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Simple contrapositive assumption-based argumentation part II: Reasoning with preferences

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ABSTRACT

Simple contrapositive assumption-based argumentation frameworks are a general setting for structured argumentation, providing a robust approach to reasoning with arguments and counter-arguments. In this paper we extend these frameworks with priorities and introduce some new results concerning the Dung's semantics of the resulting formalisms.

1. Introduction

Assumption-based frameworks (ABFs) are a well-established form of structured argumentation, aimed at reasoning in the presence of arguments and counter-arguments (see, e.g., [12,21,48]). A general form of ABFs called *simple contrapositive assumption-based frameworks* is considered in [32–35], where it is shown that this family of ABFs has many desirable properties. In this paper we extend simple contrapositive ABFs with priorities, expressing the relative strengths (or reliability) of arguments. While extensions of ABFs with priorities have already been studied in the literature (mainly in the context of ABA$^+$ frameworks, see e.g. [20,22]), several new findings on these frameworks are reported, among which are the following:

- Dung-style semantics [26] is considered for prioritized ABFs. It is shown that, like similar cases in other forms of structured argumentation, in many cases the set of naive, stable, and preferred extensions coincide. However, unlike other forms of structured argumentation (including common cases of simple contrapositive ABFs), when priorities are introduced the grounded semantics is not always unique nor does it necessarily coincide with the well-founded semantics.
- We show that under the reversibility condition (see below), all the extensions of simple contrapositive ABFs are necessarily consistent and are closed under logical consequences (properties that are not assured in general for extensions of non-prioritized ABFs in particular, and of structured argumentation frameworks in general).
- Relations to reasoning with preferred maximally consistent subsets of the premises [13] are shown.
- The impact of the underlying priority setting on the properties of the prioritized ABFs that are induced by them is analyzed in terms of rationality postulates.

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- We define conditions that assure that prioritized ABFs avoid an undesirable property of prioritized systems, known as the *drowning effect*, according to which arguments with lower priorities are excluded in the presence of unrelated arguments with higher priorities.
- Properties of the entailments induced by prioritized ABFs are examined. It is shown, in particular, that when using the max aggregation function, the corresponding entailments are cumulative in the sense of Kraus, Lehmann and Magidor [42], and satisfy the non-interference property [17].

The outcome of this work is therefore a robust approach of argumentative, preference-based reasoning with defeasible assumptions. This approach is compared in the last part of the paper to other approaches of accommodating priorities in structured argumentation.

The rest of this paper is organized as follows: In the next section we recall some basic definitions and notations concerning (simple contrapositive) ABFs. In Section 3 we define priority settings and, accordingly, prioritized ABFs. Then we consider some properties of prioritized ABFs: general argumentation-theoretic properties (Section 4), relation to reasoning with maximal consistency (Section 5), and properties that are related to preference handling (Section 6). A summary of the results in these sections appears in Table 1. In Section 7 we consider some properties of the entailments that are induced by prioritized ABFs. In Section 8 we refer to related work and in Section 9 we conclude.[1]

2. Preliminaries

We start with some background on non-prioritized (simple-contrapositive) assumption-based argumentation frameworks. The definitions in this section can be found also in the first part of the paper [35]. We recall the definitions here to make this paper self-contained, and for a convenient comparison between the basic notions and their extensions, in the next section, to the prioritized case.

In what follows we shall denote by $\mathcal{L}$ an arbitrary propositional language. Atomic formulas in $\mathcal{L}$ are denoted by p, q, r, compound formulas are denoted by ψ, ϕ, σ, and sets of formulas in $\mathcal{L}$ are denoted by Γ, Δ, Θ (possibly primed or indexed). The powerset of a set Γ is denoted by $\wp(\Gamma)$.

Definition 1. A *logic* for a language $\mathcal{L}$ is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, that is, a binary relation between sets of formulas and formulas in $\mathcal{L}$, satisfying the following conditions:

Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$.

In addition, we shall assume that $\mathfrak{L}$ satisfies the following standard conditions:

Structurality (closure under substitutions): if $\Gamma \vdash \psi$ then $\theta(\Gamma) \vdash \theta(\psi)$ for every $\mathcal{L}$-substitution θ.
Non-triviality: there are a non-empty set Γ and a formula ψ such that $\Gamma \nvdash \psi$.

The $\vdash$-transitive closure of a set Γ of $\mathcal{L}$-formulas is $Cn_{\vdash}(\Gamma) = \{\psi \mid \Gamma \vdash \psi\}$. When $\vdash$ is clear from the context or arbitrary we will sometimes just write $Cn(\Gamma)$.

Definition 2. We shall assume that the language $\mathcal{L}$ contains at least the following connectives and constant:

a $\vdash$-*negation* $\neg$, satisfying: $p \nvdash \neg p$ and $\neg p \nvdash p$ (for every atomic p).
a $\vdash$-*conjunction* $\wedge$, satisfying: $\Gamma \vdash \psi \wedge \phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$.
a $\vdash$-*disjunction* $\vee$, satisfying: $\Gamma, \phi \vee \psi \vdash \sigma$ iff $\Gamma, \phi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$.
a $\vdash$-*implication* $\supset$, satisfying: $\Gamma, \phi \vdash \psi$ iff $\Gamma \vdash \phi \supset \psi$.
a $\vdash$-*falsity* F, satisfying: $\mathsf{F} \vdash \psi$ for every formula ψ.[2]

We abbreviate $\{\neg\gamma \mid \gamma \in \Gamma\}$ by $\neg\Gamma$, and when Γ is finite we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$), the conjunction (respectively, the disjunction) of all the formulas in Γ. We shall say that ψ is $\vdash$-*tautological* if $\vdash \psi$, and that Γ is $\vdash$-*consistent* if $\Gamma \nvdash \mathsf{F}$ (otherwise Γ is $\vdash$-inconsistent).

[1] A reduced version of some parts of this paper has appeared in [10].
[2] In particular, F is not a standard atomic formula, since $\mathsf{F} \vdash \neg\mathsf{F}$.

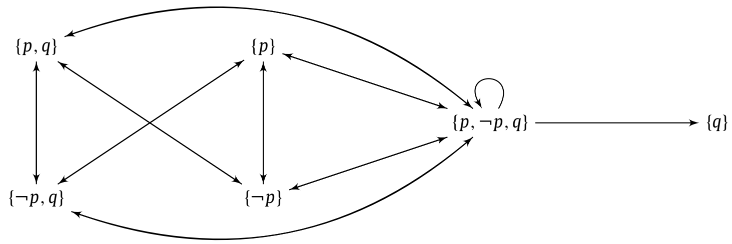

Fig. 1. An attack diagram for Example 2.

Definition 3. A logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is *explosive*, if for every $\mathcal{L}$-formula ψ the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent.[3] We say that $\mathfrak{L}$ is *contrapositive*, if (a) $\vdash \neg\mathsf{F}$ and (b) for every nonempty Γ and ψ it holds that $\Gamma \vdash \neg\psi$ iff either $\psi = \mathsf{F}$ or for every $\phi \in \Gamma$ we have that $\Gamma \setminus \{\phi\}, \psi \vdash \neg\phi$.

Example 1. Perhaps the most notable example of a logic which is both explosive and contrapositive, is classical logic, CL. Intuitionistic logic, the central logic in the family of constructive logics, and standard modal logics are other examples of well-known formalisms having these properties.

Note 1. A useful property of an explosive logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is that for every set S of $\mathcal{L}$-formulas and every $\mathcal{L}$-formulas ψ and ϕ, if $\Gamma \vdash \psi$ and $\Gamma \vdash \neg\psi$, then $\Gamma \vdash \phi$. Indeed, the assumptions imply that $\Gamma \vdash \psi \wedge \neg\psi$. Also, since $\mathfrak{L}$ is explosive, $\psi \wedge \neg\psi \vdash \mathsf{F}$. By transitivity, then, $\Gamma \vdash \mathsf{F}$. Since $\mathsf{F} \vdash \phi$, transitivity again gives $\Gamma \vdash \phi$.

We are now ready to define assumption-based argumentation frameworks (ABFs). The next definition is a generalization of the definition from [12].

Definition 4. An *assumption-based framework* is a tuple $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ where:

- $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is a propositional Tarskian logic.
- Γ (the *strict assumptions*) and Ab (the *defeasible (or abducible, or candidate) assumptions*) are distinct (and countable) sets of $\mathcal{L}$-formulas, where the former is assumed to be $\vdash$-consistent and the latter is assumed to be nonempty.
- $\sim$ is a *contrariness operator*, assigning a finite set of $\mathcal{L}$-formulas to every defeasible assumption in Ab, such that for every consistent and non-tautological formula $\psi \in Ab \setminus \{\mathsf{F}\}$ it holds that $\psi \nvdash \bigwedge\sim\psi$ and $\bigwedge\sim\psi \nvdash \psi$.

Note 2. Unlike the setting of [12], an ABF may be based on *any* Tarskian logic $\mathfrak{L}$. Also, the strict as well as the defeasible assumptions are formulas that may not be just atomic. Concerning the contrariness operator, note that it is not a connective of $\mathcal{L}$, as it is restricted only to the defeasible assumptions.

Defeasible assertions in an ABF may be attacked in the presence of a counter defeasible information. This is described in the next definition.

Definition 5. Let $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$. We say that Δ *attacks* ψ iff $\Gamma, \Delta \vdash \phi$ for some $\phi \in \sim\psi$. Accordingly, Δ attacks Θ if Δ attacks some $\psi \in \Theta$.

Example 2. Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, $Ab = \{p, \neg p, q\}$, and $\sim\psi = \{\neg\psi\}$. A corresponding attack diagram is shown in Fig. 1.[4]

Note that since in classical logic inconsistent sets of premises imply *any* conclusion, the classically inconsistent set $\{p, \neg p, q\}$ attacks all the other sets in the diagram (For instance, $\{p, \neg p, q\}$ attacks $\{q\}$, since $p, \neg p, q \vdash \neg q$).

The last definition gives rise to the following adaptation to ABFs of the usual Dung-style semantics [26] for abstract argumentation frameworks.

Definition 6. ([12]) Let $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, and let $\Delta \subseteq Ab$. Below, the maximum and the minimum are taken with respect to set inclusion. We say that:

- Δ is *closed* (in ABF) if $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$.

[3] That is, $\psi, \neg\psi \vdash \mathsf{F}$. Since $\mathsf{F} \vdash \phi$, by transitivity $\psi, \neg\psi \vdash \phi$. Thus, in explosive logics every formula follows from two $\neg$-counterassumptions.

[4] For reasons that will become apparent in the sequel we include in the diagram only *closed sets*, i.e., only subsets $\Delta \subseteq Ab$ such that $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$ (see Definition 6). Thus, the set $\{p, \neg p\}$ is omitted from the diagram.

- Δ is *conflict-free* (in ABF) iff there is no $\Delta' \subseteq \Delta$ that attacks some $\psi \in \Delta$.
- Δ is *naive* (in ABF) iff it is closed and maximally conflict-free.
- Δ *defends* (in ABF) a set $\Delta' \subseteq Ab$ iff for every closed set Θ that attacks Δ' there is $\Delta'' \subseteq \Delta$ that attacks Θ.
- Δ is *admissible* (in ABF) iff it is closed, conflict-free, and defends every $\Delta' \subseteq \Delta$.
- Δ is *complete* (in ABF) iff it is admissible and contains every $\Delta' \subseteq Ab$ that it defends.
- Δ is *well-founded* (in ABF) iff $\Delta = \bigcap\{\Theta \subseteq Ab \mid \Theta \text{ is complete}\}$.[5]
- Δ is *grounded* (in ABF) iff it is minimally complete.
- Δ is *preferred* (in ABF) iff it is maximally admissible.
- Δ is *stable* (in ABF) iff it is closed, conflict-free, and attacks every $\psi \in Ab \setminus \Delta$.

The set of the complete (respectively, the naive, grounded, well-founded, preferred, stable) extensions of ABF is denoted Cmp(ABF) (respectively, Naive(ABF), Grd(ABF), WF(ABF), Prf(ABF), Stb(ABF)). In what follows we shall denote by Sem(ABF) any of the above-mentioned sets. The entailment relations that are induced from an ABF (with respect to a certain semantics) are defined as follows:

Definition 7. Given an assumption-based framework $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Cmp}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, we denote:

- $\mathsf{ABF} \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *every* $\Delta \in \mathsf{Sem}(\mathsf{ABF})$.
- $\mathsf{ABF} \mid\!\sim^{\cup}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *some* $\Delta \in \mathsf{Sem}(\mathsf{ABF})$.

Example 3. Consider again Example 2, where $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, q\}$ (see also Fig. 1). Here, $\mathsf{Naive}(\mathsf{ABF}) = \mathsf{Prf}(\mathsf{ABF}) = \mathsf{Stb}(\mathsf{ABF}) = \{\{p, q\}, \{\neg p, q\}\}$, thus $\mathsf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} q$ for every $* \in \{\cup, \cap\}$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. Also, $\mathsf{Grd}(\mathsf{ABF}) = \mathsf{WF}(\mathsf{ABF}) = \{\emptyset\}$, since there are no unattacked arguments, thus for $* \in \{\cup, \cap\}$ and $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{WF}\}$ we have that $\mathsf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} \psi$ only if ψ is a classical tautology.

Note 3. Unlike standard consequence relations (Definition 1), which are relations between sets of formulas and formulas, the entailments in Definition 7 are relations between ABFs and formulas. This will not cause any confusion in what follows.

In the rest of the paper we shall concentrate on the following common family of ABFs[6]:

Definition 8. A *simple contrapositive* ABF is an assumption-based framework $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where $\mathfrak{L}$ is an explosive and contrapositive logic, and $\sim\psi = \{\neg\psi\}$.

Note 4. As shown in [32], for simple contrapositive ABFs the closure requirement in Definition 6 is redundant. We shall therefore disregard it in what follows (see also Section 4.3 below).

3. Priority settings and prioritized ABFs

Sometimes it is useful to extend ABFs with a numeric information for representing preferences among assumptions. The extended structures are called *prioritized ABFs*. In this section we define such structures.

Definition 9. Let Ab be a set of formulas (which, in our case, will be the defeasible assumptions of an ABF).

- An *allocation function* on Ab is a total function $g : Ab \to \mathbb{N}$.
- A *numeric aggregation function* f is a total function that maps multisets of natural numbers into non-negative real numbers, such that $\forall x \in \mathbb{N}\ f(\{x\}) = x$.[7] We also assume that an aggregation function is $\subseteq$-*coherent* in its values, namely, it is either non-decreasing with respect to the subset relation ($f(X') \leq f(X)$ whenever $X' \subseteq X$) or non-increasing with respect to the subset relation ($f(X') \geq f(X)$ whenever $X' \subseteq X$).[8]
- A *priority (or preference) setting* on Ab is a pair $\mathscr{P} = \langle g, f \rangle$, where g is an allocation function on Ab and f is a numerical aggregation function.

[5] Clearly, the well-founded extension of an ABF is unique.
[6] See [32,33,35] for a justification of this choice.
[7] In what follows, the set signs of a singleton will sometimes be omitted.
[8] Coherence is needed, for instance, in Lemma 1, which states upper and lower bounds on the p-attacking values (see below). Particular kinds of coherence are also necessary for assuring other properties, such as avoiding the drowning effect (see Proposition 16).

An allocation function makes preferences among the defeasible information. The sets $Ab_i = \{\psi \in Ab \mid g(\psi) = i\}$ form a partition of Ab, which in turn may be viewed as a stratified set. This is sometimes denoted by $Ab = Ab_1 \oplus \ldots \oplus Ab_n$. Aggregation functions are then used for aggregating the preferences. The maximum, minimum, and the summation functions are common aggregation functions.

To demonstrate how real-life situations may be represented by an ABF with preferences, we consider the following two very simple scenarios:

Example 4. A local pizzeria has a special offer in which for a fixed price one topping, mushrooms or pepper, as well as extra cheese, may be added to the pizza for no extra charge. This may be modeled as follows:

$$\mathsf{ABF} = \langle \mathsf{CL}, \{\neg\mathtt{mushroom} \vee \neg\mathtt{pepper}\}, \{\mathtt{mushroom}, \mathtt{pepper}, \mathtt{cheese}\}, \neg\rangle.$$

A preference of pepper over mushrooms, and of mushrooms and pepper over cheese is represented by the allocation function $g(\mathtt{pepper}) = 1$, $g(\mathtt{mushroom}) = 2$, $g(\mathtt{cheese}) = 3$ together with the aggregation function $f = \max$.

Example 5. Suppose that the faculty of humanities and the faculty of medicine are computing for internal funds: a doctoral program ($\mathtt{phd}$) for the former and a new piece of equipment ($\mathtt{equip}$) for the latter. As resources are limited, the funding of both proposals is hard to get ($\neg\mathtt{phd} \vee \neg\mathtt{equip}$). Thus, we have:

$$\mathsf{ABF} = \langle \mathsf{CL}, \emptyset, \{\neg\mathtt{phd} \vee \neg\mathtt{equip}, \mathtt{phd}, \mathtt{equip}\}, \neg\rangle.$$

Assuming, further, that the faculty of medicine is more prestigious than the faculty of humanities, and that such considerations are taken into account in the budgeting, we may have the following allocation function: $g(\neg\mathtt{phd} \vee \neg\mathtt{equip}) = 1$, $g(\mathtt{equip}) = 2$ and $g(\mathtt{phd}) = 3$. This time, $f = \min$ seems to be a useful aggregation function for the situation above.

To assure some nice properties of our setting, we require that the range of the allocation function should be linearly ordered (while in other frameworks, like ABA$^+$ [22], any preorder is permitted).[9] Yet, the aggregation of the allocations in our case is more general than that of ABA$^+$, for instance, which allows only comparisons by max-values (called there *weakest link*).

Notation 1. To ease the notations we will sometimes write $f(g(\Delta))$ instead of $f(\{g(\psi_1), \ldots, g(\psi_n)\})$ (where $\Delta = \{\psi_1, \ldots, \psi_n\}$). Also, we shall sometimes write $\Delta_1 \preceq_{\mathscr{P}} \Delta_2$, or just $\Delta_1 \preceq_f \Delta_2$ when g is arbitrary, to denote that $f(g(\Delta_1)) \leq f(g(\Delta_2))$. This intuitively indicates that Δ_1 is at least as preferred as Δ_2.

Next, we consider some useful properties of priority settings.

