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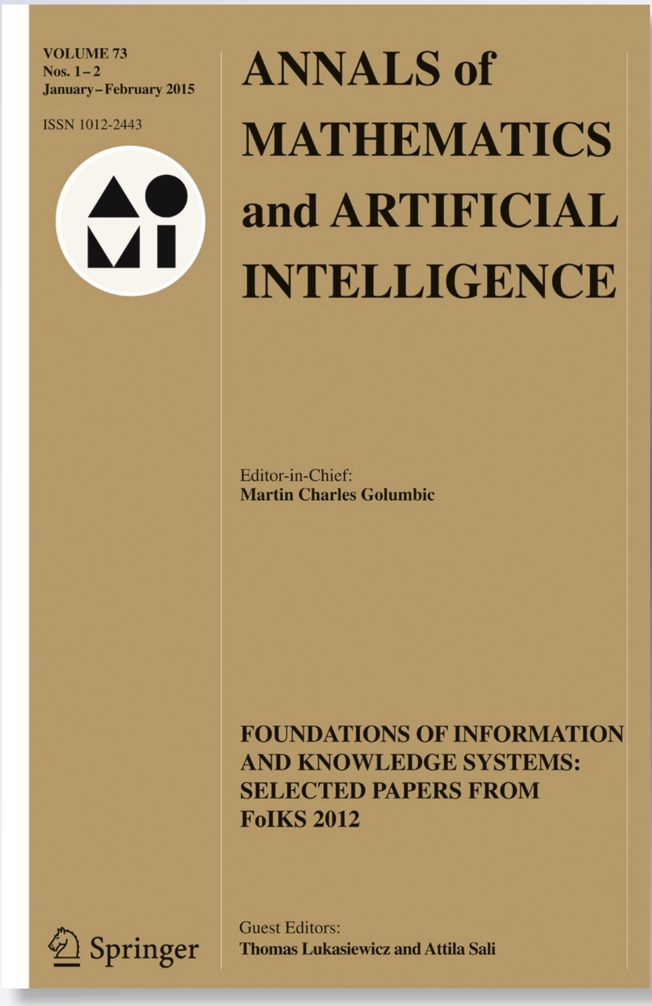

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A dissimilarity-based framework for generating inconsistency-tolerant logics

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Abstract Many commonly used logics, including classical logic and intuitionistic logic, are trivialized in the presence of inconsistency, in the sense that inconsistent premises cause the derivation of any formula. It is thus often useful to define *inconsistency-tolerant* variants of such logics, which are faithful to the original logic with respect to consistent theories but also allow for nontrivial inconsistent theories. A common way of doing so is by incorporating distance-based considerations for concrete logics. So far this has been done mostly in the context of two-valued semantics. Our purpose in this paper is to show that inconsistency-tolerance can be achieved for *any* logic that is based on a denotational semantics. For this, we need to trade distances for the more general notion of dissimilarities. We then examine the basic properties of the entailment relations that are obtained and exemplify dissimilarity-based reasoning in various forms of denotational semantics, including multi-valued semantics, non-deterministic semantics, and possible-worlds (Kripke-style) semantics. Moreover, we show that our approach can be viewed as an extension of several well-studied forms of reasoning in the context of belief revision, database integration, consistent query answering, and inconsistency maintenance in knowledge-based systems.

Keywords Reasoning with inconsistency · Dissimilarity-based entailments

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1 Introduction

A common, model-theoretic way of defining a consequence relation for a logical system S, is to require that every model of the premises would also be a model of the conclusion. Symbolically, this can be represented as follows:

$$\Gamma \vdash_{\mathsf{S}} \psi \text{ if } \mathrm{mod}_{\mathsf{S}}(\Gamma) \subseteq \mathrm{mod}_{\mathsf{S}}(\psi). \tag{1}$$

Logics that are based on this approach (including, e.g., classical logic, intuitionistic logic, and many forms of modal logics) face difficulties in handling inconsistent information, since, by (1), if Γ has no model it entails *any* conclusion.[1] This problem has long been identified, and different solutions have been proposed to it. However, many of those solutions depend on the nature of the semantics at hand, and therefore they cannot be easily adapted to other contexts.

One way of properly maintaining inconsistency, while still being faithful to (1), is to substitute $\mathrm{mod}_{\mathsf{S}}(\Gamma)$ in (1) by a non-empty set $\Delta_{\mathsf{S}}(\Gamma)$ that coincides with $\mathrm{mod}_{\mathsf{S}}(\Gamma)$ whenever the latter is non-empty. Intuitively, $\Delta_{\mathsf{S}}(\Gamma)$ consists of the semantic elements that are 'as close as possible' to satisfying Γ. This is the basic intuition behind *distance-based reasoning*, which is usually defined for standard two-valued semantics (see, e.g., [2, 8, 27, 31, 34]). In this paper, we extend the distance-based approach to *arbitrary* forms of denotational semantics by incorporating the notion of *dissimilarities*, a generalization of the notion of distances. To this end, we define in precise terms what a dissimilarity between semantic objects of a given denotational semantics is, and what properties it should satisfy in order to induce natural and useful entailments. This allows us to apply these abstract definitions, in a uniform way, on a wide range of semantics.

Given a logic L that is based on a denotational semantics, we provide a general way of constructing a logic L' that is an *inconsistency-tolerant* variant of L, in the sense that L' coincides with L with respect to consistent premises, and is nontrivial with respect to inconsistent ones. A major advantage of this approach is its uniformity: in order to construct an inconsistency-tolerant variant of one's favorite logic, one only needs to define a dissimilarity relation in this logic, and this automatically induces a corresponding inconsistency-tolerant entailment. This approach may be useful, for instance, for applying distance-based strategies for revising or merging knowledge-bases, the semantics of which is not the standard classical one (the evolutionary databases of [19] and the three-valued inference relations of [28], which are based on three-valued paraconsistent logics, are just two cases in point), or for extending standard approaches to data integration and query answering in databases, which are based on distance functions [5, 13, 37].

The rest of this paper is organized as follows: In the next section we recall some basic definitions and facts about denotational semantics and the logics induced by them. Then, in Section 3, we explain what we mean by 'inconsistency-tolerant' logics, and describe a general method of obtaining such logics by incorporating McCarthy's [41] and Shoham's [45] approach of preferential reasoning. In Section 4 we provide a concrete way of implementing preferential reasoning for any type of

[1]For languages with classical negation $\neg$, this usually means that the underlying logic is not *paraconsistent* [17]: any formula ϕ follows from ψ and $\neg\psi$ (see also [14, 16, 43]).

denotational semantics using the notion of dissimilarity. In Section 5 we examine some common properties of the entailment relations defined in Section 4, and in Section 6 we apply our framework to particular cases of denotational semantics. In Section 7 we conclude and consider some directions for future work.[2]

2 Denotational semantics and their logics

In the sequel, $\mathcal{L}$ denotes a propositional language with a countable set $\mathsf{Atoms} = \{p, q, r \ldots\}$ of atomic formulas and a (countable) set $\mathcal{F}_{\mathcal{L}} = \{\psi, \phi, \sigma, \ldots\}$ of well-formed formulas. A theory Γ is a finite set of formulas in $\mathcal{F}_{\mathcal{L}}$. The atoms appearing in the formulas of Γ and the subformulas of Γ are denoted, respectively, by $\mathsf{Atoms}(\Gamma)$ and $\mathsf{SF}(\Gamma)$. The set of all theories of $\mathcal{L}$ is denoted by $\mathcal{T}_{\mathcal{L}}$.

Definition 1 Given a language $\mathcal{L}$, a *propositional logic* for $\mathcal{L}$ is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, i.e., a binary relation between sets of formulas in $\mathcal{F}_{\mathcal{L}}$ and formulas in $\mathcal{F}_{\mathcal{L}}$, satisfying the following conditions:

Reflexivity: if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity: if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$.

A common (model-theoretical) way of defining logics for $\mathcal{L}$ is based on the notion of *denotational semantics*:

Definition 2 A *denotational semantics* for a language $\mathcal{L}$ is a pair $\mathsf{S} = \langle S, \models_S \rangle$, where S is a non-empty set (of 'interpretations'), and $\models_S$ (the 'satisfiability relation' of S) is a computable binary relation on $S \times \mathcal{F}_{\mathcal{L}}$.

Example 1 The most common case of denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$ for $\mathcal{L}$ is classical logic. In this case, $\mathcal{L}$ is a propositional language, the elements of S are (two-valued) *valuations*, i.e., functions from $\mathcal{F}_{\mathcal{L}}$ to the set $\{t, f\}$ of the classical truth values, and $\models_S$ is the ordinary satisfaction relation, defined by $\nu \models_S \Gamma$ iff $\nu(\psi) = t$ for every $\psi \in \Gamma$.

Standard generalizations of classical logic to multiple-valued logics can also be described in terms of denotational semantics and so are, e.g., the various kinds of Kripke-structures for modal and for intuitionistic logics (some of which are described in greater detail in Section 6 below), provided that the underlying satisfiability relation is computable.

Definition 3 Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics for $\mathcal{L}$, $\nu \in S$ an interpretation, and $\psi \in \mathcal{F}_{\mathcal{L}}$ a formula.

(a) If $\nu \models_S \psi$, we say that ν *satisfies* ψ and call ν an S-*model* of ψ.

[2]This is a revised and extended version of [7] and [9].

(b) The set of the S-models of ψ is denoted by $mod_{\mathsf{S}}(\psi)$. When $mod_{\mathsf{S}}(\psi)$ is the set S, ψ is called an S-*tautology*, and when $mod_{\mathsf{S}}(\psi)$ is the empty set, ψ is called an S-*contradiction*.

(c) If ν satisfies every formula ψ in a theory Γ, it is called an S-model of Γ. The set of the S-models of Γ is denoted by $mod_{\mathsf{S}}(\Gamma)$. If $mod_{\mathsf{S}}(\Gamma) \neq \emptyset$ we say that Γ is S-*consistent*, otherwise Γ is S-*inconsistent*.

In what follows we shall sometimes omit the prefix S from the above notions.

Definition 4 A denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$ is *normal*, if for each $\nu \in S$ there is a formula ψ, such that $\nu \not\models_S \psi$.

Example 2 Any denotational semantics S for which there is an S-contradiction, is normal.

A denotational semantics S induces the following relation on $\mathcal{T}_{\mathcal{L}} \times \mathcal{F}_{\mathcal{L}}$:

Definition 5 $\Gamma \vdash_{\mathsf{S}} \psi$ if $mod_{\mathsf{S}}(\Gamma) \subseteq mod_{\mathsf{S}}(\psi)$.

Proposition 1 *Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics for a propositional language $\mathcal{L}$. Then $\langle \mathcal{L}, \vdash_{\mathsf{S}} \rangle$ is a propositional logic for $\mathcal{L}$.*

Proof We have to show that $\vdash_{\mathsf{S}}$ is a Tarskian consequence relation for $\mathcal{L}$. Indeed, reflexivity and monotonicity are obvious from Definitions 3(c) and 5. For transitivity, suppose that $\Gamma \vdash \psi$, $\Gamma', \psi \vdash \phi$, and ν is an S-model of $\Gamma \cup \Gamma'$. In particular, ν is an S-model of Γ, and since $\Gamma \vdash_{\mathsf{S}} \psi$, ν is an S-model of ψ. Thus, ν is an S-model of $\Gamma' \cup \{\psi\}$, and since $\Gamma', \psi \vdash_{\mathsf{S}} \phi$, we conclude that ν is an S-model of ϕ as well. Thus $\Gamma, \Gamma' \vdash \phi$. □

3 Inconsistency tolerance by preferential reasoning

The definition given in the previous section, of a logic $\mathsf{L} = \langle \mathcal{L}, \vdash_{\mathsf{S}} \rangle$ that is induced by a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$, implies that L may not tolerate inconsistency properly. Indeed, if $mod_{\mathsf{S}}(\Gamma)$ is empty, then by Definition 5 it holds that $\Gamma \vdash_{\mathsf{S}} \psi$ for *every* formula ψ. The next definition aims at overcoming this drawback.

Definition 6 Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics for a propositional language $\mathcal{L}$. An entailment relation $\mid\!\sim_{\mathsf{S}}$ on $\mathcal{T}_{\mathcal{L}} \times \mathcal{F}_{\mathcal{L}}$ is an *inconsistency-tolerant variant of* $\vdash_{\mathsf{S}}$, if it has the following properties:

I. Faithfulness: If $mod_{\mathsf{S}}(\Gamma) \neq \emptyset$ then for all $\psi \in \mathcal{F}_{\mathcal{L}}$, $\Gamma \mid\!\sim_{\mathsf{S}} \psi$ iff $\Gamma \vdash_{\mathsf{S}} \psi$.

II. Non-Explosiveness: If $mod_{\mathsf{S}}(\Gamma) = \emptyset$ then there is a formula $\psi \in \mathcal{F}_{\mathcal{L}}$ such that $\Gamma \not\mid\!\sim_{\mathsf{S}} \psi$.

Faithfulness guarantees that $\mid\!\sim_{\mathsf{S}}$ coincides with $\vdash_{\mathsf{S}}$ with respect to S-consistent theories, and non-explosiveness assures that $\mid\!\sim_{\mathsf{S}}$ is not trivialized when the set of premises is not S-consistent. In what follows, when $\vdash_{\mathsf{S}}$ is clear from context, we just say that $\mid\!\sim_{\mathsf{S}}$ is inconsistency-tolerant.

