Chapter 4
Three-Valued Paraconsistent Propositional Logics

Ofer Arieli and Arnon Avron

Abstract Three-valued matrices provide the simplest semantic framework for introducing paraconsistent logics. This paper is a comprehensive study of the main properties of propositional paraconsistent three-valued logics in general, and of the most important such logics in particular. For each logic in the latter group, we also provide a corresponding cut-free Gentzen-type system.

Keywords Paraconsistency · 3-valued matrices · Proof systems

Mathematics Subject Classification (2000) Primary 03B53, 03B50 · Secondary 03C90, 03F05

4.1 Introduction

It is well known that classical logic is not adequate for reasoning with inconsistent information. One of the oldest and the most common approaches to overcome this shortcoming of classical logic is to enrich the set of truth-values with a third element other than the two classical ones t and f. Indeed, since their introduction by Łukasiewicz [28] (see also [29]), three-valued logics have been extensively studied for uncertainty reasoning in general, and paraconsistent reasoning in particular (see, e.g., [4, 7, 16, 20], which in turn contain references to many other works). The goal of this work is to study this approach to paraconsistency in a systematic way, as well as to present what we believe to be the most important results concerning the better

This work is supported by The Israel Science Foundation under grant agreement No. 817-15.

O. Arieli (✉)
School of Computer Science, The Academic College of Tel-Aviv, Tel Aviv, Israel
e-mail: oarieli@mta.ac.il

A. Avron
School of Computer Science, Tel-Aviv University, Tel Aviv, Israel
e-mail: aa@cs.tau.ac.il

J.-Y. Beziau et al. (eds.), *New Directions in Paraconsistent Logic*,
Springer Proceedings in Mathematics & Statistics 152,
DOI 10.1007/978-81-322-2719-9_4

accepted logics that came out from this approach. However, it should be emphasized that the scope of the material we present is limited according to the following criteria:

1. The languages that are considered in the sequel are *propositional*, as this is the heart of every paraconsistent logic ever studied so far.
2. We confine ourselves to paraconsistent propositional *logics*, in which a propositional language is equipped with a structural and nontrivial Tarskian consequence relation. In particular, no form of nonmonotonic reasoning is considered in this paper.
3. We restrict ourselves here to logics which are based on *truth-functional* three-valued semantics.[1]

The rest of the paper is organized as follows: In the next section, we review some general definitions and basic concepts that are needed in the sequel. In Sect. 4.3, we define in precise terms what paraconsistent logics are, and what additional properties they expected to have. These properties are then investigated in the context of three-valued matrices in Sect. 4.4. The most important logics that are induced by these matrices are considered in Sect. 4.5, and corresponding proof systems are discussed in Sect. 4.6.

4.2 Preliminaries

4.2.1 Propositional Logics

In what follows a propositional language with a set $\mathsf{Atoms}(\mathcal{L}) = \{P_1, P_2, \ldots\}$ of atomic formulas is denoted by $\mathcal{L}$ and use p, q, r to vary over this set. The set of the well-formed formulas of $\mathcal{L}$ is denoted by $\mathcal{W}(\mathcal{L})$ and $\varphi, \psi, \phi, \sigma$ will vary over its elements. The set $\mathsf{Atoms}(\varphi)$ denotes the atomic formulas occurring in φ. Sets of formulas in $\mathcal{W}(\mathcal{L})$ are called *theories* and are denoted by $\mathcal{T}$ or $\mathcal{T}'$. Finite theories are denoted by Γ or Δ. Following the usual convention, we shall abbreviate $\mathcal{T} \cup \{\psi\}$ by $\mathcal{T}, \psi$. More generally, we shall write $\mathcal{T}, \mathcal{T}'$ instead of $\mathcal{T} \cup \mathcal{T}'$. A *rule* in a language $\mathcal{L}$ is a pair $\langle \Gamma, \psi \rangle$, where $\Gamma \cup \{\psi\}$ is a finite set of formulas in $\mathcal{L}$. We shall henceforth denote such a rule by Γ/ψ.

Definition 4.1 A (Tarskian) *consequence relation* for a language $\mathcal{L}$ (a tcr, for short) is a binary relation $\vdash$ between theories in $\mathcal{W}(\mathcal{L})$ and formulas in $\mathcal{W}(\mathcal{L})$, satisfying the following three conditions:

[1]When truth functionality is not required, further approaches based on nondeterministic semantics [10] are available. They give rise to another brand of useful three-valued logics, which includes many of the LFIs considered in [16]. We refer the reader to [12, 13] for further information on these logics and references to related papers.

Reflexivity: if $\psi \in \mathcal{T}$ then $\mathcal{T} \vdash \psi$.
Monotonicity: if $\mathcal{T} \vdash \psi$ and $\mathcal{T} \subseteq \mathcal{T}'$, then $\mathcal{T}' \vdash \psi$.
Transitivity (cut): if $\mathcal{T} \vdash \psi$ and $\mathcal{T}', \psi \vdash \phi$ then $\mathcal{T}, \mathcal{T}' \vdash \phi$.

Let $\vdash$ be a tcr for $\mathcal{L}$. We say that $\vdash$ is

- *structural*, if for every $\mathcal{L}$-substitution θ and every $\mathcal{T}$ and ψ, if $\mathcal{T} \vdash \psi$ then $\{\theta(\varphi) \mid \varphi \in \mathcal{T}\} \vdash \theta(\psi)$.
- *nontrivial*, if there exist some nonempty theory $\mathcal{T}$ and some formula ψ such that $\mathcal{T} \nvdash \psi$.
- *finitary*, if for every theory $\mathcal{T}$ and every formula ψ such that $\mathcal{T} \vdash \psi$ there is a *finite* theory $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \vdash \psi$.

Definition 4.2 A (propositional) *logic* is a pair $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$, such that $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural and nontrivial[2] consequence relation for $\mathcal{L}$. A logic $\langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ is *finitary* if so is $\vdash_{\mathbf{L}}$.

Definition 4.3 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, and let S be a set of rules in $\mathcal{L}$. The *finitary* $\mathbf{L}$*-closure* $C_{\mathbf{L}}(S)$ of S is inductively defined as follows:

- $\langle \theta(\Gamma), \theta(\psi) \rangle \in C_{\mathbf{L}}(S)$, whenever θ is a uniform $\mathcal{L}$-substitution, Γ is a *finite* theory in $\mathcal{W}(\mathcal{L})$, and either $\Gamma \vdash \psi$ or $\Gamma/\psi \in S$.
- If the pairs $\langle \Gamma_1, \varphi \rangle$ and $\langle \Gamma_2 \cup \{\varphi\}, \psi \rangle$ are both in $C_{\mathbf{L}}(S)$, then so is the pair $\langle \Gamma_1 \cup \Gamma_2, \psi \rangle$.

Next we define what an extension of a logic means.

Definition 4.4 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, and let S be a set of rules in $\mathcal{L}$.

- A logic $\mathbf{L}' = \langle \mathcal{L}, \vdash' \rangle$ is an *extension* of $\mathbf{L}$ (in the same language) if $\vdash \subseteq \vdash'$. We say that $\mathbf{L}'$ is a *proper extension* of $\mathbf{L}$, if $\vdash \subsetneq \vdash'$.
- The *extension of* $\mathbf{L}$ *by* S is the pair $\mathbf{L}^* = \langle \mathcal{L}, \vdash^* \rangle$, where $\vdash^*$ is the binary relation between theories in $\mathcal{W}(\mathcal{L})$ and formulas in $\mathcal{W}(\mathcal{L})$, defined by: $\mathcal{T} \vdash^* \psi$ if there is a finite $\Gamma \subseteq \mathcal{T}$ such that $\langle \Gamma, \psi \rangle \in C_{\mathbf{L}}(S)$.[3]
- Extending $\mathbf{L}$ by an axiom schema φ means extending it by the rule $\emptyset/\varphi$.

The usefulness of a logic strongly depends on the question that what kind of connectives are available in it. Three particularly important types of connectives are defined next.

[2]The condition of nontriviality is not always demanded in the literature, but we find it very convenient (and natural) to include it here.

[3]Note that $\mathbf{L}^*$ is a propositional logic unless $C_{\mathbf{L}}(S)$ contains *all* the pairs of finite theories in $\mathcal{W}(\mathcal{L})$ and formulas in $\mathcal{W}(\mathcal{L})$. Moreover, $\mathbf{L}^*$ is in that case the minimal extension of $\mathbf{L}$ such that $\Gamma \vdash^* \varphi$ whenever $\Gamma/\varphi \in S$.

Definition 4.5 Let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a propositional logic.

- A binary connective $\supset$ of $\mathcal{L}$ is called an *implication for* $\mathbf{L}$ if the classical deduction theorem holds for $\supset$ and $\vdash_{\mathbf{L}}$:

$$\mathcal{T}, \varphi \vdash_{\mathbf{L}} \psi \text{ iff } \mathcal{T} \vdash_{\mathbf{L}} \varphi \supset \psi.$$

- A binary connective $\wedge$ of $\mathcal{L}$ is called a *conjunction for* $\mathbf{L}$ if it satisfies the following condition:

$$\mathcal{T} \vdash_{\mathbf{L}} \psi \wedge \varphi \text{ iff } \mathcal{T} \vdash_{\mathbf{L}} \psi \text{ and } \mathcal{T} \vdash_{\mathbf{L}} \varphi.$$

- A binary connective $\vee$ of $\mathcal{L}$ is called a *disjunction for* $\mathbf{L}$ if it satisfies the following condition:

$$\mathcal{T}, \psi \vee \varphi \vdash_{\mathbf{L}} \sigma \text{ iff } \mathcal{T}, \psi \vdash_{\mathbf{L}} \sigma \text{ and } \mathcal{T}, \varphi \vdash_{\mathbf{L}} \sigma.$$

- We say that $\mathbf{L}$ is *seminormal* if it has (at least) one of the three basic connectives defined above. We say that $\mathbf{L}$ is *normal* if it has *all* these three connectives.

The following lemma is easily verified:

Lemma 4.6 *Let* $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ *be a propositional logic.*

1. *If* $\supset$ *is an implication for* $\mathbf{L}$ *then the following three conditions hold for every* $\psi, \varphi \in \mathcal{W}(\mathcal{L})$*:*
   (a) $\varphi, \varphi \supset \psi \vdash_{\mathbf{L}} \psi$
   (b) $\vdash_{\mathbf{L}} \psi \supset \psi$
   (c) $\psi \vdash_{\mathbf{L}} \varphi \supset \psi$
2. $\wedge$ *is a conjunction for* $\mathbf{L}$ iff *the following three conditions hold for every* $\psi, \varphi \in \mathcal{W}(\mathcal{L})$*:*
   (a) $\psi \wedge \varphi \vdash_{\mathbf{L}} \psi$
   (b) $\psi \wedge \varphi \vdash_{\mathbf{L}} \varphi$
   (c) $\psi, \varphi \vdash_{\mathbf{L}} \psi \wedge \varphi$
3. *If* $\vee$ *is a disjunction for* $\mathbf{L}$ *then the following three conditions hold for every* $\psi, \varphi \in \mathcal{W}(\mathcal{L})$*:*
   (a) $\psi \vdash_{\mathbf{L}} \psi \vee \varphi$
   (b) $\varphi \vdash_{\mathbf{L}} \psi \vee \varphi$
   (c) $\varphi \vee \varphi \vdash_{\mathbf{L}} \varphi$

4.2.2 Many-Valued Matrices

The most standard semantic way of defining logics is by using the following type of structures (see, e.g., [26, 30, 39]).

Definition 4.7 A (multivalued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a nonempty set of truth-values,
- $\mathcal{D}$ is a nonempty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$, and
- $\mathcal{O}$ is a function that associates an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to \mathcal{V}$ with every n-ary connective $\diamond$ of $\mathcal{L}$.

Definition 4.8 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$, and let $\mathcal{L} \subseteq \mathcal{L}'$. A matrix $\mathcal{M}' = \langle \mathcal{V}', \mathcal{D}', \mathcal{O}' \rangle$ for $\mathcal{L}'$ is called an *expansion* of $\mathcal{M}$ to $\mathcal{L}'$ if $\mathcal{V} = \mathcal{V}'$, $\mathcal{D} = \mathcal{D}'$, and $\mathcal{O}'(\diamond) = \mathcal{O}(\diamond)$ for every connective $\diamond$ of $\mathcal{L}$.

In what follows, the elements in $\mathcal{V} \backslash \mathcal{D}$ are denoted by $\overline{\mathcal{D}}$. The set $\mathcal{D}$ is used for defining satisfiability and validity, as defined below:

Definition 4.9 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$-*valuation* for $\mathcal{L}$ is a function $\nu : \mathcal{W}(\mathcal{L}) \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}(\mathcal{L})$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_{\mathcal{M}}$ $(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote the set of all the $\mathcal{M}$-valuations by $\Lambda_{\mathcal{M}}$.
- A valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of a formula ψ (alternatively, ν $\mathcal{M}$-*satisfies* ψ), if it belongs to the set $mod_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. The $\mathcal{M}$-models of a theory $\mathcal{T}$ are the elements of the set $mod_{\mathcal{M}}(\mathcal{T}) = \cap_{\psi \in \mathcal{T}}\, mod_{\mathcal{M}}(\psi)$.
- A formula ψ is $\mathcal{M}$-*satisfiable* if $mod_{\mathcal{M}}(\psi) \neq \emptyset$. A theory $\mathcal{T}$ is $\mathcal{M}$-satisfiable if $mod_{\mathcal{M}}(\mathcal{T}) \neq \emptyset$.

In the sequel, we shall sometimes omit the prefix "$\mathcal{M}$" from the notions above. Also, when it is clear from the context, we shall omit the subscript "$\mathcal{M}$" in $\widetilde{\diamond}_{\mathcal{M}}$.

Definition 4.10 Given a matrix $\mathcal{M}$, the consequence relation $\vdash_{\mathcal{M}}$ that is *induced by* (or associated with) $\mathcal{M}$, is defined by $\mathcal{T} \vdash_{\mathcal{M}} \psi$ if $mod_{\mathcal{M}}(\mathcal{T}) \subseteq mod_{\mathcal{M}}(\psi)$. We denote by $\mathbf{L}_{\mathcal{M}}$ the pair $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$ and $\vdash_{\mathcal{M}}$ is the consequence relation induced by $\mathcal{M}$.

Proposition 4.11 [36, 37] *For every propositional language $\mathcal{L}$ and a finite matrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic. If $\mathcal{M}$ is finite, then $\vdash_{\mathcal{M}}$ is also finitary.*

We conclude this section with some simple, easily verified, results on the basic connectives (Definition 4.5) in the context of matrix-based logics.

Definition 4.12 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$ and let $\mathcal{A} \subseteq \mathcal{V}$.

- An n-ary connective $\diamond$ of $\mathcal{L}$ is called $\mathcal{A}$-*closed*, if $\tilde{\diamond}(a_1, \ldots, a_n) \in \mathcal{A}$ for every $a_1, \ldots, a_n \in \mathcal{A}$.
- An n-ary connective $\diamond$ of $\mathcal{L}$ is called $\mathcal{A}$-*limited*, if for every $a_1, \ldots, a_n \in \mathcal{V}$, if $\tilde{\diamond}(a_1, \ldots, a_n) \in \mathcal{A}$ then $a_1, \ldots, a_n \in \mathcal{A}$.

Definition 4.13 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$.

- A connective $\wedge$ in $\mathcal{L}$ is called an $\mathcal{M}$*-conjunction* if it is $\mathcal{D}$-closed and $\mathcal{D}$-limited, i.e., for every $a, b \in \mathcal{V}$, $a \wedge b \in \mathcal{D}$ iff $a \in \mathcal{D}$ and $b \in \mathcal{D}$.
- A connective $\vee$ in $\mathcal{L}$ is called an $\mathcal{M}$*-disjunction* if it is $\overline{\mathcal{D}}$-closed and $\overline{\mathcal{D}}$-limited, i.e., for every $a, b \in \mathcal{V}$, $a \vee b \in \mathcal{D}$ iff $a \in \mathcal{D}$ or $b \in \mathcal{D}$.
- A connective $\supset$ in $\mathcal{L}$ is called an $\mathcal{M}$*-implication* if for every $a, b \in \mathcal{V}$, $a \tilde{\supset} b \in \mathcal{D}$ iff either $a \notin \mathcal{D}$ or $b \in \mathcal{D}$.

Proposition 4.14 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$.*

1. *A connective of $\mathcal{L}$ is an $\mathcal{M}$-conjunction iff it is a conjunction for $\mathbf{L}_{\mathcal{M}}$.*
2. *A connective of $\mathcal{L}$ which is an $\mathcal{M}$-disjunction is also a disjunction for $\mathbf{L}_{\mathcal{M}}$.*
3. *A connective of $\mathcal{L}$ which is an $\mathcal{M}$-implication is also an implication for $\mathbf{L}_{\mathcal{M}}$.*

Corollary 4.15 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$, and let $\mathcal{M}'$ be an expansion of $\mathcal{M}$. Then*

1. *An $\mathcal{M}$-conjunction (respectively: $\mathcal{M}$-disjunction, $\mathcal{M}$-implication) is also a conjunction (respectively: disjunction, implication) of $\mathbf{L}_{\mathcal{M}'}$.*
2. *If $\mathcal{M}$ has either an $\mathcal{M}$-conjunction, or an $\mathcal{M}$-disjunction, or an $\mathcal{M}$-implication, then $\mathbf{L}_{\mathcal{M}'}$ is seminormal. If $\mathcal{M}$ has all of them then $\mathbf{L}_{\mathcal{M}'}$ is normal.*

4.3 Paraconsistent Logics

In this section, we define in precise terms the notion of *paraconsistency* which is used in this paper, as well some related desirable properties.

Definition 4.16 Let $\mathcal{L}$ be a language with a unary connective $\neg$, and let $\mathbf{L} = \langle \mathcal{L}, \vdash_{\mathbf{L}} \rangle$ be a logic for $\mathcal{L}$.

