Maximally Paraconsistent Three-Valued Logics

Ofer Arieli
School of Computer Science
The Academic College of Tel-Aviv
Israel

Arnon Avron
School of Computer Science
Tel-Aviv University
Israel

Anna Zamansky
Department of Software Engineering
Jerusalem College of Engineering
Israel

Abstract

Maximality is a desirable property of paraconsistent logics, motivated by the aspiration to tolerate inconsistencies, but at the same time retain from classical logic as much as possible. In this paper, we introduce the strongest possible notion of maximal paraconsistency, and investigate it in the context of logics that are based on deterministic or non-deterministic three-valued matrices. We first show that most of the logics that are based on properly non-deterministic three-valued matrices are not maximally paraconsistent. Then we show that in contrast, in the deterministic case all the natural three-valued paraconsistent logics *are* maximal. This includes well-known three-valued paraconsistent logics like P_1, LP, J_3, PAC and SRM_3, as well as any extension of them obtained by enriching their languages with extra three-valued connectives.

Introduction

Handling contradictory data is one of the most complex and important problems in reasoning under uncertainty. This problem is described in (Delgrande and Mylopoulos 1985) as follows:

> "It is a fact of life that large knowledge bases are inherently inconsistent, in the same way large programs are inherently buggy. Moreover, within a conventional logic, the inconsistency of a knowledge base has the catastrophic consequence that *everything* is derivable from the knowledge base."

One method of handling inconsistent information is using a *paraconsistent logic*. In contrast to classical logic, paraconsistent logics allow for non-trivial inconsistent theories. Newton da Costa, one of the main founders of paraconsistent logic, believed that one of the most important requisites of such logics should be their *maximality* with respect to classical logic: while tolerating inconsistent theories in a non-trivial way, one would still like to retain from classical logic as much as possible. Several well-known paraconsistent logics have been shown to have this property (such as Sette's P_1, Jaśkowski-D'ottaviano's J_3, etc.). However, the notions of maximality considered in the literature are based only on extending the set of *theorems* of a given logic. In this paper, we introduce a stronger notion of maximality, which is based on extending the underlying *consequence relations*: a paraconsistent logic $\mathbf{L}$ is *maximally paraconsistent* (in the strong sense), if every logic $\mathbf{L}'$ in the language of $\mathbf{L}$ that *properly* extends $\mathbf{L}$ (i.e., $\vdash_{\mathbf{L}} \subset \vdash_{\mathbf{L}'}$) is no longer paraconsistent. To show that our notion of maximality is strictly stronger than the usual one, we provide an example of a paraconsistent logic such that any extension in the same language of its set of theorems results in either classical logic or a trivial logic, yet it is not maximally paraconsistent in our sense.

In this paper, we investigate strong maximality of paraconsistent logics based on three-valued deterministic and non-deterministic matrices. The former are one of the oldest and most common ways of defining a paraconsistent logic. The latter are a recent natural generalization of the former, introduced in (Avron and Lev 2005), in which non-deterministic interpretations of connectives are allowed. Under a very minimal and natural assumption about the interpretation of negation in these matrices, we show that in the deterministic case, *all natural three-valued paraconsistent logics are maximal in the strong sense*. Our result applies to such well-known paraconsistent logics as Sette's logic P_1, Priest's LP, the semi-relevant logic SRM_3, the logics PAC and J_3, and any extension of one of these logics obtained by enriching its language with extra three-valued connectives.[1] In the non-deterministic case things are quite different, though. We show that paraconsistent logics induced by properly non-deterministic three-valued matrices are usually not maximal, except for a few special cases (which are fully characterized). However, even these exceptional cases are redundant, as we show that any maximally paraconsistent logic defined by an n-valued non-deterministic matrix can be fully characterized also by a deterministic one.

[1] Note that all these logics are monotonic; Non-monotonic paraconsistent logics are not investigated in this paper.

