}^{\mathsf{h}}(\nu, \mu)$ is the number of atoms p for which $\nu(p) \neq \mu(p)$, is a dissimilarity on, e.g., two-valued valuations (Example 1) applied to a finite number of atomic formulas. Other definitions of distance and dissimilarity functions can be found, e.g., in [4, 5, 6, 17, 18].

Note 2. Dissimilarities are a generalization of the notion of pseudo distances: If d is a pseudo distance on S (that is, if d is a symmetric total function on S that preserves identities: $\forall \nu, \mu \in S$ $\mathsf{d}(\nu, \mu) = 0$ iff $\nu = \mu$), then d is also an S-dissimilarity. However, dissimilarities are a weaker notion than distances: a function d that satisfies all the conditions in Definition 5 is not necessarily a pseudo-distance, since $\mathsf{d}(\nu, \mu) = 0$ does not necessarily mean that ν and μ are equal (similarity does not preserve identities).

To be computable, dissimilarity functions should take into consideration only *finite* fragments of the compared interpretations (which are in general infinite). This is done by restricting the comparisons to finite contexts, determined by the underlying theory.

Definition 6. Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics for $\mathcal{L}$. A *context* is a finite set of formulas (i.e., an element of $\mathcal{T}_{\mathcal{L}}$). A *context generator* (for $\mathcal{L}$) is a function $\mathcal{G} : \mathcal{T}_{\mathcal{L}} \to \mathcal{T}_{\mathcal{L}}$, producing a context for every theory.

Intuitively, $\mathcal{G}(\Gamma)$ is a relevant context for the computations about a theory Γ. In what follows we shall denote by $\mathcal{G} \subseteq \mathcal{G}'$ that $\mathcal{G}(\Gamma) \subseteq \mathcal{G}'(\Gamma)$ for every Γ.

Example 4. Common examples for context generators are, e.g., the functions $\mathcal{G}^{\mathsf{At}}$, $\mathcal{G}^{\mathsf{SF}}$, $\mathcal{G}^{\mathsf{ID}}$, respectively, defined for every theory Γ by $\mathcal{G}^{\mathsf{At}}(\Gamma) = \mathsf{Atoms}(\Gamma)$, $\mathcal{G}^{\mathsf{SF}}(\Gamma) = \mathsf{SF}(\Gamma)$, and $\mathcal{G}^{\mathsf{ID}}(\Gamma) = \Gamma$.

Dissimilarities will be used in what follows to determine how 'close' an interpretation is to satisfying a set of formulas Γ. In particular, they should differentiate between the models of Γ and non-models of Γ. However, as similarities are a weaker notion (and so more general) than pseudo-distances (Note 2), it may be the case that $\mathsf{d}(\nu, \mu) = 0$ where ν is a model of Γ while μ is not. To avoid this, the dissimilarity under consideration should be Γ-dependent. This is achieved by dissimilarity generators that generate different similarities for different theories. For defining dissimilarity generators we therefore need first the following notation and notion:

Definition 7. $\nu \sim_\Gamma \mu$ if for every $\psi \in \Gamma$, $\nu \models_S \psi$ iff $\mu \models_S \psi$.

Definition 8. We say that a formula ψ is $\mathcal{G}$*-independent* of a theory Γ, if $\mathcal{G}(\{\psi\}) \cap \mathcal{G}(\Gamma) = \emptyset$.

Definition 9. Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics for a language $\mathcal{L}$ and $\mathcal{G}$ a context generator for the same language. A $\mathcal{G}$*-dissimilarity generator for* S is a function $\mathsf{d}_{\mathcal{G}} : \mathcal{T}_{\mathcal{L}} \to (S \times S \to \mathbb{R}^+)$, such that:

1. For every $\Gamma \in \mathcal{T}_{\mathcal{L}}$, $\mathsf{d}_{\mathcal{G}}(\Gamma)$ is an S-dissimilarity.
2. For every $\nu, \mu \in S$, if $\mathsf{d}_{\mathcal{G}}(\Gamma)(\nu, \mu) = 0$, then $\nu \sim_\Gamma \mu$.

A $\mathcal{G}$-dissimilarity generator $\mathsf{d}_{\mathcal{G}}$ is called *normal*, if it satisfies the following normality condition:

– For every theory Γ, if ψ is a non S-tautological formula that is $\mathcal{G}$-independent of Γ, then for every $\nu \in mod_{\mathsf{S}}(\psi)$ there is $\mu \notin mod_{\mathsf{S}}(\psi)$ such that still $\mathsf{d}_{\mathcal{G}}(\Gamma)(\nu, \mu) = 0$.

Below, we shall sometimes write $\mathsf{d}_{\mathcal{G}(\Gamma)}$ instead of $\mathsf{d}_{\mathcal{G}}(\Gamma)$.

Note that $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$ only means that ν and μ are similar on $\mathcal{G}(\Gamma)$, but this does not imply any correspondence between ν and μ elsewhere (cf. Note 2). Intuitively, the second condition of Definition 9 makes sure that $\mathsf{d}_{\mathcal{G}(\Gamma)}$ is faithful to Γ. The normality condition assures that the measurements by $\mathsf{d}_{\mathcal{G}(\Gamma)}$ depend only on the relevant context $\mathcal{G}(\Gamma)$.