Definition 10. The following properties are defined with respect to every nonempty set Δ of formulas and a formula ϕ. A priority setting $\mathscr{P} = \langle g, f\rangle$ is called:

- *reversible*, if when $\Delta \succeq_{\mathscr{P}} \{\phi\}$, there is a $\delta \in \Delta$ such that $\Delta \cup \{\phi\} \setminus \{\delta\} \preceq_{\mathscr{P}} \{\delta\}$.
- max-*upper-bounded*, if $f(g(\Delta)) \leq \max_{\delta\in\Delta}(f(g(\delta)))$.
- max-*lower-bounded*, if $f(g(\Delta)) \geq \max_{\delta\in\Delta}(f(g(\delta)))$.

Two of these properties are in fact related:

Proposition 1. *If a priority setting is max-upper-bounded, it is also reversible.*

Proof. Suppose that $\mathscr{P} = \langle g, f\rangle$ is max-upper-bounded and that $f(g(\Delta)) \geq f(g(\phi))$. Since $\mathscr{P}$ is max-upper-bounded, for every $\delta' \in \Delta$ such that $g(\delta') = \max_{\delta\in\Delta}(g(\delta))$, $f(g(\delta')) \geq f(g(\Delta))$. Let δ_m be such an element. By transitivity of $\geq$, we get $f(g(\delta_m)) \geq f(g(\phi))$. Suppose now towards a contradiction that $f(g(\Delta \cup \{\phi\} \setminus \delta_m)) > f(g(\delta_m))$. Suppose first that $g(\phi) \neq \max(\{g(\psi) \mid \psi \in \Delta \cup \{\phi\} \setminus \delta_m\})$. Then since $g(\delta_m) = \max_{\delta\in\Delta}(g(\delta))$ and $g(\phi) \neq \max(\{g(\psi) \mid \psi \in \Delta \cup \{\phi\} \setminus \delta_m\})$, every $\psi \in \Delta \cup \{\phi\} \setminus \delta_m$ for which $g(\psi) = \max(\{g(\lambda) \mid \lambda \in \Delta \cup \{\phi\} \setminus \delta_m\})$ is an element of Δ. This means that $f(g(\delta_m)) \geq f(g(\psi))$. Since $\mathscr{P}$ is max-upper-bounded, for such a ψ we have: $f(g(\psi)) \geq f(g(\Delta \cup \{\phi\} \setminus \delta_m))$. Together with the supposition that $f(g(\Delta \cup \{\phi\} \setminus \delta_m)) > f(g(\delta_m))$, this leads to $f(g(\psi)) > f(g(\delta_m))$, which is a contradiction to $f(g(\delta_m)) \geq f(g(\psi))$. Suppose then that $g(\phi) = \max(\{g(\psi) \mid \psi \in \Delta \cup \{\phi\} \setminus \delta_m\})$. Since $\mathscr{P}$ is max-upper-bounded, $f(g(\phi)) \geq f(g(\Delta \cup \{\phi\} \setminus \delta_m))$. But then by transitivity, with the assumption that $f(g(\Delta \cup \{\phi\} \setminus \delta_m)) > f(g(\delta_m))$, we get $f(g(\phi)) > f(g(\delta_m))$, which contradicts the fact that $f(g(\delta_m)) \geq f(g(\phi))$. Thus, $f(g(\Delta \cup \{\phi\} \setminus \delta_m)) \leq f(g(\delta_m))$. □

[9] The linearity of the allocation function is inherent in the proofs of several propositions in what follows. Checking whether this assumption can be relaxed (and if so, under what conditions), is a subject of an ongoing work.

Note 5. In the proof of Proposition 1 we actually established a stronger fact: if $\mathscr{P}$ is max-upper-bounded and $\Delta \succeq_{\mathscr{P}} \phi$, there is a $\delta' \in \Delta$ such that $f(g(\delta')) = \max_{\delta\in\Delta} f(g(\delta))$, for which $\Delta \cup \{\phi\} \setminus \delta' \preceq_{\mathscr{P}} \delta'$.

Example 6. It is easy to see that for every allocation function g, the priority settings $\mathsf{Min} = \langle g, \min\rangle$ and $\mathsf{Max} = \langle g, \max\rangle$ are max-upper-bounded. Moreover, every priority setting with a non-increasing aggregation function is max-upper-bounded. Indeed, let $\delta' \in \Delta$. By the definition of a numeric aggregation function f and since it is non-increasing, we have that for every allocation function g it holds that: $f(g(\Delta)) \leq f(g(\{\delta'\})) = g(\delta') \leq \max_{\delta\in\Delta}(g(\delta)) = \max_{\delta\in\Delta}(f(g(\delta)))$.

Example 7. By Proposition 1, the priority settings in Example 6 are also reversible.

Example 8. It is easy to see that for every allocation function g, the priority setting $\mathsf{Max} = \langle g, \max\rangle$ and $\mathsf{Sum} = \langle g, \Sigma\rangle$ are max-lower-bounded. Moreover, every priority setting with a non-decreasing aggregation function is max-lower-bounded. Indeed, let $\delta' \in \Delta$. By the definition of a numeric aggregation function f and since it is non-decreasing, we have that for every allocation function g it holds that $f(g(\Delta)) \geq f(g(\{\delta'\})) = g(\delta')$. In particular, the latter holds for the element $\delta \in \Delta'$ that has the maximal g-value among all the elements in Δ, and so $f(g(\Delta)) \geq \max_{\delta\in\Delta}(g(\delta))$. Since $f(\{x\}) = x$, we get $f(g(\Delta)) \geq \max_{\delta\in\Delta}(f(g(\delta)))$.

For an example of a priority setting that is max-lower-bounded and reversible yet not max-upper-bounded, consider $\mathsf{Max}^+ = \langle g, \max^+\rangle$, where $\max^+(\{x\}) = x$ and $\max^+(\Delta) = \max(\Delta) + 1$ if Δ is not singleton.

We note, in particular, that max is the only aggregation function for which a priority setting is both max-upper-bounded and max-lower-bounded:

Proposition 2. *A priority setting $\mathscr{P} = \langle g, f\rangle$ is both max-lower-bounded and max-upper-bounded iff $f = \max$.*

Proof. Both $f(g(\Delta)) \leq \max_{\delta\in\Delta}(f(g(\delta)))$ and $f(g(\Delta)) \geq \max_{\delta\in\Delta}(f(g(\delta)))$ hold iff $f(g(\Delta)) = \max_{\delta\in\Delta}(f(g(\delta))) = \max f(g(\Delta))$. □

Prioritized ABFs are defined now as follows:

Definition 11. A *prioritized assumption-based framework* (prioritized ABF, or pABF, for short) is a pair $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$, where ABF is an assumption-based argumentation framework and $\mathscr{P}$ is a priority setting.

pABFs are similar to (non-prioritized) ABFs, except that the priorities are taken into account when defining attacks (cf. Definition 5). A prioritized ABF is called reversible (respectively, max-upper-bounded, max-lower-bounded), if so is its priority setting.

Definition 12. Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ be a prioritized ABF with $\mathscr{P} = \langle g, f\rangle$. Let also $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$. Suppose that Δ attacks ψ (in the sense of Definition 5). The *$\mathscr{P}$-attacking value* of Δ on ψ is

$$\mathsf{val}_{f,g}(\Delta, \psi) = \min\{f(g(\Delta')) \mid \Delta' \text{ is a } \subseteq\text{-minimal subset of } \Delta \text{ that attacks } \psi\}.$$

We say that Δ *p-attacks* ψ iff Δ attacks ψ (in the sense of Definition 5), and $\mathsf{val}_{f,g}(\Delta, \psi) \leq f(g(\psi))$. Again, we say that Δ p-attacks Θ if Δ p-attacks some $\psi \in \Theta$.

Thus, a set of assumptions Δ p-attacks an assumption ψ iff Δ attacks ψ and the attacking value of Δ is less than or equal to the $\mathscr{P}$-valued of ψ. The attacking value of Δ is determined according to the $\mathscr{P}$-value of the $\subseteq$-smallest subsets of Δ that attacks ψ. To always allow p-attacks on sets that are $\vdash$-inconsistent, we assume that in every priority setting it holds that $f(g(\emptyset)) = 0$ (and so, for every $\vdash$-contradictory formula ψ, $\mathsf{val}_{f,g}(\emptyset, \psi) = 0$, thus $\emptyset$ p-attacks ψ regardless of the g-value of the latter).

Example 9. Consider again the prioritized ABF from Example 4. Then, for example, $\mathsf{val}_{f,g}(\{\mathtt{pepper}\}, \mathtt{mushroom}) = 1$ and $\mathsf{val}_{f,g}(\{\mathtt{mushroom}, \mathtt{pepper}, \mathtt{cheese}\}, \mathtt{mushroom}) = 1$, while $\mathsf{val}_{f,g}(\{\mathtt{mushroom}\}, \mathtt{pepper}) = 2$. If follows, for instance, that $\{\mathtt{pepper}\}$ p-attacks $\{\mathtt{mushroom}\}$, but not vice-versa.

Note 6. A simplified version of the attacks in Definition 12 could be the following: a set $\Delta = \{\psi_1, \ldots, \psi_n\} \subseteq Ab$ p-attacks ψ iff Δ attacks ψ (in the sense of Definition 5) and $f(g(\Delta)) \leq f(g(\psi))$. However, this alternative definition of p-attacks has some unintuitive consequences. To see this, consider again the ABF of Example 2 (and Fig. 1) with the allocation function $g(p) = 1$, $g(\neg p) = 2$, $g(q) = 3$, and the aggregation function $f = \max$. Note that in this case, the set $\Delta_1 = \{\neg p\}$ does not max-attack the set $\Delta_2 = \{p, q\}$, because the attacked formula in Δ_2 (i.e., p) is of higher precedence than the attacking formula ($\neg p$) in Δ_1. Now, according to the alternative definition of p-attacks in this note, Δ_2 does not max-attack Δ_1 either, since it has a formula (q) which is of a lower precedence (that is, its g-value is higher) than the attacked formula in Δ_1. The latter seems to be counter-intuitive, since the attack of Δ_2 on Δ_1 is 'blocked' by a formula which is 'irrelevant' to the attack.

In contrast to this, Δ_2 does max-attack Δ_1 according to Definition 12, as expected, since its attacking value on Δ_1 is 1, which is smaller than the preference value (2) of the attacked formula in Δ_1 (Indeed, the attacking value of Δ_2 on $\neg p$ is affected only by the value of the attacking formula p, and so the preference value of the irrelevant formula q in Δ_2 is neutralized). Thus, the attacks in Definition 12 take into consideration only the preference values of the formulas that are relevant to the attack. A major advantage of this is considered in Lemma 2 below.

The following lemmas will be useful in what follows.

Lemma 1. *For every $\Delta \subseteq Ab$ and $\psi \in Ab$ it holds that:*

1. *if f is non-decreasing then $\mathsf{val}_{f,g}(\Delta, \psi) \leq f(g(\Delta))$,*
2. *if f is non-increasing then $f(g(\Delta)) \leq \mathsf{val}_{f,g}(\Delta, \psi)$.*

Proof. If f is non-decreasing with respect to $\subseteq$, then $f(g(\Delta')) \leq f(g(\Delta))$ for every $\Delta' \subseteq \Delta$. Item 1 of the lemma thus immediately follows from the definition of $\mathsf{val}_{f,g}(\Delta, \psi)$, as the latter is a minimum of values that none of them is bigger than $f(g(\Delta))$. If f is non-increasing with respect to $\subseteq$, then $f(g(\Delta)) \leq f(g(\Delta'))$ for every $\Delta' \subseteq \Delta$, so again, by the definition of $\mathsf{val}_{f,g}(\Delta, \psi)$, it is a minimum over a set of values that are bigger than or equal to $f(g(\Delta))$, thus $f(g(\Delta)) \leq \mathsf{val}_{f,g}(\Delta, \psi)$, which shows Item 2 of the lemma. $\square$

Lemma 2. *If Δ p-attacks Θ, so does any superset of Δ.*

Proof. Suppose that Δ p-attacks Θ. Then there is a $\psi \in \Theta$ such that Δ attacks ψ and $\mathsf{val}_{f,g}(\Delta, \psi) \leq f(g(\psi))$. Suppose now that $\Delta \subseteq \Delta'$. By the monotonicity of $\vdash$, Δ' also attacks ψ. Moreover, by Definition 12, $\mathsf{val}_{f,g}(\Delta', \psi) \leq \mathsf{val}_{f,g}(\Delta, \psi)$ (since the minimum in case of $\mathsf{val}_{f,g}(\Delta', \psi)$ is taken over a larger set that subsumes that of $\mathsf{val}_{f,g}(\Delta, \psi)$), and so $\mathsf{val}_{f,g}(\Delta', \psi) \leq f(g(\psi))$. It follows that Δ' p-attacks ψ and so Δ' also p-attacks Θ. $\square$

Lemma 3. *Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a reversible prioritized ABF, where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ is a simple contrapositive ABF, and let $\Delta \subseteq Ab$ be a conflict-free set of assumptions. Then for no $\delta \in \Delta$ it holds that $\Gamma, \Delta \vdash \neg\delta$.*

Proof. We show this by induction on the size of Δ. For the base case, let $\Delta = \{\delta\}$. Since $\Gamma, \delta \vdash \neg\delta$ and since for any aggregation function f, $f(g(\delta)) = g(\delta)$, $\{\delta\}$ p-attacks δ, a contradiction to Δ being conflict-free.

For the induction step, suppose that the lemma holds for any proper subset of Δ. Suppose towards a contradiction that $\Gamma, \Delta \vdash \neg\delta$ for some $\delta \in \Delta$. If $f(g(\Delta)) \leq f(g(\delta))$, then Δ p-attacks δ, contradicting the assumption that Δ is conflict-free. Suppose then that $f(g(\Delta)) \geq f(g(\delta))$. Since $\mathscr{P}$ is reversible, there is some $\gamma \in \Delta$ such that $f(g(\Delta \setminus \gamma \cup \delta)) \leq f(g(\gamma))$. Since ABF is contrapositive, $\Gamma, \Delta \setminus \gamma \cup \delta \vdash \neg\gamma$. Suppose first that there is no proper subset $\Theta \subset \Delta \setminus \gamma \cup \delta$ s.t. $\Gamma, \Theta \vdash \neg\gamma$. Then $f(g(\Delta \setminus \gamma \cup \delta)) = \mathsf{val}_{f,g}(\Delta \setminus \gamma \cup \delta, \gamma)$ and thus $\mathsf{val}_{f,g}(\Delta \setminus \gamma \cup \delta, \gamma) \leq f(g(\gamma))$, which means that $\Delta \setminus \gamma \cup \delta$ p-attacks γ, contradicting Δ being conflict-free. Suppose now that there is some $\Theta \subset \Delta \setminus \gamma \cup \delta$ s.t. $\Gamma, \Theta \vdash \neg\gamma$. Since $\delta \in \Delta$, it holds that $\Delta \setminus \gamma \cup \delta = \Delta \setminus \gamma$. Thus, $\Theta \cup \gamma \subset \Delta$. Since Δ is conflict-free, $\Theta \cup \gamma$ is conflict-free. But then $\Gamma, \Theta \cup \gamma \vdash \neg\gamma$ is a contradiction to the inductive hypothesis. $\square$

Lemma 4. *Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a reversible prioritized ABF, where $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$ is a simple contrapositive ABF, and let $\Delta \subseteq Ab$ be a conflict-free set of assumptions. If Δ attacks ψ, then either Δ p-attacks ψ, or there is a $\delta \in \Delta$ such that $\Delta \setminus \{\delta\} \cup \{\psi\}$ p-attacks δ.*

Proof. Since Δ attacks ψ, we have that $\Gamma, \Delta \vdash \neg\psi$. Consider a $\subseteq$-minimal set $\Delta' \subseteq \Delta$ such that $\Gamma, \Delta' \vdash \neg\psi$ and for every $\subseteq$-minimal $\Delta'' \subseteq \Delta$ such that $\Gamma, \Delta'' \vdash \neg\psi$, $f(g(\Delta'')) \geq f(g(\Delta'))$. Clearly, $f(g(\Delta')) = \mathsf{val}_{f,g}(\Delta, \psi)$. Now, if $f(g(\Delta')) \leq f(g(\psi))$ then $\mathsf{val}_{f,g}(\Delta, \psi) \leq f(g(\psi))$, so Δ p-attacks ψ and the lemma follows.

Suppose then that $f(g(\Delta')) > f(g(\psi))$. Since $\mathscr{P}$ is reversible, there is some $\delta \in \Delta'$ such that $f(g(\Delta' \cup \psi \setminus \delta)) \leq f(g(\delta))$. We now show that $\mathsf{val}_{f,g}(\Delta' \cup \psi \setminus \delta, \delta) = f(g(\Delta' \cup \psi \setminus \delta))$ by showing that there is no $\Theta \subset \Delta' \cup \psi \setminus \delta$ s.t. $\Gamma, \Theta \vdash \neg\delta$. Indeed, suppose towards a contradiction that such a Θ exists. Suppose first that $\psi \notin \Theta$. Then $\Theta \subseteq \Delta'$, which with monotonicity means that $\Gamma, \Delta' \vdash \neg\delta$. But this contradicts Lemma 3 and Δ' being conflict-free. Suppose now that $\psi \in \Theta$. Then by contraposition, $\Gamma, \Theta \cup \delta \setminus \psi \vdash \neg\psi$. Since $\Theta \subset \Delta' \cup \psi \setminus \delta$, it holds that $\Theta \cup \delta \setminus \psi \subset \Delta'$, and thus we have a contradiction to the $\subseteq$-minimality of Δ'.

Thus, we have established that $\Delta' \cup \psi \setminus \delta$ is $\subseteq$-minimal such that $\Gamma, \Delta' \cup \psi \setminus \delta \vdash \neg\delta$, and so $\mathsf{val}_{f,g}(\Delta' \cup \psi \setminus \delta, \delta) = f(g(\Delta' \cup \psi \setminus \delta))$. Since $f(g(\Delta' \cup \psi \setminus \delta)) \leq f(g(\delta))$, this means that $\mathsf{val}_{f,g}(\Delta' \cup \psi \setminus \delta, \delta) \leq f(g(\delta))$, hence $\Delta' \cup \psi \setminus \delta$ p-attacks δ. $\square$

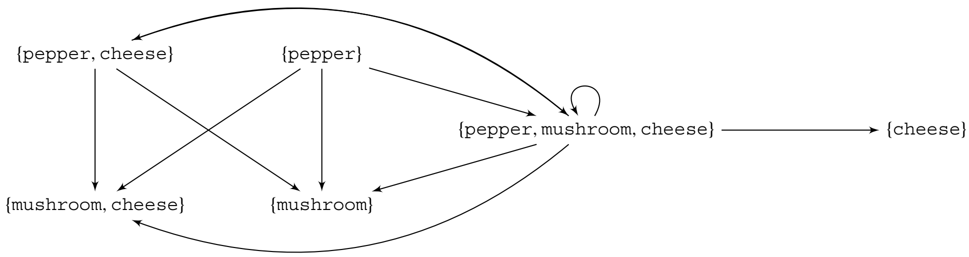

Fig. 2. An attack diagram for Example 10.

