A Sequent-Based Representation of Logical Argumentation

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Abstract. In this paper we propose a new presentation of logic-based argumentation theory through Gentzen-style sequent calculi. We show that arguments may be represented by Gentzen-type *sequents* and that attacks between arguments may be represented by *sequent elimination rules*. This framework is logic-independent, i.e., it may be based on arbitrary languages and consequence relations. Moreover, the usual conditions of minimality and consistency of support sets are relaxed, allowing for a more flexible way of expressing arguments, which also simplifies their identification. This generic representation implies that argumentation theory may benefit from incorporating techniques of proof theory and that different non-classical formalisms may be used for backing up intended argumentation semantics.

1 Introduction

Argumentation is the study of how mutually acceptable conclusions can be reached from a collection of arguments. A common dialectical approach for analyzing and evaluating arguments is based on Dung-style abstract argumentation frameworks [22], which can be seen as a diagramming of arguments and their interactions [6, 7, 32]. Logic-based formalization of argumentation frameworks (sometimes called *logical* (or *deductive*) *argumentation*; see reviews in [20, 30]) have also been extensively studied in recent years. One of the better-known approaches in this respect is Besnard and Hunter's logic-based counterpart of Dung's theory [12, 13], in which arguments are represented by classically valid entailments whose premised are consistent and minimal with respect to set inclusion (see also [3, 24, 27]).

Our purpose in this paper is to show that deductive argumentation theory can be described and represented in terms of *sequents*. The latter are logical expressions that have been introduced by Gerhard Gentzen in order to specify his famous sequent calculi [26]. We show that sequents are useful for representing logical arguments since they can be regarded as specific kinds of judgments, and that their interactions (the attack relations) can by represented by Gentzen-style rules of inference. The outcome is a general and uniform approach to deductive argumentation based on manipulations of sequents.

The introduction of sequent-based formalism in the context of logical argumentation has some important benefits. Firstly, well-studied sequent calculi may

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be incorporated for producing arguments in an automated way. Secondly, some restrictions in previous definitions of logical arguments, like minimality and consistency of support sets, may now be lifted. Finally, the sequent-based approach is general enough to accommodate different logics, including non-classical ones. This enables the use of different substructural logics, including paraconsistent logics [21] that support robust methods of handling conflicts among arguments.

The rest of this paper is organized as follows: in the next section we briefly review the basics of abstract and logical argumentation theory. In Section 3 we introduce a sequent-based representation of logical argumentation frameworks, and in Section 4 we show how argumentation semantics may be computed in this context in terms of entailment relations. In Section 5 we discuss some further advantages of using sequent calculi for argumentation frameworks, and in Section 6 we conclude.

2 Preliminaries: Abstract and Logical Argumentation

We start by recalling the terminology and some basic concepts behind Dung-style argumentation [22].

Definition 1. An *argumentation framework* [22] is a pair $\mathcal{AF} = \langle Args, Attack \rangle$, where *Args* is an enumerable set of elements, called *arguments*, and *Attack* is a binary relation on $Args \times Args$ whose instances are called *attacks*. When $(A, B) \in Attack$ we say that A *attacks* B (or that B is *attacked by* A).

The study of how to evaluate arguments based on the structures above is usually called *abstract argumentation*. Given an argumentation framework $\mathcal{AF} = \langle Args, Attack \rangle$, a key question is what sets of arguments (called *extensions*) can collectively be accepted. Different types of extensions have been considered in the literature (see, e.g., [17, 18, 22, 23]), some of them are listed below.

Definition 2. Let $\mathcal{AF} = \langle Args, Attack \rangle$ be an argumentation framework, and let $\mathcal{E} \subseteq Args$. We say that $\mathcal{E}$ *attacks* an argument A if there is an argument $B \in \mathcal{E}$ that attacks A (i.e., $(B, A) \in Attack$). The set of arguments that are attacked by $\mathcal{E}$ is denoted $\mathcal{E}^+$. We say that $\mathcal{E}$ *defends* A if $\mathcal{E}$ attacks every argument B that attacks A. The set $\mathcal{E}$ is called *conflict-free* if it does not attack any of its elements, $\mathcal{E}$ is called *admissible* if it is conflict-free and defends all of its elements, and $\mathcal{E}$ is *complete* if it is admissible and contains all the arguments that it defends.

Let $\mathcal{E}$ be a complete subset of *Args*. We say that $\mathcal{E}$ is a *grounded extension* (of $\mathcal{AF}$) iff it is the minimal complete extension of $\mathcal{AF}$,[1] a *preferred extension* iff it is a maximal complete extension of $\mathcal{AF}$, an *ideal extension* iff it is a maximal complete extension that is a subset of each preferred extension of $\mathcal{AF}$, a *stable extension* iff it is a complete extension of $\mathcal{AF}$ that attacks every argument in $Args \setminus \mathcal{E}$, a *semi-stable extension* iff it is a complete extension of $\mathcal{AF}$ where $\mathcal{E} \cup \mathcal{E}^+$ is maximal among all complete extensions of $\mathcal{AF}$, and an *eager extension* iff it is a maximal complete extension that is a subset of each semi-stable extension of $\mathcal{AF}$.

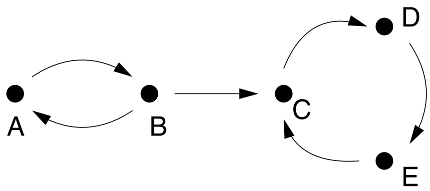
[1] In this definition the minimum and maximum are taken with respect to set inclusion.

In the context of abstract argumentation, then, the arguments themselves are usually considered as atomic objects, and argument acceptability is based on the interactions among these objects, depicted in terms of the attack relation. Acceptability of arguments (with respect to semantics like those considered above) is now defined as follows:

Definition 3. We denote by $\mathcal{E}_{\mathsf{Sem}}(\mathcal{AF})$ the set of all the Sem-extensions of an argumentation framework $\mathcal{AF} = \langle \mathit{Args}, \mathit{Attack} \rangle$, where Sem is one of the extension-based semantics considered previously. Now,

- An argument A is *skeptically accepted* by $\mathcal{AF}$ according to Sem, if $A \in \mathcal{E}$ for *every* $\mathcal{E} \in \mathcal{E}_{\mathsf{Sem}}(\mathcal{AF})$,
- An argument A is *credulously accepted* by $\mathcal{AF}$ according to Sem, if $A \in \mathcal{E}$ for *some* $\mathcal{E} \in \mathcal{E}_{\mathsf{Sem}}(\mathcal{AF})$.

Example 1. Consider the argumentation framework $\mathcal{AF}$, represented by the directed graph of Figure 1, where arguments are represented by nodes and the attack relation is represented by arrows.