Example 5. Let $\mathcal{G} = \mathcal{G}^{\mathsf{At}}$ or $\mathcal{G} = \mathcal{G}^{\mathsf{SF}}$, and let S be the standard two-valued semantics (Example 1). The following are all normal $\mathcal{G}$-dissimilarity generators for S.

a) $\mathsf{d}^{\mathsf{s}}_{\mathcal{G}}$, where for every Γ, $\mathsf{d}^{\mathsf{s}}_{\mathcal{G}(\Gamma)}(\nu,\mu)=0$ if $\nu \sim_\Gamma \mu$, otherwise $\mathsf{d}^{\mathsf{s}}_{\mathcal{G}(\Gamma)}(\nu,\mu)=1$.
b) $\mathsf{d}^{\mathsf{u}}_{\mathcal{G}}$, where for every Γ, $\mathsf{d}^{\mathsf{u}}_{\mathcal{G}(\Gamma)}(\nu,\mu)=0$ if for all $\psi \in \mathcal{G}(\Gamma)$ $\nu(\psi)=\mu(\psi)$ and otherwise $\mathsf{d}^{\mathsf{u}}_{\mathcal{G}(\Gamma)}(\nu,\mu)=1$.
c) $\mathsf{d}^{\mathsf{h}}_{\mathcal{G}}$, in which $\mathsf{d}^{\mathsf{h}}_{\mathcal{G}(\Gamma)}(\nu,\mu)$ is the number of formulas $\psi \in \mathcal{G}(\Gamma)$, for which $\nu(\psi) \neq \mu(\psi)$.

In Section 6 we consider other (normal) dissimilarity generators. Note that unlike related formalisms in standard two-values semantics (e.g., [18, 21]), it is *not necessary* to assume here that (the set of atomic formulas in) the underlying language is finite. This is because the dissimilarity calculations are made with respect to finite contexts.

Definition 10. A *(numeric) aggregation function* is a total function f, such that: (1) for every multiset of real numbers, the value of f is a real number, (2) the value of f does not decrease when the number in its multiset increases, (3) $f(\{x_1,\ldots,x_n\})=0$ iff $x_1=x_2=\ldots x_n=0$, and (4) $\forall x \in \mathbb{R}\ f(\{x\})=x$.

For non-negative values (such as dissimilarities), summation, average, median, and the maximum function, are all aggregation functions (see, e.g., [18, 23]).

Definition 11. Let S be a denotational semantics for a language $\mathcal{L}$. A (semantical) *setting* for S is a triple $\mathcal{S}=\langle \mathcal{G},\mathsf{d},f\rangle$, where $\mathcal{G}$ is a context generator (for $\mathcal{L}$), d is a $\mathcal{G}$-dissimilarity generator (for S), and f is a numeric aggregation function.

Definition 12. Let $\mathcal{S}=\langle \mathcal{G},\mathsf{d},f\rangle$ be a setting for a denotational semantics S. For a theory $\Gamma=\{\psi_1,\ldots,\psi_n\}$, define:

$$\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\psi_i)=\begin{cases} min\big\{\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu,\mu) \mid \mu \in mod_{\mathsf{S}}(\psi_i)\big\} & \text{if } mod_{\mathsf{S}}(\psi_i)\neq\emptyset,\\ 1+max\big\{\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu,\mu) \mid \nu,\mu \in S\big\} & \text{otherwise.}\end{cases}$$

$$\mathsf{M}_{\mathcal{S}}(\nu,\Gamma)=f\big(\big\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\psi_1),\ldots,\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\psi_n)\big\}\big).$$

Intuitively, $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\psi)$ is a quantitative indication for how 'close' ν is to be a model of ψ. The function $\mathsf{M}_{\mathcal{S}}(\nu,\Gamma)$ indicates how 'close' ν is to satisfy Γ. It is easy to verify that if ψ is S-consistent, then the closest elements to ψ are its models, and if ψ is *not* S-consistent, all the elements in S are equally close to ψ.

Definition 13. A setting $\mathcal{S}=\langle \mathcal{G},\mathsf{d},f\rangle$ for a denotational semantics $\mathsf{S}=\langle S,\models_S\rangle$ is *normal*, if so is d (recall Definition 9). We say that $\mathcal{S}$ is *effective*, if for every theory Γ, the set $\{\mathsf{M}_{\mathcal{S}}(\nu,\Gamma) \mid \nu \in S\}$ has a minimal element.

Reasoning by similarities is now defined as follows:

Definition 14. Given a setting $\mathcal{S}$ for a denotational semantics $\mathsf{S}=\langle S,\models_S\rangle$, the *$\mathcal{S}$-most plausible interpretations* of a (nonempty) theory Γ, are the elements of the following set:

$$\Delta_{\mathcal{S}}(\Gamma)=\big\{\nu \in S \mid \forall \mu \in S\ \mathsf{M}_{\mathcal{S}}(\nu,\Gamma)\leq \mathsf{M}_{\mathcal{S}}(\mu,\Gamma)\big\}.$$

In case that Γ is empty, we define $\Delta_{\mathcal{S}}(\emptyset)=S$.

Note 3. If $\mathcal{S}$ is effective, then $\Delta_{\mathcal{S}}(\Gamma) \neq \emptyset$ for every Γ.

Definition 15. Given a semantic setting $\mathcal{S}$ for a denotational semantics S, define: $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ iff $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod_{\mathsf{S}}(\psi)$.

5 Reasoning with $\mid\!\sim_{\mathcal{S}}$

The following example is a simple illustration of how inconsistency is handled by a dissimilarity-based entailment.

Example 6. Consider the setting $\mathcal{S} = \langle \mathcal{G}^{\mathsf{At}}, \mathsf{d}^{\mathsf{h}}, \Sigma \rangle$ for the standard two-valued semantics (Example 1), where $\mathsf{d}^{\mathsf{h}} = \mathsf{d}^{\mathsf{h}}_{\mathcal{G}^{\mathsf{At}}}$ is a dissimilarity generator in Example 5c and Σ is the summation function. Now, let $\Gamma = \{p, \neg p, q\}$. In this case, $\Delta_{\mathcal{S}}(\Gamma) = \{\nu \in S \mid \nu(q) = t\}$, thus $\Gamma \mid\!\sim_{\mathcal{S}} q$ while $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg q$. This is justified by the fact that q has nothing to do with the inconsistency of Γ (more precisely, q is not $\mathcal{G}^{\mathsf{At}}$-dependent of $\{p, \neg p\}$). On the other hand, $\Gamma \not\mid\!\sim_{\mathcal{S}} p$ and $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg p$, since both of p and $\neg p$ are 'involved' in the inconsistency of Γ.

