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# Simple contrapositive assumption-based argumentation frameworks

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A B S T R A C T

Assumption-based argumentation is one of the most prominent formalisms for logical (or structured) argumentation, with tight links to different forms of defeasible reasoning. In this paper we study the Dung semantics for extended forms of assumption-based argumentation frameworks (ABFs), based on *any* contrapositive propositional logic, and whose defeasible assumptions are expressed by *arbitrary formulas* in that logic. We show that unless the falsity propositional constant is part of the defeasible assumptions, the grounded and the well-founded semantics for ABFs lack most of the desirable properties they have in abstract argumentation frameworks (AAFs), and that for simple definitions of the contrariness operator and the attacks relations, preferred and stable semantics are reduced to naive semantics. We also show the redundancy of the closure condition in the standard definition of Dung's semantics for ABFs, and investigate the use of disjunctive attacks in this setting. Finally, we show some close relations of reasoning with ABFs to reasoning with maximally consistent sets of premises, and consider some properties of the induced entailments, such as being cumulative, preferential, or rational relations that satisfy non-interference.



## 1. Introduction

Assumption-Based Argumentation (ABA), thoroughly described in [9], was introduced in the 1990s, as a computational framework to capture and generalize default and defeasible reasoning. It was inspired by Dung's semantics for abstract argumentation and logic programming with its dialectical interpretation of the acceptability of negation-as-failure assumptions based on "no-evidence-to-the-contrary".

ABA systems are represented in different ways in the literature. A cornerstone in all of them is a distinction between two types of assumptions for the argumentation: the strict (non-revised) ones and the defeasible ones. Traditionally, the latter are usually expressed in terms of logic-programming-like expressions of the form $A_1 \wedge \ldots \wedge A_n \rightarrow B$ (intuitively understood by 'if all of $A_1, \ldots, A_n$ hold, then so does $B$'). Here we do not confine ourselves to any specific syntactical forms of the (strict or defeasible) expressions, but rather allow any propositional assertion. The logical foundation for making arguments and counter-arguments in our setting may be based on any logic respecting the contraposition rule, where the contrariness operator is of the simple and most natural form: the contrary of a formula is its negation. The outcome is what we call *simple contrapositive assumption-based (argumentation) frameworks* (simple contrapositive ABFs, for short).

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In this work we investigate the main Dung-style semantics [16] of simple contrapositive ABFs. This includes, among others, the following new findings:

1. We show that while non-flat ABFs may not have complete extensions (and so grounded and well-founded extensions may not available for them), simple contrapositive ABFs are non-flat ABFs that always have such extensions.
2. We consider the well-founded semantics for ABFs and show that under a simple condition this semantics coincides with the grounded semantics for the same ABFs.
3. We identify some conditions, without which the grounded (and the well-founded) semantics of (non-flat) ABFs lose many of their desirable properties that are assured in the context of abstract argumentation frameworks (AAFs).
4. We show that at least for the standard form of attack and simple definitions of the contrariness operator, the main types of Dung's semantics reduce to the naive semantics (a phenomenon that is known already for some specific AAFs, see [1–3]).
5. We show that for simple contrapositive ABFs the closure requirement on the frameworks' extensions is in fact redundant. As a consequence, most of the concepts that are related to such ABFs are simplified, and their computation becomes easier.
6. We show that as in the case of abstract and logical argumentation frameworks, the Dung's semantics for ABFs is tightly related to reasoning with maximal consistency [27].
7. We consider a generalization of the attack relation in ABFs, called disjunctive attacks. The use of these attacks avoids some problems of the grounded semantics under standard attacks (see Item 2 above). The consistency of extensions and correspondence to maximal consistency-based reasoning are preserved under the generalization to disjunctive attacks, which means that some of the long-standing problems that were reported by [11] for other logic-based argumentation formalisms using disjunctive attacks are avoided in our setting.
8. We study some of the properties of the entailment relations that are induced by simple contrapositive ABFs. This includes the well-known postulates introduced by Kraus, Lehmann, and Magidor (KLM) [22,23], and some other properties, such as non-interference [10].

The rest of this paper is organized as follows: in the next section we recall the main notions behind ABFs and introduce simple contrapositive ABFs. Then, in Section 3 we consider the main Dung-style semantics for such ABFs – first the preferred and the stable semantics, and then the grounded and the well-founded semantics. In Section 4 we examine some of the properties of the induced entailment relations, in particular their relations to the base logic, belonging to the KLM-defined families for non-monotonic entailments, and the satisfaction of properties that are related to inconsistency handling such as non-interference. Then we consider two generalizations of our settings: one (Section 5) is related to the removal of the closure requirement in the definition of the semantics, and the other (Section 6) involves extended attack relations, called disjunctive attacks. It is shown that in both cases most of the properties of the semantics and the entailment relations before the generalizations are preserved. In Section 7 we conclude.[1]

## 2. Simple contrapositive ABFs

We start by some preliminaries, concerning assumption-based argumentation frameworks and their ingredients. We then define what simple contrapositive ABFs are.

In what follows we shall denote by $\mathscr{L}$ an arbitrary propositional language. Atomic formulas in $\mathscr{L}$ are denoted by $p, q, r$, compound formulas are denoted by $\psi, \phi, \sigma$, and sets of formulas in $\mathscr{L}$ are denoted by $\Gamma$, $\Delta$, $\Theta$ (possibly primed or indexed). The powerset of $\mathscr{L}$ is denoted by $\wp(\mathscr{L})$.

**Definition 1.** A *logic* for a language $\mathscr{L}$ is a pair $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, where $\vdash$ is a consequence relation for $\mathscr{L}$ [29], that is, a binary relation between sets of formulas and formulas in $\mathscr{L}$, satisfying the following conditions:

*Reflexivity*: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
*Monotonicity*: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
*Transitivity*: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$.

In addition, we shall assume that $\mathfrak{L}$ satisfies the following standard conditions:

*Structurality* (closure under substitutions): if $\Gamma \vdash \psi$ then $\theta(\Gamma) \vdash \theta(\psi)$ for every $\mathscr{L}$-substitution $\theta$.
*Non-triviality*: there are a non-empty set $\Gamma$ and a formula $\psi$ such that $\Gamma \nvdash \psi$.

The $\vdash$-transitive closure of a set $\Gamma$ of $\mathscr{L}$-formulas is $Cn_{\vdash}(\Gamma) = \{\psi \mid \Gamma \vdash \psi\}$. When $\vdash$ is clear from the context, we will sometimes just write $Cn(\Gamma)$.

[1] This paper is a revised and extended version of the papers in [19–21].

**Definition 2.** We shall assume that the language $\mathscr{L}$ contains at least the following connectives and constant:

a $\vdash$-*negation* $\neg$, satisfying: $p \nvdash \neg p$ and $\neg p \nvdash p$ (for every atomic $p$).
a $\vdash$-*conjunction* $\wedge$, satisfying: $\Gamma \vdash \psi \wedge \phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$.
a $\vdash$-*disjunction* $\vee$, satisfying: $\Gamma, \phi \vee \psi \vdash \sigma$ iff $\Gamma, \phi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$.
a $\vdash$-*implication* $\supset$, satisfying: $\Gamma, \phi \vdash \psi$ iff $\Gamma \vdash \phi \supset \psi$.
a $\vdash$-*falsity* $\mathsf{F}$, satisfying: $\mathsf{F} \vdash \psi$ for every formula $\psi$.[2]

We abbreviate $\{\neg\gamma \mid \gamma \in \Gamma\}$ by $\neg\Gamma$, and when $\Gamma$ is finite we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$), the conjunction (respectively, the disjunction) of all the formulas in $\Gamma$. We shall say that $\Gamma$ is $\vdash$-*consistent* if $\Gamma \nvdash \mathsf{F}$ (otherwise $\Gamma$ is $\vdash$-inconsistent).

**Definition 3.** A logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is *explosive*, if for every $\mathscr{L}$-formula $\psi$ the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent.[3] We say that $\mathfrak{L}$ is *contrapositive*, if for every $\Gamma$ and $\psi$ it holds that $\Gamma \vdash \neg\psi$ iff either $\psi = \mathsf{F}$,[4] or for every $\phi \in \Gamma$ we have that $\Gamma \setminus \{\phi\}, \psi \vdash \neg\phi$.

**Example 1.** Perhaps the most notable example of a logic which is both explosive and contrapositive, is classical logic, $\mathsf{CL}$. Intuitionistic logic, the central logic in the family of constructive logics, and standard modal logics are other examples of well-known formalisms having these properties.

The next, simple observation, will be useful in what follows.

**Lemma 1.** *Let $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ be an explosive logic. Then $\Gamma$ is $\vdash$-inconsistent iff both $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$.*

**Proof.** If $\Gamma$ is $\vdash$-inconsistent then $\Gamma \vdash \mathsf{F}$, and since both $\mathsf{F} \vdash \psi$ and $\mathsf{F} \vdash \neg\psi$ we get by transitivity one direction. For the converse, suppose that $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$. Since $\wedge$ is a $\vdash$-conjunction, we get $\Gamma \vdash \psi \wedge \neg\psi$. Also, since $\mathfrak{L}$ is explosive, $\psi, \neg\psi \vdash \mathsf{F}$ so by the conjunction properties $\psi \wedge \neg\psi \vdash \mathsf{F}$. Thus, by transitivity, $\Gamma \vdash \mathsf{F}$, and so $\Gamma$ is $\vdash$-inconsistent. $\square$

We are now ready to define assumption-based argumentation frameworks (ABFs). The next definition is a generalization of the definition from [9].

**Definition 4.** An *assumption-based framework* is a tuple $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ where:

- $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is a propositional logic.
- $\Gamma$ (the *strict assumptions*) and $Ab$ (the *candidate/defeasible assumptions*) are distinct (countable) sets of $\mathscr{L}$-formulas, where the former is assumed to be $\vdash$-consistent and the latter is assumed to be nonempty.
- $\sim : Ab \to \wp(\mathscr{L})$ is a *contrariness operator*, assigning a finite set of $\mathscr{L}$-formulas to every defeasible assumption in $Ab$, such that for every $\vdash$-consistent $\psi \in Ab \setminus \{\mathsf{F}\}$ it holds that $\psi \nvdash \bigwedge {\sim}\psi$ and $\bigwedge {\sim}\psi \nvdash \psi$.

**Note 1.** Some remarks on the relations between assumption-based frameworks as in Definition 4 and elsewhere in the literature (e.g., [9,12]) are in order.

a) An ABF in our setting may be based on *any* propositional logic $\mathfrak{L}$ and the strict as well as the candidate assumptions consist of *arbitrary* formulas in the language of that logic. While this is the case also according to the definitions in [9,12], in practice the ABFs that are investigated in these papers are only those that are based on *atomic* (strict and defeasible) assumptions.
Concerning the contrariness operator, we note that it is not a connective of $\mathscr{L}$, as it is restricted only to the candidate assumptions. The conditions on the contrariness operator express the requirement that a formula should not imply, nor it should be implied by, its contrary.
b) Traditionally, ABFs make use of some set of domain dependent rules as known from e.g. logic programming (i.e., rules of the form $\phi_1, \ldots, \phi_n \to \phi$, as in logic programming). It is not difficult to see that our setting also applies to this subclass of ABFs by assuming that the implication $\supset$ is deductive (i.e., it is an $\vdash$-implication, see above) and treating such rules as strict premises $\bigwedge_{i=1}^{n} \phi_i \supset \phi$. Such a framework is a simple contrapositive ABF if the rules are closed under contraposition. Thus, the traditional definition of ABFs by domain dependent rules can be seen as a special case of ABFs in our setting.

[2] Note that $\mathsf{F}$ is not a standard atomic formula, since $\mathsf{F} \vdash \neg\mathsf{F}$.
[3] That is, $\psi, \neg\psi \vdash \mathsf{F}$. Thus, in explosive logics every formula follows from complementary assumptions.
[4] In particular, $\emptyset \vdash \neg\mathsf{F}$.

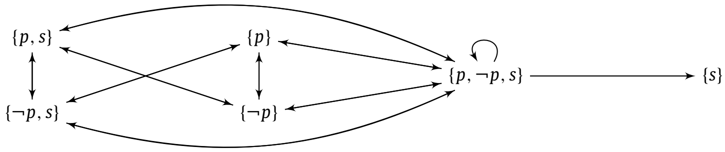


**Fig. 1.** An attack diagram for Example 2.

c) Assumption-based frameworks as defined in this paper can just as well be seen as a special class of ABFs using domain dependent rules. Indeed, given a propositional logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, one may define a domain dependent set of rules $\mathscr{R}_{\mathfrak{L}}$ as $\{\phi_1, \ldots, \phi_n \rightarrow \phi \mid \phi_1, \ldots, \phi_n \vdash \phi\}$. Such a translation views ABFs as defined in our paper as a special case of the traditional definition of assumption-based argumentation, and as such ensures that all the results, tools and concepts from assumption-based argumentation can be applied to our study of ABFs.

Defeasible assertions in an ABF may be attacked in the presence of a counter defeasible information. This is described in the next definition.

**Definition 5.** Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$. We say that $\Delta$ *attacks* $\psi$ iff $\Gamma, \Delta \vdash \phi$ for some $\phi \in \sim\psi$. Accordingly, $\Delta$ attacks $\Theta$ if $\Delta$ attacks some $\psi \in \Theta$.

**Example 2.** Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, $Ab = \{p, \neg p, s\}$, and $\sim\psi = \{\neg\psi\}$ for every formula $\psi$. A corresponding attack diagram is shown in Fig. 1.[5]

Note that since in classical logic inconsistent sets of premises imply *any* conclusion, the classically inconsistent set $\{p, \neg p, s\}$ attacks all the other sets in the diagram (for instance, $\{p, \neg p, s\}$ attacks $\{s\}$, since $p, \neg p, s \vdash \neg s$).

**Note 2.** In contrast to most of the structured accounts of argumentation (such as ASPIC$^+$ [24–26], deductive argumentation [8], DeLP [18] and sequent-based argumentation [4]), in which attacks are defined between individual arguments, in ABA systems attacks are defined between sets of assumptions. This may be viewed a higher level of abstraction, operating on equivalence classes that consist of arguments generated from the same assumptions. (There are some formulations of ABA systems that define attacks on the level of individual arguments, see for instance [17]. However, since attacks are only possible on assumptions, these formulations are equivalent to the standard ones. See also [32].)

Definition 5 gives rise to the following adaptation to ABFs of the usual Dung-style semantics [16] for abstract argumentation frameworks.

**Definition 6.** ([9])[6] Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, and let $\Delta$ be a set of defeasible assumptions. Below, maximum and minimum are taken with respect to set inclusion. We say that:

- $\Delta$ is *closed* (in **ABF**) if $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$.
- $\Delta$ is *conflict-free* (in **ABF**) iff there is no $\Delta' \subseteq \Delta$ that attacks some $\psi \in \Delta$.
- $\Delta$ is *naive* (in **ABF**) iff it is closed and maximally conflict-free (i.e., there is no conflict-free and closed $\Delta' \subseteq Ab$ such that $\Delta \subsetneq \Delta'$).
- $\Delta$ *defends* (in **ABF**) a set $\Delta' \subseteq Ab$ iff for every closed set $\Theta$ that attacks $\Delta'$ there is $\Delta'' \subseteq \Delta$ that attacks $\Theta$.
- $\Delta$ is *admissible* (in **ABF**) iff it is closed, conflict-free, and defends every $\Delta' \subseteq \Delta$.
- $\Delta$ is *complete* (in **ABF**) iff it is admissible and contains every $\Delta' \subseteq Ab$ that it defends.
- $\Delta$ is *well-founded* (in **ABF**) iff $\Delta = \bigcap\{\Theta \subseteq Ab \mid \Theta \text{ is complete}\}$.[7]
- $\Delta$ is *grounded* (in **ABF**) iff it is minimally complete (i.e., no $\Delta' \subsetneq \Delta$ is complete).
- $\Delta$ is *preferred* (in **ABF**) iff it is maximally admissible (i.e., there is no admissible $\Delta' \subseteq Ab$ such that $\Delta \subsetneq \Delta'$).
- $\Delta$ is *stable* (in **ABF**) iff it is closed, conflict-free, and attacks every $\psi \in Ab \setminus \Delta$.