Note 1 When $\mathrm{mods}_{\mathsf{S}}(\Gamma)$ is non-empty for every Γ, then $\vdash_{\mathsf{S}}$ itself is inconsistency-tolerant, but in such logics the notion of inconsistency is degenerate. In what follows we shall be interested in stronger logics (like classical logic) that do not tolerate inconsistency and so need to be refined. Another reason that $\vdash_{\mathsf{S}}$ may not be adequate, even in cases that it is inconsistency-tolerant, is that by Proposition 1, $\vdash_{\mathsf{S}}$ is monotonic, while commonsense reasoning is frequently nonmonotonic, in particular when contradictions are involved.

One way of achieving non-explosiveness is by incorporating McCarthy's [41] and Shoham's [45] idea of *preferential semantics*: Given a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$ for $\mathcal{L}$, we define an S-*preferential operator* $\Delta_{\mathsf{S}} : \mathcal{T}_{\mathcal{L}} \to 2^S$ (where 2^S is the power-set of S), that relates a theory Γ to a set $\Delta_{\mathsf{S}}(\Gamma)$ of its 'most preferred' (or 'most plausible') elements in S. The role of $\mathrm{mods}_{\mathsf{S}}(\Gamma)$ in Definition 5 is taken now by $\Delta_{\mathsf{S}}(\Gamma)$ as follows.

Definition 7 Given a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$ for $\mathcal{L}$ and a corresponding S-preferential operator $\Delta_{\mathsf{S}} : \mathcal{T}_{\mathcal{L}} \to 2^S$, we write $\Gamma \mid\!\sim_{\Delta_{\mathsf{S}}} \psi$ to denote that $\Delta_{\mathsf{S}}(\Gamma) \subseteq \mathrm{mods}_{\mathsf{S}}(\psi)$.

Note 2 By faithfulness, every two S-consistent theories that are logically equivalent with respect to $\vdash_{\mathsf{S}}$ (that is, have the same S-models), must also share the same $\mid\!\sim_{\mathsf{S}}$-conclusions. On the other hand, while in any logic defined by denotational semantics (including classical logic) *all* inconsistent theories are (trivially) logically equivalent, inconsistency-tolerant logics make a distinction between inconsistent theories, so they cannot preserve logical equivalence, and must employ other considerations. This is common to many methods for resolving inconsistencies, e.g., those that are based on information and inconsistency measures (see, e.g., [24, 25]).

Note 3 A dual way of defining entailment relations for handling inconsistencies is to make preferences among *theories* instead of interpretation, noting that every interpretation $\nu \in S$ induces a corresponding theory $\Gamma_\nu = \{\psi \mid \nu \models_S \psi\}$. This alternative approach is taken, e.g., in [48], where preference relations are defined on (logically closed) nontrivial theories.[3]

Proposition 2 *Let* $\mathsf{S} = \langle S, \models_S \rangle$ *be a normal denotational semantics. Let* Δ_{S} *be a preferential operator for* S*. If*

1. $\Delta_{\mathsf{S}}(\Gamma)$ *is non-empty for every* Γ*, and*
2. $\Delta_{\mathsf{S}}(\Gamma) = \mathrm{mods}_{\mathsf{S}}(\Gamma)$ *whenever* $\mathrm{mods}_{\mathsf{S}}(\Gamma)$ *is non-empty,*

then $\mid\!\sim_{\Delta_{\mathsf{S}}}$ *is inconsistency-tolerant.*

Proof Faithfulness follows from Condition (2). For non-explosiveness, let Γ be a theory such that $\mathrm{mods}_{\mathsf{S}}(\Gamma) = \emptyset$. By Condition (1) $\Delta_{\mathsf{S}}(\Gamma) \neq \emptyset$, and so there is $\mu \in \Delta_{\mathsf{S}}(\Gamma)$. Since S is normal, there is some $\psi \in \mathcal{F}_{\mathcal{L}}$ such that $\mu \not\models_S \psi$. Thus, by Definition 7, $\Gamma \not\mid\!\sim_{\Delta_{\mathsf{S}}} \psi$. □

[3] I.e., whose set of consequences is not the set of all well formed formulas.

Proposition 2 shows that in many cases inconsistency-tolerant entailments can be obtained from a given denotational semantics S by a proper choice of a preferential operator Δ_{S}. Frequently, such an operator can be defined in terms of a preferential function, that is, by a function P that maps every theory Γ to a strict partial order $<_\Gamma$ on S. In such cases, we have that

$$\Delta^{\mathsf{P}}_{\mathsf{S}}(\Gamma) = \{\nu \in S \mid \neg\exists \mu \in S \text{ such that } \mu <_\Gamma \nu\}, \tag{2}$$

so, intuitively, $\Delta^{\mathsf{P}}_{\mathsf{S}}(\Gamma)$ consists of the 'best' elements of S, in terms of $<_\Gamma$.

Proposition 3 *Let* S *be a normal denotational semantics, and let* P *be a preferential function, mapping every theory* Γ *to a strict partial order (i.e., irreflexive, asymmetric, transitive order)* $<_\Gamma$ *on* S. *If*

1. *for every theory* Γ, $<_\Gamma$ *is well-founded,*[4] *and*
2. *for every* S*-consistent* Γ, $\min_{<_\Gamma}(S)$ $[= \Delta^{\mathsf{P}}_{\mathsf{S}}(\Gamma)] = \mathrm{mod}_{\mathsf{S}}(\Gamma)$,

then $\mid\!\sim_{\Delta^{\mathsf{P}}_{\mathsf{S}}}$ *is inconsistency-tolerant.*

Proof Clearly, the two conditions of this proposition imply, respectively, the two conditions of Proposition 2, and so $\mid\!\sim_{\Delta^{\mathsf{P}}_{\mathsf{S}}}$ is inconsistency-tolerant. □

A preferential function P as in Proposition 3 represents *preference by satisfiability*, that is: the models of the underlying theory (if such elements exist) are preferred over the other elements in S.

Example 3 Let $\mathsf{S} = \langle S, \models_S\rangle$ be a normal denotational semantics. Define a function P that maps a theory Γ to a strict partial order $<_\Gamma$, in which for every $\nu, \mu \in S$, $\nu <_\Gamma \mu$ iff $\nu \in \mathrm{mod}_{\mathsf{S}}(\Gamma)$ and $\mu \notin \mathrm{mod}_{\mathsf{S}}(\Gamma)$. Clearly, P is a preference by satisfiability, and so $\mid\!\sim_{\Delta^{\mathsf{P}}_{\mathsf{S}}}$ is an inconsistency-tolerant variant of $\vdash_{\mathsf{S}}$. Indeed, in this case $\Delta^{\mathsf{P}}_{\mathsf{S}}(\Gamma) = \mathrm{mod}_{\mathsf{S}}(\Gamma)$ if $\mathrm{mod}_{\mathsf{S}}(\Gamma) \neq \emptyset$, and otherwise the relation $<_\Gamma$ is empty (i.e., all the elements in S are $<_\Gamma$-incomparable), thus $\Delta^{\mathsf{P}}_{\mathsf{S}}(\Gamma) = S$. It follows that both of the properties in the definition of inconsistency-tolerant entailments trivially hold.

Proposition 3 specifies natural conditions under which a strict partial-order $<_\Gamma$ induces an inconsistency-tolerant entailment. However, this proposition does not provide a method for defining such an order. In the next sections, we consider a simple and intuitive way of doing so by introducing the notion of *dissimilarity*.

4 Dissimilarity-based entailments

Distance functions often provide a subtle platform for choosing the most preferred interpretations for a given set Γ of premises. The reason for this is that such functions supply numeric estimations on how 'close' a given interpretation is to satisfy the formulas in Γ. However, as we shall see in the sequel, such estimations are not

[4]That is, every $<_\Gamma$-descending chain has a $<_\Gamma$-minimum.

adequate in some particular cases of denotational semantics, and so they need to be refined. For this, we first generalize the notion of distance between interpretations to that of *dissimilarity* between interpretations. Intuitively, dissimilarity functions provide quantitative indications on the distinction between their arguments.

Definition 8 Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics. An *S-dissimilarity* is a function $\mathsf{d} : S \times S \to \mathbb{R}^+$, satisfying the following properties for all $\nu, \mu \in S$:

Symmetry: $\mathsf{d}(\nu, \mu) = \mathsf{d}(\mu, \nu)$,
Reflexivity: $\mathsf{d}(\nu, \nu) = 0$,
Absorption: if $\mathsf{d}(\nu, \mu) = 0$ then $\mathsf{d}(\nu, \sigma) = \mathsf{d}(\mu, \sigma)$ for every $\sigma \in S$.

Example 4 The discrete (uniform, drastic) metric d^{u} on S, defined by $\mathsf{d}^{\mathsf{u}}(\nu, \mu) = 0$ if $\nu = \mu$ and $\mathsf{d}^{\mathsf{u}}(\nu, \mu) = 1$ otherwise, is an S-dissimilarity. The Hamming distance d^{h} [18], where $\mathsf{d}^{\mathsf{h}}(\nu, \mu)$ is the number of atoms p for which $\nu(p) \neq \mu(p)$, is a dissimilarity on, e.g., two-valued valuations (Example 1), restricted to a finite number of atomic formulas. Other definitions of distance and dissimilarity functions can be found, e.g., in [5–7, 27, 29].

Note 4 Dissimilarities are a generalization of the notion of pseudo distances: If d is a pseudo distance on S (that is, if d is a symmetric total function on S that preserves identities: $\forall \nu, \mu \in S\, \mathsf{d}(\nu, \mu) = 0$ iff $\nu = \mu$), then d is also an S-dissimilarity. However, dissimilarities are a weaker notion than distances: First, a function d that satisfies all the conditions in Definition 8 is not necessarily a pseudo distance, since $\mathsf{d}(\nu, \mu) = 0$ does not mean that ν and μ must be equal (dissimilarity does not preserve identities). Second, dissimilarities do not necessarily satisfy the triangular inequality.

To be computable, dissimilarity functions should take into consideration only *finite* fragments of the compared interpretations (which are in general infinite). This is done by restricting the computation to finite contexts, determined by the given set of assumptions.

Definition 9 A *context* is a finite set of formulas (i.e., an element of $\mathcal{T}_{\mathcal{L}}$). A *context generator* (for $\mathcal{L}$) is a function $\mathcal{G} : \mathcal{T}_{\mathcal{L}} \to \mathcal{T}_{\mathcal{L}}$, producing a context for every theory.

Intuitively, $\mathcal{G}(\Gamma)$ is a relevant context for the computations about a theory Γ. In what follows we shall denote by $\mathcal{G} \subseteq \mathcal{G}'$ that $\mathcal{G}(\Gamma) \subseteq \mathcal{G}'(\Gamma)$ for every Γ.

Example 5 Common examples for context generators are, e.g., the functions $\mathcal{G}^{\mathsf{At}}$, $\mathcal{G}^{\mathsf{ID}}$, $\mathcal{G}^{\mathsf{SF}}$, defined, respectively, for every theory Γ by $\mathcal{G}^{\mathsf{At}}(\Gamma) = \mathsf{Atoms}(\Gamma)$, $\mathcal{G}^{\mathsf{ID}}(\Gamma) = \Gamma$, and $\mathcal{G}^{\mathsf{SF}}(\Gamma) = \mathsf{SF}(\Gamma)$. Obviously, $\mathcal{G}^{\mathsf{At}} \subseteq \mathcal{G}^{\mathsf{SF}}$ and $\mathcal{G}^{\mathsf{ID}} \subseteq \mathcal{G}^{\mathsf{SF}}$.

Dissimilarities will be used in what follows to determine how 'close' an interpretation is to satisfying a set of formulas Γ. In particular, they should differentiate between the models of Γ and the valuations that do not satisfy Γ. However, as dissimilarities are a weaker notion (and so more general) than pseudo distances (Note 4), it may be the case that $\mathsf{d}(\nu, \mu) = 0$ where ν is a model of Γ while μ is not. To avoid this, the dissimilarity under consideration should be Γ-dependent. This is achieved by dissimilarity generators that produce a different dissimilarity for each theory. For defining dissimilarity generators we shall need the next notation and notion:

Definition 10 $\nu \sim_\Gamma \mu$ if for every $\psi \in \Gamma$, $\nu \models_S \psi$ iff $\mu \models_S \psi$.

Definition 11 We say that a formula ψ is *$\mathcal{G}$-independent* of a theory Γ, if $\mathcal{G}(\{\psi\}) \cap \mathcal{G}(\Gamma) = \emptyset$.

Definition 12 Let $\mathsf{S} = \langle S, \models_S \rangle$ be a denotational semantics for a language $\mathcal{L}$ and $\mathcal{G}$ a context generator for the same language. A *$\mathcal{G}$-dissimilarity generator for* S is a function $\mathsf{d}_\mathcal{G} : \mathcal{T}_\mathcal{L} \to (S \times S \to \mathbb{R}^+)$, such that, for every $\Gamma \in \mathcal{T}_\mathcal{L}$,

1. $\mathsf{d}_\mathcal{G}(\Gamma)$ is an S-dissimilarity,
2. there exists an $n_\Gamma \in \mathbb{R}$, such that $n_\Gamma = \max\{\mathsf{d}_\mathcal{G}(\Gamma)(\nu, \mu) \mid \nu, \mu \in S\}$,
3. for every $\nu, \mu \in S$, if $\mathsf{d}_\mathcal{G}(\Gamma)(\nu, \mu) = 0$, then $\nu \sim_\Gamma \mu$.

A $\mathcal{G}$-dissimilarity generator $\mathsf{d}_\mathcal{G}$ is called *normal*, if it satisfies the following normality condition:

– For every theory Γ, if ψ is a non S-tautological formula that is $\mathcal{G}$-independent of Γ, then for each $\nu \in mods_\mathsf{S}(\psi)$ there is $\mu \notin mods_\mathsf{S}(\psi)$ so that $\mathsf{d}_\mathcal{G}(\Gamma)(\nu, \mu) = 0$.