Next, we consider some of the main properties of $\mid\!\sim_{\mathcal{S}}$.

Theorem 1. *Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective setting for a denotational semantics S. If S is normal (Definition 3), or $\mathcal{S}$ is normal (Definition 13), then $\mid\!\sim_{\mathcal{S}}$ is an inconsistency-tolerant variant of $\vdash_{\mathsf{S}}$.*

Proof outline. It is easy to verify that for every formula ψ, $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi) = 0$ iff $\nu \in mod_{\mathsf{S}}(\psi)$. Thus, for every theory Γ, $\mathsf{M}_{\mathcal{S}}(\nu, \Gamma) = 0$ iff $\nu \in mod_{\mathsf{S}}(\Gamma)$. It follows that if Γ is S-consistent, $\Delta_{\mathcal{S}}(\Gamma) = mod_{\mathsf{S}}(\Gamma)$, and so $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ iff $\Gamma \vdash_{\mathsf{S}} \psi$. This shows faithfulness. Now, if S is normal, then non-explosiveness follows from the fact that as $\mathcal{S}$ is effective, $\Delta_{\mathcal{S}}(\Gamma)$ is nonempty (Note 3), thus for every $\nu \in \Delta_{\mathcal{S}}(\Gamma)$ there is some ψ such that $\nu \not\models_{S} \psi$, and so $\Gamma \not\mid\!\sim_{\mathsf{S}} \psi$. If $\mathcal{S}$ is normal, non-explosiveness follows from the following lemma:

Lemma 1. *Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective and normal setting for S. For every Γ and every ψ which is $\mathcal{G}$-independent of Γ, it holds that $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ iff ψ is an S-tautology.*

Proof. One direction is clear: if ψ is an S-tautology, then for every $\nu \in \Delta_{\mathcal{S}}(\Gamma)$, $\nu \models_{S} \psi$ and so $\Gamma \mid\!\sim_{\mathcal{S}} \psi$. For the converse, suppose that ψ is not an S-tautology. Since $\mathcal{S}$ is effective, by Note 3 there is a valuation $\nu \in \Delta_{\mathcal{S}}(\Gamma)$. If $\nu \not\models_{S} \psi$, we are done: $\Gamma \not\mid\!\sim_{\mathcal{S}} \psi$. Otherwise, since $\mathsf{d}_{\mathcal{G}}$ is normal, there is some $\mu \in S$ such that $\mu \not\models_{S} \psi$, and $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$. But $\mathsf{d}_{\mathcal{G}(\Gamma)}$ is an S-dissimilarity and so, by Absorption (Definition 5), for every $\psi \in \Gamma$ and every $\nu_0 \in mod_{S}(\psi)$, $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \nu_0) = \mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \nu_0)$. Hence $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi) = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu, \psi)$, and so $\mathsf{M}_{\mathcal{S}}(\nu, \Gamma) = \mathsf{M}_{\mathcal{S}}(\mu, \Gamma)$ as well. Now, since $\nu \in \Delta_{\mathcal{S}}(\Gamma)$, it holds also that $\mu \in \Delta_{\mathcal{S}}(\Gamma)$, and since $\mu \not\models_{S} \psi$, we have that $\Gamma \not\mid\!\sim_{\mathcal{S}} \psi$. $\square$

Theorem 2. *Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective setting for a denotational semantics* S*, in which f is hereditary*[2]*. Then $\mid\!\sim_{\mathcal{S}}$ is a cautious consequence relation (in the sense of [2]), i.e., it has the following properties:*

Cautious Reflexivity: if Γ is S*-satisfiable and $\psi \in \Gamma$ then $\Gamma \mid\!\sim_{\mathcal{S}} \psi$*

Cautious Monotonicity [14]: if $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ and $\Gamma \mid\!\sim_{\mathcal{S}} \phi$ then $\Gamma, \psi \mid\!\sim_{\mathcal{S}} \phi$

Cautious Transitivity [19]: if $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ and $\Gamma, \psi \mid\!\sim_{\mathcal{S}} \phi$ then $\Gamma \mid\!\sim_{\mathcal{S}} \phi$

The next proposition shows that many entailments of the form $\mid\!\sim_{\mathcal{S}}$ are *paraconsistent* [10].

Proposition 2. *Let* S *be a denotational semantics for a language $\mathcal{L}$ with a unary connective $\neg$. Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective setting for* S*, in which $\mathcal{G} \subseteq \mathcal{G}^{\mathsf{SF}}$. Then $\mid\!\sim_{\mathcal{S}}$ is $\neg$-paraconsistent: $\psi, \neg\psi \not\mid\!\sim_{\mathcal{S}} \phi$ for some formulas $\psi, \phi \in \mathcal{F}_{\mathcal{L}}$.*

Another interesting property of dissimilarity-based entailments is that their set of conclusions is always consistent.

Proposition 3. *Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective setting for a denotational semantics* S*. Then for every theory Γ the set of the formulas that are $\mid\!\sim_{\mathcal{S}}$-entailed by Γ is* S*-consistent.*

Proof. Let $\mathcal{C}_{\mathcal{S}}(\Gamma) = \{\psi \mid \Gamma \mid\!\sim_{\mathcal{S}} \psi\}$. If $\mathcal{C}_{\mathcal{S}}(\Gamma)$ is not S-consistent for some Γ, that is, if $mod_{\mathsf{S}}(\mathcal{C}_{\mathcal{S}}(\Gamma)) = \emptyset$, then since $\Delta_{\mathcal{S}}(\Gamma) \subseteq mod_{\mathsf{S}}(\psi)$ for every $\psi \in \mathcal{C}_{\mathcal{S}}(\Gamma)$, we have: $\Delta_{\mathcal{S}}(\Gamma) \subseteq \bigcap_{\psi \in \mathcal{C}_{\mathcal{S}}(\Gamma)} mod_{\mathsf{S}}(\psi) = mod_{\mathsf{S}}(\mathcal{C}_{\mathcal{S}}(\Gamma)) = \emptyset$. Thus $\Delta_{\mathcal{S}}(\Gamma) = \emptyset$, contradicting the fact that if $\mathcal{S}$ is effective, $\Delta_{\mathcal{S}}(\Gamma)$ is nonempty for every Γ (see Note 3). □

Finally, we consider the decidability of $\mid\!\sim_{\mathcal{S}}$.