- $\mathbf{L}$ is called *pre-$\neg$-paraconsistent* if there are atoms p, q such that $p, \neg p \nvdash_{\mathbf{L}} q$.
- $\mathbf{L}$ is called *boldly pre-$\neg$-paraconsistent* if there are no formula σ and an atom $p \notin Atoms(\sigma)$ such that $p, \neg p \vdash_{\mathbf{L}} \sigma$ while $\nvdash_{\mathbf{L}} \sigma$.[4]

Since $\mathbf{L}$ is a *logic*, our definition of pre-$\neg$-paraconsistency can easily be seen to be equivalent to da-Costa's definition of paraconsistency [19], which requires that there would be a theory $\mathcal{T}$ and formulas ψ, φ in $\mathcal{W}(\mathcal{L})$ such that $\mathcal{T} \vdash_{\mathbf{L}} \psi$, $\mathcal{T} \vdash_{\mathbf{L}} \neg\psi$, but $\mathcal{T} \nvdash_{\mathbf{L}} \varphi$. Both of these definitions intend to capture the idea that a contradictory set of premises should not entail every formula. However, talking about "contradictory set" makes sense only if the underlying connective $\neg$ somehow represents a "negation" operation. This is assured by the condition of "coherence with classical logic," which is defined next. Intuitively, this condition states that a logic that has such a connective should not admit entailments that do not hold in classical logic.

[4]This is a variant of a notion from [16].

Definition 4.17 Let $\mathcal{L}$ be a language with a unary connective $\neg$. A *bivalent $\neg$-interpretation for $\mathcal{L}$* is a function $\mathbf{F}$ that associates a two-valued truth-table with each connective of $\mathcal{L}$, such that $\mathbf{F}(\neg)$ is the classical truth-table for negation. We denote by $\mathcal{M}_{\mathbf{F}}$ the two-valued matrix for $\mathcal{L}$ induced by $\mathbf{F}$, that is, $\mathcal{M}_{\mathbf{F}} = \langle\{t, f\}, \{t\}, \mathbf{F}\rangle$ (see Definition 4.7).

Definition 4.18 Let $\mathbf{L} = \langle\mathcal{L}, \vdash\rangle$ be a propositional logic where $\mathcal{L}$ contains a unary connective $\neg$.

- Let $\mathbf{F}$ be a bivalent $\neg$-interpretation for $\mathcal{L}$. $\mathbf{L}$ is $\mathbf{F}$*-contained in classical logic* if the following holds for every $\varphi_1, \dots, \varphi_n, \psi \in \mathcal{W}(\mathcal{L})$: if $\varphi_1, \dots \varphi_n \vdash_{\mathbf{L}} \psi$ then $\varphi_1, \dots, \varphi_n \vdash_{\mathcal{M}_{\mathbf{F}}} \psi$.
- [3] $\mathbf{L}$ is $\neg$ *-contained in classical logic*, if it is $\mathbf{F}$-contained in it for some bivalent $\neg$-interpretation $\mathbf{F}$.
- $\mathbf{L}$ is $\neg$*-coherent with classical logic*, if it has a seminormal fragment (Definition 4.5) which is $\neg$-contained in classical logic.

Definition 4.19 Let $\mathbf{L} = \langle\mathcal{L}, \vdash\rangle$ be a propositional logic where $\mathcal{L}$ contains a unary connective $\neg$. We say that $\neg$ is a *negation* of $\mathcal{L}$ if $\mathbf{L}$ is $\neg$-coherent with classical logic.

Note 4.20 If $\neg$ is a negation of $\mathbf{L} = \langle\mathcal{L}, \vdash_{\mathbf{L}}\rangle$, then for every atom p it holds that $p \nvdash_{\mathbf{L}} \neg p$ and $\neg p \nvdash_{\mathbf{L}} p$.

Definition 4.21 Let $\mathcal{L}$ be a language with a unary connective $\neg$, and let $\mathbf{L} = \langle\mathcal{L}, \vdash_{\mathbf{L}}\rangle$ be a logic for $\mathcal{L}$.

- $\mathbf{L}$ is called $\neg$*-paraconsistent* if it is pre-$\neg$-paraconsistent and $\neg$ is a negation of $\mathbf{L}$.
- $\mathbf{L}$ is called *boldly* $\neg$ *-paraconsistent* if it is boldly pre-$\neg$-paraconsistent, and $\neg$ is a negation of $\mathbf{L}$.

Henceforth, we shall frequently omit the $\neg$ sign (if it is clear from the context), and simply refer to (*boldly*) (*pre-*) *paraconsistent logics*.

Note 4.22 It should again be emphasized that our notion of paraconsistency has *two* components. In addition to the usual demand that a formula and its negation do not imply everything, we also demand that the "negation" connective under question can indeed be taken to be a sort of *negation*.

Paraconsistent logics reject the principle of explosion (known as *Ex Contradictione Sequitur Quodlibet*: $\mathcal{T}, \psi, \neg\psi \vdash \varphi$). Bold paraconsistency is a stronger version of this property. An even stronger demand is to reject explosion in *all* circumstances:

Definition 4.23 A logic $\langle\mathcal{L}, \vdash\rangle$ is *non-exploding* if for every theory $\mathcal{T}$ such that $\mathsf{Atoms}(\mathcal{T}) \neq \mathsf{Atoms}(\mathcal{L})$ there is a formula ψ such that $\mathcal{T} \nvdash \psi$.

Note 4.24 Obviously, every non-exploding logic which is $\neg$-coherent with classical logic is boldly paraconsistent.

There are many approaches to designing paraconsistent logics. One of the oldest and best known is Newton da-Costa's approach, which has led to the family of *Logics of Formal Inconsistency* (LFIs) [16]. Now, already in the early stages of investigating this topic, it has been acknowledged by da-Costa (and others) that pre-paraconsistency by itself is not sufficient. Further properties that an "ideal" paraconsistent logic is expected to have are defined in [3]. In the rest of this section, we briefly recall (with some improvements) these properties.

A. Reasonably Strong Language. Clearly, any logic (including paraconsistent ones) should have a sufficiently expressive language. The seminormality requirement (Definition 4.5) assures that in addition to negation, a useful paraconsistent logic should provide natural counterparts for all classical connectives:

Proposition 4.25 *Let* $\mathbf{L}$ *be a logic that is* $\mathbf{F}$*-contained in classical logic for some* $\mathbf{F}$*, and let* $\mathbf{F}(\diamond) = \diamond_{\mathbf{F}}$*. Then for every* $a, b \in \{t, f\}$ *we have*

1. *If* $\diamond$ *is an implication for* $\mathbf{L}$*, then* $a \diamond_{\mathbf{F}} b = f$ *if* $a = t$ *and* $b = f$*, otherwise* $a \diamond_{\mathbf{F}} b = t$*.*
2. *If* $\diamond$ *is a conjunction for* $\mathbf{L}$*, then* $a \diamond_{\mathbf{F}} b = t$ *if* $a = t$ *and* $b = t$*, otherwise* $a \diamond_{\mathbf{F}} b = f$*.*
3. *If* $\diamond$ *is a disjunction for* $\mathbf{L}$*, then* $a \diamond_{\mathbf{F}} b = t$ *if* $a = t$ *or* $b = t$*, otherwise* $a \diamond_{\mathbf{F}} b = f$*.*

Proof Let $\mathbf{F}$ be a bivalent interpretation for which $\mathbf{L}$ is $\mathbf{F}$-contained in classical logic.

1. Suppose that $\diamond$ is an implication for $\mathbf{L}$, and let $\mathbf{F}(\diamond) = \diamond_{\mathbf{F}}$. By Item (b) of Lemma 4.6–1, $\vdash_{\mathbf{L}} p \diamond p$. Hence, $\vdash_{\mathcal{M}_{\mathbf{F}}} p \diamond p$, and so necessarily $t \diamond_{\mathbf{F}} t = f \diamond_{\mathbf{F}} f = t$. Next, $p \vdash_{\mathbf{L}} q \diamond q$, and since $\diamond$ is an implication for $\mathbf{L}$, $\vdash_{\mathbf{L}} p \diamond (q \diamond q)$. Hence also $\vdash_{\mathcal{M}_{\mathbf{F}}} p \diamond (q \diamond q)$. Since $f \diamond_{\mathbf{F}} f = t$, this implies that $f \diamond_{\mathbf{F}} t = t$. Finally, by Item (a) of Lemma 4.6–1, $p \diamond q, p \vdash_{\mathbf{L}} q$. Hence, also $p \diamond q, p \vdash_{\mathcal{M}_{\mathbf{F}}} q$, and so $t \diamond_{\mathbf{F}} f = f$ (otherwise $\nu(p) = t, \nu(q) = f$ would be a counterexample).
2. Suppose that $\diamond$ is a conjunction for $\mathbf{L}$, and let $\mathbf{F}(\diamond) = \diamond_{\mathbf{F}}$. By Lemma 4.6–2, $p \diamond q \vdash_{\mathbf{L}} p$ and so also $p \diamond q \vdash_{\mathcal{M}_{\mathbf{F}}} p$. This implies that $f \diamond_{\mathbf{F}} t = f$ and $f \diamond_{\mathbf{F}} f = f$. Similarly, since $p \diamond q \vdash_{\mathbf{L}} q$, also $p \diamond q \vdash_{\mathcal{M}_{\mathbf{F}}} q$, and so $t \diamond_{\mathbf{F}} f = f$. Finally, by Lemma 4.6–2 again, $p, q \vdash_{\mathbf{L}} p \diamond q$ and so $p, q \vdash_{\mathcal{M}_{\mathbf{F}}} p \diamond q$, which implies that $t \diamond_{\mathbf{F}} t = t$ (otherwise, $\nu(p) = \nu(q) = t$ would be a counterexample).
3. Suppose that $\diamond$ is a disjunction for $\mathbf{L}$, and let $\mathbf{F}(\diamond) = \diamond_{\mathbf{F}}$. By Lemma 4.6–3, $p \vdash_{\mathbf{L}} p \diamond q$, and $q \vdash_{\mathbf{L}} p \diamond q$. Hence also $p \vdash_{\mathcal{M}_{\mathbf{F}}} p \diamond q$, and $q \vdash_{\mathcal{M}_{\mathbf{F}}} p \diamond q$, implying that $t \diamond_{\mathbf{F}} t = t \diamond_{\mathbf{F}} f = f \diamond_{\mathbf{F}} t = t$. Finally, by Item (c) of Lemma 4.6–3 the assumption that $\diamond$ is a disjunction for $\mathbf{L}$ implies that $p \diamond p \vdash_{\mathbf{L}} p$, and so $p \diamond p \vdash_{\mathcal{M}_{\mathbf{F}}} p$. It follows that $f \diamond_{\mathbf{F}} f = f$ (otherwise, $\nu(p) = f$ would be a counterexample). □

Corollary 4.26 *Let* $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ *be a matrix for* $\mathcal{L}$ *such that* $\mathbf{L}_{\mathcal{M}}$ *is* $\mathbf{F}$*-contained in classical logic.*

1. *If* $\wedge$ *is an* $\mathcal{M}$*-conjunction then* $\mathbf{F}(\wedge)$ *is the classical conjunction.*
2. *If* $\vee$ *is an* $\mathcal{M}$*-disjunction then* $\mathbf{F}(\vee)$ *is the classical disjunction.*
3. *If* $\supset$ *is an* $\mathcal{M}$*-implication then* $\mathbf{F}(\supset)$ *is the classical implication.*

Proof This follows from Propositions 4.25 and 4.14. □

Note 4.27 Let $\mathcal{M}$ be a martix for $\mathcal{L}$ such that $\mathbf{L}_{\mathcal{M}}$ is **F**-contained in classical logic. Suppose that $\mathcal{M}$ has a connective $\diamond$ of $\mathcal{L}$ which is either an $\mathcal{M}$-conjunction, an $\mathcal{M}$-disjunction, or an $\mathcal{M}$-implication. The last corollary implies that any two-valued function is then definable in terms of $\mathbf{F}(\diamond)$ and $\mathbf{F}(\neg)$. This shows the adequacy of the expressive power of such matrices.

B. Maximal Paraconsistency. A common requirement from a paraconsistent logic, which is already realized in da-Costa's seminal paper [19], is to "retain as much of classical logic as possible, while still allowing nontrivial inconsistent theories." As observed in [3, 4], this requirement has two different interpretations, corresponding to the two aspects of this demand:

B-1. Absolute Maximal Paraconsistency. Intuitively, this means that by trying to further extend the logic (without changing the language) we lose the property of paraconsistency.

Definition 4.28 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a $\neg$-paraconsistent logic

- We say that **L** is *maximally paraconsistent*, if every extension of **L** (in the sense of Definition 4.4) whose set of theorems *properly includes* that of **L**, is not pre-paraconsistent.
- We say that **L** is *strongly maximal*, if every proper extension of **L** (in the sense of Definition 4.4) is not pre-paraconsistent.

B-2. Maximality Relative to Classical Logic. The intuitive meaning of this property is that the logic is so close to classical logic, that any attempt to further extend it should necessarily end-up with classical logic.

Definition 4.29 Let **F** be a bivalent $\neg$-interpretation for a language $\mathcal{L}$ with a unary connective $\neg$.

- An $\mathcal{L}$-formula ψ is a *classical* **F** *-tautology*, if ψ is satisfied by every two-valued valuation which respects all the truth-tables (of the form $\mathbf{F}(\diamond)$) that **F** assigns to the connectives of $\mathcal{L}$.
- A logic $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ is **F**-*complete*, if its set of theorems consists of all the classical **F**-tautologies.
- A logic **L** is **F**-*maximal relative to classical logic*, if the following hold:
  - **L** is **F**-contained in classical logic.
  - If ψ is a classical **F**-tautology not provable in **L**, then by adding ψ to **L** as a new axiom schema, an **F**-complete logic is obtained.
- A logic **L** is **F** *-maximally paraconsistent relative to classical logic*, if it is pre-paraconsistent and **F**-maximal relative to classical logic.

Definition 4.30 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic for a language with a unary connective $\neg$. We say that **L** is *maximally paraconsistent relative to classical logic* if there exists a bivalent $\neg$-interpretation **F** such that **L** is **F**-maximally paraconsistent relative to classical logic.

The two kinds of maximality are combined in the next definition.

Definition 4.31 We say that a seminormal finitary logic $\mathbf{L}$ is a *fully maximal* paraconsistent logic, if it is both maximally paraconsistent relative to classical logic and strongly maximal.

4.4 Three-Valued Paraconsistent Matrices

We now turn to the three-valued case, and investigate paraconsistent logics induced by three-valued matrices. We start with some general results.

Definition 4.32 Let $\mathcal{L}$ be a propositional language with a unary connective $\neg$. A matrix $\mathcal{M}$ for $\mathcal{L}$ is (*boldly, pre-*) $\neg$ *-paraconsistent* if so is $\mathbf{L}_{\mathcal{M}}$ (see Definitions 4.16 and 4.21).

Proposition 4.33 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language with $\neg$.*

1. *$\mathcal{M}$ is pre-paraconsistent iff there is an element $\top \in \mathcal{D}$, such that $\tilde{\neg}\top \in \mathcal{D}$.*
2. *If $\mathcal{M}$ is paraconsistent then there are three different elements t, f, and $\top$ in $\mathcal{V}$ such that $f = \tilde{\neg}t$, $f \notin \mathcal{D}$, and $\{t, \tilde{\neg}f, \top, \tilde{\neg}\top\} \subseteq \mathcal{D}$.*

Proof By its definition, $\mathcal{M}$ is pre-paraconsistent iff $p, \neg p \nvdash_{\mathcal{M}} q$. Obviously, this happens iff $\{p, \neg p\}$ has an $\mathcal{M}$-model. The latter, in turn, is possible iff there is some $\top \in \mathcal{D}$, such that $\tilde{\neg}\top \in \mathcal{D}$, as indicated in the first item of the proposition. For the second item, we may assume without loss in generality that $\mathcal{M}$ is $\neg$-contained in classical logic. We let $\mathbf{F}$ be a bivalent $\neg$-interpretation such that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic. Since $p, \neg\neg p \nvdash_{\mathcal{M}_{\mathbf{F}}} \neg p$, also $p, \neg\neg p \nvdash_{\mathcal{M}} \neg p$, and so there is some $t \in \mathcal{D}$, such that $\tilde{\neg}t \notin \mathcal{D}$, while $\tilde{\neg}\tilde{\neg}t \in \mathcal{D}$. Let $f = \tilde{\neg}t$. Then t and f have the required properties, and together with the first item we are done. □

Corollary 4.34 *Any paraconsistent matrix is boldly paraconsistent.*

Proof Suppose that $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ is a paraconsistent matrix, σ is a formula in its language such that $\nvdash_{\mathcal{M}} \sigma$, and p is an atomic formula such that $p \notin \mathsf{Atoms}(\sigma)$. Then there is a valuation ν such that $\nu(\sigma) \notin \mathcal{D}$. Let $\top$ be an element of $\mathcal{V}$ like in the first item of Proposition 4.33. Define a valuation ν' by letting $\nu'(p) = \top$, and $\nu'(q) = \nu(q)$ for every atomic formula $q \neq p$. Then $\nu'(\sigma) = \nu(\sigma) \notin \mathcal{D}$. Hence ν' is an $\mathcal{M}$-model of $\{\neg p, p\}$ which is not an $\mathcal{M}$-model of σ, and so $\{\neg p, p\} \nvdash_{\mathcal{M}} \sigma$. It follows that $\mathcal{M}$ is boldly paraconsistent. □

By the second item of Proposition 4.33, we have

Corollary 4.35 *Every paraconsistent matrix has at least two designated elements, and so no two-valued matrix can be paraconsistent.*

The last corollary vindicates the general wisdom that truth-functional semantics of a reasonable paraconsistent logic should be based on at least three truth-values. The structure of paraconsistent matrices with exactly three values is characterized next.

Proposition 4.36 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix. Then $\mathcal{M}$ isomorphic to a matrix $\mathcal{M}' = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, f, \top\}$, $\mathcal{D} = \{t, \top\}$, $\neg t = f$, $\neg f = t$ and $\neg \top \in \mathcal{D}$.*

Proof By Proposition 4.33, we only need to show that $\tilde{\neg} f \neq \top$. Assume for contradiction that $\tilde{\neg} f = \top$. This implies that $\tilde{\neg}\tilde{\neg}\tilde{\neg}\top = \top$, no matter whether $\tilde{\neg}\top = \top$ or $\tilde{\neg}\top = t$. This and the facts that $\mathcal{D} = \{t, \top\}$ and $\tilde{\neg}\top \in \mathcal{D}$ imply that $p \vdash_{\mathcal{M}} \neg\neg\neg p$, which contradicts the $\neg$-coherence of $\mathcal{M}$ with classical logic. □

In the rest of the paper, we assume that any three-valued paraconsistent matrix has the form described in Proposition 4.36.

