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Distance-based paraconsistent logics

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Abstract

We introduce a general framework that is based on distance semantics and investigate the main properties of the entailment relations that it induces. It is shown that such entailments are particularly useful for non-monotonic reasoning and for drawing rational conclusions from incomplete and inconsistent information. Some applications are considered in the context of belief revision, information integration systems, and consistent query answering for possibly inconsistent databases.

Keywords: Paraconsistent and non-monotonic reasoning; Distance semantics; Multiple-valued logics

1. Introduction

Reasoning with distance functions is a common way of giving semantics to formalisms that handle incomplete and/or inconsistent information. The basic intuition behind this approach is that, given a set of possible worlds (alternatively, interpretations) that represent the reasoner's epistemic states or the information content of different data-sources, the similarity between those worlds can be expressed quantitatively (that is, in terms of distance measurements), and thus can be evaluated by corresponding distance operators. In this respect, there is no wonder that distance semantics has played a prominent role in different paradigms for information processing. Three remarkable examples for this are the following:

- Formalisms for modelling belief revision, in which distance minimization corresponds to the idea that the difference between the reasoner's new state of belief and the old one should be kept as minimal as possible, that is, restricted only to what is really implied by the new information (see, e.g., [26,29,36,46,50,59]).
- Database integration systems [3,4,10,31,35,47] and merging operators for independent data-sources [40,41], where the basic idea is that the amalgamated information should be kept coherent and at the same time as close as possible to the collective information as it is depicted by the distributed sources.
- Different aspects of social choice theory, in which distance-based considerations are involved. This includes group decision making [43], preference representation [44], and judgment aggregation [30,51].

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The goal of this paper is to introduce similar distance considerations in the context of *paraconsistent logics*, that is: formalisms that tolerate inconsistency and do not become trivial in the presence of contradictions.[1] One could identify at least four parties with different philosophical attitudes to such logics: the *traditionalists* defend classical logics and deny any need of paraconsistent logics. On the other extreme, the *dialetheists* contend that the world is fundamentally inconsistent and hence the true logic should be paraconsistent. The *pluralists* view inconsistent structures as fundamental but provisional, and favour their replacement, at least in empirical domains, by consistent counterparts. Finally, the *reformists* defend consistency in ontological matters, but argue that human knowledge and thinking necessarily requires inconsistency, and hence that classical logic should be replaced by a paraconsistent counterpart. The underlying theme here, following the reformists, is that conflicting data is unavoidable in practice, but it corresponds to inadequate information about the real world, and therefore it should be minimized. As we show below, this intuition is nicely and easily expressed in terms of distance semantics.

The rest of this paper is organized as follows: in the next section we introduce the framework and the family of distance-based entailments that it induces. Then, in Section 3 we consider some basic properties of these entailments and in Section 4 we discuss their applications in relevant areas, such as operators for belief revision and consistent query answering in database systems. In Section 5 we examine extensions to the multiple-valued case and the incorporation of corresponding distance functions. In Section 6 we conclude.

2. Distance-based semantics and entailments

The intuition behind our approach is very simple. Suppose, for instance, that a certain set of assumptions Γ consists only of two facts p and q. In this case it seems reasonable to use the classical entailment for inferring the formulas in the transitive closure of Γ. If we learn now that $\neg p$ also holds, classical logic becomes useless, as everything classically follows from $\Gamma' = \Gamma \cup \{\neg p\}$. The decision how to maintain the inconsistent fragment of Γ' depends on the underlying formalism. For example, most of the belief revision operators prefer more recent information thus conclude $\neg p$ and exclude p in this case. Alternatively, many merging operators that view Γ and $\{\neg p\}$ as belief bases of two different sources will retract both p and $\neg p$, and so forth. It is evident, however, that $\neg q$ should *not* follow from Γ', as there is no evidence whatsoever that q is related to any contradictory information. In our context, this is captured by the fact that valuations in which q holds are 'closer' to Γ' (thus are more plausible) than valuations in which q is falsified. In what follows we formalize this idea.

Given a propositional language $\mathscr{L}$ with a finite set Atoms of atomic formulas, the space of the two-valued interpretations on Atoms is denoted Λ^2. In the sequel, we shall consider *finite multisets* of formulas in $\mathscr{L}$, called theories. Given such a theory Γ, the set $\mathsf{Atoms}(\Gamma)$ consists of the atoms that appear in the formulas of Γ. The set of the models of Γ (i.e., the valuations v s.t. $v(\psi) = \mathsf{t}$ for every $\psi \in \Gamma$) is denoted $\mathrm{mod}(\Gamma)$.

Definition 1. A total function $d : U \times U \to \mathbb{R}^+$ is called *pseudo-distance* on U if it is symmetric ($\forall u, v \in U$ $d(u, v) = d(v, u)$) and preserves identity ($\forall u, v \in U$ $d(u, v) = 0$ iff $u = v$). A *distance function* on U is a pseudo-distance on U that satisfies the triangular inequality ($\forall u, v, w \in U$ $d(u, v) \leqslant d(u, w) + d(w, v)$).

Example 1. The following functions are distances on Λ^2:

- *The Hamming distance*: $d^{H_2}(v, \mu) = |\{p \in \mathsf{Atoms} | v(p) \neq \mu(p)\}|$.[2]
- *The drastic distance*: $d^U(v, \mu) = 0$ if $v = \mu$ and $d^U(v, \mu) = 1$ otherwise.

Definition 2. A *numeric aggregation function* f is a total function that accepts a multiset of real numbers and returns a real number. In addition, (a) f is non-decreasing in the values of its argument,[3] (b) $f(\{x_1, \ldots, x_n\}) = 0$ iff $x_1 = \ldots = x_n = 0$, and (c) $\forall x \in \mathbb{R}$ $f(\{x\}) = x$.

[1] See [23,54]. Some collections of papers on this topic appear, e.g., in [14,20].
[2] That is, $d^{H_2}(v, \mu)$ is the number of atoms p such that $v(p) \neq \mu(p)$. This function is also known as the Dalal distance [24].
[3] That is, the function value is non-decreasing when an element in the multiset is replaced by a larger element.

In what follows we shall aggregate distance values, which are non-negative numbers. Thus, functions that meet the conditions in Definition 2 are, e.g., a summation or an average of distances (for mean case analysis), the maximal value among distances (which yields a worst case analysis), and so forth.

Definition 3. Given a theory $\Gamma = \{\psi_1, \ldots, \psi_n\}$, a two-valued interpretation $v \in \Lambda^2$, a pseudo-distance d on Λ^2, and an aggregation function f, define the following function on $\Lambda^2 \times \mathscr{L}$:[4]

$$d(v, \psi) = \min\{d(v, \mu) | \mu \in \mathrm{mod}(\{\psi\})\}.^5$$

Accordingly, the function $\delta_{d,f}$ is defined as follows:

$$\delta_{d,f}(v, \Gamma) = f(\{d(v, \psi_1), \ldots, d(v, \psi_n)\}).$$

The next definition captures the intuition behind distance semantics that the relevant interpretations of a theory Γ are those that are $\delta_{d,f}$-closest to Γ.

Definition 4. The *most plausible* valuations of Γ (with respect to a pseudo-distance d and an aggregation function f) are the valuations v that belong to the following set:

$$\Delta^2_{d,f}(\Gamma) = \{v \in \Lambda^2 | \forall \mu \in \Lambda^2 \ \delta_{d,f}(v, \Gamma) \leqslant \delta_{d,f}(\mu, \Gamma)\}.^6$$

Note 1. The distance-like function $\delta_{d,f}$ in Definition 3 is not invariant with respect to the notion of logical equivalence. Yet, according to Definition 4, consistent theories that are logically equivalent share *the same* most plausible models (cf. Proposition 1 below). Inconsistent theories, on the other hand, are all logically equivalent, so any definition of most plausible models that makes a distinction among such theories cannot preserve logical equivalence. Rather, it should employ some more delicate considerations.

Consider, for instance the theories $\Gamma_1 = \{p, \neg p\}$, $\Gamma_2 = \{p, p, \neg p\}$, and two valuations $v_{\mathsf{t}}, v_{\mathsf{f}} \in \Lambda^2$, for which $v_{\mathsf{t}}(p) = \mathsf{t}$ and $v_{\mathsf{f}}(p) = \mathsf{f}$. Then, when f is the summation or the average function, $\Delta^2_{d,f}(\Gamma_1) = \{v_{\mathsf{t}}, v_{\mathsf{f}}\}$, while $\Delta^2_{d,f}(\Gamma_2) = \{v_{\mathsf{t}}\}$. This captures the intuition that while Γ_1 is totally symmetric, Γ_2 contains more quantitative evidence for p than for $\neg p$ (which is a kind of a 'majority vote consideration' for resolving contradictions; see [41,47] and Section 4.3 below).

Corresponding consequence relations are now defined as follows.

Definition 5. For a pseudo-distance d and an aggregation function f, define: $\Gamma \models^2_{d,f} \psi$ if $\Delta^2_{d,f}(\Gamma) \subseteq \mathrm{mod}(\{\psi\})$. That is, conclusions should follow from *all* the most plausible valuations of the premises.

Example 2. Let $\Gamma = \{p, q, r, \neg p \vee \neg q, r \wedge s\}$. This theory is not consistent, and so everything classically follows from it, including, e.g., $\neg r$. This seems to be a very strange conclusion in our case, as r is not part of an inconsistent fragment of Γ, therefore it does not make sense here to conclude its complement. Using distance-based semantics, this anomaly can be lifted. To see this, consider the table in Fig. 1 that lists the δ-distances between the relevant valuations and Γ according to several common settings.

Here, $\Delta^2_{d^U,\Sigma}(\Gamma) = \Delta^2_{d^{H_2},\Sigma}(\Gamma) = \{v_1, v_5, v_9\}$, thus $\Gamma \models^2_{d^U,\Sigma} r$ and $\Gamma \models^2_{d^{H_2},\Sigma} r$, while $\Gamma \not\models^2_{d^U,\Sigma} \neg r$ and $\Gamma \not\models^2_{d^{H_2},\Sigma} \neg r$. The same thing happens with s, as intuitively expected. Note also that the atoms p, q that are involved in the inconsistency are not deducible from Γ, nor their complements. The entailment $\models^2_{d^{H_2},\max}$ is more cautious; it does not allow to infer neither r nor $\neg r$ (and similarly neither s nor $\neg s$ is deducible), but the weaker conclusion $r \vee s$ *is* deducible.