**Note 3.** According to Definition 6, extensions of an ABF are required to be closed. This is a standard requirement for ABFs (see, e.g., [9,31]). In many works on ABA (e.g., [14,15,32]) attention is even restricted to *flat* ABFs, which are frameworks $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ for which for no $\Delta \subseteq Ab$ it holds that $\Gamma, \Delta \vdash \phi$ for some $\phi \in Ab \setminus \Delta$. Notice that in flat ABFs, any set

[5] For reasons that will become apparent in the sequel we include in the diagram only *closed sets* (i.e., only subsets $\Delta \subseteq Ab$ such that $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$ (see Definition 6). Thus, the set $\{p, \neg p\}$ is omitted from the diagram.

[6] To be precise, the naive and grounded semantics were not defined in [9], but as they are very frequently discussed in the context of argumentation-based frameworks, we include here as well their definitions, adjusted to ABA systems.

[7] Clearly, the well-founded extension of an ABF is unique.

of assumptions is closed. We do not impose such a restriction, but require closeness in agreement with the literature on non-flat ABA, although most other frameworks for structured argumentation (such as ASPIC [25,26], sequent-based argumentation [4] or argumentation based on classical logic [7]) do not demand extensions to be closed under strict rules. In Section 5 we show under which conditions this requirement can be conservatively given up.

The set of the complete (respectively, the naive, grounded, well-founded, preferred, stable) extensions of **ABF** is denoted $\mathsf{Com}(\mathbf{ABF})$ (respectively, $\mathsf{Naive}(\mathbf{ABF})$ $\mathsf{Grd}(\mathbf{ABF})$, $\mathsf{WF}(\mathbf{ABF})$, $\mathsf{Prf}(\mathbf{ABF})$, $\mathsf{Stb}(\mathbf{ABF})$). In what follows we shall denote by $\mathsf{Sem}(\mathbf{ABF})$ any of the above-mentioned sets. The entailment relations that are induced from an ABF (with respect to a certain semantics) are defined as follows:

**Definition 7.** Given an assumption-based framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, we denote:

- $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *every* $\Delta \in \mathsf{Sem}(\mathbf{ABF})$.
- $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *some* $\Delta \in \mathsf{Sem}(\mathbf{ABF})$.

**Note 4.** Unlike standard consequence relations (Definition 1), which are relations between sets of formulas and formulas, the entailments in Definition 7 are relations between ABFs and formulas. This will not cause any confusion in what follows.

**Example 3.** Consider again Example 2, where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$ (see also Fig. 1). Here, $\mathsf{Naive}(\mathbf{ABF}) = \mathsf{Prf}(\mathbf{ABF}) = \mathsf{Stb}(\mathbf{ABF}) = \{\{p, s\}, \{\neg p, s\}\}$, and so $\mathbf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} s$ for every $* \in \{\cup, \cap\}$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

In the rest of the paper we shall investigate the semantics and entailment relations induced by the following common family of ABFs according to Definitions 6 and 7.

**Definition 8.** A *simple contrapositive* ABF is an assumption-based framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where $\mathfrak{L}$ is an explosive and contrapositive logic, and $\sim\psi = \{\neg\psi\}$.

Since in what follows we consider simple contrapositive ABFs, we shall often denote the contrariness operator by $\neg$, to emphasize that this operator is represented by negation of $\mathfrak{L}$.

## 3. Characterization results

In this section we investigate the main types of Dung's semantics for simple contrapositive ABFs. First, we consider the preferred and stable semantics (Section 3.1), and then the grounded and the well-founded semantics (Section 3.2).

### 3.1. Naive, preferred and stable semantics

We start by examining the preferred and the stable semantics of ABFs. First, we show that in simple contrapositive ABFs stable and preferred semantics actually coincide with naive semantics.

**Proposition 1.** *Let* $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive ABF. Then* $\Delta \subseteq Ab$ *is naive in* **ABF** *iff it is a stable extension of* **ABF***, iff it is a preferred extension of* **ABF***.*

**Proof.** We first show that every naive $\Delta \subseteq Ab$ is a stable extension of **ABF**. Let $\Delta \subseteq Ab$ be a naive extension of **ABF** and suppose for a contradiction that it is not stable. Since $\Delta$ is naive, it is closed, and since it is not stable, there is some $\psi \in Ab \setminus \Delta$ that is not attacked by $\Delta$, that is: $\Gamma, \Delta \nvdash \neg\psi$. Now, $\psi \notin \Delta$ means that either $\Gamma \cup \Delta \cup \{\psi\}$ is not conflict-free or $\Delta \cup \{\psi\}$ is not closed, and $Cn(\Gamma \cup \Delta \cup \{\psi\}) \cap Ab$ is not conflict-free (since $\Delta$ is maximally conflict-free). In both cases, this means that $\Gamma, \Delta, \psi \vdash \neg\phi$ for some $\phi \in \Delta \cup \{\psi\}$. Suppose first that $\phi = \psi$. Then $\Gamma, \Delta, \psi \vdash \neg\psi$, and since $\mathfrak{L}$ is contrapositive, for every $\sigma \in \Gamma \cup \Delta$, we have $(\Gamma \cup \Delta) \setminus \{\sigma\}, \psi \vdash \neg\sigma$.[8] Again, by contraposition this implies that $\Gamma, \Delta \vdash \neg\psi$, a contradiction to the assumption that $\Gamma, \Delta \nvdash \neg\psi$. Suppose now that $\phi \in \Delta$. Then again since $\mathfrak{L}$ is contrapositive, $\Gamma, \Delta \vdash \neg\psi$, again a contradiction to the assumption that $\Gamma, \Delta \nvdash \neg\psi$.

Now we show that every stable $\Delta \subseteq Ab$ is naive in **ABF**. Indeed, suppose that $\Delta \subseteq Ab$ is stable but not naive in **ABF**. Then $\Delta \subsetneq \Theta$ for some conflict-free set $\Theta \subset Ab$. Let $\phi \in \Theta \setminus \Delta$. Since $\Delta$ is stable, it attacks $\phi$, that is $\Gamma, \Delta' \vdash \neg\phi$ for some $\Delta' \subseteq \Delta$. It follows that $\Delta' \cup \{\phi\}$ is not conflict-free. Since the latter is a subset of $\Theta$, we have that $\Theta$ cannot be conflict-free either.

[8] Note that $\Gamma \cup \Delta$ is not empty, otherwise $\psi \vdash \neg\psi$, contradicting the condition on $\neg$ in Definitions 2 and 4.

Next, we show that every preferred $\Delta \subseteq Ab$ is naive in **ABF**. Indeed, let $\Delta \subseteq Ab$ be a preferred extension. Suppose for a contradiction that some strict superset $\Theta$ of $\Delta$ is closed and conflict-free. By the first case, $\Theta$ is stable and consequently admissible, contradicting that $\Delta$ is preferred.

Finally, to see that every naive extension is preferred, suppose that $\Delta$ is a naive extension. By the first case, $\Delta$ is stable. Since every stable extension is preferred [16], $\Delta$ is preferred. $\square$

Proposition 1 allow us to show that like abstract argumentation (as well as some other forms of structured argumentation, such as ASPIC [25,26], sequent-based argumentation [4], and even flat ABFs), preferred/stable extensions are complete, and maximally complete extensions are preferred/stable. This is what we show in the next proposition. This is a significant result, since for arbitrary non-flat ABFs this is not necessarily true (indeed, in such ABFs preferred extensions always exist while complete extensions do not; see e.g. [12, Examples 2.15 and 2.18]).

**Proposition 2.** *Let* **ABF** $= \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive assumption-based framework. Then:*

a) *Any preferred or stable extension of* **ABF** *is complete.*
b) *Any maximally complete extension of* **ABF** *is preferred and stable.*

**Proof.** By Proposition 1 it is sufficient to show the first item for stable extensions. So suppose for a contradiction that $\Delta$ is stable, yet some $A \in Ab \setminus \Delta$ is defended by $\Delta$. Since $\Delta$ is stable, $\Gamma, \Delta \vdash \neg A$. Since $\Delta$ defends $A$, $\Delta$ attacks itself, a contradiction to $\Delta$ being conflict-free.

For the second item, suppose that $\Delta$ is complete and there is no $\Delta \subsetneq \Delta'$ such that $\Delta'$ is complete. Since $\Delta$ is complete, it is also admissible. Suppose towards a contradiction that $\Delta$ is not preferred, i.e., there is a $\Delta \subsetneq \Delta'$ such that $\Delta'$ is admissible. Without a loss of generality, we may assume that such $\Delta'$ is maximally admissible, and so it is a preferred extension of **ABF**. By the first item of the proposition, $\Delta'$ is complete, which contradicts our assumption. It follows that $\Delta$ is preferred and by Proposition 1 it is also stable. $\square$

Next, we show the relation to reasoning with maximal consistent subsets of the premises.

**Definition 9.** Let **ABF** $= \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$. A set $\Delta \subseteq Ab$ is *maximally consistent* in **ABF**, if

a) $\Gamma, \Delta \nvdash \mathsf{F}$ and
b) $\Gamma, \Delta' \vdash \mathsf{F}$ for every $\Delta \subsetneq \Delta' \subseteq Ab$.

The set of the maximally consistent sets in **ABF** is denoted $\mathsf{MCS}(\mathbf{ABF})$. Accordingly, we denote:

- $\mathbf{ABF} \mid\!\sim^{\Cap}_{\mathsf{MCS}} \psi$ iff $\Gamma, \bigcap \mathsf{MCS}(\mathbf{ABF}) \vdash \psi$.
- $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{MCS}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *every* $\Delta \in \mathsf{MCS}(\mathbf{ABF})$.
- $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{MCS}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *some* $\Delta \in \mathsf{MCS}(\mathbf{ABF})$.

**Note 5.** Clearly, if $\mathbf{ABF} \mid\!\sim^{\Cap}_{\mathsf{MCS}} \psi$ then $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{MCS}} \psi$. To see that the converse does not hold, consider for instance the simple contrapositive ABF where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \{\neg(p \wedge q), p \supset s, q \supset s\}$ and $Ab = \{p, q\}$. Here $\mathsf{MCS}(\mathbf{ABF}) = \{\{p\}, \{q\}\}$ and $\bigcap \mathsf{MCS}(\mathbf{ABF}) = \emptyset$, thus $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{MCS}} s$ but $\mathbf{ABF} \not\mid\!\sim^{\Cap}_{\mathsf{MCS}} s$.

**Example 4.** Consider again Examples 2, where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$ (see also Fig. 1). Here, $\mathsf{MCS}(\mathbf{ABF}) = \{\{p, s\}, \{\neg p, s\}\}$, and so $\mathbf{ABF} \mid\!\sim^{*}_{\mathsf{MCS}} s$ for every $* \in \{\cup, \cap, \Cap\}$. Note the similarity to the result in Example 3. As the next theorem and corollary show, this is not a coincidence.

**Theorem 1.** *Let* **ABF** $= \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive assumption-based framework, and let* $\Delta \subseteq Ab$. *Then* $\Delta$ *is a stable extension of* **ABF**, *iff it is a preferred extension of* **ABF**, *iff it is naive in* **ABF**, *iff it is an element in* $\mathsf{MCS}(\mathbf{ABF})$.

**Proof.** Follows from Proposition 1 and the next lemma. $\square$

**Lemma 2.** *Let* **ABF** $= \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive assumption-based framework. Then* $\Delta \subseteq Ab$ *is naive in* **ABF** *iff* $\Delta \in \mathsf{MCS}(\mathbf{ABF})$.

**Proof.** [$\Rightarrow$]: Suppose that $\Delta \subseteq Ab$ is naive. Then $\Gamma, \Delta \nvdash \mathsf{F}$, otherwise since for every $\psi \in \Delta$ it holds that $\mathsf{F} \vdash \neg\psi$, by transitivity we get $\Gamma, \Delta \vdash \neg\psi$, and so $\Delta$ cannot be conflict-free. To see the maximality condition in Definition 9, suppose for a contradiction that there is some set $\Delta'$ that properly contains $\Delta$ and $\Gamma, \Delta' \nvdash \mathsf{F}$. Since $\Delta$ is naive, either $\Delta'$ is not conflict-free or $\Delta'$ is not closed and $Cn(\Delta' \cup \Gamma) \cap Ab$ is not conflict-free. In both cases, again by transitivity with $\mathsf{F} \vdash \neg\psi$ we

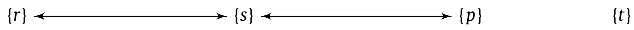


**Fig. 2.** An attack diagram for Example 5.

get $\Gamma, \Delta' \vdash \neg\phi$ for some $\phi \in \Delta'$. Since by reflexivity $\Gamma, \Delta' \vdash \phi$, Lemma 1 implies that $\Gamma \cup \Delta'$ is $\vdash$-inconsistent, a contradiction to $\Gamma, \Delta' \nvdash \mathsf{F}$. Thus $\Delta \in \mathsf{MCS}(\mathbf{ABF})$.

[$\Leftarrow$]: Suppose that $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. Then $\Delta$ is obviously conflict-free. Suppose for a contradiction that there is a set $\Delta'$ that properly contains $\Delta$ and is still conflict-free. Since $\Delta$ is a maximal consistent set in **ABF**, $\Gamma, \Delta' \vdash \mathsf{F}$. But then $\Gamma, \Delta' \vdash \neg\phi$ for any $\phi \in \Delta'$, thus $\Delta'$ cannot be conflict-free. Suppose now $\Delta$ is not closed, i.e., $\Gamma \cup \Delta \vdash \phi$ for some $\phi \in Ab \setminus \Delta$. Since $\Delta \in \mathsf{MCS}(\mathbf{ABF})$, $\Gamma, \Delta, \phi \vdash \mathsf{F}$ and consequently, $\Gamma, \Delta \vdash \neg\phi$. But since $\Gamma, \Delta \vdash \phi$, by Lemma 1 we get again that $\Gamma, \Delta \vdash \mathsf{F}$, contradicting the fact that $\Gamma, \Delta \nvdash \mathsf{F}$. $\square$

**Note 6.** Under the definition of the contrary by negation, the assumption that $\mathfrak{L}$ is explosive is essential for Lemma 2 (and so also for Theorem 1). To see this, consider a logic for which $\phi, \neg\phi \nvdash \mathsf{F}$ (e.g. Batens' CLuNs, Priest's 3-valued LP, or Dunn-Belnap's 4-valued logic). Then for $\mathbf{ABF} = \langle \mathfrak{L}, \emptyset, \{p, \neg p\}, \neg \rangle$ we have that $\mathsf{MCS}(\mathbf{ABF}) = \{\{p, \neg p\}\}$, yet $\{p\}$ attacks $\{\neg p\}$ and vice versa, i.e., the naive extensions are $\{p\}$ and $\{\neg p\}$.

By Theorem 1 we thus have:

**Corollary 1.** *Let* $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive assumption-based framework. Then:*

- $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Naive}} \psi$ *iff* $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Prf}} \psi$ *iff* $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Stb}} \psi$ *iff* $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{MCS}} \psi$.
- $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Naive}} \psi$ *iff* $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Prf}} \psi$ *iff* $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Stb}} \psi$ *iff* $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{MCS}} \psi$.

The collapsing of the preferred and stable semantics to naive semantics in simple contrapositive ABFs is not surprising. Similar results for specific AAFs are reported in [1] and [3].[9] Yet, as shown in [3], when more expressive languages, and/or attack relations, and/or entailment relations are involved, this phenomenon ceases to hold.

### *3.2. The grounded and the well-founded semantics*

We now turn to the more conservative semantics of simple contrapositive ABFs: grounded and well-founded. We start with the general case, in which we show that some of nice properties of these semantics in the context of AAFs are not guaranteed for ABFs. Then we show that by adding a simple condition, namely that $\mathsf{F} \in Ab$, those properties can be assured also for ABFs.

#### *3.2.1. The general case*

*A. The grounded semantics*

The grounded extension in *abstract* argumentation frameworks (AAFs) has many nice properties. For example, it is unique, always exists, and can be built up recursively starting from the set of unattacked arguments. The latter property stems from the following postulate, known as Dung's fundamental lemma (in short, DFL):

**DFL**: If $\Delta$ is admissible[10] and defends $\psi$, then $\Delta \cup \{\psi\}$ is also admissible.

These properties of the grounded semantics carry on to flat ABFs.[11] However, when non-flat ABFs are concerned, and thus also when the ABFs under consideration are contrapositive (not even simple ones), none of these properties is guaranteed anymore.[12] For instance, to see that the DFL fails (and so the usual iterative process for constructing grounded extensions in AAFs may fail for non-flat ABFs), consider the following example:

**Example 5.** Let $\mathfrak{L} = \mathsf{CL}$ (classical logic), $\Gamma = \{p \supset \neg s,\ s \supset \neg r,\ p \wedge r \supset t\}$, and $Ab = \{p, r, s, t\}$. A fragment of the attack diagram (for singletons only) is shown in Fig. 2.

