Preferential Logics for Reasoning with Graded Uncertainty

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Abstract. We introduce a family of preferential logics that are useful for handling information with different levels of uncertainty. The corresponding consequence relations are non-monotonic, paraconsistent, adaptive, and rational. It is also shown that any formalism in this family that is based on a well-founded ordering of the different types of uncertainty, can be embedded in a corresponding four-valued logic with at most three uncertainty levels.

1 Motivation

The ability to reason in a 'rational' way with incomplete or inconsistent information is a major challenge, and its significance should be obvious. It is well-known that classical logic is not suitable for this task, thus non-classical formalisms are usually used for handling uncertainty.[1] Such formalisms should be able, moreover, to distinguish among different types of uncertainty in the underlying data, since each kind of uncertainty may require a different treatment. The following example demonstrates such a case:

Example 1. Let $\mathcal{P} = \{p \leftarrow \texttt{true}\ ,\ \neg p \leftarrow \texttt{true}\ ,\ q \leftarrow \texttt{not}\,\neg r\ ,\ \neg q \leftarrow \texttt{not}\ r\}$. This is a 'prolog-like' program, with two kinds of negation operators: one, $\neg$, intuitively represents explicit negation, and the other, `not`, represents implicit negative information, and may be intuitively understood as a 'negation-as-failure' (to prove or verify the corresponding assertion on the basis of the available information). The meaning of the last two clauses of $\mathcal{P}$ is, therefore, that q (respectively, $\neg q$) holds provided that $\neg r$ (respectively, r) cannot be verified.

The theory above depicts several types of uncertainty: the information about r is *incomplete*, since r does not appear in a head of any clause in $\mathcal{P}$, and so no *explicit* data about it (nor about its negation) is available. This implies, in particular, that one cannot conclude that either r or $\neg r$ holds, and so, by the last two clauses, the data about q is *inconsistent*. Clearly, by the first two clauses, the information about p is inconsistent as well. Note, however, that there is a difference between the inconsistent information about p and about q: while the contradiction regarding p is based on *explicit data*, the evidence about q is

[1] See, e.g., [12,15,18,23] for some recent collections of papers on this topic.

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less 'stable', since it relies on the (possibly temporary) fact that neither r nor $\neg r$ holds. In particular, once r is validated or falsified, the information about q would not remain contradictory anymore! One may also argue that although the information about r is incomplete, there is still more knowledge about r (e.g., that it determines the validity of q) than about, say, s (about which we don't know anything whatsoever). Here, again, we have two different degrees of uncertainty.

The example above demonstrates one case in which it is natural to attach different levels of uncertainty to different assertions. This kind of information may be used, for instance, by algorithms for consistency restoration, since data with higher degree of inconsistency may be treated (i.e., eliminated) first.

In this paper we consider a framework that supports this type of considerations, and provides means to reason with different levels of uncertain information. We show that the logics that are obtained are nonmonotonic, paraconsistent [19], adaptive in the sense of Batens [10,11], and rational in the sense of Lehmann and Magidor [28]. It is also shown that under a certain assumption on the grading relations, for each one of these formalisms there is a logically equivalent four-valued logic with at most three different levels of uncertainty.

2 The Framework

2.1 Logical Lattices and Their Consequence Relations

In order to overcome the shortcomings of classical logic in properly handling uncertainty, we turn to multiple-valued logics. This is a common approach that is the basis of many formal systems (see [9] for a recent survey), including systems that are based on fuzzy logic [22], probabilistic reasoning [33], possibilistic logics [21], annotated logics [26,37], and fixpoint semantics for extended/disjunctive logic programs (see, e.g., [3,29], and a survey in [20]).

In most of the approaches mentioned above, as in the present one, the truth-values are arranged in a lattice structure. In what follows we denote by $\mathcal{L} = (L, \leq)$ a bounded lattice that has at least four elements: a $\leq$-maximal element and a $\leq$-minimal element that correspond to the classical values (denoted, respectively, by t and f), and two intermediate elements (denoted by $\top$ and $\bot$) that may intuitively be understood as representing the two basic types of uncertainty: inconsistency and incompleteness (respectively). As usual, the meet and the join operations on $\mathcal{L}$ are denoted by $\wedge$ and $\vee$. In addition, we assume that $\mathcal{L}$ has an involution operator $\neg$ (a "negation") s.t. $\neg t = f$, $\neg f = t$, $\neg\top = \top$, $\neg\bot = \bot$. We denote by $\mathcal{D}$ the set of the *designated values* of L (i.e., the set of the truth values in L that represent true assertions). We shall assume that $\mathcal{D}$ is a prime filter in $\mathcal{L}$,[2] s.t. $\top \in \mathcal{D}$ and $\bot \notin \mathcal{D}$. The pair $(\mathcal{L}, \mathcal{D})$ is called a *logical lattice* [6].

[2] In particular, $t \in \mathcal{D}$ and $f \notin \mathcal{D}$.