Note 7. By Note 5 and by the proof of the last lemma, we get the following corollary: If $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ is a max-upper bounded pABF (where ABF is a simple contrapositive ABF), and $\Delta \subseteq Ab$ is a conflict-free set that attacks ψ, then either Δ p-attacks ψ, or there is a $\delta' \in \Delta$, where $f(g(\delta')) = \max_{\delta \in \Delta} f(g(\delta))$, such that $\Delta \setminus \delta' \cup \{\psi\}$ p-attacks δ'.

The semantic notions in Definition 6 are carried on to the prioritized case in the obvious way. For instance, Δ is *conflict-free* in pABF if no $\Delta' \subseteq \Delta$ p-attacks some $\psi \in \Delta$. Similarly, Δ *defends* in pABF a set $\Delta' \subseteq Ab$ iff for every closed set Θ that p-attacks Δ' there is some $\Delta'' \subseteq \Delta$ that p-attacks Θ.

Again, we denote by $\mathsf{Sem}(\mathsf{pABF})$ the extensions of pABF according to $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, and define the entailments $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ and $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ just as in Definition 7, where pABF replaces ABF.

Example 10. Consider again the prioritized ABF of Example 4. An attack diagram of this pABF is shown in Fig. 2.[10]

Thus $\mathsf{Cmp}(\mathsf{pABF}) = \mathsf{Grd}(\mathsf{pABF}) = \mathsf{WF}(\mathsf{pABF}) = \mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \{\texttt{pepper}, \texttt{cheese}\}$.[11] It follows that $\mathsf{pABF} \mid\!\sim^{*}_{\mathsf{Sem}}$ $\texttt{pepper}$ and $\mathsf{pABF} \mid\!\sim^{*}_{\mathsf{Sem}} \texttt{cheese}$ for every semantics $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$ and every $* \in \{\cup, \cap\}$. Note that in case that the value of $\texttt{cheese}$ is smaller than those of $\texttt{pepper}$ and $\texttt{mushroom}$, the set $\{\texttt{pepper}, \texttt{mushroom}, \texttt{cheese}\}$ does not attack the sets $\{\texttt{cheese}\}$ and $\{\texttt{cheese}, \texttt{mushroom}\}$, in which case the set $\{\texttt{cheese}\}$ also belongs to $\mathsf{Cmp}(\mathsf{pABF})$. In this case $\mathsf{Grd}(\mathsf{pABF}) = \mathsf{WF}(\mathsf{pABF}) = \{\{\texttt{cheese}\}\}$, while $\mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \{\{\texttt{pepper}, \texttt{cheese}\}\}$.

In the next sections we consider some properties of prioritized ABFs: general argumentation-theoretic properties (Section 4), relation to reasoning with maximal consistency (Section 5), and properties that are related to preference handling (Section 6). In what follows, unless otherwise stated, when referring to a prioritized ABF $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ we shall assume that ABF is simple contrapositive.

4. Argumentation-theoretic properties

We start by investigating some general properties regarding the extensions of prioritized ABFs: their inter-relations (Section 4.1) and main characteristics in terms of consistency (Section 4.2) and closure (Section 4.3).

4.1. Relations between the extensions

In this subsection we investigate relations between different semantical concepts. We first examine relations between $\subseteq$-maximizing semantics such as the naive, preferred and stable semantics and then consider relations between $\subseteq$-minimal semantics such as the well-founded and the grounded semantics.

4.1.1. Naive, preferred and stable semantics

In [32,33,35] it is shown that in non-prioritized simple contrapositive ABFs, the sets of naive, preferred and stable extensions coincide (Namely, if ABF is a simple contrapositive assumption-based framework without priorities, then $\mathsf{Naive}(\mathsf{ABF}) = \mathsf{Prf}(\mathsf{ABF}) = \mathsf{Stb}(\mathsf{ABF})$). As the next examples show, when priorities are involved, this is no longer the case: Example 11 shows a situation in which the naive semantics is different than the preferred and the stable semantics, and Example 12 illustrates a case where the preferred semantics is different than the stable semantics.

Example 11. Consider again Examples 4 and 10. The set $\{\texttt{mushroom}, \texttt{cheese}\}$ is maximally conflict-free (thus naive), but it is not even admissible (not to mention preferred or stable), since it does not defend (any of) its elements. In fact, even a simpler example, without $\texttt{cheese}$, will do for our purpose: Let $Ab = \{\texttt{pepper}, \texttt{mushroom}\}$ and $\Gamma = \{\neg\texttt{pepper} \vee \neg\texttt{mushroom}\}$ with the same prioritized setting as before. Then $\{\texttt{mushroom}\}$ is naive but neither preferred nor stable.

[10] Again, we omit from the diagram sets that are not closed.

[11] Note that $\{\texttt{pepper}\}$ is not complete (thus it does not belong to any of the above-mentioned sets), since it defends $\texttt{cheese}$, which is not in $\{\texttt{pepper}\}$.

Example 12. Consider a prioritized ABF with $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \{\neg(p \wedge q \wedge s)\}$ and $Ab = \{p, q, s, \mathsf{F}\}$, where $g(\psi) = 1$ for every $\psi \in Ab$ and $f = \Sigma$. We define $\Sigma\{\emptyset\} = 0$, thus for every ψ it holds that $\mathsf{val}_{f,g}(\emptyset, \psi) = 0$. Note that the emptyset p-attacks every $\Theta \subseteq Ab$ such that $\mathsf{F} \in \Theta$, and no other subset of Ab attacks another subset of Ab.[12] This means that $\{p, q, s\}$ is the only maximally admissible subset of Ab, nevertheless it is not closed, since $\mathsf{F} \notin \{p, q, s\}$. If we restrict our attention to maximally admissible *closed* sets, there are three such sets: $\{p, q\}$, $\{s, q\}$ and $\{p, s\}$. However, these sets are not complete since they do not include an unattacked assumption. For example, $\{p, q\}$ does not include s, even though s is unattacked.

For similar reasons, neither of these sets is stable, since they do not attack the unattacked element in Ab that is not included in them (e.g., $\{p, q\}$ does not p-attack s). Thus, unless further assumptions are posed on the aggregation function (see below), a maximally admissible set might not be complete, preferred extensions might not be stable, and stable extensions might not exist.

Next, we show that, despite of the last example, the sets of preferred and stable extensions still coincide for prioritized simple contrapositive ABFs that are max-upper-bounded (see also Note 9 below).

Proposition 3. *Let* pABF *be a max-upper-bounded prioritized ABF and let* Δ *be a conflict-free set in* Ab. *Then* Δ *is maximally admissible iff it p-attacks any* $\psi \in Ab \setminus \Delta$.

Proof. One direction is clear: if a conflict-free Δ p-attacks any $\psi \in Ab \setminus \Delta$ it must be maximally admissible.[13] Let now Δ be a maximally admissible and suppose towards a contradiction that there is some $\psi \in Ab \setminus \Delta$ s.t. Δ does not p-attack ψ. Let $\{\psi_1, \dots, \psi_n\} = Ab \setminus \Delta$ s.t. $i < j$ if $g(\psi_i) < g(\psi_j)$ (i.e., ψ_1 is among the strongest assumptions that are not in Δ, ψ_2 has the same properties but has weaker or the same strength as ψ_1, and so on). We now construct an admissible set $\Delta^\star$ s.t. $\Delta \subsetneq \Delta^\star$, which contradicts the maximal admissibility of Δ. We define: $\Delta^\star = \bigcup_{i \geq 0} \Delta_i$, where: $\Delta_0 = \Delta$ and for every $0 \leq i \leq n-1$,

$$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{\psi_{i+1}\} & \text{if } \Gamma, \Delta_i \nvdash \neg\psi_{i+1}, \\ \Delta_i & \text{otherwise.} \end{cases}$$

We first show that [C1]: for no $i \geq 0$, if $\psi_i \in \Delta_i$ then $\Gamma, \Delta_i \vdash \neg\psi_i$. The case where $i = 0$ is clear, since Δ is conflict-free. Now, given any $i \geq 0$, suppose towards a contradiction that $(*)$ $\psi_{i+1} \in \Delta_{i+1}$, yet $(**)$ $\Gamma, \Delta_{i+1} \vdash \neg\psi_{i+1}$. By the construction of Δ_{i+1}, $(*)$ means that $\Gamma, \Delta_i \nvdash \neg\psi_{i+1}$. Thus $\Delta_{i+1} \neq \Delta_i$ (otherwise we get a contradiction to $(**)$), i.e., $\Delta_{i+1} = \Delta_i \cup \{\psi_{i+1}\}$, and so $(**)$ means that $\Gamma, \Delta_i, \psi_{i+1} \vdash \neg\psi_{i+1}$. By contraposition, $\Gamma, \Delta_i \setminus \delta, \psi_{i+1} \vdash \neg\delta$ for any $\delta \in \Delta_i$, and by contraposition again $\Gamma, \Delta_i, \vdash \neg\psi_{i+1}$, a contradiction to the assumption that $\Gamma, \Delta_i \nvdash \neg\psi_{i+1}$.

We now show that [C2]: for every $i \geq 0$, Δ_i is conflict-free. We show this by an induction on i. The inductive base is clear since Δ is conflict-free. Suppose now that [C2] holds for Δ_i and suppose towards a contradiction that Δ_{i+1} p-attacks some $\phi \in \Delta_{i+1}$. This means, in particular, that $\Gamma, \Delta_{i+1} \vdash \neg\phi$. If $\psi_{i+1} \notin \Delta_{i+1}$ then $\Delta_i = \Delta_{i+1}$ and by the induction hypothesis $\Delta_i = \Delta_{i+1}$ is conflict-free, so we are done. If $\psi_{i+1} \in \Delta_{i+1}$ then by C1, $\phi \neq \psi_{i+1}$, and so $\phi \in \Delta_i$ (since $\Delta_{i+1} = \Delta_i \cup \{\psi_{i+1}\}$). By contraposition, $\Gamma, (\Delta_{i+1} \setminus \psi_{i+1}), \phi \vdash \neg\psi_{i+1}$. Notice that since $\phi \neq \psi_{i+1}$, we have that $(\Delta_{i+1} \setminus \psi_{i+1}) \cup \phi = \Delta_i$ and thus the last entailment means that $\Gamma, \Delta_i \vdash \neg\psi_{i+1}$, which contradicts $\psi_{i+1} \in \Delta_{i+1}$.

We now show that [C3]: $\Delta^\star$ is admissible. Suppose towards a contradiction that some $\Theta \subseteq Ab$ p-attacks $\Delta^\star$ and $\Delta^\star$ does not p-attack Θ. Since $\Delta^\star$ does not p-attack Θ, and $\Delta \subseteq \Delta^\star$, Δ does not p-attack Θ. Since $\{\psi_1, \dots, \psi_n\}$ contains all the assumptions not p-attacked by Δ, $(\Theta \setminus \Delta^\star) \subseteq \{\psi_1, \dots, \psi_n\}$. Let $\phi \in \Theta \setminus \Delta^\star$ (Note that since by C2, $\Delta^\star$ is conflict-free, $\Theta \nsubseteq \Delta^\star$ and so such ϕ exists). Since $\phi \notin \Delta^\star$ yet $\phi = \psi_k$ for some $1 \leqslant k \leqslant n$, necessarily $\Gamma, \Delta_{k-1} \vdash \neg\phi$. By Lemma 4 (which holds since by [C2] Δ_{k-1} is conflict-free), either Δ_{k-1} p-attacks ϕ, or there is some $\sigma \in \Delta_{k-1}$ such that $\Delta_{k-1} \cup \phi \setminus \sigma$ p-attacks σ. The first case is excluded since $\phi \in \Theta$ and $\Delta^\star$ does not p-attack Θ, thus $\Delta_{k-1} \subseteq \Delta^\star$ does not p-attack Θ. Suppose then that there is some $\sigma \in \Delta_{k-1}$ such that $\Delta_{k-1} \cup \phi \setminus \sigma$ p-attacks σ. Let $\Delta^\dagger \subseteq \Delta_{k-1} \cup \phi \setminus \sigma$ be the $\subseteq$-minimal subset of $\Delta_{k-1} \cup \phi \setminus \sigma$ such that $\Gamma, \Delta^\dagger \vdash \neg\sigma$ and that is $f(g(\cdot))$-minimal among all $\subseteq$-minimal subsets Δ' of $\Delta_{k-1} \cup \phi \setminus \sigma$ such that $\Gamma, \Delta' \vdash \neg\sigma$. By Note 7 we can assume that $f(g(\sigma)) = \max\{f(g(\delta)) \mid \delta \in \Delta_{k-1}\}$. Furthermore, since $\Delta^\dagger \setminus \phi \cup \sigma \subseteq \Delta_{k-1}$, we can assume that: (†) $f(g(\sigma)) = \max\{f(g(\delta)) \mid \delta \in \Delta^\dagger \setminus \phi \cup \sigma\}$.

• Suppose first that $\sigma \notin \Delta$. By construction of Δ_{k-1} and since $\Delta_{k-1} \setminus \Delta \supseteq \Delta^\dagger \setminus (\Delta \cup \phi)$, it holds that $f(g(\phi)) \geq f(g(\sigma))$ (This is the case since $\phi = \psi_k$ for some $1 \leq k \leq n$ and, by construction of Δ_{k-1} and since $\sigma \in \Delta_{k-1}$, $\sigma = \psi_j$ for some $j < k$). We first refute the possibility that $f(g(\phi)) < f(g(\Delta^\dagger \setminus \phi \cup \sigma))$. Indeed, if this is the case, then since by max-upper-boundedness, $f(g(\Delta^\dagger \setminus \phi \cup \sigma)) \leq \max\{f(g(\delta)) \mid \delta \in \Delta^\dagger \setminus \phi \cup \sigma\}$, we get $f(g(\phi)) < \max\{f(g(\delta)) \mid \delta \in \Delta^\dagger \setminus \phi \cup \sigma\}$. Since $f(g(\sigma)) \leq f(g(\phi))$, we have $f(g(\sigma)) < \max\{f(g(\delta)) \mid \delta \in \Delta^\dagger \setminus \phi \cup \sigma\}$. But then (since by (†), $f(g(\sigma)) = \max\{f(g(\delta)) \mid \delta \in \Delta^\dagger \setminus \phi \cup \sigma\}$) we get $f(g(\sigma)) < f(g(\sigma))$, a contradiction. Thus, $f(g(\phi)) \geq f(g(\Delta^\dagger \setminus \phi \cup \sigma))$. By contraposition, $\Gamma, \Delta^\dagger \setminus \phi \cup \sigma \vdash \neg\phi$.[14] Notice

[12] Indeed, consider $\{p, q\}$. Even though $\Gamma \cup \{p, q\} \vdash \neg s$, we have that $\mathsf{val}_{f,g}(\{p, q\}, s) = f(g(\{p, q\})) = g(p) + g(q) = 2 > g(s) = 1$, thus s is not p-attacked by $\{p, q\}$. By symmetric considerations, $\{p, s\}$ does not p-attack q and $\{q, s\}$ does not p-attack p.

[13] For regular attacks, this is actually a known fact from [26].

[14] Here we rely on the fact that $\phi \in \Delta^\dagger$. This indeed is the case, since otherwise $\Delta^\dagger \subseteq \Delta_{k-1}$ which contradicts, with $\sigma \in \Delta_{k-1}$, that Δ_{k-1} is conflict-free (by [C2]).

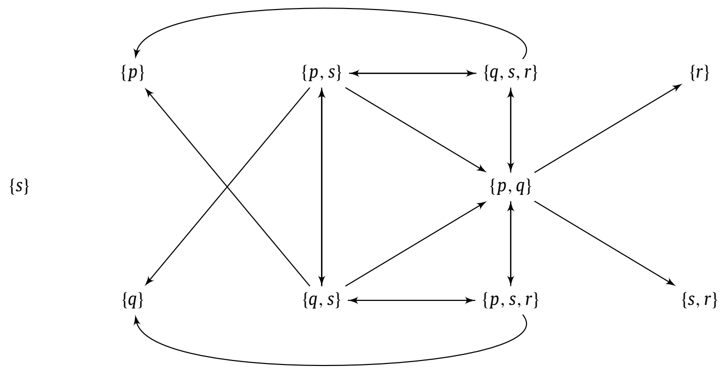

Fig. 3. An attack diagram for Example 13.

that there is no $\Delta^\ddagger \subset \Delta^\dagger \setminus \phi \cup \sigma$ s.t. $\Gamma, \Delta^\ddagger \vdash \neg\phi$. Otherwise, with contraposition, we would have that $\Gamma, \Delta^\ddagger \setminus \sigma \cup \phi \vdash \neg\sigma$. Since $\Delta^\ddagger \setminus \sigma \cup \phi \subset \Delta^\dagger$, this would contradict the minimality of $\Delta^\dagger$ which we assumed above. Thus, by definition, $\mathsf{val}_{f,g}(\Delta^\dagger \setminus \phi \cup \sigma)) = f(g(\Delta^\dagger \setminus \phi \cup \sigma))$. Since $f(g(\phi)) \geq \max(f(g(\Delta^\dagger \setminus \phi \cup \sigma))$, $\Delta^\dagger$ p-attacks ϕ and thus by Lemma 2, $\Delta^\star$ p-attacks Θ, a contradiction to the choice of Θ.

- Suppose now that $\sigma \in \Delta$. Since $\sigma \in \Delta$ and Δ is admissible, Δ p-attacks $(\Delta^\star \setminus \sigma) \cup \phi$. If Δ p-attacks $(\Delta^\star \setminus \sigma) \cup \phi$ in some $\varphi \neq \phi$, then $\Delta^\star$ would p-attack $\varphi \in \Delta^\star$, contradicting [C2]. Thus, Δ p-attacks ϕ, contradicting our assumption on ϕ.

We finally show that [C4]: $\Delta \subsetneq \Delta^\star$. Suppose towards a contradiction that $\Delta = \Delta_1$. This means that $\Gamma, \Delta \vdash \neg\psi_1$. By Lemma 4, since Δ is conflict-free and it does not p-attack ψ_1, there is a $\phi \in \Delta$ s.t. $\Delta \cup \psi_1 \setminus \phi$ p-attacks ϕ (and $\phi \neq \psi_1$). Since Δ is admissible, Δ p-attacks some $\sigma \in (\Delta \setminus \phi) \cup \psi_1$. Since Δ is conflict-free, $\sigma = \psi_1$, which contradicts the assumption that Δ does not p-attack ψ_1. We thus conclude that $\Delta \subsetneq \Delta_1 \subseteq \Delta^\star$.