Note that $\{p\}$ is admissible and that $\{p\}$ attacks $\{s\}$, which is the only attacker of $\{r\}$, thus $\{p\}$ defends $\{r\}$. However, $\{p, r\}$ is not closed and therefore it is not admissible (while $\{p, r, t\}$ is admissible).

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[9] See also the survey in [2].

[10] Recall that generally, in structured argumentation frameworks this does not mean that $\Delta$ is closed.

[11] Recall (Note 3) that these are ABFs in which no set of assumptions $\Delta \subseteq Ab$ implies an assumption $\phi \in Ab \setminus \Delta$.

[12] We note that in [9] and [12] the grounded semantics was not defined for non-flat ABA, but only for flat ABFs. Thus, one may regard this section as an answer to the question: "does it make sense to define the grounded extension also for (the non-flat) simple contrapositive ABFs?". As we show in this section, such a definition is not without problems, but these problems can be easily solved, as shown in Section 3.2.2.

The next example shows that $\mathsf{Grd}(\mathbf{ABF}) \neq \bigcap \mathsf{MCS}(\mathbf{ABF})$, thus an analogue of Theorem 1 does not hold for the grounded semantics

**Example 6.** Consider again Examples 2 and 3, where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$ (see also Fig. 1). Note that the grounded set of defeasible assumptions is the emptyset, since there are no unattacked arguments. However, here $\bigcap \mathsf{MCS}(\mathbf{ABF}) = \{s\}$. The intuitive reason for this behavior is that the inconsistent set $\{p, \neg p, s\}$ contaminates the argumentation framework, thus keeping $s$ out of the grounded set of defeasible assumptions.

The last example also demonstrates the problems of the grounded semantics in handling inconsistencies in ABFs. Indeed, in the presence of an inconsistency the whole argumentation framework may be contaminated, blocking any informative output, such as the innocent bystander $s$ in Examples 2 and 6.

Finally, we show that (unlike abstract argumentation) uniqueness is not guaranteed for grounded semantics.

**Example 7.** Let $\mathfrak{L}$ be an explosive logic, $Ab = \{p, \neg p, q\}$ and $\Gamma = \{s, s \supset q\}$. Note that the emptyset is not admissible, since it is not closed (indeed, $\Gamma \vdash q$). Also, $\{q\}$ is not admissible since $p, \neg p \vdash \neg q$. The two minimal complete extensions in this case are $\{p, q\}$ and $\{\neg p, q\}$, thus there is no *unique* grounded extension in this case.

*B. The well-founded semantics*

We now consider the well-founded semantics for ABFs. The existence of a well-founded extension for any simple contrapositive ABF follows from the following claim:

**Proposition 3.** *Any simple contrapositive ABF has a complete extension.*

**Proof.** It is clear from Definition 9 that $\mathsf{MCS}(\mathbf{ABF}) \neq \emptyset$ for every **ABF**. By Theorem 1, then, $\mathsf{Stb}(\mathbf{ABF}) \neq \emptyset$. Since every stable extension is complete (Proposition 2(a)), this concludes the proof. □

**Note 7.** As indicated in [12], non-flat ABFs may not have a complete extension (and so well-founded and grounded extensions may not be available for them; See e.g. [12, Example 2.15]). As Proposition 3 shows, this is not the case in simple contrapositive ABFs.

As already noted in Definition 6, the well-founded extension is unique and thus problems like those described in Example 7 for the grounded extension are avoided for the well-founded semantics. However, the next example shows that, as in the case of the grounded semantics, the well-founded extension of **ABF** does not always coincide with $\bigcap \mathsf{MCS}(\mathbf{ABF})$ (cf. Example 6).[13]

**Example 8.** Consider again Examples 2 and 3. We have that $\mathsf{Com}(\mathbf{ABF}) = \{\emptyset, \{p, s\}, \{\neg p, s\}\}$, therefore $\mathsf{WF}(\mathbf{ABF}) = \emptyset$. However, $\bigcap \mathsf{MCS}(\mathbf{ABF}) = \{s\}$.

*3.2.2. A more plausible case*

In the previous section we have shown that in the context of contrapositive ABFs (as well as in the context of other non-flat ABFs), the well-founded semantics and the grounded semantics do not have some of the expected properties shared by the same semantics in the context of abstract argumentation frameworks or flat assumption-based frameworks. In this section we show that these properties *can* be guaranteed also for simple contrapositive ABFs by requiring that $\mathsf{F} \in Ab$.[14] To get some intuition why this is the case, let's first observe how the addition of $\mathsf{F}$ to $Ab$ would change Examples 6 and 8.

**Example 9.** Consider the same ABF as of Example 2, except that now $\mathsf{F}$ is added to $Ab$. Note that $\{p, \neg p\} \vdash \mathsf{F}$ and consequently $\{p, \neg p\}$ is not closed, whereas $\{p, \neg p, s, \mathsf{F}\}$ is. Furthermore, $\emptyset \vdash \neg \mathsf{F}$ and consequently we have the attack diagram, shown in Fig. 3. Now the grounded as well as the well-founded set of defeasible assumptions is $\{s\}$ (cf. Example 6 and 8).

In the rest of this section we therefore assume that $\mathsf{F} \in Ab$, and check the grounded semantics and the well-founded semantics in this case.

[13] Although this is not necessarily a shortcoming of the well-founded semantics, it deviates from the usual behavior of similar semantics of structured argumentation frameworks that are based on contrapositive logics (see [2]) and related semantics for e.g. logic programming (see [33] where the well-founded semantics does allow to derive *innocent bystanders*, i.e. formulas not involved in any conflict). Furthermore, this behavior also makes for a violation of the rationality postulate of *non-interference* (see Section 4).

[14] Intuitively, this means that any inconsistent set of arguments will be attacked by the emptyset and thus it is excluded from any admissible extension.

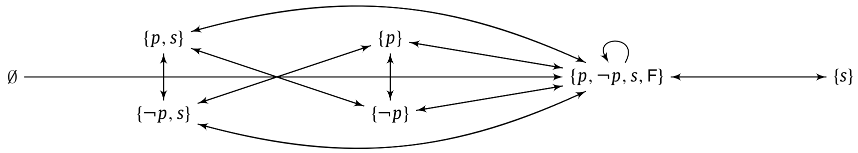


**Fig. 3.** An attack diagram for Example 9.

*A. The grounded semantics, revisited*

First we show that, despite of the failure of the DFL in simple contrapositive ABFs, when $\mathsf{F} \in Ab$ we can still get the grounded extension by the well-known iterative construction [9, Definition 5.2].

**Definition 10.** Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim\rangle$ be an assumption-based framework.

- $\mathscr{G}_0(\mathbf{ABF})$ consists of all $\phi \in Ab$ such that no $\Delta \subseteq Ab$ attacks $\phi$.
- $\mathscr{G}_{i+1}(\mathbf{ABF})$ consists of the union of $\mathscr{G}_i(\mathbf{ABF})$ and all the assumptions that are defended by $\mathscr{G}_i(\mathbf{ABF})$.
- $\mathscr{G}(\mathbf{ABF}) = \bigcup_{i\geqslant 0} \mathscr{G}_i(\mathbf{ABF})$.

When **ABF** is clear from the context we will sometimes just write $\mathscr{G}_0$, $\mathscr{G}_i$ and $\mathscr{G}$.

**Theorem 2.** *Let* **ABF** *be a simple contrapositive ABF and suppose that* $\mathsf{F} \in Ab$*. Then* $\mathsf{Grd}(\mathbf{ABF}) = \{\mathscr{G}\}$.

For the proof of Theorem 2 we first need to show a few lemmas. In all of these lemmas we assume that $\mathbf{ABF} = \langle \mathscr{L}, \vdash, \Gamma, Ab, \neg\rangle$ is a simple contrapositive ABF, and that $\mathsf{F} \in Ab$.

**Lemma 3.** *The emptyset defends every formula in Ab that follows from* $\Gamma$.

**Proof.** Suppose that $\Gamma \vdash \phi$ and some $\Theta = Cn(\Theta \cup \Gamma) \cap Ab$ attacks $\phi$, i.e., $\Gamma, \Theta \vdash \neg\phi$. By the monotonicity of $\vdash$ we thus have that $\Gamma, \Theta \vdash \phi$ and $\Gamma, \Theta \vdash \neg\phi$, so by Lemma 1, $\Gamma, \Theta \vdash \mathsf{F}$. Since $\Theta$ is closed, $\mathsf{F} \in \Theta$. But, then since $\emptyset \vdash \neg\mathsf{F}$, $\emptyset$ attacks $\Theta$. $\square$

**Lemma 4.** $\mathscr{G}_1(\mathbf{ABF})$ *is conflict-free.*

**Proof.** If $\mathscr{G}_1 = \emptyset$ then it is conflict-free by definition. Suppose for a contradiction that $\Gamma, \mathscr{G}_1 \vdash \neg\phi$ for some $\phi \in \mathscr{G}_1$. Then $\mathscr{G}_1$ attacks itself, and so $\mathscr{G}_0$ attacks some $\delta \in Cn(\mathscr{G}_1 \cup \Gamma) \cap Ab$, i.e. $\Gamma, \mathscr{G}_0 \vdash \neg\delta$. Since $\mathfrak{L}$ is contrapositive, $\Gamma, (\mathscr{G}_0 \setminus \{\psi\}), \delta \vdash \neg\psi$ for any $\psi \in \mathscr{G}_0$. But this contradicts the fact that $\mathscr{G}_0$ contains only unattacked defeasible assumptions. $\square$

**Lemma 5.** $\mathscr{G}_2(\mathbf{ABF}) = \mathscr{G}_1(\mathbf{ABF})$.

**Proof.** By Definition 10, $\mathscr{G}_1(\mathbf{ABF}) \subseteq \mathscr{G}_2(\mathbf{ABF})$. To see that $\mathscr{G}_2(\mathbf{ABF}) \subseteq \mathscr{G}_1(\mathbf{ABF})$ we have to show that every assumption that is defended by $\mathscr{G}_1$ is also defended by $\mathscr{G}_0$. For this, it is enough to show that if $\mathscr{G}_1$ attacks a closed set $\Theta$ (i.e., $\Theta = Cn(\Gamma \cup \Theta) \cap Ab$), $\mathscr{G}_0$ also attacks $\Theta$.

Suppose for a contradiction that $\Theta = Cn(\Gamma \cup \Theta) \cap Ab$ is attacked by $\mathscr{G}_1$ but not by $\mathscr{G}_0$. This means that $\Gamma, \mathscr{G}_1 \vdash \neg\phi$ for some $\phi \in \Theta$.

Suppose first that $\Gamma, (\mathscr{G}_1 \cup \mathscr{G}_0) \vdash \neg\phi$. By contraposition, there is some $\psi \in \mathscr{G}_0$ such that $\Gamma, (\mathscr{G}_0 \cup \mathscr{G}_1) \setminus \psi, \phi \vdash \neg\psi$, contradicting the fact that $\psi$ is not attacked.

Thus, $\Gamma, (\mathscr{G}_1 \setminus \mathscr{G}_0) \vdash \neg\phi$. Now, by contraposition, $\Gamma, (\mathscr{G}_1 \setminus (\mathscr{G}_0 \cup \{\psi\})), \{\phi\} \vdash \neg\psi$ for every $\psi \in \mathscr{G}_1$. Let $\psi_1 \in (\mathscr{G}_1 \setminus (\mathscr{G}_0 \cup \{\psi\})) \cup \{\phi\}$ be such a formula. Since $\psi_1 \in \mathscr{G}_1 \setminus \mathscr{G}_0$, $\psi_1$ is defended by $\mathscr{G}_0$, thus there is a $\delta \in Cn(\Gamma \cup ((\mathscr{G}_1 \setminus (\mathscr{G}_0 \cup \{\psi_1\})) \cup \{\phi\}) \cap Ab$ s.t. $\Gamma, \mathscr{G}_0 \vdash \neg\delta$. We consider three possibilities:
- if $\delta \in \mathscr{G}_1$, then $\mathscr{G}_1$ is not conflict-free, contradicting Lemma 4.
- if $\delta = \phi$, then $\mathscr{G}_0$ attacks $\Theta$, which contradicts the choice of $\Theta$.
- Suppose finally that $\delta \in Cn(\Gamma \cup (\mathscr{G}_1 \setminus (\mathscr{G}_0 \cup \{\psi_1\})) \cup \{\phi\}) \cap Ab \setminus (\mathscr{G}_1 \setminus (\mathscr{G}_0 \cup \{\psi_1\}) \cup \{\phi\})$. By contraposition, for every $\theta \in \mathscr{G}_0$, $\Gamma, (\mathscr{G}_0 \setminus \{\theta\}), \delta \vdash \neg\theta$. Since $\Gamma, (\mathscr{G}_1 \setminus (\mathscr{G}_0 \cup \{\psi_1\})), \phi) \vdash \delta$, we have that $\Gamma, \mathscr{G}_1 \vdash \neg\theta$, contradicting that $\theta$ is not attacked. $\square$

**Corollary 2.** $\mathscr{G}(\mathbf{ABF}) = \mathscr{G}_0(\mathbf{ABF}) \cup \mathscr{G}_1(\mathbf{ABF}) = \mathscr{G}_1(\mathbf{ABF})$.

**Proof.** Follows immediately from Lemma 5. $\square$

**Lemma 6.** $\mathscr{G}$ *is closed.*

**Proof.** Suppose for a contradiction that $\Gamma, \mathscr{G} \vdash \phi$ for some $\phi \in Ab$ and $\phi \notin \mathscr{G}$. In particular, $\phi \notin \mathscr{G}_0$, thus there is some $\Theta \subseteq Ab$ such that $\Gamma, \Theta \vdash \neg\phi$ and $\Gamma, \mathscr{G} \nvdash \neg\theta$ for any $\theta \in \Theta$ (otherwise $\mathscr{G}$ attacks $\Theta$, thus defends $\phi$, contradicting $\phi \notin \mathscr{G}$). Thus, $\Gamma, \mathscr{G}_1 \nvdash \neg\theta$ for any $\theta \in \Theta$ (note that by Corollary 2, $\mathscr{G} = \mathscr{G}_1$). Suppose first that $\Gamma \vdash \phi$. Then, by Lemma 3, $\emptyset$ defends $\phi$, and so also $\mathscr{G}_0$ defends $\phi$, contradicting the choice of $\phi$. Suppose now that $\Gamma \nvdash \phi$. Note that for every $\delta \in \mathscr{G}_1$, $\Gamma, \Theta, \mathscr{G}_1 \setminus \{\delta\} \vdash \neg\delta$. Thus, since $\mathscr{G}_0$ defends $\delta \in \mathscr{G}_1$, $\Gamma, \mathscr{G}_0 \vdash \neg\lambda$ for some $\lambda \in Cn(\Gamma \cup \Theta \cup \mathscr{G}_1 \setminus \{\delta\}) \cap Ab$. We consider the following possibilities:

- $\lambda \in \Theta$. This contradicts the assumption that $\Gamma, \mathscr{G}_1 \nvdash \neg\theta$ for any $\theta \in \Theta$.
- $\lambda \in \mathscr{G}_1$. This means that $\mathscr{G}_0$ attacks $\mathscr{G}_1$, contradicting $\mathscr{G}_1$ being conflict free (Lemma 4).
- $\lambda \notin \Theta \cup \mathscr{G}_1$, implying that $\Gamma, \Theta, \mathscr{G}_1 \setminus \{\delta\} \vdash \lambda$. Then since $\mathfrak{L}$ is contrapositive, $\Gamma, \Theta, \Delta, \mathscr{G}_0 \setminus \{\xi, \delta\} \vdash \neg\xi$ for any $\xi \in \mathscr{G}_0$, contradicting $\mathscr{G}_0$ containing only unattacked defeasible assumptions. $\square$

Now we can show Theorem 2.