Below, we shall sometimes write $\mathsf{d}_{\mathcal{G}(\Gamma)}$ instead of $\mathsf{d}_\mathcal{G}(\Gamma)$.

Note that $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$ only means that ν and μ are similar on $\mathcal{G}(\Gamma)$, but this does not imply any correspondence between ν and μ elsewhere (this is also indicated in Note 4 as one of the differences between dissimilarities and distances). The second condition of Definition 12 is needed for proper computations of dissimilarities (see Definition 15) and the third condition assures, intuitively, that $\mathsf{d}_{\mathcal{G}(\Gamma)}$ is faithful to Γ. The normality condition makes sure that the measurements by $\mathsf{d}_{\mathcal{G}(\Gamma)}$ depend only on the relevant context $\mathcal{G}(\Gamma)$.

Example 6 Let $\mathcal{G} = \mathcal{G}^{\mathsf{At}}$ or $\mathcal{G} = \mathcal{G}^{\mathsf{SF}}$, and let S be the standard two-valued semantics (Example 1). The following functions are all normal $\mathcal{G}$-dissimilarity generators for S.

(a) $\mathsf{d}^{\mathsf{s}}_\mathcal{G}$, where for every Γ, $\mathsf{d}^{\mathsf{s}}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$ if $\nu \sim_\Gamma \mu$, otherwise $\mathsf{d}^{\mathsf{s}}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 1$.
(b) $\mathsf{d}^{\mathsf{u}}_\mathcal{G}$, where for every Γ, $\mathsf{d}^{\mathsf{u}}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$ if for all $\psi \in \mathcal{G}(\Gamma)$ $\nu(\psi) = \mu(\psi)$ and otherwise $\mathsf{d}^{\mathsf{u}}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 1$.
(c) $\mathsf{d}^{\mathsf{h}}_\mathcal{G}$, in which $\mathsf{d}^{\mathsf{h}}_{\mathcal{G}(\Gamma)}(\nu, \mu)$ is the number of formulas $\psi \in \mathcal{G}(\Gamma)$, for which it holds that $\nu(\psi) \neq \mu(\psi)$.

The last two dissimilarity generators are generalizations of the standard discrete and Hamming distances considered in Example 4. In Section 6 we consider other (normal) dissimilarity generators.

Note that unlike related formalisms in standard two-valued semantics (see, e.g., [29, 34]), it is *not necessary* to assume here that (the set of atomic formulas of) the underlying language is finite. This is because the dissimilarity calculations are made with respect to finite contexts.

Definition 13 A *(numeric) aggregation function* is a total function f, such that: (1) for every multiset of real numbers, the value of f is a real number, (2) the value of f does not decrease when the number of elements in its multiset increases, (3) $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = x_2 = \ldots x_n = 0$, (4) $\forall x \in \mathbb{R}\ f(\{x\}) = x$.

For nonnegative numeric values (such as those provided by dissimilarities), the summation, average, median, and the maximum function, are all aggregation functions (see, e.g., [29, 36]).

Definition 14 Let S be a denotational semantics for a language $\mathcal{L}$. A (semantical) *setting* for S is a triple $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$, where $\mathcal{G}$ is a context generator (for $\mathcal{L}$), d is a $\mathcal{G}$-dissimilarity generator (for S), and f is a numeric aggregation function.

Definition 15 Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be a setting for a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$. For an interpretation $\nu \in S$ and a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, we define:

$$\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi_i) = \begin{cases} \min\left\{\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) \mid \mu \in \mathrm{mod}_{\mathsf{S}}(\psi_i)\right\} & \text{if } \mathrm{mod}_{\mathsf{S}}(\psi_i) \neq \emptyset, \\ 1 + \max\left\{\mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \sigma) \mid \mu, \sigma \in S\right\} & \text{otherwise.}^5 \end{cases}$$

$$\mathsf{M}_{\mathcal{S}}(\nu, \Gamma) = f\left(\left\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi_1), \ldots, \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi_n)\right\}\right).$$

Intuitively, $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi)$ is a quantitative indication for how 'close' ν is to be a model of ψ. The function $\mathsf{M}_{\mathcal{S}}(\nu, \Gamma)$ indicates how 'close' ν is to be a model of Γ. It is easy to verify that if ψ is S-consistent, then the closest elements to ψ are its models, and if ψ is *not* S-consistent, all the elements in S are equally close to ψ.

Definition 16 A setting $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ for a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$ is *normal*, if so is d (recall Definition 12). We say that $\mathcal{S}$ is *effective*, if for every theory Γ, the set $\{\mathsf{M}_{\mathcal{S}}(\nu, \Gamma) \mid \nu \in S\}$ has a minimal element.

Reasoning by dissimilarities is now defined as follows:

Definition 17 Given a setting $\mathcal{S}$ for a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$, the $\mathcal{S}$-*most plausible interpretations* of a (non-empty) theory Γ, are the elements of the following set:

$$\Delta_{\mathcal{S}}(\Gamma) = \left\{\nu \in S \mid \forall \mu \in S\ \mathsf{M}_{\mathcal{S}}(\nu, \Gamma) \leq \mathsf{M}_{\mathcal{S}}(\mu, \Gamma)\right\}.$$

In case that Γ is empty, we define $\Delta_{\mathcal{S}}(\emptyset) = S$.

Note 5 If $\mathcal{S}$ is effective, then $\Delta_{\mathcal{S}}(\Gamma) \neq \emptyset$ for every Γ.

Definition 18 Given a semantic setting $\mathcal{S}$ for a denotational semantics S, the dissimilarity-based entailment $\mid\!\sim_{\mathcal{S}}$ is defined by: $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ iff $\Delta_{\mathcal{S}}(\Gamma) \subseteq \mathrm{mod}_{\mathsf{S}}(\psi)$.

[5] Note that $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi_i)$ is well-defined by the second property in Definition 12.

5 Reasoning with $\mid\!\sim_{\mathcal{S}}$

The following example is a simple illustration of how inconsistency is handled by a dissimilarity-based entailment.

Example 7 Consider the setting $\mathcal{S} = \langle \mathcal{G}^{\mathsf{At}}, \mathsf{d}^{\mathsf{h}}, \Sigma \rangle$ for the standard two-valued semantics (Example 1), where $\mathsf{d}^{\mathsf{h}} = \mathsf{d}^{\mathsf{h}}_{\mathcal{G}^{\mathsf{At}}}$ is the dissimilarity generator computing for each Γ the Hamming distance on $\mathcal{G}^{\mathsf{At}}(\Gamma)$ from Example 6(c) and Σ is the summation function. Now, let $\Gamma = \{p, \neg p, q\}$. In this case, $\Delta_{\mathcal{S}}(\Gamma) = \{\nu \in S \mid \nu(q) = t\}$, thus $\Gamma \mid\!\sim_{\mathcal{S}} q$ while $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg q$. Intuitively, this is justified by the fact that q is 'unrelated' to the inconsistency of Γ (more precisely, q is $\mathcal{G}^{\mathsf{At}}$-independent of $\{p, \neg p\}$). On the other hand, $\Gamma \not\mid\!\sim_{\mathcal{S}} p$ and $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg p$, since both of p and $\neg p$ are 'related' to the inconsistency of Γ.

Next, we consider some important properties of $\mid\!\sim_{\mathcal{S}}$.

Theorem 1 *Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective setting for a denotational semantics* S. *If* S *is normal (Definition 4), or $\mathcal{S}$ is normal (Definition 16), then $\mid\!\sim_{\mathcal{S}}$ is an inconsistency-tolerant variant of* $\vdash_{\mathsf{S}}$.

Proof We show that the two properties in Definition 6 are satisfied in this case. First, we show that $\mid\!\sim_{\mathcal{S}}$ is faithful to $\vdash_{\mathsf{S}}$. For this, consider the following lemma:

Lemma 1 *Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective setting for a denotational semantics* $\mathsf{S} = \langle S, \models_S \rangle$. *Then for every theory Γ and every $\psi \in \Gamma$ it holds that $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi) = 0$ iff $\nu \in$* $\mathrm{mods}_{\mathsf{S}}(\psi)$.

Proof If $\nu \in \mathrm{mods}_{\mathsf{S}}(\psi)$, then by Reflexivity (Definition 8) and since $\mathsf{d}_{\mathcal{G}}(\Gamma)$ is an S-dissimilarity (Definition 12, Property (1)), we have that $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi) = \mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \nu) = 0$. For the converse, suppose that $\nu \notin \mathrm{mods}_{\mathsf{S}}(\psi)$. If ψ is *not* S-satisfiable, we are done, as $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi) = 1 + \max\{\mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \sigma) \mid \mu, \sigma \in S\} > 0$. Otherwise, let $\mu \in \mathrm{mods}_{\mathsf{S}}(\psi)$. Since $\mu \models_S \psi$ while $\nu \not\models_S \psi$, it holds that $\nu \not\sim_{\Gamma} \mu$ and so, by Property (3) in Definition 12, $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) > 0$. Hence, $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi)$ is a minimum of a set of strictly positive numbers, and so $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi) > 0$ as well. □

By Lemma 1 and since f is an aggregation function, for every theory Γ it holds that $\mathsf{M}_{\mathcal{S}}(\nu, \Gamma) = 0$ iff $\nu \in \mathrm{mods}_{\mathsf{S}}(\Gamma)$. Thus, for every S-consistent theory Γ and every $\nu \in S$, we have that $\nu \in \mathrm{mods}_{\mathsf{S}}(\Gamma)$ iff $\mathsf{M}_{\mathcal{S}}(\nu, \Gamma) = 0$, iff $\forall \mu \in S\ \mathsf{M}_{\mathcal{S}}(\nu, \Gamma) \leq \mathsf{M}_{\mathcal{S}}(\mu, \Gamma)$, iff $\nu \in \Delta_{\mathcal{S}}(\Gamma)$. It follows that if Γ is S-consistent, $\Delta_{\mathcal{S}}(\Gamma) = \mathrm{mods}_{\mathsf{S}}(\Gamma)$, and so $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ iff $\Gamma \vdash_{\mathsf{S}} \psi$. This shows faithfulness.

Next, we show non-explosiveness. If S is normal, then non-explosiveness follows from the fact that as $\mathcal{S}$ is effective, $\Delta_{\mathcal{S}}(\Gamma)$ is non-empty (Note 5), thus for every $\nu \in \Delta_{\mathcal{S}}(\Gamma)$ there is some ψ such that $\nu \not\models_S \psi$, and so $\Gamma \not\mid\!\sim_{\mathsf{S}} \psi$. If $\mathcal{S}$ is normal, non-explosiveness follows from the following lemma:

Lemma 2 *Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective and normal setting for* S. *For every Γ and every ψ which is $\mathcal{G}$-independent of Γ, it holds that $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ iff ψ is an* S*-tautology.*

Proof One direction is clear: if ψ is an S-tautology, then for every $\nu \in \Delta_{\mathcal{S}}(\Gamma)$, $\nu \models_S \psi$ and so $\Gamma \mid\!\sim_{\mathcal{S}} \psi$. For the converse, suppose that ψ is not an S-tautology. Since $\mathcal{S}$ is effective, by Note 5 there is an interpretation $\nu \in \Delta_{\mathcal{S}}(\Gamma)$. If $\nu \not\models_S \psi$, we are done: $\Gamma \not\mid\!\sim_{\mathcal{S}} \psi$. Otherwise, since $\mathsf{d}_{\mathcal{G}}$ is normal, there is some $\mu \in S$ such that $\mu \not\models_S \psi$, and $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$. But $\mathsf{d}_{\mathcal{G}(\Gamma)}$ is an S-dissimilarity and so, by Absorption (Definition 8), for every $\psi \in \Gamma$ and every $\nu_0 \in mod_S(\psi)$, $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \nu_0) = \mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \nu_0)$. Hence $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi) = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu, \psi)$, and so $\mathsf{M}_{\mathcal{S}}(\nu, \Gamma) = \mathsf{M}_{\mathcal{S}}(\mu, \Gamma)$ as well. Now, since $\nu \in \Delta_{\mathcal{S}}(\Gamma)$, it holds also that $\mu \in \Delta_{\mathcal{S}}(\Gamma)$, and since $\mu \not\models_S \psi$, we have that $\Gamma \not\mid\!\sim_{\mathcal{S}} \psi$. □

This concludes the proof of Theorem 1. □

It is important to note that while $\mid\!\sim_{\mathcal{S}}$ is faithful to $\vdash_{\mathsf{S}}$ (as implied by Theorem 1), unlike $\vdash_{\mathsf{S}}$, entailments of the form $\mid\!\sim_{\mathcal{S}}$ are usually *not* consequence relations (in the sense of Definition 1). In fact, as shown e.g. in [2], each property in Definition 1 may be violated already by distance-based entailments for classical logic. In the context of nonmonotonic reasoning, however, it is usual to consider the following weaker conditions that guarantee a 'proper behaviour' of nonmonotonic entailments in the presence of inconsistency (see, e.g., [3, 30, 33, 38]):

Theorem 2 *Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$ be an effective setting for a denotational semantics* S, *in which f is hereditary*[6]. *Then $\mid\!\sim_{\mathcal{S}}$ is a cautious consequence relation (in the sense of* [3]*), i.e., it has the following properties:*

Cautious Reflexivity:	*if Γ is* S*-satisfiable and $\psi \in \Gamma$ then $\Gamma \mid\!\sim_{\mathcal{S}} \psi$*
Cautious Monotonicity [21]*:*	*if $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ and $\Gamma \mid\!\sim_{\mathcal{S}} \phi$ then $\Gamma, \phi \mid\!\sim_{\mathcal{S}} \psi$*
Cautious Transitivity [30]*:*	*if $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ and $\Gamma, \psi \mid\!\sim_{\mathcal{S}} \phi$ then $\Gamma \mid\!\sim_{\mathcal{S}} \phi$*

Proof Cautious reflexivity follows from the facts that $\mid\!\sim_{\mathcal{S}}$ is faithful to $\vdash_{\mathsf{S}}$ and that $\vdash_{\mathsf{S}}$ is a Tarskian consequence relation, thus it is in particular reflexive.

