

Nonmonotonic and Paraconsistent Reasoning: From Basic Entailments to Plausible Relations

Ofer Arieli and Arnon Avron

Department of Computer Science, School of Mathematical Sciences,
Tel-Aviv University, Tel-Aviv 69978, Israel
{ofer,aa}@math.tau.ac.il

Abstract. In this paper we develop frameworks for logical systems which are able to reflect not only nonmonotonic patterns of reasoning, but also paraconsistent reasoning. For this we consider a sequence of generalizations of the pioneering works of Gabbay, Kraus, Lehmann, Magidor and Makinson. Our sequence of frameworks culminates in what we call plausible, nonmonotonic, multiple-conclusion consequence relations (which are based on a given monotonic one). Our study yields intuitive justifications for conditions that have been proposed in previous frameworks, and also clarifies the connections among some of these systems. In addition, we present a general method for constructing plausible nonmonotonic relations. This method is based on a multiple-valued semantics, and on Shoham's idea of preferential models.

1 Introduction

Our main goal in this paper is to get a better understanding of the conditions that a useful relation for nonmonotonic and paraconsistent [5] reasoning should satisfy. For this we consider a sequence of generalizations the pioneering works of Gabbay [7], Kraus, Lehmann, Magidor [8] and Makinson [12]. These generalizations are based on the following ideas:

- Each nonmonotonic logical system is based on some underlying monotonic one.
- The underlying monotonic logic should not necessarily be classical logic, but should be chosen according to the intended application. If, for example, inconsistent data is not to be totally rejected, then an underlying paraconsistent logic might be a better choice than classical logic.
- The more significant logical properties of the main connectives of the underlying monotonic logic, especially conjunction and disjunction (which have crucial roles in monotonic consequence relations), should be preserved as far as possible.
- On the other hand, the conditions that define a certain class of nonmonotonic systems should not assume anything concerning the language of the system (in particular, the existence of appropriate conjunction or disjunction should *not* be assumed).

Our sequence of generalizations culminates in what we call (following Lehmann [9]) cautious plausible consequence relations (which are based on a given monotonic one.¹)

Our study yields intuitive justifications for conditions that have been proposed in previous frameworks and also clarifies the connections among some of these systems. Moreover, while the logic behind most of the systems which were proposed so far is supraclassical (i.e., every first-degree inference rule that is classically sound remains valid in the resulting logics), the consequence relations considered here are also capable of drawing conclusions from incomplete and inconsistent theories in a nontrivial way.

In the last part of this paper we present a general method for constructing such plausible nonmonotonic and paraconsistent relations. This method is based on a multiple-valued semantics, and on Shoham's idea of preferential models [21].²

2 General Background

We first briefly review the original treatments of [8] and [12]. The language they use is based on the standard propositional one. Here, \rightarrow denotes the material implication and \sim denotes the corresponding equivalence operator. The classical propositional language, with the connectives \neg , \vee , \wedge , \rightarrow , \sim , and with a propositional constant t , is denoted here by Σ_{cl} .

Definition 1. [8] Let \vdash_{cl} be the classical consequence relation. A binary relation³ \sim' between formulae in Σ_{cl} is called *cumulative* if it is closed under the following inference rules:

<i>reflexivity:</i>	$\psi \sim' \psi$.
<i>cautious monotonicity:</i>	if $\psi \sim' \phi$ and $\psi \sim' \tau$, then $\psi \wedge \phi \sim' \tau$.
<i>cautious cut:</i>	if $\psi \sim' \phi$ and $\psi \wedge \phi \sim' \tau$, then $\psi \sim' \tau$.
<i>left logical equivalence:</i>	if $\vdash_{cl} \psi \sim \phi$ and $\psi \sim' \tau$, then $\phi \sim' \tau$.
<i>right weakening:</i>	if $\vdash_{cl} \psi \rightarrow \phi$ and $\tau \sim' \psi$, then $\tau \sim' \phi$.