The admissible sets of $\mathcal{AF}$ are $\emptyset$, $\{A\}$, $\{B\}$ and $\{B, D\}$, its complete extensions are $\emptyset$, $\{A\}$, and $\{B, D\}$, the grounded extension is $\emptyset$, the preferred extensions are $\{A\}$ and $\{B, D\}$, the ideal extension is $\emptyset$, the stable extension is $\{B, D\}$, and this is also the only semi-stable extension and eager extension of $\mathcal{AF}$. Thus, e.g., B is credulously accepted by $\mathcal{AF}$ according to the preferred semantics and it is skeptically accepted by $\mathcal{AF}$ according to the stable semantics.

A wealth of research has been conducted on formalizing deductive argumentation, in which arguments can be expressed in terms of formal languages and acceptance of arguments can be determined by logical entailments. This is usually called *logical argumentation*. One of the better-known works in this context is that of Besnard and Hunter [12], sketched below.

Definition 4. Let $\mathcal{L}$ be a standard propositional language, Σ a finite set of formulas in $\mathcal{L}$, and $\vdash_{cl}$ the consequence relation of classical logic (for $\mathcal{L}$). An *argument in the sense of Besnard and Hunter* [12] (BH-argument, for short), formed by Σ, is a pair $A = \langle \Gamma, \psi \rangle$, where ψ is a formula in $\mathcal{L}$ and Γ is a minimally consistent subset of Σ (where minimization is with respect to set inclusion), such that $\Gamma \vdash_{cl} \psi$. Here, Γ is called the *support set* of the argument A and ψ is its *consequent*.[2]

[2] A similar definition of arguments for defeasible reasoning goes back to [33]; We refer, e.g., to [13] for a comparison between the two approaches.

Different attack relations have been considered in the literature for logical argumentation frameworks. Below we recall those that are considered in [27] (see also [1, 2, 12, 28, 29]).

Definition 5. Let $A_1 = \langle \Gamma_1, \psi_1 \rangle$ and $A_2 = \langle \Gamma_2, \psi_2 \rangle$ be two BH-arguments.

- A_1 is a *defeater* of A_2 if $\psi_1 \vdash_{cl} \neg \bigwedge_{\gamma \in \Gamma_2} \gamma$.
- A_1 is a *direct defeater* of A_2 if there is $\gamma \in \Gamma_2$ such that $\psi_1 \vdash_{cl} \neg\gamma$.
- A_1 is an *undercut* of A_2 if there is $\Gamma_2' \subseteq \Gamma_2$ such that ψ_1 is logically equivalent to $\neg \bigwedge_{\gamma \in \Gamma_2'} \gamma$.
- A_1 is a *direct undercut* of A_2 if there is $\gamma \in \Gamma_2$ such that ψ_1 is logically equivalent to $\neg\gamma$.
- A_1 is a *canonical undercut* of A_2 if ψ_1 is logically equivalent to $\neg \bigwedge_{\gamma \in \Gamma_2} \gamma$.
- A_1 is a *rebuttal* of A_2 if ψ_1 is logically equivalent to $\neg\psi_2$.
- A_1 is a *defeating rebuttal* of A_2 if $\psi_1 \vdash_{cl} \neg\psi_2$.

Let $Args_{\mathsf{BH}}(\Sigma)$ be the (countably infinite) set of BH-arguments formed by Σ. Each condition in Definition 5 induces a corresponding attack relation *Attack* on $Args_{\mathsf{BH}}(\Sigma)$. For instance, one may define that $(A_1, A_2) \in Attack$ iff A_1 is a defeater of A_2. In turn, Σ and *Attack* induce the (abstract) argumentation framework $\mathcal{AF}(\Sigma) = \langle Args_{\mathsf{BH}}(\Sigma), Attack \rangle$. By this, one may draw conclusions from Σ with respect to each of the abstract argumentation semantics considered in Definition 2, by incorporating Definition 3:

Definition 6. Let $\mathcal{AF}(\Sigma) = \langle Args_{\mathsf{BH}}(\Sigma), Attack \rangle$ be a logical argumentation framework and Sem one of the extension semantics considered in Definition 2.

- A formula ψ is *skeptically entailed* by Σ according to Sem, if there is an argument $\langle \Gamma, \psi \rangle \in Args_{\mathsf{BH}}(\Sigma)$ that is skeptically accepted by $\mathcal{AF}(\Sigma)$ according to Sem.
- A formula ψ is *credulously entailed* by Σ according to Sem, if there is an argument $\langle \Gamma, \psi \rangle \in Args_{\mathsf{BH}}(\Sigma)$ that is credulously accepted by $\mathcal{AF}(\Sigma)$ according to Sem.

3 Sequent-Based Logical Argumentation

The setting described in the previous section is a basis of several works on logical argumentation (e.g., [1, 2, 3, 12, 13, 14, 24, 27]). In this section we re-examine some of its basic concepts.

3.1 Arguments as Sequents

First, we consider the notion of a logical argument. We argue that the minimality and consistency requirements in Definition 4 not only cause complications in the evaluation and the construction of arguments, but also may not be really necessary for capturing the intended meaning of this notion.

– **Minimality.** Minimization of supports is not an essential principle for defining arguments. For instance, mathematical proofs are usually not required to be minimal in order to validate their claim. For a more specific example, consider a framework in which supports are expressed only by literals (atomic formulas or their negation). Then $\langle\{p,q\}, p \vee q\rangle$ is excluded due to minimality considerations, although one may consider $\{p,q\}$ as a *stronger* support for $p \vee q$ than, say, $\{p\}$. Indeed, the former contains *several* pieces of evidence for $p \vee q$ (this may be relevant when, e.g., majority votes or other quantitative considerations are taken into account).[3]

– **Consistency.** The requirement that the support set Γ of an argument $\langle\Gamma, \psi\rangle$ should be consistent may be irrelevant for some logics, at least when consistency is defined by satisfiability. Indeed, in logics such as Priest's LP [31] or Belnap's four-valued logic [10], *every* set of formulas in the language of $\{\neg, \vee, \wedge\}$ is satisfiable. What really matters in these cases is the consequence relation of the underlying logic. Thus, e.g., in opposed to classical logic, when intuitionistic logic is concerned, $\langle\{\neg\neg\psi\}, \psi\rangle$ shouldn't be considered as a legitimate argument, although $\neg\neg\psi \in \Gamma$ is (minimally) consistent in Γ when ψ is consistent.

– **Complexity.** From a more pragmatic point of view, the involvement of minimally consistent subsets of the underlying knowledge-base poses serious questions on the computational viability of identifying arguments and generating them. Indeed, deciding the existence of a minimal subset of formulas that implies the consequent is already at the second level of the polynomial hierarchy (see [25]).