Definition 16. A setting $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}_{\mathcal{G}}, f \rangle$ is *computable*, if the following conditions hold:

1. The functions $\mathcal{G}$, f, $\mathsf{d}_{\mathcal{G}}$ and $\mathsf{d}_{\mathcal{G}(\Gamma)}$ for every theory Γ, are all computable.
2. $\mathsf{d}_{\mathcal{G}(\Gamma)}$ is bounded: for every Γ there is $n_\Gamma \in \mathbb{N}$ such that $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) \leq n_\Gamma$ for all $\nu, \mu \in S$.
3. For every theory Γ, formula ψ, and number $i \in \mathbb{N}$, checking whether $\{\nu \in S \mid \mathsf{M}_{\mathcal{S}}(\nu, \Gamma) = i\} \subseteq mod_{\mathsf{S}}(\psi)$ is decidable.

Example 7. The setting $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}_{\mathcal{G}}, f \rangle$ for standard two-valued semantics is computable for every context generator $\mathcal{G}$ in Example 4, all the similarity generators $\mathsf{d}_{\mathcal{G}}$ in Example 5, and every computable aggregation function f.

Theorem 3. *If $\mathcal{S}$ is computable then $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ is decidable.*

6 Applications

We now demonstrate the usefulness (and generality) of dissimilarity reasoning for defining a variety of inconsistency-tolerant logics based on different types of denotational semantics.

[2] An aggregation function f is called hereditary, if $f(\{x_1, \ldots, x_n\}) < f(\{y_1, \ldots, y_n\})$ implies that $f(\{x_1, \ldots, x_n, z_1, \ldots, z_m\}) < f(\{y_1, \ldots, y_n, z_1, \ldots, z_m\})$ (see [1]).

6.1 Multi-valued Logics

The most standard way of defining multi-valued logics (including, of course, classical logic), is by the following structures (see, e.g., [25]):

Definition 17. A *(multi-valued) matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and $\mathcal{O}$ contains an interpretation $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$ for each n-ary connective of $\mathcal{L}$.

Given a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we shall assume that $\mathcal{V}$ includes at least the two classical values t and f, and that only the former belongs to the set $\mathcal{D}$ of the *designated elements* in $\mathcal{V}$ (representing 'true assertions'). The set $\mathcal{O}$ contains the interpretations (the 'truth tables') of each connective in $\mathcal{L}$. The associated semantical notions are now defined as usual: An $\mathcal{M}$*-valuation* is a function $\nu : \mathcal{F}_{\mathcal{L}} \to \mathcal{V}$ so that, for every connective $\diamond$ in $\mathcal{L}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. The set of all $\mathcal{M}$-valuations is denoted by $\Lambda_{\mathcal{M}}$. We say that ν is a *model* of ψ, denoted $\nu \models_{\mathcal{M}} \psi$, if $\nu(\psi) \in \mathcal{D}$.

Note that the pair $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ is a denotational semantics in the sense of Definition 2. In what follows we shall sometimes identify this semantics with the matrix $\mathcal{M}$ that defines it. In particular, we shall say that $\mathcal{M}$ is normal if so is the denotational semantics $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ that it induces. By Proposition 1 we have, then, that:

Proposition 4. *The relation $\vdash_{\mathcal{M}}$, induced from a matrix $\mathcal{M}$ by Definition 4, is a Tarskian consequence relation.*

Given a matrix-based denotational semantics $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ and a corresponding consequence relation $\vdash_{\mathcal{M}}$, one may define inconsistency-tolerant variants of $\vdash_{\mathcal{M}}$ by a dissimilarity-based setting $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$, just as in Definition 15.

Example 8. Kleene's three-valued logic [16] is obtained from the matrix $\mathcal{M}^3_K = \langle \{t, f, \perp\}, \{t\}, \mathcal{O}_K \rangle$, in which $\mathcal{O}_K$ contains the following connective interpretations:

	$\tilde{\neg}$
f	t
$\perp$	$\perp$
t	f

$\tilde{\wedge}$	f	$\perp$	t
f	f	f	f
$\perp$	f	$\perp$	$\perp$
t	f	$\perp$	t

$\tilde{\vee}$	f	$\perp$	t
f	f	$\perp$	t
$\perp$	$\perp$	$\perp$	t
t	t	t	t

Let $\mathcal{S} = \langle \mathcal{G}^{\mathsf{At}}, \mathsf{d}, \Sigma \rangle$ be a setting, where $\mathcal{G}^{\mathsf{At}}$ is the atom-based context generator, Σ is a summation function, and d is a $\mathcal{G}^{\mathsf{At}}$-dissimilarity generator, producing for each Γ the following dissimilarity:

$$\mathsf{d}(\Gamma)(\nu, \mu) = \Sigma \, \{d_\Sigma(\nu(p), \mu(p)) \mid p \in \mathcal{G}^{\mathsf{At}}(\Gamma)\}.$$

Here, d_Σ is an extended Hamming distance on $\{t, f, \perp\}$, where $d_\Sigma(t, f) = 1$ and $d_\Sigma(t, \perp) = d_\Sigma(f, \perp) = 0.5$ (see [1, 12]). Let $\Gamma = \{\neg p, \neg q, p \vee q\}$. Clearly, Γ is not $\mathcal{M}^3_K$-satisfiable. We compute its most plausible interpretations with respect to $\mathcal{S}$:

	p	q	$\neg p$	$\neg q$	$p \vee q$	1	2	3	$\mathsf{M}_{\mathcal{S}}(\nu_i, \Gamma)$
ν_1	t	t	f	f	t	1	1	0	2
ν_2	t	f	f	t	t	1	0	0	**1**
ν_3	t	$\bot$	f	$\bot$	t	1	0.5	0	1.5
ν_4	f	t	t	f	t	0	1	0	**1**
ν_5	f	f	t	t	f	0	0	1	**1**
ν_6	f	$\bot$	t	$\bot$	$\bot$	0	0.5	0.5	**1**
ν_7	$\bot$	t	$\bot$	f	t	0.5	1	0	1.5
ν_8	$\bot$	f	$\bot$	t	$\bot$	0.5	0	0.5	**1**
ν_9	$\bot$	$\bot$	$\bot$	$\bot$	$\bot$	0.5	0.5	0.5	1.5

Legend. $1 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg p)$, $2 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg q)$, $3 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, p \vee q)$.