**Proof.** It is clear from the construction of $\mathscr{G}$ that it is unique and that $\phi \in \mathscr{G}$ iff $\phi$ is defended by $\mathscr{G}$. Moreover, by Lemma 4 and Corollary 2, $\mathscr{G}$ is conflict-free. By Lemma 6, $\mathscr{G}$ is also closed. Thus $\mathscr{G}$ it is complete. It remains to show that $\mathscr{G}$ is minimal among the complete sets of **ABF**. If $\mathscr{G}$ is empty we are done. Otherwise, suppose for a contradiction that there is some complete $\Delta \subsetneq \mathscr{G}$, and let $\phi \in \mathscr{G} \setminus \Delta$. If $\phi \in \mathscr{G}_0$, then $\phi$ has no attackers and consequently $\phi$ is (vacuously) defended by $\Delta$, in which case $\Delta$ cannot be complete. Thus $\phi \notin \mathscr{G}_0$ and $\mathscr{G}_0 \subseteq \Delta$. Suppose now that $\phi \in \mathscr{G}_1$. Then $\Delta$ defends $\phi$ since $\mathscr{G}_0 \subseteq \Delta$. Again, this contradicts the completeness of $\Delta$. Thus, $\mathscr{G}_1 \subseteq \Delta$. By Corollary 2, $\mathscr{G} = \mathscr{G}_1$ and consequently, $\mathscr{G} \subseteq \Delta$, contradicting the assumption that $\Delta \subsetneq \mathscr{G}$. $\square$

The following is the counterpart, for the grounded semantics, of Theorem 1.

**Theorem 3.** *Let* **ABF** $= \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive assumption-based framework in which* $\mathsf{F} \in Ab$. *Then* $\mathsf{Grd}(\mathbf{ABF}) = \{\bigcap \mathsf{MCS}(\mathbf{ABF})\}$.

**Proof.** By Theorem 2 it suffices to show that $\mathscr{G}(\mathbf{ABF}) = \bigcap \mathsf{MCS}(\mathbf{ABF})$.

To see that $\mathscr{G}(\mathbf{ABF}) \subseteq \bigcap \mathsf{MCS}(\mathbf{ABF})$, let $\Theta \in \mathsf{MCS}(\mathbf{ABF})$. By Theorem 1, $\Theta$ is stable, and so $\mathscr{G}(\mathbf{ABF}) \subseteq \Theta$. (Indeed, suppose otherwise. Then there is $\psi \in \mathscr{G} \setminus \Theta$, and since $\Theta$ is stable, it attacks $\psi$. Since $\psi \in \mathscr{G}$, by Lemma 5, $\mathscr{G}_0$ attacks $\Theta$ (note that $\mathscr{G} = \mathscr{G}_1$ by Corollary 2). Since obviously $\mathscr{G}_0 \subseteq \Theta$, this contradicts the fact that $\Theta$ is conflict-free.)

To see that $\bigcap \mathsf{MCS}(\mathbf{ABF}) \subseteq \mathscr{G}(\mathbf{ABF})$, suppose for a contradiction that there is $\phi \in \bigcap \mathsf{MCS}(\mathbf{ABF})$ yet $\phi \notin \mathscr{G}(\mathbf{ABF})$. By Lemma 5, this means that some $\Theta = Cn(\Gamma \cup \Theta) \cap Ab$ attacks $\phi$ but $\mathscr{G}_0(\mathbf{ABF})$ does not attack $\Theta$. Since $\phi \in \bigcap \mathsf{MCS}(\mathbf{ABF})$, $\Theta \notin \mathsf{MCS}(\mathbf{ABF})$. Suppose first that $\Theta \cup \Gamma \vdash \mathsf{F}$. Then $\mathsf{F} \in \Theta$ and consequently $\mathscr{G}_0(\mathbf{ABF})$ attacks $\Theta$, which is a contradiction. Suppose then that $\Theta \subsetneq \Theta'$ for some $\Theta' \in \mathsf{MCS}(\mathbf{ABF})$. In this case, by monotonicity $\Theta' \cup \Gamma \vdash \neg\phi$, thus $\phi \notin \Theta'$ (otherwise, $\Theta' \cup \Gamma \vdash \mathsf{F}$ and so $\Theta'$ cannot be in $\mathsf{MCS}(\mathbf{ABF})$). This contradicts the assumption that $\phi \in \bigcap \mathsf{MCS}(\mathbf{ABF})$. $\square$

From Theorems 2 and 3 the following result is obtained (cf. Corollary 1):

**Corollary 3.** *Let* **ABF** $= \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive assumption-based framework in which* $\mathsf{F} \in Ab$. *Then:* $\mathbf{ABF} \mathrel{|\!\sim}^{\cap}_{\mathsf{Grd}} \psi$ *iff* $\mathbf{ABF} \mathrel{|\!\sim}^{\cup}_{\mathsf{Grd}} \psi$ *iff* $\mathbf{ABF} \mathrel{|\!\sim}^{\Cap}_{\mathsf{MCS}} \psi$.

**Proof.** The equality of $\mathrel{|\!\sim}^{\cap}_{\mathsf{Grd}}$ and $\mathrel{|\!\sim}^{\cup}_{\mathsf{Grd}}$ follows from Theorem 2, since it shows that under the conditions of the corollary $\mathsf{Grd}(\mathbf{ABF})$ is a singleton. The equality of $\mathrel{|\!\sim}^{\cap}_{\mathsf{Grd}}$ and $\mathrel{|\!\sim}^{\Cap}_{\mathsf{MCS}}$ follows from Theorem 3. $\square$

*B. The well-founded semantics, revisited*

In what follows we show that as in the case of the grounded extension (see Theorems 2 and 3), the correspondence of the well-founded extension to the (intersection of the) maximally consistent sets of assumptions can be guaranteed by requiring that $\mathsf{F} \in Ab$ (cf. Example 8).

**Proposition 4.** *Let* **ABF** $= \langle \mathscr{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive ABF. If* $\mathsf{F} \in Ab$ *then* $\mathsf{WF}(\mathbf{ABF}) = \{\bigcap \mathsf{MCS}(\mathbf{ABF})\}$.

**Proof.** By Theorem 2 and since $\mathsf{F} \in Ab$, there exists a unique grounded extension for any ABF. From this it follows that $\bigcup \mathsf{Grd}(\mathbf{ABF}) \subseteq \Delta$ for any $\Delta \in \mathsf{Com}(\mathbf{ABF})$. This implies that $\bigcap \mathsf{Com}(\mathbf{ABF}) = \bigcup \mathsf{Grd}(\mathbf{ABF})$, that is: $\mathsf{WF}(\mathbf{ABF}) = \mathsf{Grd}(\mathbf{ABF})$. $\square$

By Theorem 3 and Proposition 4 we thus have:

**Corollary 4.** *Let* **ABF** $= \langle \mathscr{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive ABF. If* $\mathsf{F} \in Ab$ *then* $\mathsf{WF}(\mathbf{ABF}) = \mathsf{Grd}(\mathbf{ABF})$.

**Note 8.** In relation to Corollary 4, we note the following:

a) The fact that (unlike flat ABFs) the grounded semantics and the well-founded semantics may not be the same in non-flat ABFs is already known (see, e.g., [12, Example 2.16]). However, to the best of our knowledge, Corollary 4 is the first one that introduces a condition under which these two semantics coincide in non-flat ABFs.
b) Examples 7 and 8 show that the condition that $\mathsf{F} \in Ab$ is indeed necessary for the last corollary.

## 4. Properties of the induced entailments $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ and $\mid\!\sim^{\cup}_{\mathsf{Sem}}$

In this section we consider some further properties of the entailment relations introduced in Definition 7, and which are induced from simple contrapositive ABFs. Below, when $\mathbf{ABF} \mid\!\sim \psi$ for some $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ (where $\mid\!\sim$ is a relation defined in Definition 7), we shall just write $\Gamma, Ab \mid\!\sim \psi$.[15]

### 4.1. Relations to the base logic

First, we note the following relations between $\mid\!\sim$ and the consequence relation $\vdash$ of the base logic:

**Proposition 5.** *If $\Gamma \cup Ab$ is $\vdash$-consistent then for every relation $\mid\!\sim$ in Definition 7, $\Gamma, Ab \mid\!\sim \psi$ iff $\Gamma, Ab \vdash \psi$.*[16]

**Proof.** When $\Gamma \cup Ab$ is $\vdash$-consistent, $\mathsf{WF}(\mathbf{ABF}) = \mathsf{Grd}(\mathbf{ABF}) = \mathsf{Prf}(\mathbf{ABF}) = \mathsf{Stb}(\mathbf{ABF}) = \{Ab\}$, so the claim immediately follows from Definition 7. $\square$

**Proposition 6.** *For every relation $\mid\!\sim$ in Definition 7 it holds that:*

- *If $\Gamma, Ab \mid\!\sim \psi$ then $\Gamma, Ab \vdash \psi$.*
- *If $\vdash \psi$ then $\Gamma, Ab \mid\!\sim \psi$ for every $\Gamma$ and $Ab$.*

**Proof.** For the first item, note that if $\Gamma, Ab \mid\!\sim \psi$ then there is at least one subset $\Delta \subseteq Ab$ for which $\Gamma, \Delta \vdash \psi$. By the monotonicity of $\vdash$, then, $\Gamma, Ab \vdash \psi$. For the second item, note that if $\vdash \psi$, then for *every* $\Delta \subseteq Ab$ it holds that $\Gamma, \Delta \vdash \psi$, thus $\Gamma, Ab \mid\!\sim \psi$. $\square$

### 4.2. Cumulativity, preferentiality and rationality

Next, we consider the entailments in Definition 7 in the context of the reasoning patterns introduced by Kraus, Lehmann and Magidor in [22] and [23].

**Definition 11.** A relation $\mid\!\sim$ between ABFs and formulas in their languages is called *cumulative*, if the following conditions are satisfied:

- *Cautious Reflexivity* (CR): For every $\vdash$-consistent formula $\psi$ it holds that $\psi \mid\!\sim \psi$.
- *Cautious Monotonicity* (CM): If $\Gamma, Ab \mid\!\sim \phi$ and $\Gamma, Ab \mid\!\sim \psi$ then $\Gamma, Ab, \phi \mid\!\sim \psi$.
- *Cautious Cut* (CC): If $\Gamma, Ab \mid\!\sim \phi$ and $\Gamma, Ab, \phi \mid\!\sim \psi$ then $\Gamma, Ab \mid\!\sim \psi$.
- *Left Logical Equivalence* (LLE): If $\phi \vdash \psi$ and $\psi \vdash \phi$ then $\Gamma, Ab, \phi \mid\!\sim \rho$ iff $\Gamma, Ab, \psi \mid\!\sim \rho$.
- *Right Weakening* (RW): If $\phi \vdash \psi$ and $\Gamma, Ab \mid\!\sim \phi$ then $\Gamma, Ab \mid\!\sim \psi$.

A cumulative relation is called *preferential*, if it satisfies the following condition:

- *Distribution* (OR): If $\Gamma, Ab, \phi \mid\!\sim \rho$ and $\Gamma, Ab, \psi \mid\!\sim \rho$ then $\Gamma, Ab, \phi \vee \psi \mid\!\sim \rho$.

A cumulative entailment is called *rational*, if it satisfies the following condition[17]:

- *Rational Monotonicity* (RM): If $\Gamma, Ab \mid\!\sim \rho$ and $\Gamma, Ab \not\mid\!\sim \neg\psi$ then $\Gamma, Ab, \psi \mid\!\sim \rho$.

**Theorem 4.** *Let $\mathbf{ABF} = \langle \mathcal{L}, \Gamma, Ab, \neg \rangle$ be a simple contrapositive ABF. Then $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ is preferential for* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

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[15] Note that this writing is somewhat ambiguous, since, e.g. when $\Gamma, Ab, \phi$ are the premises, $\phi$ may be either a strict or a defeasible assumption. Yet, we chose this notation because usually it won't make a difference whether $\phi$ is a strict or a defeasible assumption, so this notation covers both cases. When it does make a difference, we shall indicate this explicitly.

[16] Note, in particular, that skeptical and credulous reasoning coincide in this case.

[17] Notice that we do not require rational entailment to be preferential, but merely cumulative.

**Proof.** We show preferentiality for $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ where $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$, based on Corollary 1. In the proofs below, when $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$, we shall sometimes write $\mathsf{MCS}(\Gamma, Ab)$ instead of $\mathsf{MCS}(\mathbf{ABF})$.

CR: This property holds by Proposition 5 and the reflexivity of $\vdash$ (thus $\psi \vdash \psi$).[18]

CM: Since $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \psi$, by Corollary 1 we have that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\Gamma, Ab)$, and so, by monotonicity, $(*)$ $\Gamma, \Delta, \phi \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\Gamma, Ab)$. Also, since $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \phi$, we have that $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}(\Gamma, Ab)$, thus $(**)$ $\mathsf{MCS}(\Gamma, Ab, \phi) = \{\Delta \cup \{\phi\} \mid \Delta \in \mathsf{MCS}(\Gamma, Ab)\}$. By $(*)$ and $(**)$, then, $\Gamma, \Delta' \vdash \psi$ for every $\Delta' \in \mathsf{MCS}(\Gamma, Ab, \phi)$, and by Corollary 1 again $\Gamma, Ab, \phi \mid\!\sim^{\cap}_{\mathsf{sem}} \psi$.

CC: Suppose that $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{Sem}} \phi$. By Corollary 1 we have that, $(*)$ $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. Let $\mathbf{ABF}' = \langle \mathscr{L}, \Gamma, Ab \cup \{\phi\}, \neg \rangle$ or $\mathbf{ABF}' = \langle \mathscr{L}, \Gamma \cup \{\phi\}, Ab, \neg \rangle$. Then, in the first case, $\mathsf{MCS}(\mathbf{ABF}') = \{\Delta \cup \{\phi\} \mid \Delta \in \mathsf{MCS}(\mathbf{ABF})\}$, and since $\Gamma, Ab, \phi \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$, by Corollary 1 we have in both cases that $(**)$ $\Gamma, \Delta, \phi \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. Thus, by Transitivity on $(*)$ and $(**)$ we have that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$, and by Corollary 1, $\Gamma, Ab \mid\!\sim \psi$.

LLE: Suppose that $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \psi$. By Corollary 1 we have that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. Thus, by transitivity with $\psi \vdash \phi$, it holds that $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. By Corollary 1 again, $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \phi$. The converse is dual.

RW: Suppose that $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \phi$. By Corollary 1 we have that $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$. By transitivity with $\phi \vdash \psi$ we get that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}(\mathbf{ABF})$, and by Corollary 1 again, $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \psi$.

OR: We first show the case where the primary formulas $\psi$, $\phi$, and $\psi, \phi \vee \psi$ are defeasible. For $\lambda \in \{\phi, \psi, \phi \vee \psi\}$ we denote $\mathbf{ABF}^{\lambda} = \langle \mathscr{L}, \Gamma, Ab \cup \{\lambda\}, \neg \rangle$. Suppose for a contradiction that $\mathbf{ABF}^{\phi} \mid\!\sim \rho$ and $\mathbf{ABF}^{\psi} \mid\!\sim \rho$, however $\mathbf{ABF}^{\phi \vee \psi} \not\mid\!\sim \rho$. By Corollary 1, there is some $\Delta \in \mathsf{MCS}(\Gamma, Ab, \psi \vee \phi)$ such that $\Gamma, \Delta \nvdash \rho$. We consider two cases:

- If $\psi \vee \phi \notin \Delta$, then $\Delta \in \mathsf{MCS}(\Gamma, Ab)$ and so in particular $\Delta$ is a consistent subset of $\Gamma \cup Ab \cup \{\psi\}$. We show that it is a maximally consistent set of the latter. Indeed, for every $\delta \in Ab \setminus \Delta$ the set $\Gamma, \Delta \cup \{\delta\}$ is not consistent, otherwise $\Delta \notin \mathsf{MCS}(\Gamma, Ab)$. Moreover, $\Delta \cup \{\psi\}$ is not consistent either, otherwise $\Gamma, \Delta, \psi \nvdash \mathsf{F}$, and since $\vee$ is a disjunction, by Definition 2 we have that $\Gamma, \Delta, \phi \vee \psi \nvdash \mathsf{F}$, thus $\Delta \cup \{\phi \vee \psi\}$ is a consistent subset of $\Gamma \cup Ab \cup \{\phi \vee \psi\}$ that properly includes $\Delta$, contradicting that $\Delta \in \mathsf{MCS}(\Gamma, Ab, \phi \vee \psi)$. It follows, then, that $\Delta \in \mathsf{MCS}(\Gamma, Ab, \psi)$. By Corollary 1, this contradicts the assumption that $\Gamma, Ab, \psi \mid\!\sim^{\cap}_{\mathsf{sem}} \rho$.
- If $\Delta = \Delta' \cup \{\psi \vee \phi\}$ where $\Delta'$ is some consistent subset of $\Gamma \cup Ab$, then $\Delta', \psi \vee \phi \nvdash \rho$ and since $\vee$ is a disjunction, by Definition 2 we have that either $\Delta', \psi \nvdash \rho$ or $\Delta', \phi \nvdash \rho$. Thus, either $\Delta' \cup \{\psi\}$ is not maximally consistent in $\Gamma \cup Ab \cup \{\psi\}$ or $\Delta' \cup \{\phi\}$ is not maximally consistent in $\Gamma \cup Ab \cup \{\phi\}$ (otherwise, by Corollary 1 we get a contradiction to one of the assumptions). Without loss of generality, suppose the former. Then $\Delta' \in \mathsf{MCS}(\Gamma, Ab, \psi)$, and so, by Corollary 1, $\Gamma, Ab, \psi \not\mid\!\sim^{\cap}_{\mathsf{sem}} \rho$, a contradiction again.