Preliminaries

Maximally Paraconsistent Logics

In the sequel, $\mathcal{L}$ denotes a propositional language with a set $\mathcal{A}_\mathcal{L}$ of atomic formulas and a set $\mathcal{W}_\mathcal{L}$ of well-formed formulas. We denote the elements of $\mathcal{A}_\mathcal{L}$ by p, q, r (possibly with subscripted indexes), and the elements of $\mathcal{W}_\mathcal{L}$ by ψ, ϕ, σ. Sets of formulas in $\mathcal{W}_\mathcal{L}$ are called *theories* and are denoted by Γ or Δ. Following the usual convention, we shall abbreviate $\Gamma \cup \{\psi\}$ by Γ, ψ. More generally, we shall write Γ, Δ instead of $\Gamma \cup \Delta$.

Definition 1 A (Tarskian) *consequence relation* for a language $\mathcal{L}$ (a tcr, for short) is a binary relation $\vdash$ between theories in $\mathcal{W}_\mathcal{L}$ and formulas in $\mathcal{W}_\mathcal{L}$, satisfying the following three conditions:

Reflexivity:	if $\psi \in \Gamma$ then $\Gamma \vdash \psi$.
Monotonicity:	if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$, then $\Gamma' \vdash \psi$.
Transitivity:	if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$.

Let $\vdash$ be a tcr for $\mathcal{L}$.

- We say that $\vdash$ is *structural*, if for every uniform $\mathcal{L}$-substitution θ and every Γ and ψ, if $\Gamma \vdash \psi$ then $\theta(\Gamma) \vdash \theta(\psi)$. (Where $\theta(\Gamma) = \{\theta(\gamma) \mid \gamma \in \Gamma\}$).
- We say that $\vdash$ is *consistent* (or *non-trivial*), if there exist some non-empty theory Γ and some formula ψ such that $\Gamma \not\vdash \psi$.
- We say that $\vdash$ is *finitary*, if for every theory Γ and every formula ψ such that $\Gamma \vdash \psi$ there is a *finite* theory $\Gamma' \subseteq \Gamma$ such that $\Gamma' \vdash \psi$.

Definition 2 A (propositional) *logic* is a pair $\langle \mathcal{L}, \vdash \rangle$, so that $\mathcal{L}$ is a propositional language, and $\vdash$ is a structural, consistent, and finitary consequence relation for $\mathcal{L}$.

Note 3 The conditions of being consistent and finitary are usually not required in the definitions of propositional logics. However, consistency is convenient for excluding trivial logics (those in which every formula follows from every theory, or every formula follows from every non-empty theory). The other property is assumed since we believe that it is essential for practical reasoning, where a conclusion is always derived from a finite set of premises. In particular, every logic that has a decent proof system is finitary.

A useful property of a propositional logic is that it admits the following stronger version of Transitivity (referring to a cut on multiple formulas):

Lemma 4 *Let $\langle \mathcal{L}, \vdash \rangle$ be a propositional logic. If $\Gamma \vdash \psi_i$ for every $\psi_i \in \Gamma'$, and $\Gamma, \Gamma' \vdash \phi$, then $\Gamma \vdash \phi$.*

Sketch of proof. Follows from Transitivity, by induction on the number of formulas in Γ' in case that Γ' is finite. The case in which Γ' is not finite is proved using the finitariness assumption on the logic. □

In this paper we are interested in consequence relations that tolerate inconsistent theories in a non-trivial way. This property, known as *paraconsistency*, is defined next.

Definition 5 (da Costa 1974; Jaśkowski 1999) A logic $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a language with a unary connective $\neg$, and $\vdash$ is a tcr for $\mathcal{L}$, is called $\neg$-*paraconsistent*, if there are formulas ψ, ϕ in $\mathcal{W}_\mathcal{L}$, such that $\psi, \neg\psi \not\vdash \phi$.

In what follows, when it is clear from the context, we shall sometimes omit the '$\neg$' symbol and simply refer to paraconsistent logics.

Note 6 As $\vdash$ is structural, it is enough to require in Definition 5 that there are *atoms* p, q such that $p, \neg p \not\vdash q$. The original definition is adequate also for non-structural consequence relations.