Next, we provide an effective necessary and sufficient criterion for checking which paraconsistent matrix is also non-exploding.

Proposition 4.37 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a paraconsistent 3-valued matrix. Then $\mathbf{L}_{\mathcal{M}}$ is non-exploding iff every connective $\diamond$ of $\mathcal{M}$ is $\{\top\}$-closed (i.e., $\tilde{\diamond}(\top, \ldots, \top) = \top$).*

Proof Suppose that every connective of $\mathcal{M}$ is $\{\top\}$-closed. Let $\mathcal{T}$ be a theory and q an atomic formula such that $q \notin \mathsf{Atoms}(\mathcal{T})$. Let ν be an assignment in $\mathcal{M}$ such that $\nu(p) = \top$ for every $p \in \mathsf{Atoms}(\mathcal{T})$, while $\nu(q) = f$. Since every connective of $\mathcal{M}$ is $\{\top\}$-closed, $\nu(\varphi) = \top$ for every $\varphi \in \mathcal{T}$. Hence ν is a model of $\mathcal{T}$ which is not a model of q. It follows that $\mathcal{T} \nvdash_{\mathcal{M}} q$.

For the converse, assume that there is an n-ary connective $\diamond$ of the language of $\mathcal{M}$ such that $\tilde{\diamond}$ is not $\{\top\}$-closed. Then $S = \{P_1, \neg P_1, \diamond(P_1, \ldots, P_1), \neg\diamond(P_1, \ldots, P_1)\}$ has no models in $\mathcal{M}$, and so $S \vdash_{\mathcal{M}} \varphi$ for every φ. Hence $\mathbf{L}_{\mathcal{M}}$ is not non-exploding. □

By Proposition 4.36, it follows that there are exactly two possible definitions for negation connectives in three-valued paraconsistent matrices:

- Kleene's negation [27], in which $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg}\top = \top$, and
- Sette's negation [35], in which $\tilde{\neg} t = f$, $\tilde{\neg} f = t$, $\tilde{\neg}\top = t$.

The other basic connectives are characterized by the following proposition.

Proposition 4.38 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a paraconsistent three-valued matrix.*

1. *A connective $\wedge$ is a conjunction for $\mathbf{L}_{\mathcal{M}}$ iff it is an $\mathcal{M}$-conjunction.*
2. *A connective $\vee$ is a disjunction for $\mathbf{L}_{\mathcal{M}}$ iff it is an $\mathcal{M}$-disjunction.*
3. *A connective $\supset$ is an implication for $\mathbf{L}_{\mathcal{M}}$ iff it is an $\mathcal{M}$-implication.*

Proof In all cases, the "if" direction is shown in Proposition 4.14. Below we prove the "only if" directions.

1. Immediate from Proposition 4.14.
2. Assume that $\vee$ is a disjunction for $\mathbf{L}_{\mathcal{M}}$. By Lemma 4.6–3, $\varphi \vdash_{\mathbf{L}_{\mathcal{M}}} \varphi \vee \psi$ and $\psi \vdash_{\mathbf{L}_{\mathcal{M}}} \varphi \vee \psi$. This implies that if either $a \in \mathcal{D}$ or $b \in \mathcal{D}$ then $a \mathrel{\tilde{\vee}} b \in \mathcal{D}$. On the other hand, Item (c) of Lemma 4.6–3 entails that $\varphi \vee \varphi \vdash_{\mathbf{L}_{\mathcal{M}}} \varphi$, implying that $f \mathrel{\tilde{\vee}} f = f$. It follows that $\vee$ is an $\mathcal{M}$-disjunction.
3. Assume that $\supset$ is an implication for $\mathbf{L}_{\mathcal{M}}$. By Item (a) in Lemma 4.6–1, $a \mathrel{\tilde{\supset}} f = f$ for $a \in \mathcal{D}$, by Item (b) of the same lemma $f \mathrel{\tilde{\supset}} f \in \mathcal{D}$, and by Item (c), $b \mathrel{\tilde{\supset}} a \in \mathcal{D}$ for $a \in \mathcal{D}$. $\square$

Corollary 4.39 *If $\mathcal{M}$ is a paraconsistent three-valued matrix, then $\mathbf{L}_{\mathcal{M}'}$ is paraconsistent for every expansion $\mathcal{M}'$ of $\mathcal{M}$. Moreover, if $\supset$ (respectively, if $\vee$, $\wedge$) is an implication (respectively, a disjunction, conjunction) for $\mathbf{L}_{\mathcal{M}}$, then it is also an implication (respectively, a disjunction, conjunction) for $\mathbf{L}_{\mathcal{M}'}$.*

Proof Immediate from Proposition 4.38 and Corollary 4.15. $\square$

Corollary 4.40 *If $\mathcal{M}$ is a paraconsistent three-valued matrix, then $\mathbf{L}_{\mathcal{M}}$ is seminormal.*

Proof By definition of paraconsistency, if $\mathcal{M}$ is a paraconsistent then it has a paraconsistent seminormal fragment. Hence the claim follows from Corollary 4.39. $\square$

We now give some general characterizations of logics which are induced by three-valued paraconsistent matrices, with particular emphasis on those which are actually $\neg$-contained in classical logic (and not just $\neg$-coherent with it). Our first result is the following:

Theorem 4.41 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a 3-valued $\neg$-paraconsistent matrix for a language $\mathcal{L}$. If $\mathcal{M}$ is $\neg$-contained in classical logic then $\mathcal{M}$ is classically closed (i.e., $\{t, f\}$-closed).*

Proof By Proposition 4.36, $\mathcal{V} = \{t, f, \top\}$, $\mathcal{D} = \{t, \top\}$, $f = \tilde{\neg} t$, $t = \tilde{\neg} f$, and $\tilde{\neg} \top \in \{t, \top\}$. Therefore, we have two cases to consider.

$\tilde{\neg}_{\mathcal{M}}$ is **Sette's negation**: Assume for contradiction that $\mathcal{M}$ is not classically closed, then $\tilde{\neg} \top = t$ and there is a connective $\diamond$ and $a_1, \ldots, a_n \in \{t, f\}$ such that $\tilde{\diamond}(a_1, \ldots, a_n) = \top$. For $i = 1, \ldots, n$ let $r_i = p_i$ if $a_i = t$ and $r_i = \neg p_i$ if $a_i = f$. Then, for every valuation $\nu \in \Lambda_{\mathcal{M}}$, if $\nu(p_i) = t$ for every $1 \le i \le n$ then $\nu(\diamond(r_1, \ldots, r_n)) = \top$. Let now $S = \{p_1, \neg\neg p_1, p_2, \neg\neg p_2, \ldots p_n, \neg\neg p_n\}$. Then $v \models_{\mathcal{M}} S$ iff $\nu(p_1) = \cdots = \nu(p_n) = t$. It follows that $S \vdash_{\mathcal{M}} \diamond(r_1, \ldots, r_n)$ and $S \vdash_{\mathcal{M}} \neg \diamond (r_1, \ldots, r_n)$. Since $\mathcal{M}$ is $\neg$-contained in classical logic, $S \vdash_{\mathcal{M}_{\mathbf{F}}} \diamond(r_1, \ldots, r_n)$ and $S \vdash_{\mathcal{M}_{\mathbf{F}}} \neg \diamond (r_1, \ldots, r_n)$ for some bivalent $\neg$-interpretation $\mathbf{F}$ for $\mathcal{L}$. This means that S is classically unsatisfiable, but this is false.

$\tilde{\neg}_{\mathcal{M}}$ is **Kleene's negation**: First, we show that the fact that $\mathcal{M}$ is seminormal (Corollary 4.40) entails in this case that it has an $\mathcal{M}$-disjunction. We do this by considering all the three possible cases.

- Suppose that $\mathbf{L}_{\mathcal{M}}$ has a disjunction connective $\vee$. Then $\vee$ is also an $\mathcal{M}$-disjunction by Proposition 4.38.
- Suppose that $\mathbf{L}_{\mathcal{M}}$ has a an $\mathcal{M}$-implication $\supset$. Then $\supset$ is an $\mathcal{M}$-implication by Proposition 4.38. This easily implies that the connective $\vee$ defined by $\varphi \vee \psi = (\varphi \supset \psi) \supset \psi$ is an $\mathcal{M}$-disjunction.
- Suppose that $\mathbf{L}_{\mathcal{M}}$ has a conjunction $\wedge$. Then $\wedge$ is an $\mathcal{M}$-conjunction by Proposition 4.38, and so we have $(*)$ $a \tilde{\wedge} b = f$ iff either $a = f$ or $b = f$.
  First, we prove that $t \tilde{\wedge} t = t$. Assume otherwise. Then $t \tilde{\wedge} t = \top$ by $(*)$ above. Hence, if $\nu \in \Lambda_{\mathcal{M}}$ then $\nu(\neg(p \wedge p)) \in \mathcal{D}$ in case $\nu(p) = t$. By $(*)$ again, this implies that $\nu(\neg(p \wedge p)) \in \mathcal{D}$ in case $\nu(p) \in \{t, f\}$. On the other hand, if $\nu(p) = \top$ then $\nu(p) = \nu(\neg p)$, and so $\nu(\neg(p \wedge p)) = \nu(\neg(p \wedge \neg p))$. It follows that $\neg(p \wedge \neg p) \vdash_{\mathcal{M}} \neg(p \wedge p)$. Since $\mathcal{M}$ is $\neg$-contained in classical logic, $\neg(p \wedge \neg p) \vdash_{\mathcal{M}_\mathbf{F}} \neg(p \wedge p)$, which is false.
  Next, we show that using $\neg$ and $\wedge$, it is possible to define in $\mathcal{L}$ an $\mathcal{M}$-disjunction $\vee$. We have two cases to consider:

  – $\top \tilde{\wedge} \top = t$:
    In this case we take $\varphi \vee \psi =_{Df} \neg(\neg(\varphi \wedge \varphi) \wedge \neg(\psi \wedge \psi))$. The fact that $t \tilde{\wedge} t = \top \tilde{\wedge} \top = t$ and (*) easily imply that this formula has the required property.
  – $\top \tilde{\wedge} \top = \top$:
    In this case we first let $\mathbf{t}_{\varphi,\psi}$ abbreviate $\neg(\varphi \wedge \neg\varphi \wedge \psi \wedge \neg\psi)$ (where association of conjunction is taken to the right,). Then $\nu(\mathbf{t}_{\varphi,\psi}) = \top$ in case that $\nu(\varphi) = \nu(\psi) = \top$, and $\nu(\mathbf{t}_{\varphi,\psi}) = t$ otherwise. Now, we take:

$$\varphi \vee \psi =_{Df} \neg(\neg(\mathbf{t}_{\varphi,\psi} \wedge \varphi \wedge \mathbf{t}_{\varphi,\psi}) \wedge \neg(\mathbf{t}_{\varphi,\psi} \wedge \psi \wedge \mathbf{t}_{\varphi,\psi})).$$

    We show that this formula has in this case the required property:

    * Suppose first that $\nu(\varphi) = \nu(\psi) = f$. Since for every x, $x \tilde{\wedge} f = f \tilde{\wedge} x = f$, we have that $\nu(\mathbf{t}_{\varphi,\psi} \wedge \varphi \wedge \mathbf{t}_{\varphi,\psi}) = \nu(\mathbf{t}_{\varphi,\psi} \wedge \psi \wedge \mathbf{t}_{\varphi,\psi}) = f$. Since $\tilde{\neg} f = t$, $t \tilde{\wedge} t = t$, and $\tilde{\neg} t = f$, it follows that in this case $\nu(\varphi \vee \psi) = f$.
    * Suppose that $\nu(\varphi) = t$. Then $\nu(\mathbf{t}_{\varphi,\psi}) = t$. Since $t \tilde{\wedge} t = t$, $\nu(\mathbf{t}_{\varphi,\psi} \wedge \varphi \wedge \mathbf{t}_{\varphi,\psi}) = t$. Again, since $\tilde{\neg} t = f$, $f \tilde{\wedge} x = f$, and $\tilde{\neg} f = t$, we conclude that in this case $\nu(\varphi \vee \psi) = t$.
    * Suppose that $\nu(\psi) = t$. Then again $\nu(\mathbf{t}_{\varphi,\psi}) = t$. Like in the previous case, this implies that $\nu(\varphi \vee \psi) = t$.
    * Suppose that $\nu(\varphi) = \nu(\psi) = \top$. Then $\nu(\mathbf{t}_{\varphi,\psi}) = \top$. Since $\tilde{\neg}\top = \top$ and $\top \tilde{\wedge} \top = \top$, $\nu(\sigma) = \top$ for every sub-formula σ of $\varphi \vee \psi$. Hence $\nu(\varphi \vee \psi) = \top$ as well.
    * Suppose that $\nu(\varphi) = f$, $\nu(\psi) = \top$. Then $\nu(\mathbf{t}_{\varphi,\psi}) = t$, and so we haver that $\nu(\varphi \vee \psi) = \tilde{\neg}(t \tilde{\wedge} \tilde{\neg}((t \tilde{\wedge} \top) \tilde{\wedge} t))$. If $(t \tilde{\wedge} \top) \tilde{\wedge} t) = t$ (which is the case if either $t \tilde{\wedge} \top = t$ or $\top \tilde{\wedge} t = t$) then $\nu((\varphi \vee \psi) = t$, and if $(t \tilde{\wedge} \top) \tilde{\wedge} t) = \top$ (which is the case if $t \tilde{\wedge} \top = \top \tilde{\wedge} t = \top$) then $\nu(\varphi \vee \psi) = \top$. In both cases we are done.
    * The case where $\nu(\varphi) = \top$ and $\nu(\psi) = f$ is similar to the previous case.

We therefore have shown that $\mathcal{M}$ has an $\mathcal{M}$-disjunction. We show that this implies that $\mathcal{M}$ is classically closed. Assume for contradiction that it is not, then there is a connective $\diamond$ and elements $a_1, \ldots, a_n \in \{t, f\}$, such that $\tilde{\diamond}(a_1, \ldots, a_n) = \top$. For $i = 1, \ldots, n$ let $r_i = p_i$ if $a_i = t$, $r_i = \neg p_i$ if $a_i = f$. Then for every $\nu \in \Lambda_{\mathcal{M}}$, if $\nu(p_i) = t$ for every $1 \leq i \leq n$ then $\nu(\diamond(r_1, \ldots, r_n)) = \tilde{\diamond}(a_1, \ldots, a_n) = \top$. Hence $\tilde{\neg}\tilde{\diamond}(a_1, \ldots, a_n)$ is in $\{t, \top\}$. These two facts imply:

$$p_1, \ldots, p_n \vdash_{\mathcal{M}} \neg p_1 \vee \cdots \vee \neg p_n \vee \diamond(r_1, \ldots, r_n)$$
$$p_1, \ldots, p_n \vdash_{\mathcal{M}} \neg p_1 \vee \cdots \vee \neg p_n \vee \neg \diamond (r_1, \ldots, r_n)$$

Indeed, let ν be a model of $\{p_1, \ldots, p_n\}$. If $\nu(p_i) \neq t$ for some i then $\nu(\neg p_i) \in \mathcal{D}$, and so ν is a model of the disjunctions on the right-hand sides. If $\nu(p_i) = t$ for all i then ν is a model of both $\tilde{\diamond}(r_1, \ldots, r_n)$ and $\tilde{\neg}\tilde{\diamond}(r_1, \ldots, r_n)$, and so again ν is a model of both right-hand sides. Now, since $\mathcal{M}$ is $\neg$-contained in classical logic, Corollary 4.26 entails that the above two facts remain true if we replace $\vdash_{\mathcal{M}}$ by $\vdash_{\mathcal{M}_{\mathbf{F}}}$ and interpret $\vee$ and $\neg$ as the classical disjunction and negation (respectively). However, this is impossible for any two-valued interpretation of $\diamond$. □

The next theorems characterize all the three-valued matrices which induce paraconsistent logics that are $\neg$-contained in classical logic and show how to construct all such matrices which induce (semi)normal logics in a language that contains an implication $\supset$, a conjunction $\wedge$, and a disjunction $\vee$.

Theorem 4.42 *There are exactly* 2^{13} *(8192) distinct normal paraconsistent logics in the language* $\mathcal{L}_{CL} = \{\neg, \wedge, \vee, \supset\}$ *which are* $\neg$*-contained in classical logic, induced by three-valued matrices, and in which* $\supset$ *is an implication,* $\wedge$*—a conjunction, and* $\vee$*—a disjunction. The corresponding matrices are those that belong to the following family* 8Kb *of matrices from* [16][5] *(where the notation "*$x \wr y$*" means that* x *and* y *are two optional values):*

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$t \wr \top$
f	f	f	f
$\top$	$t \wr \top$	f	$t \wr \top$

$\tilde{\vee}$	t	f	$\top$
t	t	t	$t \wr \top$
f	t	f	$t \wr \top$
$\top$	$t \wr \top$	$t \wr \top$	$t \wr \top$

$\tilde{\supset}$	t	f	$\top$
t	t	f	$t \wr \top$
f	t	t	$t \wr \top$
$\top$	$t \wr \top$	f	$t \wr \top$

	$\tilde{\neg}$
t	f
f	t
$\top$	$t \wr \top$

Proof That the matrices above indeed exhaust all the possible cases follows from Propositions 4.36, 4.38, and Theorem 4.41. That all of them induce paraconsistent logics which are $\neg$-contained in classical logic easily follows from Proposition 4.33

[5]In [16] the language is extended with a consistency operator $\circ$, defined by $\tilde{\circ}t = t$, $\tilde{\circ}f = t$, and $\tilde{\circ}\top = f$.

(second part) and the fact that the $\{t, f\}$-reductions of the connectives yield bivalent $\neg$-interpretations. That they are all normal follows from Proposition 4.14.