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[4] As usual, we identify the language $\mathscr{L}$ with its set of formulas. Also, to reduce the amount of notations, we use the same symbol (d) to denote the pseudo-distance on Λ^2 and the induced distance-like function on $\Lambda^2 \times \mathscr{L}$. The exact meaning of d will be clear from the context.

[5] Below, we exclude classical contradictions in the premises. Alternatively, if ψ is a contradiction, one may set, for every $v \in \Lambda^2$, $d(v, \psi) = |\mathsf{Atoms}|$.

[6] The superscript '2' denotes the standard two-valued semantics. This indication will be useful in Section 5, where we consider multiple-valued structures.

	p	q	r	s	$\delta_{d^U,\Sigma}$	$\delta_{d^{H_2},\Sigma}$	$\delta_{d^{H_2},\max}$
ν_1	t	t	t	t	**1**	**1**	**1**
ν_2	t	t	t	f	2	2	**1**
ν_3	t	t	f	t	3	3	**1**
ν_4	t	t	f	f	3	4	2
ν_5	t	f	t	t	**1**	**1**	**1**
ν_6	t	f	t	f	2	2	**1**
ν_7	t	f	f	t	3	3	**1**
ν_8	t	f	f	f	3	4	2

	p	q	r	s	$\delta_{d^U,\Sigma}$	$\delta_{d^{H_2},\Sigma}$	$\delta_{d^{H_2},\max}$
ν_9	f	t	t	t	**1**	**1**	**1**
ν_{10}	f	t	t	f	2	2	**1**
ν_{11}	f	t	f	t	3	3	**1**
ν_{12}	f	t	f	f	3	4	2
ν_{13}	f	f	t	t	2	2	**1**
ν_{14}	f	f	t	f	3	3	**1**
ν_{15}	f	f	f	t	4	4	**1**
ν_{16}	f	f	f	f	4	5	2

Fig. 1. Interpretations for Γ (Example 2) and their δ-distances from Γ.

3. Reasoning with $\models^2_{d,f}$

The principle of uncertainty minimization by distance semantics, depicted in Definition 5, is in fact a preference criterion among different interpretations of the premises. In this respect, the formalisms that are defined here may be considered as a certain kind of *preferential logics* [48,49,57,58], as only 'preferred' valuations (those that are 'as close as possible' to the premises) are taken into consideration for drawing conclusions from the premises.

When a theory is classically consistent, its set of models is not empty, so it seems natural to choose these valuations as the preferred (i.e., most plausible) ones. The following proposition shows that the models of a theory Γ are indeed closest to Γ.

Proposition 1. *Let Γ be a consistent theory. For every pseudo-distance d and aggregation function f, $\Delta^2_{d,f}(\Gamma) = mod(\Gamma)$.*[7]

Proof. Let $\Gamma = \{\psi_1, \ldots, \psi_n\}$. If v is a model of Γ, then $d(v, \psi_i) = 0$ for every $1 \leqslant i \leqslant n$, and so $\delta_{d,f}(v, \Gamma) = 0$ as well. Now, since for every valuation μ, $\delta_{d,f}(\mu, \Gamma) \geqslant 0$, necessarily $v \in \Delta^2_{d,f}(\Gamma)$.

For the converse, suppose that v is not a model of Γ. Then v does not satisfy ψ_j for some $1 \leqslant j \leqslant n$, and so $d(v, \psi_j) > 0$. Now, by conditions (a) and (b) in Definition 2, f is strictly positive whenever it has at least one strictly positive argument and the other arguments are non-negative. We have, then, that $\delta_{d,f}(v, \Gamma) > 0$. On the other hand, as $\text{mod}(\Gamma) \neq \emptyset$ there is a model μ of Γ, for which $\delta_{d,f}(\mu, \Gamma) = 0$. It follows, then, that $v \notin \Delta^2_{d,f}(\Gamma)$. $\square$

Denote by $\models^2$ the standard entailment of classical logic (that is, $\Gamma \models^2 \psi$ if every model of Γ satisfies ψ). Then:

Corollary 1. *For every classically consistent theory Γ and for every formula ψ, $\Gamma \models^2 \psi$ iff $\Gamma \models^2_{d,f} \psi$.*

Proof. Immediate from Proposition 1. $\square$

A characteristic property of distance-based entailments is that they are paraconsistent [23,54], namely: unlike classical entailment, contradictory premises do *not* entail everything (thus they do not have an explosive character):

Proposition 2. *For every pseudo-distance d and aggregation function f, $\models^2_{d,f}$ is paraconsistent.*

Proof. Follows from the fact that for every theory Γ, $\Delta^2_{d,f}(\Gamma) \neq \emptyset$ (as the minimal $\delta_{d,f}$-distance from Γ over a finite space of interpretations is always obtained). Thus, for every formula ψ such that there exists a valuation $v \in \Delta^2_{d,f}(\Gamma)$ for which $v(\psi) = \mathsf{f}$, it holds that $\Gamma \not\models^2_{d,f} \psi$. $\square$

[7] Clearly, the converse is also true: if Γ is not consistent then $\Delta^2_{d,f}(\Gamma) \neq \text{mod}(\Gamma)$, since $\text{mod}(\Gamma) = \emptyset$ while $\Delta^2_{d,f}(\Gamma) \neq \emptyset$.

Paraconsistency also follows from the fact that the set of conclusions of a (not necessarily consistent) theory is always consistent:

Proposition 3. *For every pseudo-distance d, aggregation function f, theory Γ and formula ψ, we have that if $\Gamma \models^2_{d,f} \psi$ then $\Gamma \not\models^2_{d,f} \neg\psi$.*

Proof. Suppose for a contradiction that there is a formula ψ such that $\Gamma \models^2_{d,f} \psi$ and $\Gamma \models^2_{d,f} \neg\psi$ at the same time. Then $\Delta^2_{d,f}(\Gamma) \subseteq mod(\{\psi\})$ and $\Delta^2_{d,f}(\Gamma) \subseteq mod(\{\neg\psi\})$. Thus, $\Delta^2_{d,f}(\Gamma) \subseteq mod(\{\psi\}) \cap \mathrm{mod}(\{\neg\psi\}) = \emptyset$, a contradiction to the fact that for every Γ, $\Delta^2_{d,f}(\Gamma) \neq \emptyset$ (see the proof of Proposition 2). $\square$

Corollary 1 and Proposition 2 (or Proposition 3) imply the following desirable property of $\models^2_{d,f}$:

Corollary 2. *For every pseudo-distance d and aggregation function f, $\models^2_{d,f}$ is the same as the classical entailment with respect to consistent premises, and is non-trivial otherwise.*

For the next propositions we concentrate on unbiased distances:

Definition 6. A (pseudo) distance d is called *unbiased*, if for every formula ψ and interpretations $v_1, v_2 \in \Lambda^2$ so that $v_1(p) = v_2(p)$ for every $p \in \mathsf{Atoms}(\{\psi\})$, it holds that $d(v_1,\psi) = d(v_2,\psi)$.

The last property assures that, given a pseudo-distance on Λ^2, the distance-like function that it induces on $\Lambda^2 \times \mathscr{L}$ depends only on the atoms that appear in the formula, and so this function is not 'biased' by irrelevant atoms. Note, e.g., that the distances in Example 1 are unbiased.

Unbiasedness will be useful in what follows for assuring some desirable properties of the distance-based consequence relations. First, we consider a useful condition for unbiasedness, specified in terms of (pseudo) distances on Λ^2:

Definition 7. An (unnormalized) *fuzzy measure* [44] on a finite set $\mathscr{S}$ is a mapping $\mathscr{F} : 2^{\mathscr{S}} \to \mathbb{R}^+$, such that $\mathscr{F}(\emptyset) = 0$, and for every $A, B \subseteq \mathscr{S}$ if $A \subseteq B$ then $\mathscr{F}(A) \leqslant \mathscr{F}(B)$.[8]

Proposition 4. *A pseudo-distance d on Λ^2 is unbiased, if for all $v_1, v_2 \in \Lambda^2$ it holds that $d(v_1, v_2) = \mathscr{F}(\mathsf{Diff}(v_1, v_2))$, where $\mathscr{F}$ is a fuzzy measure on* Atoms *and* $\mathsf{Diff}(v_1, v_2) = \{p \in \mathsf{Atoms} | v_1(p) \neq v_2(p)\}$.

Proof. Given a formula ψ in $\mathscr{L}$ and two valuations $v_1, v_2 \in \Lambda^2$ such that for every $p \in \mathsf{Atoms}(\{\psi\})$ $v_1(p) = v_2(p)$, we shall show that $d(v_1,\psi) = d(v_2,\psi)$. Indeed, let $\mu_1^{\min} \in mod(\{\psi\})$ be a valuation so that $\forall \mu \in mod(\{\psi\})\ d(v_1, \mu_1^{\min}) \leqslant d(v_1, \mu)$ and $\mu_2^{\min} \in \mathrm{mod}(\{\psi\})$ a valuation so that $\forall \mu \in \mathrm{mod}(\{\psi\})\ d(v_2, \mu_2^{\min}) \leqslant d(v_2, \mu)$. Consider also the valuations $\mu_1, \mu_2 \in \Lambda^2$, defined for every $p \in \mathsf{Atoms}$ as follows:

$$\mu_1(p) = \begin{cases} \mu_1^{\min}(p) & \text{if } p \in \mathsf{Atoms}(\{\psi\}), \\ v_1(p) & \text{if } p \notin \mathsf{Atoms}(\{\psi\}), \end{cases}$$

and

$$\mu_2(p) = \begin{cases} \mu_1^{\min}(p) & \text{if } p \in \mathsf{Atoms}(\{\psi\}), \\ v_2(p) & \text{if } p \notin \mathsf{Atoms}(\{\psi\}). \end{cases}$$