By [C3] and [C4] we get a contradiction to the maximal admissibility of Δ. □

4.1.2. Well-founded and grounded semantics

We now turn to the grounded and the well-founded semantics. As the next example shows, the grounded semantics is not always unique (unlike, e.g., in abstract argumentation frameworks), and so it does not necessarily coincide with the well-founded semantics (which is unique by its definition).

Example 13. Consider a pABF with $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \{p \wedge q \supset \neg s, r \supset s, s \supset r\}$, $Ab = \{s, p, q, r\}$, $g(s) = 1$, $g(p) = g(q) = 2$, $g(r) = 3$ and $f = \max$. The p-attack diagram of this pABF is shown in Fig. 3.

In this example, there is no unique minimal complete extension: $\{s\}$ is not attacked, but it is not closed since $\Gamma, \{s\} \vdash r$. Also, $\{r, s\}$ does not defend itself from $\{p, q\}$. This pABF has two minimal complete extensions, $\{p, s, r\}$ and $\{q, s, r\}$, which are also preferred and stable. It follows that the well-founded extension in this case is $\{r, s\}$, and so this example also shows that in pABFs the grounded extensions and the well-founded extensions do now always coincide.

Even though there may exist more than one grounded extension for a pABF, the (unique) well-founded extension of a pABF equals to the intersection of all the grounded extensions:

Proposition 4. *Let* pABF *be a prioritized ABF. Then* $\mathsf{WF}(\mathsf{pABF}) = \bigcap \mathsf{Grd}(\mathsf{pABF})$.

Proof. The fact that $\mathsf{WF}(\mathsf{pABF}) = \bigcap \mathsf{Cmp}(\mathsf{pABF}) \subseteq \bigcap \mathsf{Grd}(\mathsf{pABF})$ immediately follows from the fact that by definition, $\mathsf{Grd}(\mathsf{pABF}) \subseteq \mathsf{Cmp}(\mathsf{pABF})$. For the converse, note that every element in $\bigcap \mathsf{Grd}(\mathsf{pABF})$ belongs to every $\subseteq$-minimal complete extension of pABF, and so it belongs to every complete extension (not necessarily minimal) of pABF, thus $\bigcap \mathsf{Grd}(\mathsf{pABF}) \subseteq \bigcap \mathsf{Cmp}(\mathsf{pABF}) = \mathsf{WF}(\mathsf{pABF})$. □

By Proposition 4 we thus have the following result:

Corollary 1. *The grounded and the well-founded semantics of* pABF *coincide iff* pABF *has a unique grounded extension.*

Note 8. In [32,33,35] it is shown that in the non-prioritized case, when $\mathsf{F} \in Ab$, the grounded and the well-founded semantics coincide and are unique. As Example 12 shows, in prioritized ABFs this is no longer the case.

We conclude this subsection by a few words on the construction of grounded extensions: As is well-known, in abstract argumentation frameworks the (unique) grounded extension can be constructed by first computing the set $\mathscr{G}_0(\mathsf{ABF})$ of the

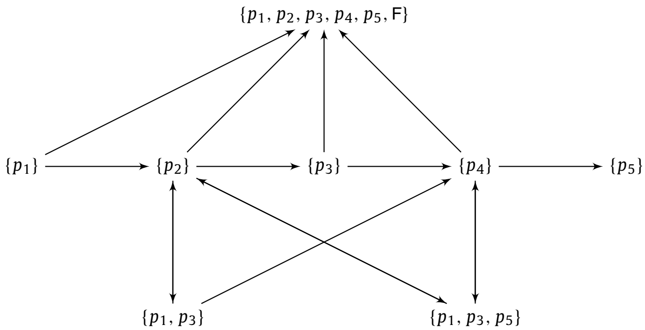

Fig. 4. An attack diagram for Example 14.

non-attacked arguments and then, for every $i \geq 0$, iteratively computing the set $\mathscr{G}_{i+1}(\mathsf{ABF})$ that is the union of $\mathscr{G}_i(\mathsf{ABF})$ and whatever is defended by $\mathscr{G}_i(\mathsf{ABF})$, until reaching a fixpoint. The validity of this process is based on what is known as 'Dung's fundamental lemma', stating that if Δ is admissible and defends ψ, then $\Delta \cup \{\psi\}$ is also admissible. In [32,33,35] it is shown that when $\mathsf{F} \in Ab$, this process is valid also for (non-prioritized) simple contrapositive ABFs, and moreover, it requires no more than two iterative steps (so $\mathscr{G}_1(\mathsf{ABF})$ is already the grounded semantics of ABF). However,

1. As Example 12 shows, in general Dung fundamental lemma does not hold for prioritized ABFs.
2. As the next example shows, the construction of grounded extensions may require more than two iterative steps.

Example 14. Let $Ab = \{p_1, p_2, p_3, p_4, p_5, \mathsf{F}\}$, $\Gamma = \{\neg(p_1 \wedge p_2), \neg(p_2 \wedge p_3), \neg(p_3 \wedge p_4), \neg(p_4 \wedge p_5)\}$, and let $g(p_i) = i$, $g(\mathsf{F}) = 1$, $f = \max$. Part of the corresponding p-attack diagram is shown in Fig. 4.[15] Here, $\mathscr{G}_0(\mathsf{pABF}) = \{p_1\}$, $\mathscr{G}_1(\mathsf{pABF}) = \{p_1, p_3\}$ and $\mathscr{G}_2(\mathsf{pABF}) = \{p_1, p_3, p_5\}$. The latter is the grounded extension in this case.

3. It is an open question whether there is an iterative procedure for constructing all the grounded extensions of a prioritized ABF, which is similar to the fixpoint construction of the grounded extension of an ABF (when $\mathsf{F} \in Ab$).

4.2. Consistency of extensions

We now show that when prioritized ABFs are reversible, the extensions are consistent.

Proposition 5. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a reversible prioritized ABF. Then* pABF *satisfies the following consistency postulate [16]: There is no conflict-free set* $\Delta \subseteq Ab$ *such that* $\Gamma, \Delta \vdash \neg\psi$ *for some* $\psi \in \Delta$.

Proof. Suppose for a contradiction that $\Gamma, \Delta \vdash \neg\psi$ for some conflict free $\Delta \subseteq Ab$ and $\psi \in \Delta$. By Lemma 4, either Δ p-attacks ψ or there is a $\delta \in \Delta$ such that $\Delta \setminus \delta \cup \{\psi\}$ p-attacks δ. Since $\Delta \setminus \delta \cup \{\psi\} \subset \Delta$, in both cases we get a contradiction to the assumption that Δ is conflict-free. □

Consistency now immediately follows from the last proposition:

Corollary 2. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a reversible prioritized ABF and let* Δ *be a conflict-free subset of* AB*. Then* $\Gamma, \Delta \nvdash \mathsf{F}$.

Proof. Follows from Proposition 5: If $\Gamma, \Delta \vdash \mathsf{F}$ then since $\mathsf{F} \vdash \neg\psi$ for every $\psi \in \Delta$, by transitivity we get that $\Gamma, \Delta \vdash \neg\psi$ for every such ψ, contradicting Proposition 5. □

The next example shows that the reversibility requirement from the aggregation function in Proposition 5 (and in Lemma 3) is indeed necessary.

Example 15. Consider a variation of Example 12 where F is removed from Ab, namely: $\Gamma = \{\neg(p \wedge q \wedge s)\}$, $Ab = \{p, q, s\}$, $g(\delta) = 1$ for every $\delta \in Ab$, and $f = \Sigma$. Clearly, $\mathscr{P} = \langle g, f \rangle$ is not reversible (for instance, $\{p, q\} \succeq_{\mathscr{P}} s$, yet neither $\{p, s\} \preceq_{\mathscr{P}} q$ nor $\{q, s\} \preceq_{\mathscr{P}} p$). Also, similar considerations as in Example 12 show that there is no p-attack in this example. Thus, there is one maximally admissible set: Ab. However, this set is not consistent. Thus, consistency can be violated when f is not reversible.

[15] To keep the figure readable, we do not explicitly mention all the $2^{|Ab|}$ possible sets, but only those that are most relevant for the example (a similar remark applies to other figures along the paper).

4.3. Closure of extensions

Next, we consider the closure requirement from extensions (see Definition 6). As Example 13 shows, this requirement is in general *not* redundant in prioritized ABFs. However, as we show below, under the assumption that the aggregation function is reversible, the closure requirement may be lifted. This result generalizes similar results shown in [32,33,35] for simple contrapositive ABFs without priorities (see also Note 4).

Proposition 6. *Let* pABF $= \langle$ABF$, \mathscr{P}\rangle$ *be a reversible prioritized ABF. Then the closure requirement is redundant in the definition of stable extensions (Definition 6): Any conflict-free $\Delta \subseteq Ab$ that p-attacks every $A \in Ab \setminus \Delta$ is closed.*

Proof. Suppose that Δ p-attacks every $\psi \in Ab \setminus \Delta$, yet $\Gamma, \Delta \vdash \phi$ for some $\phi \in Ab \setminus \Delta$. Since Δ p-attacks ϕ, it holds that $\Gamma, \Delta \vdash \neg\phi$. Thus, by Note 1, we have that $\Gamma, \Delta \vdash$ F, in a contradiction to Corollary 2. □

Proposition 7. *Let* pABF $= \langle$ABF$, \mathscr{P}\rangle$ *be a max-upper-bounded prioritized ABF. Then the closure requirement is redundant in the definition of preferred extensions (Definition 6): Any $\Delta \subseteq Ab$ that is conflict free and maximally admissible is closed.*

Proof. Suppose that $\Delta \subseteq Ab$ is conflict free and maximally admissible. By Proposition 3, Δ attacks every $A \in Ab \setminus \Delta$. By Proposition 6 (which holds in our case by Proposition 1), this means that Δ is closed. □

Note 9. By Propositions 6, 7 and 1, Proposition 3 may be restated as follows: The stable extensions and the preferred extensions of a max-upper-bounded pABF coincide.

5. Representations in terms of preferred maximally-consistent subsets

We now consider the relation between reasoning with prioritized ABFs and reasoning with prioritized maximal consistency, as defined next. In this section we assume that the set Ab of the defeasible assumptions is finite.

Definition 13. Let pABF $= \langle$ABF$, \mathscr{P}\rangle$ be a prioritized ABF.

- $\Delta \subseteq Ab$ is a *maximally consistent set* (MCS) in ABF, if (a) $\Gamma, \Delta \nvdash$ F and (b) $\Gamma, \Delta' \vdash$ F for every $\Delta \subsetneq \Delta' \subseteq Ab$.[16] The set of the maximally consistent sets in ABF is denoted MCS(ABF).
- $\Delta \subseteq Ab$ is a *preferred (or prioritized) maximally consistent set* (pMCS) in pABF, if $\Delta \in$ MCS(ABF) and there is no $\Theta \in$ MCS(ABF) that is $\prec_{\mathscr{P}}$-preferred than Δ. The set of the preferred maximally consistent sets in pABF is denoted $\mathsf{MCS}_{\prec_{\mathscr{P}}}$(ABF) (or just $\mathsf{MCS}_{\mathscr{P}}$(ABF)).

The relation between prioritized argumentation frameworks and reasoning with preferred maximally consistent subsets of the premises has been investigated in several different contexts (see, e.g. [6,8,46] and [31, Chapter 7]). In this section we first show that under certain assumptions, prioritized assumption-based argumentation can represent Brewka's preferred sub-theories [13]. Then we consider some necessary and sufficient conditions on the preference relations for assuring that they can be represented by prioritized ABFs.

5.1. Brewka's preferred sub-theories

First, we consider the case where the priority setting in Definition 13 is defined by Brewka's preference order [13]:

Definition 14. Let $Ab = Ab_1 \oplus \ldots \oplus Ab_n$ (that is, Ab is stratified according to the allocation function g; Recall Section 3), and let $\Delta, \Theta \subseteq Ab$. We say that Δ is *preferred* over Θ (with respect to g), denoted $\Delta \sqsubset_g \Theta$ (or just $\Delta \sqsubset \Theta$ when g is known or arbitrary), iff there is an $1 \leq i \leq n$ such that $Ab_j \cap \Delta = Ab_j \cap \Theta$ for every $1 \leq j < i$, and $Ab_i \cap \Delta \supsetneq Ab_i \cap \Theta$.

Thus, in the notation of Definition 14, Δ is preferred over Θ when both sets have the same $i-1$ stratifications with the g-most preferred formulas, and the i-th stratification of Δ properly contains that of Θ. This is a kind of lexicographic preference in term of the g-values. In turn, this preference can be posed on the maximally consistent subsets of Ab.

Example 16. Consider again the prioritized ABF from Example 4 (see also Example 10). In this case, we have: $\mathsf{MCS}_{\sqsubset_{max}}(\mathsf{ABF}) = \{\{\mathtt{pepper}, \mathtt{cheese}\}\}$. Indeed, $\mathsf{MCS} = \{\{\mathtt{pepper}, \mathtt{cheese}\}, \{\mathtt{mushroom}, \mathtt{cheese}\}\}$ and $\{\mathtt{pepper}, \mathtt{cheese}\} \sqsubset_{max} \{\mathtt{mushroom}, \mathtt{cheese}\}$ since:

$$Ab_1 \cap \{\mathtt{pepper}, \mathtt{cheese}\} = \{\mathtt{pepper}\} \supsetneq \emptyset = Ab_1 \cap \{\mathtt{mushroom}, \mathtt{cheese}\}.$$

[16] In what follows, (a) is called the consistency condition and (b) is the maximality condition.

To see the relation between prioritized argumentation frameworks and reasoning with preferred maximally consistent subsets of the premises we first show two lemmas:

Lemma 5. *Let* pABF *be a max-lower-bounded and reversible pABF with* $\mathscr{P} = \langle g, f\rangle$*, and let* Δ *be a stable extension of* pABF*. Then* $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$.

Proof. We first show that $\Delta \in \mathsf{MCS}(\mathsf{ABF})$. To see that Δ is consistent, suppose for a contradiction that $\Gamma, \Delta \vdash \mathsf{F}$. Since $\mathsf{F} \vdash \neg\delta$ for every $\delta \in \Delta$, we get by transitivity and contraposition, that $\Gamma, \Delta \setminus \delta \vdash \neg\delta$ for every $\delta \in \Delta$. Since Δ is conflict-free (because it is stable), by Lemma 4, this means that either $\Delta \setminus \delta$ p-attacks δ or there is some $\delta' \in \Delta$ s.t. $\Delta \setminus \delta'$ p-attacks δ'. In any case, Δ p-attacks itself, which contradicts Δ being conflict-free.

We now show that Δ is maximally consistent. Indeed, since Δ is stable, $\Gamma, \Delta \vdash \neg\psi$ for every $\psi \in Ab \setminus \Delta$. By monotonicity, $\Gamma, \Delta, \neg\mathsf{F} \vdash \neg\psi$, and by contraposition, $\Gamma, \Delta, \psi \vdash \mathsf{F}$ for every $\psi \in Ab \setminus \Delta$. Thus every $\Delta' \subseteq Ab$ that properly contains Δ is not consistent.

We show now that Δ is $\sqsubset$-preferred, i.e., for no $\Theta \in \mathsf{MCS}(\mathsf{ABF})$, $\Theta \sqsubset \Delta$. Suppose for a contradiction that there is such a Θ. This means that there is some $j \geq 1$ such that for every $1 \leq i < j$, $Ab_i \cap \Delta = Ab_i \cap \Theta$, and $Ab_j \cap \Delta \subset Ab_j \cap \Theta$. Let $\psi \in Ab_j \cap (\Theta \setminus \Delta)$. Since $\psi \notin \Delta$ and Δ is stable, Δ attacks ψ. This means that $\Gamma, \Delta \vdash \neg\psi$ and there is some $\Delta' \subseteq \Delta$ such that $\Gamma, \Delta' \vdash \neg\psi$ and $f(g(\Delta')) = \mathsf{val}_{f,g}(\Delta, \psi) < f(g(\psi)) = g(\psi) = j$. Now, by the max-lower-boundedness of $\langle g, f\rangle$ we have that $f(g(\Delta')) \geq \max(g(\Delta'))$, thus $j \geq \max(g(\Delta'))$. It follows that $\Delta' \subseteq (\bigcup_{1 \leq i \leq j} Ab_i \cap \Delta) \subseteq (\bigcup_{1 \leq i \leq j} Ab_i \cap \Theta) \subseteq \Theta$. But then $\Gamma, \Theta \vdash \neg\psi$. On the other hand, since $\psi \in \Theta$, necessarily $\Gamma, \Theta \vdash \psi$. Thus, by Note 1, $\Gamma, \Theta \vdash \mathsf{F}$, a contradiction to $\Theta \in \mathsf{MCS}(\mathsf{ABF})$. $\square$

Lemma 6. *Let* pABF *be a max-upper-bounded pABF with* $\mathscr{P} = \langle g, f\rangle$*, and let* $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$*. Then* Δ *is a stable extension of* pABF.

Proof. We first show that Δ is conflict-free. Suppose towards a contradiction that Δ p-attacks some $\delta \in \Delta$. This means, in particular, that $\Gamma, \Delta \vdash \neg\delta$. But then, since by reflexivity $\Gamma, \Delta \vdash \delta$, Note 1 implies that $\Gamma, \Delta \vdash \mathsf{F}$, a contradiction to $\Delta \in \mathsf{MCS}_{\sqsubset}(\mathsf{ABF}) \subseteq \mathsf{MCS}(\mathsf{ABF})$.

We now show that Δ p-attacks every $\psi \in Ab \setminus \Delta$. Let $\psi \in Ab \setminus \Delta$. Since $\Delta \in \mathsf{MCS}_{\sqsubset}(\mathsf{ABF}) \subseteq \mathsf{MCS}(\mathsf{ABF})$, $\Delta \cup \{\psi\}$ is inconsistent in pABF. Thus, $\Gamma, \Delta, \psi \vdash \mathsf{F}$, and since in particular $\mathsf{F} \vdash \neg\delta$ for $\delta \in \Delta$, by transitivity, monotonicity and contraposition we get $\Gamma, \Delta \vdash \neg\psi$. If $\mathsf{val}_{f,g}(\Delta, \psi) \leq f(g(\psi))$ we are done. Suppose now that $\mathsf{val}_{f,g}(\Delta, \psi) > f(g(\psi))$. We show that this leads to a contradiction. Let $\Delta_1, \ldots, \Delta_n$ be all the subsets of Δ such that $\Gamma, \Delta_i \vdash \neg\psi$ and for no $\Delta'_i \subset \Delta_i$ it holds that $\Gamma, \Delta'_i \vdash \neg\psi$. (Note that since Ab is finite so is Δ and thus there are a finite number of such sets). Let $\delta_i \in \{\sigma_i \in \Delta_i \mid g(\sigma_i) = \max(g(\Delta_i)\}$ for $i = 1, \ldots, n$.