It is also not difficult to show that all of the logics in 8Kb are indeed different. For instance, suppose that $\vdash_1$ and $\vdash_2$ are two consequence relations induced by matrices with different interpretations for a disjunction. Below we check the possible cases for such different interpretations and show that in each case the logics that are obtained are indeed different. First, note that by Theorem 4.41 and Corollary 4.26, the two matrices coincide on $a \tilde{\vee} b$ whenever $a \in \{t, f\}$ and $b \in \{t, f\}$. Now,

1. Suppose that $\top \tilde{\vee}_1 \top = \top$ while $\top \tilde{\vee}_2 \top = t$.
   In this case $p, \neg p, q, \neg q \vdash_1 \neg(p \vee q)$, but this is not true for $\vdash_2$ (since in both cases all the models of the right-hand side assign $\top$ to p and q).
2. Suppose that $\top \tilde{\vee}_1 \top = \top \tilde{\vee}_2 \top \in \{t, \top\}$ and that $f \tilde{\vee}_1 \top = \top$ while $f \tilde{\vee}_2 \top = t$.
   Then $q, \neg q, \neg(p \vee q) \vdash_2 p$ (a model of the right-hand side must assign $\top$ to q, and since $f \tilde{\vee}_2 \top = t$ it cannot assign f to p), while this is not true for $\vdash_1$ (a counter-model in this case assigns f to p and $\top$ to q).
3. The remaining cases are dual to the ones in the previous cases. □

Theorem 4.43 *Let $\mathcal{M}$ be a three-valued matrix for a language with a unary connective $\neg$.*

1. *$\mathcal{M}$ induces a $\neg$-paraconsistent logic which is $\neg$-contained in classical logic iff it is isomorphic to a matrix of the form $\langle\{t, f, \top\}, \{t, \top\}, \mathcal{O}\rangle$ which satisfies the following conditions:*

   (a) It has as its interpretation of $\neg$ one of the two tables for $\neg$ given in Theorem 4.42;
   (b) It has a (possibly definable) connective whose interpretation is either one of the 2^3 possible interpretations for conjunction ($\wedge$) given in Theorem 4.42, or one of the 2^5 interpretations for disjunction ($\vee$) given there, or one of the 2^4 interpretations for implication ($\supset$) given there;
   (c) All its connectives are classically closed: $\tilde{\diamond}(a_1, \ldots, a_n) \in \{t, f\}$ for all $a_1, \ldots, a_n \in \{t, f\}$.

2. *$\mathcal{M}$ induces a $\neg$-paraconsistent logic iff it is isomorphic to a matrix of the form $\langle\{t, f, \top\}, \{t, \top\}, \mathcal{O}\rangle$ which satisfies Conditions (a) and (b) above.*
3. *$\mathcal{M}$ induces a* normal *$\neg$-paraconsistent logic which is $\neg$-contained in classical logic iff it is isomorphic to a matrix of the form $\langle\{t, f, \top\}, \{t, \top\}, \mathcal{O}\rangle$ which satisfies the following conditions:*

   (a) It has its interpretation of $\neg$ one of the two tables for $\neg$ given in Theorem 4.42;
   (b) It has a (possibly definable) connective whose interpretation is one of the 2^3 possible interpretations for $\wedge$ given in Theorem 4.42, and a connective whose interpretation is one of the 2^5 interpretations for $\vee$ given there, and a connective whose interpretation is one of the 2^4 interpretations for $\supset$ given there;
   (c) All its connectives are classically closed (i.e., $\{t, f\}$-closed).

Proof That the conditions in Parts 1 and 3 are necessary again follows from Propositions 4.36, 4.38, and Theorem 4.41. That they are sufficient again follows from Proposition 4.33 (second part), the fact that the $\{t, f\}$-reductions of the connectives yield bivalent $\neg$-interpretations and Proposition 4.14. Part 2 follows from Part 1 and Corollary 4.39. □

Note 4.44 Although all the logics which are induced by the matrices in the family 8Kb are different from each other, some of them have the same expressive power. For instance, consider any paraconsistent matrix for the language of $\{\neg, \supset\}$ in which $\tilde{\neg}\top = \top$ and $\tilde{\supset}$ is D'Ottaviano and da-Costa's implication [19, 21], defined as follows:

$$a \supset b = \begin{cases} b & \text{if } a \neq f, \\ t & \text{if } a = f. \end{cases}$$

In this language, the formulas $\neg(\varphi \supset \neg\psi)$ and $\neg(\psi \supset \neg\varphi)$ define two different conjunctions. Hence the corresponding matrices in the family 8Kb are equivalent in their expressive power.

We conclude this section with a theorem about the desirable maximal paraconsistency properties (Sect. 4.3) that three-valued $\neg$-paraconsistent logics enjoy:

Theorem 4.45 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix. Then*

1. $\mathbf{L}_{\mathcal{M}}$ *is strongly maximal.*
2. *If $\mathcal{M}$ is $\neg$-contained in classical logic then it is also maximally paraconsistent relative to classical logic (and so it is fully maximal).*

Note 4.46 The first part of Theorem 4.45 is a generalization of [3, Theorem 2] (see also [4, Theorem 3.2]) and the second part of the theorem is a generalization of [3, Theorem 1]. In both cases, the proofs given below are similar to the ones given in [3]. To keep this paper complete, we repeat those proofs and adjust them to the more general case considered here.

Proof Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$. To see the first item of the theorem, note first that Theorem 4.43 implies that $\mathcal{M}$ has a classically closed binary connective $\diamond$ (from those listed in Theorem 4.42), which is either an $\mathcal{M}$-disjunction, or an $\mathcal{M}$-conjunction, or an $\mathcal{M}$-implication. Let $\Psi(p)$ be $\neg p \diamond p$ in the first case, $\neg(\neg p \diamond p)$ in the second one, and $p \diamond p$ in the third case. Then for all $\nu \in \Lambda_{\mathcal{M}}$, $\nu(\Psi) = t$ if $\nu(p) \neq \top$.

Now let $\langle \mathcal{L}, \vdash \rangle$ be a proper extension of $\mathbf{L}_{\mathcal{M}}$ by some set of rules. We show that $\langle \mathcal{L}, \vdash \rangle$ is not pre-paraconsistent. Let Γ be a finite theory and ψ a formula in $\mathcal{L}$ such that $\Gamma \vdash \psi$ but $\Gamma \nvdash_{\mathcal{M}} \psi$. In particular, there is a valuation $\nu \in mod_{\mathcal{M}}(\Gamma)$ such that $\nu(\psi) = f$. Consider the substitution θ, defined for every $p \in \mathsf{Atoms}(\Gamma \cup \{\psi\})$ by

$$\theta(p) = \begin{cases} q_0 & \text{if } \nu(p) = t, \\ \neg q_0 & \text{if } \nu(p) = f, \\ p_0 & \text{if } \nu(p) = \top, \end{cases}$$

where p_0 and q_0 are two different atoms in $\mathcal{L}$. Note that $\theta(\Gamma)$ and $\theta(\psi)$ contain (at most) the variables p_0, q_0, and that for every valuation $\mu \in \Lambda_{\mathcal{M}}$ where $\mu(p_0) = \top$ and $\mu(q_0) = t$ it holds that $\mu(\theta(\phi)) = \nu(\phi)$ for every formula ϕ such that $\mathsf{Atoms}(\{\phi\}) \subseteq \mathsf{Atoms}(\Gamma \cup \{\psi\})$. Thus,

($\star$) *any* $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$ and $\mu(q_0) = t$ is an $\mathcal{M}$-model of $\theta(\Gamma)$ but not of $\theta(\psi)$.

Now, consider the following two cases:

Case I There is a formula $\phi(p, q)$ (i.e., $\mathsf{Atoms}(\phi) = \{p, q\}$, where $p \neq q$) such that for every $\mu \in \Lambda_{\mathcal{M}}$, $\mu(\phi) \neq \top$ if $\mu(p) = \mu(q) = \top$.
In this case, let $\mathsf{tt} = \Psi(\phi(p_0, p_0))$. Note that $\mu(\mathsf{tt}) = t$ for every $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$. Now, as $\vdash$ is structural, $\Gamma \vdash \psi$ implies that

$$\theta(\Gamma)\ [\mathsf{tt}/q_0] \vdash \theta(\psi)\ [\mathsf{tt}/q_0]. \tag{4.1}$$

Also, by the above property of tt and by ($\star$), any $\mu \in \Lambda_{\mathcal{M}}$ for which $\mu(p_0) = \top$ is a model of $\theta(\Gamma)\ [\mathsf{tt}/q_0]$ but does not $\mathcal{M}$-satisfy $\theta(\psi)\ [\mathsf{tt}/q_0]$. Thus,

- $p_0, \neg p_0 \vdash_{\mathcal{M}} \theta(\gamma)\ [\mathsf{tt}/q_0]$ for every $\gamma \in \Gamma$. As $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, this implies that
$$p_0, \neg p_0 \vdash \theta(\gamma)\ [\mathsf{tt}/q_0] \text{ for every } \gamma \in \Gamma. \tag{4.2}$$
- The set $\{p_0,\ \neg p_0,\ \theta(\psi)[\mathsf{tt}/q_0]\}$ is not $\mathcal{M}$-satisfiable, thus $p_0, \neg p_0, \theta(\psi)\ [\mathsf{tt}/q_0] \vdash_{\mathcal{M}} q_0$. Again, as $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, we have that
$$p_0,\ \neg p_0,\ \theta(\psi)\ [\mathsf{tt}/q_0] \vdash q_0. \tag{4.3}$$

By (4.1)–(4.3) $p_0, \neg p_0 \vdash q_0$, thus $\langle \mathcal{L}, \vdash \rangle$ is not pre-paraconsistent.

Case II For every formula $\phi(p, q)$ and for every $\mu \in \Lambda_{\mathcal{M}}$, if $\mu(p) = \mu(q) = \top$ then $\mu(\phi) = \top$.
Again, as $\vdash$ is structural, and since $\Gamma \vdash \psi$,

$$\theta(\Gamma)\ [\Psi(q_0)/q_0] \vdash \theta(\psi)\ [\Psi(q_0)/q_0]. \tag{4.4}$$

In addition, ($\star$) above entails that any valuation $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$ and $\mu(q_0) \in \{t, f\}$ is a model of $\theta(\Gamma)\ [\Psi(q_0)/q_0]$ which is not a model of $\theta(\psi)\ [\Psi(q_0)/q_0]$. Thus, the only $\mathcal{M}$-model of $\{p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0)/q_0]\}$ is the one in which both of p_0 and q_0 are assigned the value $\top$. It follows that $p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0)/q_0] \vdash_{\mathcal{M}} q_0$. Thus,

$$p_0, \neg p_0, \theta(\psi)\ [\Psi(q_0)/q_0] \vdash q_0. \tag{4.5}$$

By using ($\star$) again (for $\mu(q_0) \in \{t, f\}$) and the condition of Case II (for $\mu(q_0) = \top$), we have

$$p_0, \neg p_0 \vdash \theta(\gamma)\ [\Psi(q_0)/q_0] \text{ for every } \gamma \in \Gamma. \tag{4.6}$$

Again, by (4.4)–(4.6) above we have that $p_0, \neg p_0 \vdash q_0$, and so $\langle \mathcal{L}, \vdash \rangle$ is not pre-paraconsistent in this case either.

For the second part of the theorem we need the following lemma.

Lemma 4.47 *Let $\mathcal{M}$ be a paraconsistent three-valued matrix, and suppose that there is some bivalent $\neg$-interpretation $\mathbf{F}$ such that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-contained in classical logic, but $\mathbf{L}_{\mathcal{M}}$ is not $\mathbf{F}$-maximal relative to classical logic. Then $\mathcal{M}$ is classically closed.*

Proof The assumption about $\mathbf{F}$ implies that there is some classical $\mathbf{F}$-tautology (Definition 4.29) ψ_0 which is not provable in $\mathbf{L}_{\mathcal{M}}$, and by adding it as an axiom to $\mathbf{L}_{\mathcal{M}}$ we get a logic $\mathbf{L}^*$ that is *not* $\mathbf{F}$-complete. Since $\mathbf{L}_{\mathcal{M}}$ is strongly maximal by the first part of this theorem (and $\mathbf{L}^*$ is an extension by a rule of $\mathbf{L}_{\mathcal{M}}$), $\varphi, \neg\varphi \vdash_{\mathbf{L}^*} \phi$ for every φ, ϕ. It follows that

$$\mathcal{S}^*, \varphi, \neg\varphi \vdash_{\mathcal{M}} \phi \text{ for every } \varphi, \phi \tag{4.7}$$

where $\mathcal{S}^*$ is the set of all substitution instances of ψ_0. Now, let σ be some classical $\mathbf{F}$-tautology not provable in $\mathbf{L}^*$. So $\nvdash_{\mathbf{L}^*} \sigma$, and so $\mathcal{S}^* \nvdash_{\mathcal{M}} \sigma$. Hence there is a valuation $\nu \in \Lambda_{\mathcal{M}}$ which is a model of $\mathcal{S}^*$, but $\nu(\sigma) = f$. We show that there is no formula ψ for which $\nu(\psi) = \top$. Assume for contradiction that this is not the case for some ψ. Since ν is a model of $\mathcal{S}^*$, it is also a model of $\mathcal{S}^* \cup \{\psi, \neg\psi\}$, and so it is a model of σ by (4.7) above. This contradicts the fact that $\nu(\sigma) = f$. It follows that $\nu(\psi) \in \{t, f\}$ for all ψ. We show that this implies that all the operations of $\mathcal{M}$ are classically closed. Let $\diamond$ be some n-ary connective of $\mathcal{L}$ and let $a_1, \ldots, a_n \in \{t, f\}$. For $i = 1, \ldots, n$, define $\varphi_i = P_i$ if $\nu(p_i) = a_i$, and $\varphi_i = \neg P_i$ otherwise. Thus $\nu(\varphi_i) = a_i$, and $\tilde{\diamond}(a_1, \ldots, a_n) = \tilde{\diamond}(\nu(\varphi_1), \ldots, \nu(\varphi_n)) = \nu(\diamond(\varphi_1, \ldots, \varphi_n)) \in \{t, f\}$. □

Now we can show the second part of Theorem 4.45. The assumption that $\mathcal{M}$ is $\neg$-contained in classical logic entails that it is $\mathbf{F}$-contained in classical logic for some $\mathbf{F}$. If $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}$-maximal relative to classical logic, then we are done. Otherwise, $\mathcal{M}$ is classically closed by Lemma 4.47, and so we can consider the bivalent $\neg$-interpretation induced by $\mathcal{M}$, defined by $\mathbf{F}_{\mathcal{M}}(\diamond) = \tilde{\diamond}_{\mathcal{M}}/\{t, f\}^n$ (where n is the arity of $\diamond$, and $\tilde{\diamond}_{\mathcal{M}}/\{t, f\}^n$ is the reduction of $\tilde{\diamond}_{\mathcal{M}}$ to $\{t, f\}^n$). As the next lemma shows, $\mathbf{F}$ must be identical to this interpretation.

Lemma 4.48 $\mathbf{F} = \mathbf{F}_{\mathcal{M}}$

Proof Suppose otherwise.Then there is some n-ary connective $\diamond$ of $\mathcal{L}$ such that $\tilde{\diamond}/\{t, f\} = \mathbf{F}_{\mathcal{M}}(\diamond) \neq \mathbf{F}(\diamond)$. Hence there are some elements $a_1, \ldots, a_n \in \{t, f\}$ such that $\tilde{\diamond}(a_1, \ldots, a_n) \neq \mathbf{F}(\diamond)(a_1, \ldots, a_n)$. Because $\mathbf{F}$ and $\mathbf{F}_{\mathcal{M}}$ are both bivalent $\neg$-interpretations, we may assume without loss of generality that $\mathbf{F}(\diamond)(a_1, \ldots, a_n) = t$ and $\tilde{\diamond}(a_1, \ldots, a_n) = f$ (otherwise we consider $\neg\diamond$ instead of $\diamond$). Next, for $i = 1, \ldots, n$ we define $\varphi_i = p$ if $a_i = t$ and $\varphi_i = \neg p$ otherwise, thus $p, \diamond(\varphi_1, \ldots, \varphi_n) \vdash_{\mathcal{M}} \neg p$, while $p, \diamond(\varphi_1, \ldots, \varphi_n) \nvdash_{\mathcal{M}_{\mathbf{F}}} \neg p$ (because $\nu(p) = t$ provides a counterexample). This contradicts the $\mathbf{F}$-containment of $\mathbf{L}_{\mathcal{M}}$ in classical logic. □

Now, by the lemma above, $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}_{\mathcal{M}}$-contained in classical logic. We end by showing that $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}_{\mathcal{M}}$-maximal relative to classical logic. The proof of this is very similar to the proof of Lemma 4.47: Let ψ' be a classical $\mathbf{F}_{\mathcal{M}}$-tautology not provable in $\mathbf{L}_{\mathcal{M}}$, and let $\mathcal{S}'^*$ be the set of all of its substitution instances. Let $\mathbf{L}'^*$ be the logic obtained by adding ψ' as a new axiom to $\mathbf{L}_{\mathcal{M}}$. Then for every theory $\mathcal{T}$ we have that $\mathcal{T} \vdash_{\mathbf{L}'^*} \phi$ iff $\mathcal{T}, \mathcal{S}'^* \vdash_{\mathcal{M}} \phi$. In particular, since $\mathbf{L}_{\mathcal{M}}$ is strongly maximal, Condition (4.7) from the proof of Lemma 4.47 holds for $\mathcal{S}'^*$. Suppose for contradiction that there is some classical $\mathbf{F}_{\mathcal{M}}$-tautology σ not provable in $\mathbf{L}'^*$. Since $\nvdash_{\mathbf{L}'^*} \sigma$, also $\mathcal{S}'^* \nvdash_{\mathcal{M}} \sigma$. Hence, there is a valuation $\nu \in \Lambda_{\mathcal{M}}$ which is a model of $\mathcal{S}'^*$, but $\nu(\sigma) = f$. If there is some ψ such that $\nu(\psi) = \top$, then since ν is a model of $\mathcal{S}'^*$, it is also a model of $\mathcal{S}'^* \cup \{\psi, \neg\psi\}$, and so by (4.7) it is a model of σ, in contradiction to the fact that $\nu(\sigma) = f$. Otherwise, $\nu(\psi) \in \{t, f\}$ for all ψ, and so ν is an $\mathcal{M}_{\mathbf{F}_{\mathcal{M}}}$-valuation, which assigns f to σ. This contradicts the fact that $\vdash_{\mathcal{M}_{\mathbf{F}_{\mathcal{M}}}} \sigma$. Hence, all classical $\mathbf{F}_{\mathcal{M}}$-tautologies are provable in $\mathbf{L}'^*$, and so $\mathbf{L}_{\mathcal{M}}$ is $\mathbf{F}_{\mathcal{M}}$-maximal relative to classical logic. □

Note 4.49 Suppose that $\mathcal{M}$ is a three-valued paraconsistent matrix which is $\neg$-contained in classical logic. Then any three-valued expansion of it which is obtained by enriching the language of $\mathcal{M}$ with extra classically closed connectives necessarily has the same properties (see Theorem 4.43). It follows that not only is $\mathbf{L}_{\mathcal{M}}$ fully maximal, but so must be also all the logics induced by its expansions that are so obtained.