While paraconsistency is characterized by a 'negation connective', there is no general agreement about the properties that such a connective should satisfy.[2] Below, we assume some *very minimal* requirements that a negation connective should satisfy.

Definition 7 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic, where $\mathcal{L}$ is a language with a unary connective $\neg$.

- We say that $\neg$ is a *pre-negation* (for $\mathbf{L}$), if $p \not\vdash \neg p$ for atomic p.
- A pre-negation $\neg$ is a *weak negation* (for $\mathbf{L}$), if $\neg p \not\vdash p$ for atomic p.[3]

In what follows, when referring to $\neg$-paraconsistency we shall assume that $\neg$ is a pre-negation.

Definition 8 Let $\mathbf{L} = \langle \mathcal{L}, \vdash \rangle$ be a $\neg$-paraconsistent logic (where $\neg$ is a pre-negation for $\mathbf{L}$).

- We say that $\mathbf{L}$ is *maximally paraconsistent in the weak sense*, if every logic $\langle \mathcal{L}, \Vdash \rangle$ that extends $\mathbf{L}$ (i.e., $\vdash \subseteq \Vdash$) and whose set of theorems *properly includes* that of $\mathbf{L}$, is not $\neg$-paraconsistent.
- We say that $\mathbf{L}$ is *maximally paraconsistent in the strong sense*, if every logic $\langle \mathcal{L}, \Vdash \rangle$ that *properly extends* $\mathbf{L}$ (i.e., $\vdash \subset \Vdash$) is not $\neg$-paraconsistent.

Maximal paraconsistency in the weak sense is considered, e.g., in (Carnielli, Marcos, and de Amo 2000; Marcos 2005a) and in (Karpenko 2000), where it is noted, respectively, that Jaśkowski–D'ottaviano logic J_3 (D'Ottaviano 1985) and Sette's logic P_1 (Sette 1973) have this property. To the best of our knowledge, maximal paraconsitency in the strong sense has not been considered before. Clearly, maximal paraconsistency in the strong sense implies maximal paraconsistency in the weak sense. As we show next, the converse is not true: the notion of maximal paraconsistency in the weak sense, which is based only on extending the underlying set of *theorems*, is indeed weaker than our notion of maximal paraconsistency in the strong sense, that is based on extending the underlying *consequence relation*.

Example 9 What is usually known as Sobociński's three-valued logic (1952) has been motivated by the matrix $\mathcal{S} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\to}, \tilde{\neg}\} \rangle$ (see Definition 10), where $\tilde{\neg} t =$

[2] See, e.g., the papers collection in (Gabbay and Wansing 1999) that is devoted to this issue.

[3] Similar properties are considered, e.g., in (Marcos 2005b).

f, $\tilde{\neg} f = t$, $\tilde{\neg}\top = \top$, and the implication is interpreted as follows:

$$a \tilde{\rightarrow} b = \begin{cases} \top & \text{if } a = b = \top, \\ f & \text{if } a >_{\mathsf{t}} b \text{ (where } t >_{\mathsf{t}} \top >_{\mathsf{t}} f), \\ t & \text{otherwise.} \end{cases}$$

In (Sobociński 1952), the *set of valid sentences* of $\mathcal{S}$ was axiomatized by a Hilbert-type system $\mathsf{H}_{\mathcal{S}}$ with Modus Ponens as the single inference rule and it is shown that ψ is provable in $\mathsf{H}_{\mathcal{S}}$ iff ψ is valid in $\mathcal{S}$. In (Avron 1984) it is shown that the corresponding logic $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}} \rangle$ is maximally paraconsistent in the *weak* sense: any extension of the *set of theorems* of $\mathsf{H}_{\mathcal{S}}$ by a non-provable axiom yields either classical logic or a trivial logic. On the other hand, the logic $\langle \mathcal{L}, \vdash_{\mathsf{H}_{\mathcal{S}}} \rangle$ is *not* maximally $\neg$-paraconsistent in the *strong* sense, as $\vdash_{\mathcal{S}}$ (see Definition 12 below) is a proper extension of $\vdash_{\mathsf{H}_{\mathcal{S}}}$. Indeed, it holds that $\neg(p \rightarrow q) \vdash_{\mathcal{S}} p$ while $\neg(p \rightarrow q) \not\vdash_{\mathsf{H}_{\mathcal{S}}} p$.