Definition 2. [8] A cumulative relation \sim' is called *preferential* if it is closed under the following rule:

left \vee -introduction (Or) if $\psi \sim' \tau$ and $\phi \sim' \tau$, then $\psi \vee \phi \sim' \tau$.

The conditions above might look a little-bit ad-hoc. For example, one might ask why \rightarrow is used on the right, while the stronger \sim is used on the left. A

¹ See [4] for another formalism with non-classical monotonic consequence relations as the basis for nonmonotonic consequence relations.

² Due to a lack of space, proofs of propositions are omitted; They will be given in the full version of the paper.

³ A “conditional assertion” in terms of [8].

discussion and some justification appears in [8, 11].⁴ A stronger intuitive justification will be given below, using more general frameworks.

In what follows we consider several generalizations of the basic relations presented above:

1. Allowing the use of nonclassical logics (for example: paraconsistent logics) as the basis, instead of classical logic.
2. Allowing the use of a set of premises rather than a single one.
3. Allowing the use of multiple conclusions relations rather than single conclusion ones.

The key logical concepts which stand behind these generalizations are the following:

Definition 3.

a) An ordinary *Tarskian* consequence relation (*tcr*, for short) [22] is a binary relation \vdash between sets of formulae and formulae that satisfies the following conditions:⁵

- s-TR** *strong T-reflexivity*: $\Gamma \vdash \psi$ for every $\psi \in \Gamma$.
TM *T-monotonicity*: if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \psi$.
TC *T-cut*: if $\Gamma_1 \vdash \psi$ and $\Gamma_2, \psi \vdash \phi$ then $\Gamma_1, \Gamma_2 \vdash \phi$.

b) An ordinary *Scott* consequence relation (*scr*, for short) [19, 20] is a binary relation \vdash between sets of formulae that satisfies the conditions below:

- s-R** *strong reflexivity*: if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \vdash \Delta$.
M *monotonicity*: if $\Gamma \vdash \Delta$ and $\Gamma \subseteq \Gamma'$, $\Delta \subseteq \Delta'$, then $\Gamma' \vdash \Delta'$.
C *cut*: if $\Gamma_1 \vdash \psi, \Delta_1$ and $\Gamma_2, \psi \vdash \Delta_2$ then $\Gamma_1, \Gamma_2 \vdash \Delta_1, \Delta_2$.

Definition 4.

a) Let \sim be a relation between sets of formulae.

- A connective \wedge is called *internal conjunction* (w.r.t. \sim) if:

$$[\wedge \sim]_I \quad \frac{\Gamma, \psi, \phi \sim \Delta}{\Gamma, \psi \wedge \phi \sim \Delta} \qquad [\wedge \sim]_E \quad \frac{\Gamma, \psi \wedge \phi \sim \Delta}{\Gamma, \psi, \phi \sim \Delta}$$

- A connective \wedge is called *combining conjunction* (w.r.t. \sim) if:

$$[\sim \wedge]_I \quad \frac{\Gamma \sim \psi, \Delta \quad \Gamma \sim \phi, \Delta}{\Gamma \sim \psi \wedge \phi, \Delta} \qquad [\sim \wedge]_E \quad \frac{\Gamma \sim \psi \wedge \phi, \Delta}{\Gamma \sim \psi, \Delta} \quad \frac{\Gamma \sim \psi \wedge \phi, \Delta}{\Gamma \sim \phi, \Delta}$$

- A connective \vee is called *internal disjunction* (w.r.t. \sim) if:

$$[\sim \vee]_I \quad \frac{\Gamma \sim \psi, \phi, \Delta}{\Gamma \sim \psi \vee \phi, \Delta} \qquad [\sim \vee]_E \quad \frac{\Gamma \sim \psi \vee \phi, \Delta}{\Gamma \sim \psi, \phi, \Delta}$$

⁴ Systems that satisfy the conditions of Definitions 1, 2, as well as other related systems, are also considered in [6, 13, 18, 10].