We now show the case where the primary formulas are strict. For $\lambda \in \{\phi, \psi, \phi \vee \psi\}$ we denote $\mathbf{ABF}^{\lambda} = \langle \mathscr{L}, \Gamma \cup \{\lambda\}, Ab, \neg \rangle$. Suppose towards a contradiction that $\mathbf{ABF}^{\phi} \mid\!\sim \rho$ and $\mathbf{ABF}^{\psi} \mid\!\sim \rho$, but $\mathbf{ABF}^{\phi \vee \psi} \not\mid\!\sim \rho$. By Corollary 1, there is some $\Delta \in \mathsf{MCS}(\Gamma \cup \{\psi \vee \phi\}, Ab)$ such that $\Gamma \cup \{\psi \vee \phi\}, \Delta \nvdash \rho$. We show that this implies that $\Delta \in \mathsf{MCS}(\Gamma \cup \{\phi\}, Ab)$ or $\Delta \in \mathsf{MCS}(\Gamma \cup \{\psi\}, Ab)$, contradicting the assumption that $\mathbf{ABF}^{\phi} \mid\!\sim \rho$ and $\mathbf{ABF}^{\psi} \mid\!\sim \rho$. Indeed, by Definition 2, $\Gamma \cup \{\psi \vee \phi\}, \Delta \nvdash \mathsf{F}$ implies that $\Gamma \cup \{\psi\}, \Delta \nvdash \mathsf{F}$ or $\Gamma \cup \{\phi\}, \Delta \nvdash \mathsf{F}$. Without loss of generality, suppose the former. If $\Delta \notin \mathsf{MCS}(\Gamma \cup \{\psi\}, Ab)$, there is a $\Delta' \in \mathsf{MCS}(\Gamma \cup \{\psi\}, Ab)$ such that $\Delta' \supsetneq \Delta$. For this $\Delta'$, still $\Gamma \cup \{\psi\}, \Delta' \nvdash \mathsf{F}$. By Definition 2 again, this means that $\Gamma \cup \{\psi \vee \phi\}, \Delta' \nvdash \mathsf{F}$, contradicting the assumption that $\Delta \in \mathsf{MCS}(\Gamma \cup \{\psi \vee \phi\}, Ab)$. Thus, $\Delta \in \mathsf{MCS}(\Gamma \cup \{\psi\}, Ab)$. □

Unlike $\mid\!\sim^{\cap}_{\mathsf{Sem}}$, the credulous entailments $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ are *not* preferential, since they do not satisfy the postulate OR. This is shown in the next example.

**Example 10.** Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{r \wedge (q \supset p), \neg r \wedge (\neg q \supset p)\}$. Then $Ab, q \mid\!\sim p$ and $Ab, \neg q \mid\!\sim p$ but $Ab, q \vee \neg q \not\mid\!\sim p$ for every entailment $\mid\!\sim$ of the form $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ where $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

As the next proposition shows, the entailments $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$ are still cumulative.

**Theorem 5.** *Let $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \neg \rangle$ be a simple contrapositive ABF. Then $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ is cumulative for every $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.*

**Proof.** Similar to that of Theorem 4. □

We now turn to the grounded and well-founded semantics.

**Theorem 6.** *Let $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \neg \rangle$ be a simple contrapositive ABF where $\mathsf{F} \in Ab$. Then $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ is preferential for $\mathsf{Sem} \in \{\mathsf{WF}, \mathsf{Grd}\}$.*

**Proof.** It is rather a simple exercise to modify the proof of Theorem 4 to a proof of Theorem 6 for $\mathsf{Sem} = \mathsf{Grd}$, using Corollary 3 instead of Corollary 1. Then, by Corollary 4 we get also the case where $\mathsf{Sem} = \mathsf{WF}$. We therefore show only CM for $\mid\!\sim^{\cap}_{\mathsf{Grd}}$, leaving the other cases to the reader.

[18] If $\psi$ is a strict assumption, this property can be strengthened as follows: $\Gamma \mid\!\sim \psi$ for every $\psi \in \Gamma$. Note that this strengthening ceases to hold for defeasible assumptions: if $Ab = \{\mathsf{F}, p, \neg p\}$ then $Ab \not\mid\!\sim^{\cap}_{\mathsf{sem}} p$ and $Ab \not\mid\!\sim^{\cap}_{\mathsf{sem}} \neg p$.

Since $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{Grd}} \psi$, by Corollary 3 we have that $\Gamma, \cap\mathsf{MCS}(\Gamma, Ab) \vdash \psi$, and so, by monotonicity, we have that $(*)$ $\Gamma, \cap\mathsf{MCS}(\Gamma, Ab), \phi \vdash \psi$. Also, since $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{Grd}} \phi$, by Corollary 3 it holds that $\Gamma, \cap\mathsf{MCS}(\Gamma, Ab) \vdash \phi$, which implies that $\phi \in \cap\mathsf{MCS}(\Gamma, Ab, \phi)$ (since, by monotonicity, $\Gamma, \Theta \vdash \phi$ for every $\Theta \in \mathsf{MCS}(\Gamma, Ab)$), and so $(**)$ $\cap\,\mathsf{MCS}(\Gamma, Ab) \cup \{\phi\} \subseteq \cap\mathsf{MCS}(\Gamma, Ab, \phi)$. Now, by $(*)$, $(**)$, and monotonicity, $\Gamma, \cap\mathsf{MCS}(\Gamma, Ab, \phi) \vdash \psi$. Thus, by Corollary 3 again, $\Gamma, Ab, \phi \mid\!\sim^{\cap}_{\mathsf{Grd}} \psi$. $\square$

**Note 9.** By Theorem 2, if $\mathsf{F} \in Ab$ then $\mid\!\sim^{\cup}_{\mathsf{Grd}} = \mid\!\sim^{\cap}_{\mathsf{Grd}}$, and so, by Theorem 6, $\mid\!\sim^{\cup}_{\mathsf{Grd}}$ is preferential. Similarly, by Corollary 4, $\mid\!\sim^{\cup}_{\mathsf{WF}}$ is preferential.

We now turn to rationality. The next example shows that RM does not hold for skeptical entailments based on the naive, preferred and stable semantics, as well as for the grounded and well-founded semantics when $\mathsf{F} \notin Ab$:

**Example 11** *([28]).* Let $\mathbf{ABF} = \langle \mathsf{CL}, \emptyset, Ab, \neg \rangle$ be a simple contrapositive assumption-based framework in which $Ab = \{r,\ p \wedge q \wedge \neg r,\ (p \wedge r) \supset \neg q,\ \neg p \wedge q\}$. By Theorem 1, when $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$ we can just consider $\mathsf{MCS}(\mathbf{ABF}) = \{\{r, (p \wedge r) \supset \neg q, \neg p \wedge q\}, \{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q\}\}$. Note that none of the two elements of $\mathsf{MCS}(\mathbf{ABF})$ implies $\neg p$, while both of them imply $q$.

Now, consider $\mathbf{ABF}' = \langle \mathsf{CL}, \emptyset, Ab \cup \{p\}, \neg \rangle$. We get: $\mathsf{MCS}(\mathbf{ABF}') = \{\{r, (p \wedge r) \supset \neg q, \neg p \wedge q\}, \{p \wedge q \wedge \neg r, (p \wedge r) \supset \neg q, p\}, \{r, p, (p \wedge r) \supset \neg q\}\}$. Since $\{r, p, (p \wedge r) \supset \neg q\} \nvdash_{\mathsf{CL}} q$, we have that $\emptyset, Ab, p \not\mid\!\sim^{\cap}_{\mathsf{Sem}} q$ for every $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. Thus, rational monotonicity does not hold for $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ for such Sem.

Note, further, that in this example $\mathsf{MCS}(\mathbf{ABF})$ also coincides with the grounded extensions of $\mathsf{MCS}(\mathbf{ABF})$ and the minimally complete subsets of $Ab$. Thus, this example shows that RM is not satisfied also for $\mid\!\sim^{\cap}_{\mathsf{Grd}}$ and $\mid\!\sim^{\cap}_{\mathsf{WF}}$.

The next example considers rationality of entailments based on the grounded and well-founded semantics when $\mathsf{F} \in Ab$.

**Example 12.** Let $\mathbf{ABF} = \langle \mathsf{CL}, \Gamma, Ab, \neg \rangle$ be a simple contrapositive assumption-based framework in which $\Gamma = \{\neg(p \wedge q \wedge s), \neg(p \wedge r)\}$ and $Ab = \{p, q, r, \mathsf{F}\}$. By Theorem 3 we can just consider $\bigcap \mathsf{MCS}(\mathbf{ABF}) = \{q\}$ (since $\mathsf{MCS}(\mathbf{ABF}) = \{\{p, q\}, \{r, q\}\}$). We thus have that $\Gamma, \bigcap \mathsf{MCS}(\mathbf{ABF}) \vdash q$ and $\Gamma, \bigcap \mathsf{MCS}(\mathbf{ABF}) \nvdash \neg s$.

Now, consider the framework $\mathbf{ABF}' = \langle \mathsf{CL}, \Gamma, Ab \cup \{s\}, \neg \rangle$. We get: $\mathsf{MCS}(\mathbf{ABF}') = \{\{p, q\}, \{r, q, s\}, \{s, p\}\}$ and thus $\bigcap \mathsf{MCS}(\mathbf{ABF}') = \emptyset$, so now $\bigcap \mathsf{MCS}(\mathbf{ABF}') \nvdash q$.

It follows that rational monotonicity does not hold for $\mid\!\sim^{\cap}_{\mathsf{Grd}} = \mid\!\sim^{\cup}_{\mathsf{Grd}} = \mid\!\sim^{\cap}_{\mathsf{WF}} = \mid\!\sim^{\cup}_{\mathsf{WF}}$ even if $\mathsf{F} \in Ab$.

**Note 10.** The results in Theorems 4, 6, and Examples 11, 12 resemble similar results for other formalisms. For instance, in [5,6] is it shown that reasoning with (preferred) maximally consistent subsets of a knowledge-base yields, under classical logic, preferential but not rational relations.

For the credulous entailments based on the naive, preferred and stable semantics, however, RM does hold:

**Proposition 7.** *Let $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \neg \rangle$ be a simple contrapositive ABF. Then $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ satisfies RM for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.*

**Proof.** By Theorem 1 it suffices to show that if $\Delta \in \mathsf{MCS}(\Gamma, Ab)$ and $\Delta \vdash \phi$ and $\Delta \nvdash \neg\psi$, there is a $\Theta \in \mathsf{MCS}(\Gamma, Ab, \psi)$ s.t. $\Theta \vdash \phi$. Indeed, suppose that $\Delta \in \mathsf{MCS}(\Gamma, Ab)$ and $\Delta \vdash \phi$ and $\Delta \nvdash \neg\psi$. Suppose for a contradiction that $\Delta, \psi \vdash \mathsf{F}$. This means that $\Delta, \psi \vdash \neg\psi$, so by contraposition for any $\delta \in \Delta$ it holds that $\Delta \setminus \{\delta\}, \psi \vdash \neg\delta$. Again, by contraposition, $\Delta \vdash \neg\psi$, contradicting the assumption. Thus, $\Delta \cup \{\psi\}$ is consistent. Since $\Delta \in \mathsf{MCS}(\Gamma, Ab)$, necessarily $\Delta \cup \{\psi\} \in \mathsf{MCS}(\Gamma, Ab, \psi)$. Since $\Delta \vdash \phi$, by monotonicity, $\Delta, \psi \vdash \phi$. $\square$

### *4.3. Non-interference*

The following is an adaptation to ABFs of the property of non-interference (NI), introduced in [10]. It assures a proper handling of contradictory assumptions.

**Definition 12.** Given a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, let $\Gamma_i$ $(i = 1, 2)$ be two sets of countable $\mathscr{L}$-formulas, and let $\mathbf{ABF}_i = \langle \mathfrak{L}, \Gamma_i, Ab_i, \sim_i \rangle$ $(i = 1, 2)$ be two ABF based on $\mathfrak{L}$.

- We denote by $\mathsf{Atoms}(\Gamma_i)$ $(i = 1, 2)$ the set of all atoms occurring in $\Gamma_i$.
- We say that $\Gamma_1$ and $\Gamma_2$ are *syntactically disjoint* if $\mathsf{Atoms}(\Gamma_1) \cap \mathsf{Atoms}(\Gamma_2) = \emptyset$.
- $\mathbf{ABF}_1$ and $\mathbf{ABF}_2$ are *syntactically disjoint* if so are $\Gamma_1 \cup Ab_1$ and $\Gamma_2 \cup Ab_2$.
- We denote: $\mathbf{ABF}_1 \cup \mathbf{ABF}_2 = \langle \mathfrak{L}, \Gamma_1 \cup \Gamma_2, Ab_1 \cup Ab_2, \sim_1 \cup \sim_2 \rangle$.

An entailment $\mid\!\sim$ satisfies *non-interference*, if for every two syntactically disjoint assumption-based frameworks $\mathbf{ABF}_1 = \langle \mathfrak{L}, \Gamma_1, Ab_1, \sim_1 \rangle$ and $\mathbf{ABF}_2 = \langle \mathfrak{L}, \Gamma_2, Ab_2, \sim_2 \rangle$ where $\Gamma_1 \cup \Gamma_2$ is consistent, it holds that $\mathbf{ABF}_1 \mid\!\sim \psi$ iff $\mathbf{ABF}_1 \cup \mathbf{ABF}_2 \mid\!\sim \psi$ for every $\mathscr{L}$-formula $\psi$ such that $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\Gamma_1 \cup Ab_1)$.

**Table 1**
Properties of $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ and $\mid\!\sim^{\cup}_{\mathsf{Sem}}$.

| Entailment | Cumulativity | Preferentiality | Rationality | Non-Interference |
|---|---|---|---|---|
| $\mid\!\sim^{\cap}_{\mathsf{Naive}} = \mid\!\sim^{\cap}_{\mathsf{Prf}} = \mid\!\sim^{\cap}_{\mathsf{Stb}}$ | Theorem 4 | Theorem 4 | Example 11 | Theorem 7 |
| $\mid\!\sim^{\cup}_{\mathsf{Naive}} = \mid\!\sim^{\cup}_{\mathsf{Prf}} = \mid\!\sim^{\cup}_{\mathsf{Stb}}$ | Theorem 5 | Example 10 | Proposition 7 | Theorem 7 |
| $\mid\!\sim_{\mathsf{Grd}} = \mid\!\sim_{\mathsf{WF}}$ | Theorem 6* | Theorem 6* | Example 12* | Theorem 8* |

**Theorem 7.** *For* Sem $\in$ {Naive, Prf, Stb}, *both* $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ *and* $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ *satisfy non-interference with respect to simple contrapositive assumption-based frameworks.*

**Proof.** By Theorem 1 and the fact that if $\mathbf{ABF}_1$ and $\mathbf{ABF}_2$ are syntactically disjoint, then $\mathsf{MCS}(\mathbf{ABF}_1 \cup \mathbf{ABF}_2) = \{\Delta_1 \cup \Delta_2 \mid \Delta_1 \in \mathsf{MCS}(\mathbf{ABF}_1), \Delta_2 \in \mathsf{MCS}(\mathbf{ABF}_2)\}$. $\square$

As the next example shows, non-interference is not satisfied with respect to $\mid\!\sim_{\mathsf{Grd}}$ (either for $\mid\!\sim_{\mathsf{Grd}} = \mid\!\sim^{\cap}_{\mathsf{Grd}}$ or $\mid\!\sim_{\mathsf{Grd}} = \mid\!\sim^{\cup}_{\mathsf{Grd}}$) and $\mid\!\sim_{\mathsf{WF}}$.