4.5 The Most Important Paraconsistent Three-Valued Logics

As shown in the previous section, there are exactly eight ways of defining conjunctions in three-valued paraconsistent matrices. Of these eight operations, only four are symmetric. Of these four, only two are $\{\top\}$-closed, and to the best of our knowledge, only three (including these two) have been seriously investigated in the literature. In this section we examine in greater detail the properties of the most important (and famous) three-valued paraconsistent logics that are based on these three symmetric conjunctions and the two possible negations. Then in the next section, we shall show that each of these logics has a corresponding cut-free Gentzen-type system, which is very close to the classical one.

Our main criterion here for "importance" of three-valued paraconsistent matrices is having a *natural* set of connectives that can be characterized by a combination of potentially desirable properties. The most important such property is of course $\{t, f\}$-closure, which by Theorem 4.43 is equivalent to $\neg$-containment in classical logic. Another important property is $\{\top\}$-closure, which by Proposition 4.37 is equivalent to being non-exploding. Other properties are introduced and used in the sequel.

4.5.1 The Logic $\mathbf{P_1}$

Sette's logic $\mathbf{P_1} = \langle \mathcal{L}_{\mathsf{P}_1}, \vdash_{\mathsf{P}_1} \rangle$ [35] is induced by the matrix $\mathsf{P}_1 = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\wedge}, \tilde{\neg}\} \rangle$,[6] where the operations are defined as follows:

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

Proposition 4.50 $\mathbf{P_1}$ *is boldly paraconsistent, normal, $\neg$-contained in classical logic, and fully maximal.*

Proof Define $\psi \vee \phi = \neg(\neg(\psi \wedge \psi) \wedge \neg(\phi \wedge \phi))$ and $\psi \supset \phi = \neg((\psi \wedge \psi) \wedge \neg(\phi \wedge \phi))$. The corresponding interpretations are the following:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\supset}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

Therefore, Item 3 of Theorem 4.43 implies that $\mathbf{P_1}$ is a normal paraconsistent logic, which is $\neg$-contained in classical logic. The other properties follow from Corollary 4.34 and Theorem 4.45. □

Note 4.51 As far as we know, $\mathbf{P_1}$ was the first paraconsistent logic for which a maximality property has been stated and proved (in [35]). Therefore, it is frequently referred to as "Sette maximal paraconsistent logic." However, the results in Sect. 4.4 show that there is nothing special about $\mathbf{P_1}$ in this respect. Its maximality is just one (out of thousands) instances of Theorem 4.45.

The next theorem characterizes the expressive power of the language of $\mathbf{P_1}$.

Theorem 4.52 *A function $g : \{t, f, \top\}^n \to \{t, f, \top\}$ is representable in $\mathcal{L}_{\mathsf{P}_1}$ iff its range is $\{t, f\}$.*

Proof Obviously, the condition is necessary. To show that it is also sufficient, define:

$$\psi_a(p) = \begin{cases} p \wedge \neg\neg p & \text{if } a = t \\ \neg(p \wedge p) & \text{if } a = f \\ p \wedge \neg p & \text{if } a = \top \end{cases}$$

It is easy to check that if ν is a valuation in $\mathbf{P_1}$, then $\nu \models_{\mathbf{P_1}} \psi_a(p)$ iff $\nu(p) = a$. Now, given a function $g : \{t, f, \top\}^n \to \{t, f\}$, it is not difficult to see that g is represented in

[6] Note that in our notations $\mathbf{P_1}$ is also denoted $\mathbf{L}_{\mathsf{P}_1}$.

$\mathcal{L}_{\mathsf{P}_1}$ by the disjunction (as defined in the proof of Proposition 4.50) of all the formulas of the form $\psi_{a_1}(P_1) \wedge \psi_{a_2}(P_2) \wedge \cdots \wedge \psi_{a_n}(P_n)$ such that $g(a_1, a_2, \ldots, a_n) = t$ (and by the formula $\neg\neg P_1 \wedge \neg P_1$ if no such $a_1, a_2, \ldots, a_n$ exist). $\square$

Corollary 4.53 *The connectives defined in the proof of Proposition 4.50 are the* only *disjunction and implication definable in* $\mathbf{P_1}$.

Proof This easily follows from Theorems 4.42 and 4.52. $\square$

Note 4.54 As noted previously, the Logic $\mathbf{P_1}$ has all the desirable properties mentioned in the previous section. Nevertheless, $\mathbf{P_1}$ also has the following two severe drawbacks:

- It is paraconsistent only with respect to atomic formulas (that is, for a nonatomic formula ψ we have that $\psi, \neg\psi \vdash_{\mathsf{P}_1} \varphi$, since nonatomic formulas get only values in $\{t, f\}$).
- The conjunction–negation combination does not always behave as expected, e.g., $\neg p \nvdash_{\mathsf{P}_1} \neg(p \wedge q)$.

The main source of these problematic features is the fact that Sette's negation (which is the negation used in $\mathbf{P_1}$) has the following drawbacks in comparison to Kleene's negation:

- It is explosive with respect to negated data: $\neg\varphi, \neg\neg\varphi \vdash_{\mathsf{P}_1} \psi$ for every φ, ψ.
- It is not right involutive: $p \nvdash_{\mathsf{P}_1} \neg\neg p$.

These drawbacks should be the reason why $\mathbf{P_1}$ is (to the best of our knowledge) the only three-valued paraconsistent logic considered in the literature whose negation is Sette's negation. Accordingly, all the other logics described in this section use Kleene's negation.

4.5.2 The Logic $\mathbf{SRM}_{\underset{\to}{\sim}}$

Another conjunction of the eight possible conjunctions listed in Sect. 4.4 has (implicitly) been used by Sobociński in his three-valued matrix [38]. This is the matrix $\mathcal{A}_1 = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\otimes}, \tilde{\neg}\}\rangle$ for the language $\mathcal{IL} = \{\neg, \otimes\}$ in which $\tilde{\neg}$ is Kleene's negation, and $\tilde{\otimes}$ is Sobociński's conjunction, defined below:

$\tilde{\otimes}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

We denote by $\mathbf{SRM}_{\underset{\to}{\sim}}$ (or $\mathbf{SRMI}^{\mathbf{1}}_{\underset{\to}{\sim}}$) the logic that is induced by $\mathcal{A}_1$.

Note 4.55 The official language that was used in [38] (as well as in the literature on relevance logic) is $\{\neg, \rightarrow\}$, and the interpretation of $\rightarrow$ there was the following *Sobociński's implication*:

$$a \rightarrow_S b \ = \ \begin{cases} \top & \text{if } a = b = \top, \\ f & \text{if } a = t \text{ and } b \neq t, \text{ or } b = f \text{ and } a \neq f, \\ t & \text{otherwise.} \end{cases}$$

It is easy to see that $a \rightarrow_S b = \tilde{\neg}(a \tilde{\otimes} \tilde{\neg} b)$, while $a \tilde{\otimes} b = \tilde{\neg}(a \rightarrow_S \tilde{\neg} b)$. Hence, $\mathcal{IL}$ and $\mathcal{A}_1$ are equivalent to Sobociński's original language and matrix (respectively).

Note 4.56 It should be emphasized that $\mathbf{SRM}_{\tilde{\rightarrow}}$ is *not* identical to the logic introduced by Sobociński in [38]. That logic has only been *motivated* by the matrix $\mathcal{A}_1$. What Sobociński actually did in [38] is to axiomatize *the set of valid formulas of* $\mathcal{A}_1$ using a Hilbert-type system with Modus Ponens for $\rightarrow$ as the single rule of inference. In other words: his system is only *weakly* complete for $\mathcal{A}_1$. Thus, one cannot derived in it φ from $\varphi \otimes \psi$, even though $\varphi \otimes \psi \vdash_{\mathbf{SRM}_{\tilde{\rightarrow}}} \varphi$.[7]

The connective $\rightarrow$ of $\mathbf{SRM}_{\tilde{\rightarrow}}$ is not an implication for that logic (since $\varphi \rightarrow (\psi \rightarrow \varphi)$ is not valid in $\mathcal{A}_1$). Despite this we have

Proposition 4.57 $\mathbf{SRM}_{\tilde{\rightarrow}}$ *is non-exploding, normal, $\neg$-contained in classical logic, and fully maximal.*

Proof Define $\varphi \supset \psi = \varphi \rightarrow (\varphi \otimes \psi)$, where (as above) $\varphi \rightarrow \psi = \neg(\varphi \otimes \neg\psi)$. Then $\supset$ has in $\mathcal{A}_1$ the following interpretation:

$\tilde{\supset}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	$\top$

It follows that $\supset$ is an $\mathcal{A}_1$-implication. This implies that the connective $\vee$, defined by $\psi \vee \varphi = (\psi \supset \varphi) \supset \varphi$, is an $\mathcal{A}_1$-disjunction. Finally, $\otimes$ is an $\mathcal{A}_1$-conjunction. Therefore, Item 3 of Theorem 4.43 implies that $\mathbf{SRM}_{\tilde{\rightarrow}}$ is a normal paraconsistent logic which is $\neg$-contained in classical logic. The other properties follow from Theorem 4.45 and Proposition 4.37. □

The following theorem characterizes the expressive power of the language of $\mathbf{SRM}_{\tilde{\rightarrow}}$:

Theorem 4.58 [9] *The connectives that are definable in the language of* $\mathbf{SRM}_{\tilde{\rightarrow}}$ *are those that are both $\{\top\}$-closed and $\{\top\}$-limited (Definition 4.12).*

Note that by the last theorem, it follows that Kleene's conjunction (see next section) is *not* definable in the language of $\mathbf{SRM}_{\tilde{\rightarrow}}$ (since Kleene's conjunction is not $\{\top\}$-limited).

[7] Meyer has shown (see [1]) that Sobociński's system induces the $\{\neg, \rightarrow, \otimes\}$-fragment of the semi-relevant logic **RM**.

4.5.3 The Logic **LP** *and Its Main Monotonic Expansions*

The most popular conjunction used in three-valued paraconsistent logics and three-valued logics in general is Kleene's (strong) conjunction (see the truth-table below), and the most basic paraconsistent logic which is based on it is Asenjo–Priest's three-valued logic **LP** [5, 31–33]. This is the logic induced by the three-valued matrix $\mathsf{LP} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\wedge}, \tilde{\neg}\}\rangle$, where the truth-tables for $\neg$ and $\wedge$ are the following:

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

The matrix LP also has a disjunction, defined by $\psi \vee \varphi = \neg(\neg\psi \wedge \neg\varphi)$. What is obtained is one of the possible interpretations of disjunction given in Theorem 4.43: the strong Kleene's disjunction, whose truth-table is the following:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

Note 4.59 A common way of defining and understanding the disjunction, conjunction, and negation of LP is with respect to total order $\leq_{\mathsf{t}}$ on $\{t, f, \top\}$, in which t is the maximal element and f is the minimal one. This order may be intuitively understood as reflecting differences in the amount of *truth* that each element exhibits. Here, $\tilde{\wedge}$ and $\tilde{\vee}$ are the meet and the join (respectively) of $\leq_{\mathsf{t}}$, and $\tilde{\neg}$ is order reversing with respect to $\leq_{\mathsf{t}}$.

Next, we introduce the simplest expansions of **LP**: those that are obtained by adding to its language the propositional constants which correspond to the truth-values that are used. We shall denote by f the one for which $\forall\nu \in \Lambda\ \nu(\mathsf{f}) = f$ and by $\top$ the constant for which $\forall\nu \in \Lambda\ \nu(\top) = \top$. (There is no need to consider also a constant for t, because such a constant and f are definable in terms of each other and $\neg$.)

Definition 4.60

- $\mathbf{LP}^{\mathsf{f}}$ is the logic induced by the expansion of the matrix LP to the language $\{\neg, \wedge, \vee, \mathsf{f}\}$ (or just $\{\neg, \wedge, \mathsf{f}\}$).
- $\mathbf{LP}^{\top}$ is the logic induced by the expansion of the matrix LP to the language $\{\neg, \wedge, \vee, \top\}$.
- $\mathbf{LP}^{\mathsf{f},\top}$ is the logic induced by the expansion of the matrix LP to the language $\{\neg, \wedge, \vee, \mathsf{f}, \top\}$

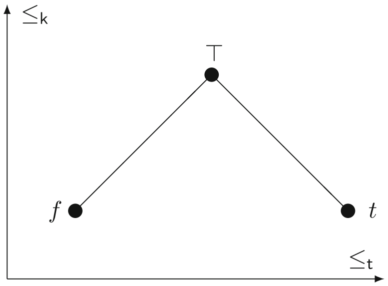

Fig. 4.1 $\mathcal{THREE}$

For characterizing the expressive power of the languages of **LP** and its above expansions, it is convenient to order the truth-values in a partial order $\leq_k$ that intuitively reflects differences in the amount of *knowledge* (or *information*) that the truth values convey. According to this relation $\top$ is the maximal element, while neither of the remaining truth-values is greater than the other. Therefore, $\langle \mathcal{V}, \leq_k \rangle$ is an upper semilattice. A Double-Hasse diagram representing the structure $\mathcal{THREE}$ which is induced by $\leq_k$ and $\leq_t$ (Note 4.59) is given in Fig. 4.1. In this diagram b is an immediate $\leq_t$-successor of a iff b is on the right-hand side of a, and there is an edge between them; Similarly, b is an immediate $\leq_k$-successor of a iff b is above a, and there is an edge between them.[8]

Definition 4.61 A function $g : \{t, f, \top\}^n \to \{t, f, \top\}$ is $\leq_k$*-monotonic* if $g(a_1, \ldots, a_n) \leq_k g(b_1, \ldots, b_n)$ in case $a_i \leq_k b_i$ for every $1 \leq i \leq n$.

Now we are able to characterize the expressive power of **LP** and its expansions:

Theorem 4.62 [8] *Let* $g : \{t, f, \top\}^n \to \{t, f, \top\}$.

1. g *is representable in the language of* $\mathbf{LP}^{\mathsf{f},\top}$ *iff it is* $\leq_k$*-monotonic.*
2. g *is representable in the language of* $\mathbf{LP}^{\top}$ *iff it is* $\leq_k$*-monotonic and* $\{\top\}$*-closed.*
3. g *is representable in the language of* $\mathbf{LP}^{\mathsf{f}}$ *iff it is* $\leq_k$*-monotonic and classically (i.e.,* $\{t, f\}$*-) closed.*
4. g *is representable in the language of* **LP** *iff it is* $\leq_k$*-monotonic,* $\{\top\}$*-closed, and classically closed.*

Next we turn to the main properties of the four logics considered in Theorem 4.62.

Proposition 4.63

1. **LP**, $\mathbf{LP}^{\mathsf{f}}$, $\mathbf{LP}^{\top}$, *and* $\mathbf{LP}^{\top,\mathsf{f}}$ *are all boldly paraconsistent and strongly maximal paraconsistent logics.*
2. **LP** *is* $\neg$*-contained in classical logic and fully maximal. The same is true for* $\mathbf{LP}^{\mathsf{f}}$ *(but not for* $\mathbf{LP}^{\top}$ *or* $\mathbf{LP}^{\top,\mathsf{f}}$*).*
3. **LP** *is non-exploding. The same is true for* $\mathbf{LP}^{\top}$*, but not for* $\mathbf{LP}^{\mathsf{f}}$ *or* $\mathbf{LP}^{\top,\mathsf{f}}$.

[8] We refer to [2, 15, 23, 25] for further motivation and discussions on algebraic structures that combine order relations about truth and knowledge.

Proof Immediate from Corollary 4.34, Theorems 4.43, 4.45, 4.62, and Proposition 4.37. □

Perhaps the most remarkable property of **LP** (and $\mathbf{LP}^{\mathsf{f}}$) is given in the next proposition.

Proposition 4.64 [31] *The tautologies of* **LP** *and* $\mathbf{LP}^{\mathsf{f}}$ *are the same as those of classical logic in their languages: if* ψ *is a formula in the language of* $\{\neg, \wedge, \vee\}$ $(\{\neg, \wedge, \vee, \mathsf{f}\})$ *then* $\vdash_{\mathsf{LP}} \psi$ $(\vdash_{\mathbf{LP}^{\mathsf{f}}} \psi)$ *iff* $\vdash_{\mathcal{M}_{CL}} \psi$, *where* $\mathcal{M}_{CL}$ *is the two-valued matrix for classical logic.*

Proof One direction is trivial. For the converse, suppose, e.g., that ν is an LP-valuation (the proof in the case of $\mathbf{LP}^{\mathsf{f}}$ is similar). Let μ be the $\mathcal{M}_{CL}$-valuation such that for every $p \in \mathsf{Atoms}$, $\mu(p) = t$ iff $\nu(p) \in \{t, \top\}$. It is easy to prove by induction on the complexity of ψ that if $\mu(\psi) = t$ then $\nu(\psi) \in \{t, \top\}$, and if $\mu(\psi) = f$ then $\nu(\psi) \in \{f, \top\}$. It follows that if for every $\mathcal{M}_{CL}$-valuation μ it holds that $\mu(\psi) = t$, then for every LP-valuation ν, $\nu(\psi)$ is designated. □

Note 4.65 Despite of having the same set of valid formulas, **LP** is paraconsistent, while classical logic (in the language of $\{\neg, \wedge, \vee\}$) is not. The difference between the two is due to their *consequence relations*.

The main drawback of **LP** and the other logics studied in this section is given in the next proposition.