We show that $(\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi$ is consistent. Suppose towards a contradiction that $\Gamma, (\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi \vdash \mathsf{F}$ and let $\delta \in \Delta$. Then $\mathsf{F} \vdash \neg\delta$, and by transitivity, $\Gamma, (\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi \vdash \neg\delta$. Thus, by contraposition, $\Gamma, \Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i \vdash \neg\psi$. Thus, there is some $\Delta' \subseteq \Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i$ s.t. $\Gamma, \Delta' \vdash \neg\psi$ and for no $\Delta'' \subseteq \Delta'$ it holds that $\Gamma, \Delta'' \vdash \neg\psi$. Since $\Delta' \subseteq \Delta$, necessarily $\Delta' = \Delta_i$ for some $1 \leqslant i \leqslant n$. But this contradicts $\delta_i \notin \Delta' \subseteq \Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i$. Thus, $(\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi$ is consistent.

Let now $\Theta \in \mathsf{MCS}(\mathsf{ABF})$ be a set such that $(\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi \subseteq \Theta$. We show that $\Theta \sqsubset \Delta$. If there is a $\phi \in \Theta \setminus (\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi$ s.t. $g(\phi) \leq g(\psi)$ we are done (since then for the $\phi \in \Theta \setminus (\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi$ with minimal g-value, $\Delta \cap Ab_i = \Theta \cap Ab_i$ for every $i < g(\phi)$ and $\Delta \cap Ab_{g(\phi)} \subset \Theta \cap Ab_{g(\phi)}$). Suppose then that: (†) there is no $\phi \in \Theta \setminus (\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi$ s.t. $g(\phi) \leq g(\psi)$. In that case, observe that since:

a) we assumed that $\mathsf{val}_{f,g}(\Delta, \psi) > f(g(\psi))$,
b) by max-upper-boundedness, $\max_{\delta \in \Delta_i}(g(\delta)) \geq f(g(\Delta_i))$ (for every $1 \leqslant i \leqslant n$), and
c) by Definition 12, $f(g(\Delta_i)) \geq \mathsf{val}_{f,g}(\Delta, \psi)$ (for every $1 \leqslant i \leqslant n$, since $\Delta_i \subseteq \Delta$),

by transitivity of $>$ we have that $\max_{\delta \in \Delta_i}(g(\delta)) > f(g(\psi))$ for every $1 \leqslant i \leqslant n$. Since $g(\delta_i) \in \max_{\delta \in \Delta_i}(g(\delta))$ for every $1 \leqslant i \leqslant n$, we thus obtain that for every $1 \leqslant i \leqslant n$, $f(g(\delta_i)) = g(\delta_i) > f(g(\psi))$. This means (with (†)) that for every $i < f(g(\psi))$, $\Delta \cap Ab_i = \Theta \cap Ab_i$ and $\Delta \cap Ab_{g(\psi)} \subset \Theta \cap Ab_{g(\psi)} = (\Delta \cap Ab_{g(\psi)}) \cup \psi$. Thus, $\Delta \sqsupset \Theta \in \mathsf{MCS}(\mathsf{ABF})$, which contradicts the assumptions that $\Delta \in \mathsf{MCS}_{\sqsubset}(\mathsf{ABF})$. $\square$

When the aggregation is by the maximum function we can combine the two lemmas above to get a full characterization of the preferred and the stable semantics in terms of preferred maximally consistent sets.

Proposition 8. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ *be a prioritized ABF in which* $\mathscr{P} = \langle g, \max\rangle$ *for some allocation function* g*. Then* $\mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$.

Proof. Since $\mathscr{P} = \langle g, \max\rangle$ is max-upper-bounded, by Proposition 3, $\mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF})$ (see also Note 9). Since it is also max-lower-bounded, by Lemmas 5 and 6, $\mathsf{Stb}(\mathsf{pABF}) = \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$. Altogether, we get the proposition. $\square$

5.2. Necessary and sufficient conditions for representations by pABFs

Next, we consider general conditions on the preference relations that are necessary or sufficient for representation by preferred MCSs. For this, the priority setting of the pABF should clearly be related to the priority order that is supposed to be represented:

Definition 15. A priority setting that is *induced by* a preference relation $\preceq$ on $\wp(Ab)$ is a pair $\mathscr{P}_{\preceq} = \langle g_{\preceq}, f\rangle$, where for every $\psi, \phi \in Ab$ $g(\phi) \leq g(\psi)$ iff $\{\psi\} \preceq \{\phi\}$.[17] We say that $\mathscr{P}_{\preceq}$ is non-decreasing if so is f.

For the results in this section we need the following monotonicity properties:

- $\prec$-monotonicity: If $\Theta = (\Delta \setminus \Delta_1) \cup \Delta_2$ and $\forall \delta_1 \in \Delta_1\ \delta_2 \in \Delta_2\ \{\delta_1\} \prec \{\delta_2\}$, then $\Delta \prec \Theta$.
- $\subseteq$-monotonicity: if $\Delta_1 \subseteq \Delta_2$ then $\Delta_1 \preceq \Delta_2$.
- $\vdash$-monotonicity: if $\Delta_1 \preceq \Delta_2$ then $\{\psi \mid \Delta_1 \vdash \psi\} \subseteq \{\psi \mid \Delta_2 \vdash \psi\}$.

$\prec$-monotonicity intuitively means that the relative $\preceq$-preference of a set Δ cannot downsize by replacing elements in Δ by $\preceq$-preferred ones. $\subseteq$-monotonicity indicates that $\preceq$ cannot decrease by increasing the size of the set, and $\vdash$-monotonicity indicates a correspondence between $\preceq$ and logical consequences. We call *monotonic* a preference relation $\preceq$ on $\wp(Ab)$ that satisfies all the three monotonicity conditions above.

Proposition 9. *Let* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim\rangle$ *be a simple contrapositive ABF and let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}_{\preceq}\rangle$ *be a prioritized ABF, where* $\mathscr{P}_{\preceq}$ *is a priority setting that is induced by a* $\prec$*-monotonic and* $\subseteq$*-monotonic preference order* $\preceq$ *on* $\wp(Ab)$*. If* $\Delta \in \mathsf{MCS}_{\preceq}(\mathsf{pABF})$ *then* Δ *is a stable extension of* pABF*.*

Proof. We first show that Δ is conflict-free. Suppose towards a contradiction that Δ p-attacks some $\delta \in \Delta$. This means, in particular, that $\Gamma, \Delta \vdash \neg\delta$. But then, since by reflexivity $\Gamma, \Delta \vdash \delta$, by Note 1 we have that $\Gamma, \Delta \vdash \mathsf{F}$, which contradicts the fact that $\Delta \in \mathsf{MCS}_{\sqsubset}(\mathsf{pABF}) \subseteq \mathsf{MCS}(\mathsf{ABF})$.

We now show that Δ p-attacks every $\psi \in Ab \setminus \Delta$. Let $\psi \in Ab \setminus \Delta$. Since $\Delta \in \mathsf{MCS}_{\sqsubset}(\mathsf{pABF}) \subseteq \mathsf{MCS}(\mathsf{ABF})$, $\Delta \cup \{\psi\}$ is inconsistent in pABF. Thus, $\Gamma, \Delta, \psi \vdash \mathsf{F}$, and since in particular $\mathsf{F} \vdash \neg\delta$ for $\delta \in \Delta$, by transitivity, monotonicity and contraposition we get $\Gamma, \Delta \vdash \neg\psi$. If $\mathsf{val}_{f,g}(\Delta, \psi) \leq f(g(\psi))$ we are done. Suppose now that $\mathsf{val}_{f,g}(\Delta, \psi) > f(g(\psi))$. We show that this leads to a contradiction. Let $\Delta_1, \ldots, \Delta_n$ be all the subsets of Δ such that $\Gamma, \Delta_i \vdash \neg\psi$ and for no $\Delta'_i \subset \Delta_i$, $\Gamma, \Delta'_i \vdash \neg\psi$. (Note that since Ab is finite so is Δ and thus there are a finite number of such sets). Let $\delta_i \in \{\sigma_i \in \Delta_i \mid g(\sigma_i) = \max(g(\Delta_i)\}$ for $i = 1, \ldots, n$.

First, we show that $(\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi$ is consistent. Suppose towards a contradiction that $\Gamma, (\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi \vdash \mathsf{F}$ and let $\delta \in \Delta$. Then $\mathsf{F} \vdash \neg\delta$, and by transitivity, $\Gamma, (\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi \vdash \neg\delta$. Thus, by contraposition, $\Gamma, \Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i \vdash \neg\psi$. Thus, there is some $\Delta' \subseteq \Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i$ s.t. $\Gamma, \Delta' \vdash \neg\psi$ and for no $\Delta'' \subseteq \Delta'$, it holds that $\Gamma, \Delta'' \vdash \neg\psi$. Since $\Delta' \subseteq \Delta$, necessarily $\Delta' = \Delta_i$ for some $1 \leqslant i \leqslant n$. But this contradicts $\delta_i \notin \Delta' \subseteq \Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i$. Thus, $(\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi$ is consistent.

Let now $\Theta \in \mathsf{MCS}(\mathsf{ABF})$ be a set such that $(\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi \subseteq \Theta$. We show that $\Delta \prec \Theta$. Indeed, by max-upper-boundedness we have that for every δ_i $(1 \leq i \leq n)$,

$$g(\delta_i) = f(g(\Delta_i) \geq \max_{\delta \in \Delta_i}(g(\delta)) \geq f(g(\Delta_i) \geq \mathsf{val}_{f,g}(\Delta, \psi) > f(g(\psi)) = g(\psi).$$

Thus, by the definition of g, $\{\delta_i\} \preceq \{\psi\}$ for every δ_i $(1 \leq i \leq n)$. By $\prec$-monotonicity, $\Delta \prec (\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi$, and by $\subseteq$-monotonicity we get $(\Delta \setminus \bigcup_{1 \leqslant i \leqslant n} \delta_i) \cup \psi \preceq \Theta$. Together, we conclude that $\Delta \prec \Theta$. Since $\Theta \in \mathsf{MCS}(\mathsf{ABF})$, this contradicts the assumption that $\Delta \in \mathsf{MCS}_{\preceq}(\mathsf{pABF})$. $\square$

Proposition 10. *Let* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim\rangle$ *be a simple contrapositive ABF and let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}_{\preceq}\rangle$ *be a reversing pABF, where* $\mathscr{P}_{\preceq}$ *is a non-decreasing priority setting that is induced by a* $\subseteq$*-monotonic and* $\vdash$*-monotonic preference order* $\preceq$ *on* $\wp(Ab)$*. If* Δ *is a stable extension of* pABF *then* $\Delta \in \mathsf{MCS}_{\preceq}(\mathsf{pABF})$*.*

Proof. For a stable extension Δ of pABF we have to show that it is consistent, maximally consistent and $\preceq$-preferred over the sets in $\mathsf{MCS}(\mathsf{ABF})$:

To see that Δ is consistent, suppose otherwise that $\Gamma, \Delta \vdash \mathsf{F}$. Since $\mathsf{F} \vdash \neg\delta$ for every $\delta \in \Delta$, we get by transitivity and contraposition, that $\Gamma, \Delta \setminus \delta \vdash \neg\delta$ for every $\delta \in \Delta$. Since f in non decreasing, for every $\delta \in \Delta$ it holds that $f(g(\Delta)) \geq f(g(\delta))$, and by the reversibility of $\mathscr{P}$, there is a $\delta' \in \Delta$ such that $f(g(\Delta \setminus \delta')) \leqslant f(g(\delta'))$. By Item 1 of Lemma 1, $\mathsf{val}_{f,g}(\Delta \setminus \delta') \leqslant f(g(\delta'))$, and so Δ is not conflict-free. It follows that Δ cannot be stable.

[17] Thus, $g_{\preceq}$ is an allocation function that reflects $\preceq$: ϕ is g-preferred over ψ iff $\{\phi\}$ is $\preceq$-preferred over $\{\psi\}$.

We now show that Δ is maximally consistent. Indeed, since Δ is stable, $\Gamma, \Delta \vdash \neg\psi$ for every $\psi \in Ab \setminus \Delta$. By monotonicity, $\Gamma, \Delta, \neg \mathsf{F} \vdash \neg\psi$, and by contraposition, $\Gamma, \Delta, \psi \vdash \mathsf{F}$ for every $\psi \in Ab \setminus \Delta$. Thus, every $\Delta' \subseteq Ab$ that properly contains Δ is not consistent.

It remains to show that Δ is $\preceq$-preferred in $\mathsf{MCS}(\mathsf{ABF})$. Suppose for a contradiction that there is a $\Theta \in \mathsf{MCS}(\mathsf{ABF})$ such that $\Theta \succ \Delta$. Let $\psi \in \Theta \setminus \Delta$ (such a formula exists, since otherwise $\Theta \subseteq \Delta$ and by $\subseteq$-monotonicity, $\Theta \preceq \Delta$). Since $\psi \notin \Delta$ and Δ is stable, Δ attacks ψ. This means that $\Gamma, \Delta \vdash \neg\psi$, and since $\Theta \succ \Delta$, by $\vdash$-monotonicity also $\Gamma, \Theta \vdash \neg\psi$. On the other hand, since $\psi \in \Theta$, necessarily $\Gamma, \Theta \vdash \psi$. Thus, by Note 1, $\Gamma, \Theta \vdash \mathsf{F}$, a contradiction to $\Theta \in \mathsf{MCS}(\mathsf{ABF})$. $\square$

By Propositions 9 and 10 we have:

Corollary 3. *Let* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF and let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}_{\preceq} \rangle$ *be a reversing pABF, where* $\mathscr{P}_{\preceq}$ *is a non-decreasing priority setting that is induced by a monotonic preference order* $\preceq$ *on* $\wp(Ab)$. *Then* $\mathsf{Stb}(\mathsf{pABF}) = \mathsf{MCS}_{\preceq}(\mathsf{pABF})$.

Note that all the assumptions in the last corollary can be simultaneously satisfied, e.g., when $\preceq$ is the subset relation $\subseteq$.[18]

Note 10. In Proposition 10, the $\vdash$-monotonicity of $\preceq$ may be traded by the following condition, called *$\mathscr{P}$-alignment*: if $\mathsf{val}_{f,g}(\Delta, \psi) < f(g(\psi))$ then $\Theta \prec \Delta$ for any $\Theta \subseteq Ab$ such that $\psi \in \Theta$.

Proposition 11. *Let* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF and let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}_{\preceq} \rangle$ *be a reversing pABF, where* $\mathscr{P}_{\preceq}$ *is a non-decreasing priority setting that is induced by a* $\subseteq$*-monotonic and* $\mathscr{P}$*-aligned preference order* $\preceq$ *on* $\wp(Ab)$. *If* Δ *is a stable extension of* pABF *then* $\Delta \in \mathsf{MCS}_{\preceq}(\mathsf{pABF})$.

Proof. Let Δ be a stable extension of pABF. The proof that Δ is in $\mathsf{MCS}(\mathsf{ABF})$ is the same as that in the proof of Proposition 10. To see that Δ is $\preceq$-preferred in $\mathsf{MCS}(\mathsf{ABF})$, suppose for a contradiction that there is a $\Theta \in \mathsf{MCS}(\mathsf{ABF})$ such that $\Theta \succ \Delta$. Let $\psi \in \Theta \setminus \Delta$ (Again, such a formula exists, since otherwise $\Theta \subseteq \Delta$ and by $\subseteq$-monotonicity, $\Theta \preceq \Delta$). Since $\psi \notin \Delta$ and since Δ is stable, Δ attacks ψ. This means in particular that $\mathsf{val}_{f,g}(\Delta, \psi) \leq \psi$. Since $\preceq$ is aligned with $\langle g, f \rangle$ and since $\psi \in \Theta$, necessarily $\Theta \preceq \Delta$, a contradiction with $\Delta \prec \Theta$. $\square$

By Propositions 9 and 11 we thus have:

Corollary 4. *Let* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ *be a simple contrapositive ABF and let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a reversing pABF, where* $\mathscr{P}$ *is a non-decreasing priority setting that is induced by a* $\prec$*-monotonic,* $\subseteq$*-monotonic, and* $\mathscr{P}$*-aligned preference order* $\preceq$ *on* $\wp(Ab)$. *Then* $\mathsf{Stb}(\mathsf{pABF}) = \mathsf{MCS}_{\preceq}(\mathsf{pABF})$.

To summarize, we have shown that a representation by prioritized reversing ABFs is possible for a priority setting that is:

- non-decreasing and induced by a monotonic preference order $\preceq$, or
- non decreasing and induced by a $\prec$- and $\subseteq$-monotonic $\mathscr{P}$-aligned preference order $\preceq$.

6. Preference handling properties

6.1. Preference-related postulates

In this section we consider a series of postulates that are concerned with the handling of preferences in prioritized ABFs. In particular, we show how the properties of the priority setting affect the properties of the resulting prioritized ABF. Recall that in all the results below we implicitly assume that the underlying ABF is simple contrapositive.

6.1.1. Degenerated preferences

We start with two postulates that relate extensions of prioritized ABFs and extensions of their non-prioritized fragment. The first one (introduced in [2,15]) refers to situations in which the priority setting is degenerated.

Empty preferences (for Sem): If $\mathscr{P}$ is a degenerated priority setting (i.e., if g is a uniform allocation function), then $\mathsf{Sem}(\mathsf{pABF}) = \mathsf{Sem}(\mathsf{ABF})$.

[18] However, in this case $g_{\subseteq}$ is degenerated, assigning the same value to all the formulas, and $\mathsf{MCS}_{\subseteq}(\mathsf{pABF}) = \mathsf{MCS}(\mathsf{ABF})$, so this example boils down to the non-prioritized case. The existence of more interesting cases for the corollary remains an open question.

The empty preferences postulate is satisfied by every prioritized ABF in which the aggregation function is invariant to multiple occurrences, namely: if S is a set and S' is a multiset with the same elements as S (so S' may have multiple instances of the same element in S, but there is no element in S' that is not in S), then $f(S) = f(S')$. This is the case, e.g., when $f = \min$ or $f = \max$.

Proposition 12. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized simple contrapositive ABF with a priority setting* $\mathscr{P} = \langle g, f \rangle$*. If* f *is invariant to multiple occurrences then* pABF *satisfies the empty preferences postulate for every* Sem.