Now, as μ_1 and $\mu_1^{\min}$ are identical on $\mathsf{Atoms}(\{\psi\})$, we have that $\mu_1 \in \mathrm{mod}(\{\psi\})$, thus $\mathscr{F}(\mathsf{Diff}(v_1, \mu_1^{\min})) = d(v_1, \mu_1^{\min}) \leqslant d(v_1, \mu_1) = \mathscr{F}(\mathsf{Diff}(v_1, \mu_1))$. On the other hand, by the definitions of these interpretations, $\mathsf{Diff}(v_1, \mu_1) \subseteq \mathsf{Diff}(v_1, \mu_1^{\min})$ and as $\mathscr{F}$ is a fuzzy measure, $\mathscr{F}(\mathsf{Diff}(v_1, \mu_1)) \leqslant \mathscr{F}(\mathsf{Diff}(v_1, \mu_1^{\min}))$. Combining these two facts we have, therefore, that $\mathscr{F}(\mathsf{Diff}(v_1, \mu_1)) = \mathscr{F}(\mathsf{Diff}(v_1, \mu_1^{\min}))$. Denote now by $v^{\downarrow S}$ the restriction of v to the atomic formulas in S. Then, by the considerations above, we have:

[8] Normalized fuzzy measures are obtained by setting their range to the unit interval and requiring that $\mathscr{F}(\mathscr{S}) = 1$. As in [44], we find the unnormalized version of this concept more convenient.

$$\begin{aligned} d(\nu_1,\psi) &= d(\nu_1,\mu_1^{\min}) \\ &= \mathscr{F}(\mathsf{Diff}(\nu_1,\mu_1^{\min})) = \mathscr{F}(\mathsf{Diff}(\nu_1,\mu_1)) \\ &= \mathscr{F}(\mathsf{Diff}(\nu_1^{\downarrow \mathsf{Atoms}(\{\psi\})},\mu_1^{\downarrow \mathsf{Atoms}(\{\psi\})})) \\ &= \mathscr{F}(\mathsf{Diff}(\nu_2^{\downarrow \mathsf{Atoms}(\{\psi\})},\mu_1^{\downarrow \mathsf{Atoms}(\{\psi\})})) \\ &= \mathscr{F}(\mathsf{Diff}(\nu_2^{\downarrow \mathsf{Atoms}(\{\psi\})},\mu_2^{\downarrow \mathsf{Atoms}(\{\psi\})})) \\ &= \mathscr{F}(\mathsf{Diff}(\nu_2,\mu_2)) = d(\nu_2,\mu_2) \\ &\geqslant d(\nu_2,\mu_2^{\min}) = d(\nu_2,\psi). \end{aligned}$$

Similarly, by symmetric considerations, $d(\nu_2,\psi) \geqslant d(\nu_1,\psi)$. Thus, $d(\nu_1,\psi) = d(\nu_2,\psi)$. □

Unbiasedness allows us to strengthen Proposition 2, as in this case a non-tautological formula never follows from a theory unless they share propositional atoms.

Proposition 5. *Let d be an unbiased pseudo-distance and f an aggregation function. Suppose that Γ is a theory in $\mathscr{L}$ and ψ is a non-tautological formula in $\mathscr{L}$ such that $\mathsf{Atoms}(\Gamma) \cap \mathsf{Atoms}(\{\psi\}) = \emptyset$. Then $\Gamma \not\models^2_{d,f} \psi$.*

Proof. Let $\Gamma = \{\psi_1,\ldots,\psi_n\}$ and $\nu \in \Delta^2_{d,f}(\Gamma)$. If $\nu(\psi) = \mathsf{f}$ we are done. Otherwise, consider a valuation μ that is the same as ν on $\mathsf{Atoms}(\Gamma)$ and $\mu(\psi) = \mathsf{f}$. Such a valuation exists, of-course, since ψ is not a tautology, the value of $\mu(\psi)$ depends only on the assignments of μ on $\mathsf{Atoms}(\{\psi\})$, and $\mathsf{Atoms}(\Gamma) \cap \mathsf{Atoms}(\{\psi\}) = \emptyset$. Now, as d is unbiased, $d(\nu,\psi_i) = d(\mu,\psi_i)$ for every $\psi_i \in \Gamma$, thus $\delta_{d,f}(\nu,\Gamma) = \delta_{d,f}(\mu,\Gamma)$, and so $\mu \in \Delta^2_{d,f}(\Gamma)$ as well. Hence, $\Gamma \not\models^2_{d,f} \psi$. □

Note 2. Unbiasedness of the pseudo-distance is indeed a necessary condition for assuring Proposition 5. To see this, consider the following function on Λ^2:

$$d^U_q(\nu,\mu) = \begin{cases} 0 & \text{if } \nu = \mu, \\ 1 & \text{if } \nu \neq \mu \text{ and } \nu(q) = \mu(q) = \mathsf{t}, \\ 5 & \text{otherwise.} \end{cases}$$

Clearly, d^U_q is a (biased) pseudo-distance on Λ^2, and $\{p,\neg p\} \models^2_{d^U_q,\Sigma} q$.

Another characteristic property of $\models^2_{d,f}$ is its non-monotonic nature. According to $\models^2_{d,f}$, a formula that is entailed by a certain theory Γ might not necessarily be a consequence of a superset of Γ (cf. monotonicity in Definition 9).[9]

Proposition 6. *For every pseudo-distance d and aggregation function f, $\models^2_{d,f}$ is non-monotonic.*

Proof. By Corollary 1, $p \models^2_{d,f} p$ and $\neg p \models^2_{d,f} \neg p$. By Proposition 3, on the other hand, either $p,\neg p \not\models^2_{d,f} p$ or $p,\neg p \not\models^2_{d,f} \neg p$ (or both). Hence, the set of conclusions does not monotonically grow with respect to the size of the premises, and so $\models^2_{d,f}$ is non-monotonic. □

Next we show that in many cases non-monotonicity goes along with *rationality* [45], that is: a reasoner does not have to retract any previous conclusion when learning about a new fact that has no influence on the premises. Borrowing the example in [45], suppose that we know that a certain bird b can fly, and then we learn that b is a red bird. As the color of a bird should not affect its flying ability, we still want to retain our previous conclusion that b can fly.

Note 3. It is important to note that we are using here the notion of rationality in the particular sense mentioned above, which is one of the requirements for 'rational closure' in the sense of Lehmann and Magidor (see [45]). Note, also, that our terminology should not be confused with the notion of rationality in the literature of non-monotonic reasoning, which is concerned with the satisfaction of the rationality postulate.

[9] This corresponds to the fact that humans tend to change their mind in light of new information that contradicts previous conclusion(s).

Definition 8. An aggregation function f is *hereditary*, if $f(\{x_1,\ldots,x_n\}) < f(\{y_1,\ldots,y_n\})$ entails $f(\{x_1,\ldots,x_n,z_1,\ldots,z_m\}) < f(\{y_1,\ldots,y_n,z_1,\ldots,z_m\})$.

Remark that hereditary, unlike monotonicity, is defined by strict inequalities. Thus, for instance, summation of distances is hereditary (as distances are non-negative), while the maximum function is not.

Proposition 7. *Let d be an unbiased pseudo-distance and f a hereditary aggregation function. If $\Gamma \models^2_{d,f} \psi$ then $\Gamma, \phi \models^2_{d,f} \psi$ for every formula ϕ such that* $\mathsf{Atoms}(\Gamma \cup \{\psi\}) \cap \mathsf{Atoms}(\{\phi\}) = \emptyset$.

Intuitively, the condition on ϕ in Proposition 7 guarantees that ϕ is 'irrelevant' for Γ and ψ. The intuitive meaning of Proposition 7 is, therefore, that the reasoner does not have to retract ψ when learning that ϕ holds.

Proof of Proposition 7. If ψ is a tautology then the proposition obviously holds. Otherwise, let μ be a valuation such that $\mu(\psi) = \mathsf{f}$. As $\Gamma \models^2_{d,f} \psi$, necessarily $\mu \notin \Delta^2_{d,f}(\Gamma)$, and so there is a valuation $\nu \in \Delta^2_{d,f}(\Gamma)$ for which $\delta_{d,f}(\nu,\Gamma) < \delta_{d,f}(\mu,\Gamma)$. Now, assuming that $\Gamma = \{\psi_1,\ldots,\psi_n\}$, we have that $f(\{d(\nu,\psi_1),\ldots,d(\nu,\psi_n)\}) < f(\{d(\mu,\psi_1),\ldots,d(\mu,\psi_n)\})$. Again, $\Gamma \models^2_{d,f} \psi$ implies that $\nu(\psi) = \mathsf{t}$. Now, consider a valuation σ, defined as follows:

$$\sigma(p) = \begin{cases} \nu(p) & \text{if } p \in \mathsf{Atoms}(\Gamma \cup \{\psi\}), \\ \mu(p) & \text{otherwise.} \end{cases}$$

Note that $\sigma(p) = \nu(p)$ for every $p \in \mathsf{Atoms}(\{\psi\})$, and so $\sigma(\psi) = \mathsf{t}$ as well. Also, as $\mathsf{Atoms}(\Gamma \cup \{\psi\}) \cap \mathsf{Atoms}(\{\phi\}) = \emptyset$, $\sigma(p) = \mu(p)$ for every $p \in \mathsf{Atoms}(\{\phi\})$, thus $\sigma(\phi) = \mu(\phi)$. Now, since d is unbiased and f is hereditary, we have that

$$\begin{aligned} \delta_{d,f}(\sigma,\Gamma\cup\{\phi\}) &= f(\{d(\sigma,\psi_1),\ldots,d(\sigma,\psi_n),d(\sigma,\phi)\}) \\ &= f(\{d(\nu,\psi_1),\ldots,d(\nu,\psi_n),d(\mu,\phi)\}) \\ &< f(\{d(\mu,\psi_1),\ldots,d(\mu,\psi_n),d(\mu,\phi)\}) \\ &= \delta_{d,f}(\mu,\Gamma\cup\{\phi\}). \end{aligned}$$

Thus, for every valuation μ such that $\mu(\psi) = \mathsf{f}$ there is a valuation σ such that $\sigma(\psi) = \mathsf{t}$ and $\delta_{d,f}(\sigma,\Gamma\cup\{\phi\}) < \delta_{d,f}(\mu,\Gamma\cup\{\phi\})$. It follows that the elements of $\Delta^2_{d,f}(\Gamma\cup\{\phi\})$ must satisfy ψ, and so $\Gamma, \phi \models^2_{d,f} \psi$. □