Proposition 4.66 [3] *Suppose that* $\mathcal{M}$ *is a three-valued paraconsistent matrix which has only* $\leq_k$*-monotonic connectives. Then* $\mathbf{L}_{\mathcal{M}}$ *does not have an implication connective.*

Proof Suppose for contradiction that $\supset$ is a definable implication for $\mathbf{L}_{\mathcal{M}}$. By Lemma 4.6 this implies that (i) $\vdash_{\mathcal{M}} p \supset p$, and (ii) $p, p \supset q \vdash_{\mathcal{M}} q$. Now, (i) entails that $\tilde{\supset}(f, f) \in \{t, \top\}$. Therefore it follows from the $\leq_k$-monotonicity of $\supset$ that $\tilde{\supset}(\top, f) \in \{t, \top\}$. This contradicts (ii), since it is refuted by any assignment ν such that $\nu(p) = \top$ and $\nu(q) = f$. □

Corollary 4.67 *The logics* **LP**, $\mathbf{LP}^{\top}$, $\mathbf{LP}^{\mathsf{f}}$, *and* $\mathbf{LP}^{\mathsf{f},\top}$ *are not normal, but only semi-normal.*

Proof Immediate from Corollary 4.40, Theorem 4.62, and Proposition 4.66. □

4.5.4 The Logics PAC ($\mathbf{RM_3}$) *and Its Main Expansions*

The most straightforward way to turn **LP** into a normal logic is to extend LP by an implication connective. A natural candidate for this is D'Ottaviano and da-Costa's implication [19, 21], considered in Note 4.44. Because of its nice properties (to be presented below), this is the main implication connective (in the sense of Definition 4.5) which has been used in three-valued paraconsistent logics. The logic that is obtained by extending LP with $\supset$ is called **PAC** (also known as $\mathbf{RM_3}$) [6, 7, 14, 20, 22, 34]. Thus, **PAC** is the logic which is induced by the three-valued matrix $\mathsf{PAC} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\wedge}, \tilde{\vee}, \tilde{\supset}, \tilde{\neg}\}\rangle$, where $\tilde{\wedge}$, $\tilde{\vee}$, and $\tilde{\neg}$ are like in LP, while $\tilde{\supset}$ is given by the following truth-table:

$\tilde{\supset}$	t	f	$\top$
t	t	f	$\top$
f	t	t	t
$\top$	t	f	$\top$

Note 4.68 Since $a \tilde{\to}_S b = (a \tilde{\supset} b) \tilde{\wedge} (\tilde{\neg} b \tilde{\supset} \tilde{\neg} a)$, while $a \tilde{\supset} b = b \tilde{\vee} (a \tilde{\to}_S b)$, another way that leads to **PAC** is to extend $\mathcal{A}_1$, and with it $\mathbf{SRM}_{\tilde{\to}}$, with Kleene's conjunction (which, as indicated at the end of Sect. 4.5.2, is not definable in their language).

Again, the simplest expansions of **PAC** are those that are obtained by adding to its language the propositional constants $\top$ and f.

Definition 4.69

- $\mathbf{J_3}$ [20, 22] is the logic induced by the expansion of the matrix PAC to the language $\{\neg, \wedge, \vee, \supset, \mathsf{f}\}$.
- $\mathbf{PAC}^\top$ is the logic induced by the expansion of the matrix PAC to the language $\{\neg, \wedge, \vee, \supset, \top\}$.
- $\mathbf{J_3^\top}$ is the logic induced by the expansion of the matrix PAC to the language $\{\neg, \wedge, \vee, \supset, \mathsf{f}, \top\}$

Note 4.70 Instead of the propositional constant f it is common in the literature on $\mathbf{J_3}$ to use as the extra connective the *consistency* operator $\circ$, whose interpretation $\tilde{\circ}$ is given by: $\tilde{\circ}(t) = \tilde{\circ}(f) = t$, and $\tilde{\circ}(\top) = f$. This does not make much difference, since $\tilde{\circ}(a) = (a \tilde{\wedge} \tilde{\neg} a) \tilde{\supset} \mathsf{f}$, while $\mathsf{f} = \tilde{\circ}(a) \tilde{\wedge} \tilde{\neg}\tilde{\circ}(a)$. As a logic in the language of $\{\neg, \wedge, \vee, \supset, \circ\}$, $\mathbf{J_3}$ is *the strongest logic* in the family of LFIs (Logics of Formal Inconsistency, [16]) in this language. Recently, $\mathbf{J_3}$ and its weaker versions have also been considered in the context of epistemic logics, where in [17, 18] it is shown that these logics can be encoded in a simple fragment of the modal logic **KD**, containing only modal formulas without nesting.

The following theorem characterizes the expressive power of the languages of PAC and its expansions:

Theorem 4.71 [8] *Let* $g : \{t, f, \top\}^n \to \{t, f, \top\}$.

1. *g is representable in the language of* $\mathbf{J_3^\top}$.
2. *g is representable in the language of* $\mathbf{J_3}$ *iff it is* $\{t, f\}$*-closed (i.e., iff it is classically closed).*

3. *g is representable in the language of* $\mathbf{PAC}^{\top}$ *iff it is* $\{\top\}$*-closed.*
4. *g is representable in the language of* $\mathbf{PAC}$ *iff it is both* $\{t, f\}$*-closed and* $\{\top\}$*-closed.*

Note 4.72 In [8] it is also shown that by adding to PAC *any* classically closed connective not available in it, we get a matrix in which exactly the classically closed connectives are available. Similarly, by adding to PAC *any* $\{\top\}$-closed connective not available in it, we get a matrix in which exactly the $\{\top\}$-closed connectives are available. It follows that there is no intermediate expansion of $\mathbf{PAC}$ between $\mathbf{PAC}$ and $\mathbf{J_3}$, or between $\mathbf{PAC}$ and $\mathbf{PAC}^{\top}$. From the results of [8], it also follows that there is no intermediate expansion of $\mathbf{J_3}$ or $\mathbf{PAC}^{\top}$ between these logics and $\mathbf{J_3^{\top}}$.

The main properties of the four logics discussed above are considered next.

Proposition 4.73

1. $\mathbf{PAC}$*,* $\mathbf{J_3}$*,* $\mathbf{PAC}^{\top}$*, and* $\mathbf{J_3^{\top}}$ *are all normal, boldly paraconsistent, and strongly maximal paraconsistent logics.*
2. $\mathbf{PAC}$ *and* $\mathbf{J_3}$ *are* $\neg$*-contained in classical logic and fully maximal. This is false for* $\mathbf{PAC}^{\top}$ *and* $\mathbf{J_3^{\top}}$*.*
3. $\mathbf{PAC}$ *and* $\mathbf{PAC}^{\top}$ *are non-exploding. This is false for* $\mathbf{J_3}$ *and* $\mathbf{J_3^{\top}}$*.*

Proof Follows from Corollary 4.34, Theorems 4.71, 4.43, 4.45, and Proposition 4.37. □

Corollary 4.74

1. *Every three-valued paraconsistent logic can be embedded in* $\mathbf{J_3^{\top}}$*.*
2. $\mathbf{J_3}$ *is the strongest three-valued paraconsistent logic which is* $\neg$*-contained in classical logic (i.e., every other logic with these properties, like* $\mathbf{P_1}$*, can be embedded in it).*
3. $\mathbf{PAC}^{\top}$ *is the strongest three-valued paraconsistent logic which is non-exploding.*
4. $\mathbf{PAC}$ *is the strongest three-valued paraconsistent logic which is both* $\neg$*-contained in classical logic and non-exploding.*

Proof Immediate from Theorems 4.71 and 4.43, and from Propositions 4.73 and 4.37. □

4.5.5 The Logic $\mathbf{PAC}_{\supset}^{\neg}$

One more interesting paraconsistent three-valued logic is given by the $\{\neg, \supset\}$-fragment of $\mathbf{PAC}$. We call this fragment $\mathbf{PAC}_{\supset}^{\neg}$, and it is the logic induced by the matrix $\mathsf{PAC}_{\supset}^{\neg} = \langle\{t, f, \top\}, \{t, \top\}, \{\tilde{\supset}, \tilde{\neg}\}\rangle$, where $\tilde{\supset}$ and $\tilde{\neg}$ are like in PAC.

Proposition 4.75 [6] *The matrix* $\mathsf{PAC}_{\tilde{\supset}}$ *is (equivalent to) a proper expansion of* $\mathcal{A}_1$.

Proof The matrix $\mathsf{PAC}_{\tilde{\supset}}$ is (equivalent to) an expansion of $\mathcal{A}_1$, since:

$$a \mathbin{\tilde{\otimes}} b = \tilde{\neg}(\tilde{\neg}(a \mathbin{\tilde{\supset}} \tilde{\neg} b) \mathbin{\tilde{\supset}} (\tilde{\neg}(\tilde{\neg} b \mathbin{\tilde{\supset}} a))).$$

The expansion is proper by Proposition 4.58 and by the fact that $\tilde{\supset}$ is not $\{\top\}$-limited. $\square$

Corollary 4.76 $\mathbf{PAC}_{\tilde{\supset}}$ *is non-exploding, normal,* $\neg$*-contained in classical logic, and fully maximal.*

Proof The normality of $\mathbf{PAC}_{\tilde{\supset}}$ follows from Propositions 4.75, 4.57, and Corollary 4.39. The other properties follow, as usual, from Theorems 4.43, 4.45, and Proposition 4.37. $\square$

Next, we characterize the expressive power of the language of $\mathbf{PAC}_{\tilde{\supset}}$.

Theorem 4.77 *A function* $g : \{t, f, \top\}^n \to \{t, f, \top\}$ *is representable in the language of* $\mathbf{PAC}_{\tilde{\supset}}$ *iff it is* $\{\top\}$*-closed, and there is* $1 \leq i \leq n$ *such that* $g(a_1, \ldots, a_n) = \top$ *only if* $a_i = \top$.

Proof For a formula φ in the language of $\{\neg, \supset\}$, we define $\varphi_\top$ recursively as follows: $p_\top = p$ if p is atomic, $(\neg\psi)_\top = \psi_\top$, and $(\varphi \supset \psi)_\top = \psi_\top$. It is easy to verify that for every φ, $\varphi_\top$ is an atom such that $\nu(\varphi_\top) = \top$ whenever ν is a valuation in $\mathsf{PAC}_{\tilde{\supset}}$ such that $\nu(\varphi) = \top$. This easily implies that if g representable in the language of $\mathbf{PAC}_{\tilde{\supset}}$ then it satisfies the condition given above. Obviously, such g is also $\{\top\}$-closed. This prove the "only if" part of the proposition.

For the converse, let $\mathsf{f}_n = \neg P_1 \otimes P_1 \otimes \neg P_2 \otimes P_2 \otimes \cdots \otimes \neg P_n \otimes P_n$. For $a \in \{t, f, \top\}$ we define:

$$\psi_a(p) = \begin{cases} \neg p \supset \mathsf{f}_n & \text{if } a = t, \\ p \supset \mathsf{f}_n & \text{if } a = f, \\ p \otimes \neg p & \text{if } a = \top. \end{cases}$$

It is easy to check that for every valuation ν such that $\nu(P_j) \neq \top$ for some $1 \leq j \leq n$, it holds that $\nu(\psi_a(p)) \neq f$ iff $\nu(p) = a$. Next, for $\mathbf{a} = (a_1, \ldots, a_n) \in \{t, f, \top\}^n$ we let $\psi_{\mathbf{a}} = \psi_{a_1}(P_1) \otimes \cdots \otimes \psi_{a_n}(P_n)$. Then for every valuation ν, $\nu(\psi_{\mathbf{a}}) \neq f$ iff $\nu(P_i) = a_i$ for every $1 \leq i \leq n$, or $\nu(P_i) = \top$ for every $1 \leq i \leq n$.

Now, suppose that $g : \{t, f, \top\}^n \to \{t, f, \top\}$ has the above two properties, and let $1 \leq i \leq n$ have the property that $g(a_1, \ldots, a_n) = \top$ only if $a_i = \top$. It is not difficult to check that g is represented by the $\otimes$-conjunction of all the formulas which either has the form $\psi_{\mathbf{a}} \supset \mathsf{f}_n$ where $g(\mathbf{a}) = f$, or the form $\psi_{\mathbf{a}} \supset (P_i \supset P_i)$ where $g(\mathbf{a}) = \top$. (Note that since g is $\{\top\}$-closed, there is at least one formula of the latter form). $\square$

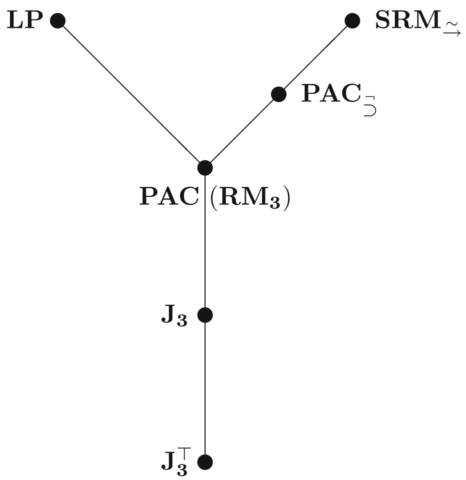

Fig. 4.2 Relative strength of some logics with Kleene's negation

Corollary 4.78 **PAC** *is a proper expansion of* $\mathbf{PAC}_{\stackrel{\neg}{\supset}}$.

Proof This follows from the previous proposition and the fact that Kleene's conjunction does not satisfy the second condition given there. □

Figure 4.2 shows the relative expressive power of six of the three-valued logics with Kleene's negation which are considered in this section (in the figure, if two logics are connected, the lower one is the stronger).

4.6 Proof Systems

4.6.1 Gentzen-Type Systems

In this section, we provide an explicit and concise presentation of Gentzen-type systems which correspond to the logics discussed in Sect. 4.5, as well as direct proofs of their completeness and the admissibility of the cut rule in them.[9] We start by recalling the notions of derivation and provability in a Gentzen-type sequent calculi. Below, we denote a sequent in a language $\mathcal{L}$ by s, or more explicitly by $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$ and $\Rightarrow$ is a new symbol, not used in $\mathcal{L}$.

Definition 4.79 Let G be a Gentzen-type sequent calculus.

- A *proof* (or *derivation*) in G of a sequent s from a set $\mathcal{S}$ of sequents is a finite sequence of sequents which ends with s, and every element in it either belongs to

[9] In [11] a general algorithm has been given for deriving sound and complete, cut-free Gentzen-type systems for finite-valued logics which have sufficiently expressive languages. That algorithm in fact works for *all* three-valued paraconsistent logics, but we shall not describe it here.

Axioms: $\psi \Rightarrow \psi$

Structural Rules:

Weakening: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Cut: $$\frac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$$[\wedge\Rightarrow]\ \frac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta} \qquad [\Rightarrow\wedge]\ \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$$

$$[\vee\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta} \qquad [\Rightarrow\vee]\ \frac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$$

$$[\supset\Rightarrow]\ \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta} \qquad [\Rightarrow\supset]\ \frac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$$

Fig. 4.3 The proof system LK^+

Axioms: $\varphi \Rightarrow \varphi$

Rules: All the rules of LK^+, and the following rules for negation:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\varphi}$$

$$\frac{\Gamma, \varphi \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi, \psi}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\varphi}{\Gamma, \neg\neg\varphi \Rightarrow \Delta}$$

Fig. 4.4 The proof system $G_{\mathbf{P_1}}$

$\mathcal{S}$, or is an axiom of G, or is obtained from previous elements of the sequence by one of the rules of G.

- We say that s *follows* from $\mathcal{S}$ in G (notation: $\mathcal{S} \vdash_{\mathsf{G}} s$), if there is a proof in G of s from $\mathcal{S}$.
- A sequent s is *provable* in G (notation: $\vdash_{\mathsf{G}} s$), if it follows in G from the empty set of sequents.
- The tcr $\vdash_{\mathsf{G}}$ induced by G is defined by $\mathcal{T} \vdash_{\mathsf{G}} \varphi$, if there exists a finite Γ such that $\vdash_{\mathsf{G}} \Gamma \Rightarrow \varphi$, and Γ consists only of elements of $\mathcal{T}$.[10]

In what follows, Fig. 4.3 presents a well-known version of Gentzen's proof system LK^+ for positive classical logic [24], on which all the Gentzen-type calculi presented here are based. Figure 4.4 describes a Gentzen-type system $G_{\mathbf{P_1}}$ for Sette's logic $\mathbf{P_1}$,

[10]Although the notation $\vdash_{\mathsf{G}}$ is overloaded in this definition, this should not cause any confusion in what follows.

Axioms: $\varphi \Rightarrow \varphi \qquad \Rightarrow \neg\varphi, \varphi$

Rules: All the rules of LK^+, and the following rules for $\neg$, f, and $\top$:

$$[\neg\neg\Rightarrow]\ \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \qquad [\Rightarrow\neg\neg]\ \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

$$[\neg\wedge\Rightarrow]\ \frac{\Gamma, \neg\varphi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \wedge \psi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\wedge]\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \wedge \psi)}$$

$$[\neg\vee\Rightarrow]\ \frac{\Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \vee \psi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\vee]\ \frac{\Gamma \Rightarrow \Delta, \neg\varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \vee \psi)}$$

$$[\neg\supset\Rightarrow]\ \frac{\Gamma, \varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \supset \psi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\supset]\ \frac{\Gamma \Rightarrow \varphi, \Delta \quad \Gamma \Rightarrow \neg\psi, \Delta}{\Gamma \Rightarrow \neg(\varphi \supset \psi), \Delta}$$

$$[\mathsf{f}\Rightarrow]\ \Gamma, \mathsf{f} \Rightarrow \Delta \qquad [\Rightarrow\neg\mathsf{f}]\ \Gamma \Rightarrow \Delta, \neg\mathsf{f}$$

$$[\Rightarrow\top]\ \Gamma \Rightarrow \Delta, \top$$

$$[\Rightarrow\neg\top]\ \Gamma \Rightarrow \Delta, \neg\top$$

Fig. 4.5 The proof system $G_{\mathbf{J}_3^\top}$

and Fig. 4.5 describes a Gentzen-type system $G_{\mathbf{J}_3^\top}$ for $\mathbf{J}_3^\top$. A Gentzen-type system $G_{\mathbf{L}}$ for every $\mathbf{L} \in \{\mathbf{LP}, \mathbf{LP}^{\mathsf{f}}, \mathbf{LP}^\top, \mathbf{LP}^{\mathsf{f},\top}, \mathbf{PAC}, \mathbf{J_3}, \mathbf{PAC}^\top, \mathbf{PAC}_{\overset{\neg}{\supset}}\}$ is obtained from $G_{\mathbf{J}_3^\top}$ by deleting from it the irrelevant rules (e.g., the rules for $\supset$ and f in the case of $\mathbf{LP}^\top$). Finally, Fig. 4.6 describes a Gentzen-type system $G_{\mathbf{SRM}_{\overset{\sim}{\to}}}$ for $\mathbf{SRM}_{\overset{\sim}{\to}}$ in the primitive language of this logic.