Hence, $\Delta_{\mathcal{S}}(\Gamma) = \{\nu_2, \nu_4, \nu_5, \nu_6, \nu_8\}$, and so, for instance, $\Gamma \mid\!\sim_{\mathcal{S}} \neg p \vee \neg q$ (even though $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg p$ and $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg q$).

Let us now describe a general method of defining dissimilarity generators in multi-valued semantics.

Proposition 5. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$, d a pseudo-distance on $\mathcal{V}$, g an aggregation function, and $\mathcal{G}$ a context generator for $\mathcal{L}$. We define, for every $\Gamma \in \mathcal{T}_{\mathcal{L}}$, a function $\mathsf{d}^g_{\mathcal{G}(\Gamma)} : \Lambda_{\mathcal{M}} \times \Lambda_{\mathcal{M}} \to \mathbb{R}^+$ by:*

$$\mathsf{d}^g_{\mathcal{G}(\Gamma)}(\nu, \mu) = g\big(\{d(\nu(\psi), \mu(\psi)) \mid \psi \in \mathcal{G}(\Gamma)\}\big).$$

Then $\mathsf{d}^g_{\mathcal{G}((\Gamma)}$ is a dissimilarity function. Moreover,

- *if $\mathcal{G}^{\mathsf{At}} \subseteq \mathcal{G}$ or $\mathcal{G}^{\mathsf{ID}} \subseteq \mathcal{G}$, $\mathsf{d}^g_{\mathcal{G}}$ is a $\mathcal{G}$-dissimilarity generator,*
- *if $\mathcal{G}^{\mathsf{At}} \subseteq \mathcal{G}$ and $\mathcal{G}^{\mathsf{At}} \circ \mathcal{G} \subseteq \mathcal{G}$,[3] then $\mathsf{d}^g_{\mathcal{G}}$ is a normal $\mathcal{G}$-dissimilarity generator.*

Given a propositional logic $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, Proposition 5 yields a simple and a general way of defining entailments that are inconsistency-tolerant variants of $\vdash_{\mathcal{M}}$. We note that many of the inconsistency-tolerant entailments that have been considered in the literature (e.g., those in [15] and [18]) are particular cases of this construction, where $\mathcal{M}$ is a two-valued matrix and $\mathcal{G} = \mathcal{G}^{\mathsf{At}}$.

6.2 Non-deterministic Logics

Matrix-based semantics is truth-functional in the sense that the truth-value of a complex formula is determined by the truth-values of its subformulas. Such a semantics is not useful in capturing non-deterministic phenomena. This leads to the idea of non-deterministic matrices [7], allowing non-deterministic evaluations of formulas. This kind of semantics has a variety of applications for reasoning under uncertainty (see, e.g., [8]).

[3] I.e., $\mathcal{G}^{\mathsf{At}}(\mathcal{G}(\Gamma)) \subseteq \mathcal{G}(\Gamma)$ for every Γ.

Definition 18. A *non-deterministic matrix* (*Nmatrix*) for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and $\mathcal{O}$ contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective of $\mathcal{L}$.

An $\mathcal{M}$-*valuation* is a function $\nu : \mathcal{F}_{\mathcal{L}} \to \mathcal{V}$ such that for every connective $\diamond$ in $\mathcal{L}$,

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)).$$

The set of all $\mathcal{M}$-valuations is denoted by $\Lambda_{\mathcal{M}}$. Again, $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of ψ (denoted $\nu \models_{\mathcal{M}} \psi$), if $\nu(\psi) \in \mathcal{D}$.

Ordinary matrices can be thought of as Nmatrices whose interpretations return singletons of truth-values. Again, for an Nmatrix $\mathcal{M}$, the pair $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ is a denotational semantics and it induces a Tarskian consequence relation $\vdash_{\mathcal{M}}$.

As before, dissimilarity-based entailments can be applied for non-deterministic matrices. However, as was shown in [5], some dissimilarity generators (and the respective settings) that are definable with respect to standard matrices, are *not* applicable in the non-deterministic case. This is due to the fact that non-deterministic valuations are not truth functional, so they can agree on atomic formulas, but may make different non-deterministic choices on complex formulas. This is also the reason why Proposition 5 is not extendable to non-deterministic semantics. Yet, a stricter version of that proposition does hold also in the non-deterministic case:

Proposition 6. *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language $\mathcal{L}$, d a pseudo-distance on $\mathcal{V}$, g an aggregation function, and $\mathcal{G}$ a context generator for $\mathcal{L}$. We define, for each $\Gamma \in \mathcal{T}_{\mathcal{L}}$, a function $\mathsf{d}^g_{\mathcal{G}(\Gamma)} : \Lambda_{\mathcal{M}} \times \Lambda_{\mathcal{M}} \to \mathbb{R}^+$ by:*

$$\mathsf{d}^g_{\mathcal{G}(\Gamma)}(\nu, \mu) = g\big(\{d(\nu(\psi), \mu(\psi)) \mid \psi \in \mathcal{G}(\Gamma)\}\big).$$

Then $\mathsf{d}^g_{\mathcal{G}((\Gamma)}$ is a dissimilarity function. Moreover,