⁵ The prefix “T” reflects the fact that these are Tarskian rules.

- A connective \vee is called *combining disjunction* (w.r.t. \sim) if:

$$[\vee \sim]_I \quad \frac{\Gamma, \psi \sim \Delta \quad \Gamma, \phi \sim \Delta}{\Gamma, \psi \vee \phi \sim \Delta} \quad [\vee \sim]_E \quad \frac{\Gamma, \psi \vee \phi \sim \Delta}{\Gamma, \psi \sim \Delta} \quad \frac{\Gamma, \psi \vee \phi \sim \Delta}{\Gamma, \phi \sim \Delta}$$

b) Let \sim be a relation between sets of formulae and formulae. The notions of combining conjunction, internal conjunction, and combining disjunction are defined for \sim exactly like in case (a).

Note: If \vdash is an scr (tcr) then \wedge is an internal conjunction for \vdash iff it is a combining conjunction for \vdash . The same is true for \vee in case \vdash is an scr. This, however, is not true in general.

3 Tarskian Cautious Consequence Relations

Definition 5. A Tarskian *cautious* consequence relation (*tccr*, for short) is a binary relation \sim between sets of formulae and formulae in a language Σ that satisfies the following conditions:⁶

- s-TR** *strong T-reflexivity:* $\Gamma \sim \psi$ for every $\psi \in \Gamma$.
- TCM** *T-cautious monotonicity:* if $\Gamma \sim \psi$ and $\Gamma \sim \phi$, then $\Gamma, \psi \sim \phi$.
- TCC** *T-cautious cut:* if $\Gamma \sim \psi$ and $\Gamma, \psi \sim \phi$, then $\Gamma \sim \phi$.

Proposition 6. Any tccr \sim is closed under the following rules for every n :

- TCM^[n]** if $\Gamma \sim \psi_i$ ($i=1, \dots, n$) then $\Gamma, \psi_1, \dots, \psi_{n-1} \sim \psi_n$.
- TCC^[n]** if $\Gamma \sim \psi_i$ ($i=1, \dots, n$) and $\Gamma, \psi_1, \dots, \psi_n \sim \phi$, then $\Gamma \sim \phi$.

We now generalize the notion of a cumulative entailment relation. We first do it for Tarskian consequence relations \vdash that have an internal conjunction \wedge .

Definition 7. A tccr \sim is called $\{\wedge, \vdash\}$ -*cumulative* if it satisfies the following conditions:

- if $\psi \vdash \phi$ and $\phi \vdash \psi$ and $\psi \sim \tau$, then $\phi \sim \tau$. (*weak left logical equivalence*)
- if $\psi \vdash \phi$ and $\tau \sim \psi$, then $\tau \sim \phi$. (*weak right weakening*)
- \wedge is also an internal conjunction w.r.t. \sim .

If, in addition, \vdash has a combining disjunction \vee , then \sim is called $\{\vee, \wedge, \vdash\}$ -*preferential* if it also satisfies the single-conclusion version of $[\vee \sim]_I$.

Proposition 8. Suppose \sim is \vdash_{cl} -cumulative [\vdash_{cl} -preferential]. Let $\psi \sim' \phi$ iff $\psi \sim \phi$. Then w.r.t. Σ_{cl} , \sim' is cumulative [preferential] in the sense of [8]. Conversely: if \sim' is cumulative [preferential] in the sense of [8] and we define $\psi_1, \dots, \psi_n \sim \phi$ iff $\psi_1 \wedge \dots \wedge \psi_n \sim' \phi$, then \sim is \vdash_{cl} -cumulative [\vdash_{cl} -preferential].

⁶ This set of conditions was first proposed in [7].

We next generalize the definition of a cumulative tccr to make it independent of the existence of an internal conjunction.