Proof. The empty preferences postulate assumes that g is uniform. Thus, under the condition on f, we have that $f(g(\Delta))$ is the same for every $\Delta \subseteq Ab$. It follows that p-attacks coincide with attacks, and so $\mathsf{Sem}(\mathsf{pABF}) = \mathsf{Sem}(\mathsf{ABF})$ for every semantics Sem. $\square$

6.1.2. Preferences as criteria for selecting extensions

The next property also relates the extensions of a prioritized ABF to the extension of its ABF. This postulate is taken from [47]. Intuitively, it may be understood by the fact that priorities allow to select the 'best' extensions according to some preference criteria, in the sense that any extension of a pABF is an extension of the corresponding ABF.[19]

Extensions selection (for Sem**):** If $\mathscr{E} \in \mathsf{Sem}(\mathsf{pABF})$ then $\mathscr{E} \in \mathsf{Sem}(\mathsf{ABF})$.

Proposition 13. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a max-upper-bounded prioritized ABF. Then* pABF *satisfies the extensions selection postulate for every* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.[20]

Proof. We first show that if Δ is conflict-free in pABF then it is conflict-free in ABF. Suppose towards a contradiction that Δ attacks some $\delta \in \Delta$. This means that $\Gamma, \Delta \vdash \neg\delta$. If $\mathsf{val}_{f,g}(\Delta, \delta) \leq f(g(\delta))$, this would contradict the assumption that Δ is conflict-free in pABF. Suppose therefore that $\mathsf{val}_{f,g}(\Delta, \delta) > f(g(\delta))$. Since $\mathscr{P}$ is max-upper-bounded, by Proposition 1 it is also reversing, so there is a $\psi \in \Delta$ such that $\Delta \cup \{\delta\} \setminus \psi \preceq_{\mathscr{P}} \psi$. By contraposition, $\Gamma, \Delta \cup \{\delta\} \setminus \psi \vdash \neg\psi$, and thus Δ p-attacks $\psi \in \Delta$, a contradiction again to the assumption that Δ is conflict-free in pABF.

We now show that if Δ is stable in pABF then it is stable in ABF. We have already shown above that Δ is conflict-free in ABF. Now, since Δ is stable in pABF, Δ p-attacks every $\psi \in Ab \setminus \Delta$, which in particular means that $\Gamma, \Delta \vdash \neg\psi$ for every such ψ. Thus, Δ attacks every $\psi \in Ab \setminus \Delta$, and so it is stable in ABF.

We now show that if Δ is preferred in pABF then it is preferred in ABF. Indeed, suppose for a contradiction that Δ is not preferred in ABF. By [32, Proposition 1], Δ is not stable. By the previous case, this means that Δ is not stable in pABF. By Proposition 3 (or Note 9), this implies that Δ is not preferred in pABF, a contradiction.

It remains to show that if Δ is naive in pABF then it is naive in ABF. We already know that Δ is conflict-free in ABF. Suppose now that there is some $\Delta \subsetneq \Delta' \subseteq Ab$ such that Δ' is conflict-free. Since Δ' is not conflict-free in pABF (due to the assumption that Δ is naive in pABF), there is some $\psi \in \Delta'$ such that $\Gamma, \Delta' \vdash \neg\psi$ and $\mathsf{val}_{f,g}(\Delta', \psi) > f(g(\psi))$. But then Δ' also attacks ψ in ABF, a contradiction to the assumption that Δ' is conflict-free (in ABF). $\square$

6.1.3. Conflict preservation

The next postulate is considered, e.g., in [2,4,44]. It requires that conflicts between sets of assumptions are preserved by the semantics, in the sense that if an attack occurs between two sets of assumptions, then there is no extension that contains both the attacked and the attacking sets of assumptions.

Conflict preservation (for Sem**):** If $\mathscr{E} \in \mathsf{Sem}(\mathsf{pABF})$ and Δ p-attacks Θ, either $\Delta \not\subseteq \mathscr{E}$ or $\Theta \not\subseteq \mathscr{E}$.

Conflict preservation follows in our case from the fact that every $\mathscr{E} \in \mathsf{Sem}(\mathsf{pABF})$ is conflict-free. This property is not so obvious in other formalisms in which attacks are sometimes discarded due to preference over arguments (see [20] for some examples).

6.1.4. Inclusion of the most preferred arguments

The next principle is concerned with the inclusion in extensions of the 'strongest' arguments (see [4,20]). In our case this means that assumptions with a minimal g-value will be included in every extension (according to some semantics).

Preferred arguments (for Sem**):** $\mathsf{Min}_g(Ab) = \{\psi \in Ab \mid \nexists\, \phi \in Ab \text{ s.t. } g(\phi) < g(\psi)\} \subseteq \mathscr{E}$ for every $\mathscr{E} \in \mathsf{Sem}(\mathsf{pABF})$.

[19] In a way, this resembles what is called in [3] *refining argumentation frameworks by preferences*, where priorities are used for selecting extensions rather than for defining attacks.

[20] Note that for the proof for naive and stable semantics, it is enough to assume that pABF is reversible.

Clearly, the principle of preferred arguments cannot hold in our setting unless $\mathsf{Min}_g(Ab)$ itself is $\vdash$-consistent (otherwise $\mathcal{E}$ is not conflict free). A sufficient condition for assuring this principle for stable semantics in max-lower-bounded and reversible pABFs is given next:

Proposition 14. *Let* pABF *be a max-lower-bounded and reversible pABF. If* $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\sqsubset_{\mathscr{P}}}(\mathsf{pABF})$ *then* pABF *satisfies the principle of preferred arguments for the stable semantics.*

Proof. Let $\mathcal{E}$ be a stable extension of pABF. By Lemma 5, $\mathcal{E} \in \mathsf{MCS}_{\sqsubset_{\mathscr{P}}}(\mathsf{pABF})$. Now, since $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\sqsubset_{\mathscr{P}}}(\mathsf{pABF})$, we get that $\mathsf{Min}_g(Ab) \subseteq \mathcal{E}$. $\square$

Note that, by Proposition 8, when $\mathscr{P} = \langle g, \max\rangle$, the condition that $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\sqsubset_{\mathscr{P}}}(\mathsf{pABF})$ is also necessary for assuring the satisfaction of the preferred argument postulate for stable and preferred semantics.

6.1.5. Brewka-Eiter principle

The next postulate is taken from [14]. We denote it by its founders (see also [20]). This postulate asserts that if two sets of assumptions of a prioritized ABF differ in only in one formula, the set with the less preferred additional assumption cannot be an extension of the pABF.

BE principle (for Sem): If $\Delta = \Lambda \cup \{\phi\} \in \mathsf{Sem}(\mathsf{ABF})$ and $\Theta = \Lambda \cup \{\psi\} \in \mathsf{Sem}(\mathsf{ABF})$ (where $\phi, \psi \notin \Lambda$) and $g(\psi) < g(\phi)$, then $\Delta \notin \mathsf{Sem}(\mathsf{pABF})$.

This principle doesn't hold for prioritized ABFs in general, as demonstrated by the following example:

Example 17. Consider again Example 15 (i.e., where $\Gamma = \{\neg(p \wedge q \wedge s)\}$ and $Ab = \{p, q, s\}$), but this time with $g(p) = 1$, $g(q) = 2$, $g(s) = 3$ and $f = \min$. It can be verified that $\mathsf{Stb}(\mathsf{pABF}) = \{\{p, q\}, \{p, s\}\}$ and $\mathsf{Stb}(\mathsf{ABF}) = \{\{p, q\}, \{p, s\}, \{q, s\}\}$. This constitutes a violation of the BE-principle, since $\{p, q\}, \{p, s\} \in \mathsf{Stb}(\mathsf{ABF})$ and $g(q) < g(s)$, yet $\{p, s\} \in \mathsf{Stb}(\mathsf{pABF})$.

This example can also be used to demonstrate that for naive semantics, even a max-lower-bounded priority setting does not respect the BE-principle. Indeed, let $\mathsf{pABF}' = \langle \mathsf{ABF}, \mathscr{P}'\rangle$ where $\mathscr{P}' = \langle g, \max\rangle$. It is not hard to see that $\mathsf{Naive}(\mathsf{pABF}') = \{\{p, q\}, \{p, s\}, \{q, s\}\} = \mathsf{Naive}(\mathsf{ABF})$.

For reversing pABFs, however, the BE-principle holds.

Proposition 15. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ *be a reversing pABF that is max-lower-bounded. Then* pABF *satisfies the BE-principle for the stable semantics.*

Proof. Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ be as in the proposition. Let $\Delta, \Theta \in \mathsf{Stb}(\mathsf{ABF})$ and $\Lambda \cup \{\phi, \psi\} \subseteq Ab$ s.t. $\phi, \psi \notin \Lambda$ and $\Delta = \Lambda \cup \{\phi\}$ and $\Theta = \Lambda \cup \{\psi\}$ and $g(\psi) < g(\phi)$. Since $\Delta, \Theta \in \mathsf{Stb}(\mathsf{ABF})$, by [32, Theorem 1], $\Delta, \Theta \in \mathsf{MCS}(\mathsf{ABF})$. Also, $\Theta \sqsubset \Delta$ (recall Definition 14), and so $\Delta \notin \mathsf{MCS}_{\sqsubset}(\mathsf{pABF})$. By Lemma 5, $\Delta \notin \mathsf{Stb}(\mathsf{pABF})$. $\square$

Note 11. Let pABF be a reversing pABF that is max-lower-bounded. If pABF is also max-upper-bound (and so necessarily $f = \max$, recall Proposition 2), we have by Proposition 3 that $\mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF})$, and so in this case the BE-principle holds for the preferred semantics as well.

6.1.6. Principle of tolerance

The last postulate that we consider is the *principle of tolerance*, which requires that if a non-prioritized framework ABF admits extensions, there will be extensions also for the pABFs that are obtained from ABF.

Principle of tolerance (for Sem): If $\mathsf{Sem}(\mathsf{ABF}) \neq \emptyset$ then $\mathsf{Sem}(\mathsf{pABF}) \neq \emptyset$ as well.

The principle of tolerance for complete and preferred semantics is clear by the fact that pABF is in particular an argumentation framework, and so $\mathsf{Cmp}(\mathsf{pABF})$ and $\mathsf{Prf}(\mathsf{pABF})$ are not empty. This principle for stable semantics holds for max-upper-bounded pABF by Proposition 3, and for max-lower-bounded and reversible pABF by Lemma 6. (As noted in [20], when the prioritized assumption-based framework ABA^+ is concerned (see [22]), the principle of tolerance does not hold for the stable semantics).

6.2. Avoidance of the drowning effect

A desirable property of prioritized information systems in general, and pABFs in particular, is that their conclusions shouldn't be altered when lower priority information arrives. In this section we consider this property in our context.

Definition 16. Let $\mathsf{pABF}' = \langle \mathsf{ABF}', \mathscr{P}' \rangle$ be a prioritized ABF that is obtained from $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ by adding to ABF some defeasible assumptions whose priorities are lower than those in Ab, namely:

- if $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ then $\mathsf{ABF}' = \langle \mathfrak{L}, \Gamma, Ab \cup Ab', \sim \rangle$ for some $Ab' \neq \emptyset$.
- if $\mathscr{P} = \langle g, f \rangle$ then $\mathscr{P}' = \langle g', f \rangle$, where $g'(\psi) > \max\{g(\varphi) \mid \varphi \in Ab\}$ if $\psi \in Ab'$ and $g'(\psi) = g(\psi)$ otherwise.

In this case we say that pABF' is *an extension of* pABF *by least-preferred assumptions*.

Some simple facts about the relations between a prioritized framework and its extensions by least-preferred assumptions are given below. In what follows we use the notations of Definition 16 and assume that pABF' is an extension of pABF by least-preferred assumptions (in particular, the defeasible assumptions in Ab are extended by lower-prioritized assumptions in Ab').

Lemma 7. *If Δ p-attacks ψ in* pABF *then Δ p-attacks ψ in* pABF'.

Proof. Since $\Delta \cup \{\psi\} \subseteq Ab$ (that is, the formulas in the attacking set, as well as the attacked formula, are all defeasible assumptions in pABF), for every $\varphi \in \Delta \cup \{\psi\}$ it holds that $g'(\varphi) = g(\varphi)$. This immediately implies that the conditions for the p-attack of Δ on ψ in pABF' are met. $\square$

Lemma 8. *If f is non-decreasing and Δ p-attacks $\psi \in Ab$ in* pABF' *then $\Delta \cap Ab$ p-attacks ψ in* pABF.

Proof. Since Δ p-attacks ψ, there must be a subset $\Delta' \subseteq \Delta$ such that Δ' attacks ψ and $f(g'(\Delta')) \leq f(g'(\psi))$. Note that $\Delta' \subseteq Ab$, since otherwise there is a non-preferred formula $\phi \in \Delta' \cap Ab'$ and since f is non-decreasing and $\psi \in Ab$, we get $f(g'(\Delta')) \geq f(g'(\phi)) = g'(\phi) > \max\{g(\varphi) \mid \varphi \in Ab\} \geq g(\psi) = g'(\psi) = f(g'(\psi))$, in a contradiction to the assumption that $f(g'(\Delta')) \leq f(g'(\psi))$. It follows that $\Delta' \subseteq \Delta \cap Ab$ p-attacks ψ in pABF'. Since $\Delta' \cup \{\psi\} \subseteq Ab$, for every $\varphi \in \Delta' \cup \{\psi\}$ it holds that $g'(\varphi) = g(\varphi)$, and so $\Delta' \subseteq \Delta \cap Ab$ p-attacks ψ also in pABF. By Lemma 2, $\Delta \cap Ab$ p-attacks ψ in pABF. $\square$

Note 12. The requirement in Lemma 8 that f is non-decreasing is indeed necessary. To see this, let pABF be a prioritized ABF with $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, $Ab = \{p, q\}$ and $g(p) = g(q) = 1$, and let pABF' be a prioritized ABF with $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, $Ab = \{p, q\}$, $Ab' = \{\neg p \vee \neg q\}$ and $g'(p) = 1$, $g'(q) = 1$, $g'(\neg p \vee \neg q) = 2$, where in both cases $f = \min$. Clearly, pABF' extended pABF with the least-preferred assumption $\neg p \vee \neg q$. Moreover, $\Delta = \{p, q, \neg p \vee \neg q\}$ p-attacks p in pABF', but $\Delta \cap Ab = \{p, q\}$ does not p-attacks p in pABF, simply because $p \nvdash \neg p$ and $q \nvdash p$.

By the last two lemmas we have:

Corollary 5. *Let f be a non-decreasing aggregation function and let $\psi \in Ab$. Then Δ p-attacks ψ in* pABF' *iff $\Delta \cap Ab$ p-attacks ψ in* pABF.

Proof. One direction is Lemma 8. For the other direction, suppose that $\Delta \cap Ab$ p-attacks ψ in pABF. By Lemma 7, $\Delta \cap Ab$ p-attacks ψ also in pABF', and by Lemma 2, Δ p-attacks ψ in pABF'. $\square$

Lemma 9. *Let* $\mathsf{pABF}' = \langle \mathsf{ABF}', \mathscr{P}' \rangle$ *be an extension of* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *by least-preferred assumptions, that is:* $\mathsf{ABF}' = \langle \mathfrak{L}, \Gamma, Ab \cup Ab', \sim \rangle$, $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, $\mathscr{P}' = \langle g', f \rangle$, *and* $\mathscr{P} = \langle g, f \rangle$ *are as in Definition 16. Suppose further that f is a non-decreasing aggregation function.*

a) If Δ is a complete extension in pABF' *then $\Delta' = \Delta \cap Ab$ is a complete extension in* pABF.
b) If Δ is a preferred extension in pABF' *then $\Delta' = \Delta \cap Ab$ is a preferred extension in* pABF.
c) If Δ is a sable extension in pABF' *then $\Delta' = \Delta \cap Ab$ is a stable extension in* pABF.

Proof. We show that Δ' meets all the required conditions.

First, for Item (a), we show that Δ' is complete in pABF. Indeed, Δ' is conflict free, since it is a subset of Δ, which is conflict free.

To see that Δ' is admissible, suppose that some $\Delta'' \subseteq \Delta'$ is p-attacked by some $\Theta \subseteq Ab$. Then there is a $\psi \in \Delta''$ that is p-attacked by Θ. Since $\Delta'' \subseteq \Delta$ (because $\Delta'' \subseteq \Delta'$ and $\Delta' \subseteq \Delta$), and since Δ is admissible in pABF' (because it is complete in pABF'), Δ p-attacks Θ, that is: there is some $\phi \in \Theta$ that is p-attacked by Δ. But $\phi \in Ab$ (because $\Theta \subseteq Ab$), so by Lemma 8, using the assumption that f is non-decreasing, ϕ is p-attacked by Δ' as well (in pABF). It follows that Δ' p-attacks Θ, and so it defends Δ''.

For Item (a) it remains to shows that Δ' contains all the subsets that it defends. So suppose that Δ' defends (in pABF) some $\Delta'' \subseteq Ab$. Since $\Delta' \subseteq \Delta$, by Lemma 2, Δ also defends Δ'' (in pABF'). But Δ is complete, thus $\Delta'' \subseteq \Delta$. It follows that $\Delta'' \subseteq \Delta \cap Ab = \Delta'$.

We now turn to Item (b). Suppose for a contradiction that there is an admissible set $\Delta'' \subseteq Ab$ such that $\Delta' \subsetneq \Delta''$. In particular, $\Delta'' \setminus \Delta' \neq \emptyset$ and $\Delta'' \setminus \Delta \neq \emptyset$. Let $\Delta^* = \Delta \cup \Delta''$. Then $\Delta \subsetneq \Delta^*$. We show that Δ^* is admissible in pABF', in a contradiction to the assumption that Δ is preferred (and so maximally admissible) in pABF':

Δ^* is conflict-free since both Δ and Δ'' are conflict free, and neither Δ attacks Δ'' nor Δ'' attacks Δ. The latter holds since otherwise, by Corollary 5, either $\Delta' = \Delta \cap Ab$ attacks Δ'' or Δ'' attacks $\Delta' = \Delta \cap Ab$, but $\Delta' \subset \Delta''$, so this contradicts the assumption that Δ'' is conflict-free.

To see that Δ^* defends its elements suppose that Θ p-attacks (in pABF') some $\psi \in \Delta^*$. If $\psi \in \Delta$ then since Δ is admissible in pABF', it defends ψ, i.e., Δ p-attacks Θ and by Lemma 2 also Δ^* p-attacks Θ, i.e., it defends ψ. If $\psi \in \Delta''$ then since $\psi \in Ab$, by Lemma 8 $\Theta \cap Ab$ p-attacks ψ (in pABF). Since Δ'' is admissible in pABF it p-attacks $\Theta \cap Ab$ in pABF and by Lemma 7 also in pABF'. Again, by Lemma 2 Θ^* p-attacks $\Theta \cap Ab$ so it defends ψ in this case as well.