- *if $\mathcal{G}^{\mathsf{ID}} \subseteq \mathcal{G}$ then $\mathsf{d}^g_{\mathcal{G}}$ is a $\mathcal{G}$-dissimilarity generator,*
- *if $\mathcal{G}^{\mathsf{SF}} \subseteq \mathcal{G}$ and $\mathcal{G}^{\mathsf{SF}} \circ \mathcal{G} \subseteq \mathcal{G}$, then $\mathsf{d}^g_{\mathcal{G}}$ is a normal $\mathcal{G}$-dissimilarity generator.*

Note 4. Clearly, there are useful dissimilarity generators other than those that are covered by the construction of Proposition 6. One of them is the dissimilarity generator $\mathsf{d}^{\mathsf{n}}_{\mathcal{G}}$, in which $\mathsf{d}^{\mathsf{n}}_{\mathcal{G}(\Gamma)}(\nu, \mu)$ is the number of formulas $\psi \in \mathcal{G}(\Gamma)$, for which (i) $\nu(\psi) \neq \mu(\psi)$, and (ii) if $\psi = \diamond(\varphi_1, \ldots, \varphi_n)$, then for all $1 \leq i \leq n$: $\nu(\varphi_i) = \mu(\varphi_i)$. This function is one of the pseudo-distances that are introduced in [5] for distance-based reasoning for non-deterministic matrices. It can be verified that $\mathsf{d}^{\mathsf{n}}_{\mathcal{G}}$ is also a $\mathcal{G}$-dissimilarity generator in our sense.

Example 9. Consider a transmission protocol for a system with three transmitters T_1, T_2 and T_3, where the first two are connected to a bus through an arbiter A, and the third one is connected directly to the bus. The bus has a line Msg for the transmitted message, and a line Busy, which is turned on whenever the transmission occurs. When one of the transmitters T_1 or T_2 has a message to transmit, it signals to the arbiter by turning on the line M_1 or M_2 respectively. The arbiter then turns on the line Busy, and T_i transmits its message on the line Msg. As for the third transmitter, whenever T_3 wants to transmit a message, it turns on M_3 and transmits the message on Msg. A schematic presentation of this circuit (excluding some issues about the logic of Msg, which are not relevant for this example) is shown in Figure 1.

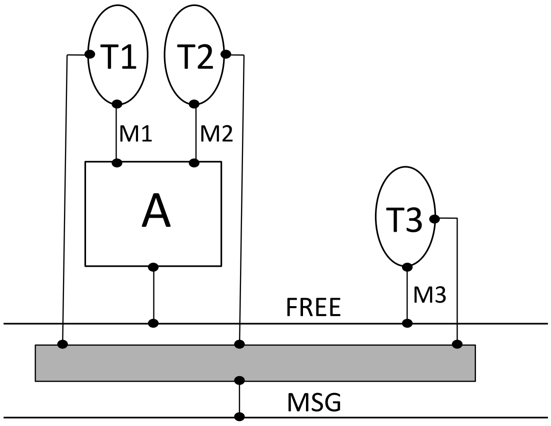

Fig. 1. The system of Example 9

Suppose now that the arbiter has no synchronization method, and whenever T_1 and T_2 request the line at the same time, the result is unpredictable: the line Busy can either stay on or be turned off. This non-deterministic behavior of the arbiter can be described using the following interpretation of the connective $\odot$

$\tilde{\odot}$	f	t
f	$\{f\}$	$\{t\}$
t	$\{t\}$	$\{t, f\}$

Let $\mathcal{M}$ be the two-valued Nmatrix for the language of $\{\neg, \vee, \odot\}$, which includes the above non-deterministic interpretation for $\odot$ and the standard interpretations of negation $\neg$ and disjunction $\vee$.

Next, suppose that we observe the following unexpected behavior of the arbiter: although T_1 has a message to transmit, while T_2 has none, the line Busy is not turned on. This can be captured by the following theory:

$$\Gamma = \{M_1, \neg M_2, \neg \mathsf{Busy}\}$$

where Busy is an abbreviation of the formula $(M_1 \odot M_2) \vee M_3$, representing the normal behavior of the line Busy. Obviously, this theory is not $\mathcal{M}$-satisfiable. For reasoning with this abnormality, we use the setting $\mathcal{S} = \langle \mathcal{G}^{\mathsf{SF}}, \mathsf{d}^{\mathsf{n}}_{\mathcal{G}^{\mathsf{SF}}}, \Sigma \rangle$, where $\mathsf{d}^{\mathsf{n}}_{\mathcal{G}^{\mathsf{SF}}}$

is the dissimilarity generator defined in Note 4. The dissimilarity computations for this case are represented in the table below:

	M_1	M_2	M_3	$\neg M_2$	$M_1 \odot M_2$	Busy	$\neg$Busy	1	2	3	$\mathsf{M}_{\mathcal{S}}(\nu_i, \Gamma)$
ν_1	t	t	t	f	t	t	f	0	1	2	3
ν_2	t	t	t	f	f	t	f	0	1	1	2
ν_3	t	t	f	f	t	t	f	0	1	1	2
ν_4	t	t	f	f	f	f	t	0	1	0	**1**
ν_5	t	f	t	t	t	t	f	0	0	2	2
ν_6	t	f	f	t	t	t	f	0	0	1	**1**
ν_7	f	t	t	f	t	t	f	1	1	2	4
ν_8	f	t	f	f	t	t	f	1	1	1	3
ν_9	f	f	t	t	f	t	f	1	0	1	2
ν_{10}	f	f	f	t	f	f	t	1	0	0	**1**

Legend. $1 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, M_1)$, $2 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg M_2)$, $3 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg\mathsf{Busy})$.

Hence, $\Delta_{\mathcal{S}}(\Gamma) = \{\nu_4, \nu_6, \nu_{10}\}$, and so, for instance, $\Gamma \mid\!\sim_{\mathcal{S}} \neg M_3$, even though neither M_1 nor $\neg M_2$ are $\mid\!\sim_{\mathcal{S}}$-inferrable from Γ. Hence, although one can say nothing about T_1 and T_2, it is still possible to conclude in this case that T_3 has no message to transmit.