Note 4.80 Here are some important remarks about the Gentzen-type systems presented in this section:

- The last four rules in Fig. 4.4 can be combined into one rule: Infer $\neg\varphi, \Gamma \Rightarrow \Delta$ from $\Gamma \Rightarrow \Delta, \varphi$ (which is the rule $[\neg\Rightarrow]$, introducing negation on the left-hand side, of Gentzen's system LK for classical logic) with the constraint that the active formula (φ) should not be atomic.
- It is possible to take as axioms of $G_{\mathbf{J}_3^\top}$ only $p \Rightarrow p$, $\neg p \Rightarrow \neg p$, and $\Rightarrow p, \neg p$, where p is atomic (and the rules for f and $\top$, which are really axioms). All other instances of the axioms are then derivable using the logical rules of the system. The same is true for $G_{\mathbf{SRM}_{\overset{\sim}{\to}}}$ and for the various fragments of $G_{\mathbf{J}_3^\top}$.
- The first rule for $G_{\mathbf{P_1}}$ shown in Fig. 4.4 (which is also the rule $[\Rightarrow\neg]$ of LK, introducing negation on the right-hand sides of sequents) is valid for every logic which is induced by a three-valued paraconsistent matrix, and the extra axioms of $G_{\mathbf{J}_3^\top}$ are derivable by it from the standard identity axioms. Therefore, we could have included this rule in the definition of $G_{\mathbf{J}_3^\top}$ instead of its new axioms (note that this rule is derivable from these axioms using a cut). We prefer our official formulation in Fig. 4.5, because all of its logical rules are *invertible* (see Lemma 4.88). This

Axioms: $\Gamma, \varphi \Rightarrow \Delta, \varphi \qquad \Gamma \Rightarrow \Delta, \varphi, \neg\varphi$

Rules: Exchange, Contraction, and the following logical rules:

$$\frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \neg\neg\varphi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \neg\neg\varphi}$$

$$\frac{\Gamma, \varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \otimes \psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \varphi \otimes \psi}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \psi \quad \Gamma, \neg\varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \otimes \psi) \Rightarrow \Delta} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta, \neg\psi \quad \Gamma, \psi \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \neg(\varphi \otimes \psi)}$$

$$\frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma, \neg\varphi, \psi \Rightarrow \Delta}{\Gamma, \varphi \to \psi \Rightarrow \Delta} \qquad \frac{\Gamma, \varphi \Rightarrow \Delta, \psi \quad \Gamma, \neg\psi \Rightarrow \Delta, \neg\varphi}{\Gamma \Rightarrow \Delta, \varphi \to \psi}$$

$$\frac{\Gamma, \varphi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\varphi \to \psi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \varphi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\varphi \to \psi)}$$

Fig. 4.6 The proof system $G_{\mathbf{SRM}_{\tilde{\to}}}$

is a very useful property in proof search and for other goals (as the proofs given below show).

- Actually, we could have formulated $G_{\mathbf{P_1}}$ too by using only invertible rules. This can be done by adding to it the new axioms of $G_{\mathbf{J}_3^{\top}}$, and limiting the applications of $[\Rightarrow\neg]$ to the case where the active formula is not atomic. Again, we can have only $p \Rightarrow p$, $\neg p \Rightarrow \neg p$, and $\Rightarrow p, \neg p$ as axioms in these versions of the system, where p is atomic.

Our next goal is to show the strong soundness and completeness of all these Gentzen-type systems. Our first step toward this goal is to define the semantics of sequents in the context of matrices.

Definition 4.81 Let $\mathcal{M}$ be a matrix for $\mathcal{L}$ and let $\nu \in \Lambda_{\mathcal{M}}$.

- We say that ν is an $\mathcal{M}$-*model* of a sequent $\Gamma \Rightarrow \Delta$, or that ν $\mathcal{M}$-*satisfies* $\Gamma \Rightarrow \Delta$ (notation: $\nu \models_{\mathcal{M}} \Gamma \Rightarrow \Delta$) if $\nu \not\models_{\mathcal{M}} \varphi$ for some φ in Γ, or $\nu \models_{\mathcal{M}} \psi$ for some ψ in Δ.
- We say that a sequent s $\mathcal{M}$-*follows* from a set $\mathcal{S}$ of sequents (notation: $\mathcal{S} \vdash_{\mathcal{M}} s$) if every $\mathcal{M}$-model of $\mathcal{S}$ is also an $\mathcal{M}$-model of s.
- A sequent s is $\mathcal{M}$-valid (notation: $\vdash_{\mathcal{M}} s$) if $\nu \models_{\mathcal{M}} s$ for every $\nu \in \Lambda_{\mathcal{M}}$ (i.e., if $\emptyset \vdash_{\mathcal{M}} s$).

By Definition 4.81 and Proposition 4.36, we have

Proposition 4.82 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix, and let ν be an assignment in $\mathcal{M}$. Then $\nu \models_{\mathcal{M}} \Gamma \Rightarrow \Delta$ (where $\Gamma \Rightarrow \Delta$ is a sequent in the language* *of $\mathcal{M}$) iff either $\nu(\varphi) = f$ for some $\varphi \in \Gamma$, or $\nu(\psi) \neq f$ (i.e., $\nu(\psi) \in \{t, \top\}$) for some $\psi \in \Delta$.*

Note 4.83 It is easy to see that $\vdash_{\mathcal{M}} \Gamma \Rightarrow \psi$ iff $\Gamma \vdash_{\mathcal{M}} \psi$.

Definition 4.84 Let $\mathbf{L} = \mathbf{L}_{\mathcal{M}}$ be one of the logics discussed in Sect. 4.5, and let $G_{\mathbf{L}}$ be the corresponding Gentzen-type calculus. We say that $G_{\mathbf{L}}$ is (strongly) sound and complete for $\mathbf{L}$ if for every $\mathcal{T}$ and ψ it holds that $\mathcal{T} \vdash_{G_{\mathbf{L}}} \psi$ (Definition 4.79) iff $\mathcal{T} \vdash_{\mathbf{L}} \psi$.

To show soundness and completeness of our various systems, we first need some lemmas.

Lemma 4.85 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix, and let $\Gamma \Rightarrow \Delta$ be a sequent which consists of literals (i.e., atomic formulas or negations of atomic formulas).*

1. *$\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$ iff either $\Gamma \cap \Delta \neq \emptyset$, or there is an atomic formula p such that $\{p, \neg p\} \subseteq \Delta$, or $\mathsf{f} \in \Gamma$, or $\neg\mathsf{f} \in \Delta$, or $\top \in \Delta$, or $\neg\top \in \Delta$.*
2. *If $\mathbf{L}$ is one of the logics discussed in Sect. 4.5 and $\vdash_{\mathbf{L}} \Gamma \Rightarrow \Delta$, then $\vdash_{G_{\mathbf{L}}} \Gamma \Rightarrow \Delta$.*

Proof Suppose Γ and Δ consist only of literals.

1. From Proposition 4.36, it follows that if Γ and Δ satisfies one of the six conditions, then $\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$. Suppose now that $\Gamma \Rightarrow \Delta$ does not satisfy any of them. Define
$$\nu(p) = \begin{cases} f & \text{if } p \in \Delta, \\ t & \text{if } \neg p \in \Delta, \\ \top & \text{otherwise.} \end{cases}$$
Then ν is well-defined, and $\nu \not\models_{\mathcal{M}} \Gamma \Rightarrow \Delta$. Hence $\not\vdash_{\mathcal{M}} \Gamma \Rightarrow \Delta$ in this case.
2. This follows from the first part and the fact that every sequent which satisfies the condition given in that part is obviously provable in $G_{\mathbf{L}}$ (except in the case that $\mathbf{L} = \mathbf{P_1}$ such a sequent is simply an axiom of $G_{\mathbf{L}}$. In the case of $\mathbf{P_1}$ we use the rule for introducing negation on the right). □

Lemma 4.86 *Let $\mathbf{L}$ be one of the logics discussed in Sect. 4.5, and let $\mathcal{M}$ be the three-valued paraconsistent matrix which induces $\mathbf{L}$. Then every logical rule of $G_{\mathbf{L}}$ is strongly sound for $\vdash_{\mathcal{M}}$: if $\mathcal{S}$ is the set of premises of (an application of) such a rule, and s is its conclusion, then $\mathcal{S} \vdash_{\mathcal{M}} s$.*

Proof Easy. As an example, we show the case of the rule of $G_{\mathbf{SRM}_{\tilde{\to}}}$ for introducing $\neg(\varphi \otimes \psi)$ on the right. So assume that $\nu \models_{\mathcal{A}_1} \Gamma, \varphi \Rightarrow \Delta, \neg\psi$ and $\nu \models_{\mathcal{A}_1} \Gamma, \psi \Rightarrow \Delta, \neg\varphi$. We show that $\nu \models_{\mathcal{A}_1} \Gamma \Rightarrow \Delta, \neg(\varphi \otimes \psi)$. If $\nu \models_{\mathcal{A}_1} \Gamma \Rightarrow \Delta$ we are done. Otherwise, we have that either $\nu(\varphi) = f$ or $\nu(\psi) \neq t$, and either $\nu(\psi) = f$ or $\nu(\varphi) \neq t$. This gives us four possibilities, and it is easy to check that in all of them $\nu(\varphi \otimes \psi) \neq t$, i.e., $\nu(\neg(\varphi \otimes \psi)) \neq f$. □

Lemma 4.87 *Let* $\mathbf{L}$ *be one of the logics discussed in Sect.* 4.5*. Then* $G_{\mathbf{L}}$ *is strongly sound for* $\mathbf{L}$*.*

Proof By Lemma 4.86 we need only to check that the axioms of $G_{\mathbf{L}}$ are valid in $\mathbf{L}$. This is obvious. □

Lemma 4.88 *Let* $\mathbf{L}$ *and* $\mathcal{M}$ *be like in Lemma* 4.86*, and let* r *be a logical rule of* $G_{\mathbf{L}}$*. If* r *is not the rule of* $G_{\mathbf{P_1}}$ *for introducing* $\neg$ *on the right, then* r *is strongly invertible in* $\vdash_{\mathcal{M}}$*: If* s_c *is the conclusion of* r *and* s_p *is any of its premises, then* $s_c \vdash_{\mathcal{M}} s_p$*. This is true also for every application of the exceptional rule in which the active formula is not atomic (I.e., if* φ *is not atomic then* $\Gamma \Rightarrow \Delta, \neg\varphi \vdash_{\mathsf{P}_1} \varphi, \Gamma \Rightarrow \Delta$*).*

Proof Again we do as an example the case in which r is the rule of $G_{\mathbf{SRM}_{\tilde{\rightarrow}}}$ for introducing $\neg(\varphi \otimes \psi)$ on the right. So assume that $\nu \models_{\mathcal{A}_1} \Gamma \Rightarrow \Delta, \neg(\varphi \otimes \psi)$. We show, e.g., that $\nu \models_{\mathcal{A}_1} \Gamma, \psi \Rightarrow \Delta, \neg\varphi$. If $\nu \models_{\mathcal{A}_1} \Gamma \Rightarrow \Delta$ we are done. Otherwise $\nu(\neg(\varphi \otimes \psi)) \neq f$, and so $\nu(\varphi \otimes \psi) \neq t$. This implies that either $\nu(\varphi) = f$, or $\nu(\psi) = f$, or $\nu(\varphi) = \nu(\psi) = \top$, and so either $\nu(\psi) = f$ or $\nu(\neg\varphi) \neq f$. In both cases, we have that $\nu \models_{\mathcal{A}_1} \Gamma, \psi \Rightarrow \Delta, \neg\varphi$.

As for the exceptional rule, suppose that φ is not atomic, and $\nu \models_{\mathsf{P}_1} \Gamma \Rightarrow \Delta, \neg\varphi$. We show that $\nu \models_{\mathsf{P}_1} \varphi, \Gamma \Rightarrow \Delta$. If $\nu \models_{\mathsf{P}_1} \Gamma \Rightarrow \Delta$ we are done. Otherwise $\nu(\neg\varphi) \neq f$, and so $\nu(\varphi) \in \{f, \top\}$. Since φ is not atomic, $\nu(\varphi) \neq \top$. It follows that $\nu(\varphi) = f$, and so $\nu \models_{\mathsf{P}_1} \varphi, \Gamma \Rightarrow \Delta$. □

Lemma 4.89 *Let* $\mathbf{L}$ *and* $\mathcal{M}$ *be like in Lemma* 4.86*, and let* s *be a sequent in the language of* $\mathbf{L}$*. If* $\vdash_{\mathcal{M}} s$ *then* s *has a cut-free proof in* $G_{\mathbf{L}}$*.*

Proof It is easy to check that by applying the logical rules of $G_{\mathbf{L}}$ backward, and using Lemma 4.88, we can construct for every sequent s a finite set $\mathcal{S}(s)$ with the following properties:

1. Each element of $\mathcal{S}(s)$ is a sequent which consists only of literals.
2. $s \vdash_{\mathcal{M}} s'$ for every element s' of $\mathcal{S}(s)$.
3. There is a cut-free proof of s from $\mathcal{S}(s)$.

Suppose now that $\vdash_{\mathcal{M}} s$. By Lemma 4.88 and the second property of $\mathcal{S}(s)$ this implies that $\vdash_{\mathcal{M}} s'$ for every element s' of $\mathcal{S}(s)$. By Lemma 4.85 and the first and third properties of $\mathcal{S}(s)$ it follows that s has a cut-free proof in $G_{\mathbf{L}}$. □

Now we are ready to prove the two main results of this section.

Theorem 4.90 *Let* $\mathbf{L}$ *be one of the logics discussed in Sect.* 4.5*.*

1. $G_{\mathbf{L}}$ *is sound and complete for* $\mathbf{L}$*.*
2. $\mathcal{T} \vdash_{G_{\mathbf{L}}} \psi$ *iff* $\mathcal{T} \vdash_{\mathbf{L}} \psi$*.*

Proof The first part is immediate from Lemmas 4.87 and 4.89; The second part follows from the first part and the fact that by Proposition 4.11, $\mathbf{L}$ is finitary. □

Inference Rule: [MP] $\dfrac{\psi \quad \psi \supset \varphi}{\varphi}$

Axioms:

$[\supset 1]$	$\psi \supset \varphi \supset \psi$
$[\supset 2]$	$(\psi \supset \varphi \supset \tau) \supset (\psi \supset \varphi) \supset (\psi \supset \tau)$
$[\supset 3]$	$((\psi \supset \varphi) \supset \psi) \supset \psi$
$[\wedge \supset]$	$\psi \wedge \varphi \supset \psi, \quad \psi \wedge \varphi \supset \varphi$
$[\supset \wedge]$	$\psi \supset \varphi \supset \psi \wedge \varphi$
$[\supset \vee]$	$\psi \supset \psi \vee \varphi, \quad \varphi \supset \psi \vee \varphi$
$[\vee \supset]$	$(\psi \supset \tau) \supset (\varphi \supset \tau) \supset (\psi \vee \varphi \supset \tau)$

Fig. 4.7 The proof system HCL^+

Theorem 4.91 *Let* **L** *be one of the logics discussed in Sect. 4.5. Then* $G_{\mathbf{L}}$ *admits cut-elimination (i.e., every sequent that is provable in* $G_{\mathbf{L}}$ *has a proof in which the cut rule is not used).*

Proof Suppose that $\vdash_{G_{\mathbf{L}}} s$. By Lemma 4.87, $\vdash_{\mathbf{L}} s$. By Lemma 4.89 this implies that s has a cut-free proof in $G_{\mathbf{L}}$. □

4.6.2 *Hilbert-Type Systems*

To complete the picture, in this final subsection we present Hilbert-type proof systems with MP for $\supset$ as the sole rule of inference for all the logics studied in Sect. 4.5 in which $\supset$ is a primitive connective.[11] Again, these systems are based on some sound and complete proof system of the same type for positive classical logic ($\mathbf{CL}^+$). Such a system, denoted HCL^+, is presented in Fig. 4.7.[12]

Definition 4.92 Figures 4.8 and 4.9 contain Hilbert-type proof systems for the logic $\mathbf{P_1}$ and the logics **PAC** and $\mathbf{J_3}$, respectively. Hilbert-type proof systems $H_{\mathbf{PAC}^\top}$ and $H_{\mathbf{J}_3^\top}$ for the logics $\mathbf{PAC}^\top$ and $\mathbf{J}_3^\top$ (respectively) are obtained by adding to $H_{\mathbf{PAC}}$ and $H_{\mathbf{J_3}}$ (respectively) the axioms $\top$ and $\neg\top$. A Hilbert-type proof system $H_{\mathbf{PAC}_{\supset}}$ for $\mathbf{PAC}_{\supset}$ is obtained from $H_{\mathbf{PAC}}$ by replacing [t] with either $(\neg\varphi \supset \varphi) \supset \varphi$ or $(\psi \supset \varphi) \supset (\neg\psi \supset \varphi) \supset \varphi$, changing $[\Rightarrow\neg\supset]$ to $\varphi \supset (\neg\psi \supset \neg(\varphi \supset \psi))$, and deleting all axioms that mention $\wedge$ or $\vee$.

[11] Note that by Proposition 4.66, the four $\leq_k$-monotonic expansions of **LP** (including **LP** itself) have no implication, and so they cannot have a corresponding Hilbert-type system of the above type. In contrast, by Proposition 4.57 $\mathbf{SRM}_{\tilde{\to}}$ can be defined using such a system, but the resulting system does not look very natural. A natural Hilbert-type system for $\mathbf{SRM}_{\tilde{\to}}$ in its primitive language (but with two inference rules) can be found in [9].