Our conclusion, then, is that what really matters for an argument, is that (i) its consequent would logically follow, according to the underlying logic, from the support set, and that (ii) there would be an effective way of constructing and identifying it. In what follows we therefore adhere the following principles:

1. Supports and consequents of arguments are solely determined by the *logic*.
2. Arguments are syntactical objects that are *effectively computable* by a formal system that is related to the logic, and are *refutable* by the attack relation of the argumentation system.

For the first item we indicate what a logic is (Definition 7). The first part of the second item corresponds to the primary goal of proof theory, so notations and machinery are borrowed from that area (Definitions 8 and 9).

[3] Another argument that is sometimes pleaded for set-inclusion minimization is that it reduces the number of attacks. Again, it is disputable whether set-inclusion minimization is the right principle for assuring this property, since, for instance, the singletons $S_1 = \{p_1\}$ and $S_2 = \{p_2 \wedge \ldots \wedge p_n\}$, supporting (e.g., in classical logic) the claim $p_1 \vee \ldots \vee p_n$, are incomparable w.r.t. set-inclusion (and moreover they even do not share any atomic formula), but it is obvious that as n becomes larger S_2 becomes more exposed to attacks than S_1.

Definition 7. Let $\mathcal{L}$ be a propositional language. A (propositional) *logic* for $\mathcal{L}$ is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, i.e., a binary relation between sets of formulas and formulas in $\mathcal{L}$, satisfying the following conditions:

Cautious Reflexivity:	$\psi \vdash \psi$.
Monotonicity:	if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity:	if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \varphi$ then $\Gamma, \Gamma' \vdash \varphi$.

Definition 8. Let $\mathcal{L}$ be a propositional language, and let $\Rightarrow$ be a symbol that does not appear in $\mathcal{L}$. An *$\mathcal{L}$-sequent* (or just a sequent) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of formulas in $\mathcal{L}$.

Proof systems that operate on sequents are called *sequent calculi* [26]. We shall say that a logic $\mathfrak{L}$ is *effective*, if it has a sound and complete sequent calculus. For an effective logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, then, there is an effective way of drawing entailments: $\Gamma \vdash \psi$ iff there is a proof of the sequent $\Gamma \Rightarrow \psi$ in the corresponding sequent calculus. In what follows we shall always assume that the underlying logics are effective.

Definition 9. Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be an effective logic with a corresponding sequent calculus $\mathfrak{C}$, and let Σ be a set of formulas in $\mathcal{L}$. An *$\mathfrak{L}$-argument* based on Σ is an $\mathfrak{L}$-sequent of the form $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \Sigma$, that is provable in $\mathfrak{C}$. The set of all the $\mathfrak{L}$-arguments that are based on Σ is denoted $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$.

In the notation of Definition 9, we have that:

Proposition 1. *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be an effective propositional logic. Then $\Gamma \Rightarrow \psi$ is in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ iff $\Gamma \vdash \psi$ for $\Gamma \subseteq \Sigma$.*

Example 2. When the underlying logic is classical logic $\mathfrak{CL}$, one may use Gentzen's well-known sequent calculus LK, which is sound and complete for $\mathfrak{CL}$ [26]. In this case we have, for instance, that the sequent $\psi \supset \phi \Rightarrow \neg\psi \vee \phi$ is derivable in LK and so it belongs to $\mathsf{Arg}_{\mathfrak{CL}}(\Sigma)$ whenever Σ contains the formula $\psi \supset \phi$. Note, however, that this sequent is not derivable by any sequent calculus that is sound and complete for intuitionistic logic $\mathfrak{IL}$ (e.g., Gentzen's LJ), thus it is not in $\mathsf{Arg}_{\mathfrak{IL}}(\Sigma)$ for any Σ.

Proposition 2. *For every effective logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and a finite set Σ of formulas in $\mathcal{L}$, the set $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ is closed under the following rules:*[4]

Σ-Reflexivity:	*For every $\Gamma \subseteq \Sigma$ and $\psi \in \Gamma$ it holds that $\Gamma \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$.*
Σ-Monotonicity:	*If $\Gamma \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ and $\Gamma \subseteq \Gamma' \subseteq \Sigma$ then $\Gamma' \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$*
Σ-Transitivity:	*If $\Gamma \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ and $\Gamma', \psi \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ then also $\Gamma, \Gamma' \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$.*

[4] Following the usual convention we use commas in a sequent for denoting the union operation.

Proof. By Proposition 1, Σ-Reflexivity follows from the cautious reflexivity and the monotonicity of $\vdash$, Σ-Transitivity follows from the transitivity of $\vdash$, and Σ-monotonicity follows from the monotonicity of $\vdash$. $\square$

Note 1. The set $Args_{\mathsf{BH}}(\Sigma)$ of the BH-arguments is not closed under any rule in Proposition 2. To see this consider for instance the set $\Sigma = \{p, q, \neg p \vee q, \neg q \vee p\}$. Then $\langle \{p, \neg p \vee q\}, q \rangle \in Args_{\mathsf{BH}}(\Sigma)$ and $\langle \{q, \neg q \vee p\}, p \rangle \in Args_{\mathsf{BH}}(\Sigma)$, however $\langle \{p, \neg p \vee q, \neg q \vee p\}, p \rangle \notin Args_{\mathsf{BH}}(\Sigma)$, since its support set is not minimal. Thus $Args_{\mathsf{BH}}(\Sigma)$ is not Σ-transitive. The fact that $\langle \{p, \neg p \vee q, \neg q \vee p\}, p \rangle \notin Args_{\mathsf{BH}}(\Sigma)$ (while $\langle \{p\}, p \rangle \in Args_{\mathsf{BH}}(\Sigma)$) also shows that $Args_{\mathsf{BH}}(\Sigma)$ is not Σ-monotonic and that it is not Σ-reflexive.[5]

Note 2. Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be an effective logic and Σ a finite set of formulas in $\mathcal{L}$. Then Σ-Transitivity can be strengthened as follows:

If $\Gamma \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ and $\Gamma', \psi \vdash \phi$ for $\Gamma' \subseteq \Sigma$, then $\Gamma, \Gamma' \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$.

This rule implies that for generating $\mathfrak{L}$-arguments based on Σ it is enough to consider only formulas in Σ.

3.2 Attacks as Sequent Elimination Rules

In order to represent attack relations we introduce rules for excluding arguments (i.e., sequents) in the presence of counter arguments. We call such rules *sequent elimination rules*, or *attack rules*. The obvious advantage of representing attacks by sequent elimination rules is that the form of such rules is similar to that of the construction rules, and both types of rules are expressed by the same syntactical objects. This allows us to uniformly identify and generate arguments and attacks by the same sequent-manipulation systems.