Item (c) immediately follows from Corollary 5. Indeed, suppose for a contradiction that Δ is stable in pABF' but Δ' is not stable in pABF. Then there is a $\psi \in Ab \setminus \Delta'$ that is not p-attacked (in pABF) by $\Delta' = \Delta \cap Ab$. Corollary 5 implies in this case that Δ does not attack ψ in pABF'. But $\psi \in Ab \setminus \Delta' = Ab \setminus (\Delta \cap Ab)$, and Δ is conflict-free, thus $\psi \in Ab \setminus \Delta$ and so $\psi \in (Ab \cup Ab') \setminus \Delta$, in a contradiction that Δ is stable in pABF'. $\square$

The next property assures that conclusions of a prioritized ABF are preserved under extensions of the prioritized ABF by least-preferred assumptions.

Definition 17. An aggregation function f (and so every priority setting $\mathscr{P} = \langle g, f \rangle$ that is obtained from it) *avoids the drowning effect* with respect to $\mid\!\sim$, if for every $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ with $\mathscr{P} = \langle g, f \rangle$, and for every extension pABF' of pABF by least-preferred assumptions, $\mathsf{pABF} \mid\!\sim \psi$ implies that $\mathsf{pABF}' \mid\!\sim \psi$ (for every formula ψ).

Proposition 16. *Let f be a non-decreasing aggregation function. Then it avoids the drowning effect with respect to $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ for every $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}\}$.*

Proof. We show the case $\mathsf{Sem} = \mathsf{Cmp}$; The other cases are similar.

Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a prioritized ABF with $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ and $\mathscr{P} = \langle g, f \rangle$ where f is non-decreasing, and let pABF' be an extension of pABF by least-preferred assumptions. Suppose for a contradiction that for some formula ψ it holds that $\mathsf{pABF} \mid\!\sim^{\cap}_{\mathsf{Cmp}} \psi$ but $\mathsf{pABF}' \not\mid\!\sim^{\cap}_{\mathsf{Cmp}} \psi$. The latter means that there is some complete extension Δ of pABF' for which $\Delta \nvdash_{\mathfrak{L}} \psi$. By the monotonicity of $\vdash_{\mathfrak{L}}$ it holds that $\Delta' \nvdash_{\mathfrak{L}} \psi$ for every $\Delta' \subseteq \Delta$. In particular, $\Delta' \nvdash_{\mathfrak{L}} \psi$ when $\Delta' = \Delta \cap Ab$. But by Lemma 9, Δ' is a complete extension of pABF, in contradiction to the assumption that $\mathsf{pABF} \mid\!\sim^{\cap}_{\mathsf{Cmp}} \psi$. $\square$

Corollary 6. *Let $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}\}$.*

1. *Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a prioritized ABF with $\mathscr{P} = \langle g, \max \rangle$. Then for every extension pABF' of pABF by least-preferred assumptions and for every formula ψ, if $\mathsf{pABF} \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ then $\mathsf{pABF}' \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ as well.*
2. *Let $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ be a prioritized ABF with $\mathscr{P} = \langle g, \Sigma \rangle$. Then for every extension pABF' of pABF by least-preferred assumptions and for every formula ψ, if $\mathsf{pABF} \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ then $\mathsf{pABF}' \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ as well.*

Proof. By Proposition 16, since both the maximum function and the summation function are non-decreasing. $\square$

Note 13. Let $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Prf}, \mathsf{Stb}\}$. Since $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ is non-monotonic, in general extending a prioritized ABF with extra assumptions does not guarantee the preservation of its conclusions. Indeed, consider for instance the prioritized framework pABF_1 that is based on CL with $\Gamma = \{\neg\mathtt{mushroom} \vee \neg\mathtt{pepper}\}$, $Ab = \{\mathtt{mashroom}, \mathtt{cheese}\}$, and where the allocation function is $g(\mathtt{mashroom}) = 2$, $g(\mathtt{cheese}) = 3$, and the aggregation function is $f = \max$. Clearly, $\mathsf{pABF}_1 \mid\!\sim^{\cap}_{\mathsf{Sem}} \mathtt{mushroom}$. Now, let pABF_2 be a prioritized ABF that is obtained by adding to Ab of pABF_1 the assumption $\mathtt{pepper}$ with $g(\mathtt{pepper}) = 1$. This is the prioritized ABF considered in Example 4 (see also Fig. 2), and as it is shown there, $\mathsf{pABF}_2 \not\mid\!\sim^{\cap}_{\mathsf{Sem}} \mathtt{mushroom}$ (in fact, even $\mathsf{pABF}_2 \not\mid\!\sim^{\cup}_{\mathsf{Sem}} \mathtt{mushroom}$).[21]

Table 1 summarizes the results in Sections 4–6 with respect to the stable semantics.

7. Properties of the entailments induced by pABFs

Next, we consider some properties of the entailment relations which are induced by prioritized simple contrapositive ABFs, namely: entailments like those in Definition 7, but where pABFs replace ABFs. Obviously, priorities have an effect on such properties only in the presence of conflicts:

[21] For another counterexample see Example 19 in what follows.

Table 1
Summary of the postulates for the stable semantics.

Property of the pABF	Conditions on the priority setting
Consistency	Reversible
Closure	Reversible
$\mathsf{Stb} \subseteq \mathsf{MCS}_{\sqsubset}$	Max-lower-bounded
$\mathsf{Stb} \supseteq \mathsf{MCS}_{\sqsubset}$	Max-upper-bounded
Empty preferences	Invariance of multiple-occurrence
Extension selection	Reversible
Conflict preservation	–
Preferred assumptions	Reversible & Max-lower-bounded
Brewka-Eiter postulate	Reversible & Max-lower-bounded
Tolerance	Reversible & Max-lower-bounded, or Max-upper-bounded
No drowning effect	Non-decreasing

Proposition 17. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF for* $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$*. If* $\Gamma \cup Ab$ *is* $\vdash$*-consistent, then for every relation* $\mid\!\sim$ *that is induced from* pABF *according to Definition 7,* $\mathsf{pABF} \mid\!\sim \psi$ *iff* $\mathsf{ABF} \vdash \psi$*.*[22]

Proof. When $\Gamma \cup Ab$ is $\vdash$-consistent, $\mathsf{WF}(\mathsf{pABF}) = \mathsf{Grd}(\mathsf{pABF}) = \mathsf{Prf}(\mathsf{pABF}) = \mathsf{Stb}(\mathsf{pABF}) = \{Ab\}$, so the claim immediately follows from Definition 7. $\square$

The following properties were introduced by Kraus, Lehmann and Magidor in [42] and [43], and their formulations are adjusted to our setting. Some of the properties (CM, CC, and LLE) take into account also the priority setting. In such cases, the original formulation in [42] is obtained just by ignoring the conditions about the allocation function.

Below, instead of $\mathsf{ABF} \mid\!\sim \psi$, where $\mid\!\sim$ is a relation defined in Definition 7 and $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg \rangle$, we just write $\Gamma, Ab \mid\!\sim \psi$.

Definition 18. A relation $\mid\!\sim$ between pABFs and formulas in their languages is called *cumulative*, if the following conditions are satisfied:

- *Cautious Reflexivity* (CR): For every $\vdash$-consistent formula ψ it holds that $\psi \mid\!\sim \psi$.
- *Cautious Monotonicity* (CM): If $\Gamma, Ab \mid\!\sim \phi$ where $g(\phi) \leq \min\{g(\varphi) \mid \varphi \in Ab\}$, and $\Gamma, Ab \mid\!\sim \psi$, then $\Gamma, Ab, \phi \mid\!\sim \psi$.
- *Cautious Cut* (CC): If $\Gamma, Ab \mid\!\sim \phi$ where $g(\phi) \leq \min\{g(\varphi) \mid \varphi \in Ab\}$, and $\Gamma, Ab, \phi \mid\!\sim \psi$, then $\Gamma, Ab \mid\!\sim \psi$.
- *Left Logical Equivalence* (LLE): If $\phi \vdash \psi$ and $\psi \vdash \phi$ and $g(\phi) = g(\psi)$, then $\Gamma, Ab, \phi \mid\!\sim \rho$ iff $\Gamma, Ab, \psi \mid\!\sim \rho$.
- *Right Weakening* (RW): If $\phi \vdash \psi$ and $\Gamma, Ab \mid\!\sim \phi$ then $\Gamma, Ab \mid\!\sim \psi$.

A cumulative relation is called *preferential*, if it satisfies the following condition:

- *Distribution* (OR): If $\Gamma, Ab, \phi \mid\!\sim \rho$ and $\Gamma, Ab, \psi \mid\!\sim \rho$ then $\Gamma, Ab, \phi \vee \psi \mid\!\sim \rho$.

A cumulative entailment is called *rational*, if it satisfies the following condition[23]:

- *Rational Monotonicity* (RM): If $\Gamma, Ab \mid\!\sim \rho$ and $\Gamma, Ab \not\mid\!\sim \neg\psi$ then $\Gamma, Ab, \psi \mid\!\sim \rho$.

Another property that is useful for reasoning with conflicts is the following:

Definition 19. Given a logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, let Γ_i $(i = 1, 2)$ be two sets of countable $\mathscr{L}$-formulas, and let $\mathsf{ABF}_i = \langle \mathfrak{L}, \Gamma_i, Ab_i, \sim_i \rangle$ $(i = 1, 2)$ be two ABFs based on $\mathfrak{L}$.

- We denote by $\mathsf{Atoms}(\Gamma_i)$ $(i = 1, 2)$ the set of all atoms occurring in Γ_i.
- We say that Γ_1 and Γ_2 are *syntactically disjoint* if $\mathsf{Atoms}(\Gamma_1) \cap \mathsf{Atoms}(\Gamma_2) = \emptyset$.
- ABF_1 and ABF_2 are *syntactically disjoint* if so are $\Gamma_1 \cup Ab_1$ and $\Gamma_2 \cup Ab_2$.
- We denote: $\mathsf{ABF}_1 \cup \mathsf{ABF}_2 = \langle \mathfrak{L}, \Gamma_1 \cup \Gamma_2, Ab_1 \cup Ab_2, \sim_1 \cup \sim_2 \rangle$.

An entailment $\mid\!\sim$ satisfies *non-interference* [17], if for every two syntactically disjoint assumption-based frameworks $\mathsf{ABF}_1 = \langle \mathfrak{L}, \Gamma_1, Ab_1, \sim_1 \rangle$ and $\mathsf{ABF}_2 = \langle \mathfrak{L}, \Gamma_2, Ab_2, \sim_2 \rangle$ where $\Gamma_1 \cup \Gamma_2$ is consistent, it holds that $\mathsf{ABF}_1 \mid\!\sim \psi$ iff $\mathsf{ABF}_1 \cup \mathsf{ABF}_2 \mid\!\sim \psi$ for every $\mathscr{L}$-formula ψ such that $\mathsf{Atoms}(\psi) \subseteq \mathsf{Atoms}(\Gamma_1 \cup Ab_1)$.

[22] Note, in particular, that skeptical and credulous reasoning coincide in this case.
[23] Notice that we do not require rational entailment to be preferential, but merely cumulative.

Table 2
Properties of $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ and $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ for non-prioritized ABFs.

Entailment	Cumulativity	Preferentiality	Rationality	Non-interference
$\mid\!\sim^{\cap}_{\mathsf{Naive}}$, $\mid\!\sim^{\cap}_{\mathsf{Prf}}$, $\mid\!\sim^{\cap}_{\mathsf{Stb}}$	✓	✓	–	✓
$\mid\!\sim^{\cup}_{\mathsf{Naive}}$, $\mid\!\sim^{\cup}_{\mathsf{Prf}}$, $\mid\!\sim^{\cup}_{\mathsf{Stb}}$	✓	–	✓	✓
$\mid\!\sim_{\mathsf{Grd}}$, $\mid\!\sim_{\mathsf{WF}}$	✓	✓	–	✓

For extending non-interference to the prioritized case, we further suppose that we are equipped with priority settings $\mathscr{P}_i = \langle g_i, f\rangle$ over Ab_i $(i = 1, 2)$. When ABF_1 and ABF_2 are syntactically disjoint, we can define a priority setting $\mathscr{P} = \langle g, f\rangle$ over $Ab_1 \cup Ab_2$, where g coincides with g_i on Ab_i. In such a case, non-interference is defined as in the non-prioritized case, except that now $\mathsf{pABF}_1 \mid\!\sim \psi$ iff $\mathsf{p}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2) \mid\!\sim \psi$, where $\mathsf{p}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2) = \langle \mathsf{ABF}_1 \cup \mathsf{ABF}_2, \mathscr{P}\rangle$.

Table 2 summarizes the results for the *non-prioritized case*, proved in [35]. The results concerning $\mid\!\sim_{\mathsf{Grd}}$ and $\mid\!\sim_{\mathsf{WF}}$ hold only for simple contrapositive ABFs in which $\mathsf{F} \in Ab$.

Turning to the prioritized case, we first note the following:

Note 14. By the result in Section 6.1.1, if the allocation function is uniform and the aggregation function is invariant to multiple occurrences, the priority-based entailments coincide with the entailments that are induced by the corresponding non-prioritized ABFs. In such a case, then, the properties in Table 2 carry on to the relevant prioritized ABFs.

In 'non-degenerated' pABFs, however, many of the properties above cannot be always guaranteed. For instance, Example 19 below shows the failure of CM in some prioritized ABFs. The next example shows that non-interference may be violated as well:

Example 18. Let $\mathsf{pABF}_i = \langle \mathsf{ABF}_i, \mathscr{P}_i\rangle$ $(i = 1, 2)$ be two prioritized simple contrapositive pABFs, where $\mathfrak{L}_1 = \mathfrak{L}_2 = \mathsf{CL}$, $\Gamma_1 = \Gamma_2 = \emptyset$, $Ab_1 = \{p, \neg p\}$, and $Ab_2 = \{q\}$. Suppose further that $g_1(p) = 2$, $g_1(\neg p) = 3$, and $g_2(q) = 1$. Clearly, ABF_1 and ABF_2 are syntactically disjoint, and $\mathsf{ABF}_1 \mid\!\sim_{\mathsf{Grd}} p$ as well as $\mathsf{ABF}_1 \mid\!\sim_{\mathsf{WF}} p$. However, $\mathsf{Grd}(\mathsf{p}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2)) = \mathsf{WF}(\mathsf{p}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2)) = \{\{q\}\}$ and so $\mathsf{p}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2) \not\mid\!\sim_{\mathsf{Sem}} p$ for either $\mathsf{Sem} = \mathsf{WF}$ or $\mathsf{Sem} = \mathsf{Grd}$.

The next two propositions describe a case in which cumulativity and non-interference are satisfied.

Proposition 18. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ *be a prioritized simple contrapositive ABF in which* $\mathscr{P} = \langle g, \max\rangle$ *for some allocation function* g. *Then both* $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ *and* $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ *are cumulative for* $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.

Proof. We show the proposition for $\mid\!\sim^{\cap}_{\mathsf{Sem}}$; the proof for $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ is similar. In the proofs below, when $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P}\rangle$ and $\mathsf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \neg\rangle$, $\mathscr{P} = \langle g, f\rangle$, we shall sometimes write $\mathsf{MCS}_{\sqsubset_g}(\Gamma, Ab)$ instead of $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$.

- CR: This property holds by Proposition 17 and the reflexivity of $\vdash$ (thus $\psi \vdash \psi$).[24]
- CM: Since $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \psi$, by Proposition 8 we have that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\Gamma, Ab)$, and so, by monotonicity, $(*)$ $\Gamma, \Delta, \phi \vdash \psi$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\Gamma, Ab)$. Also, since $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \phi$, we have that $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\Gamma, Ab)$, and since ϕ is at least as preferred as any formula in Ab, we have that $(**)$ $\mathsf{MCS}_{\sqsubset_g}(\Gamma, Ab, \phi) = \{\Delta \cup \{\phi\} \mid \Delta \in \mathsf{MCS}_{\sqsubset_g}(\Gamma, Ab)\}$. By $(*)$ and $(**)$, then, $\Gamma, \Delta' \vdash \psi$ for every $\Delta' \in \mathsf{MCS}_{\sqsubset_g}(\Gamma, Ab, \phi)$, and by Proposition 8 again, $\Gamma, Ab, \phi \mid\!\sim^{\cap}_{\mathsf{sem}} \psi$.
- CC: Suppose that $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{Sem}} \phi$. By Proposition 8 we have that, $(*)$ $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$. Let $\mathsf{ABF}' = \langle \mathscr{L}, \Gamma, Ab \cup \{\phi\}, \neg\rangle$ or $\mathsf{ABF}' = \langle \mathscr{L}, \Gamma \cup \{\phi\}, Ab, \neg\rangle$. Then, since ϕ is at least as preferred as any formula in Ab, in the first case, $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF}') = \{\Delta \cup \{\phi\} \mid \Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})\}$, and since $\Gamma, Ab, \phi \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$, by Proposition 8 we have in both cases that $(**)$ $\Gamma, \Delta, \phi \vdash \psi$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$. Thus, by transitivity on $(*)$ and $(**)$ we have that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$, and by Proposition 8, $\Gamma, Ab \mid\!\sim \psi$.
- LLE: Suppose that $\Gamma, Ab, \phi \mid\!\sim \rho$. By Proposition 8 we have that $\Gamma, \Delta \vdash \rho$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_{\mathsf{g}}}(\Gamma, Ab, \phi)$. Since $\phi \vdash \psi$ and $\psi \vdash \phi$ and $g(\phi) = g(\psi)$, it holds that $\Delta \in \mathsf{MCS}_{\sqsubset_{\mathsf{g}}}(\Gamma, Ab, \phi)$ iff $\Delta \setminus \{\phi\} \cup \{\psi\} \in \mathsf{MCS}_{\sqsubset_{\mathsf{g}}}(\Gamma, Ab, \psi)$. It follows that $\Gamma, \Delta \vdash \rho$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_{\mathsf{g}}}(\Gamma, Ab, \psi)$. By Proposition 8 again, we get $\Gamma, Ab, \psi \mid\!\sim \rho$. The proof of the other direction is dual (replacing the roles of ϕ and ψ).
- RW: Suppose that $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \phi$. By Proposition 8 we have that $\Gamma, \Delta \vdash \phi$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$. By transitivity with $\phi \vdash \psi$ we get that $\Gamma, \Delta \vdash \psi$ for every $\Delta \in \mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF})$, and by Proposition 8 again, $\Gamma, Ab \mid\!\sim^{\cap}_{\mathsf{sem}} \psi$. □

[24] If ψ is a strict assumption, this property can be strengthened as follows: $\Gamma \mid\!\sim \psi$ for every $\psi \in \Gamma$. Note that this strengthening ceases to hold for defeasible assumptions: if $Ab = \{\mathsf{F}, p, \neg p\}$ and g is a uniform allocation, then $Ab \not\mid\!\sim^{\cap}_{\mathsf{sem}} p$ and $Ab \not\mid\!\sim^{\cap}_{\mathsf{sem}} \neg p$.