Proposition 9. Let \vdash be a tcr, and let \sim be a tccr in the same language. The following connections between \vdash and \sim are equivalent:

- TCum** *T-cumulativity* for every $\Gamma \neq \emptyset$, if $\Gamma \vdash \psi$ then $\Gamma \sim \psi$.
- TLLE** *T-left logical equiv.* if $\Gamma, \psi \vdash \phi$ and $\Gamma, \phi \vdash \psi$ and $\Gamma, \psi \sim \tau$, then $\Gamma, \phi \sim \tau$.
- TRW** *T-right weakening* if $\Gamma, \psi \vdash \phi$ and $\Gamma \sim \psi$, then $\Gamma \sim \phi$.
- TMiC** *T-mixed cut:* for every $\Gamma \neq \emptyset$, if $\Gamma \vdash \psi$ and $\Gamma, \psi \sim \phi$, then $\Gamma \sim \phi$.

Definition 10. Let \vdash be a tcr. A tccr \sim in the same language is called \vdash -*cumulative* if it satisfies any of the conditions of Proposition 9. If \vdash has a combining disjunction \vee , and \sim satisfies $[\vee \sim]_I$, then \sim is called $\{\vee, \vdash\}$ -*preferential*.

Note: Since $\Gamma \vdash \psi$ for every $\psi \in \Gamma$, TCum implies s-TR, and so a binary relation that satisfies TCum, TCM, and TCC is a \vdash -cumulative tccr.

Proposition 11. Suppose that \vdash is a tcr with an internal conjunction \wedge . A tccr \sim is a $\{\wedge, \vdash\}$ -cumulative iff it is \vdash -cumulative. If \vdash has also a combining disjunction \vee , then \sim is $\{\vee, \wedge, \vdash\}$ -preferential iff it is $\{\vee, \vdash\}$ -preferential.

Proposition 12. Let \sim be a \vdash -cumulative relation, and let \wedge be an internal conjunction w.r.t. \vdash . Then \wedge is both an internal conjunction and a combining conjunction w.r.t. \sim .

4 Scott Cautious Consequence Relations

Definition 13. A Scott *cautious* consequence relation (*sccr*, for short) is a binary relation \sim between nonempty⁷ sets of formulae that satisfies the following conditions:

- s-R** *strong reflexivity:* if $\Gamma \cap \Delta \neq \emptyset$ then $\Gamma \sim \Delta$.
- CM** *cautious monotonicity:* if $\Gamma \sim \psi$ and $\Gamma \sim \Delta$ then $\Gamma, \psi \sim \Delta$.
- CC^[1]** *cautious 1-cut:* if $\Gamma \sim \psi$ and $\Gamma, \psi \sim \Delta$ then $\Gamma \sim \Delta$.

A natural requirement from a Scott cumulative consequence relation is that its single-conclusion counterpart will be a Tarskian cumulative consequence relation. Such a relation should also use disjunction on the r.h.s. like it uses conjunction on the l.h.s. The following definition formalizes these requirements.

Definition 14. Let \vdash be an scr with an internal disjunction \vee . A relation \sim between nonempty finite sets of formulae is called $\{\vee, \vdash\}$ -*cumulative sccr* if it

⁷ The condition of non-emptiness is just technically convenient here. It is possible to remove it with the expense of complicating somewhat the definitions and propositions. It is preferable instead to employ (whenever necessary) the propositional constants t and f to represent the empty l.h.s. and the empty r.h.s., respectively.

is an sccr that satisfies the following two conditions:

- a) Let \vdash_T and \sim_T be, respectively, the single-conclusion counterparts of \vdash and \sim . Then \sim_T is a \vdash_T -cumulative tccr.
- b) \vee is an internal disjunction w.r.t. \sim_T as well.

Following the line of what we have done in the previous section, we next specify conditions that are equivalent to those of Definition 14, but are independent of the existence of *any* specific connective in the language.