Note that Proposition 7 holds, in particular, when Γ is not consistent. If consistency is assumed, a more general result is obtained:

Proposition 8. *Let d be a pseudo-distance and f an aggregation function. If $\Gamma \models^2_{d,f} \psi$ and $\Gamma \cup \{\phi\}$ is consistent, then $\Gamma, \phi \models^2_{d,f} \psi$.*

Proof. If $\Gamma \cup \{\phi\}$ is consistent, then so is Γ. Thus, by Corollary 1, $\Gamma \models^2_{d,f} \psi$ implies that $\Gamma \models^2 \psi$, which implies that $\Gamma, \phi \models^2 \psi$, and so (Corollary 1 again) $\Gamma, \phi \models^2_{d,f} \psi$. □

Corollary 3. *Let d be a pseudo-distance and f an aggregation function. If $\Gamma \models^2_{d,f} \psi$ and $\Gamma \not\models^2 \neg\phi$, then $\Gamma, \phi \models^2_{d,f} \psi$.*

Another useful property of $\models^2_{d,f}$ is known as *adaptivity* [12,13]. This property is concerned with the ability to handle contradictory theories in a non-trivial way and at the same time to presuppose a consistency of all the formulas 'unless and until proven otherwise'. Consequence relations with this property *adapt* to the *specific* inconsistencies that occur in the theories. For instance, the Disjunctive Syllogism should *not* be applied for concluding q from $\{p, \neg p, \neg p \vee q\}$. On the other hand, in the case of $\{p, \neg p, r, \neg r \vee q\}$, applying the Disjunctive Syllogism on r and $\neg r \vee q$ may be justified by the fact that the subset of formulas on which the Disjunctive Syllogism is applied is not affected by the inconsistency of the whole theory, therefore inference rules that are classically valid can be applied to it.

The following proposition shows that $\models^2_{d,f}$ is adaptive when d is unbiased and f is hereditary: if a given theory can be split up to a consistent and an inconsistent parts, then every assertion that is not related to the inconsistent part, and that classically follows from the consistent part, is entailed by the whole theory.

Proposition 9. *Let d be an unbiased pseudo-distance and f a hereditary aggregation function. Suppose that Γ is a theory that can be represented as $\Gamma' \cup \Gamma''$, where Γ' is a classically consistent theory and*

$\mathsf{Atoms}(\Gamma') \cap \mathsf{Atoms}(\Gamma'') = \emptyset$. *Then for every formula* ψ *such that* $\mathsf{Atoms}(\{\psi\}) \cap \mathsf{Atoms}(\Gamma'') = \emptyset$, *it holds that if* $\Gamma' \models^2 \psi$ *then* $\Gamma \models^2_{d,f} \psi$.

Proof. If $\Gamma' \models^2 \psi$, then by Corollary 1, $\Gamma' \models^2_{d,f} \psi$. Now, as $\mathsf{Atoms}(\Gamma' \cup \{\psi\}) \cap \mathsf{Atoms}(\Gamma'') = \emptyset$, we have, by Proposition 7, that $\Gamma \models^2_{d,f} \psi$. $\square$

Note 4. The condition on the aggregation function in Proposition 9 is indeed necessary. To see this consider, for instance, the theory Γ in Example 2. This theory can be partitioned to a consistent subtheory $\Gamma' = \{r, r \wedge s\}$ and an inconsistent subtheory $\Gamma'' = \{p, q, \neg p \vee \neg q\}$. Also, $\mathsf{Atoms}(\Gamma') \cap \mathsf{Atoms}(\Gamma'') = \emptyset$. Yet, although $\Gamma' \models^2 r$, we have that $\Gamma \not\models^2_{d^H,\max} r$, since max is not a hereditary function. This example also shows that the condition in Proposition 7 on the heredity of f is necessary for assuring rationality.

Example 3. By Proposition 9, $\models^2_{d^H,\Sigma}$ is adaptive, while by Note 4, $\models^2_{d^H,\max}$ is not adaptive.

We conclude this section by checking to what extent $\models^2_{d,f}$ may be considered as a consequence relation.

Definition 9. A (Tarskian) *consequence relation* [60] is a relation $\vdash$ between multisets of formulae and formulae, that satisfies the following conditions:

reflexivity: $\Gamma \vdash \psi$ for every $\psi \in \Gamma$.
monotonicity: *if* $\Gamma \vdash \psi$ *and* $\Gamma \subseteq \Gamma'$ *then* $\Gamma' \vdash \psi$.
cut: if $\Gamma_1 \vdash \psi$ and $\Gamma_2, \psi \vdash \phi$ then $\Gamma_1, \Gamma_2 \vdash \phi$.

By what we have shown so far about $\models^2_{d,f}$ it is easy to see that these entailments do not satisfy any property in Definition 9. Indeed, for any d and f, Proposition 3 shows that either $p, \neg p \not\models^2_{d,f} p$ or $p, \neg p \not\models^2_{d,f} \neg p$, and so $\models^2_{d,f}$ is not reflexive, Proposition 6 shows that $\models^2_{d,f}$ is not monotonic, and it is easy to see that the cut rule is violated as well: indeed, if $p, \neg p \not\models^2_{d,f} q$, the cut rule is falsified by the facts that, by Corollary 1, $p \models^2_{d,f} \neg p \to q$ and $\neg p, \neg p \to q \models^2_{d,f} q$; Otherwise, if $p, \neg p \models^2_{d,f} q$, then by Proposition 3, $p, \neg p \not\models^2_{d,f} \neg q$, and this, together with the facts that $p \models^2_{d,f} \neg p \to \neg q$ and $\neg p, \neg p \to \neg q \models^2_{d,f} \neg q$ (Corollary 1 again) provide a counterexample for the cut rule. However,

(1) By Corollary 1, $\models^2$ and $\models^2_{d,f}$ are identical with respect to consistent premises, and the former *is* a Tarskian consequence relation.
(2) Although $\models^2_{d,f}$ is not a consequence relation in the usual sense, it does satisfy the weaker conditions in Definition 10 below, which guarantee a 'proper behaviour' of non-monotonic entailments in the presence of inconsistency.

Notation 1. Denote by $\Gamma = \Gamma' \oplus \Gamma''$ that Γ can be partitioned to two disjoint subtheories Γ' and Γ'' (i.e., $\Gamma = \Gamma' \cup \Gamma''$ and $\mathsf{Atoms}(\Gamma') \cap \mathsf{Atoms}(\Gamma'') = \emptyset$).

Definition 10. A *cautious* consequence relation is a relation $|\!\sim$ between multisets of formulae and formulae, that satisfies the following conditions:

cautious reflexivity: if $\Gamma = \Gamma' \oplus \Gamma''$ and Γ' is consistent, then $\Gamma |\!\sim \psi$ for all $\psi \in \Gamma'$.
cautious monotonicity [34]: if $\Gamma |\!\sim \psi$ and $\Gamma |\!\sim \phi$, then $\Gamma, \psi |\!\sim \phi$.
cautious cut [42]: if $\Gamma |\!\sim \psi$ and $\Gamma, \psi |\!\sim \phi$, then $\Gamma |\!\sim \phi$.

Proposition 10. *For every unbiased pseudo-distance* d *and monotonic hereditary aggregation function* f, $\models^2_{d,f}$ *is a cautious consequence relation.*

Proof. Cautious reflexivity follows from Proposition 9. For cautious monotonicity, let $\Gamma = \{\gamma_1, \ldots, \gamma_n\}$ and suppose that $\Gamma \models^2_{d,f} \psi$, $\Gamma \models^2_{d,f} \phi$, and $\nu \in \Delta^2_{d,f}(\Gamma \cup \{\psi\})$. We show that $\nu \in \Delta^2_{d,f}(\Gamma)$ and since $\Gamma \models^2_{d,f} \phi$ this implies that $\nu \in \mathsf{mod}(\{\phi\})$. Indeed, if $\nu \notin \Delta^2_{d,f}(\Gamma)$, there is a valuation $\mu \in \Delta^2_{d,f}(\Gamma)$ so that $\delta_{d,f}(\mu, \Gamma) < \delta_{d,f}(\nu, \Gamma)$, i.e., $f(\{d(\mu, \gamma_1), \ldots, d(\mu, \gamma_n)\}) < f(\{d(\nu, \gamma_1), \ldots, d(\nu, \gamma_n)\})$. Also, as $\Gamma \models^2_{d,f} \psi$, $\mu \in \mathsf{mod}(\{\psi\})$, thus $d(\mu, \psi) = 0$. By these facts, then,

$$\begin{aligned}\delta_{d,f}(\mu,\Gamma\cup\{\psi\}) &= f(\{d(\mu,\gamma_1),\ldots,d(\mu,\gamma_n),0\})\\ &< f(\{d(\nu,\gamma_1),\ldots,d(\nu,\gamma_n),0\})\\ &\leqslant f(\{d(\nu,\gamma_1),\ldots,d(\nu,\gamma_n),d(\nu,\psi)\}) = \delta_{d,f}(\nu,\Gamma\cup\{\psi\}),\end{aligned}$$

a contradiction to $\nu \in \Delta^2_{d,f}(\Gamma\cup\{\psi\})$.