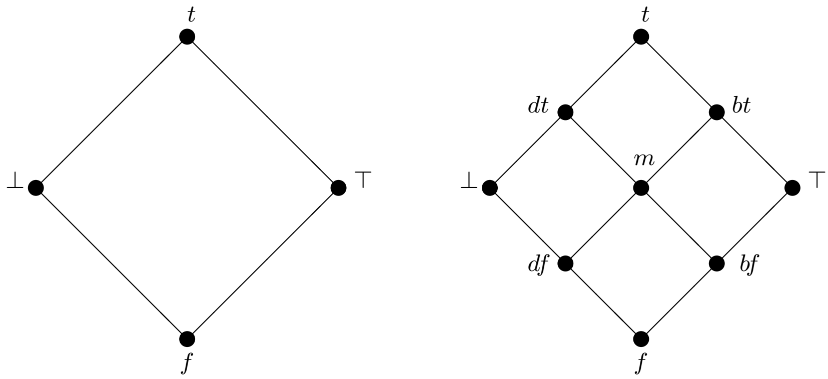

Fig. 1. $\mathcal{FOUR}$ and $\mathcal{NINE}$

Example 2. The smallest logical lattice is shown in Fig. 1 (left). We denote it by $\mathcal{FOUR}$. This lattice, together with the set $\mathcal{D} = \{t, \top\}$ of designated values, is the algebraic structure behind Belnap's well-known four-valued logic [13,14], and it will play an important role here as well (see Sect. 3). $\mathcal{NINE}$ (Fig. 1, right), may be viewed as an extended version of $\mathcal{FOUR}$ for default reasoning (dt = true by default, bt = 'biased' for t, etc.). This lattice depicts three main levels of uncertainty: incomplete data ($\bot$), inconsistent data ($\top$), and a middle level of uncertainty (m). The latter kind of uncertainty sometimes follows from contradictory default assumptions, so it may be retracted when further information arrives. The decision whether to view m as designated is one of the differences between the two logical lattices that $\mathcal{NINE}$ induces, namely ($\mathcal{NINE}$, $\{t, bt, \top\}$) and ($\mathcal{NINE}$, $\{t, dt, bt, bf, m, \top\}$).

The set $\mathcal{U} = \{(x, y) \mid 0 \leq x \leq 1,\ 0 \leq y \leq 1\}$ with $(x_1, y_1) \vee (x_2, y_2) = (\max(x_1, x_2), \min(y_1, y_2))$ and $(x_1, y_1) \wedge (x_2, y_2) = (\min(x_1, x_2), \max(y_1, y_2))$ is an infinite lattice, and $(\mathcal{U}, \{(1, x) \mid 0 \leq x \leq 1\})$ is a logical lattice with $t = (1, 0)$, $f = (0, 1)$, $\top = (1, 1)$, and $\bot = (0, 0)$. One way to intuitively understand the meaning of an element $(x, y) \in \mathcal{U}$ is such that x represents the amount of belief for the underlying assertion, and y represents the amount of belief against it. Following this intuition, every element $(x, x) \in \mathcal{U}$ may be associated with a different degree of inconsistency.

Given a logical lattice $(\mathcal{L}, \mathcal{D})$, the standard semantical notions are natural generalizations of the classical ones: a (multiple-valued) *valuation* ν is a function that assigns an element of L to each atomic formula. The set of valuations onto L is denoted by $\mathcal{V}^L$. Extension to complex formulae is done in the usual way. A valuation is a *model* of a set of assertions Γ if it assigns a designated value to every formula in Γ. The set of all the models of Γ is denoted by $mod(\Gamma)$.

The language considered here is a propositional one. Note that there are no tautologies in the language of $\{\neg, \vee, \wedge\}$, since if all the atomic formulae that appear in a formula ψ are assigned $\bot$ by a valuation ν, then $\nu(\psi) = \bot$ as well. It follows that the definition of the material implication $p \leadsto q$ as $\neg p \vee q$ is not adequate for representing entailments in our semantics. Instead, we use another connective, which does function as an implication in our setting:

Definition 1. [4,8] Let $(\mathcal{L}, \mathcal{D})$ be a logical lattice. Define: $x \rightarrow y = y$ if $x \in \mathcal{D}$, and $x \rightarrow y = t$ otherwise.[3]

The language of $\{\neg, \vee, \wedge, \rightarrow\}$ together with the propositional constants $t, f, \top$ and $\bot$, will be denoted by Σ. Given a set of formulae Γ in Σ, we shall denote by $\mathcal{A}(\Gamma)$ the set of the atomic formulae that appear in some formula of Γ.

Now, a natural definition of a lattice-based consequence relation is the following:

Definition 2. Let $(\mathcal{L}, \mathcal{D})$ be a logical lattice, Γ a set of formulae, and ψ a formula. Denote $\Gamma \models^{\mathcal{L},\mathcal{D}} \psi$ if every model of Γ is a model of ψ.