Definition 15. Let \vdash be an scr. An sccr \sim in the same language is called *weakly \vdash -cumulative* if it satisfies the following conditions:

- Cum** *cumulativity:* if $\Gamma, \Delta \neq \emptyset$ and $\Gamma \vdash \Delta$, then $\Gamma \sim \Delta$.
- RW^[1]** *right weakening:* if $\Gamma, \psi \vdash \phi$ and $\Gamma \sim \psi, \Delta$, then $\Gamma \sim \phi, \Delta$.
- RM** *right monotonicity:* if $\Gamma \sim \Delta$ then $\Gamma \sim \psi, \Delta$.

Proposition 16. Let \vdash and \vee be as in Definition 14. A relation \sim is a $\{\vee, \vdash\}$ -cumulative sccr iff it is a weakly \vdash -cumulative sccr.

Proposition 17. If \vdash has an internal disjunction, then \sim is a weakly \vdash -cumulative sccr if it satisfies Cum, CM, CC^[1], and RW^[1].

We turn now to examine the role of conjunction in the present context.

Proposition 18. Let \vdash be an scr with an internal conjunction \wedge , and let \sim be a weakly \vdash -cumulative sccr. Then:

- a) \wedge is an internal conjunction w.r.t. \sim .
- b) \wedge is a “half” combining conjunction w.r.t. \sim . I.e, it satisfies $[\sim \wedge]_E$.

Definition 19. Suppose that an scr \vdash has an internal conjunction \wedge . A weakly \vdash -cumulative sccr \sim is called *$\{\wedge, \vdash\}$ -cumulative* if \wedge is also a combining conjunction w.r.t. \sim .

As usual, we provide an equivalent notion in which one does not have to assume that an internal conjunction is available:

Definition 20. A weakly \vdash -cumulative sccr \sim is called *\vdash -cumulative* if for every finite n the following condition is satisfied:

- RW^[n]** if $\Gamma \sim \psi_i, \Delta$ ($i=1, \dots, n$) and $\Gamma, \psi_1, \dots, \psi_n \vdash \phi$ then $\Gamma \sim \phi, \Delta$.

Proposition 21. Let \wedge be an internal conjunction for \vdash . An sccr \sim is $\{\wedge, \vdash\}$ -cumulative iff it is \vdash -cumulative.

Corollary 22. If \vdash is an scr with an internal conjunction \wedge and \sim is a \vdash -cumulative sccr, then \wedge is a combining conjunction and an internal conjunction w.r.t. \sim .

Let us return now to disjunction, examining it this time from its combining aspect. Our first observation is that unlike conjunction, one direction of the combining disjunction property for \sim of \vee yields monotonicity of \sim :

Lemma 23. Suppose that \vee is an internal disjunction for \vdash and \sim is a weakly \vdash -cumulative sccr in which $[\vee \sim]_E$ is satisfied. Then \sim is (left) monotonic.

It follows that requiring $[\vee \sim]_E$ from a weakly \vdash -cumulative sccr is too strong. It is reasonable, however, to require its converse.

Definition 24. A weakly \vdash -cumulative sccr \sim is called *weakly $\{\vee, \vdash\}$ -preferential* if it satisfies $[\vee \sim]_I$.

Unlike the Tarskian case, this time we are able to provide an equivalent condition in which one does not have to assume that a disjunction is available:

Definition 25. Let \vdash be an sccr. A weakly \vdash -cumulative sccr is called *weakly \vdash -preferential* if it satisfies the following rule:

CC *cautious cut*: if $\Gamma \sim \psi, \Delta$ and $\Gamma, \psi \sim \Delta$ then $\Gamma \sim \Delta$.

Proposition 26. Let \vdash be an scr with an internal disjunction \vee . An sccr \sim is weakly $\{\vee, \vdash\}$ -preferential iff it is weakly \vdash -preferential.

Some characterizations of weak \vdash -preferentiality are given in the following proposition:

Proposition 27. Let \vdash be an scr.