**Example 13.** Consider the syntactically disjoint simple contrapositive frameworks $\mathbf{ABF_1} = \langle \mathsf{CL}, \emptyset, \{s\}, \neg\rangle$ and $\mathbf{ABF_2} = \langle \mathsf{CL}, \emptyset, \{p, \neg p\}, \neg\rangle$. Clearly, $\mathbf{ABF_1} \mid\!\sim_{\mathsf{Grd}} s$, but by Example 6, $\mathbf{ABF_1} \cup \mathbf{ABF_2} \not\mid\!\sim_{\mathsf{Grd}} s$. The same holds for $\mid\!\sim_{\mathsf{WF}}$.

Again, the addition of $\mathsf{F}$ to $Ab$ guarantees non-interference for $\mid\!\sim_{\mathsf{Grd}}$ and $\mid\!\sim_{\mathsf{WF}}$:

**Theorem 8.** $\mid\!\sim_{\mathsf{Grd}}$ *and* $\mid\!\sim_{\mathsf{WF}}$ *satisfy non-interference with respect to simple contrapositive ABFs in which* $\mathsf{F} \in Ab$.

**Proof.** By Theorem 3 and the fact that if $\mathbf{ABF}_1$ and $\mathbf{ABF}_2$ are syntactically disjoint, then $\bigcap \mathsf{MCS}(\mathbf{ABF}_1 \cup \mathbf{ABF}_2) = \bigcap \mathsf{MCS}(\mathbf{ABF}_1) \cup \bigcap \mathsf{MCS}(\mathbf{ABF}_2)$. Thus, by Corollary 4, non-interference holds also for $\mid\!\sim_{\mathsf{WF}}$. $\square$

Table 1 summarizes the results of this section. An asterisk indicates that the property holds only for ABFs in which $\mathsf{F} \in Ab$. Cells with a gray background indicate that the relevant properties *do not hold* (and contain references to counterexamples). The other cells contain references to the proofs of the relevant properties.

## 5. Lifting the closure requirement

According to Definition 6, extensions of an ABF are required to be closed. This is a standard requirement for ABFs (see, e.g., [9,13,31]). In this section we show that in fact the closure condition is not necessary for simple contrapositive ABFs.

**Definition 13.** Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim\rangle$ be an assumption-based framework, a let $\Delta \subseteq Ab$. We say that:

- $\Delta$ is *weakly admissible* (in **ABF**) iff it is conflict-free, and defends every $\Delta' \subseteq \Delta$,
- $\Delta$ is *weakly complete* (in **ABF**) iff it is weakly admissible and contains every $\Delta' \subseteq Ab$ that it defends.

Weak admissibility (respectively, weak completeness) is thus admissibility (respectively, completeness) without the closure requirement.

Below, we fix a simple contrapositive argumentation framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg\rangle$. First, we show that closure is redundant in the definition of stable extensions for **ABF**.

**Proposition 8.** *If* $\Delta \subseteq Ab$ *is conflict-free and attacks every* $\psi \in Ab \setminus \Delta$ *then it is closed.*

**Proof.** Suppose that $\Delta$ is conflict-free and attacks every $\psi \in Ab \setminus \Delta$, yet $\Gamma, \Delta \vdash \phi$ for some $\phi \in Ab \setminus \Delta$. In this case $\Gamma, \Delta \vdash \neg\phi$, and by Lemma 1, $\Gamma, \Delta \vdash \mathsf{F}$. By the property of $\mathsf{F}$ and transitivity, $\Gamma, \Delta \vdash \neg\delta$ for every $\delta \in \Delta$, thus $\Delta$ is not conflict-free – a contradiction. $\square$

By Proposition 8 the following corollary immediately follows.

**Corollary 5.** *A set* $\Delta \subseteq Ab$ *is stable iff it is conflict-free and attacks every* $\psi \in Ab \setminus \Delta$.

Next we show that closure is redundant for naive semantics as well.

**Proposition 9.** *If* $\Delta \subseteq Ab$ *is maximally conflict-free, it is closed.*

**Proof.** Suppose that $\Delta$ is maximally conflict-free and $\Gamma, \Delta \vdash \phi$. Suppose for a contradiction that $\phi \in Ab \setminus \Delta$. Since $\Delta$ is maximally conflict-free, this means that $\Gamma, \Delta, \phi \vdash \neg\psi$ for some $\psi \in \Delta \cup \{\phi\}$. Suppose first that $\psi = \phi$.[19] Then $\Gamma, \Delta, \phi \vdash \neg\phi$ and by transitivity $\Gamma, \Delta \vdash \neg\phi$. In this case, then, $\Gamma, \Delta \vdash \phi$ and $\Gamma, \Delta \vdash \neg\phi$, thus $\Gamma, \Delta \vdash \mathsf{F}$ (by Lemma 1), which implies, by the property of $\mathsf{F}$ and transitivity, that $\Gamma, \Delta \vdash \neg\delta$ for every $\delta \in \Delta$. Thus $\Delta$ is not conflict-free – a contradiction. Suppose now that $\psi \in \Delta$. Then by contraposition, $\Gamma, \Delta \vdash \neg\phi$, and the same arguments as in the previous case lead to a contradiction. $\square$

The next corollary immediately follows from the last proposition.

**Corollary 6.** *$\Delta \subseteq Ab$ is naive iff it is maximally conflict-free.*

We now turn to preferential semantics. For this (Proposition 10 and Corollary 7), we first need the following lemma:

**Lemma 7.** *A maximal conflict-free $\Delta$ attacks every $\phi \in Ab \setminus \Delta$.*

**Proof.** Suppose that $\Delta$ is maximally conflict-free and let $\phi \in Ab \setminus \Delta$. Since $\Delta$ is maximally conflict-free, $\Gamma, \Delta, \phi \vdash \neg\psi$ for some $\psi \in \Delta \cup \{\phi\}$. Either when $\psi = \phi$ or $\psi \in \Delta$, as in to the proof of Proposition 9 we get $\Gamma, \Delta \vdash \neg\phi$. $\square$

**Proposition 10.** *If $\Delta$ is maximal weakly admissible then it is closed.*

**Proof.** By Proposition 9 it suffices to show that a maximal weakly admissible set is maximally conflict-free. Indeed, suppose for a contradiction that $\Delta$ is maximal weakly admissible, yet for some $\phi \in Ab \setminus \Delta$, the set $\Delta \cup \{\phi\}$ is still conflict-free. Then $\Delta \subset \Theta$ for some maximally conflict-free $\Theta$. By Lemma 7, $\Theta$ attacks every $\psi \in Ab \setminus \Theta$. This means that $\Theta$ is weakly admissible, contradicting the assumption that $\Delta$ is maximal weakly admissible. $\square$

We thus get the following corollary:

**Corollary 7.** *A set $\Delta \subseteq Ab$ is preferred iff it is maximal weakly admissible.*

**Proof.** By Proposition 10, since it implies that a set is maximal weakly admissible iff it is maximally admissible. $\square$

Finally, we consider the grounded and the well-founded semantics. First, we note that when $\mathsf{F} \notin Ab$ the closure condition is not superfluous. To see this, let $\Gamma = \{s, s \supset q\}$, $Ab = \{p, \neg p, q\}$, and let classical logic be the base logic. Without the closure requirement, the emptyset is minimally complete in $Ab$.[20] However, the closure requirement excludes the emptyset from being a complete extension in this case, since it is not closed (indeed, $\Gamma \vdash q$).

When $\mathsf{F} \in Ab$ the closure condition *is* superfluous:

**Proposition 11.** *If $\mathsf{F} \in Ab$, a set $\Delta \subseteq Ab$ is grounded and well-founded iff it is minimal weakly complete.*

**Proof.** When $\mathsf{F} \in Ab$, we know by Lemma 6 that the grounded extension is closed. By Corollary 4, so is the well-founded extension. $\square$

## 6. Using disjunctive attacks

In this section we extend the pointed attacks in Definition 5 to disjunctive ones. First (Section 6.1) we motivate this generalization and provide some intuition for it. Then (Sections 6.2 and 6.3), we examine Dung semantics under this generalization, and finally (Section 6.4) we show that the properties of the induced entailments are preserved under this generalization.

### *6.1. Motivation and intuition*

The following definition is a conservative extension of the definition of (pointed) attacks (see Definition 5).

**Definition 14.** Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ be a simple contrapositive ABF. We say that a set $\Delta \subseteq Ab$ *attacks* a set $\Theta \subseteq Ab$ if there is a finite subset $\Theta' \subseteq \Theta$ such that $\Gamma, \Delta \vdash \bigvee \neg\Theta'$.

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[19] Note that this assumption implies in particular that $\Delta \neq \emptyset$. Otherwise, the assumption that $\Gamma, \Delta \vdash \phi$ means that $\Gamma \vdash \phi$, and $\Gamma, \Delta, \phi \vdash \neg\psi$ with the assumption that $\psi = \phi$ implies that $\Gamma, \phi \vdash \neg\phi$, so by transitivity with $\Gamma \vdash \phi$ we get $\Gamma \vdash \neg\phi$. This means, together with $\Gamma \vdash \phi$, that $\Gamma$ is not consistent (recall Lemma 1) – a contradiction.

[20] In particular, it does not defend $q$ from the attack $p, \neg p \vdash \neg q$.

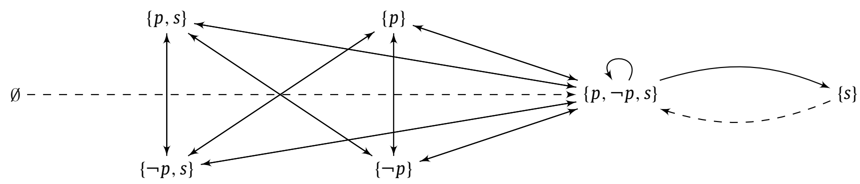


**Fig. 4.** An attack diagram for Example 14.

**Note 11.** When the ABF is not simple (that is, when the contrariness operator is defined by sets of formulas), disjunctive attacks may be defined as follows: we let $\sim\theta' = \{\sim\nu \mid \nu \in \theta'\}$ and say that a set $\Delta \subseteq Ab$ attacks a set $\Theta \subseteq Ab$ if there is a finite subset $\Theta' \subseteq \Theta$ such that $\Gamma, \Delta \vdash \bigvee_{\theta' \in \Theta'} \bigvee_{\sigma' \in \Sigma' \subseteq \sim\theta'} \sigma'$.

The next example demonstrates the differences between 'standard' (pointed) attacks and disjunctive attacks.

**Example 14.** Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$. A corresponding attack diagram is shown in Fig. 4,[21] where the strict lines represent standard attacks (Definition 5), and the dashed lines represent attacks that are applicable only according to the disjunctive version of attacks (Definition 14).

Note that the 'contaminating' set $\{p, \neg p, s\}$ attacks the set $\{s\}$. However, when disjunctive attacks are allowed the attacking set $\{p, \neg p, s\}$ is counter-attacked by the emptyset (since $\emptyset \vdash \neg p \vee \neg\neg p$), thus $\{s\}$ is defended by $\emptyset$ (which is not the case when only 'standard' attacks are allowed, cf. Example 2).

In what follows we again fix some simple contrapositive assumption-based argumentation framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$, this time with disjunctive attacks as in Definition 14. We further assume that the base logic $\mathfrak{L}$ respects the following de-Morgan rules:

$$(\star) \qquad \textit{de-Morgan I: } \bigvee \neg\Delta \vdash \neg \bigwedge \Delta, \qquad \textit{de-Morgan II: } \neg \bigwedge \Delta \vdash \bigvee \neg\Delta.$$

One benefit of using disjunctive attacks is that the notion of defense in Definition 6 can be independent of closed sets (see also Section 5). Indeed, the following definition is the same as Definition 6, but without any reference to closed sets.

**Definition 15.** We say that $\Delta$ *purely defends* $\Delta' \subseteq Ab$ iff for every $\Theta$ that attacks $\Delta'$, there is some $\Delta'' \subseteq \Delta$ that attacks $\Theta$.

**Proposition 12.** *When disjunctive attacks are used, the notions of defense and pure defense coincide.*

**Proof.** Clearly, pure defense implies defense, since if a set is capable of counter-attacking any attacker of its subset, then in particular it is capable of attacking any attacker of its subset, which is closed. The converse follows from the following lemma:

**Lemma 8.** *Let* $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive ABF with disjunctive attacks. If* $\Delta$ *attacks* $Cn(\Theta \cup \Gamma) \cap Ab$*, then it attacks* $\Theta$.

**Proof.** Suppose that $\Delta$ attacks $Cn(\Theta \cup \Gamma) \cap Ab$, i.e., there is some $\Theta' \subseteq Cn(\Theta \cup \Gamma) \cap Ab$ such that $\Gamma, \Delta \vdash \bigvee \neg\Theta'$. By de-Morgan I, $\Gamma, \Delta \vdash \neg \bigwedge \Theta'$. By the definition of conjunction, $\Gamma, \bigwedge \Delta \vdash \neg \bigwedge \Theta'$. By contraposition, $\Gamma, \bigwedge \Theta' \vdash \neg \bigwedge \Delta$. Since $\Theta' \subseteq Cn(\Theta \cup \Gamma) \cap Ab$, we know that $\Gamma, \Theta \vdash \bigwedge \Theta'$ and thus by transitivity and the definition of conjunction, $\Gamma, \bigwedge \Theta \vdash \neg \bigwedge \Delta$. By contraposition, $\Gamma, \bigwedge \Delta \vdash \neg \bigwedge \Theta$. By de-Morgan II, $\Gamma, \bigwedge \Delta \vdash \bigvee \neg\Theta$ and by the definition of conjunction, $\Gamma, \Delta \vdash \bigvee \neg\Theta$ which implies that $\Delta$ attacks $\Theta$. $\square$

It remains to show the following simple, technical lemma:

**Lemma 9.** $\Gamma, \bigwedge \Delta \vdash \psi$ *iff* $\Gamma, \Delta \vdash \psi$.

**Proof.** Suppose first that $\Gamma, \bigwedge \Delta \vdash \psi$. By reflexivity, $\Gamma, \Delta \vdash \delta$ for any $\delta \in \Delta$, thus, since $\wedge$ is a conjunction, $\Gamma, \Delta \vdash \bigwedge \Delta$. By transitivity, $\Gamma, \Delta \vdash \psi$. For the converse, suppose that $\Gamma, \Delta \vdash \psi$. By reflexivity, $\Gamma, \bigwedge \Delta \vdash \bigwedge \Delta$, thus by definition of conjunction, $\Gamma, \bigwedge \Delta \vdash \delta$ for any $\delta \in \Delta$. By transitivity (applied $|\Delta|$-times), $\Gamma, \bigwedge \Delta \vdash \psi$. $\square$

This concludes the proof of Proposition 12. $\square$

[21] Again, we refer only to closed sets, thus the set $\{p, \neg p\}$ does not appear in the diagram.

**Note 12.** To see that the condition of having disjunctive attacks is indeed necessary for Proposition 12, consider again Example 14. As indicated in that example, when only standard attacks are used, $\{s\}$ cannot be purely defended from the attacking set $\{p, \neg p\}$. On the other hand, $\{s\}$ *is* defended according to Definition 6, simply because any attacker of $\{s\}$ not containing $\mathsf{F}$ is not closed (e.g., $\{p, \neg p\}$ is not closed since $\{p, \neg p\} \vdash \mathsf{F}$).[22]

*6.2. Naive, preferred and stable semantics with disjunctive attacks*

We now consider Dung's semantics for ABFs with disjunctive attacks. In this section we treat naive, preferred, and stable semantics. In the next section we turn to the grounded and well-founded semantics, in which case another benefit of using disjunctive attacks will become evident (see, e.g., Example 15).