For cautious cut, let again $\Gamma=\{\gamma_1,\ldots,\gamma_n\}$ and suppose that $\Gamma\models^2_{d,f}\psi, \Gamma,\psi\models^2_{d,f}\phi$, and $\nu\in\Delta^2_{d,f}(\Gamma)$. We have to show that $\nu\in \mathrm{mod}(\{\phi\})$. Indeed, since $\nu\in\Delta^2_{d,f}(\Gamma)$, we have that for every $\mu\in\Lambda^2$, $f(\{d(\nu,\gamma_1),\ldots,d(\nu,\gamma_n)\})\leqslant f(\{d(\mu,\gamma_1),\ldots,d(\mu,\gamma_n)\})$. Moreover, since $\Gamma\models^2_{d,f}\psi$, $\nu\in\mathrm{mod}(\{\psi\})$, and so $d(\nu,\psi)=0\leqslant d(\mu,\psi)$. It follows, then, that for every $\mu\in\Lambda^2$,

$$\begin{aligned}\delta_{d,f}(\nu,\Gamma\cup\{\psi\}) &= f(\{d(\nu,\gamma_1),\ldots,d(\nu,\gamma_n),d(\nu,\psi)\})\\ &\leqslant f(\{d(\mu,\gamma_1),\ldots,d(\mu,\gamma_n),d(\nu,\psi)\})\\ &\leqslant f(\{d(\mu,\gamma_1),\ldots,d(\mu,\gamma_n),d(\mu,\psi)\}) = \delta_{d,f}(\mu,\Gamma\cup\{\psi\}).\end{aligned}$$

Thus, $\nu\in\Delta^2_{d,f}(\Gamma\cup\{\psi\})$, and since $\Gamma,\psi|\sim\phi$, necessarily $\nu\in\mathrm{mod}(\{\phi\})$. □

4. Applications

The general form of the distance-based reasoning considered in the previous sections allows us to apply it in several areas. Below, we demonstrate this in the context of three basic operations in information systems: repair (Section 4.1), revision (Section 4.2) and merging (Section 4.3).

4.1. Database repair

Definition 11. A *database* $\mathscr{DB}$ is a pair $(\mathfrak{D},\mathscr{C})$, where $\mathfrak{D}$ (the *database instance*) is a finite set of atoms, and $\mathscr{C}$ (the *integrity constraints*) is a finite and consistent set of formulas in $\mathscr{L}$.

The meaning of $\mathfrak{D}$ is usually determined by the conjunction of its facts, augmented with Reiter's closed world assumption [55], stating that each atomic formula that does not appear in $\mathfrak{D}$ is false: $\mathsf{CWA}(\mathfrak{D})=\{\neg p | p\notin\mathfrak{D}\}$. A database $\mathscr{DB}=(\mathfrak{D},\mathscr{C})$ is thus associated with the following theory:

$$\Gamma_{\mathscr{DB}} = \mathfrak{D}\cup\mathsf{CWA}(\mathfrak{D})\cup\mathscr{C}.^{10}$$

A database $(\mathfrak{D},\mathscr{C})$ is *consistent* if all the integrity constraints are satisfied by the database instance, that is: $\mathfrak{D}\cup\mathsf{CWA}(\mathfrak{D})\models^2\psi$ for every $\psi =\in\mathscr{C}$. When a database is not consistent, at least one integrity constraint is violated, and so it is usually required to 'repair' the database, i.e., restore its consistency. Clearly, the repaired database instance should be consistent and at the same time as close as possible to $\mathfrak{D}$. This can be formally described in our framework as follows: given a pseudo-distance d and an aggregation function f, we consider for every database $\mathscr{DB}$ the following set of (most plausible) interpretations:

$$\Delta^2_{d,f}(\Gamma_{\mathscr{DB}}) = \{\nu\in\mathrm{mod}(\mathscr{C}) | \forall\mu\in\mathrm{mod}(\mathscr{C})\ \delta_{d,f}(\nu,\mathfrak{D}\cup\mathsf{CWA}(\mathfrak{D}))\leqslant\delta_{d,f}(\mu,\mathfrak{D}\cup\mathsf{CWA}(\mathfrak{D}))\}.$$

This definition is an obvious reproduction of Definition 4, where Λ^2 is replaced by $\mathrm{mod}(\mathscr{C})$. The requirement that the most plausible interpretations of $\Gamma_{\mathscr{DB}}$ should satisfy $\mathscr{C}$ reflects the superior position of the integrity constraints over the facts in $\mathfrak{D}$. Again, we denote by $\mathscr{DB}\models^2_{d,f}\psi$ that $\Delta^2_{d,f}(\Gamma_{\mathscr{DB}})\subseteq\mathrm{mod}(\{\psi\})$.

The definition above of $\models^2_{d,f}$ is a conservative extension of the usual notion of database entailment, used for defining query answering. Indeed, as it is easily verified, if a database $\mathscr{DB}$ is consistent, then $\Delta^2_{d,f}(\Gamma_{\mathscr{DB}})=\mathrm{mod}(\Gamma_{\mathscr{DB}})$, so in this case $\mathscr{DB}\models^2\psi$ iff $\mathscr{DB}\models^2_{d,f}\psi$.

Now we can also represent the concept of consistent query answering [3,4,21] in our framework:

[10] In case that $\mathscr{L}$ is a first-order language it is usual to add to $\Gamma_{\mathscr{DB}}$ also the unique name axioms and the domain closure axioms (see [1]).

Definition 12. Let $\mathscr{DB}$ be a (possibly inconsistent) database, and let ψ be a formula in $\mathscr{L}$.

- ψ *credulously follows* from $\mathscr{DB}$ if $\Delta^2_{d,f}(\Gamma_{\mathscr{DB}}) \cap mod(\{\psi\}) \neq \emptyset$ (i.e., ψ is satisfied by *some* most plausible interpretation of $\Gamma_{\mathscr{DB}}$).
- ψ *conservatively follows* from $\mathscr{DB}$ if $\mathscr{DB} \models^2_{d,f} \psi$ (i.e., ψ is satisfied by *all* the most plausible interpretations of $\Gamma_{\mathscr{DB}}$).

Example 4. Let $\mathfrak{D} = \{p, r\}$ and $\mathscr{C} = \{p \to q\}$. Here, $\Gamma_{\mathscr{DB}} = \{p, r, \neg q, p \to q\}$. When d is the drastic distance and f is the summation function, $\Delta^2_{d,f}(\Gamma_{\mathscr{DB}}) = \{v_1, v_2\}$, where $v_1(p) = \mathsf{t}$, $v_1(q) = \mathsf{t}$, $v_1(r) = \mathsf{t}$, $v_2(p) = \mathsf{f}$, $v_2(q) = \mathsf{f}$, $v_2(r) = \mathsf{t}$.
In terms of distance entailments, then, $\Gamma_{\mathscr{DB}} \models^2_{d^U,\Sigma} r$ but $\Gamma_{\mathscr{DB}} \not\models^2_{d^U,\Sigma} p$ and $\Gamma_{\mathscr{DB}} \not\models^2_{d^U,\Sigma} \neg q$.

This can be justified as follows: there are two optimal ways (in terms of minimal amount of changes in $\mathfrak{D}$) of restoring the consistency of $\mathscr{DB}$: one 'repair' is obtained by removing p from the database instance $\mathfrak{D}$, and the other one is obtained by inserting q to $\mathfrak{D}$. These repairs support, respectively, the fact that $\Gamma_{\mathscr{DB}}$ does not entail p nor $\neg q$ (although $p, \neg q \in \Gamma_{\mathscr{DB}}$). Note, also, that in both cases r remains in the database instance. Indeed, there is no reason to remove r from $\mathfrak{D}$, as this will not contribute to the consistency restoration of $\mathscr{DB}$. This intuitively justifies the fact that for r we do have that $\Gamma_{\mathscr{DB}} \models^2_{d^U,\Sigma} r$.

It follows, then, that r conservatively (and so credulously) follows from $\mathscr{DB}$, while p, q, and their complements, credulously (but not conservatively) follow from $\mathscr{DB}$. The same results are obtained by the formalisms for querying inconsistent databases, considered e.g. in [3,4,9,10,19,31,35].

4.2. Belief revision

A belief revision theory describes how a belief state is obtained by revising a belief state $\mathscr{B}$ by some new information, ψ (which is not a contradiction). The new belief state, denoted $\mathscr{B} \circ \psi$, is usually characterized by the 'closest' worlds to $\mathscr{B}$ in which ψ holds. Clearly, this principle of minimal change is derived by distance considerations, so it is not surprising that it can be expressed in our framework. Indeed, if we assume that a belief state is represented by a set of formulas, then given a pseudo-distance d and an aggregation function f, the most plausible representations of the new belief state may be defined as follows:

$$\Delta^2_{d,f}(\mathscr{B} \circ \psi) = \{v \in \mathrm{mod}(\psi) | \forall \mu \in \mathrm{mod}(\psi)\ \delta_{d,f}(v, \mathscr{B}) \leqslant \delta_{d,f}(\mu, \mathscr{B})\}. \tag{1}$$

The revised conclusions of the reasoner may now be represented, again, by a distance entailment:

$$\mathscr{B} \circ \psi \models^2_{d,f} \phi \text{ iff } \Delta^2_{d,f}(\mathscr{B} \circ \psi) \subseteq \mathrm{mod}(\{\phi\}).$$

Example 5. The revision operator $\Delta^2_{d^{H_2},\Sigma}$ is the same as the one considered by Dalal in [24]. It is well-known that this operator satisfies the AGM postulates [2] of belief revision.

Below we consider some basic postulates of the operator defined in (1).[11]

Definition 13. Denote by $\Gamma_1 \equiv^2 \Gamma_2$ that for every ψ, $\Gamma_1 \models^2 \psi$ iff $\Gamma_2 \models^2 \psi$. Similarly, $\Gamma_1 \equiv^2_{d,f} \Gamma_2$ denotes that for every ψ, $\Gamma_1 \models^2_{d,f} \psi$ iff $\Gamma_2 \models^2_{d,f} \psi$.

Proposition 11. *Let d be an unbiased pseudo-distance, f an aggregation function, $\mathscr{B}$ a belief state, and ψ a non-contradictory formula in $\mathscr{L}$. Then $\circ$ satisfies the following postulates*:

[Succ] $\mathscr{B} \circ \psi \models^2_{d,f} \psi$.

[Cons] $\mathscr{B} \circ \psi$ is consistent.

[Opt] if $\mathscr{B} \cup \{\psi\}$ is consistent, then $\mathscr{B} \circ \psi \equiv^2_{d,f} \mathscr{B} \cup \{\psi\}$.

[IS] if $\psi \equiv^2 \psi'$ then $\mathscr{B} \circ \psi \equiv^2_{d,f} \mathscr{B} \circ \psi'$.

[11] A full analysis of our approach from a belief revision point of view is outside the scope of this paper. In particular, we do not intend to give here an exhaustive list of properties for $\circ$, but just to mention some of the better known, generally accepted postulates that it satisfies. For a more detailed analysis see [41, Section 5].