Since the underlying logic may not be classical and its language may not be the standard propositional one, we shall have to make the following assumptions on the availability of particular connectives in the language:

- To generalize attack relations that are defined by the classical conjunction, we assume that the underlying language contains a $\vdash$-conjunctive connective $\wedge$, for which $\Gamma \vdash \psi \wedge \phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$. In these cases we shall denote by $\wedge\Gamma$ the conjunction of all the elements in Γ.
- To generalize attack relations that are defined by logical equivalence, we assume that in addition to the $\vdash$-conjunctive connective, the underlying language also contains a $\vdash$-deductive implication $\supset$, for which $\Gamma, \psi \vdash \phi$ iff $\Gamma \vdash \psi \supset \phi$. In these cases we shall abbreviate the formula $(\psi \supset \phi) \wedge (\phi \supset \psi)$ by $\psi \leftrightarrow \phi$.

Let us now show how the attack relations in Definition 5 can be described in terms of corresponding sequent elimination rules. Typical conditions of such

[5] Note that $Args_{\mathsf{BH}}(\Sigma)$ is *cautiously Σ-reflexive*: $\langle \{\psi\}, \psi \rangle \in Args_{\mathsf{BH}}(\Sigma)$ for a consistent formula $\psi \in \Sigma$.

rules consist of three sequents: the attacking argument, the attacked argument, and the condition for the attack. Conclusions of sequent elimination rules will be the elimination of the attacked argument. In the sequel, we denote by $\Gamma \not\Rightarrow \psi$ the elimination of the argument $\Gamma \Rightarrow \psi$.

In what follows we say that a sequent elimination rule $\mathcal{R}$ is *applicable* with respect to a logic $\mathfrak{L}$, if all of its conditions are valid for $\mathfrak{L}$, that is, every condition of $\mathcal{R}$ is provable in a corresponding sound and complete sequent calculus for $\mathfrak{L}$.[6]

Attacks by defeaters: In terms of an arbitrary logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and $\mathfrak{L}$-arguments in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$, an argument $\Gamma_1 \Rightarrow \psi_1$ is an $\mathfrak{L}$-defeater of an argument $\Gamma_2 \Rightarrow \psi_2$ if $\psi_1 \vdash \neg \bigwedge \Gamma_2$. In the presence of a $\vdash$-deductive implication $\supset$ in $\mathcal{L}$, this means that $\vdash \psi_1 \supset \neg \bigwedge \Gamma_2$, and so $\Rightarrow \psi_1 \supset \neg \bigwedge \Gamma_2$ is an $\mathfrak{L}$-argument in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$. It follows that attacks by defeaters may be represented by the following sequent elimination rule (relative to $\mathfrak{L}$):

$$\text{Defeat:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \supset \neg \bigwedge \Gamma_2 \qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$$

In the particular case where the underlying logic is classical logic $\mathfrak{CL}$, this rule is a sequent-based encoding of a defeater attack in the sense of Definition 5:

Proposition 3. *Let $A_1 = \langle \Gamma_1, \psi_1 \rangle$ and $A_2 = \langle \Gamma_2, \psi_2 \rangle$ be two BH-arguments. Then A_1 is a defeater of A_2 in the sense of Definition 5 iff the Defeat rule $\mathcal{R}$, in which $\Gamma_2 \Rightarrow \psi_2$ is attacked by $\Gamma_1 \Rightarrow \psi_1$, is $\mathfrak{CL}$-applicable.*

Proof. Since A_i is a BH-argument it holds that $\Gamma_i \Rightarrow \psi_i$ is $\mathfrak{CL}$-valid ($i = 1, 2$). Moreover, since A_1 is a defeater of A_2, the attack condition of $\mathcal{R}$ is also $\mathfrak{CL}$-valid. It follows that $\mathcal{R}$ is $\mathfrak{CL}$-applicable. Conversely, suppose that $A_1 = \langle \Gamma_1, \psi_1 \rangle$ and $A_2 = \langle \Gamma_2, \psi_2 \rangle$ are BH-arguments so that the Defeat rule in which $\Gamma_2 \Rightarrow \psi_2$ is attacked by $\Gamma_1 \Rightarrow \psi_1$ is $\mathfrak{CL}$-applicable. Then the attacking condition of this rule is $\mathfrak{CL}$-valid, and so A_1 is a defeater of A_2 in the sense of Definition 5. $\square$

Note 3. The following sequent elimination rule may be viewed as a generalized form of Defeat, which moreover does not assume the availability of a deductive implication in the language.

$$\text{Strong Defeat:} \quad \frac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2 \qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$$

Proposition 4. *Strong Defeat implies Defeat.*

Proof. Assume that the three conditions of Defeat hold. Since $\Rightarrow \psi_1 \supset \neg \bigwedge \Gamma_2$ is derivable and $\mathfrak{L}$ is effective, it holds that $\vdash \psi_1 \supset \neg \bigwedge \Gamma_2$. Thus, since $\supset$ is a $\vdash$-deductive implication, $\psi_1 \vdash \neg \bigwedge \Gamma_2$. This, together with the assumption that $\Gamma_1 \Rightarrow \psi_1$ is derivable (and so it is an argument in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$), imply by Note 2 that $\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2$ is an argument in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$, and so it is derivable in the underlying sequent calculus. By Strong Defeat, then, $\Gamma_2 \not\Rightarrow \psi_2$, which is also the conclusion of Defeat. $\square$

[6] Semantically, this usually means that for every condition $\Gamma \Rightarrow \psi$ of $\mathcal{R}$, any $\mathfrak{L}$-model of (all the formulas in) Γ is an $\mathfrak{L}$-model of ψ.

Attacks by direct defeaters: Direct defeat with respect to an arbitrary logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ and a set $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ of $\mathfrak{L}$-arguments based on Σ, means that $\Gamma_1 \Rightarrow \psi_1$ is an $\mathfrak{L}$-direct defeater of $\Gamma_2 \Rightarrow \psi_2$ if $\psi_1 \vdash \neg\gamma$ for some $\gamma \in \Gamma_2$. Thus, a direct defeat attack may be expressed by the following sequent elimination rule:

$$\text{Direct Defeat:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \supset \neg\phi \qquad \Gamma_2, \phi \Rightarrow \psi_2}{\Gamma_2, \phi \not\Rightarrow \psi_2}$$

Thus, an argument should be withdrawn in case that the negation of an element in its support set is implied by a consequent of another argument.