The relation $\models^{\mathcal{L},\mathcal{D}}$ of Definition 2 is a consequence relation in the standard sense of Tarski [38]. In [4] it is shown that this relation is monotonic, compact, paraconsistent [19], and has a corresponding sound and complete cut-free Gentzen-type system. The major drawbacks of $\models^{\mathcal{L},\mathcal{D}}$ are that it is strictly weaker than classical logic even for consistent theories (e.g., $\psi \not\models^{\mathcal{L},\mathcal{D}} \neg\phi \vee \phi$), and that it always invalidates some intuitively justified inference rules, like the Disjunctive Syllogism (that is, $\psi,\ \neg\psi \vee \phi \not\models^{\mathcal{L},\mathcal{D}} \phi$). In the next section we consider a family of logics that overcome these drawbacks.

2.2 Preferential Reasoning and the Consequence Relation $\models_{c}^{\mathcal{L},\mathcal{D}}$

In order to recapture within our many-valued framework classical reasoning (where its use is appropriate), as well as standard non-monotonic and paraconsistent methods, we incorporate a concept first introduced by McCarthy [32] and later considered by Shoham [36], according to which inferences from a given theory are made w.r.t. a subset of the models of that theory (and not w.r.t. every model of the theory; see also [24,27,30,31,35]). This set of *preferential models* is determined according to some conditions that can be specified by a set of (usually second-order) propositions [7], or by some order relation on the models of the theory [4,5]. This relation should reflect some kind of preference criterion on the models of the set of premises. In our case the idea is to give precedence to those valuations that minimize the amount of uncertain information in the set of premises. The truth values are therefore arranged according to an order relation that reflects differences in the amount of uncertainty that each one of them exhibits. Then we choose those valuations that minimize the amount of uncertainty w.r.t. this order. The intuition behind this approach is that incomplete or contradictory data corresponds to inadequate information about the real world, and therefore it should be minimized. Next we formalize this idea.

[3] Note that on $\{t, f\}$ the material implication ($\leadsto$) and the new implication ($\rightarrow$) are identical, and both of them are generalizations of the classical implication.

Definition 3. A partial order $<$ on a set L is called *modular* if $y < x_2$ for every $x_1, x_2, y \in L$ s.t. $x_1 \not< x_2$, $x_2 \not< x_1$, and $y < x_1$.

Proposition 1. [28] *Let $<$ be a partial order on L. The following conditions are equivalent:*
a) *$<$ is modular.*
b) *For every $x_1, x_2, y \in L$, if $x_1 < x_2$ then either $y < x_2$ or $x_1 < y$.*
c) *There is a totally ordered set L' with a strict order $\prec$ and a function $g : L \to L'$ s.t. $x_1 < x_2$ iff $g(x_1) \prec g(x_2)$.*

Definition 4. An *inconsistency order* $<_c^{\mathcal{L},\mathcal{D}}$ on a logical lattice $(\mathcal{L}, \mathcal{D})$ is a well-founded modular order on L, with the following properties:
a) t and f are minimal and $\top$ is maximal w.r.t. $<_c^{\mathcal{L},\mathcal{D}}$,
b) if $\{x, \neg x\} \subseteq \mathcal{D}$ while $\{y, \neg y\} \not\subseteq \mathcal{D}$, then $x \not<_c^{\mathcal{L},\mathcal{D}} y$,
c) x and $\neg x$ are either equal or $<_c^{\mathcal{L},\mathcal{D}}$-incomparable.

Inconsistency orders are used here for grading uncertainty in general, and inconsistency in particular. Intuitively, the meaning of $x <_c^{\mathcal{L},\mathcal{D}} y$ is that formulae that are assigned x are more definite than formulae with a truth value y. Modularity is needed for assuring a proper grading of the truth values.[4] Condition (b) in Definition 4 assures that truth values that intuitively represent inconsistent data will not be considered as more consistent than those ones that correspond to consistent data. The last condition makes sure that any truth value and its negation have the same degree of (in)consistency.

Example 3. $\mathcal{FOUR}$ has four inconsistency orders:
a) The degenerated order, $<_{c_0}^4$, in which $t, f, \bot, \top$ are all incomparable.
b) $<_{c_1}^4$, in which $\bot$ is considered as minimally inconsistent: $\{t, f, \bot\} <_{c_1}^4 \top$.
c) $<_{c_2}^4$, in which $\bot$ is maximally inconsistent: $\{t, f\} <_{c_2}^4 \{\top, \bot\}$.
d) $<_{c_3}^4$, in which $\bot$ is an intermediate level of inconsistency: $\{t, f\} <_{c_3}^4 \bot <_{c_3}^4 \top$.

In the rest of the paper we shall continue to use the notations of Example 3 for denoting the inconsistency orders in $\mathcal{FOUR}$.