The four postulates above, denoted (IC0)–(IC3) in [41], are also the basic postulates considered in [27] as the "four basic, undebatable properties of [flat] merging". [Succ] means that the revision process succeeds and the last piece of information is always believed; [Cons] assures that a belief state is always consistent; [Opt] states that observations are all accepted if they are jointly consistent, and [IS] is the principle of irrelevancy of syntax.

Proof of Proposition 11. [Succ] holds since $\Delta^2_{d,f}(\mathscr{B} \circ \psi)$ is a subset of $\mathrm{mod}(\{\psi\})$. [Cons] follows from the facts that $\mathrm{mod}(\{\psi\}) \neq \emptyset$ and Λ^2 is finite (because Atoms is finite), thus $\emptyset \neq \Delta^2_{d,f}(\mathscr{B} \circ \psi) \subseteq mod(\{\psi\})$. [Opt] follows from the fact that if $\mathscr{B} \cup \{\psi\}$ is consistent, then $\Delta^2_{d,f}(\mathscr{B} \circ \psi) = \mathrm{mod}(\mathscr{B} \cup \{\psi\})$ (cf. Proposition 1). Finally, [IS] holds by the fact that if $\psi \equiv {}^2\psi'$ then $\mathrm{mod}(\{\psi\}) = \mathrm{mod}(\{\psi'\})$, thus $\Delta^2_{d,f}(\mathscr{B} \circ \psi) = \Delta^2_{d,f}(\mathscr{B} \circ \psi')$. □

Note 5. The principle of syntax independence ([IS]) cannot be tailored to equivalent belief bases. That is, if $\mathscr{B} \equiv^2 \mathscr{B}'$ and $\psi \equiv {}^2\psi'$, it is *not necessarily true* that $\mathscr{B} \circ \psi \equiv^2_{d,f} \mathscr{B}' \circ \psi'$. To see this, let $\mathscr{B} = \{p, q\}$ and $\mathscr{B}' = \{p, \neg p \vee q\}$. Clearly, $\mathscr{B} \equiv^2 \mathscr{B}'$, but $\mathscr{B} \circ \{\neg q\} \not\equiv^2_{d^{H_2},\Sigma} \mathscr{B}' \circ \{\neg q\}$. Indeed, let $v_1(p) = \mathsf{t}$, $v_1(q) = \mathsf{f}$, and $v_2(p) = v_2(q) = \mathsf{f}$. Then $\Delta^2_{d^{H_2},\Sigma}(\mathscr{B} \circ \neg q) = \{v_1\}$ and $\Delta^2_{d^{H_2},\Sigma}(\mathscr{B}' \circ \neg q) = \{v_1, v_2\}$. Thus, for instance, $\mathscr{B} \circ \neg q \models^2_{d^{H_2},\Sigma} p$, while $\mathscr{B}' \circ \neg q \not\models^2_{d^{H_2},\Sigma} p$. This may be intuitively justified by the adaptive character of $\models^2_{d^{H_2},\Sigma}$ (see Section 3). Indeed, in $\mathscr{B}'$, p is connected to q, and the latter becomes unreliable in light of the new data, $\neg q$. Thus, no reliable information about p can be extracted from $\mathscr{B}' \circ \neg q$. In $\mathscr{B}$, on the other hand, the information about p is *not related* to q, so the revision by $\neg q$ does not involve p.

4.3. Information integration

Integration of autonomous data-sources under global integrity constraints (see [41]) is also applicable in our framework. Given n independent data-sources $\Gamma_1, \ldots, \Gamma_n$ and a consistent set of global integrity constraints $\mathscr{C}$, the sources should be merged to a theory Γ that reflects the collective information of the local sources in a coherent way (that is, $\Gamma \models^2 \psi$ for every $\psi \in \mathscr{C}$). Clearly, the union of the distributed information might not preserve $\mathscr{C}$, and in such cases the intuitive idea is to minimize the overall distance between Γ and Γ_i $(1 \leqslant i \leqslant n)$. This can be done by the following straightforward extension of Definition 4:

Definition 14. Let $\overline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_n\}$ be a set of n finite theories in $\mathscr{L}$, d a pseudo-distance function, and f,g two aggregation functions. For an interpretation v and a theory Γ, let $\delta_{d,f}(v, \Gamma)$ be the same function as in Definition 3. Now, define:

$$\delta_{d,f,g}(v, \overline{\Gamma}) = g(\{\delta_{d,f}(v, \Gamma_1), \ldots, \delta_{d,f}(v, \Gamma_n)\}).$$

The *most plausible valuations* (with respect to d, f, g) of the integration of the elements in $\overline{\Gamma}$ under the constraints in $\mathscr{C}$, are the elements of the following set:

$$\Delta^2_{d,f,g}(\overline{\Gamma}, \mathscr{C}) = \{v \in \mathrm{mod}(\mathscr{C}) | \forall \mu \in \mathrm{mod}(\mathscr{C})\ \delta_{d,f,g}(v, \overline{\Gamma}) \leqslant \delta_{d,f,g}(\mu, \overline{\Gamma})\}.$$

Information integration is now definable as a direct extension of Definition 5:

Definition 15. $\overline{\Gamma}, \mathscr{C} \models^2_{d,f,g} \psi$ iff $\Delta^2_{d,f,g}(\overline{\Gamma}, \mathscr{C}) \subseteq \mathrm{mod}(\{\psi\})$.

Example 6 [41]. Four flat co-owners discuss the construction of a swimming pool (s), a tennis-court (t) and a private car-park (p). It is also known that any investment in two or more items will increase the rent (r), otherwise the rent will not be changed. The opinions of the owners are represented by the following data-sources: $\Gamma_1 = \Gamma_2 = \{s, t, p\}$, $\Gamma_3 = \{\neg s, \neg t, \neg p, \neg r\}$, and $\Gamma_4 = \{t, p, \neg r\}$;[12] The impact on the rent may be represented by the integrity constraint $\mathscr{C} = \{r \leftrightarrow ((s \wedge t) \vee (s \wedge p) \vee (t \wedge p))\}$.[13]

[12] Here, $q \in \Gamma_i$ (respectively, $\neg q \in \Gamma_i$) denotes that owner i supports (respectively, is against) q.

[13] Note that although the opinion of owner 4 violates the integrity constraint (while the solution must preserve the constraint), it is still taken into account.

	s	t	p	r	$\delta_{d^U,\Sigma,\Sigma}(\cdot,\overline{\Gamma})$	$\delta_{d^U,\Sigma,\max}(\cdot,\overline{\Gamma})$
ν_1	t	t	t	t	**5**	4
ν_2	t	t	f	t	7	3
ν_3	t	f	t	t	7	3
ν_4	t	f	f	f	7	**2**
ν_5	f	t	t	t	7	3
ν_6	f	t	f	f	6	**2**
ν_7	f	f	t	f	6	**2**
ν_8	f	f	f	f	8	3

Fig. 2. The models of $\mathscr{C}$ and their distances to $\overline{\Gamma}$ (Example 6).

Consider now two merging strategies: in both of them $d = d^U$ and $f = \Sigma$, but one minimizes the summation of the distances to the data-sources (i.e., $g = \Sigma$), while the other minimizes the maximal distance to the sources (that is, $g = \max$). The models of $\mathscr{C}$ and their distances to $\overline{\Gamma} = \{\Gamma_1, \ldots, \Gamma_4\}$ according to both strategies are given in Fig. 2.

The most plausible interpretations in each merging context are determined by the minimal values in the two right-most columns. It follows that according to the first context v_1 is the (unique) most-plausible interpretation for the merging, thus $\overline{\Gamma}, \mathscr{C} \models^2_{d^U,\Sigma,\Sigma} s \wedge t \wedge p$, and so the owners decide to build all the three facilities (and the rent increases). In the other context we have three optimal interpretations, as $\Delta^2_{d^U,\Sigma,\max}(\overline{\Gamma}, \mathscr{C}) = \{v_4, v_6, v_7\}$. This implies that only one out of the three facilities will be built, and so the rent will remain the same.[14]

The choice of the merging parameters (d,f,g) is frequently determined by the nature of the mediator system. For instance, $\delta_{d^U,\Sigma,\Sigma}$ poses *merging by majority* [47]. This means, intuitively, that if a formula follows from a sufficiently large amount of sources in $\overline{\Gamma}$, it will also follow from $\overline{\Gamma}$.[15] This is demonstrated in Example 6, where the flat owner represented by Γ_3 has to accept the majority wish, although it is opposed to his or her own opinion. In contrast, $\delta_{d^U,\Sigma,\max}$ is useful in situations where the sources represent competitive parties, and agreement among all the members is necessary for the merging. See [40,41] for detailed discussions on operators for merging constraint belief-bases and corresponding complexity results. Some interesting properties of aggregation-based merging operators in the context of social choice theory are discussed in [51].

5. Extensions to multiple-valued semantics

Our framework can be extended in a natural way to multiple-valued semantics, so that multiple-valued logics are incorporated and new distance functions are introduced. For this, we first consider a general setting of multiple-valued structures and the entailment relations that they induce (Section 5.1), then we consider distance functions for the multiple-valued setting (Section 5.2). This allows us to define a family of distance-based entailments (Section 5.3), which are a conservative extension of the distance-based entailments considered above for the two-valued semantics. It is also shown that many properties in the two-valued setting are carried on to the multiple-valued setting.

5.1. Basic multiple-valued entailments

We continue to denote by $\mathscr{L}$ a propositional language with a finite set Atoms of atomic formulas.

[14] The decision which facility to choose involves further preference criteria. Summation of distances, for instance, prefers v_6 and v_7 over v_4, hence according to this criterion t and p are preferred over s.

[15] Formally, denote by Γ^n a multiset that consists of n copies of Γ. Then majority vote means that if $\Gamma_0 \models^2_{d,f,g} \psi$, then for every $\overline{\Gamma}$ there is an n such that $\overline{\Gamma}, \Gamma_0^n \models^2_{d,f,g} \psi$.