As in the case of attacks by defeaters, we have the following relation between attacks by direct defeaters in classical logic (Definition 5) and the above sequent-based formalization:

Proposition 5. *Let $A_1 = \langle \Gamma_1, \psi_1 \rangle$ and $A_2 = \langle \Gamma_2, \psi_2 \rangle$ be BH-arguments. Then A_1 is a direct defeater of A_2 in the sense of Definition 5 iff the Direct Defeat sequent elimination rule, in which $\Gamma_2 \Rightarrow \psi_2$ is attacked by $\Gamma_1 \Rightarrow \psi_1$, is $\mathfrak{CL}$-applicable.*

Proof. Similar to that of Proposition 3. □

Note 4. Again, it is possible to express a stronger form of the rule above, which does not mention an implication connective:

$$\text{Strong Direct Defeat:} \quad \frac{\Gamma_1 \Rightarrow \neg\phi \qquad \Gamma_2, \phi \Rightarrow \psi_2}{\Gamma_2, \phi \not\Rightarrow \psi_2}$$

Proposition 6. *Strong Direct Defeat implies Direct Defeat.*

Proof. As in the proof of Proposition 4, by Note 2 and the fact that $\supset$ is a $\vdash$-deductive implication, the availability of $\Gamma_1 \Rightarrow \psi_1$ and $\Rightarrow \psi_1 \supset \neg\phi$ implies that $\Gamma_1 \Rightarrow \neg\phi$ is an element in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$. Thus, Strong Direct Defeat may be applied to conclude that $\Gamma_2, \phi \not\Rightarrow \psi_2$, which is also the conclusion of Direct Defeat. □

Attacks by undercuts: For expressing undercuts with respect to a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ we first have to define logical equivalence in $\mathfrak{L}$. A natural way to do so is to require that ψ and ϕ are logically equivalent in $\mathfrak{L}$ iff $\psi \vdash \phi$ and $\phi \vdash \psi$. Using a $\vdash$-deductive implication $\supset$ and a $\vdash$-conjunctive connective $\wedge$, this means that $\vdash (\psi \supset \phi) \wedge (\phi \supset \psi)$, i.e., that $\psi \leftrightarrow \phi$ is a theorem of $\mathfrak{L}$. It follows that attacks by undercuts are represented by the following sequent elimination rule:

$$\text{Undercut:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2 \qquad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$$

Again, one may show that an attack by undercuts in the sense of Definition 5 is a particular case, for classical logic, of the rule above (cf. Propositions 3 and 5).

Proposition 7. *Let $A_1 = \langle \Gamma_1, \psi_1 \rangle$ and $A_2 = \langle \Gamma_2, \psi_2 \rangle$ be BH-arguments. Then A_1 is an undercut of A_2 iff the Undercut rule in which $\Gamma_2 \Rightarrow \psi_2$ is attacked by $\Gamma_1 \Rightarrow \psi_1$ is $\mathfrak{CL}$-applicable.*

Attacks by direct and canonical undercuts: Using the same notations as those for attacks by undercuts, and under the same assumptions on the language, attacks by direct undercuts may be represented by the following elimination rule:

$$\text{Direct Undercut:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \leftrightarrow \neg\gamma_2 \qquad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2}$$

Similarly, attacks by canonical undercuts may be represented as follows:

$$\text{Canonical Undercut:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2 \qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$$

The rules above may be justified by propositions that are similar to 3, 5, and 7.

Attacks by rebuttal and defeating rebuttal: By the discussion above it is easy to see that attacks by rebuttal and defeating rebuttal are also represented by sequent elimination rules. Indeed, these two attacks are represented as follows:

$$\text{Rebuttal:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \leftrightarrow \neg\psi_2 \qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$$

$$\text{Defeating Rebuttal:} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \supset \neg\psi_2 \qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$$

Again, these rules are justifiable by propositions that are similar to 3, 5, and 7.

As the next proposition shows, the relations between the attacks in Definition 5, indicated in [27], carry on to our sequent elimination rules.

Proposition 8. *Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be an effective propositional logic and suppose that $\mathcal{L}$ has a $\vdash$-conjunction $\wedge$ and a $\vdash$-deductive implication $\supset$. Then: (a) Defeating Rebuttal implies Rebuttal, (b) Undercut implies Canonical Undercut and Direct Undercut, (c) Direct Defeat implies Direct Undercut.*

Proof. Part (a) follows from the fact that the conditions of Rebuttal are stronger than those of Defeating Rebuttal. More specifically, suppose that the conditions of Rebuttal hold, i.e., $\Gamma_1 \Rightarrow \psi_1$ and $\Rightarrow \psi_1 \leftrightarrow \neg\psi_2$ and $\Gamma_2 \Rightarrow \psi_2$ are derivable in the underlying sequent calculus. Since $\mathfrak{L}$ is effective, it holds that $\vdash \psi_1 \leftrightarrow \neg\psi_2$, i.e., $\vdash (\psi_1 \supset \neg\psi_2) \wedge (\neg\psi_2 \supset \psi_1)$. Since $\wedge$ is a $\vdash$-conjunction, $\vdash \psi_1 \supset \neg\psi_2$, thus by the effectiveness of $\mathfrak{L}$ again, the sequent $\Rightarrow \psi_1 \supset \neg\psi_2$ is derivable in the calculus. By Defeating Rebuttal, $\Gamma_2 \not\Rightarrow \psi_2$, which is also the conclusion of Rebuttal.

Part (b) follows from the fact that Undercut holds in particular when Γ_2 is a singleton (in which case Direct Undercut is obtained) and when Γ_2 is the whole support set of the sequent (in which case Canonical Undercut is obtained).

To see Part (c), note that the conditions of Direct Undercut are stronger than those of Direct Defeat (taking $\gamma_2 = \phi$). □

Note 5. Further relations between the elimination rules introduced above may be obtained under further assumptions on the underlying logics. For instance, when $\mathfrak{L}$ is classical logic, Defeat implies Direct Defeat, since in LK the sequent $\Rightarrow \psi \supset \neg \bigwedge \Gamma$ is derivable from $\Rightarrow \psi \supset \neg\gamma$ for any $\gamma \in \Gamma$. Similar considerations show that in this case Defeat also implies Undercut and Defeating Rebuttal.

4 Argumentation by Sequent Processing

In light of the previous section, a logical argumentation framework for a set of formulas Σ, based on a logic $\mathfrak{L}$, consists of a set of arguments $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ and a set of sequent elimination rules *Attack*. The arguments in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ may be constructed by a sequent calculus which is sound and complete for $\mathfrak{L}$, while the rules in *Attack* allow to discard arguments that are attacked according to some attacking policy. Semantics of such a logical framework $\mathcal{AF}_{\mathfrak{L}}(\Sigma)$ are therefore determined by a process involving constructions and eliminations of sequents (logical arguments). Below, we describe and exemplify this process.

Definition 10. We say that an argument $\Gamma \Rightarrow \psi$ is *discarded* by an argument $\Gamma' \Rightarrow \psi'$, if there is a rule $\mathcal{R} \in Attack$ in which $\Gamma' \Rightarrow \psi'$ attacks $\Gamma \Rightarrow \psi$, that is, $\Gamma \Rightarrow \psi$ and $\Gamma' \Rightarrow \psi'$ appear in the conditions of $\mathcal{R}$ and $\Gamma \not\Rightarrow \psi$ is the conclusion of $\mathcal{R}$.