The main results of this section is that, again, when disjunctive attacks are involved:

a) preferential and stable semantics are reducible to naive semantics, and
b) the correspondence to reasoning with maximally consistent subsets is preserved.

To show these results we first indicate that also when switching to the more generalized (i.e., disjunctive) attacks, the closure requirement in the definitions of naive, preferred, and stable extensions (Definition 6) remains redundant. Namely:

**Proposition 13.** *For a set $\Delta \subseteq Ab$, we have:*

1. *$\Delta$ is stable iff it is conflict-free in* **ABF** *and attacks every $\psi \in Ab \setminus \Delta$.*
2. *$\Delta$ is naive iff it is maximally conflict-free in* **ABF**.
3. *$\Delta$ is preferred iff it is maximal weakly admissible in* **ABF**.

The proofs of Items 1–3 above are similar to those of Corollaries 5–7 (respectively) in Section 5, except that instead of the fact that $\Gamma, \Delta, \psi \vdash \neg\psi$ implies $\Gamma, \Delta \vdash \neg\psi$ (when $\Delta \neq \emptyset$), we use the following adjustment to disjunctive attacks:

**Lemma 10.** *If $\Gamma, \Delta, \psi \vdash \bigvee \neg\Delta'$ for some $\Delta' \subseteq \Delta \cup \{\psi\}$, then $\Gamma, \Delta \vdash \neg\psi$.*

**Proof.** First, we show the following lemma:

**Lemma 11.** $\Gamma, \Delta \vdash \neg\phi$ *iff* $\Gamma, \phi \vdash \bigvee \neg\Delta$.

**Proof.** Suppose that $\Gamma, \Delta \vdash \neg\phi$. Then by Lemma 9 we have that $\Gamma, \bigwedge\Delta \vdash \neg\phi$. By contraposition, $\Gamma, \phi \vdash \neg\bigwedge\Delta$. By de-Morgan II and transitivity, $\Gamma, \phi \vdash \bigvee\neg\Delta$. Conversely: if $\Gamma, \phi \vdash \bigvee\neg\Delta$ then by de-Morgan I and transitivity, $\Gamma, \phi \vdash \neg\bigwedge\Delta$. By contraposition $\Gamma, \bigwedge\Delta \vdash \neg\phi$, and by Lemma 9 we get, $\Gamma, \Delta \vdash \neg\phi$. □

Back to the proof of Lemma 10: Suppose that $\Gamma, \Delta, \psi \vdash \bigvee \neg\Delta'$ for some $\Delta' \subseteq \Delta \cup \{\psi\}$. By Lemma 11 and since $\Delta' \subseteq \Delta \cup \{\psi\}$, we get that $\Gamma, \Delta \vdash \neg\psi$. □

We now show (see Proposition 14 below) that when disjunctive attacks are incorporated in simple contrapositive ABFs, preferential and stable semantics collapse to naive semantics (just as in the case of standard attacks, cf. Proposition 1). For the proof of this result we first need the following two lemmas.

**Lemma 12.** *If $\Delta$ is maximally conflict-free it attacks every $\psi \in Ab \setminus \Delta$.*

**Proof.** Suppose that $\Delta$ is maximally conflict-free and $\psi \in Ab \setminus \Delta$. Since $\Delta$ is maximally conflict-free, $\Gamma, \Delta, \psi \vdash \bigvee \neg\Delta'$ for some $\Delta' \subseteq \Delta \cup \{\psi\}$. By Lemma 10, $\Gamma, \Delta \vdash \neg\psi$. □

**Lemma 13.** *If $\Delta$ is maximal weakly admissible then it is also maximally conflict-free.*

**Proof.** Suppose that $\Delta$ is maximal weakly admissible, yet for some $\phi \in Ab \setminus \Delta$ the set $\Delta \cup \{\phi\}$ is conflict-free. Then there is a proper superset $\Theta$ of $\Delta$ that is maximally conflict-free. By Lemma 12, $\Theta$ attacks every $\psi \in Ab \setminus \Theta$. This means that $\Theta$ is weakly admissible, contradicting the assumption that $\Delta$ is maximal weakly admissible. □

[22] This is exactly the reason why the restriction to closed sets is imposed when standard attacks are used, while for disjunctive attacks this is not necessary.

**Proposition 14.** *A set $\Delta \subseteq Ab$ is naive in* **ABF** *iff it is stable, iff it is preferred.*

**Proof.** By Lemma 12, together with Items 1 and 2 in Proposition 13, a naive set in $Ab$ is also stable. By Items 1 and 3 of Proposition 13, a stable extension is also preferred (indeed, a stable extension $\Delta$ is conflict-free, thus every attacker of $\Delta$ contains an argument in $Ab \setminus \Delta$, and the latter is attacked by $\Delta$, since $\Delta$ is stable thus attacks every element in $Ab \setminus \Delta$). Finally, by Lemma 13 together with Items 2 and 3, in Proposition 13, a preferred extension is also naive. □

We now show the relation to maximally consistent sets.

**Proposition 15.** *$\Delta$ is naive in* **ABF** *iff it is in* $\mathsf{MCS}(\mathbf{ABF})$.

**Proof.** [$\Rightarrow$]: Let $\Delta$ be a naive set in $Ab$. Suppose for a contradiction that $\Gamma, \Delta \vdash \mathsf{F}$. By the property of $\mathsf{F}$ and transitivity, this means that $\Gamma, \Delta \vdash \bigvee \neg \Delta'$ for any $\Delta' \subseteq \Delta$, contradicting the conflict-freeness of $\Delta$. Thus $\Delta$ is consistent. To see that $\Delta$ is maximally consistent in **ABF**, note that since $\Delta$ maximally conflict-free, for every proper superset $\Delta'$ of $\Delta$ there is some $\Theta \subseteq \Delta'$ such that $\Gamma, \Delta' \vdash \bigvee \neg \Theta$. By de-Morgan I and transitivity, then, $\Gamma, \Delta' \vdash \neg \bigwedge \Theta$. On the other hand, $\Theta \subseteq \Delta'$, and so $\Gamma, \Delta' \vdash \bigwedge \Theta$. By Lemma 1, then, $\Gamma, \Delta' \vdash \mathsf{F}$. Thus, $\Delta$ is maximally consistent in **ABF**.

[$\Leftarrow$]: Let $\Delta \in \mathsf{MCS}(\mathbf{ABF})$ and suppose for a contradiction that $\Gamma, \Delta \vdash \bigvee \neg \Delta'$ for some $\Delta' \subseteq \Delta$. Again, by de-Morgan condition and transitivity we get on one hand that $\Gamma, \Delta \vdash \neg \bigwedge \Delta'$, and since $\Delta' \subseteq \Delta$, by reflexivity we get on the other hand that $\Gamma, \Delta \vdash \bigwedge \Delta$, which together contradict the assumption that $\Gamma, \Delta \nvdash \mathsf{F}$. Thus $\Delta$ is conflict-free. To see that $\Delta$ is maximally conflict-free, suppose for a contradiction that $\Delta \cup \{\phi\}$ is conflict-free for some $\phi \in Ab \setminus \Delta$. Since $\Delta$ is maximally consistent, $\Gamma, \Delta, \phi \vdash \mathsf{F}$, thus by the property of $\mathsf{F}$ and transitivity, $\Gamma, \Delta, \phi \vdash \neg \delta$ for every $\delta \in \Delta \cup \{\phi\}$, contradicting the assumption that $\Delta \cup \{\phi\}$ is conflict-free. □

By Propositions 14 and 15 we have the following counterpart of Theorem 1:

**Theorem 9.** *Let $\mathfrak{L}$ be a logic in which de-Morgan's rules in ($\star$) are satisfied, and let* $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive ABF with disjunctive attacks. Then $\Delta$ is a stable extension of* **ABF**, *iff it is a preferred extension of* **ABF**, *iff it is naive in* **ABF**, *iff it is an element in* $\mathsf{MCS}(\mathbf{ABF})$.

### *6.3. Grounded and well-founded semantics with disjunctive attacks*

We now turn to the use of disjunctive attacks with the grounded and the well-founded semantics. The next example helps to appreciate the role of disjunctive attacks in such cases.

**Example 15.** Recall Examples 2, 6, and 14 (together with, respectively, Figs. 1 and 4), in which $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, s\}$. As indicated in these examples, when only standard attacks are allowed, the grounded extension is the emptyset, while when disjunctive attacks are allowed the grounded and the well-founded extension are both the set $\{s\}$ (which is defended by the emptyset). As $s$ should not be contaminated by the inconsistency about $p$ and $\neg p$, having $\{s\}$ as the grounded extension makes much more sense in this case, and – what is more – it holds that $\mathsf{Grd}(\mathbf{ABF}) = \mathsf{WF}(\mathbf{ABF}) = \{\{s\}\} = \{\bigcap \mathsf{MCS}(\mathbf{ABF})\}$ (see also Theorems 10 and 11 below).

In what follows we shall show that the grounded extension (which, as we shall show, is also the well-founded extension) is well-behaved for disjunctive attacks, even without requiring that $\mathsf{F} \in Ab$ (cf. Theorem 2).

**Theorem 10.** *Let $\mathfrak{L}$ be a logic in which de-Morgan's rules in ($\star$) are satisfied. If* **ABF** *is a simple contrapositive ABF with disjunctive attacks, then* $\mathsf{Grd}(\mathbf{ABF}) = \{\mathscr{G}\}$.[23]

The proof of Theorem 10 is similar to that of Theorem 2, with some adjustments to disjunctive attacks. In the sequel we shall use the notations of Definition 10. Again, we first need a few lemmas. In what follows we suppose that $\mathbf{ABF} = \langle \mathscr{L}, \Gamma, Ab, \neg \rangle$ is a simple contrapositive ABF with disjunctive attacks and that $\Delta \subseteq Ab$.

**Lemma 14.** *Let $\mathfrak{L}$ be a logic that satisfies the two de-Morgan rules. If $\Delta \vdash \bigvee \neg (\Theta \cup \Delta')$ and $\Delta' \subseteq \Delta$ then $\Delta \vdash \bigvee \neg \Theta$.*

**Proof.** Suppose $\Delta \vdash \bigvee \neg (\Theta \cup \Delta')$. By de-Morgan I, $\Delta \vdash \neg \bigwedge (\Theta \cup \Delta')$. By Lemma 11, de-Morgan I and transitivity, $\bigwedge (\Theta \cup \Delta') \vdash \neg \bigwedge \Delta$. By the definition of conjunction and transitivity, $\bigwedge \Theta, \Delta' \vdash \neg \bigwedge \Delta$. By Lemma 11, $\Delta', \bigwedge \Delta \vdash \neg \bigwedge \Theta$. Since $\Delta' \subseteq \Delta$, by the definition of conjunction and transitivity, $\Delta \vdash \neg \bigwedge \Theta$. By de-Morgan II, $\Delta \vdash \bigvee \neg \Theta$. □

[23] Recall that $\mathscr{G}$ is defined in Definition 10.

**Lemma 15.** *$\mathscr{G}_0(\mathbf{ABF})$ is (disjunctively) unattacked.*

**Proof.** Suppose for a contradiction that there is some $\Theta \subseteq \mathscr{G}_0(\mathbf{ABF})$ and some $\Delta \subseteq Ab$ such that $\Delta$ attacks $\Theta$, i.e., $\Gamma, \Delta \vdash \bigvee \neg\Theta$. By de-Morgan I and definition of conjunction, $\Gamma, \bigwedge \Delta \vdash \neg \bigwedge \Theta$. By contraposition, $\Gamma, \bigwedge \Theta \vdash \neg \bigwedge \Delta$. By definition of conjunction and transitivity, $\Gamma, \Theta \vdash \neg \bigwedge \Delta$. Again by contraposition, definition of conjunction and transitivity, $\Gamma, \Delta, \Theta \setminus \phi \vdash \neg\phi$ for any $\phi \in \Theta$. However, since $\phi \in \mathscr{G}_0(\mathbf{ABF})$, by Definition 10 $\phi$ cannot be (pointed) attacked. $\square$

**Lemma 16.** *If $\mathscr{G}_1$ attacks a closed set $\Theta$ then $\mathscr{G}_0$ also attacks $\Theta$.*

**Proof.** Suppose that $\mathscr{G}_1$ attacks a closed set $\Theta$ and assume for a contradiction that $\mathscr{G}_0, \Gamma \nvdash \bigvee \neg\Theta'$ for any $\Theta' \subseteq \Theta$. Since $\mathscr{G}_1$ attacks $\Theta$, there is a $\Theta' \subseteq \Theta$ s.t. $\Gamma, \mathscr{G}_1 \vdash \bigvee \neg\Theta'$. Suppose first that $\Gamma, \mathscr{G}_0 \cup \mathscr{G}_1 \vdash \bigvee \neg\Theta'$. In that case, by de-Morgan I and transitivity, $\Gamma, (\mathscr{G}_1 \cup \mathscr{G}_0) \vdash \neg\bigwedge\Theta'$. By contraposition and Lemma 9, $\Gamma, ((\mathscr{G}_1 \cup \mathscr{G}_0) \setminus \{\delta\}), \Theta' \vdash \neg\delta$ for any $\delta \in \mathscr{G}_0$, thus $\mathscr{G}_0$ is attacked, in a contradiction to Lemma 15.

Thus, $\Gamma, (\mathscr{G}_1 \setminus \mathscr{G}_0) \vdash \bigvee \neg\Theta'$. By de-Morgan I and transitivity, $\Gamma, (\mathscr{G}_1 \setminus \mathscr{G}_0) \vdash \neg\bigwedge\Theta'$. Now, by contraposition, $\Gamma, (\mathscr{G}_1 \setminus \mathscr{G}_0) \cup \{\psi\}, \bigwedge \Theta' \vdash \neg\psi$ for every $\psi \in \mathscr{G}_1$. Let $\psi_1 \in \mathscr{G}_1$ be such a formula. Since $\psi_1 \in \mathscr{G}_1 \setminus \mathscr{G}_0$, $\psi_1$ is defended by $\mathscr{G}_0$, there is a $\Lambda \subseteq Cn(\Gamma \cup ((\mathscr{G}_1 \setminus (\mathscr{G}_0 \cup \{\psi_1\})) \cup \Theta') \cap Ab$ such that $\Gamma, \mathscr{G}_0 \vdash \bigvee \neg\Lambda$. By Lemma 9, $\Gamma, \bigwedge \mathscr{G}_0 \vdash \bigvee \neg\Lambda$. By de-Morgan I and contraposition, $\Gamma, \bigwedge \Lambda \vdash \neg \bigwedge \mathscr{G}_0$. By de-Morgan II and Lemma 9 $\Gamma, \Lambda \vdash \bigvee \neg\mathscr{G}_0$, contradicting the fact that $\mathscr{G}_0$ is not attacked (Lemma 15 again). $\square$

**Lemma 17.** $\mathscr{G}_2(\mathbf{ABF}) = \mathscr{G}_1(\mathbf{ABF})$.

**Proof.** By Definition 10, $\mathscr{G}_1(\mathbf{ABF}) \subseteq \mathscr{G}_2(\mathbf{ABF})$. To see that $\mathscr{G}_2(\mathbf{ABF}) \subseteq \mathscr{G}_1(\mathbf{ABF})$, we have to show that every assumption that is defended by $\mathscr{G}_1$ is also defended by $\mathscr{G}_0$. This follows from Lemma 16. $\square$

**Corollary 8.** $\mathscr{G}(\mathbf{ABF}) = \mathscr{G}_0(\mathbf{ABF}) \cup \mathscr{G}_1(\mathbf{ABF}) = \mathscr{G}_1(\mathbf{ABF})$.

**Proof.** Follows from Lemma 17. $\square$

**Lemma 18.** *$\Delta \subseteq \mathscr{G}(\mathbf{ABF})$ iff $\Delta$ is defended by $\mathscr{G}(\mathbf{ABF})$.*

**Proof.** [$\Rightarrow$]: Suppose that some $\Theta = Cn(\Theta \cup \Gamma) \cap Ab$ attacks $\Delta \subseteq \mathscr{G}(\mathbf{ABF})$. First, we assume that $\Theta$ and $\Delta$ are minimal, i.e., for no $\Theta' \subset \Theta$ and no $\Delta' \subset \Delta$, $\Theta'$ attacks $\Delta'$. Note that since $\Theta$ attacks $\Delta$, it holds that $\Gamma, \Theta \vdash \bigvee \neg\Delta$.