Given an inconsistency order $<_c^{\mathcal{L},\mathcal{D}}$ on a logical lattice $(\mathcal{L}, \mathcal{D})$, it induces an equivalence relation on L, in which two elements in L are equivalent iff they are equal or $<_c^{\mathcal{L},\mathcal{D}}$-incomparable. For every $x \in \mathcal{L}$, we denote by $[x]$ the equivalence class of x with respect to this equivalence relation. I.e.,

$$[x] = \{y \mid y = x, \text{ or } x \text{ and } y \text{ are } <_c^{\mathcal{L},\mathcal{D}}\text{-incomparable}\}.$$

The order relation on these classes is defined as usual by representatives: we denote $[x] \leq_c^{\mathcal{L},\mathcal{D}} [y]$ iff either $x <_c^{\mathcal{L},\mathcal{D}} y$, or x and y are $<_c^{\mathcal{L},\mathcal{D}}$-incomparable.[5] It is

[4] That is, to eliminate orders such as $\{\ \{t\},\ \{f \prec \bot \prec \top\}\ \}$, in which $\top$ and $\bot$ are not comparable with t, while they are comparable with $\neg t$.

[5] As usual, we use the same notation to denote the order relation among equivalence classes and the order relation among their elements.

easy to verify that this definition is proper, i.e. it does not depend on the choice of the representatives.

An inconsistency order on $(\mathcal{L},\mathcal{D})$ induces the following pre-order on $\mathcal{V}^L$:

Definition 5. Let $<_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order on $(\mathcal{L},\mathcal{D})$, and let $\nu_1,\nu_2 \in \mathcal{V}^L$.
a) $\nu_1 \leq_c^{\mathcal{L},\mathcal{D}} \nu_2$ iff for every atom p, $[\nu_1(p)] \leq_c^{\mathcal{L},\mathcal{D}} [\nu_2(p)]$.
b) $\nu_1 <_c^{\mathcal{L},\mathcal{D}} \nu_2$ if $\nu_1 \leq_c^{\mathcal{L},\mathcal{D}} \nu_2$ and there is an atom q s.t. $[\nu_1(q)] <_c^{\mathcal{L},\mathcal{D}} [\nu_2(q)]$.

Definition 6. Let $<_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order in a logical lattice $(\mathcal{L},\mathcal{D})$. The set of the *c-most consistent models* of a set Γ of formulae in Σ (abbreviation: the *c-mcms* of Γ) are the minimal inconsistent models of Γ, i.e.:

$$!(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}}) = \{\nu \in mod(\Gamma) \mid \neg\exists \mu \in mod(\Gamma) \text{ s.t. } \mu <_c^{\mathcal{L},\mathcal{D}} \nu\}.$$

Now we can refine the inference process, defined by the lattice-based consequence relation $\models^{\mathcal{L},\mathcal{D}}$ (Definition 2). Instead of considering every possible model of the premises, we take into account only the c-most consistent ones.

Definition 7. Let $<_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order on a logical lattice $(\mathcal{L},\mathcal{D})$. Denote: $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \psi$ if every c-mcm of Γ is a model of ψ.

Example 4. Consider one direction of the barber paradox:[6]

$$\Gamma = \{\neg\texttt{shaves}(\texttt{x},\texttt{x}) \to \texttt{shaves}(\texttt{Barber},\texttt{x})\}.$$

Denote by ν_1, ν_2, and ν_3 the valuations that assign t, $\bot$, and $\top$ (respectively) to the assertion $\texttt{shaves}(\texttt{Barber},\texttt{Barber})$. Using $\mathcal{FOUR}$ as the underlying logical lattice, we have that $!(\Gamma, \leq_{c_2}^4) = !(\Gamma, \leq_{c_3}^4) = \{\nu_1\}$, $!(\Gamma, \leq_{c_1}^4) = \{\nu_1,\nu_2\}$, and $!(\Gamma, \leq_{c_0}^4) = \{\nu_1,\nu_2,\nu_3\}$. Thus, $\Gamma \not\models_{c_i}^4 \texttt{shaves}(\texttt{Barber},\texttt{Barber})$ when $i = 0,1$, while $\Gamma \models_{c_i}^4 \texttt{shaves}(\texttt{Barber},\texttt{Barber})$ when $i=2,3$.

3 Embedding in Four-Valued Logics

Four-valued reasoning may be traced back to the 1950's, where is has been investigated by a number of people, including Bialynicki-Birula [16], Rasiowa [17], and Kalman [25]. Later, Belnap [13,14] introduced a corresponding four-valued algebraic structure (denoted here by $\mathcal{FOUR}$) for paraconsistent reasoning. Theorem 1 below, which is our main result here, shows that this structure is canonical for reasoning with graded uncertainty. Following [5], this is another evidence for the robustness of four-valued logics as representing commonsense reasoning.

[6] Here we assume that formulae with variables are universally quantified. Consequently, a set of assertions Γ, containing a non-grounded formula, ψ, is viewed as representing the corresponding set of ground formulae, formed by substituting for each variable that appears in ψ, every element in the relevant Herbrand universe.