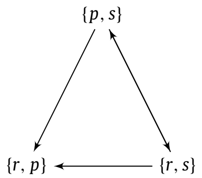

Fig. 5. An attack diagram for pABF' in Example 19.

Note 15. The enhancements of some of the properties of cumulativity with requirements on the allocation function are indeed necessary in the prioritized case. To see this for LLE, for instance, consider a prioritized ABF based on classical logic, in which $\Gamma = \emptyset$, $Ab = \{p, \neg p\}$ and $g(p) = 1, g(\neg p) = 2$. Clearly, $\{p\}$ is the only preferred and stable extension in this case, thus p follows from this prioritized ABF (by any of the entailment relations considered in Proposition 18). If we now replace p by $\neg\neg p$ and let $g(\neg\neg p) = 3$, then $\{\neg p\}$ becomes the single preferred and stable extension of the revised pABF, and so p is not inferred anymore.

The next example shows that cumulativity may fail for aggregation functions other than max.

Example 19. Let $\Gamma = \{r \supset s;\ p \supset s;\ p, s \supset \neg r\}$, $Ab = \{p, r\}$ and $\mathscr{P} = \langle g, \min\rangle$, where $g(p) = 3$ and $g(r) = 2$. Here, $\{r\}$ p-attacks $\{p\}$ but $\{p\}$ does not p-attack $\{r\}$. It follows that $\Gamma, r, p \mathrel{|\!\sim}^{\cap}_{\mathsf{Sem}} s$ and $\Gamma, r, p \mathrel{|\!\sim}^{\cap}_{\mathsf{Sem}} r$ for $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.

Consider now $\mathsf{pABF}' = \langle \mathsf{ABF}', \mathscr{P}' \rangle$ where $\mathsf{ABF}' = \langle \mathfrak{L}, \Gamma, Ab \cup \{s\}, \neg \rangle$ and $g(s) = 1$ (the g-values of the other formulas remain the same as before). The defeat diagram of pABF' is presented in Fig. 5.

pABF' has two preferred and stable extensions: $\{p, s\}$ and $\{r, s\}$, and so for each $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$ it holds that $\Gamma, r, p, s \mathrel{|\!\not\sim}^{\cap}_{\mathsf{Sem}} r$. It follows that cautious monotony is not satisfied in this case.

When max is the aggregation function of the priority setting, non-interference is also assured.

Proposition 19. *Let* $\mathsf{pABF} = \langle \mathsf{ABF}, \mathscr{P} \rangle$ *be a prioritized simple contrapositive ABF in which* $\mathscr{P} = \langle g, \max \rangle$ *for some allocation function* g. *Then both* $\mathrel{|\!\sim}^{\cup}_{\mathsf{Sem}}$ *and* $\mathrel{|\!\sim}^{\cap}_{\mathsf{Sem}}$ *satisfy non-interference for* $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$.

Proof. By Proposition 8 and the fact that if ABF_1 and ABF_2 are syntactically disjoint, then $\mathsf{MCS}_{\sqsubset_g}(\mathsf{ABF}_1 \cup \mathsf{ABF}_2) = \{\Delta_1 \cup \Delta_2 \mid \Delta_1 \in \mathsf{MCS}_{\sqsubset_{g_1}}(\mathsf{ABF}_1), \Delta_2 \in \mathsf{MCS}_{\sqsubset_{g_2}}(\mathsf{ABF}_2)\}$.[25] □

8. Related work

This work extends to the prioritized case the works in [32–34], summarized in [35], where (non-prioritized) simple contrapositive assumption-based argumentation frameworks are considered. As was argued in these papers, unlike similar works on logical argumentation (e.g. [1,9,19,49]), simple contrapositive ABFs give rise to a finite number of arguments for a finite set Ab of defeasible assumptions, and moreover they do not require to make assumptions of minimality or consistency on arguments. As such, the current paper allows us to incorporate priorities in a formalism for logic-based argumentation with some clear benefits over other works.

Priorities have been integrated in all the major approaches to structured argumentation frameworks, including the assumption-based argumentation formalism ABA^+ (see [20,22]), ASPIC-based systems [18,37,38,45,46], sequent-based argumentation frameworks [8,11], and dialectical argumentation frameworks [23,24].[26] We remark that even though there is a lot of work on logical structured argumentation and on prioritized structured argumentation, there are only a few works that combine the two. Below, we first discuss works on prioritized assumption-based argumentation and then make a comparison of our work to works on dialectical argumentation frameworks and to works on ASPIC-like formalisms.

8.1. ABA and ABA$^+$

Apart of ABA^+, the incorporation of priorities in all of the above-mentioned settings is similar: for the attack to take place the attacking argument should be at least as preferred as the attacked argument (i.e., the former should have an equal or higher priority than the latter). The ABA^+ system, in contrast, is based on the idea of *reverse defeats*: A set of assumptions Δ reverse defeats a set of assumptions Θ if either Δ attacks Θ and Δ is not less preferred than Θ, or Θ attacks Δ and Θ is (strictly) less preferred than Δ.[27] The use of reverse defeats is required for avoiding some violations of rationality postulates

[25] Recall that g coincides with g_i on Ab_i $(i = 1, 2)$.

[26] Axiomatic approaches to structured argumentation with preferences are considered by Dung and his coauthors in some recent papers, e.g. [27–29].

[27] See [40] for the use of similar principles in the context of abstract argumentation frameworks.

such as consistency (see [22] for more details). However, in [31, Chapter 7] it is shown that such reverse defeats are actually superfluous when assuming that the deducibility relation is closed under contraposition (as in our case), and when using the max-attacks (see Definition 12). Proposition 5 is a generalization of this result, showing that contraposition together with reversibility of the preference function is sufficient to guarantee consistency (and thus reverse defeats are superfluous).

Another difference between the present work and the one in [22] is related to the priority setting: while we concentrate on linear preference orders, in [22] any preference relation that is a partial order is allowed. However, the formalism in [22] is adequate only for the weakest link principle (i.e., max-attacks) for comparing arguments, while we do not confine ourselves to a particular priority setting.

8.2. Dialectical argumentation frameworks

Another formalism that allows for logic-based argumentation with priorities is that of *dialectical argumentation frameworks* [23,24]. In these frameworks arguments are conceived as support-conclusion pairs, where the supports of the arguments are split to two disjoint sets, intuitively understood as follows:

> "An argument entails a conclusion from assumptions regarded as premises assumed to be true, and assumptions that are supposed true for the 'sake of argument' (i.e., those premises that an interlocutor commits to)" [24].

In the structures that are obtained in this way, Brewka's order on preferred subtheories [13] (recall Definition 13) can be represented by the preferred and stable semantics. In order to avoid the problem of having to deal with a possibly infinite set of support-conclusion pairs even when considering a finite set of defeasible assumptions, the formalism in [23,24] allows for the use of a so-called *depth-bounded logics*. Such depth-bounded logics are proof-theoretically defined as sub-systems of classical logic, which restrict the depth of a proof. In this way, a hierarchy of logics is constructed, whose deductive power increases monotonically with the allowed depth of proofs and converges to classical logic.

It turns out that dialectical argumentation using these depth-bounded logics can capture preferred sub-theories in dialectical argumentation, and give rise to finite argumentation frameworks given a finite set of defeasible assumptions. However, [23,24] only study preferred sub-theories based on classical logic whereas we show that preferred-subtheories based on *any* contrapositive Tarskian logic can be represented by simple contrapositive prioritized ABFs. On the other hand, it is shown in [24] that dialectical argumentation satisfies the closure and consistency postulates for any lifting (i.e., preference relation) principle and any proof theory for classical logic or depth-bounded logic approximating classical logic, whereas for pABFs these postulates only hold when the priority setting is assumed to be reversible.

Non-interference is shown to hold for any complete-based semantics under the assumption that the preference relation over arguments is *dialectically coherent*, which is a requirement on the interaction between preferences of arguments and premises and assumptions in the support of arguments (see [23, Definition 20]). As such, dialectical coherence is hard to interpret in our setting, and thus the conditions under which non-interference is satisfied in both approaches are hard to compare. In view of the strong relation between preferred sub-theories and dialectical classical logic under the weakest link lifting [24, Theorem 4], we conjecture that a result analogous to Proposition 16 holds for dialectical argumentation and so dialectical argumentation (under the weakest link lifting) avoids the drowning effect as well. Likewise, we conjecture that cumulativity and preferentiality of the resulting inference relation are satisfied. However, these statements remain, to the best of our knowledge, to be proven. It would be interesting to check whether the sufficient conditions for the satisfaction of the properties studied here are also sufficient for the satisfaction of the same properties in the context of dialectical argumentation.

8.3. ASPIC systems

ASPIC-like formalisms such as those considered in [18,37,38,45,46] are argumentation-based systems built up from a formal language $\mathscr{L}$ and contain the following ingredients:

- Strict rules of the form $\psi_1, \ldots, \psi_n \to \phi$ (for formulas $\psi_1, \ldots, \psi_n, \phi$ in $\mathscr{L}$), which are deductive in the sense that the truth of their premises $\psi_1, \ldots, \psi_n$ necessarily implies the truth of their antecedent ϕ. Just as in ABA, this may be realized in several ways. One is directed by a logic $\mathfrak{L}$ with an associated consequence relation $\vdash_{\mathfrak{L}}$, so that $\psi_1, \ldots, \psi_n \to \phi$ holds when $\psi_1, \ldots, \psi_n \vdash_{\mathfrak{L}} \phi$. Another way is to treat strict rules as *domain dependent* rules, often used in the context of logic programming.
- Defeasible rules of the form $\psi_1, \ldots, \psi_n \Rightarrow \phi$ (again, where $\psi_1, \ldots, \psi_n, \phi$ are $\mathscr{L}$-formulas), which unlike strict rules warrant the truth of their conclusion only provisionally: the application of a defeasible rule can be retracted in case counter-arguments are encountered. This is clearly a difference w.r.t. assumption-based argumentation, where the only defeasible element in an ABF is the set of assumptions.
- A set of strict premises $\mathscr{K}_s$ and a set of defeasible premises $\mathscr{K}_d$, corresponding (respectively) to the set of premises Γ and the set of assumptions Ab of an ABF.
- A contrariness function specifying, as in ABA, conflicts between elements of the language.

- A preorder $\leq$ over the defeasible rules and the defeasible premises, expressing relative preferences over the defeasible elements.

Thus, the only difference between prioritized ABFs and ASPIC argumentation systems is that the latter additionally allow to formulate defeasible rules. However, in [36] it has been shown that, at least in the absence of priorities, defeasible rules can be defined as defeasible premises.[28]

A further difference between ASPIC-like formalisms and ABA is that *closure* under strict rules is not a requirement of any of the semantical concepts used in ASPIC-like formalisms. This is due to the fact that in ASPIC systems arguments are constructed as proof-trees on the basis of the argumentation system, as opposed to sets of assumptions in ABA. Furthermore, different ASPIC systems use different notions of attacks. Arguably, the notion of attack closest to that of ABA is *undermining-defeat*, used in ASPIC$^+$ [45,46], according to which an argument A undermining-defeats an argument B if the conclusion of A is a contrary of a defeasible premise used in the construction of B, and A is not strictly less preferred than B. Preference relations between arguments are obtained on the basis of a *lifting principle*, which specifies how to obtain preferences over arguments on the basis of preferences over defeasible rules. Such lifting principles can thus be seen as an analogue to our aggregation functions. Indeed, just as in this paper, in ASPIC-like formalisms often a wide variety of lifting principles is considered, and various conditions on lifting principles have been formulated to ensure desirable behavior of the resulting argumentative consequence relations (see, e.g., [30,37,38,45,46]). However, since in ASPIC-like formalisms defeasible rules give a proof-tree structure to arguments, further variations between lifting principles may occur. For instance, the *last link principle* [45,46] states that only the last defeasible rules that have been applied are relevant when comparing the strength of arguments. Once an appropriate lifting principle has been chosen, a defeat relation over the arguments is obtained (just as p-attack relations are obtained in our case). On the basis of the set of arguments and a selected notion of defeat, an argumentation graph can then be constructed, which is then evaluated using Dung's *abstract argumentation semantics*. These are very similar to the semantics from Definition 6, with the only difference that closure is not required. However, as shown in [45], when using a contrapositive strict rule base (which is a primary assumption in this paper as well), closure is satisfied for any complete (and therefore for also for any stable, preferred and grounded) set of arguments.

Moving to representational results, in [45] it is shown that for ASPIC$^+$-based frameworks including priorities and when classical logic is the strict base logic, the preferred and stable extensions coincide and correspond to the set of preferred sub-theories of the set of premises under consideration (see [13] and Definition 13). In [39] it is shown that ASPIC$^+$ without undercut can represent a generalization of preferred subtheories to sets of defeasible rules.[29] However, we note that in ASPIC$^+$ a finite set of (defeasible) assumptions gives rise to an infinite set of arguments. The fact that for simple contrapositive ABFs the size of an argumentation graph is bounded by the size of the powerset of the defeasible assumptions is therefore a great benefit in comparison to the other argumentation-based approaches.

Another interesting point of comparison is the satisfaction of the properties that were studied in this paper. In this respect, we note the following:

- In general, for ASPIC$^+$ under the weakest link principle, the drowning effect cannot be avoided, as shown in [39, Example 20]. However, it should be noted that the counterexample in [39] is based on the special way that ASPIC$^+$ treats non-symmetric contraries. It is an open question whether the drowning effect can be avoided for some specific classes of ASPIC systems.
- The closure and consistency postulates are well-studied in ASPIC-like formalisms. Some comprehensive studies of the satisfaction of these postulates are given in [27,28,46], where several sufficient conditions for the satisfaction of the closure and consistency postulates are given, including closure under contraposition and the related closure under transposition.
- Non-interference has been studied in depth as well. In the presence of priorities, there are only three variants of ASPIC that satisfy non-interference while preserving consistency and closure: under the grounded semantics, ASPIC$^\ominus$ [37,38] satisfies all rationality postulates, and it is shown in [39] that under the preferred and stable semantics, a fragment of ASPIC$^+$ without undercut satisfies all rationality postulates when using the weakest link lifting. D-ASPIC$^+$, a variant of ASPIC$^+$ based on *dialectical argumentation* (discussed above) is shown in [25] to satisfy all rationality postulates for any complete-based semantics.

To the best of our knowledge, the other properties considered in this paper, in particular, empty preferences, extension selection, conflict-preservation, preferred assumptions, the Brewka-Either postulate and tolerance, have not been studied yet in relation to ASPIC systems. A further discussion on the relations between ABA and ASPIC systems can be found in [7].

[28] It is an open question whether in the presence of priorities this translation is still adequate.

[29] Similar results concerning the representation of Brewka's preferred subtheories using sequent-based argumentation frameworks are shown in [11] (see also [6]).

9. Conclusion

The enhancement with priorities of simple contrapositive assumption-based argumentation frameworks strengthens their expressivity and provides additional layer to their inference process. The choice of a proper priority setting for reflecting priorities among arguments should be taken with care, though. As we have shown in this paper, for assuring certain properties of the corresponding ABFs, the priority settings, and in particular the aggregation functions, should meet certain requirements. Let us give a last demonstration of this, using a variation of an example from [41] (see also [5] and [8]).

Example 20. A flat owner negotiates the construction of a swimming pool (s), a tennis-court (t) and a private car-park (p) with the tenants. It is known that investing in all of these facilities will increase the rent (r), otherwise the rent will not be changed. The tenants do not have a particular preference among these options, but if they have to make a choice, they prefer not to have two sport facilities (s and t) and definitely do not want to increase the rent. Based on these inputs, the flat owner has to reach a conclusion about the facilities to be constructed.

Abbreviating $(\phi \supset \psi) \wedge (\psi \supset \phi)$ by $\phi \leftrightarrow \psi$, the consideration that the rent increases if all the facilities are constructed can be represented by the formula $\psi_1 = r \leftrightarrow (s \wedge t \wedge p)$. The preferences of the tenants not to increase the rent and not to have two sport facilities are modeled by $\neg r$ and by $\psi_2 = s \supset \neg t$ and $\psi_3 = t \supset \neg s$, respectively. This situation may be represented by a CL-based prioritized ABF with $\Gamma = \{\psi_1\}$ and $Ab = \{s, t, p, \neg r, \psi_2, \psi_3\}$, where $g(\neg r) = 1$, $g(\psi_2) = g(\psi_3) = 2$ and $g(s) = g(t) = g(p) = 3$, together with the aggregation function $f = \max$.

By Proposition 8, there are two preferred extensions of this prioritized ABF: $\mathcal{E}_1 = \{\psi_2, \psi_3, r, s, p\}$ and $\mathcal{E}_2 = \{\psi_2, \psi_3, r, t, p\}$. These are also the stable extensions in this case. It follows that $\mathsf{pABF} \mid\!\sim^*_{\mathsf{Sem}} r$ and $\mathsf{pABF} \mid\!\sim^*_{\mathsf{Sem}} p$ and $\mathsf{pABF} \mid\!\sim^*_{\mathsf{Sem}} s \vee t$, but $\mathsf{pABF} \not\mid\!\sim^*_{\mathsf{Sem}} s \wedge t$ for every $\mathsf{Sem} \in \{\mathsf{Prf}, \mathsf{Stb}\}$ and every $* \in \{\cup, \cap\}$. This means that according to the preferred and the stable semantics, the rent will not be increased in this case and a parking lot will be built as well as (exactly) one sport facility. In order to decide whether to invest in a swimming pool or in a tennis-court, a discrimination between the allocation values of these facilities should be made.

We note, further, that by Proposition 19, an unrelated new information that is added as a strict premise or as a defeasible assumption with a high priority (say, that there is already a central heating (h) in the house, where $g(h) = 1$) will not affect the above-mentioned conclusions. Likewise, by Proposition 16, adding information with a low priority, e.g. that there is still some low preference for having the three facilities ($s \wedge t \wedge p$ with $g(s \wedge t \wedge p) = 4$), will not have any effect on the conclusions derived above.

In future work, we plan to consider several generalizations of the frameworks presented in this paper. For example, we shall investigate priorities induced by a (non-necessarily total) preorder over the defeasible assumptions. Also, it may be useful to allow for *conditional preferences*, namely, defeasible assumptions that are stronger than other assumptions only when certain conditions are satisfied (see, e.g., [29]).

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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