6.3 Modal Logics

Next, we consider a denotational semantics that is based on a many-valued extension of standard Kripke semantics (see [13]), where the logical connectives are interpreted by a matrix $\mathcal{M}$,[4] and qualifications of the truth of a judgement are expressed by the necessitation operator "$\Box$". In case of the standard two-valued matrix we get the usual Kripke-style (possible worlds) semantics.

Definition 19. Let $\mathcal{L}$ be a propositional language.

- A *frame* for $\mathcal{L}$ is a triple $\mathcal{F} = \langle W, R, \mathcal{M} \rangle$, where W is a non-empty set (of "worlds"), R (the "accessibility relation") is a binary relation on W, and $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a matrix for $\mathcal{L}$. We say that a frame is finite if so is W.
- Let $\mathcal{F} = \langle W, R, \mathcal{M} \rangle$ be a frame for $\mathcal{L}$. An $\mathcal{F}$*-valuation* is a function $\nu : W \times F_{\mathcal{L}} \to \mathcal{V}$ that assigns truth values to the $\mathcal{L}$-formulas at each world in W according to the following conditions:
  - For every connective $\diamond$, $\nu(w, \diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}_{\mathcal{M}}(\nu(w, \psi_1), \ldots, \nu(w, \psi_n))$,
  - $\nu(w, \Box\psi) \in \mathcal{D}$ iff $\nu(w', \psi) \in \mathcal{D}$ for all w' such that $R(w, w')$.

  The set of $\mathcal{F}$-valuations is denoted by $\Lambda_{\mathcal{F}}$. The set of $\mathcal{F}$-valuations that satisfy a formula ψ in a world $w \in W$ is $mod^{w}_{\mathcal{F}}(\psi) = \{\nu \in \Lambda_{\mathcal{F}} \mid \nu(w, \psi) \in \mathcal{D}\}$.
- A *frame interpretation* is a pair $I = \langle \mathcal{F}, \nu \rangle$, in which $\mathcal{F} = \langle W, R, \mathcal{M} \rangle$ is a frame and ν is an $\mathcal{F}$-valuation. We say that I *satisfies* ψ (or that I is a model of ψ), if $\nu \in mod^{w}_{\mathcal{F}}(\psi)$ for every $w \in W$. We say that I satisfies Γ if it satisfies every $\psi \in \Gamma$.

[4] This framework can be extended to Nmatrices as well, but for simplicity we stick to deterministic matrices.

Let $\mathcal{I}$ be a nonempty set of frame interpretations. We define a satisfaction relation $\models_{\mathcal{I}}$ on $\mathcal{I} \times F_{\mathcal{L}}$ by $I \models_{\mathcal{I}} \psi$ iff I satisfies ψ. Note that $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}} \rangle$ is a denotational semantics in the sense of Definition 2. By Proposition 1, then, the induced relation $\vdash_{\mathfrak{I}}$ is a Tarskian consequence relation for $\mathcal{L}$.

Given a possible-world semantics $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}} \rangle$, it is possible to define an inconsistency-tolerant variant of $\vdash_{\mathfrak{I}}$ by introducing a dissimilarity-based setting $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ and applying the definitions in Section 4. As frame interpretations are more complicated semantic structures than those considered in the previous sections, defining intuitive and simple dissimilarity generators is more challenging in this case. Below, we consider a simple and useful case: a set of frame interpretations in which all the frames share the same set of worlds and accessibility relation. In this case, dissimilarity between frame interpretations may be defined by comparing valuations in each world and then aggregating over the worlds:

Proposition 7. *Let $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}} \rangle$ be a possible world semantics, where $\mathcal{I} = \langle \langle W, R, \mathcal{M} \rangle, \nu_i \rangle$ for a finite W, and let $\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}$ be a $\mathcal{G}$-dissimilarity generator for $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ (e.g., of the form considered in Proposition 5). We define, for an aggregation function g, a function $\mathsf{d}_{\mathcal{G}}^{g}$, such that for any Γ and frame interpretations $I_1 = \langle \langle W, R, \mathcal{M} \rangle, \nu_1 \rangle$ and $I_2 = \langle \langle W, R, \mathcal{M} \rangle, \nu_2 \rangle \in \mathcal{I}$,*

$$\mathsf{d}_{\mathcal{G}}^{g}(\Gamma)(I_1, I_2) = g\big(\{\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}(\Gamma)(\nu_1(w), \nu_2(w)) \mid w \in W\}\big)$$

where, for $i = 1, 2$, $\nu_i(w)$ is the 'restriction' of ν_i to the world w, that is: for every formula ψ, $\nu_i(w)(\psi) = \nu_i(w, \psi)$. Then:

a) $\mathsf{d}_{\mathcal{G}}^{g}$ is a $\mathcal{G}$-dissimilarity generator for $\mathfrak{I}$,
b) if $\mathcal{I}$ is $\mathcal{M}$-closed (that is, if for each $\langle \mathcal{F}, \nu \rangle \in \mathcal{I}$ and $\mu \in \Lambda_{\mathcal{M}}$ there is $\langle \mathcal{F}, \mu \rangle \in \mathcal{I}$), then $\mathsf{d}_{\mathcal{G}}^{g}$ is a normal $\mathcal{G}$-dissimilarity generator.

Example 10. A committee of three people, P_1, P_2 and P_3 should nominate two or less candidates among a and b for a governmental position. A committee member P_i may consult with any other member. Moreover, P_i will vote for $\mathsf{x} \in \{\mathsf{a}, \mathsf{b}\}$ only if P_i believes that x is qualified, and no member that P_i consults with believes otherwise. A candidate is recommended only upon a consensus.

A journalist j wants to predict the committee's recommendation, based on his partial knowledge about the committee and some leaking rumors. Suppose that he knows that P_1 believes that both candidates are qualified and that P_2 believes that b is qualified. Moreover, j knows that P_1 and P_2 consult with P_3, but P_3 never asks anyone else for advice.