Definition 8. $\mathcal{V}^L$ is *stoppered* w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$ if for every Γ and every $\nu \in mod(\Gamma)$, either $\nu \in\, !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$, or there is an $\nu' \in\, !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$ s.t. $\nu' <_c^{\mathcal{L},\mathcal{D}} \nu$.[7]

Note that in case that $\mathcal{V}^L$ is well-founded w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$ (i.e., $\mathcal{V}^L$ does not have an infinitely descending chain w.r.t. $\leq_c^{\mathcal{L},\mathcal{D}}$), then it is in particular stoppered.

Theorem 1. *Let $\leq_c^{\mathcal{L},\mathcal{D}}$ be an inconsistency order on a logical lattice $(\mathcal{L},\mathcal{D})$ such that $\mathcal{V}^L$ is stoppered (with respect to the induced order on valuations). Then $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \psi$ iff $\Gamma \models_{c_i}^4 \psi$ for some $0 \leq i \leq 3$.*

In the rest of this section we prove Theorem 1. First, we consider some notations and definitions.

Definition 9. Given a logical lattice $(\mathcal{L},\mathcal{D})$, its elements may be divided into the following four sets:

$$\mathcal{T}_t^{\mathcal{L},\mathcal{D}} = \{x \in L \mid x \in \mathcal{D}, \neg x \notin \mathcal{D}\}, \qquad \mathcal{T}_f^{\mathcal{L},\mathcal{D}} = \{x \in L \mid x \notin \mathcal{D}, \neg x \in \mathcal{D}\},$$

$$\mathcal{T}_\top^{\mathcal{L},\mathcal{D}} = \{x \in L \mid x \in \mathcal{D}, \neg x \in \mathcal{D}\}, \qquad \mathcal{T}_\bot^{\mathcal{L},\mathcal{D}} = \{x \in L \mid x \notin \mathcal{D}, \neg x \notin \mathcal{D}\}.$$

Henceforth we shall usually omit the superscripts, and write $\mathcal{T}_t, \mathcal{T}_f, \mathcal{T}_\top$, $\mathcal{T}_\bot$.

Definition 10. Let $(\mathcal{L},\mathcal{D})$ be a logical lattice. Denote:

$$\min\nolimits_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x = \{y \in \mathcal{T}_x \mid \neg\exists y' \in \mathcal{T}_x \text{ s.t. } y' <_c^{\mathcal{L},\mathcal{D}} y\} \quad (x \in \{t, f, \top, \bot\})$$

$$\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}} = \min\nolimits_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_t \cup \min\nolimits_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_f \cup \min\nolimits_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\bot \cup \min\nolimits_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top$$

Definition 11. Let $(\mathcal{L}_1,\mathcal{D}_1)$ and $(\mathcal{L}_2,\mathcal{D}_2)$ be two logical lattices. Suppose that x_i is some element in L_i and ν_i is a valuation onto L_i $(i=1,2)$.
a) x_1 and x_2 are *similar* if $x_1 \in \mathcal{T}_y^{\mathcal{L}_1,\mathcal{D}_1}$ implies that $x_2 \in \mathcal{T}_y^{\mathcal{L}_2,\mathcal{D}_2}$ $(y \in \{t, f, \top, \bot\})$.
b) ν_1 and ν_2 are *similar* if for every atom p, $\nu_1(p)$ and $\nu_2(p)$ are similar.

Proposition 2. *Let $(\mathcal{L}_1,\mathcal{D}_1)$ and $(\mathcal{L}_2,\mathcal{D}_2)$ be two logical lattices and suppose that ν_1 and ν_2 are two similar valuations on L_1 and L_2 (respectively). Then for every formula ψ, $\nu_1(\psi)$ and $\nu_2(\psi)$ are similar.*

Proof. By an induction on the structure of ψ.[8]

Proof (of Theorem 1). We shall denote by m_x some element in $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x^{\mathcal{L},\mathcal{D}}$ $(x \in \{t, f, \top, \bot\})$, and by $\omega : L \to \{t, f, \top, \bot\}$ the "categorization" function: $\omega(y) = x$ iff $y \in \mathcal{T}_x$. Also, in the rest of this proof we shall abbreviate $[y] \cap \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ by $[y]$ (thus we shall refer here to classes that consist only of elements in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$).

Lemma 1. *If $M \in\, !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$ then for every atom p, $M(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$.*

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[7] The notion "stopperdness" is due to Mackinson [31]. In [27] the same property is called *smoothness*.

[8] Note that the fact that $\mathcal{D}$ is a *prime* filter is crucial here.