For cautious monotonicity, let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and suppose that $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, $\Gamma \mid\!\sim_{\mathcal{S}} \phi$, and $\nu \in \Delta_{\mathcal{S}}(\Gamma \cup \{\psi\})$. We show that $\nu \in \Delta_{\mathcal{S}}(\Gamma)$ and since $\Gamma \mid\!\sim_{\mathcal{S}} \phi$ this implies that $\nu \in mod_{\mathsf{S}}(\{\phi\})$. Indeed, if $\nu \notin \Delta_{\mathcal{S}}(\Gamma)$, there is an element $\mu \in S$ such that $\mu \in \Delta_{\mathcal{S}}(\Gamma)$ so that $\mathsf{M}_{\mathcal{S}}(\mu, \Gamma) < \mathsf{M}_{\mathcal{S}}(\nu, \Gamma)$, i.e., $f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu, \gamma_1), \ldots, \mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu, \gamma_n)\}) < f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \gamma_1), \ldots, \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \gamma_n)\})$. Also, as $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, $\mu \in mod_{\mathsf{S}}(\{\psi\})$, thus $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu, \psi) = 0$. By these facts, and since f is hereditary,

$$\begin{aligned}
\mathsf{M}_{\mathcal{S}}(\mu, \Gamma \cup \{\psi\}) &= f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu, \gamma_1), \ldots, \mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu, \gamma_n), 0\}) \\
&< f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \gamma_1), \ldots, \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \gamma_n), 0\}) \\
&\leq f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \gamma_1), \ldots, \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \gamma_n), \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi)\}) \\
&= \mathsf{M}_{\mathcal{S}}(\nu, \Gamma \cup \{\psi\}),
\end{aligned}$$

a contradiction to $\nu \in \Delta_{\mathcal{S}}(\Gamma \cup \{\psi\})$.

For cautious transitivity, let again $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and assume that $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, $\Gamma, \psi \mid\!\sim_{\mathcal{S}} \phi$, and $\nu \in \Delta_{\mathcal{S}}(\Gamma)$. We show that $\nu \in mod_{\mathsf{S}}(\{\phi\})$. Indeed, since $\nu \in \Delta_{\mathcal{S}}(\Gamma)$, for

[6]An aggregation function f is called hereditary, if $f(\{x_1, \ldots, x_n\}) < f(\{y_1, \ldots, y_n\})$ implies that $f(\{x_1, \ldots, x_n, z_1, \ldots, z_m\}) < f(\{y_1, \ldots, y_n, z_1, \ldots, z_m\})$ (see [2]).

all $\mu \in S$, $f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\gamma_1),\ldots,\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\gamma_n)\}) \leq f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu,\gamma_1),\ldots,\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu,\gamma_n)\})$. Moreover, since $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, $\nu \in \mathrm{mod}_{\mathsf{S}}(\{\psi\})$, and so $\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\psi) = 0 \leq \mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu,\psi)$. It follows, then, that for every $\mu \in S$,

$$\begin{aligned}
\mathsf{M}_{\mathcal{S}}(\nu,\Gamma\cup\{\psi\}) =\ & f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\gamma_1),\ldots,\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\gamma_n),\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\psi)\}) \\
\leq\ & f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu,\gamma_1),\ldots,\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu,\gamma_n),\mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu,\psi)\}) \\
\leq\ & f(\{\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu,\gamma_1),\ldots,\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu,\gamma_n),\mathsf{m}^{\Gamma}_{\mathcal{S}}(\mu,\psi)\}) \\
=\ & \mathsf{M}_{\mathcal{S}}(\mu,\Gamma\cup\{\psi\}).
\end{aligned}$$

Thus, $\nu \in \Delta_{\mathcal{S}}(\Gamma\cup\{\psi\})$, and since $\Gamma,\psi \mid\!\sim_{\mathcal{S}} \phi$, necessarily $\nu \in \mathrm{mod}_{\mathsf{S}}(\{\phi\})$. □

The next proposition shows that many dissimilarity-based entailments of the form $\mid\!\sim_{\mathcal{S}}$ are *paraconsistent*.

Proposition 4 *Let* S *be a denotational semantics for a language* $\mathcal{L}$ *with a unary connective* $\neg$. *Let* $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f\rangle$ *be an effective and normal setting for* S, *in which* $\mathcal{G} \subseteq \mathcal{G}^{\mathsf{SF}}$. *Then* $\mid\!\sim_{\mathcal{S}}$ *is* $\neg$*-paraconsistent:* $\psi, \neg\psi \not\mid\!\sim_{\mathcal{S}} \phi$ *for some formulas* $\psi, \phi \in \mathcal{F}_{\mathcal{L}}$.

Proof For the proof, we observe the following lemma:

Lemma 3 Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f\rangle$ be an effective and normal setting for S, in which $\mathcal{G} \subseteq \mathcal{G}^{\mathsf{SF}}$, and let ψ be a formula that is $\mathcal{G}^{\mathsf{SF}}$-independent of a theory Γ. Then $\Gamma \mid\!\sim_{\mathcal{S}} \psi$ iff ψ is an S-tautology.

Proof Since $\mathcal{G}(\Gamma) \subseteq \mathsf{SF}(\Gamma)$ and $\mathcal{G}(\{\psi\}) \subseteq \mathsf{SF}(\psi)$, and since ψ is $\mathcal{G}^{\mathsf{SF}}$-independent of Γ, ψ is also $\mathcal{G}$-independent of Γ. The lemma now follows from Lemma 2. □

Now, let $p, q \in \mathsf{Atoms}$. Since $\{p, \neg p\}$ and q do not share variables, and since q is not an S-tautology, by Lemma 3 it follows that $p, \neg p \not\mid\!\sim_{\mathcal{S}} q$ and so $\mid\!\sim_{\mathcal{S}}$ is $\neg$-paraconsistent. □

Another interesting property of dissimilarity-based entailments is that their set of conclusions is always consistent (even for inconsistent sets of premises).

Proposition 5 *Let* $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f\rangle$ *be an effective setting for a denotational semantics* S. *Then for every theory* Γ *the set of the formulas that are* $\mid\!\sim_{\mathcal{S}}$*-entailed by* Γ *is* S*-consistent.*

Proof Let $\mathcal{C}_{\mathcal{S}}(\Gamma) = \{\psi \mid \Gamma \mid\!\sim_{\mathcal{S}} \psi\}$. If $\mathcal{C}_{\mathcal{S}}(\Gamma)$ is not S-consistent for some Γ, that is, if $\mathrm{mod}_{\mathsf{S}}(\mathcal{C}_{\mathcal{S}}(\Gamma)) = \emptyset$, then since $\Delta_{\mathcal{S}}(\Gamma) \subseteq \mathrm{mod}_{\mathsf{S}}(\psi)$ for every $\psi \in \mathcal{C}_{\mathcal{S}}(\Gamma)$, we have: $\Delta_{\mathcal{S}}(\Gamma) \subseteq \bigcap_{\psi\in\mathcal{C}_{\mathcal{S}}(\Gamma)} \mathrm{mod}_{\mathsf{S}}(\psi) = \mathrm{mod}_{\mathsf{S}}(\mathcal{C}_{\mathcal{S}}(\Gamma)) = \emptyset$. Thus $\Delta_{\mathcal{S}}(\Gamma) = \emptyset$, contradicting the fact that if $\mathcal{S}$ is effective, $\Delta_{\mathcal{S}}(\Gamma)$ is non-empty for every Γ (see Note 5). □

Finally, we consider the decidability of $\mid\!\sim_{\mathcal{S}}$. Checking whether $\Gamma \mid\!\sim_{\mathcal{S}} \psi$, i.e., whether the most plausible interpretations of Γ are models of ψ, may not be feasible for several reasons. For instance, interpretations may be infinite objects, and there can be infinitely many of them, so the dissimilarities among them may not be computable. To guarantee decidability one has to show that entailments in our framework can be reduced to terminating computations. This can be formalized as follows.

Definition 19 Let $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}_{\mathcal{G}}, f \rangle$ be a setting for a denotational semantics $\mathsf{S} = \langle S, \models \rangle$. We say that $\mathcal{S}$ is *computable*, if it satisfies the following conditions for every theory Γ:[7]

1. The functions $\mathcal{G}$, f, $\mathsf{d}_{\mathcal{G}}$ and $\mathsf{d}_{\mathcal{G}(\Gamma)}$ are computable.
2. There is a surjective function $\xi_\Gamma : S \to S_{\mathsf{fin}}$, where S_{fin} is some finite set of elements satisfying the following conditions:
   (a) For all $\nu, \mu \in S$: $\xi_\Gamma(\nu) = \xi_\Gamma(\mu)$ iff $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$.
   (b) Denote by $\xi_\Gamma^{-1}[s]$ the preimage of s, i.e., $\xi_\Gamma^{-1}[s] = \{\nu \in S \mid \xi(\nu) = s\}$. Then for every formula ψ and every $s \in S_{\mathsf{fin}}$ there is an effective way to decide whether $\xi_\Gamma^{-1}[s] \subseteq mods_{\mathsf{S}}(\psi)$.
3. There is a computable function $\mathsf{Fd}_{\mathcal{G}(\Gamma)} : S_{\mathsf{fin}} \times S_{\mathsf{fin}} \to \mathbb{R}^+$, such that for all $\nu, \mu \in S$, $\mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), \xi_\Gamma(\mu)) = \mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu)$.

Example 8 It can be verified that the setting $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}_{\mathcal{G}}, f \rangle$ for the classical two-valued semantics is computable for every context generator $\mathcal{G}$ in Example 5, all the dissimilarity generators $\mathsf{d}_{\mathcal{G}}$ in Example 6, and every computable aggregation function f. We show this for the case of $\mathsf{d}_{\mathcal{G}} = \mathsf{d}^{\mathsf{h}}_{\mathcal{G}}$ (where $\mathsf{d}^{\mathsf{h}}_{\mathcal{G}(\Gamma)}(\nu, \mu)$ is the number of formulas $\psi \in \mathcal{G}(\Gamma)$ for which $\nu(\psi) \neq \mu(\psi)$): Given a theory Γ, we let S_{fin} be the set of all the restrictions of classical two-valued valuations to $\mathcal{G}(\Gamma)$. For this Γ, we define the function $\xi_\Gamma : S \to S_{\mathsf{fin}}$ so that $\xi_\Gamma(\mu)$ is the restriction of μ to $\mathcal{G}(\Gamma)$. Clearly, $\xi_\Gamma(\nu) = \xi_\Gamma(\mu)$ iff $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$, since the latter means that ν and μ agree on all formulas in $\mathcal{G}(\Gamma)$. Now, for every $s \in S_{\mathsf{fin}}$ and every formula ψ, we can decide whether $\xi_\Gamma^{-1}[s]$ is contained in the set of the (classical) models of ψ by going over all possible extensions s' of s to $\mathsf{SF}(\mathcal{G}(\Gamma) \cup \{\psi\})$, which respect the classical truth tables, and checking whether $s'(\psi) = t$. Next, we define $\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s, s')$ as the number of formulas ψ in $\mathcal{G}(\Gamma)$, for which $s(\psi) \neq s'(\psi)$. Clearly, $\mathsf{Fd}_{\mathcal{G}(\Gamma)}$ is computable, and for every two classical valuations ν, μ it holds that $\mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), \xi_\Gamma(\mu)) = \mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu)$. Hence, the functions ξ_Γ and $\mathsf{Fd}_{\mathcal{G}(\Gamma)}$ defined above satisfy the conditions in Definition 19, and so $\mathcal{S}$ is computable.

Theorem 3 *Let $\mathcal{S}$ be a setting for a denotational semantics $\mathsf{S} = \langle S, \models_S \rangle$. If $\mathcal{S}$ is computable, then checking whether $\Gamma \mid\!\sim_{\mathcal{S}} \phi$ is decidable.*

[7] Below, when saying that a mathematical object that is related to Γ is 'computable', we actually mean that it is 'uniformly computable in Γ', in the sense that there is an effective way to determine if (and how) this object is computed (see also [44]).