This situation can be represented by the classical matrix and a modal language $\mathcal{L} = \{\Box, \wedge, \neg\}$. The atoms $\mathsf{q_a}$ and $\mathsf{q_b}$ respectively represent the belief that a and b are qualified, and the formula $\mathsf{Q_x} = \Box \mathsf{q_x}$ indicates that x is in the list of qualified candidates. Each world is associated with a committee member: $W = \{P_1, P_2, P_3\}$. Accessibility between worlds indicates a consulting relation between the members, thus: $R = \{\langle P_1, P_1 \rangle, \langle P_2, P_2 \rangle, \langle P_3, P_3 \rangle, \langle P_1, P_3 \rangle, \langle P_2, P_3 \rangle\}$. The corresponding frame is shown in Figure 2.

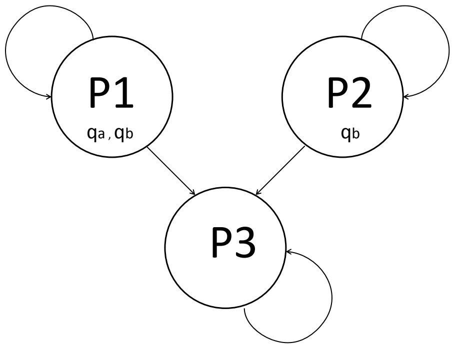

Fig. 2. The frame of Example 10

Next, suppose that two contradictory rumors are brought to j's attention: according to one, the list includes the names of both a and b. According to the other, at least one of the members disqualified a. This information may be represented by $\Gamma = \{\neg\Box\mathsf{q_a}, \Box\mathsf{q_a} \wedge \Box\mathsf{q_b}\}$. For maintaining this contradictory theory, j uses $\mid\!\sim_{\mathcal{S}}$, induced by the setting $\mathcal{S} = \langle \mathcal{G}^{\mathsf{At}}, \mathsf{d}^{\Sigma}, \Sigma \rangle$, where d^{Σ} is the dissimilarity-generator defined like in Proposition 7 for $g = \Sigma$ and $d^{\mathcal{M}}_{\mathcal{G}} = d^{\mathsf{h}}_{\mathcal{G}}$. Note that since $\mathfrak{I}$ is normal, by Theorem 1 and Proposition 7, $\mid\!\sim_{\mathcal{S}}$ is inconsistency-tolerant.

The dissimilarity calculations for the frame interpretations that correspond to the partial knowledge of j are given below (where $\psi^{\mathsf{i}}_{\mathsf{x}}$, for $\psi \in \{\mathsf{q}, \mathsf{Q}\}$, $\mathsf{x} \in \{\mathsf{a}, \mathsf{b}\}$ and $1 \leq i \leq 3$, denotes the value of the formula ψ_{x} in the world P_i).

	q^1_a	q^2_a	q^3_a	q^1_b	q^2_b	q^3_b	Q^1_a	Q^2_a	Q^3_a	Q^1_b	Q^2_b	Q^3_b	1	2	3
I_1	t	t	t	t	t	t	t	t	t	t	t	t	1	0	1
I_2	t	t	t	t	t	f	t	t	t	f	f	f	1	1	2
I_3	t	t	f	t	t	t	f	f	f	t	t	t	0	1	1
I_4	t	t	f	t	t	f	f	f	f	f	f	f	0	2	2
I_5	t	f	t	t	t	t	t	f	t	t	t	t	0	1	1
I_6	t	f	t	t	t	f	t	f	t	f	f	f	0	2	2
I_7	t	f	f	t	t	t	f	f	f	t	t	t	0	2	2
I_8	t	f	f	t	t	f	f	f	f	f	f	f	0	3	3

Legend. $1 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(I_i, \neg\mathsf{Q}_a), 2 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(I_i, \mathsf{Q}_a \wedge \mathsf{Q}_b)), 3 = \mathsf{M}_{\mathcal{S}}(i, \Gamma)$.

Thus, $\Delta_{\mathcal{S}}(\Gamma) = \{I_1, I_3, I_5\}$, and so $\Gamma \mid\!\sim_{\mathcal{S}} \mathsf{Q_b}$ while $\Gamma \not\mid\!\sim_{\mathcal{S}} \mathsf{Q_a}$, $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg\mathsf{Q_a}$. Based on his knowledge, then, j may assume that b will be nominated, while nothing can be predicted about a.

7 Conclusion

We have introduced a general method of supplementing different logics, based on denotational semantics, with extra apparatus assuring a proper tolerance of inconsistency, based on the notion of dissimilarity. As shown in [6], the properties of dissimilarities and of their generators guarantee that the logics that are obtained are inconsistency-tolerant. However, while the work in [6] investigates *specific* ways of defining dissimilarity-based logics for concrete forms of denotational semantics, here we provide a *generic and abstract* form of dissimilarity-based reasoning, suitable for *any type* of denotational semantics.

It should also be noted that, unlike many other frameworks of inconsistency handling, nothing is assumed here about the underlying semantics, apart of it being denotational. This, and the fact that dissimilarity is a more general notion than (pseudo) distances, imply that our approach may be used, e.g., for extending traditional distance-related methodologies in the context of revision and merging systems [17, 18, 21], cardinality-based methods for database repair [4, 9, 24], and forgetting-based approaches to reasoning with inconsistency [20, 22]. As a consequence, the standard two-valued propositional framework, which is the common platform for these methodologies, may be replaced by other non-classical, decidable platforms, involving many-valuedness, possible-worlds, and non-determinism.

An important subject for future research is a comparative study concerning the different entailment relations that are induced by different dissimilarity-based settings. This involves extensions to the non-monotonic case of works such as that in [3] (which introduces a list of desirable properties that paraconsistent consequence relations should have[5]).

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[5] Unlike our case, then, the work in [3] refers to *monotonic* logics (see Definition 1).

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