Note 6. If $\mathfrak{L}$ is a logic in which any formula follows from a contradiction (in particular, if $\mathfrak{L} = \mathfrak{CL}$), *any* sequent is discarded when Σ is contradictory. It follows that either the consistency requirement from support sets of arguments should be restored, or the underlying logic should be paraconsistent [21]. Since our goal here is to avoid the first option, in what follows we consider argumentation frameworks that are based on paraconsistent logics.[7] Here we chose Priest's three-valued logic $\mathcal{LP}$ [31], which is one of the most famous and simplest paraconsistent logics in the literature. A sound and complete sequent calculus for $\mathcal{LP}$ is given in Figure 1 (see also [4]).

Example 3. Consider the argumentation framework for the set $\Sigma = \{\neg p, p, q\}$, based on $\mathcal{LP}$, in which attacks are by Undercut. Then, while $q \Rightarrow q \vee p$ is not discarded by any argument in $\mathsf{Arg}_{\mathcal{LP}}(\Sigma)$, the argument $q, p \Rightarrow q \vee p$ is discarded by, e.g., $\neg p \Rightarrow \neg p$. The intuition behind this is that the support set of the argument $q, p \Rightarrow q \vee p$, unlike that of the argument $q \Rightarrow q \vee p$, contains a formula (p) which is controversial in Σ (because it is contradictory).

The arguments that are not discarded by any argument are those that are not attacked according to the attack rules in *Attack*. This conflict-free set of arguments may define a semantics for $\mathcal{AF}_{\mathfrak{L}}(\Sigma)$ as follows:

Definition 11. Let $\mathcal{AF}_{\mathfrak{L}}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}}(\Sigma), Attack \rangle$ be a logical argumentation framework for a set of formulas Σ based on a logic $\mathfrak{L}$. We denote $\mathsf{Arg}_{\mathfrak{L}}(\Sigma) \Vdash_{\mathsf{ND}} \psi$ if there is a set of formulas $\Gamma \subseteq \Sigma$ such that $\Gamma \Rightarrow \psi$ is an argument in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ that is not discarded by any argument in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ according to the rules in *Attack*.

[7] Paraconsistent logics may also be helpful in preventing contamination in defeasible argumentation (see, for instance, [16, 18]). This is beyond the scope of the current paper.

Axioms: $\psi \Rightarrow \psi \qquad \Rightarrow \psi, \neg\psi$

Structural Rules:

Weakening: $$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$$

Cut: $$\frac{\Gamma_1, \psi \Rightarrow \Delta_1 \quad \Gamma_2 \Rightarrow \Delta_2, \psi}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$$

Logical Rules:

$$[\neg\neg\Rightarrow]\ \frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \neg\neg\phi \Rightarrow \Delta} \qquad [\Rightarrow\neg\neg]\ \frac{\Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \neg\neg\phi}$$

$$[\wedge\Rightarrow]\ \frac{\Gamma, \phi, \psi \Rightarrow \Delta}{\Gamma, \phi \wedge \psi \Rightarrow \Delta} \qquad [\Rightarrow\wedge]\ \frac{\Gamma \Rightarrow \Delta, \phi \quad \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \wedge \psi}$$

$$[\neg\wedge\Rightarrow]\ \frac{\Gamma, \neg\phi \Rightarrow \Delta \quad \Gamma, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\phi \wedge \psi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\wedge]\ \frac{\Gamma \Rightarrow \Delta, \neg\phi, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\phi \wedge \psi)}$$

$$[\vee\Rightarrow]\ \frac{\Gamma, \phi \Rightarrow \Delta \quad \Gamma, \psi \Rightarrow \Delta}{\Gamma, \phi \vee \psi \Rightarrow \Delta} \qquad [\Rightarrow\vee]\ \frac{\Gamma \Rightarrow \Delta, \phi, \psi}{\Gamma \Rightarrow \Delta, \phi \vee \psi}$$

$$[\neg\vee\Rightarrow]\ \frac{\Gamma, \neg\phi, \neg\psi \Rightarrow \Delta}{\Gamma, \neg(\phi \vee \psi) \Rightarrow \Delta} \qquad [\Rightarrow\neg\vee]\ \frac{\Gamma \Rightarrow \Delta, \neg\phi \quad \Gamma \Rightarrow \Delta, \neg\psi}{\Gamma \Rightarrow \Delta, \neg(\phi \vee \psi)}$$

Fig. 1. A sequent calculus for $\mathcal{LP}$

By Definition 11, ψ is a $\Vdash_{\mathsf{ND}}$-consequence of $\mathcal{AF}_{\mathfrak{L}}(\Sigma)$ if it is a consequent of an unattacked (and so, non-discarded) argument in $\mathcal{AF}_{\mathfrak{L}}(\Sigma)$. Thus, the set of these arguments is clearly admissible (and in particular conflict-free).

In what follows, when the underlying logical framework is fixed and known, we shall abbreviate $\mathsf{Arg}_{\mathfrak{L}}(\Sigma) \Vdash_{\mathsf{ND}} \psi$ by $\Sigma \Vdash_{\mathsf{ND}} \psi$.

Example 4. By Example 3, in an argumentation framework based on $\mathcal{LP}$ and Undercut, $\{\neg p, p, q\} \Vdash_{\mathsf{ND}} q \vee p$. It is easy to see that in the same framework $\{\neg p, p, q\} \Vdash_{\mathsf{ND}} q$ but $\{\neg p, p, q\} \not\Vdash_{\mathsf{ND}} p$ and $\{\neg p, p, q\} \not\Vdash_{\mathsf{ND}} \neg p$.

Example 5. As indicated, e.g., in [19], abstract argumentation frameworks face difficulties in handling n-ary conflicts for $n \geq 3$. As far as consequences are defined by entailment relations, such conflicts are easily maintained in logical argumentation frameworks. Using the canonical example from [19], it holds that in an argumentation framework for $\Sigma = \{p, q, \neg p \vee \neg q\}$ that is based, for instance, on $\mathcal{LP}$ and Undercut, the argument $p \Rightarrow p$ is discarded, e.g., by the argument $q, \neg p \vee \neg q \Rightarrow \neg p$. Similarly, the arguments $q \Rightarrow q$, and $\neg p \vee \neg q \Rightarrow \neg p \vee \neg q$ are discarded by other arguments based on Σ, and so neither of the consequents of these arguments is derivable from Σ according to $\Vdash_{\mathsf{ND}}$.[8]

[8] Assertions that are not related to the inconsistency in Σ are still inferrable, though. For instance, $\Sigma' \Vdash_{\mathsf{ND}} r$ when $\Sigma' = \Sigma \cup \{r\}$.

Some interesting properties of $\Vdash_{\mathsf{ND}}$ are considered next.

Proposition 9. *Let ψ be a theorem of a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$. Then for every set Σ of formulas and every set Attack of elimination rules considered in Section 3.2,* $\mathsf{Arg}_{\mathfrak{L}}(\Sigma) \Vdash_{\mathsf{ND}} \psi$.