Proof Since $\mathcal{S}$ is computable, for every theory Γ there are functions $\xi_\Gamma : S \to S_{\mathsf{fin}}$ and $\mathsf{Fd}_{\mathcal{G}(\Gamma)} : S_{\mathsf{fin}} \times S_{\mathsf{fin}} \to \mathbb{R}^+$ that satisfy the properties in Definition 19. For every $s \in S_{\mathsf{fin}}$ and $\psi \in \Gamma$, we define:

$$\mathsf{Fm}^\Gamma_{\mathcal{S}}(s, \psi) = \begin{cases} \min\{\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s, s') \mid \xi_\Gamma^{-1}[s'] \subseteq \mathrm{mod}_{\mathsf{S}}(\psi)\} & \text{if } \mathrm{mod}_{\mathsf{S}}(\psi) \neq \emptyset, \\ 1 + \max\{\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s', s'') \mid s', s'' \in S_{\mathsf{fin}}\} & \text{otherwise.} \end{cases}$$

$$\mathsf{FM}_{\mathcal{S}}(s, \Gamma) = f\big(\{\mathsf{Fm}^\Gamma_{\mathcal{S}}(s, \psi_1), \ldots, \mathsf{Fm}^\Gamma_{\mathcal{S}}(s, \psi_n)\}\big).$$

Accordingly, we define:

$$\Delta^{\mathsf{fin}}_{\mathcal{S}}(\Gamma) = \begin{cases} \{s \mid \forall s' \in S_{\mathsf{fin}}\ \mathsf{FM}_{\mathcal{S}}(s, \Gamma) \leq \mathsf{FM}_{\mathcal{S}}(s', \Gamma)\} & \text{if } \Gamma \neq \emptyset, \\ S_{\mathsf{fin}} & \text{otherwise.} \end{cases}$$

Lemma 4 *For every theory Γ, formula $\psi \in \Gamma$, and interpretation $\nu \in S$,*

$$\{\mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), s') \mid \xi_\Gamma^{-1}[s'] \subseteq \mathrm{mod}_{\mathsf{S}}(\psi)\} = \{\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) \mid \mu \in \mathrm{mod}_{\mathsf{S}}(\psi)\}.$$

Proof We denote the set on the left-hand side of the equation by S_1 and the set on the right-hand side of the equation by S_2. To see that $S_1 \subseteq S_2$, let $n \in S_1$. Then there is some $s' \in S_{\mathsf{fin}}$, such that $\mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), s') = n$ and $\xi_\Gamma^{-1}[s'] \subseteq \mathrm{mod}_{\mathsf{S}}(\psi)$. Let $\mu_0 \in \xi_\Gamma^{-1}[s']$ (the set $\xi_\Gamma^{-1}[s']$ is non-empty as the function ξ_Γ is surjective). Then $\mu_0 \in \mathrm{mod}_{\mathsf{S}}(\psi)$. We have that $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu_0) = \mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), \xi_\Gamma(\mu_0)) = \mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), s') = n$, and so $n \in S_2$. To see that $S_2 \subseteq S_1$, let $n \in S_2$. Then there is some $\mu \in \mathrm{mod}_{\mathsf{S}}(\psi)$, such that $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) = n$. Let $s' = \xi_\Gamma(\mu)$ and $\mu_0 \in \xi_\Gamma^{-1}[s']$. Then $\xi_\Gamma(\mu) = \xi_\Gamma(\mu_0)$, and by Property 2a of Definition 19, $\mathsf{d}_{\mathcal{G}(\Gamma)}(\mu_0, \mu) = 0$. By Property 3 of Definition 12, $\mu \sim_\Gamma \mu_0$, and so $\mu_0 \in \mathrm{mod}_{\mathsf{S}}(\psi)$ as well. Hence, $\xi_\Gamma^{-1}[s'] \subseteq \mathrm{mod}_{\mathsf{S}}(\psi)$, and it follows that $\mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), s') = \mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), \xi_\Gamma(\mu)) = \mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) = n$. Thus, $n \in S_1$. □

Lemma 5 *For every theory Γ and formula $\psi \in \Gamma$,*

$$\{\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s', s'') \mid s', s'' \in S_{\mathsf{fin}}\} = \{\mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \sigma) \mid \mu, \sigma \in S\}.$$

Proof Let $n \in \{\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s', s'') \mid s', s'' \in S_{\mathsf{fin}}\}$, i.e., $\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s', s'') = n$ for some $s', s'' \in S_{\mathsf{fin}}$. Let $\nu', \nu'' \in S$ such that $s' = \xi_\Gamma(\nu')$ and $s'' = \xi_\Gamma(\nu'')$ (their existence is guaranteed by the surjectiveness of ξ_Γ). Then we have that $\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu', \nu'') = \mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu'), \xi_\Gamma(\nu'')) = \mathsf{Fd}_{\mathcal{G}(\Gamma)}(s', s'') = n$, and so $n \in \{\mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \sigma) \mid \mu, \sigma \in S\}$. Conversely, suppose that $n \in \{\mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \sigma) \mid \mu, \sigma \in S\}$, i.e., $\mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \sigma) = n$ for some $\mu, \sigma \in S$. Let $s' = \xi_\Gamma(\mu)$ and $s'' = \xi_\Gamma(\sigma)$. Then $\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s', s'') = \mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\mu), \xi_\Gamma(\sigma)) = \mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \sigma) = n$, and so $n \in \{\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s', s'') \mid s', s'' \in S_{\mathsf{fin}}\}$. □

Corollary 1 *For every $\Gamma \in \mathcal{T}_{\mathcal{L}}$, $\psi \in \Gamma$ and $\nu \in S$, $\mathsf{Fm}^\Gamma_{\mathcal{S}}(\xi_\Gamma(\nu), \psi) = \mathsf{m}^\Gamma_{\mathcal{S}}(\nu, \psi)$.*

Proof If $\mathrm{mod}_{\mathsf{S}}(\psi) \neq \emptyset$, then by Lemma 4,

$$\begin{aligned} \mathsf{Fm}^\Gamma_{\mathcal{S}}(\xi_\Gamma(\nu), \psi) &= \min\{\mathsf{Fd}_{\mathcal{G}(\Gamma)}(\xi_\Gamma(\nu), s') \mid \xi_\Gamma^{-1}[s'] \subseteq \mathrm{mod}_{\mathsf{S}}(\psi)\} \\ &= \min\{\mathsf{d}_{\mathcal{G}(\Gamma)}(\nu, \mu) \mid \mu \in \mathrm{mod}_{\mathsf{S}}(\psi)\} = \mathsf{m}^\Gamma_{\mathcal{S}}(\nu, \psi). \end{aligned}$$

Otherwise, if $\text{mods}_{\mathsf{S}}(\psi) = \emptyset$, by Lemma 5,

$$\begin{aligned}\mathsf{Fm}^{\Gamma}_{\mathcal{S}}(\xi_\Gamma(\nu), \psi) &= 1 + \max\{\mathsf{Fd}_{\mathcal{G}(\Gamma)}(s', s'') \mid s', s'' \in S_{\mathsf{fin}}\}\\ &= 1 + \max\{\mathsf{d}_{\mathcal{G}(\Gamma)}(\mu, \sigma) \mid \mu, \sigma \in S\} = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu, \psi).\end{aligned}$$

□

By Corollary 1 it now follows that for every theory Γ and interpretation $\nu \in S$, $\mathsf{FM}_{\mathcal{S}}(\xi_\Gamma(\nu), \Gamma) = \mathsf{M}_{\mathcal{S}}(\nu, \Gamma)$. This implies that $\nu \in \Delta_{\mathcal{S}}(\Gamma)$ iff $\xi_\Gamma(\nu) \in \Delta^{\mathsf{fin}}_{\mathcal{S}}(\Gamma)$. Hence, the question whether $\Gamma \mid\!\sim_{\mathcal{S}} \phi$ is reducible to the question whether for each $s \in \Delta^{\mathsf{fin}}_{\mathcal{S}}(\Gamma)$, it holds that $\xi^{-1}[s] \subseteq \text{mods}_{\mathsf{S}}(\phi)$. It is easy to see that $\Delta^{\mathsf{fin}}_{\mathcal{S}}(\Gamma)$ is finite and computable (using the facts that $\mathcal{G}$, f, $\mathsf{d}_{\mathcal{G}}$ and $\mathsf{d}_{\mathcal{G}(\Gamma)}$ are computable). Also, by Property 2b of Definition 19, the question whether $\xi^{-1}[s] \subseteq \text{mods}_{\mathsf{S}}(\phi)$ is decidable for each $s \in \Delta^{\mathsf{fin}}_{\mathcal{S}}(\Gamma)$. Thus, checking whether $\Gamma \mid\!\sim_{\mathcal{S}} \phi$ is decidable. This proves Theorem 3. □

6 Applications

In this section we demonstrate the usefulness (and generality) of dissimilarity-based reasoning for defining a variety of inconsistency-tolerant logics based on different types of denotational semantics. In particular, we consider normal and effective settings, so the properties discussed in the previous section hold for these logics.

6.1 Multi-valued logics

The most standard way of defining multi-valued logics (including, of course, classical logic), is by the following structures (see, e.g., [23, 39, 49]):

Definition 20 A *(multi-valued) matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and $\mathcal{O}$ contains an interpretation $\tilde{\diamond} : \mathcal{V}^n \to \mathcal{V}$ for each n-ary connective $\diamond$ of $\mathcal{L}$.

Henceforth, we shall consider only *finite* matrices, i.e., matrices in which the set of truth values is finite. Given a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, we shall assume that $\mathcal{V}$ includes at least the two classical values t and f, and that only the former belongs to the set $\mathcal{D}$ of the *designated elements* in $\mathcal{V}$. Intuitively, $\mathcal{D}$ consists of the truth values that are assigned to 'true' assertions. The set $\mathcal{O}$ contains the interpretations (the 'truth tables') of each connective in $\mathcal{L}$. The associated semantical notions are now defined as usual: An $\mathcal{M}$*-valuation* is a function $\nu : \mathcal{F}_{\mathcal{L}} \to \mathcal{V}$ so that, for every connective $\diamond$ in $\mathcal{L}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n))$. The set of all $\mathcal{M}$-valuations is denoted by $\Lambda_{\mathcal{M}}$. We say that a valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$*-model* of ψ, denoted $\nu \models_{\mathcal{M}} \psi$, if $\nu(\psi) \in \mathcal{D}$. An $\mathcal{M}$-valuation ν is an $\mathcal{M}$-model of a theory Γ, if $\nu \models_{\mathcal{M}} \psi$ for every $\psi \in \Gamma$. When the matrix $\mathcal{M}$ is clear from the context we shall sometimes omit the prefix $\mathcal{M}$ from the notions defined above.

Note that the pair $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ is a denotational semantics in the sense of Definition 2. In what follows, we shall sometimes identify this semantics with the matrix $\mathcal{M}$ that defines it. In particular, we shall say that $\mathcal{M}$ is normal if so is the denotational semantics $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ that it induces. By Proposition 1 we have, then, that:

Proposition 6 *The relation $\vdash_{\mathcal{M}}$, induced by a matrix $\mathcal{M}$ by Definition 5, is a Tarskian consequence relation.*

Example 9 The most common matrix-based logics are induced by two-valued matrices. Thus, for instance, when $\mathcal{L}$ is the standard propositional language, $\mathcal{V} = \{t, f\}$, $\mathcal{D} = \{t\}$, and $\mathcal{O}$ consists of the standard interpretations of the connectives in $\mathcal{L}$, the pair $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ for $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is the classical propositional logic.

Three-valued logics are obtained by adding to $\mathcal{V}$ a third element. For instance, Kleene's logic [26] and McCarthy's logic [40] are obtained, respectively, from the matrices $\mathcal{M}_K^{3_\bot} = \langle \{t, f, \bot\}, \{t\}, \mathcal{O}_K \rangle$ and $\mathcal{M}_M^{3_\bot} = \langle \{t, f, \bot\}, \{t\}, \mathcal{O}_M \rangle$, in which the disjunction and conjunction are interpreted differently:

(Kleene)

	$\tilde{\neg}$
f	t
$\bot$	$\bot$
t	f

$\tilde{\wedge}$	f	$\bot$	t
f	f	f	f
$\bot$	f	$\bot$	$\bot$
t	f	$\bot$	t

$\tilde{\vee}$	f	$\bot$	t
f	f	$\bot$	t
$\bot$	$\bot$	$\bot$	t
t	t	t	t

(McCarthy)

$\tilde{\wedge}$	f	$\bot$	t
f	f	f	f
$\bot$	$\bot$	$\bot$	$\bot$
t	f	$\bot$	t

$\tilde{\vee}$	f	$\bot$	t
f	f	$\bot$	t
$\bot$	$\bot$	$\bot$	$\bot$
t	t	t	t

Priest's logic LP [42] is similar to Kleene's logic, but the third (middle) element is designated, so we denote it by $\top$ rather than $\bot$. This logic is induced by the matrix $\mathcal{M}_P^{3_\top} = \langle \{t, f, \top\}, \{t, \top\}, \mathcal{O}_P \rangle$, where $\mathcal{O}_P$ is obtained from $\mathcal{O}_K$ by replacing $\bot$ by $\top$.

Given a matrix-based denotational semantics $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ and a corresponding consequence relation $\vdash_{\mathcal{M}}$, one may define inconsistency-tolerant variants of $\vdash_{\mathcal{M}}$ by a dissimilarity-based setting $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f \rangle$, just as in Definition 18. Next, we describe such a construction.

Proposition 7 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a finite matrix for a propositional language $\mathcal{L}$, d a pseudo distance on $\mathcal{V}$, g an aggregation function, and $\mathcal{G}$ a context generator for $\mathcal{L}$. We define, for every $\Gamma \in \mathcal{T}_{\mathcal{L}}$, a function $\mathsf{d}^g_{\mathcal{G}(\Gamma)} : \Lambda_{\mathcal{M}} \times \Lambda_{\mathcal{M}} \to \mathbb{R}^+$ by:*

$$\mathsf{d}^g_{\mathcal{G}(\Gamma)}(\nu, \mu) = g\big(\{d(\nu(\psi), \mu(\psi)) \mid \psi \in \mathcal{G}(\Gamma)\}\big).$$

Then $\mathsf{d}^g_{\mathcal{G}((\Gamma)}$ is a dissimilarity function. Moreover,

– *if $\mathcal{G}^{\mathsf{At}} \subseteq \mathcal{G}$ or $\mathcal{G}^{\mathsf{ID}} \subseteq \mathcal{G}$, $\mathsf{d}^g_{\mathcal{G}}$ is a $\mathcal{G}$-dissimilarity generator,*[8]

[8]It is interesting to note that while context generators are needed only for defining normality of dissimilarity generators (see Definition 12), here the context $\mathcal{G}$ is crucial also for guaranteeing the other properties of a dissimilarity genertator.