Proof. Suppose that there is some atom p_0 s.t. $M(p_0) \notin \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$. Then, assuming that $M(p_0) \in \mathcal{T}_x$, there is an element $m_x \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$ s.t. $m_x <_c^{\mathcal{L},\mathcal{D}} M(p_0)$. Consider the following valuation:

$$N(p) = \begin{cases} m_x & \text{if } p = p_0 \\ M(p) & \text{if } p \neq p_0 \end{cases}$$

N is similar to M, and so, by Proposition 2, N is also a model of Γ. Moreover, $N <_c^{\mathcal{L},\mathcal{D}} M$, thus $M \notin !(\Gamma, \leq_c^{\mathcal{L},\mathcal{D}})$. □

Now, since $\leq_c^{\mathcal{L},\mathcal{D}}$ is well-founded and since $\mathcal{T}_x$ is nonempty for every $x \in \{t, f, \top, \bot\}$, $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$ is nonempty as well, and so there is at least one element of the form m_x for every $x \in \{t, f, \top, \bot\}$. Also, it is clear that for every $m_x, m'_x \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$, $[m_x] = [m'_x]$ (otherwise either $m_x <_c^{\mathcal{L},\mathcal{D}} m'_x$ or $m_x >_c^{\mathcal{L},\mathcal{D}} m'_x$, and so either $m'_x \notin \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$ or $m_x \notin \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_x$). It follows, therefore, that there are no more than three equivalence classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$:

$$\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_t \cup \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_f \subseteq [t], \quad \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\bot \subseteq [m_\bot], \quad \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top \subseteq [m_\top],$$

where $m_\bot$ is some element of $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\bot$, and $m_\top$ is some element of $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top$. By Definition 4, $[t]$ must be a minimal inconsistency class among those in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$, and $[m_\top]$ must be a maximal one. It follows, then, that the inconsistency classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ are arranged in one of the following orders:

0. $[t] = [m_\bot] = [m_\top]$,
1. $[t] = [m_\bot] <_c^{\mathcal{L},\mathcal{D}} [m_\top]$,
2. $[t] <_c^{\mathcal{L},\mathcal{D}} [m_\bot] = [m_\top]$,
3. $[t] <_c^{\mathcal{L},\mathcal{D}} [m_\bot] <_c^{\mathcal{L},\mathcal{D}} [m_\top]$.

If the order relation among the inconsistency classes in $\Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ corresponds to case i above $(0 \leq i \leq 3)$ we say that the inconsistency order $\leq_c^{\mathcal{L},\mathcal{D}}$ is *of type i*.[9]

Lemma 2. *If $\leq_c^{\mathcal{L},\mathcal{D}}$ is an inconsistency order of type i, then for every $m, m' \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$, $[m] <_c^{\mathcal{L},\mathcal{D}} [m']$ iff $[\omega(m)] <_{c_i}^4 [\omega(m')]$.*

Proof. Immediate from the definition of inconsistency order of type i, and the definition of $\leq_{c_i}^4$. □

Lemma 3. *If $\leq_c^{\mathcal{L},\mathcal{D}}$ is an inconsistency order of type i in $(\mathcal{L}, \mathcal{D})$, then $\models_c^{\mathcal{L},\mathcal{D}}$ is the same as $\models_{c_i}^4$.*

Proof. Suppose that $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \psi$ but $\Gamma \not\models_{c_i}^4 \psi$. Then there is a c_i^4-mcm M^4 of Γ s.t. $M^4(\psi) \notin \{t, \top\}$. Now, for every atom p let $M^L(p)$ be some element in $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_{M^4(p)}$. Thus $\omega \circ M^L = M^4$, and M^L is similar to M^4. By Proposition 2, M^L is a model of Γ and it is not a model of ψ. To get a contradiction to $\Gamma \models_c^{\mathcal{L},\mathcal{D}} \psi$, it remains to show, then, that M^L is a c-mcm of Γ in $(\mathcal{L}, \mathcal{D})$. Indeed, otherwise by stopperdness there is a c-mcm N^L of Γ s.t. $N^L <_c^{\mathcal{L},\mathcal{D}} M^L$. So for every atom

[9] In particular, for every $0 \leq i \leq 3$, the inconsistency order $\leq_{c_i}^4$ in $\mathcal{FOUR}$ is of type i.

p, $[N^L(p)] \leq_c^{\mathcal{L},\mathcal{D}} [M^L(p)]$, and there is an atom p_0 s.t. $[N^L(p_0)] <_c^{\mathcal{L},\mathcal{D}} [M^L(p_0)]$. Let $N^4 = \omega \circ N^L$. Again, N^4 is similar to N^L, therefore it is a (four-valued) model of Γ. Also, by the definition of M, for every atom p, $M^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ and by Lemma 1, $\forall p\ N^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$. Thus, by Lemma 2,

$$[N^4(p)] = [\omega \circ N^L(p)] \leq^4_{c_i} [\omega \circ M^L(p)] = [M^4(p)].$$

Also, by the same lemma,

$$[N^4(p_0)] = [\omega \circ N^L(p_0)] <^4_{c_i} [\omega \circ M^L(p_0)] = [M^4(p_0)].$$

It follows that $N^4 <^4_{c_i} M^4$, but this contradicts the assumption that M^4 is a c_i^4-mcm of Γ.