Proof. Since ψ is an $\mathfrak{L}$-theorem, we have that $\vdash \psi$, and so $\Rightarrow \psi$ is an element in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$. Since the support set of this argument is empty, it is not discarded by an argument in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ according to a rule in *Attack*, thus $\mathsf{Arg}_{\mathfrak{L}}(\Sigma) \Vdash_{\mathsf{ND}} \psi$. □

Proposition 10. $\Vdash_{\mathsf{ND}}$ *is nonmonotonic in the size of the underlying knowledge-bases: Let $\mathcal{AF}_{\mathfrak{L}}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}}(\Sigma), Attack \rangle$ and $\mathcal{AF}_{\mathfrak{L}}(\Sigma') = \langle \mathsf{Arg}_{\mathfrak{L}}(\Sigma'), Attack \rangle$ be two argumentation frameworks such that $\Sigma \subseteq \Sigma'$ and Attack contains (at least) one of the elimination rules considered in Section 3.2. Then the fact that $\mathsf{Arg}_{\mathfrak{L}}(\Sigma) \Vdash_{\mathsf{ND}} \psi$ does* not *necessarily imply that $\mathsf{Arg}_{\mathfrak{L}}(\Sigma') \Vdash_{\mathsf{ND}} \psi$ as well.*

Proof. Consider, for instance, $\Sigma = \{p\}$. Since $\mathfrak{L}$ is a logic, $p \Rightarrow p \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma)$, and so $\mathsf{Arg}_{\mathfrak{L}}(\{p\}) \Vdash_{\mathsf{ND}} p$. From the same reason, $\neg p \Rightarrow \neg p \in \mathsf{Arg}_{\mathfrak{L}}(\Sigma')$ where $\Sigma' = \Sigma \cup \{\neg p\}$. It follows that every argument in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma')$ whose consequent is p, is discarded by $\neg p \Rightarrow \neg p$, and so $\mathsf{Arg}_{\mathfrak{L}}(\{p, \neg p\}) \nVdash_{\mathsf{ND}} p$. □

Note 7. An interesting property of $\Vdash_{\mathsf{ND}}$ is that arguments that hold in a stronger logic cannot be discharged by weaker logics. This may be useful in agent negotiation as described below: Consider two agents G_1 and G_2, relying on the same knowledge-base Σ and referring to the same attack rules, but using different logics $\mathfrak{L}_1 = \langle \mathcal{L}, \vdash_1 \rangle$ and $\mathfrak{L}_2 = \langle \mathcal{L}, \vdash_2 \rangle$, respectively. In this case each agent has its own logical argumentation framework, which can be represented, respectively, by $\mathcal{AF}_{\mathfrak{L}_1}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}_1}(\Sigma), Attack \rangle$ and $\mathcal{AF}_{\mathfrak{L}_2}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}_2}(\Sigma), Attack \rangle$. Now, suppose that the logic used by G_2 is at least as strong as the logic used by G_1, i.e., $\vdash_1 \subseteq \vdash_2$. Then $\Gamma \vdash_1 \psi$ implies that $\Gamma \vdash_2 \psi$ and so $\mathsf{Arg}_{\mathfrak{L}_1}(\Sigma) \subseteq \mathsf{Arg}_{\mathfrak{L}_2}(\Sigma)$. Suppose now that $\mathsf{Arg}_{\mathfrak{L}_2}(\Sigma) \Vdash_{\mathsf{ND}} \psi$. Then there is an argument $\Gamma \Rightarrow \psi$ in $\mathsf{Arg}_{\mathfrak{L}_2}(\Sigma)$ that is not discarded by any argument in $\mathsf{Arg}_{\mathfrak{L}_2}(\Sigma)$. In particular, this sequent is not discarded by any argument in $\mathsf{Arg}_{\mathfrak{L}_1}(\Sigma)$. It follows that in this case G_2 has an argument in favor of ψ, which may not be producible by G_1 (since ψ may not follow from any subset of Σ according to $\vdash_1$), yet it cannot be discharged by G_1. In this setting, then, claims of agents with stronger logical sources may not be verified but cannot be dismissed by agents with weaker sources.

Other entailment relations, similar to $\Vdash_{\mathsf{ND}}$, may be defined by other semantics just like in Definition 3, provided that the underlying semantics is computable in terms of the rules in *Attack*. For instance, the grounded extension of $\mathcal{AF}_{\mathfrak{L}}(\Sigma)$, denoted by $\mathsf{GE}(\mathcal{AF}_{\mathfrak{L}}(\Sigma))$, contains all the arguments which are not attacked as well as the arguments which are directly or indirectly defended by non-attacked arguments. Thus, $\mathsf{GE}(\mathcal{AF}_{\mathfrak{L}}(\Sigma))$ is computable as follows: First, the non-attacked arguments in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ are added to $\mathsf{GE}(\mathcal{AF}_{\mathfrak{L}}(\Sigma))$. Then, the rules in *Attack* are applied on $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ and the discarded arguments are removed. Denote the modified set of arguments by $\mathsf{Arg}^1_{\mathfrak{L}}(\Sigma)$. Again, the non-attacked arguments in $\mathsf{Arg}^1_{\mathfrak{L}}(\Sigma)$ are added to the set $\mathsf{GE}(\mathcal{AF}_{\mathfrak{L}}(\Sigma))$ and those that are discarded by rules in *Attack* are removed. This defines a new set, $\mathsf{Arg}^2_{\mathfrak{L}}(\Sigma)$, and so forth. Now, entailment by grounded semantics is defined by: $\mathcal{AF}_{\mathfrak{L}}(\Sigma) \Vdash_{\mathsf{GE}} \psi$ if there is an argument of the form $\Gamma \Rightarrow \psi$ in $\mathsf{GE}(\mathcal{AF}_{\mathfrak{L}}(\Sigma))$ for some $\Gamma \subseteq \Delta$.

We conclude this section with some simple observations regarding the general entailment relations that are obtained in our framework.

Definition 12. Let $\mathcal{AF}_{\mathfrak{L}}(\Sigma) = \langle \mathsf{Arg}_{\mathfrak{L}}(\Sigma), \mathit{Attack} \rangle$ be a logical argumentation framework for a set of formulas Σ based on an effective logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$. Let Sem be one of the extension-based semantics considered in Definition 2 and $\mathcal{E}_{\mathsf{Sem}}(\mathcal{AF}_{\mathfrak{L}}(\Sigma))$ the corresponding Sem-extensions (Definition 3).