– *if* $\mathcal{G}^{\mathsf{At}} \subseteq \mathcal{G}$ *and* $\mathcal{G}^{\mathsf{At}} \circ \mathcal{G} \subseteq \mathcal{G}$,[9] *then* $\mathsf{d}^g_{\mathcal{G}}$ *is a normal* $\mathcal{G}$*-dissimilarity generator.*

Proof Let us first show that $\mathsf{d}^g_{\mathcal{G}(\Gamma)}$ is a dissimilarity function. Indeed, since $d(\nu(\psi), \nu(\psi)) = 0$ and $g(\{0, \ldots, 0\}) = 0$, we have that $\mathsf{d}^g_{\mathcal{G}(\Gamma)}(\nu, \nu) = 0$, so $\mathsf{d}^g_{\mathcal{G}(\Gamma)}$ is reflexive. Clearly, it is also symmetric (since so is the pseudo distance d). For Absorption, suppose that $\mathsf{d}^g_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$. This necessarily means that $\nu(\psi) = \mu(\psi)$ for every $\psi \in \mathcal{G}(\Gamma)$ and so, for every $\sigma \in \Lambda_{\mathcal{M}}$,

$$\begin{aligned}\mathsf{d}^g_{\mathcal{G}(\Gamma)}(\nu, \sigma) = {} & g\big(\{d(\nu(\psi), \sigma(\psi)) \mid \psi \in \mathcal{G}(\Gamma)\}\big) = \\ = {} & g\big(\{d(\mu(\psi), \sigma(\psi)) \mid \psi \in \mathcal{G}(\Gamma)\}\big) = \mathsf{d}^g_{\mathcal{G}(\Gamma)}(\mu, \sigma).\end{aligned}$$

To see that $\mathsf{d}^g_{\mathcal{G}}$ is a $\mathcal{G}$-dissimilarity generator it remains to show Properties (2) and (3) in Definition 12. Indeed, Property (2) in that definition is guaranteed by the first condition in Definition 13 and the fact that the number of truth values of $\mathcal{M}$ is finite. For Property (3), note that if $\mathsf{d}^g_{\mathcal{G}(\Gamma)}(\nu, \mu) = 0$ then $\nu(\psi) = \mu(\psi)$ for every $\psi \in \mathcal{G}(\Gamma)$. Thus, if $\mathsf{Atoms}(\Gamma) = \mathcal{G}^{\mathsf{At}}(\Gamma) \subseteq \mathcal{G}(\Gamma)$, we have that $\nu(p) = \mu(p)$ for every $p \in \mathsf{Atoms}(\Gamma)$, which means that $\nu(\psi) = \mu(\psi)$ for every $\psi \in \Gamma$, and so $\nu \sim_\Gamma \mu$. Otherwise, if $\mathcal{G}^{\mathsf{ID}} \subseteq \mathcal{G}$ then again since $\nu(\psi) = \mu(\psi)$ for every $\psi \in \mathcal{G}(\Gamma)$ we have that $\nu(\psi) = \mu(\psi)$ for every $\psi \in \Gamma$, and so $\nu \sim_\Gamma \mu$.

To see that $\mathsf{d}^g_{\mathcal{G}}$ is normal, let ψ be a non-tautological formula that is $\mathcal{G}$-independent of Γ and suppose that $\sigma \notin \mathrm{mod}_{\mathcal{M}}(\psi)$. Since $\mathsf{Atoms}(\mathcal{G}(\Gamma)) \subseteq \mathcal{G}(\Gamma)$ and $\mathsf{Atoms}(\psi) \subseteq \mathcal{G}(\psi)$, this means that $\mathsf{Atoms}(\mathcal{G}(\Gamma)) \cap \mathsf{Atoms}(\psi) = \emptyset$. Thus, for every $\nu \in \mathrm{mod}_{\mathcal{M}}(\psi)$ there is a valuation $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p) = \nu(p)$ if $p \in \mathsf{Atoms}(\mathcal{G}(\Gamma))$ and $\mu(p) = \sigma(p)$ if $p \in \mathsf{Atoms}(\psi)$. Clearly, $\mu \notin \mathrm{mod}_{\mathcal{M}}(\psi)$, but since $\nu(\psi) = \mu(\psi)$ for every $\psi \in \mathcal{G}(\Gamma)$, still $\mathsf{d}^g_{\mathcal{G}}(\Gamma)(\nu, \mu) = 0$. □

Given a propositional logic $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, Proposition 7 yields a simple and general way of defining entailments that are inconsistency-tolerant variants of $\vdash_{\mathcal{M}}$. We note that many of the inconsistency-tolerant entailments that have been considered in the literature (e.g., those in [8, 22, 27, 29]) are particular cases of this construction, where $\mathcal{M}$ is a two-valued matrix and $\mathcal{G} = \mathcal{G}^{\mathsf{At}}$.

Example 10 Let $\mathcal{S} = \langle \mathcal{G}^{\mathsf{At}}, \mathsf{d}, \Sigma \rangle$ be a setting for Kleene's three-valued logic $\mathcal{M}^{3_\perp}_K$ (Example 9), where $\mathcal{G}^{\mathsf{At}}$ is the atom-based context generator, Σ is a summation function, and d is a $\mathcal{G}^{\mathsf{At}}$-dissimilarity generator, producing for each Γ the following dissimilarity:

$$\mathsf{d}^{\Sigma}_{\mathcal{G}^{\mathsf{At}}(\Gamma)}(\nu, \mu) = \Sigma\ \{d_0(\nu(p), \mu(p)) \mid p \in \mathcal{G}^{\mathsf{At}}(\Gamma)\}.$$

Here, d_0 is a pseudo distance on $\{t, f, \perp\}$, for which $d_0(t, f) = 1$ and $d_0(t, \perp) = d_0(f, \perp) = 0.5$ (see also [2, 19]). Note that the fact that d is a $\mathcal{G}^{\mathsf{At}}$-dissimilarity generator follows by Proposition 7.

[9] I.e., $\mathcal{G}^{\mathsf{At}}(\mathcal{G}(\Gamma)) \subseteq \mathcal{G}(\Gamma)$ for every Γ.

Now, consider the theory $\Gamma = \{\neg p, \neg q, p \vee q\}$. Clearly, Γ is not $\mathcal{M}_K^{3_\perp}$-satisfiable. We compute its most plausible interpretations with respect to $\mathcal{S}$:

	p	q	$\neg p$	$\neg q$	$p \vee q$	1	2	3	$\mathsf{M}_{\mathcal{S}}(\nu_i, \Gamma)$
ν_1	t	t	f	f	t	1	1	0	2
ν_2	t	f	f	t	t	1	0	0	**1**
ν_3	t	$\perp$	f	$\perp$	t	1	0.5	0	1.5
ν_4	f	t	t	f	t	0	1	0	**1**
ν_5	f	f	t	t	f	0	0	1	**1**
ν_6	f	$\perp$	t	$\perp$	$\perp$	0	0.5	0.5	**1**
ν_7	$\perp$	t	$\perp$	f	t	0.5	1	0	1.5
ν_8	$\perp$	f	$\perp$	t	$\perp$	0.5	0	0.5	**1**
ν_9	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	0.5	0.5	0.5	1.5

Legend. $1 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg p)$, $2 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg q)$, $3 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, p \vee q)$.

Hence, $\Delta_{\mathcal{S}}(\Gamma) = \{\nu_2, \nu_4, \nu_5, \nu_6, \nu_8\}$, and so, for instance, $\Gamma \mid\!\sim_{\mathcal{S}} \neg p \vee \neg q$ (even though $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg p$ and $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg q$).

6.2 Non-deterministic logics

Matrix-based semantics is truth-functional in the sense that the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas. However, real-world information is inescapably incomplete, uncertain, vague, imprecise or inconsistent, and these phenomena are in an obvious conflict with the principle of truth-functionality. One possible solution to this problem is to relax this principle by borrowing from automata and computability theory the idea of *non-deterministic* computations, and apply it in evaluations of truth-values of formulas. This leads to the idea of non-deterministic matrices [11], allowing non-deterministic evaluations of formulas. This kind of semantics has a variety of applications for reasoning under uncertainty (see, e.g., [12]).

Definition 21 A *non-deterministic matrix* (*Nmatrix*) for $\mathcal{L}$ is a tuple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where $\mathcal{V}$ is a non-empty set of truth values, $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, and $\mathcal{O}$ contains an interpretation function $\tilde{\diamond} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$. Again, we say that $\mathcal{M}$ is *finite* if so is $\mathcal{V}$.

An $\mathcal{M}$-*valuation* is a function $\nu : \mathcal{F}_{\mathcal{L}} \to \mathcal{V}$ such that for every connective $\diamond$ in $\mathcal{L}$,

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)).$$

The set of all $\mathcal{M}$-valuations is denoted by $\Lambda_{\mathcal{M}}$. Again, $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of ψ (denoted $\nu \models_{\mathcal{M}} \psi$), if $\nu(\psi) \in \mathcal{D}$.

Ordinary matrices can be thought of as Nmatrices whose interpretations return singletons of truth-values. Again, for an Nmatrix $\mathcal{M}$, the pair $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ is a denotational semantics and it induces a Tarskian consequence relation $\vdash_{\mathcal{M}}$.

Example 11 It is well-known that McCarthy's three-valued logics (Example 9) is appropriate for describing sequential (lazy) computations with errors. Its asymmetric conjunction and disjunction correspond to computation processes that halt after encountering the first error, evaluated from left to right. Thus, e.g., when $\nu(\psi) = \bot$, we have that $\nu(\psi \vee \phi) = \bot$ as well, regardless of $\nu(\phi)$. In turn, Kleene's three-valued logics (see again Example 9) is more appropriate for describing parallel computations with errors, since it has symmetric conjunction and disjunction.

Consider now a situation in which it is not known whether a certain system performs sequential or parallel computations, and that in each particular case it may apply a different kinds of computations. This scenario can be captured by the following non-deterministic matrix, combining Kleene's and McCarthy's three-valued interpretations of the connectives:

	$\tilde{\neg}$
f	$\{t\}$
$\bot$	$\{\bot\}$
t	$\{f\}$

$\tilde{\wedge}$	f	$\bot$	t
f	$\{f\}$	$\{f\}$	$\{f\}$
$\bot$	$\{f, \bot\}$	$\{\bot\}$	$\{\bot\}$
t	$\{f\}$	$\{\bot\}$	$\{t\}$

$\tilde{\vee}$	f	$\bot$	t
f	$\{f\}$	$\{\bot\}$	$\{t\}$
$\bot$	$\{\bot\}$	$\{\bot\}$	$\{t, \bot\}$
t	$\{t\}$	$\{t\}$	$\{t\}$

The suitability of the above Nmatrix for reasoning about computation errors is shown in [10]. In Example 12 below we shall demonstrate its use for dissimilarity-based reasoning.

The dissimilarities-based approach can be applied to the framework of Nmatrices in a way which is quite similar to the deterministic case. However, as was shown in [6], it is important to note that some dissimilarity generators (and the respective settings) that are definable with respect to standard matrices, are *not* applicable in the non-deterministic case. This is due to the fact that non-deterministic valuations are not truth-functional, so they can agree on atomic formulas, but may make different non-deterministic choices on complex formulas. This is also the reason why Proposition 7 is not extendable to non-deterministic semantics. Yet, a stricter version of that proposition does hold also in the non-deterministic case:

Proposition 8 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a finite Nmatrix for a propositional language $\mathcal{L}$, d a pseudo distance on $\mathcal{V}$, g an aggregation function, and $\mathcal{G}$ a context generator for $\mathcal{L}$. For each $\Gamma \in \mathcal{T}_{\mathcal{L}}$ define a function $\mathsf{d}^g_{\mathcal{G}(\Gamma)} : \Lambda_{\mathcal{M}} \times \Lambda_{\mathcal{M}} \to \mathbb{R}^+$ by:*

$$\mathsf{d}^g_{\mathcal{G}(\Gamma)}(\nu, \mu) = g\big(\{d(\nu(\psi), \mu(\psi)) \mid \psi \in \mathcal{G}(\Gamma)\}\big).$$

Then $\mathsf{d}^g_{\mathcal{G}((\Gamma)}$ is a dissimilarity function. Moreover,

- *if $\mathcal{G}^{\mathsf{ID}} \subseteq \mathcal{G}$ then $\mathsf{d}^g_{\mathcal{G}}$ is a $\mathcal{G}$-dissimilarity generator,*
- *if $\mathcal{G}^{\mathsf{SF}} \subseteq \mathcal{G}$ and $\mathcal{G}^{\mathsf{SF}} \circ \mathcal{G} \subseteq \mathcal{G}$, then $\mathsf{d}^g_{\mathcal{G}}$ is a normal $\mathcal{G}$-dissimilarity generator.*

Proof Similar to that of Proposition 7, leaving only the case that $\mathcal{G}^{\mathsf{ID}} \subseteq \mathcal{G}$ in the proof that $\mathsf{d}^g_{\mathcal{G}}$ is a $\mathcal{G}$-dissimilarity generator (note that this case covers also the condition that $\mathcal{G}^{\mathsf{SF}} \subseteq \mathcal{G}$), and replacing $\mathsf{Atoms}(\Gamma)$ by $\mathsf{SF}(\Gamma)$ in the proof of normality. □

Example 12 Let $\mathcal{S} = \langle \mathcal{G}^{\mathsf{SF}}, \mathsf{d}, \Sigma \rangle$ be a setting for the denotational semantics $\mathcal{M}^{3_\bot}_{KM}$, induced by the Nmatrix of Example 11, combining Kleene's and McCarthy's three-valued logics. Here, $\mathcal{G}^{\mathsf{SF}}$ is the context generator by subformulas, Σ is the summation function, and d is a $\mathcal{G}^{\mathsf{SF}}$-dissimilarity generator, producing for each Γ the following dissimilarity:

$$\mathsf{d}^{\Sigma}_{\mathcal{G}^{\mathsf{SF}(\Gamma)}}(\nu, \mu) = \Sigma \{d_0(\nu(\psi), \mu(\psi)) \mid \psi \in \mathsf{SF}(\Gamma)\},$$

where $d_0 : \{t, f, \perp\} \times \{t, f, \perp\} \to \{0, 0.5, 1\}$ is defined as in Example 10. The fact that d is indeed a $\mathcal{G}^{\mathsf{SF}}$-dissimilarity generator follows by Proposition 8.