For the converse, suppose that $\Gamma \models^4_{c_i} \psi$, but $\Gamma \not\models_c^{\mathcal{L},\mathcal{D}} \psi$. Then there is a c-mcm M^L of Γ in $(\mathcal{L},\mathcal{D})$ s.t. $M^L(\psi) \notin \mathcal{D}$. Define, for every atom p, $M^4(p) = \omega \circ M^L(p)$. By the definition of ω, M^4 is similar to M^L and so M^4 is a model of Γ in $\mathcal{FOUR}$, but it is not a model of ψ. It remains to show, then, that M^4 is a c_i^4-mcm of Γ. Indeed, otherwise there is a model N^4 of Γ s.t. $N^4 <^4_{c_i} M^4$, that is, for every atom p $[N^4(p)] \leq^4_{c_i} [M^4(p)]$, and there is an atom p_0 for which this inequality is strict: $[N^4(p_0)] <^4_{c_i} [M^4(p_0)]$. Now, for every atom p, let $N^L(p)$ be some element in $\min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_{N^4(p)}$. Thus $\omega \circ N^L = N^4$, and N^L is similar to N^4. By Proposition 2, N^L is in particular a model of Γ in $(\mathcal{L},\mathcal{D})$. Moreover, for every atom p,

$$[\omega \circ N^L(p)] = [N^4(p)] \leq^4_{c_i} [M^4(p)] = [\omega \circ M^L(p)].$$

Now, by the definition of N^L we have that for every atom p, $N^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$, and by Lemma 1, $M^L(p) \in \Omega_{\leq_c^{\mathcal{L},\mathcal{D}}}$ as well. Hence, by Lemma 2, $[N^L(p)] \leq_c^{\mathcal{L},\mathcal{D}} [M^L(p)]$. Similarly,

$$[\omega \circ N^L(p_0)] = [N^4(p_0)] <^4_{c_i} [M^4(p_0)] = [\omega \circ M^L(p_0)]$$

and again this entails that $[N^L(p_0)] <_c^{\mathcal{L},\mathcal{D}} [M^L(p_0)]$. It follows that $N^L <_c^{\mathcal{L},\mathcal{D}} M^L$, but this contradicts the assumption that M^L is a c-mcm of Γ in $(\mathcal{L},\mathcal{D})$.
This concludes the proof of Lemma 3 and Theorem 1. □

Note 1. The proof of Theorem 1 also induces a simple algorithm for determining which one of the four-valued consequence relations is the same as a given consequence relation of the form $\models_c^{\mathcal{L},\mathcal{D}}$: given an inconsistency order $\leq_c^{\mathcal{L},\mathcal{D}}$ in $(\mathcal{L},\mathcal{D})$, choose some $m_\perp \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\perp$ and $m_\top \in \min_{\leq_c^{\mathcal{L},\mathcal{D}}} \mathcal{T}_\top$. If $[m_\top] = [t]$ then $\models_c^{\mathcal{L},\mathcal{D}} = \models^4_{c_0}$. Otherwise, if $[m_\perp] = [t]$, then $\models_c^{\mathcal{L},\mathcal{D}} = \models^4_{c_1}$. Otherwise, if $[m_\top] = [m_\perp]$, then $\models_c^{\mathcal{L},\mathcal{D}} = \models^4_{c_2}$. Otherwise, $\models_c^{\mathcal{L},\mathcal{D}} = \models^4_{c_3}$.

4 Reasoning with $\models_c^{\mathcal{L},\mathcal{D}}$

We conclude by briefly considering some basic properties of $\models_c^{\mathcal{L},\mathcal{D}}$.[10] In what follows we assume stopperdness, and so, by Theorem 1, it is sufficient to consider

[10] Most of the propositions in this section easily follow from similar results concerning modular preferential relations, considered in [1]. Due to space limitations, corresponding proofs are omitted.

$\mathcal{FOUR}$ and the four corresponding consequence relations $\models^4_{c_i}$ (i=0,..., 3). First, we consider the relative strength of these logics:

Proposition 3. *Let Γ be a set of formulae and ψ a formula in Σ.*
a) *The consequence relations $\models^4_{c_i}$, $0 \leq i \leq 3$, are all different.*
b) *For every $1 \leq i \leq 3$, if $\Gamma \models^4_{c_0} \psi$ then $\Gamma \models^4_{c_i} \psi$.*
c) *No one of $\models^4_{c_1}$, $\models^4_{c_2}$, and $\models^4_{c_3}$, is stronger than the other.*

In what follows we shall write $\models^2$ for the classical consequence relation, and $\models^4_c$ for any one of $\models^4_{c_i}$, $0 \leq i \leq 3$. As the next proposition shows, reasoning with $\models^4_c$ does not reduce to triviality when the set of premises is not consistent.