Definition 16. A *multiple-valued structure* for $\mathscr{L}$ is a triple $\mathscr{S} = \langle \mathscr{V}, \mathscr{O}, \mathscr{D} \rangle$, where $\mathscr{V}$ is set of elements (truth values), $\mathscr{O}$ is a set of operations on $\mathscr{V}$ that correspond[16] to the connectives of $\mathscr{L}$, and $\mathscr{D}$ is a nonempty proper subset of $\mathscr{V}$.

The set $\mathscr{D}$ consists of the *designated* values of $\mathscr{V}$, i.e., those that represent true assertions. In what follows we shall assume that $\mathscr{V}$ contains at least the classical values t, f, and that $\mathsf{t} \in \mathscr{D}$, $\mathsf{f} \notin \mathscr{D}$.

Definition 17. Let $\mathscr{S} = \langle \mathscr{V}, \mathscr{O}, \mathscr{D} \rangle$ be a multiple-valued structure for $\mathscr{L}$.

(a) A (multiple-valued) *valuation* v is a function that assigns an element of $\mathscr{V}$ to each atomic formula in $\mathscr{L}$. Extensions to complex formulas are defined as usual. In what follows we shall sometimes write $v = \{p_1\colon x_1, \ldots, p_n\colon x_n\}$ to denote that $v(p_i) = x_i$ for $i = 1, \ldots, n$. The set of valuations on $\mathscr{V}$ is denoted by $\Lambda^{\mathscr{V}}$.
(b) A valuation v *satisfies* a formula ψ if $v(\psi) \in \mathscr{D}$.
(c) A valuation v is a *model* of a multiset Γ of formulas in $\mathscr{L}$, if v satisfies every formula in Γ. The set of the models of Γ in $\mathscr{S}$ is denoted by $\mathrm{mod}^{\mathscr{S}}(\Gamma)$.

Definition 18. Let $\mathscr{S} = \langle \mathscr{V}, \mathscr{O}, \mathscr{D} \rangle$ be a multiple-valued structure for a language $\mathscr{L}$. A *basic $\mathscr{S}$-entailment* is a relation $\models^{\mathscr{S}}$ between multisets of formulas in $\mathscr{L}$ and formulas in $\mathscr{L}$, defined by $\Gamma \models^{\mathscr{S}} \psi$ if every model of Γ satisfies ψ.

Example 7. In many cases there is a 'natural' lattice ordering of the truth values in $\mathscr{V}$, and so it is usual to include in $\mathscr{O}$ (at least) the basic lattice operations. In such cases, a conjunction in $\mathscr{L}$ is associated with the meet, the disjunction corresponds to the join, and the negation operation is defined according to a negation on the lattice (which is usually an order reversing involution). In what follows, we shall use these definitions for the operators in $\mathscr{O}$. Now, the two-valued structure TWO is defined by the two-valued lattice, and is obtained by taking $\mathscr{V} = \{\mathsf{t}, \mathsf{f}\}$ and $\mathscr{D} = \{\mathsf{t}\}$. The corresponding entailment was denoted above by $\models^2$. For three-valued structures we take $\mathscr{V} = \{\mathsf{t}, \mathsf{f}, \mathsf{m}\}$, the lattice operators in $\mathscr{O}$ are defined with respect to the total order in which m is the middle element, that is: $\mathsf{f} < \mathsf{m} < \mathsf{t}$, and $\mathscr{D}$ is either $\{\mathsf{t}\}$ or $\{\mathsf{t},\mathsf{m}\}$. The structure with $\mathscr{D} = \{\mathsf{t}\}$ is denoted here by $\mathsf{THREE}_\perp$. The associated entailment, $\models^{3_\perp}$, corresponds to Kleene's three-valued logic [39]. The other three-valued structure, $\mathsf{THREE}_\top$, corresponds to Priest's logic LP [52,53].[17] We denote its entailment by $\models^{3_\top}$. Note that by different choices of the operators in $\mathscr{O}$ other three-valued logics are obtained, like weak Kleene logic, strong Kleene logic, and Łukasiewicz's logic (see, e.g., [11,33]). In the four-valued case there are usually two middle elements, denoted here by b (both) and n (neither).[18] In this context it is usual to take t and b as the designated values. The corresponding structure is known as Belnap's bilattice (see [15,16] as well as [7]), and it is denoted here by FOUR. The basic entailment of FOUR is denoted by $\models^4$. Entailments in which $\mathscr{V}$ is the unit interval and $\mathscr{D} = \{1\}$ are common in the context of fuzzy logic (see, e.g., [38]). In this context it is usual to consider different kinds of operations on the unit interval (T-norms, T-conorms, residual implications, etc.), and this is naturally supported in our framework as well. The simplest case is obtained by associating $\wedge$ and $\vee$ with the meet and the join operators on the unit interval, which in this case are the same as the minimum and the maximum functions (respectively), and relating negation to the involutive operator $\neg$, defined for every $0 \leqslant x \leqslant 1$ by $\neg x = 1 - x$.

5.2. Distance functions

The shift to multiple-valued semantics opens the door to many new opportunities for defining the distance functions under consideration. While in the multiple-valued setting it is possible to apply the same distance functions as in the two-valued case, e.g., those that are given in Example 1, it is also possible to introduce

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16 An n-ary operator $\widetilde{\diamond}$ on $\mathscr{V}^n$ corresponds to an n-ary connective $\diamond$ of $\mathscr{L}$, if for every $v \in \Lambda^{\mathscr{V}}$ and every n formulas $\psi_1, \ldots, \psi_n$ in $\mathscr{L}$, $v(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}(v(\psi_1), \ldots, v(\psi_n))$.

17 Also known as J_3, RM_3, and PAC (see [11,28,56] and chapter IX of [32]).

18 The notations of these elements reflect their intuitive meanings as representing conflicts (both true and false) and partial information (neither true nor false).

generalized versions of the distance functions, capturing the multiple-valued nature of the underlying semantics. For instance, for three-valued logics, such as Kleene's and Priest's logics considered above, it is possible to extend the Hamming distance so that the distance between the extreme elements t and f will be strictly bigger than the distances between each one of them and the middle element. In this case, t is associated with the value 1, f is associated with 0, and the middle element m corresponds to $\frac{1}{2}$. The generalized Hamming distance is then defined as follows:

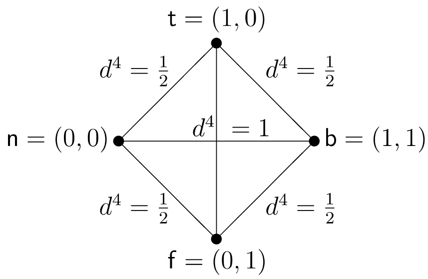
$$d^{H_3}(\nu,\mu) = \sum_{p \in \mathsf{Atoms}} |\nu(p) - \mu(p)|.$$

This function is used, e.g., in [25] as part of the semantics behind its three-valued database integration systems.

For four-valued interpretations there is also a natural generalization of the Hamming distance. The idea here is that each one of the four truth values is associated with a pair of two-valued components as follows: $\mathsf{t} = (1,0)$, $\mathsf{f} = (0,1)$, $\mathsf{n} = (0,0)$, $\mathsf{b} = (1,1)$. This pairwise representation preserves Belnap's original four-valued structure (see [5,6,8]), and so it is a valid rewriting of the truth values. Now, the distance between two values $x = (x_1, x_2)$ and $y = (y_1, y_2)$ in this pairwise representation is given by

$$d^4(x,y) = \frac{|x_1 - y_1| + |x_2 - y_2|}{2},$$

so the graphic representation of d^4 on the four-valued structure is the following:



Now, the generalized Hamming distance between two four-valued interpretations ν, μ is defined by

$$d^{H_4}(\nu,\mu) = \sum_{p \in \mathsf{Atoms}} d^4(\nu(p),\mu(p)).$$

Clearly, this definition may be applied on any lattice whose elements have a pairwise representation (see [5,6]).

It is not difficult to verify that all the functions defined above satisfy the conditions in Definition 1. Note also that, given two interpretations ν, μ into $\{\mathsf{t},\mathsf{f}\}$, it holds that $d^{H_4}(\nu,\mu) = d^{H_3}(\nu,\mu) = d^{H_2}(\nu,\mu)$, thus d^{H_3} and d^{H_4} are indeed generalizations of the standard Hamming distance.

5.3. Distance-based entailments for multiple-valued semantics

By their definition, basic $\mathscr{S}$-entailments are monotonic. In addition, some of them are trivial in the presence of contradictions (e.g., $p, \neg p \models^2 q$ and $p, \neg p \models^{3_\perp} q$), or exclude classically valid rules (e.g., $p, \neg p \vee q \not\models^{3_\top} q$ and $p, \neg p \vee q \not\models^4 q$). Common-sense reasoning and human thinking, on the other hand, is frequently non-monotonic and tolerant to inconsistency. For assuring such properties we consider in what follows distance-based derivatives of the basic entailments. We do so in a way that is completely analogous to our approach in the two-valued case (cf. Section 2):

Definition 19. Given a multiple-valued structure $\mathscr{S} = \langle \mathscr{V}, \mathscr{O}, \mathscr{D} \rangle$, a distance function d on the set $\Lambda^{\mathscr{V}}$ of $\mathscr{V}$-valued valuations, and an aggregation function f, the *most plausible* $\mathscr{S}$-valuations of a theory Γ (with respect to d and f) are the elements of the set

$$\Delta^{\mathscr{S}}_{d,f}(\Gamma) = \{\nu \in \Lambda^{\mathscr{V}} \mid \forall \mu \in \Lambda^{\mathscr{V}}\ \delta_{d,f}(\nu,\Gamma) \leqslant \delta_{d,f}(\mu,\Gamma)\}.$$

Definition 20. For a multiple-valued structure $\mathscr{S} = \langle \mathscr{V}, \mathscr{O}, \mathscr{D} \rangle$, a distance function d and an aggregation function f, let $\Gamma \models^{\mathscr{S}}_{d,f} \psi$ if $\Delta^{\mathscr{S}}_{d,f}(\Gamma) \subseteq \text{mod}^{\mathscr{S}}(\psi)$. That is, the most plausible $\mathscr{S}$-valuations of the premises satisfy the conclusion.