Now, as in Example 10, we consider the theory $\Gamma = \{\neg p, \neg q, p \vee q\}$. It is easy to verify that Γ is not $\mathcal{M}^{3_\perp}_{KM}$-satisfiable, so we compute its most plausible interpretations. Note that this time, in addition to the nine valuations computed in Example 10, we now have an additional valuation, which stems from the fact that in $\mathcal{M}^{3_\perp}_{KM}$ the value of $\perp \tilde{\vee} t$ may be either t or $\perp$, so we need two valuations to represent this (denoted ν_{7a} and ν_{7b} in the table below) instead of just one valuation (ν_7 in the table of Example 10), as in the deterministic case. So we now have:

	p	q	$\neg p$	$\neg q$	$p \vee q$	1	2	3	$\mathsf{M}_{\mathcal{S}}(\nu_i, \Gamma)$
ν_1	t	t	f	f	t	2	2	0	4
ν_2	t	f	f	t	t	3	0	0	**3**
ν_3	t	$\perp$	f	$\perp$	t	2.5	1	0	3.5
ν_4	f	t	t	f	t	0	3	0	**3**
ν_5	f	f	t	t	f	0	0	3	**3**
ν_6	f	$\perp$	t	$\perp$	$\perp$	0	1.5	1.5	**3**
ν_{7a}	$\perp$	t	$\perp$	f	t	1	2.5	0	3.5
ν_{7b}	$\perp$	t	$\perp$	f	$\perp$	1.5	2	0.5	4
ν_8	$\perp$	f	$\perp$	t	$\perp$	1.5	0	1.5	**3**
ν_9	$\perp$	$\perp$	$\perp$	$\perp$	$\perp$	1	1	1.5	3.5

Legend. $1 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg p)$, $2 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg q)$, $3 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, p \vee q)$.

Thus, $\Delta_{\mathcal{S}}(\Gamma) = \{\nu_2, \nu_4, \nu_5, \nu_6, \nu_8\}$. Note that, restricted to $\mathsf{SF}(\Gamma)$, these are exactly the same most plausible interpretations of Γ as those obtained in Example 10 (for the deterministic matrix $\mathcal{M}^{3_\perp}_{K}$).

Note 6 Clearly, there are useful dissimilarity generators other than those that are covered by the construction of Proposition 8. One of them is the dissimilarity generator $\mathsf{d}^{\mathsf{n}}_{\mathcal{G}}$, in which $\mathsf{d}^{\mathsf{n}}_{\mathcal{G}(\Gamma)}(\nu, \mu)$ is the number of formulas $\psi \in \mathcal{G}(\Gamma)$, for which (i) $\nu(\psi) \neq \mu(\psi)$, and (ii) if $\psi = \diamond(\varphi_1, \ldots, \varphi_n)$, then for all $1 \leq i \leq n$: $\nu(\varphi_i) = \mu(\varphi_i)$. This function is one of the pseudo distances that are introduced in [6] for distance-based reasoning for non-deterministic matrices. It can be verified that $\mathsf{d}^{\mathsf{n}}_{\mathcal{G}}$ is also a $\mathcal{G}$-dissimilarity generator.

Example 13 Consider a transmission protocol for a system with three transmitters T_1, T_2 and T_3, where the first two are connected to a bus through an arbiter A, and the third one is connected directly to the bus. The bus has a line Msg for the transmitted message, and a line Busy, which is turned on whenever a transmission occurs. When one of the transmitters T_1 or T_2 has a message to transmit, it signals to the arbiter by turning on the line M_1 or M_2 respectively. The arbiter then turns on the line Busy, and T_i transmits its message on the line Msg. As for the third transmitter, whenever T_3 wants to transmit a message, it turns on M_3 and transmits the message

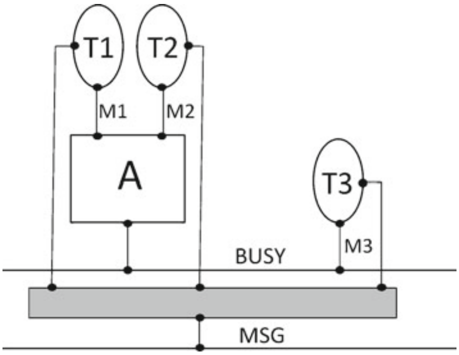

Fig. 1 The system of Example 13

on Msg. A schematic presentation of this circuit (excluding some details of the logic of Msg, which are not relevant for this example) is shown in Fig. 1.

Suppose now that the arbiter has no synchronization method, and whenever T_1 and T_2 request the line at the same time, the result is unpredictable: the line Busy can either stay on or be turned off. This non-deterministic behavior of the arbiter can be described using the following interpretation of the connective $\odot$

$\tilde{\odot}$	f	t
f	$\{f\}$	$\{t\}$
t	$\{t\}$	$\{t, f\}$

Let $\mathcal{M}$ be the two-valued Nmatrix for the language of $\{\neg, \vee, \odot\}$, which includes the above non-deterministic interpretation for $\odot$ and the standard interpretations of negation $\neg$ and disjunction $\vee$.

Next, suppose that we observe the following unexpected behavior of the arbiter: although T_1 has a message to transmit, while T_2 has none, the line Busy is not turned on. This can be captured by the following theory:

$$\Gamma = \{M_1, \neg M_2, \neg \mathsf{Busy}\}$$

where Busy is an abbreviation of the formula $(M_1 \odot M_2) \vee M_3$, representing the normal behavior of the line Busy. Obviously, this theory is not $\mathcal{M}$-satisfiable. For reasoning with this abnormality, we use the setting $\mathcal{S} = \langle \mathcal{G}^{\mathsf{SF}}, \mathsf{d}^{\mathsf{n}}_{\mathcal{G}^{\mathsf{SF}}}, \Sigma \rangle$, where $\mathsf{d}^{\mathsf{n}}_{\mathcal{G}^{\mathsf{SF}}}$ is the dissimilarity generator defined in Note 6. The dissimilarity computations for this case are represented in the table below:

	M_1	M_2	M_3	$\neg M_2$	$M_1 \odot M_2$	Busy	$\neg$Busy	1	2	3	$\mathsf{M}_{\mathcal{S}}(\nu_i, \Gamma)$
ν_1	t	t	t	f	t	t	f	0	1	2	3
ν_2	t	t	t	f	f	t	f	0	1	1	2
ν_3	t	t	f	f	t	t	f	0	1	1	2
ν_4	t	t	f	f	f	f	t	0	1	0	**1**
ν_5	t	f	t	t	t	t	f	0	0	2	2
ν_6	t	f	f	t	t	t	f	0	0	1	**1**
ν_7	f	t	t	f	t	t	f	1	1	2	4
ν_8	f	t	f	f	t	t	f	1	1	1	3
ν_9	f	f	t	t	f	t	f	1	0	1	2
ν_{10}	f	f	f	t	f	f	t	1	0	0	**1**

Legend. $1 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, M_1)$, $2 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg M_2)$, $3 = \mathsf{m}^{\Gamma}_{\mathcal{S}}(\nu_i, \neg \mathsf{Busy})$.

Hence, $\Delta_{\mathcal{S}}(\Gamma) = \{\nu_4, \nu_6, \nu_{10}\}$, and so, for instance, $\Gamma \mid\!\sim_{\mathcal{S}} \neg M_3$, even though neither M_1 nor $\neg M_2$ are $\mid\!\sim_{\mathcal{S}}$-inferrable from Γ. Hence, although one can say nothing about T_1 and T_2, it is still possible to conclude in this case that T_3 has no message to transmit.

6.3 Modal logics

Next, we consider a denotational semantics that is based on a many-valued extension of standard Kripke semantics (see [20]), where the logical connectives are interpreted by a finite matrix $\mathcal{M}$,[10] and qualifications of the truth of a judgement are expressed by the necessitation operator "$\Box$". In case of the standard two-valued matrix we get the usual Kripke-style (possible worlds) semantics.

Definition 22 Let $\mathcal{L}$ be a propositional language.

- A *frame* for $\mathcal{L}$ is a triple $\mathcal{F}r = \langle W, R, \mathcal{M}\rangle$, where W is a non-empty set (of "worlds"), R (the "accessibility relation") is a binary relation on W, and $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}\rangle$ is a matrix for $\mathcal{L}$. We say that a frame is finite if so is W.
- Let $\mathcal{F}r = \langle W, R, \mathcal{M}\rangle$ be a frame for $\mathcal{L}$. An *$\mathcal{F}r$-valuation* is a function $\nu : W \times F_{\mathcal{L}} \to \mathcal{V}$ that assigns truth values to the $\mathcal{L}$-formulas at each world in W according to the following conditions:
  - For every connective $\diamond$, $\nu(w, \diamond(\psi_1, \ldots, \psi_n)) = \tilde{\diamond}_{\mathcal{M}}(\nu(w, \psi_1), \ldots, \nu(w, \psi_n))$,
  - $\nu(w, \Box\psi) \in \mathcal{D}$ iff $\nu(w', \psi) \in \mathcal{D}$ for all w' such that $R(w, w')$.

  The set of $\mathcal{F}r$-valuations is denoted by $\Lambda_{\mathcal{F}r}$. The set of $\mathcal{F}r$-valuations that satisfy a formula ψ in a world $w \in W$ is $\mathrm{mod}^{w}_{\mathcal{F}r}(\psi) = \{\nu \in \Lambda_{\mathcal{F}r} \mid \nu(w, \psi) \in \mathcal{D}\}$.
- A *frame interpretation* is a pair $I = \langle \mathcal{F}r, \nu\rangle$, in which $\mathcal{F}r = \langle W, R, \mathcal{M}\rangle$ is a frame and ν is an $\mathcal{F}r$-valuation. We say that I *satisfies* ψ (or that I is a model of ψ), if $\nu \in \mathrm{mod}^{w}_{\mathcal{F}r}(\psi)$ for every $w \in W$. We say that I satisfies Γ if it satisfies every $\psi \in \Gamma$.

Definition 23 A set $\mathcal{I} = \{\langle\langle W_i, R_i, \mathcal{M}\rangle, \nu_i\rangle \mid i = 1, 2, \ldots\}$ of frame interpretations is called $\mathcal{M}$-closed, if for each interpretation $\langle \mathcal{F}r, \nu\rangle \in \mathcal{I}$ and $\mu \in \Lambda_{\mathcal{M}}$ there is an interpretation $\langle \mathcal{F}r, \mu\rangle \in \mathcal{I}$.

Let $\mathcal{I}$ be a non-empty set of frame interpretations. We define a satisfaction relation $\models_{\mathcal{I}}$ on $\mathcal{I} \times F_{\mathcal{L}}$ by $I \models_{\mathcal{I}} \psi$ iff I satisfies ψ. Note that $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}}\rangle$ is a denotational semantics in the sense of Definition 2. By Proposition 1, then, the induced relation $\vdash_{\mathfrak{I}}$ is a Tarskian consequence relation for $\mathcal{L}$.

Given a possible-world semantics $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}}\rangle$, it is possible to define an inconsistency-tolerant variant of $\vdash_{\mathfrak{I}}$ by introducing a dissimilarity-based setting $\mathcal{S} = \langle \mathcal{G}, \mathsf{d}, f\rangle$ and applying the definitions in Section 4. As frame interpretations are more complicated semantic structures than those considered in the previous sections, defining intuitive and simple dissimilarity generators is more challenging in this case. Below, we consider a simple and useful case: a set of frame interpretations in which all the frames share the same set of worlds and accessibility relation. In this case, dissimilarity between frame interpretations may be defined by comparing valuations in each world and then aggregating over the worlds:

[10] This framework can be extended to Nmatrices as well, but for simplicity we stick to deterministic matrices.