[12] As usual, in the formulation of the axioms of the systems the association of nested implications is taken to the right.

Inference Rule: [MP] $\dfrac{\psi \quad \psi \supset \varphi}{\varphi}$

Axioms: The axioms of HCL^+ and:

[t]	$\neg\psi \vee \psi$
$[\neg\supset\Rightarrow]$	$(\varphi \supset \psi) \supset \neg(\varphi \supset \psi) \supset \tau$
$[\neg\vee\Rightarrow]$	$(\varphi \vee \psi) \supset \neg(\varphi \vee \psi) \supset \tau$
$[\neg\wedge\Rightarrow]$	$(\varphi \wedge \psi) \supset \neg(\varphi \wedge \psi) \supset \tau$
$[\neg\Rightarrow]$	$\neg\varphi \supset \neg\neg\varphi \supset \tau$

Fig. 4.8 The proof system $H_{\mathbf{P_1}}$

Inference Rule: [MP] $\dfrac{\psi \quad \psi \supset \varphi}{\varphi}$

Axioms of $H_{\mathbf{PAC}}$: The axioms of HCL^+ and:

[t]	$\neg\psi \vee \psi$
$[\neg\neg\Rightarrow]$	$\neg\neg\varphi \supset \varphi$
$[\Rightarrow\neg\neg]$	$\varphi \supset \neg\neg\varphi$
$[\neg\supset\Rightarrow 1]$	$\neg(\varphi \supset \psi) \supset \varphi$
$[\neg\supset\Rightarrow 2]$	$\neg(\varphi \supset \psi) \supset \neg\psi$
$[\Rightarrow\neg\supset]$	$(\varphi \wedge \neg\psi) \supset \neg(\varphi \supset \psi)$
$[\neg\vee\Rightarrow 1]$	$\neg(\varphi \vee \psi) \supset \neg\varphi$
$[\neg\vee\Rightarrow 2]$	$\neg(\varphi \vee \psi) \supset \neg\psi$
$[\Rightarrow\neg\vee]$	$(\neg\varphi \wedge \neg\psi) \supset \neg(\varphi \vee \psi)$
$[\neg\wedge\Rightarrow]$	$\neg(\varphi \wedge \psi) \supset (\neg\varphi \vee \neg\psi)$
$[\Rightarrow\neg\wedge 1]$	$\neg\varphi \supset \neg(\varphi \wedge \psi)$
$[\Rightarrow\neg\wedge 2]$	$\neg\psi \supset \neg(\varphi \wedge \psi)$

Axioms of $H_{\mathbf{J_3}}$: The axioms of $H_{\mathbf{PAC}}$ and:

$[\mathsf{f}\supset]$	$\mathsf{f} \supset \psi$
$[\supset\mathsf{f}]$	$\psi \supset \neg\mathsf{f}$

Fig. 4.9 The proof systems $H_{\mathbf{PAC}}$ and $H_{\mathbf{J_3}}$

Theorem 4.93 *Let* $\mathbf{L} \in \{\mathbf{P_1}, \mathbf{PAC}_{\bar{\supset}}, \mathbf{PAC}, \mathbf{PAC}^{\top}, \mathbf{J_3}, \mathbf{J_3^{\top}}\}$. *Then* $\vdash_{H_{\mathbf{L}}} = \vdash_{G_{\mathbf{L}}}$.

Proof Using cuts and the fact that $\vdash_{LK^+} \psi, \psi \supset \varphi \Rightarrow \varphi$, it is easy to show by induction on length of proofs in $H_{\mathbf{L}}$ that if $\Gamma \vdash_{H_{\mathbf{L}}} \varphi$ (where Γ is finite) then $\Gamma \vdash_{G_{\mathbf{L}}} \varphi$. All one needs to do is to show that $\vdash_{G_{\mathbf{L}}} \varphi$ for every axiom φ of $H_{\mathbf{L}}$, and this is a straightforward exercise. It immediately follows that $\vdash_{H_{\mathbf{L}}} \subseteq \vdash_{G_{\mathbf{L}}}$.

For the converse, it would be more convenient to use the versions of the Gentzen-type systems which employ lists of formulas rather than finite sets,[13] and to treat each of the six logics separately.

$\mathbf{L} = \mathbf{PAC}$. In this case, it is easy to prove (either syntactically, using the cut-elimination theorem for $G_{\mathbf{PAC}}$, or semantically, using the soundness theorem for it)

[13] In such a case we need also the structural rules of Permutation, Contraction, and Expansion that assure that the underlying consequence relation remains the same.

that a sequent $s = \varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m$ is provable in $G_{\mathbf{PAC}}$ only if $m > 0$. For each such sequent s we define a translation $Tr_{\mathbf{L}}(s)$ by $Tr_{\mathbf{L}}(s) = \varphi_1 \wedge \cdots \wedge \varphi_n \supset \psi_1 \vee \cdots \vee \psi_m$ (in particular: $Tr_{\mathbf{L}}(\Rightarrow \psi_1, \ldots, \psi_m) = \psi_1 \vee \cdots \vee \psi_m$). Obviously, to show that $\vdash_{G_{\mathbf{L}}} \subseteq \vdash_{H_{\mathbf{L}}}$ it suffices to prove that if $\vdash_{G_{\mathbf{L}}} s$ then $\vdash_{H_{\mathbf{L}}} Tr_{\mathbf{L}}(s)$. We prove this claim by induction on length of proofs in $G_{\mathbf{L}}$. This is a routine (though tedious) induction, and here we shall do as examples three of the various possible cases that should be considered.

- Suppose s is an axiom of the form $\Rightarrow \neg\varphi, \varphi$. Then $Tr_{\mathbf{L}}(s)$ is an instance of the axiom [**t**] of **L** (= **PAC**).
- Suppose s is inferred from s_1 and s_2 using $[\supset\Rightarrow]$. Then there are formulas φ, ψ, τ_2, and (perhaps) τ_1 such that $Tr_{\mathbf{L}}(s) = \tau_1 \wedge (\varphi \supset \psi) \supset \tau_2$, $Tr_{\mathbf{L}}(s_1) = \tau_1 \supset \tau_2 \vee \varphi$, and $Tr_{\mathbf{L}}(s_2) = \tau_1 \wedge \psi \supset \tau_2$ (the case where $Tr_{\mathbf{L}}(s) = (\varphi \supset \psi) \supset \tau_2$, $Tr_{\mathbf{L}}(s_1) = \tau_2 \vee \varphi$, and $Tr_{\mathbf{L}}(s_2) = \psi \supset \tau_2$ is similar, but easier). By induction hypothesis, $\vdash_{H_{\mathbf{L}}} Tr_{\mathbf{L}}(s_1)$ and $\vdash_{H_{\mathbf{L}}} Tr_{\mathbf{L}}(s_2)$. Now

  $$P_1 \supset P_2 \vee P_3,\ P_1 \wedge P_4 \supset P_2 \vdash_{\mathbf{CL}^+} P_1 \wedge (P_3 \supset P_4) \supset P_2.$$

  Since HCL^+ is complete for $\mathbf{CL}^+$ and $H_{\mathbf{L}}$ is an extension of HCL^+, it follows (by substituting τ_1 for P_1, τ_2 for P_2, φ for P_3, and ψ for P_4) that $Tr_{\mathbf{L}}(s_1), Tr_{\mathbf{L}}(s_2) \vdash_{H_{\mathbf{L}}} Tr_{\mathbf{L}}(s)$. Hence $\vdash_{H_{\mathbf{L}}} Tr_{\mathbf{L}}(s)$.
- Suppose s is inferred from s_1 using $[\neg\supset\Rightarrow]$. Then there are formulas φ, ψ, τ_2, and (perhaps) τ_1 such that $Tr_{\mathbf{L}}(s) = \tau_1 \wedge \neg(\varphi \supset \psi) \supset \tau_2$, while $Tr_{\mathbf{L}}(s_1) = \tau_1 \wedge \varphi \wedge \neg\psi \supset \tau_2$ (again the case where there is no τ_1 is easier). By induction hypothesis, $\vdash_{H_{\mathbf{L}}} Tr_{\mathbf{L}}(s_1)$. Now

  $$P_5 \supset P_3,\ P_5 \supset P_4,\ P_1 \wedge P_3 \wedge P_4 \supset P_2 \vdash_{\mathbf{CL}^+} P_1 \wedge P_5 \supset P_2.$$

  Since HCL^+ is complete for $\mathbf{CL}^+$ and $H_{\mathbf{L}}$ is an extension of HCL^+, it follows (by substituting τ_1 for P_1, τ_2 for P_2, φ for P_3, $\neg\psi$ for P_4, and $\neg(\varphi \supset \psi)$ for P_5) that

  $$\neg(\varphi \supset \psi) \supset \varphi,\ \neg(\varphi \supset \psi) \supset \neg\psi,\ Tr_{\mathbf{L}}(s_1) \vdash_{H_{\mathbf{L}}} Tr_{\mathbf{L}}(s).$$

  Using the axioms $[\neg\supset\Rightarrow 1]$ and $[\neg\supset\Rightarrow 2]$ of $H_{\mathbf{L}}$, it follows from the induction hypothesis for s_1 that $\vdash_{H_{\mathbf{L}}} Tr_{\mathbf{L}}(s)$.

The proofs in the other cases are similar. One should only note that in some of the cases (e.g., when s is inferred from s_1 using weakening on the right) there are four subcases to consider (rather than just two as in the cases handled above): that we have both τ_1 and τ_2; that we have τ_1 but not τ_2; that we have τ_2 but not τ_1; and that we have neither τ_1 nor τ_2.

L $=$ **PAC**$^{\top}$. The proof in this case is very similar to that in the previous one, and is left to the reader.

L $=$ **J**$_3$. The proof in this case is again similar to that in case of **PAC**. The main difference is that now also sequents of the form $\Gamma \Rightarrow$ may be proved in $G_{\mathbf{L}}$, and so the translation of sequents into formulas should be extended to these type of sequents. This is done by letting $Tr_{\mathbf{L}}(\varphi_1, \ldots, \varphi_n \Rightarrow)$ be $\varphi_1 \wedge \cdots \wedge \varphi_n \supset \mathsf{f}$. Details are left to the reader.

$\mathbf{L} = \mathbf{J_3^{\top}}$. The proof in this case is very similar to that in the case of $\mathbf{J_3}$, and is left to the reader.

$\mathbf{L} = \mathbf{P_1}$. The proof in this case is similar to the case $\mathbf{L} = \mathbf{J_3}$, but instead of f we use $\neg P_1 \wedge \neg\neg P_1$ (say).

$\mathbf{L} = \mathbf{PAC}_{\underset{\supset}{\neg}}$. This time there is another problem: $\wedge$ and $\vee$ are not included in the language of $\mathbf{PAC}_{\underset{\supset}{\neg}}$, and so we cannot employ the translation function that was used in the case of **PAC**. However, we can use the facts that $\varphi \vee \psi$ is equivalent in $\mathbf{CL}^+$ to $(\varphi \supset \psi) \supset \psi$ and $\varphi \wedge \psi \supset \tau$ is equivalent in $\mathbf{CL}^+$ to $\varphi \supset \psi \supset \tau$. With the help of this fact we can transform the definition of $Tr_{\mathbf{PAC}}$ into an equivalent (in $\mathbf{CL}^+$) definition in which $\wedge$ and $\vee$ are not used:

$$
\begin{aligned}
& Tr_{\mathbf{PAC}_{\underset{\supset}{\neg}}}(\varphi_1, \ldots, \varphi_n \Rightarrow \psi_1, \ldots, \psi_m) \\
& = \varphi_1 \supset \ldots \varphi_n \supset (\ldots((\psi_1 \supset \psi_2) \supset \psi_2) \supset \cdots \supset \psi_m) \supset \psi_m
\end{aligned}
$$

With this definition, and using instead of HCL^+ the Hilbert-type system consisting only of [MP], [$\supset$1], [$\supset$2], and [$\supset$3] (this proof system is sound and complete with respect to the $\{\supset\}$-fragment of classical logic), one can proceed in a way which is very similar to that used in the case $\mathbf{L} = \mathbf{PAC}$. □

Theorem 4.94 *For every logic* $\mathbf{L} \in \{\mathbf{P_1}, \mathbf{PAC}_{\underset{\supset}{\neg}}, \mathbf{PAC}, \mathbf{PAC}^{\top}, \mathbf{J_3}, \mathbf{J_3^{\top}}\}$, *the proof system* $H_{\mathbf{L}}$ *is strongly sound and complete for* $\mathbf{L}$, *i.e.,* $\mathcal{T} \vdash_{H_{\mathbf{L}}} \psi$ *iff* $\mathcal{T} \vdash_{\mathbf{L}} \psi$ *for each such* $\mathbf{L}$.

Proof This is a direct corollary of Theorems 4.93 and 4.90. □

References

1. Anderson, A., Belnap, N.: Entailment, vol. 1. Princeton University Press (1975)
2. Arieli, O., Avron, A.: The value of the four values. Artif. Intell. **102**(1), 97–141 (1998)
3. Arieli, O., Avron, A., Zamansky, A.: Ideal paraconsistent logics. Stud. Logica. **99**(1–3), 31–60 (2011)
4. Arieli, O., Avron, A., Zamansky, A.: Maximal and premaximal paraconsistency in the framework of three-valued semantics. Stud. Logica. **97**(1), 31–60 (2011)
5. Asenjo, F.G.: A calculus of antinomies. Notre Dame J. Formal Logic **7**, 103–106 (1966)
6. Avron, A.: On an implication connective of RM. Notre Dame J. Formal Logic **27**, 201–209 (1986)
7. Avron, A.: Natural 3-valued logics: characterization and proof theory. J. Symbolic Logic **56**(1), 276–294 (1991)
8. Avron, A.: On the expressive power of three-valued and four-valued languages. J. Logic Comput. **9**(6), 977–994 (1999)
9. Avron, A.: Combining classical logic, paraconsistency and relevance. J. Appl. Logic **3**, 133–160 (2005)

10. Avron, A., Lev, I.: Non-deterministic multi-valued structures. J. Logic Comput. **15**, 241–261 (2005)
11. Avron, A., Ben-Naim, J., Konikowska, B.: Cut-free ordinary sequent calculi for logics having generalized finite-valued semantics. Log. Univers. **1**, 41–69 (2006)
12. Avron, A., Konikowska, B., Zamansky, A.: Cut-free sequent calculi for C-systems with generalized finite-valued semantics. J. Logic Comput. **23**, 517–540 (2013)
13. Avron, A., Zamansky, A.: Non-deterministic semantics for logical systems—a survey. In: Gabbay, D., Guenther, F. (eds) Handbook of Philosophical Logic, vol. 16, pp. 227–304. Springer (2011)
14. Batens, D.: Paraconsistent extensional propositional logics. Logique et Analyse **90–91**, 195–234 (1980)
15. Belnap, N.: How a computer should think. In: Ryle, G. (ed.) Contemporary Aspects of Philosophy, pp. 30–56. Oriel Press (1977)
16. Carnielli, W., Coniglio, M., Marcos, J.: Logics of formal inconsistency. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, 2nd edn., pp, vol. 14, pp. 1–93. Springer (2007)
17. Ciucci, D., Dubois, D.: A modal theorem-preserving translation of a class of three-valued logics of incomplete information. J. Appl. Non-Class. Logics **23**(4), 321–352 (2013)
18. Ciucci, D., Dubois, D.: From possibility theory to paraconsistency (2015). A chapter in this book
19. da Costa, N.: On the theory of inconsistent formal systems. Notre Dame J. Formal Logic **15**, 497–510 (1974)
20. D'Ottaviano, I.: The completeness and compactness of a three-valued first-order logic. Revista Colombiana de Matematicas **XIX**(1–2), 31–42 (1985)
21. D'Ottaviano, I., da Costa, N.C.: Sur un problèm de Jakowśki. C. R. Acad Sc. Paris **270**(Sèrie A), 1349–1353 (1970)
22. Epstein, R.L.: The Semantic Foundation of Logic, vol. I, Propositional Logics. Kluwer (1990)
23. Fitting, M.: Kleene's three valued logics and their children. Fundamenta Informaticae **20**(1–3), 113–131 (1994)
24. Gentzen, G.: Investigations into Logical Deduction (1934). In: Szabo, M.E. (ed.) In German. An English translation appears in 'The Collected Works of Gerhard Gentzen'. North-Holland (1969)
25. Ginsberg, M.: Multi-valued logics: a uniform approach to reasoning in AI. Comput. Intell. **4**, 256–316 (1988)
26. Gottwald, S.: A treatise on many-valued logics. In: Studies in Logic and Computation, vol. 9. Research Studies Press, Baldock (2001)
27. Kleene, S.C.: Introduction to Metamathematics. Van Nostrand (1950)
28. Łukasiewicz, J.: On 3-valued logic. Ruch Filosoficzny **5**, 169–171 (1920). English translation: McCall, S. (ed.) Polish Logic 1920–1939, pp. 15–18. Oxford University Press, Oxford (1967)
29. Łukasiewicz, J.: Selected Works. North Holland (1970). L. Borkowski (ed)
30. Malinowski, G.: Many-Valued Logics. Clarendon Press (1993)
31. Priest, G.: Logic of paradox. J. Philos. Logic **8**, 219–241 (1979)
32. Priest, G.: Reasoning about truth. Artif. Intell. **39**, 231–244 (1989)
33. Priest, G.: Minimally inconsistent LP. Stud. Logica. **50**, 321–331 (1991)
34. Rozoner, L.I.: On interpretation of inconsistent theories. Inf. Sci. **47**, 243–266 (1989)
35. Sette, A.M.: On propositional calculus P_1. Mathematica Japonica **16**, 173–180 (1973)
36. Shoesmith, D.J., Smiley, T.J.: Deducibility and many-valuedness. J. Symbolic Logic **36**, 610–622 (1971)
37. Shoesmith, D.J., Smiley, T.J.: Multiple Conclusion Logic. Cambridge University Press (1978)
38. Sobociński, B.: Axiomatization of a partial system of three-value calculus of propositions. J. Comput. Syst. **1**, 23–55 (1952)
39. Urquhart, A.: Many-valued logic. In: Gabbay, D., Guenthner, F. (eds.) Handbook of Philosophical Logic, 2nd edn., vol. II, pp. 249–295. Kluwer (2001)