- a) \sim is a weakly \vdash -preferential sccr iff it satisfies Cum, CM, CC, and RM.
- b) \sim is a weakly \vdash -preferential sccr iff it is a weakly \vdash -cumulative sccr and for every finite n it satisfies *cautious n -cut*:

CC^[n] if $\Gamma, \psi_i \sim \Delta$ ($i=1, \dots, n$) and $\Gamma \sim \psi_1, \dots, \psi_n$, then $\Gamma \sim \Delta$.

Note: By Proposition 6, the single conclusion counterpart of CC^[n] is valid for *any* sccr (not only the cumulative or preferential ones).

We are now ready to introduce our strongest notions of nonmonotonic Scott consequence relation:

Definition 28. Let \vdash be an scr. A relation \sim is called *\vdash -preferential* iff it is both \vdash -cumulative and weakly \vdash -preferential.

Proposition 29. Let \vdash be an scr. \sim is \vdash -preferential iff it satisfies Cum, CM, CC, RM, and RW^[n] for every n .

Proposition 30. Let \vdash be an scr and let \sim be a \vdash -preferential sccr.

- a) An internal conjunction \wedge w.r.t. \vdash is also an internal conjunction and a combining conjunction w.r.t. \sim .
- b) An internal disjunction \vee w.r.t. \vdash is also an internal disjunction and “half” combining disjunction w.r.t. \sim .⁸

⁸ I.e., \sim satisfies $[\vee \sim]_I$ (but *not* necessarily $[\vee \sim]_E$).

$CC^{[n]}$ ($n \geq 1$), which is valid for \vdash -preferential sccrs, is a natural generalization of cautious cut. A dual generalization, which seems equally natural, is given in the following rule from [9]:

$$LCC^{[n]} \quad \frac{\Gamma \vdash \psi_1, \Delta \quad \dots \quad \Gamma \vdash \psi_n, \Delta, \quad \Gamma, \psi_1, \dots, \psi_n \vdash \Delta}{\Gamma \vdash \Delta}$$

Definition 31. [9] A binary relation \vdash is a *plausibility logic* if it satisfies Inclusion ($\Gamma, \psi \vdash \psi$), CM, RM, and $LCC^{[n]}$ ($n \geq 1$).

Definition 32. Let \vdash be an scr. A relation \vdash is called \vdash -*plausible* if it is a \vdash -preferential sccr and a plausibility logic.

A more concise characterization of a \vdash -plausible relation is given in the following proposition:

Proposition 33. Let \vdash be an scr. A relation \vdash is \vdash -plausible iff it satisfies Cum, CM, RM, and $LCC^{[n]}$ for every n .

Proposition 34. Let \vdash be an scr with an internal conjunction \wedge . A relation \vdash is \vdash -preferential iff it is \vdash -plausible.

Table 1 and Figure 1 summarize the various types of Scott relations considered in this section and their relative strengths. \vdash is assumed there to be an scr, and \vee, \wedge are internal disjunction and conjunction (respectively) w.r.t. \vdash , whenever they are mentioned.

Table 1. Scott relations

consequence relation	general conditions
	valid conditions with \wedge and \vee
sccr	s-R, CM, $CC^{[1]}$
weakly \vdash-cumulative sccr	Cum, CM, $CC^{[1]}$, $RW^{[1]}$, RM
	$[\wedge \sim]_I, [\wedge \sim]_E, [\sim \wedge]_E, [\sim \vee]_I, [\sim \vee]_E$
\vdash-cumulative sccr	Cum, CM, $CC^{[1]}$, $RW^{[n]}$, RM
	$[\wedge \sim]_I, [\wedge \sim]_E, [\sim \wedge]_I, [\sim \wedge]_E, [\sim \vee]_I, [\sim \vee]_E$
weakly \vdash-preferential sccr	Cum, CM, CC, RM
	$[\wedge \sim]_I, [\wedge \sim]_E, [\sim \wedge]_E, [\vee \sim]_I, [\sim \vee]_I, [\sim \vee]_E,$
\vdash-preferential sccr	Cum, CM, CC, $RW^{[n]}$, RM
	$[\wedge \sim]_I, [\wedge \sim]_E, [\sim \wedge]_I, [\sim \wedge]_E, [\vee \sim]_I, [\sim \vee]_I, [\sim \vee]_E$
\vdash-plausible sccr	Cum, CM, $LCC^{[n]}$, RM
	$[\wedge \sim]_I, [\wedge \sim]_E, [\sim \wedge]_I, [\sim \wedge]_E, [\vee \sim]_I, [\sim \vee]_I, [\sim \vee]_E$
scr extending \vdash	Cum, M, C
	$[\wedge \sim]_I, [\wedge \sim]_E, [\sim \wedge]_I, [\sim \wedge]_E, [\vee \sim]_I, [\vee \sim]_E, [\sim \vee]_I, [\sim \vee]_E$