- We denote $\Sigma \Vdash_{\mathsf{Sem}} \psi$ if there is an argument $\Gamma \Rightarrow \psi$ in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ that is an element of every $\mathcal{E} \in \mathcal{E}_{\mathsf{Sem}}(\mathcal{AF}_{\mathfrak{L}}(\Sigma))$.
- We denote $\Sigma \vdash_{\mathsf{Sem}} \psi$ if every Sem-extension $\mathcal{E} \in \mathcal{E}_{\mathsf{Sem}}(\mathcal{AF}_{\mathfrak{L}}(\Sigma))$ contains an argument in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ whose consequent is ψ.

Proposition 11. *In the notations of Definition 12 we have that:*

1. *If $\Sigma \Vdash_{\mathsf{Sem}} \psi$ then $\Sigma \vdash_{\mathsf{Sem}} \psi$.*
2. *If $\Sigma \Vdash_{\mathsf{Sem}} \psi$ or $\Sigma \vdash_{\mathsf{Sem}} \psi$ then $\Sigma \vdash \psi$.*
3. *If $\vdash$ is paraconsistent, so are $\Vdash_{\mathsf{Sem}}$ and $\vdash_{\mathsf{Sem}}$.*
4. *If $\vdash$ has the variable sharing property,*[9] *so do $\Vdash_{\mathsf{Sem}}$ and $\vdash_{\mathsf{Sem}}$.*

Proof. Item 1 holds because the condition defining $\Vdash_{\mathsf{Sem}}$ is stronger than the one defining $\vdash_{\mathsf{Sem}}$. The condition of Item 2 assures that there is an argument of the form $\Gamma \Rightarrow \psi$ in $\mathsf{Arg}_{\mathfrak{L}}(\Sigma)$ and so by Proposition 1, $\Gamma \vdash \psi$ for some $\Gamma \subseteq \Sigma$. Since $\vdash$ is monotonic (because $\mathfrak{L}$ is a logic), also $\Sigma \vdash \psi$. For Item 3, note that if $p, \neg p \nvdash q$ then by Item 2 $p, \neg p \nVdash_{\mathsf{Sem}} q$ and $p, \neg p \nvdash_{\mathsf{Sem}} q$. Similar argument holds for Item 4: if $\Sigma \nvdash \psi$ whenever Σ and ψ do not share any atomic formula, so by Item 2 we have that in this case $\Sigma \nVdash_{\mathsf{Sem}} \psi$ and $\Sigma \nvdash_{\mathsf{Sem}} \psi$ either. $\square$

5 Further Utilizations of Arguments as Sequents

Apart of the obvious benefits of viewing arguments as sequents, such as the ability to incorporate well-established and general methods for representing arguments and automatically generating new arguments from existing ones, the use of sequents also allows to make some further enhancements in the way arguments are traditionally captured. Below, we mention two such enhancements.

- We used Gentzen-type systems which employ finite sets of formulas. However, one could follow Gentzen's original formulation and use *sequences* instead. This would allow, for instance, to encode *prefenrences* in the arguments, where the order in a sequence represents priorities. In this way one would be able to argue, for example, that $\Gamma \Rightarrow p$ for any sequence Γ

[9] That is, $\Sigma \nvdash \psi$ unless Σ and ψ share some atomic formula.

of literals that contains p and in which the first appearance of p precedes any appearance of $\neg p$. Another possibility is to employ *multisets* in the sequents, e.g. for representing majority considerations. Thus, one may state that $\Gamma \Rightarrow p$ holds whenever the number of appearances of p in a multiset Γ of literals is strictly bigger than the number of appearances of $\neg p$ in the same multiset. Of-course, the opposite may also be stated when incorporating mathematical objects other than (finite) sets. That is, it is possible to explicitly indicate that the order and/or the number of appearances of formulas do *not* matter, by introducing (either of) the following standard structural rules:

$$\text{Permutation:} \quad \frac{\Gamma_1, \psi, \varphi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \varphi, \psi, \Gamma_2 \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta_1, \psi, \varphi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \varphi, \psi, \Delta_2}$$

$$\text{Contraction:} \quad \frac{\Gamma_1, \psi, \psi, \Gamma_2 \Rightarrow \Delta}{\Gamma_1, \psi, \Gamma_2 \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \Delta_1, \psi, \psi, \Delta_2}{\Gamma \Rightarrow \Delta_1, \psi, \Delta_2}$$

– The incorporation of more complex forms of sequents, such as nested sequents [15] or hypersequents [5], allows to express more sophisticated forms of argumentation, such as argumentation by counterfactuals or case-based argumentation. For instance, the nested sequent $\Gamma_1 \Rightarrow (\Gamma_2 \Rightarrow \psi)$ may be intuitively understood by "if Γ_1 were true, one would argue that $\Gamma_2 \Rightarrow \psi$" and the hypersequent $\Gamma_1 \Rightarrow \psi_1 \mid \Gamma_2 \Rightarrow \psi_2$ may be understood (again, intuitively) as a disjunction, at the meta-level, of the arguments $\Gamma_1 \Rightarrow \psi_1$ and $\Gamma_2 \Rightarrow \psi_2$.

6 Conclusion and Further Work

The contribution of this paper is mainly conceptual. It raises some basic questions on the definition of arguments in the context of logic-based argumentation, and claims that sequent-based representation and reasoning is an appropriate setting for logic-based modeling of argumentation systems. Among others, this approach enables a general and natural way of expressing arguments and implies that well-studied techniques and methodologies may be borrowed from proof theory and applied in the context of argumentation theory.

The starting point of this paper is Besnard and Hunter's approach to logical argumentation, which we believe is a successful way of representing deductive reasoning in argumentation-based environments (Comparisons to other logic-based approaches, such as those that are based on defeasible logics [30, 33], can be found e.g. in [13]). Our work extends this approach in several ways: first, the usual conditions of minimality and consistency of supports are abandoned. This offers a simpler way of producing arguments and identifying them (also for systems that are not formulated in a Gentzen-type style). Second, arguments are produced and are withdrawn by rules of the same style, allowing for a more uniform way of representing the frameworks and computing their extensions. What is more, as noted previously, the representation of arguments as inferences suggests that techniques of proof theory may be incorporated.[10] Third, our approach

[10] Other techniques for generating arguments are considered, e.g., in [11] and [24].

is logic-independent. This allows in particular to rely on a classical as well as on a non-classical logic, and so, for instance, paraconsistent formalisms may be used for improving consistency-maintenance. Logic independence also implies that our approach is appropriate for multi-agent environments, involving different logics for different agents.

Much is still need to be done in order to tighten the links between abstract and logical argumentation theories. For instance, it would be interesting to investigate what *specific* logics and attack relations yield useful frameworks, and whether the argumentation semantics that they induce give intuitively acceptable solutions to practical problems. Another important issue for further exploration is the computational aspect of our approach, which so far remains mainly on the representation level. This requires an automated machinery that not only produces sequents, but is also capable of eliminating them, as well as their consequences. Here, techniques like those used in the context of dynamic proof theory for adaptive logics may be useful (see, e.g., [8, 9]).

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