Example 8. Consider the three-valued structure $\mathscr{S} = \mathsf{THREE}_\perp$ in which the middle element is not designated, and let $f = \Sigma$. Consider now two different distance functions for this setting: the standard Hamming distance d^{H_2} and its three-valued extension d^{H_3}. The most plausible interpretations of $\Gamma = \{p, \neg p\}$ are different with respect to these two contexts. Indeed,

$$\Delta^{\mathsf{THREE}_\perp}_{d^{H_2},\Sigma}(\Gamma) = \{\{p : \mathsf{t}\}, \{p : \mathsf{f}\}\},$$
$$\Delta^{\mathsf{THREE}_\perp}_{d^{H_3},\Sigma}(\Gamma) = \{\{p : \mathsf{t}\}, \{p : \mathsf{f}\}, \{p : \mathsf{m}\}\}.$$

Thus, for instance, $\Gamma \models^{3_\perp}_{d^{H_2},\Sigma} p \vee \neg p$, while $\Gamma \not\models^{3_\perp}_{d^{H_3},\Sigma} p \vee \neg p$.[19]

Next we examine some of the properties of $\models^{\mathscr{S}}_{d,f}$. As it turns out, most of the properties of the two-valued case considered in Section 3 hold also in arbitrary multiple-valued semantics. In what follows, unless otherwise stated, we fix some multiple-valued structure $\mathscr{S} = \langle \mathscr{V}, \mathscr{O}, \mathscr{D} \rangle$, a pseudo-distance d on $\Lambda^{\mathscr{V}}$, and an aggregation function f.

Proposition 12. *Let Γ be a theory in $\mathscr{L}$ such that $mod^{\mathscr{S}}(\Gamma) \neq \emptyset$. Then $\Delta^{\mathscr{S}}_{d,f}(\Gamma) = mod^{\mathscr{S}}(\Gamma)$.*

Proof. The same as the proof of Proposition 1, since that proof relies only on the fact that d is a pseudo-distance and f is an aggregation function. □

As in the two-valued case, Proposition 12 implies the following result about the relation between basic multiple-valued entailments and distance-based ones:

Corollary 4. *Let Γ be a theory in $\mathscr{L}$ such that $mod^{\mathscr{S}}(\Gamma) \neq \emptyset$. Then for every formula ψ in $\mathscr{L}$, $\Gamma \models^{\mathscr{S}} \psi$ iff $\Gamma \models^{\mathscr{S}}_{d,f} \psi$.*

Note that Corollary 1 immediately follows from Corollary 4 (when $\mathscr{S} = \mathsf{TWO}$), since every classically consistent theory has a model. Another straightforward consequence of Corollary 4 is the following:

Corollary 5. *Let $\mathscr{L}$ be the standard propositional language, based on the connectives $\{\neg, \vee, \wedge\}$, and let $\mathscr{S} = \langle \mathscr{V}, \mathscr{O}, \mathscr{D} \rangle$ be a multiple-valued structure for $\mathscr{L}$ in which and there is an element $x \in \mathscr{V}$ such that both x and $\neg x$ are in $\mathscr{D}$. Then for every multiset Γ of formulas in $\mathscr{L}$ and every formula ψ in $\mathscr{L}$, $\Gamma \models^{\mathscr{S}} \psi$ iff $\Gamma \models^{\mathscr{S}}_{d,f} \psi$.*

Proof. By induction on the structure of the formulas in $\mathscr{L}$ it is easy to verify that if $v(p) = x$ for every atom $p \in \mathsf{Atoms}(\{\psi\})$, then $v(\psi) = x$ as well. Thus, an $\mathscr{S}$-valuation that assigns x to every atom is a model of every theory in $\mathscr{L}$. The corollary thus follows from Corollary 4. □

Example 9. Let $\mathscr{L}$ be the standard propositional language. Then, for every multiset Γ of formulas in $\mathscr{L}$ and every formula ψ in $\mathscr{L}$,

(a) $\Gamma \models^{3_\top} \psi$ iff $\Gamma \models^{3_\top}_{d,f} \psi$,
(b) $\Gamma \models^{4} \psi$ iff $\Gamma \models^{4}_{d,f} \psi$.

Indeed, both claims follow from the last corollary, where x is the designated middle element of the relevant structure ($\mathscr{S} = \mathsf{THREE}_\top$ in the first case and $\mathscr{S} = \mathsf{FOUR}$ in the second case).

Non-trivial reasoning is another property that is preserved in the multiple-valued case:

Proposition 13. *Let $\mathscr{S} = \langle \mathscr{V}, \mathscr{O}, \mathscr{D} \rangle$ be a multiple-valued structure in which $\mathscr{V}$ is finite. Then, for every pseudo-distance d on $\Lambda^{\mathscr{V}}$ and for every aggregation function f, the distance-based entailment $\models^{\mathscr{S}}_{d,f}$ is paraconsistent.*

[19] This is so since $v(p \vee \neg p) = \mathsf{m}$ when $v(p) = \mathsf{m}$, and in $\mathsf{THREE}_\perp$ the middle element is not designated.

Proof. Similar to the proof of Proposition 2, based on the fact that as $\mathcal{V}$ is finite, so is $\Lambda^{\mathcal{V}}$, thus every theory Γ has a non-empty set of most plausible $\mathcal{S}$-valuations. It follows, then, that for every formula ψ such that there exists a valuation $v \in \Delta^{\mathcal{S}}_{d,f}(\Gamma)$ for which $v(\psi) \notin \mathcal{D}$, we have that $\Gamma \not\models^{\mathcal{S}}_{d,f} \psi$. $\square$

Proposition 14. *Let d be an unbiased pseudo-distance on $\Lambda^{\mathcal{V}}$ and f an aggregation function. Suppose that Γ is a theory in $\mathcal{L}$ and ψ is a non-tautological formula in $\mathcal{L}$ such that* $\mathsf{Atoms}(\Gamma) \cap \mathsf{Atoms}(\{\psi\}) = \emptyset$. *Then* $\Gamma \not\models^{\mathcal{S}}_{d,f} \psi$.

Proof. Similar to that of Proposition 5. $\square$

It is easy to verify that several other properties of $\models^2_{d,f}$ are satisfied in the general case as well. Two such properties are rationality and adaptivity under unbiased distances and hereditary functions:

Proposition 15. *Let d be an unbiased pseudo-distance and f a hereditary aggregation function. If* $\Gamma \models^{\mathcal{S}}_{d,f} \psi$ *then* $\Gamma, \phi \models^{\mathcal{S}}_{d,f} \psi$ *for every formula ϕ such that* $\mathsf{Atoms}(\Gamma \cup \{\psi\}) \cap \mathsf{Atoms}(\{\phi\}) = \emptyset$.

Proposition 16. *Let d be an unbiased pseudo-distance and f a hereditary aggregation function. Suppose that Γ can be represented as $\Gamma' \cup \Gamma''$, where $mod^{\mathcal{S}}(\Gamma') \neq \emptyset$ and* $\mathsf{Atoms}(\Gamma') \cap \mathsf{Atoms}(\Gamma'') = \emptyset$. *Then for every formula ψ such that* $\mathsf{Atoms}(\{\psi\}) \cap \mathsf{Atoms}(\Gamma'') = \emptyset$, *it holds that if* $\Gamma' \models^{\mathcal{S}} \psi$ *then* $\Gamma \models^{\mathcal{S}}_{d,f} \psi$.

The proofs of Propositions 15 and 16 are completely analogous to those of Propositions 7 and 9, respectively. Similarly, we have the dual result (with a similar proof) of Proposition 8:

Proposition 17. *Let d be a pseudo-distance and f an aggregation function. If* $\Gamma \models^{\mathcal{S}}_{d,f} \psi$ *and* $\mathrm{mod}^{\mathcal{S}}(\Gamma \cup \{\phi\}) \neq \emptyset$, *then* $\Gamma, \phi \models^{\mathcal{S}}_{d,f} \psi$.

Not every property of $\models^2_{d,f}$ is preserved in the general case, though. Proposition 3 is one example for this. Another property that is determined by the underlying multiple-valued structure is monotonicity. Indeed, $\models^2_{d,f}$ is not monotonic (Proposition 6), but as Example 9 shows, in the standard propositional language $\models^{3_\top}_{d,f}$ and $\models^4_{d,f}$ *are* monotonic for every d and f (since they are equivalent to the monotonic consequence relations $\models^{3_\top}$ and $\models^4$, respectively).

6. Conclusion

The principle of minimal change is a primary motif in many contexts of reasoning with incomplete and inconsistent information, such as formalisms for modelling belief revision (e.g., [17,18,26,29,36,37,46,50,59]), decision making in the context of social choice theory [30,43,44,51], database integration systems [3,4,9,21,22,47], and operators for merging independent constraint data-sources [40,41]. In this paper, we introduced a simple and natural framework for representing this principle in an explicit way. It is shown that the entailment relations that are obtained can be incorporated in a variety of deductive systems, mediators of distributed databases, consistent query answering engines, and formalisms for belief revision.

The primary goal of this paper was to consider some of the main *logical properties* of those distance-based entailments. It is shown that within our framework it is possible to define cautious consequence relations that are paraconsistent (inconsistent information is tolerated in a non-trivial way), non-monotonic (conclusions may be revised), and obey the law of inertia (irrelevant facts do not affect existing conclusions). A characteristic property of the underlying entailments is that to a large extent they retain consistency. Indeed, in a two-valued (respectively, $\mathcal{S}$-valued) semantics, these entailments are identical to the classical logic entailment (respectively, are identical to the corresponding basic $\mathcal{S}$-entailment), as long as the set of premises is kept consistent. Moreover, even when the set of premises becomes inconsistent, the conclusions of the fragment of the theory that is not related to the 'core' of the inconsistency are the same as those obtained by the basic entailment of the two-valued (respectively, the $\mathcal{S}$-valued) logic, whenever only this consistent fragment is taken into account. In contrast to the entailment of classical logic, however, our formalisms are not degenerated in the presence of contradictions, so the set of conclusions is not 'exploded' in such cases.

Future work on this subject involves a development of proof systems for automated reasoning with distance-based considerations, and the incorporation of off-the-shelf computational tools for practical applications of distance-based reasoning.

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