Proposition 4. *$\models^4_c$ is paraconsistent.*

Proposition 5. *If $\Gamma \models^4_c \psi$ then $\Gamma \models^2 \psi$.*

The converse of Proposition 5 is not true in general. For instance, excluded middle is not valid w.r.t. $\models^4_{c_0}$ and $\models^4_{c_1}$. However, with respect to the other basic four-valued consequence relations, the converse of Proposition 5 does hold.

Proposition 6. *Let Γ be a classically consistent theory. Then for every formula ψ in Σ we have that $\Gamma \models^2 \psi$ iff $\Gamma \models^4_{c_2} \psi$ iff $\Gamma \models^4_{c_3} \psi$.*

By Propositions 4 and 6, it follows that with (any consequence relation of the form $\models^{\mathcal{L},\mathcal{D}}_c$ that is equivalent to) $\models^4_{c_2}$ and $\models^4_{c_3}$ one can draw classical conclusions from (classically) consistent theories, while the set of conclusions is not "exploded" when the theory becomes inconsistent. Batens [10] describes this property as an "oscillation" between some lower limit (paraconsistent) logic and an upper limit (classical) logic.

Proposition 7. *$\models^4_{c_0}$ is a monotonic consequence relation, while $\models^4_{c_i}$, $i = 1, 2, 3$, are nonmonotonic relations.*

The last proposition implies that unless the inconsistency order is degenerated, $\models^{\mathcal{L},\mathcal{D}}_c$ is not monotonic, thus it is not a consequence relation in the sense of Tarski [38]. In such cases it is usual to require a weaker condition:

Proposition 8. [24,27] *$\models^4_c$ satisfies cautious left monotonicity: if $\Gamma \models^4_c \psi$ and $\Gamma \models^4_c \phi$, then $\Gamma, \psi \models^4_c \phi$.*

A desirable property of non-monotonic consequence relations is the ability to preserve any conclusion when learning about a new fact that has no influence on the set of premises. Consequence relations that satisfy this property are called *rational* [28]. The next proposition shows that $\models^4_{c_i}$ (i=0, ..., 3) are rational.

Proposition 9. *If $\Gamma \models^4_c \psi$ and $\mathcal{A}(\Gamma \cup \{\psi\}) \cap \mathcal{A}(\phi) = \emptyset$, then $\Gamma, \phi \models^4_c \psi$.*[11]

[11] Recall that $\mathcal{A}(\Gamma)$ is the set of atomic formulae that appear in some formula of Γ.

Intuitively, the second condition in Proposition 9 guarantees that ϕ is 'irrelevant' for Γ and ψ. The intuitive meaning of Proposition 9 is, therefore, that the reasoner does not have to retract ψ when learning that ϕ holds.

Note 2. In order to assure rationality, Lehmann and Magidor [28] introduced the rule of *rational monotonicity:* if $\Gamma \mid\!\sim \psi$ then $\Gamma, \phi \mid\!\sim \psi$, unless $\Gamma \mid\!\sim \neg\phi$.

Rational monotonicity may be considered as too strong for assuring rationality, and many general patterns of nonmonotonic reasoning do not satisfy this rule. For instance, $\models^4_{c_1}$ is rational (Proposition 9), but it does not satisfy rational monotonicity (consider, e.g., $\Gamma = \{p,\ q \to \neg p\}$, $\psi = \neg p \to \neg q$, and $\phi = q$).

In terms of Batens [10,11], $\models^4_{c_2}$ and $\models^4_{c_3}$ are also *adaptive*, i.e.: if it is possible to distinguish between a consistent part and an inconsistent part of a given theory, then every assertion that classically follows from the consistent part, and is not related to the inconsistent part, is also a $\models^4_{c_i}$-consequence ($i = 2, 3$) of the whole theory. Thus, as the following proposition shows, $\models^4_{c_2}$ and $\models^4_{c_3}$ presuppose a consistency of all the assertions 'unless and until proven otherwise'.

Proposition 10. *Let $\Gamma = \Gamma' \cup \Gamma''$ be a set of formulae in Σ s.t. Γ' is classically consistent and $\mathcal{A}(\Gamma') \cap \mathcal{A}(\Gamma'') = \emptyset$. Then for every ψ s.t. $\mathcal{A}(\psi) \cap \mathcal{A}(\Gamma'') = \emptyset$, the fact that $\Gamma' \models^2 \psi$ entails that $\Gamma \models^4_{c_2} \psi$ and $\Gamma \models^4_{c_3} \psi$.*

We conclude by noting that consequence relations of the form $\models_c^{\mathcal{L},\mathcal{D}}$ naturally generalize some related formalisms such as the consequence relations $\models_{\mathcal{I}_1}^{\mathcal{L},\mathcal{D}}$, $\models_{\mathcal{I}_2}^{\mathcal{L},\mathcal{D}}$, introduced in [5,6], and the logic LPm of Priest [34].[12]

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[12] See [2] for a proof of this claim.

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