Suppose first that $\Delta \cap \mathscr{G}_0 \neq \emptyset$. Since $\Gamma, \Theta \vdash \bigvee \neg\Delta$, by Lemma 14, $\Gamma, \Theta, \Delta \setminus \mathscr{G}_0 \vdash \bigvee \neg(\Delta \cap \mathscr{G}_0)$. But this contradicts the fact that $\mathscr{G}_0$ is unattacked (by Lemma 15).

We can thus suppose that $\Delta \cap \mathscr{G}_0 = \emptyset$. Since $\Gamma, \Theta \vdash \bigvee \neg\Delta$, by de-Morgan I, definition of conjunction and transitivity $\Gamma, \bigwedge \Theta \vdash \neg \bigwedge \Delta$. By contraposition, definition of conjunction and transitivity $\Gamma, \Delta \vdash \neg \bigwedge \Theta$. Again by contraposition, definition of conjunction and transitivity, for every $\phi \in \Delta$ we have that $\Gamma, \Theta, \Delta \setminus \{\phi\} \vdash \neg\phi$. Since $\phi \in \mathscr{G}_1 \setminus \mathscr{G}_0$, it follows that $\mathscr{G}_0$ defends $\phi$ from $\Theta \cup \Delta \setminus \{\phi\}$. This means that for some $\Theta' \subseteq \Theta$ and some $\Delta' \subseteq \Delta \setminus \{\phi\}$ it holds that $\Gamma, \mathscr{G}_0 \vdash \bigvee \neg(\Delta' \cup \Theta')$.

- Suppose first that $\Delta' \neq \emptyset$.
  - Suppose that $\Gamma \nvdash \bigvee \neg(\Delta' \cup \Theta')$. By de-Morgan I, $\Gamma, \mathscr{G}_0 \vdash \neg \bigwedge(\Delta' \cup \Theta')$. By Lemma 11, the definition of conjunction and transitivity, $\Gamma, \Delta' \cup \Theta' \vdash \neg \bigwedge \mathscr{G}_0$. By de-Morgan II, $\Gamma, \Delta' \cup \Theta' \vdash \bigvee \neg\mathscr{G}_0$. This contradicts that fact that $\mathscr{G}_0$ is unattacked (Lemma 15).
  - Suppose that $\Gamma \vdash \bigvee \neg(\Delta' \cup \Theta')$. By Lemma 14, $\Gamma, \Theta' \vdash \bigvee \neg\Delta'$, in contradiction to the minimality of $\Theta$ and $\Delta$.
- Suppose now that $\Delta' = \emptyset$. In that case, $\Gamma, \mathscr{G}_0 \vdash \bigvee \neg\Theta'$ and thus $\Delta$ is defended by $\mathscr{G}_0$.

Suppose now that $\Theta$ is not minimal, i.e., there is some $\Theta' \subsetneq \Theta$ that attacks $\Delta$. Without loss of generality, we can now suppose that $\Theta'$ is minimal, that is, for no $\Theta'' \subset \Theta'$ does $\Theta''$ attack $\Delta$. In that case, we have established above that $\mathscr{G}_0$ attacks $\Theta'$. But then $\mathscr{G}_0$ also attacks $\Theta$. Thus $\Delta$ is defended by $\mathscr{G}(\mathbf{ABF})$.

Suppose finally that $\Delta$ is not minimal, i.e., there is some $\Delta' \subsetneq \Delta$ that is attacked by $\Theta$. We have already established above that we can suppose $\Theta$ to be minimal. Without loss of generality, then, suppose that $\Delta' \subset \Delta$ is minimal, i.e., for no $\Delta'' \subset \Delta$ does $\Theta$ attack $\Delta''$. As we have shown above, $\mathscr{G}_0$ attacks $\Theta$. Thus, $\mathscr{G}_0$ also defends $\Delta$ from $\Theta$, and so, again, $\Delta$ is defended by $\mathscr{G}(\mathbf{ABF})$.

[$\Leftarrow$]: Suppose that $\Delta$ is defended by $\mathscr{G}(\mathbf{ABF})$. Then clearly every $\delta \in \Delta$ is defended by $\mathscr{G}(\mathbf{ABF})$. But then by Definition 10, for every $\delta \in \Delta$, $\delta \in \mathscr{G}(\mathbf{ABF})$. Thus, $\Delta \subseteq \mathscr{G}(\mathbf{ABF})$. $\square$

**Lemma 19.** *$\mathscr{G}$ is closed.*

**Proof.** Suppose for a contradiction that $\Gamma, \mathscr{G} \vdash \phi$ for some $\phi \in Ab \setminus \mathscr{G}$. In particular, $\phi \notin \mathscr{G}_0$, thus there is some $\Theta \subseteq Ab$ such that, $\Gamma, \Theta \vdash \neg\phi$. By Lemma 11 we get $\Gamma, \phi \vdash \bigvee \neg\Theta$. But then $\Gamma, \mathscr{G} \vdash \bigvee \neg\Theta$ by transitivity, and so $\mathscr{G}$ defends $\phi$ from $\Theta$. By Lemma 18, this means that $\phi \in \mathscr{G}$, a contradiction. □

Now we can show Theorem 10.

**Proof.** It is clear from the construction of $\mathscr{G}$ that it is unique. By Lemma 18, $\phi \in \mathscr{G}$ iff $\phi$ is defended by $\mathscr{G}$, thus $\mathscr{G}$ is complete. By Lemma 19, $\mathscr{G}$ is closed. It therefore remains to show the minimality of $\mathscr{G}$ among the complete sets of **ABF**. If $\mathscr{G}$ is empty we are done. Otherwise, suppose for a contradiction that there is some complete proper subset $\Delta$ of $\mathscr{G}$, and let $\Theta \subseteq \mathscr{G} \setminus \Delta$. If $\Theta \subseteq \mathscr{G}_0$, then $\Theta$ has no attackers by Lemma 15 and consequently $\Theta$ is (vacuously) defended by $\Delta$, in which case $\Delta$ cannot be complete. Thus $\Theta \nsubseteq \mathscr{G}_0$ and $\mathscr{G}_0 \subseteq \Delta$. Suppose now that $\Theta \subseteq \mathscr{G}_1$, i.e., $\Theta \subseteq \mathscr{G}_1 \setminus \mathscr{G}_0$. By Definition 10, $\Theta \subseteq \Xi$ for every set $\Xi$ that is maximally defended by $\mathscr{G}_0$. Thus $\Theta$ is defended by $\mathscr{G}_0$. But since $\mathscr{G}_0 \subseteq \Delta$ this means that $\Delta$ defends $\Theta$. Again, this contradicts the completeness of $\Delta$. Thus, $\mathscr{G}_1 \subseteq \Delta$. By Corollary 8 $\mathscr{G} = \mathscr{G}_1$, and consequently $\mathscr{G} \subseteq \Delta$, contradicting the assumption that $\Delta \subsetneq \mathscr{G}$. □

The correspondence of the grounded semantics to reasoning with maximally consistent subsets is also carried on from the standard case (cf. Theorem 3).

**Theorem 11.** *Let $\mathfrak{L}$ be a logic in which de-Morgan's rules in ($\star$) are satisfied, and let* **ABF** $= \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ *be a simple contrapositive assumption-based framework with disjunctive attacks. Then* $\mathsf{Grd}(\mathbf{ABF}) = \{\bigcap \mathsf{MCS}(\mathbf{ABF})\}$.

**Proof.** By Theorem 10 it suffices to show that $\mathscr{G}(\mathbf{ABF}) = \bigcap \mathsf{MCS}(\mathbf{ABF})$.

To see that $\mathscr{G}(\mathbf{ABF}) \subseteq \bigcap \mathsf{MCS}(\mathbf{ABF})$, let $\Theta \in \mathsf{MCS}(\mathbf{ABF})$. We show that $\mathscr{G}(\mathbf{ABF}) \subseteq \Theta$, thus $\mathscr{G}(\mathbf{ABF}) \subseteq \bigcap \mathsf{MCS}(\mathbf{ABF})$. Indeed, suppose otherwise that $\mathscr{G} \nsubseteq \Theta$. Then there is $\phi \in \mathscr{G} \setminus \Theta$, and since $\Theta$ is stable (by Theorem 9), it attacks $\phi$. Since $\phi \in \mathscr{G}$, by Lemma 17, $\mathscr{G}_0$ attacks $\Theta$ (note that $\mathscr{G} = \mathscr{G}_1$ by Corollary 8). Since obviously $\mathscr{G}_0 \subseteq \Theta$, this contradicts the fact that $\Theta$ is conflict-free.

We now show that $\bigcap \mathsf{MCS}(\mathbf{ABF}) \subseteq \mathscr{G}(\mathbf{ABF})$. Indeed, suppose that $\Delta \subseteq \bigcap \mathsf{MCS}(\mathbf{ABF})$. Suppose also that some $\Theta \subseteq Ab$ attacks $\Delta$. Thus, $\Gamma, \Theta \vdash \bigvee \neg\Delta$. Suppose first that $\Gamma, \Theta \nvdash \mathsf{F}$. By de-Morgan rules, the definition of conjunction and contraposition, we have (‡): $\Gamma, \Theta, \Delta \vdash \mathsf{F}$, i.e., for no consistent set of defeasible assumptions $\Theta \subseteq Ab$, $\Theta \cup \Delta \subseteq Ab$. Since $\Gamma, \Theta \nvdash \mathsf{F}$, there is a $\Theta' \in \mathsf{MCS}(\mathbf{ABF})$ such that $\Theta' \subseteq \Theta$. But by (‡), $\Delta \nsubseteq \Theta'$, a contradiction to $\Delta \subseteq \bigcap \mathsf{MCS}(\mathbf{ABF})$. Thus, (†): for any attacker $\Theta$ of $\Delta$, $\Gamma, \Theta \vdash \mathsf{F}$. By Lemma 11, this means that $\Gamma, \neg\mathsf{F} \vdash \bigvee \neg\Theta$. Since $\Gamma \vdash \neg\mathsf{F}$, by transitivity, $\Gamma \vdash \bigvee \neg\Theta$. But then $\Delta$ is defended by $\emptyset \subseteq \mathscr{G}_0$. We have thus shown that $\Delta$ is defended by $\mathscr{G}_0$ from every attacker, and so by Lemma 18 it follows that $\Delta \subseteq \mathscr{G}$. □

We now turn to the well-founded semantics.

**Proposition 16.** *Let* **ABF** *be a simple contrapositive ABF with disjunctive attacks. Then* $\mathsf{WF}(\mathbf{ABF}) = \{\bigcap \mathsf{MCS}(\mathbf{ABF})\}$.

**Proof.** Similar to that of Proposition 4, using Theorem 10 instead of Theorem 2. □

By Theorem 11 and Proposition 16 we thus have:

**Corollary 9.** *Let* **ABF** *be a simple contrapositive ABF with disjunctive attacks. Then* $\mathsf{WF}(\mathbf{ABF}) = \mathsf{Grd}(\mathbf{ABF})$.

### *6.4. Properties of the induced entailments*

Given a simple contrapositive assumption-based framework **ABF** $= \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ with disjunctive attacks and a logic $\mathfrak{L}$ satisfying de-Morgan's rules in ($\star$), most of the properties of the induced entailment relations (Definition 7) remain the same as those of the entailments that are induced from ABFs with pointed attacks. Below, we list some of them (Cf. Section 4. Since the proofs are similar to those for pointed attacks, in most case we just present the results).

**Reasoning with MCS:**

By Theorem 9, we have (cf. Corollary 1):

- $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Prf}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Stb}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Naive}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{MCS}} \psi$.
- $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Prf}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Stb}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Naive}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{MCS}} \psi$.

Similarly, by Theorem 11 and Corollary 9, we have (cf. Corollary 3):

- $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Grd}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Grd}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{WF}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{WF}} \psi$ iff $\mathbf{ABF} \mid\!\sim^{\Cap}_{\mathsf{MCS}} \psi$.

**Cumulativity, Preferentiality and Rationality:**

- $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ is preferential for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Grd}, \mathsf{WF}, \mathsf{Prf}, \mathsf{Stb}\}$.
- $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ is cumulative for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.[24,25]

  The proof is ultimately the same as that of Theorem 4.
- $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ is rational for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

  The proof is similar to that of Proposition 7.[26]

**Non-interference:**

- $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ and $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ satisfy non-interference for $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Grd}, \mathsf{WF}, \mathsf{Prf}, \mathsf{Stb}\}$.

  The proof of the last claim is similar to that of Theorem 7.

## 7. Summary and conclusion, in view of related work

In this paper we have considered the main Dung-style semantics of assumption-based argumentation frameworks based on contrapositive logics. Different perspectives were considered:

- We have shown that some of the problems of Dung's semantics for structured argumentation frameworks that are reported in [1] are carried on to (simple contrapositive) ABFs. Moreover, we delineated a class of anomalies in the application of the grounded semantics for non-flat ABFs, and specified conditions under which these anomalies can be avoided. Similar phenomena are discussed in [12], but to the best of our knowledge this paper is the first one[27] where a solution (adding $\mathsf{F}$ to $Ab$) is suggested.
- Some rationality postulates are considered. The closure and consistency postulates have also been studied for certain assumption-based argumentation frameworks in [30], but this paper is the first investigation of the property of non-interference in assumption-based argumentation.
- The relation between Dung's semantics for ABFs and other general patterns of non-monotonic reasoning were investigated. In particular, we have studied the connections to approaches based on maximal consistency [27] and to the KLM cumulative, preferential and rational entailments [22,23]. While the relations between Dung-style semantics and reasoning with maximal consistency have been investigated before in, e.g., [1,3,11,34], none of these works have considered ABFs. Thus, this paper closes a gap in the literature and shows the exact relation between MCS approaches, KLM semantics, and assumption-based argumentation based on contrapositive logics. Moreover, while all of these approaches give rise to an infinite number of arguments even for a finite set $Ab$ of defeasible assumptions, our approach avoids this problem by considering *sets* of assumptions (as opposed to derivations of a specific conclusion) as nodes in the argumentation graph, whose size is bounded by the size of the powerset of $Ab$.
- We showed that for simple contrapositive ABFs the closure requirement from the framework's extensions is in fact redundant. As a consequence, most of the concepts that are related to such ABFs were simplified, and their computation became easier. To the best of our knowledge, this is the first time that such a question has been asked and answered for assumption-based argumentation.
- We considered a generalization of the attack relation in ABFs, called disjunctive attacks. The use of these kind of attacks guarantees some desirable properties of the grounded semantics without the extra-condition that is required when standard attacks are used (see the first item in this list). Concerning the other types of semantics, we have shown that (as in the case of ordinary attacks), preferential and stable semantics are reducible to naive semantics.
- We have shown that the entailment relations induced from the generalized ABFs still correspond to reasoning with maximal consistency. They are proved preferential for skeptical reasoning, and cumulative and rational for credulous reasoning. For both of these kinds of entailments the property of non-interference is satisfied. These results again resemble similar findings concerning other forms of structured argumentation, presented e.g. in [2,3]. Since our formalism preserves consistency and the correspondence to maximal consistency-based reasoning even when using disjunctive attacks, it avoids some of the long-standing problems that were reported by [11] for other logic-based argumentation formalisms using disjunctive attacks.

Future work includes, among others, the incorporation of more expressive languages, involving preferences among arguments, and the introduction of other kinds of contrariness operators as well as further forms of attacks.

[24] Example 10 shows that $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ is not preferential even for ABFs with standard (pointed, non-disjunctive) attacks.
[25] Note that by Theorem 10, $\mid\!\sim^{\cup}_{\mathsf{Grd}} = \mid\!\sim^{\cap}_{\mathsf{Grd}}$, and so $\mid\!\sim^{\cup}_{\mathsf{Grd}}$ is not only cumulative, but also preferential. The same holds for $\mid\!\sim^{\cup}_{\mathsf{WF}}$.
[26] Example 11 provides a counterexample for the skeptical entailments.
[27] Except, of-course, of the conference papers [19–21] on which this paper is based.

## Declaration of competing interest

We wish to confirm that there are no known conflicts of interest associated with this publication and there has been no significant financial support for this work that could have influenced its outcome.

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