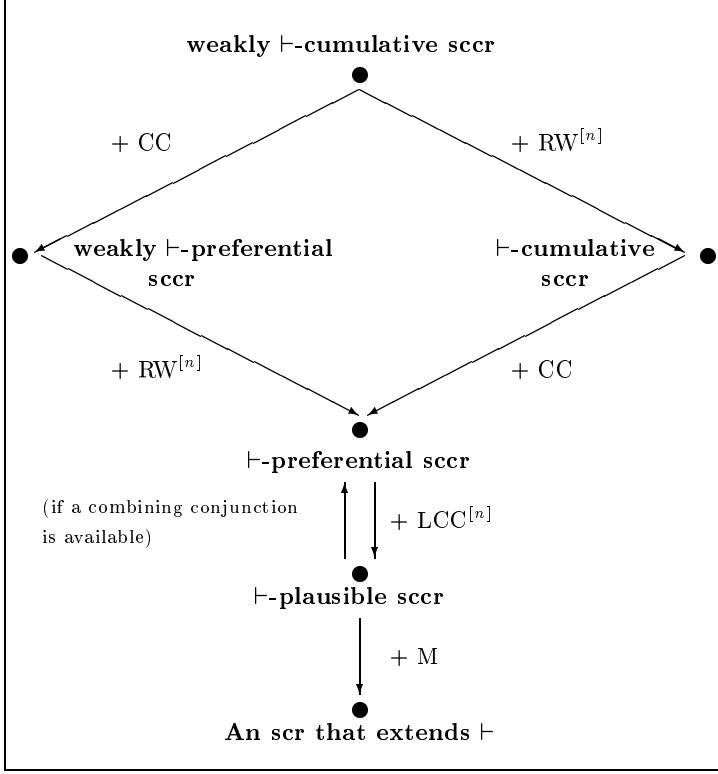


Fig. 1. Relative strength of the Scott relations

5 A Semantical Point of View

In this section we present a general method of constructing nonmonotonic consequence relations of the strongest type considered in the previous section, i.e., preferential and plausible sccrs. Our approach is based on a multiple-valued semantics. This will allow us to define in a natural way consequence relations that are not only nonmonotonic, but also paraconsistent (see examples below).

A basic idea behind our method is that of using a set of *preferential models* for making inferences. Preferential models were introduced by McCarthy [14] and later by Shoham [21] as a generalization of the notion of circumscription. The essential idea is that only a subset of its models should be relevant for making inferences from a given theory. These models are the most preferred ones according to some conditions or preference criteria.

Definition 35. Let Σ be an arbitrary propositional language. A *preferential multiple-valued structure* for Σ (pms, for short) is a quadruple $(\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$, where \mathcal{L} is set of elements (“truth values”), \mathcal{F} is a nonempty proper subset of \mathcal{L} , \mathcal{S} is

a set of operations on \mathcal{L} that correspond to the connectives in Σ , and \leq is a well-founded partial order on \mathcal{L} .

The set \mathcal{F} consists of the *designated* values of \mathcal{L} , i.e., those that represent true assertions. In what follows we shall assume that \mathcal{L} contains at least the classical values t, f , and that $t \in \mathcal{F}$, $f \notin \mathcal{F}$.