Proposition 9 *Let $\mathfrak{I} = \langle \mathcal{I}, \models_{\mathcal{I}} \rangle$ be a possible world semantics, where $\mathcal{I} = \{\langle\langle W, R, \mathcal{M}\rangle, \nu_i\rangle \mid i = 1, 2, \ldots\}$ and W is finite. Let $\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}$ be a $\mathcal{G}$-dissimilarity generator for $\langle \Lambda_{\mathcal{M}}, \models_{\mathcal{M}} \rangle$ (e.g., of the form considered in Proposition 7). We define, for an aggregation function g, a function $\mathsf{d}_{\mathcal{G}}^{g}$, such that for any Γ and frame interpretations $I_1 = \langle\langle W, R, \mathcal{M}\rangle, \nu_1\rangle$ and $I_2 = \langle\langle W, R, \mathcal{M}\rangle, \nu_2\rangle \in \mathcal{I}$,*

$$\mathsf{d}_{\mathcal{G}}^{g}(\Gamma)(I_1, I_2) = g\big(\{\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}(\Gamma)(\nu_1(w), \nu_2(w)) \mid w \in W\}\big)$$

where, for $i = 1, 2$, $\nu_i(w)$ is the 'restriction' of ν_i to the world w, that is: for every formula ψ, $\nu_i(w)(\psi) = \nu_i(w, \psi)$. Then:

(a) *$\mathsf{d}_{\mathcal{G}}^{g}$ is a $\mathcal{G}$-dissimilarity generator for $\mathfrak{I}$,*
(b) *if $\mathcal{I}$ is $\mathcal{M}$-closed and $\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}$ is normal, then $\mathsf{d}_{\mathcal{G}}^{g}$ is a normal $\mathcal{G}$-dissimilarity generator for $\mathfrak{I}$.*

Proof The fact that for every theory Γ the function $\mathsf{d}_{\mathcal{G}}^{g}(\Gamma)$ is a $\mathcal{G}$-dissimilarity follows by the facts that $\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}(\Gamma)$ is a $\mathcal{G}$-dissimilarity function for $\mathcal{M}$ and that g is an aggregation function. These facts also assure the second condition in Definition 12. To see that $\mathsf{d}_{\mathcal{G}}^{g}$ satisfies also the last condition in that definition, suppose that $\mathsf{d}_{\mathcal{G}}^{g}(\Gamma)(I_1, I_2) = 0$. Then $\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}(\Gamma)(\nu_1(w), \nu_2(w)) = 0$ for all $w \in W$ and so $\nu_1(w) \sim_{\Gamma} \nu_2(w)$ for every $w \in W$. Since the frame interpretations have the same accessibility relation, for every $\psi \in \Gamma$ and every $w \in W$, $\nu_1(w) \in \mathrm{mod}_{\mathcal{M}}(\psi)$ iff $\nu_2(w) \in \mathrm{mod}_{\mathcal{M}}(\psi)$. It follows that for every $\psi \in \Gamma$, $I_1 \models_{\mathcal{I}} \psi$ iff $I_2 \models_{\mathcal{I}} \psi$, and so $I_1 \sim_{\Gamma} I_2$.

For Item (b), let ϕ be a non-tautological formula that is $\mathcal{G}$-independent of Γ and let $I_1 = \langle\langle W, R, \mathcal{M}\rangle, \nu_1\rangle \in \mathrm{mod}_{\mathfrak{I}}(\phi)$. Then $\nu_1(w) \in \mathrm{mod}_{\mathcal{M}}(\phi)$ for every $w \in W$. Now, since $\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}$ is a normal $\mathcal{G}$-dissimilarity generator for $\mathcal{M}$, for every $w \in W$ there is some $\mu_2(w) \in \Lambda_{\mathcal{M}}$ such that $\mu_2(w) \notin \mathrm{mod}_{\mathcal{M}}(\phi)$ but still $\mathsf{d}_{\mathcal{G}}^{\mathcal{M}}(\Gamma)(\nu_1(w), \mu_2(w)) = 0$. Let $\nu_2 \in \Lambda_{\mathcal{M}}$ be any valuation such that for every world $w \in W$ and formula σ, $\nu_2(w, \sigma) = \mu_2(w)(\sigma)$. Since $\mathcal{I}$ is $\mathcal{M}$-closed, $I_2 = \langle\langle W, R, \mathcal{M}\rangle, \nu_2\rangle \in \mathcal{I}$. Moreover, $I_2 \not\models_{\mathcal{I}} \phi$, but still $\mathsf{d}_{\mathcal{G}}^{g}(\Gamma)(I_1, I_2) = 0$. Thus $\mathsf{d}_{\mathcal{G}}^{g}$ is normal. □

Example 14 A committee of three people, P_1, P_2 and P_3 should nominate two or less candidates among a and b for a governmental position. A committee member P_i may consult with any other member. Moreover, P_i will vote for $\mathsf{x} \in \{\mathsf{a}, \mathsf{b}\}$ only if P_i believes that x is qualified, and no member that P_i consults with believes otherwise. A candidate is recommended only upon a consensus.

A journalist J wants to predict the committee's recommendation, based on his partial knowledge about the committee and some leaking rumors. Suppose that he knows that P_1 believes that both candidates are qualified and that P_2 believes that b is qualified. Moreover, J knows that P_1 and P_2 consult with P_3, but P_3 never asks anyone else for advice.

This situation can be represented by the classical matrix and a modal language $\mathcal{L} = \{\Box, \wedge, \neg\}$. The atoms $\mathsf{q_a}$ and $\mathsf{q_b}$ respectively represent the belief that a and b are qualified, and the formula $\mathsf{Q_x} = \Box \mathsf{q_x}$ indicates that x is in the list of qualified candidates. Each world is associated with a committee member: $W = \{P_1, P_2, P_3\}$. Accessibility between worlds indicates a consulting relation between the members, thus: $R = \{\langle P_1, P_1\rangle, \langle P_2, P_2\rangle, \langle P_3, P_3\rangle, \langle P_1, P_3\rangle, \langle P_2, P_3\rangle\}$. The corresponding frame is shown in Fig. 2.

Next, suppose that two contradictory rumors are brought to J's attention: according to one, the list includes the names of both a and b. According to the other,

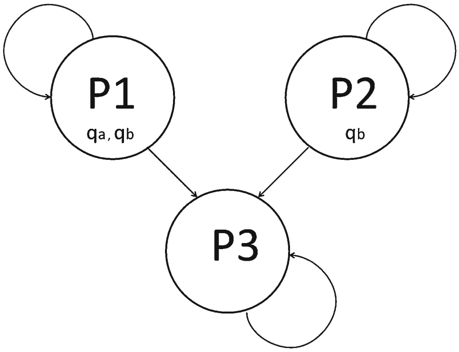

Fig. 2 The frame of Example 14

at least one of the members disqualified a. This information may be represented by $\Gamma = \{\neg\Box \mathsf{q_a}, \Box \mathsf{q_a} \wedge \Box \mathsf{q_b}\}$. For maintaining this contradictory theory, J uses $\mid\!\sim_{\mathcal{S}}$, induced by the setting $\mathcal{S} = \langle \mathcal{G}^{\mathsf{At}}, \mathsf{d}^{\Sigma}, \Sigma \rangle$, where d^{Σ} is the dissimilarity-generator defined like in Proposition 9 for $g = \Sigma$ and $d_{\mathcal{G}}^{\mathcal{M}} = d_{\mathcal{G}}^{\mathsf{h}}$. Note that since $\mathfrak{I}$ is normal, by Theorem 1 and Proposition 9, $\mid\!\sim_{\mathcal{S}}$ is inconsistency-tolerant.

The dissimilarity calculations for the frame interpretations that correspond to the partial knowledge of J are given below (where $\psi_{\mathsf{x}}^{\mathsf{i}}$, for $\psi \in \{\mathsf{q}, \mathsf{Q}\}$, $\mathsf{x} \in \{\mathsf{a}, \mathsf{b}\}$ and $1 \leq i \leq 3$, denotes the value of the formula ψ_{x} in the world P_i).

	$\mathsf{q_a^1}$	$\mathsf{q_a^2}$	$\mathsf{q_a^3}$	$\mathsf{q_b^1}$	$\mathsf{q_b^2}$	$\mathsf{q_b^3}$	$\mathsf{Q_a^1}$	$\mathsf{Q_a^2}$	$\mathsf{Q_a^3}$	$\mathsf{Q_b^1}$	$\mathsf{Q_b^2}$	$\mathsf{Q_b^3}$	1	2	3
I_1	*t*	*t*	*t*	*t*	*t*	*t*	*t*	*t*	*t*	*t*	*t*	*t*	1	0	**1**
I_2	*t*	*t*	*t*	*t*	*t*	*f*	*t*	*t*	*t*	*f*	*f*	*f*	1	1	2
I_3	*t*	*t*	*f*	*t*	*t*	*t*	*f*	*f*	*f*	*t*	*t*	*t*	0	1	**1**
I_4	*t*	*t*	*f*	*t*	*t*	*f*	*f*	*f*	*f*	*f*	*f*	*f*	0	2	2
I_5	*t*	*f*	*t*	*t*	*t*	*t*	*t*	*f*	*t*	*t*	*t*	*t*	0	1	**1**
I_6	*t*	*f*	*t*	*t*	*t*	*f*	*t*	*f*	*t*	*f*	*f*	*f*	0	2	2
I_7	*t*	*f*	*f*	*t*	*t*	*t*	*f*	*f*	*f*	*t*	*t*	*t*	0	2	2
I_8	*t*	*f*	*f*	*t*	*t*	*f*	*f*	*f*	*f*	*f*	*f*	*f*	0	3	3

Legend. $1 = \mathsf{m}_{\mathcal{S}}^{\Gamma}(I_i, \neg \mathsf{Q}_a)$, $2 = \mathsf{m}_{\mathcal{S}}^{\Gamma}(I_i, \mathsf{Q}_a \wedge \mathsf{Q}_b))$, $3 = \mathsf{M}_{\mathcal{S}}(i, \Gamma)$.

Thus, $\Delta_{\mathcal{S}}(\Gamma) = \{I_1, I_3, I_5\}$, and so $\Gamma \mid\!\sim_{\mathcal{S}} \mathsf{Q_b}$ while $\Gamma \not\mid\!\sim_{\mathcal{S}} \mathsf{Q_a}$, $\Gamma \not\mid\!\sim_{\mathcal{S}} \neg \mathsf{Q_a}$. Based on his knowledge, then, J may assume that b will be nominated, while nothing can be predicted about a.

7 Conclusion

We have introduced an abstract and modular framework of supplementing different logics, based on denotational semantics, with some extra apparatus of inconsistency tolerance. To obtain an inconsistency-tolerant variant of one's favorite logic (defined by some denotational semantics S), one simply needs to choose an appropriate semantic setting for S according to some application-specific considerations. This automatically induces an operator that relates each theory to its most plausible interpretations, and so a corresponding inconsistency-tolerant variant of the original logic is available. Our framework is schematically depicted in Fig. 3.

Similar methods for generating such logics were already introduced, e.g., in [2] for deterministic matrices, and in [6] for two-valued non-deterministic matrices. However, all these methods heavily depend on their underlying semantics. The

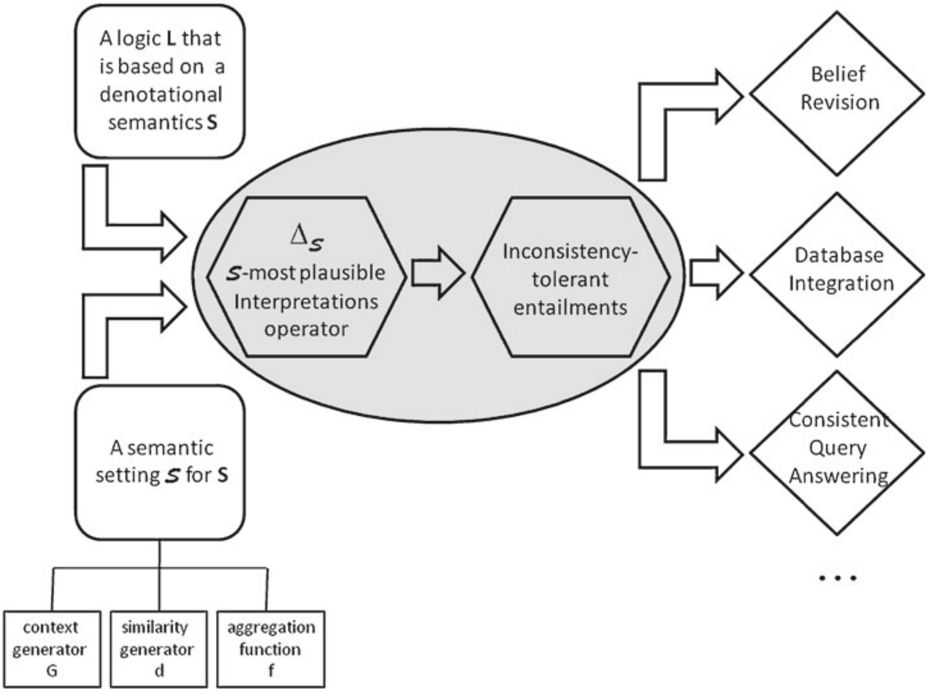

Fig. 3 The dissimilarity-based framework

framework presented here, however, is substantially more general—it assumes nothing about the underlying semantics, except for its being denotational. This is done by identifying what a "similarity" between abstract semantic entities is, and what properties it should satisfy. In particular, we have found that the standard notion of a distance is not adequate for our purpose, and that a more general notion is required. The generalized notion of a dissimilarity admits the definition of some preferential logics that are not even cumulative (the weakest family of preferential logics considered in the well-known framework of Makinson [38] and Kraus-Lehmann-Magidor [30]), but which still have some merit for AI applications. We have shown, moreover, that our approach may be used for extending traditional distance-related methodologies in the context of revision and merging systems [27, 29, 34], cardinality-based methods for database repair [5, 13, 37] and consistent query answering [1, 47], and forgetting-based approaches to reasoning with inconsistency [32, 35].

An important subject for future research is a comparative study of the different entailment relations that are induced by different dissimilarity-based settings. This involves extensions to the nonmonotonic case of works such as that in [4] (which introduces a list of desirable properties that paraconsistent consequence relations should have[11]). A better understanding of the relationships between semantic settings and the entailment relations that they induce will be helpful in providing guidelines on how to adapt semantic settings to application-specific needs. Other

[11] Unlike our case, then, the work in [4] refers to *monotonic* logics (see Definition 1).

directions for future work include the extension of our framework to the first-order case, and the incorporation of semantics that are not denotational, such as the one in [15], which is induced by ordinal conditional functions [46].

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