Definition 36. Let $(\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$ be a pms.

- a) A (multiple-valued) *valuation* ν is a function that assigns an element of \mathcal{L} to each atomic formula. Extensions to complex formulae are done as usual.
- b) A valuation ν *satisfies* a formula ψ if $\nu(\psi) \in \mathcal{F}$.
- c) A valuation ν is a *model* of a set Γ of formulae, if ν satisfies every formula in Γ . The set of the models of Γ is denoted by $\text{mod}(\Gamma)$.

Definition 37. Let $(\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$ be a pms. Denote $\Gamma \vdash^{\mathcal{L}, \mathcal{F}} \Delta$ if every model of Γ satisfies some formula in Δ .

Proposition 38. $\vdash^{\mathcal{L}, \mathcal{F}}$ is an scr.

Definition 39. Let $(\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$ be a pms for a language Σ .

- a) An operator \wedge ⁹ is *conjunctive* if $\forall x, y \in \mathcal{L}$, $x \wedge y \in \mathcal{F}$ iff $x \in \mathcal{F}$ and $y \in \mathcal{F}$.
- b) An operator \vee is *disjunctive* if $\forall x, y \in \mathcal{L}$, $x \vee y \in \mathcal{F}$ iff $x \in \mathcal{F}$ or $y \in \mathcal{F}$.

Proposition 40. Let $(\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$ be a pms for Σ , and let \wedge (\vee) be in Σ . If the operation which corresponds to \wedge (\vee) is conjunctive (disjunctive), then \wedge (\vee) is both an internal and a combining conjunction (disjunction) w.r.t. $\vdash^{\mathcal{L}, \mathcal{F}}$.

Definition 41. Let \mathcal{P} be a pms and Γ a set of formulae in Σ . A valuation $M \in \text{mod}(\Gamma)$ is a \mathcal{P} -*preferential model* of Γ if there is no other valuation $M' \in \text{mod}(\Gamma)$ s.t. for every atom p , $M'(p) \leq M(p)$. The set of all the \mathcal{P} -preferential models of Γ is denoted by $!(\Gamma, \mathcal{P})$.

Definition 42. Let \mathcal{P} be a pms. A set of formulae Γ \mathcal{P} -*preferentially entails* a set of formulae Δ (notation: $\Gamma \vdash_{\leq}^{\mathcal{L}, \mathcal{F}} \Delta$) if every $M \in !(\Gamma, \mathcal{P})$ satisfies some $\delta \in \Delta$.

Proposition 43. Let $(\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$ be a pms. Then $\vdash_{\leq}^{\mathcal{L}, \mathcal{F}}$ is $\vdash^{\mathcal{L}, \mathcal{F}}$ -plausible.

Corollary 44. Let $\mathcal{P} = (\mathcal{L}, \mathcal{F}, \mathcal{S}, \leq)$ be a pms for a language Σ .

- a) If \wedge is a conjunctive connective of Σ (relative to \mathcal{P}), then it is a combining conjunction and an internal conjunction w.r.t. $\vdash_{\leq}^{\mathcal{L}, \mathcal{F}}$.
- b) If \vee is a disjunctive connective of Σ (relative to \mathcal{P}), then it is an internal disjunction w.r.t. $\vdash_{\leq}^{\mathcal{L}, \mathcal{F}}$, which also satisfies $[\vee \models]_{\mathcal{I}}$.

Examples. Many known formalisms can be viewed as based on preferential multiple-valued structures. Among which are classical logic, Reiter's closed-world assumption [17], the paraconsistent logic LPm of Priest [15, 16], and the paraconsistent bilattice-based logics $\vdash_k^{\mathcal{L}, \mathcal{F}}$ and $\vdash_{\mathcal{I}}^{\mathcal{L}, \mathcal{F}}$ [1, 2].

⁹ We use here the same symbol for a connective and its corresponding operation in \mathcal{S} .

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