In what follows, when referring to 'maximal paraconsistency' we shall mean the strong sense of this notion.

Matrices and Their Consequence Relations

The most standard semantic (model-theoretical) way of defining a consequence relation (and so a logic) is by using the following type of structures (see, e.g., (Gottwald 2001) or (Urquhart 2001)).

Definition 10 A (multi-valued) *matrix* for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set of truth values,
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$, called the *designated* elements of $\mathcal{V}$, and
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \rightarrow \mathcal{V}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

The set $\mathcal{D}$ is used for defining satisfiability and validity, as defined below:

Definition 11 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for $\mathcal{L}$.

- An $\mathcal{M}$-*valuation* for $\mathcal{L}$ is a function $\nu : \mathcal{W}_{\mathcal{L}} \rightarrow \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$, $\nu(\diamond(\psi_1, \ldots, \psi_n)) = \widetilde{\diamond}_{\mathcal{M}}(\nu(\psi_1), \ldots, \nu(\psi_n))$. We denote the set of all the $\mathcal{M}$-valuations by $\Lambda_{\mathcal{M}}$.
- A valuation $\nu \in \Lambda_{\mathcal{M}}$ is an $\mathcal{M}$-*model* of a formula ψ (alternatively, ν $\mathcal{M}$-*satisfies* ψ), if it belongs to the set $mod_{\mathcal{M}}(\psi) = \{\nu \in \Lambda_{\mathcal{M}} \mid \nu(\psi) \in \mathcal{D}\}$. The $\mathcal{M}$-models of a theory Γ are the elements of the set $mod_{\mathcal{M}}(\Gamma) = \cap_{\psi \in \Gamma}\, mod_{\mathcal{M}}(\psi)$.
- A formula ψ is $\mathcal{M}$-*satisfiable* if $mod_{\mathcal{M}}(\psi) \neq \emptyset$. A theory Γ is $\mathcal{M}$-satisfiable (or $\mathcal{M}$-*consistent*) if $mod_{\mathcal{M}}(\Gamma) \neq \emptyset$.

In what follows we shall sometimes omit the prefix '$\mathcal{M}$' from the notions above. Also, when it is clear from the context, we shall omit the subscript '$\mathcal{M}$' in $\widetilde{\diamond}_{\mathcal{M}}$.

Definition 12 Given a matrix $\mathcal{M}$, the relation $\vdash_{\mathcal{M}}$ that is *induced by* (or associated with) $\mathcal{M}$, is defined by: $\Gamma \vdash_{\mathcal{M}} \psi$ if $mod_{\mathcal{M}}(\Gamma) \subseteq mod_{\mathcal{M}}(\psi)$. We denote by $\mathbf{L}_{\mathcal{M}}$ the pair $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, where $\mathcal{M}$ is a matrix for $\mathcal{L}$ and $\vdash_{\mathcal{M}}$ is the relation induced by $\mathcal{M}$.

Example 13 Propositional classical logic is induced from the two-valued matrix $\langle \{t, f\}, \{t\}, \{\tilde{\wedge}, \tilde{\neg}\} \rangle$ with the standard two-valued interpretations for $\wedge$ and $\neg$.

The following proposition has been proven in (Shoesmith and Smiley 1971; 1978).

Proposition 14 *For every propositional language $\mathcal{L}$ and a finite matrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.*[4]

The next proposition is straightforward.

Proposition 15 *Let $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ be a logic induced by a matrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for a language $\mathcal{L}$ with a unary connective $\neg$. Denote $\overline{\mathcal{D}} = \mathcal{V} \setminus \mathcal{D}$. Then:*

a) *$\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$, iff there is an element $x \in \mathcal{D}$ such that $\tilde{\neg} x \in \overline{\mathcal{D}}$.*

b) *$\neg$ is a weak negation for $\mathbf{L}_{\mathcal{M}}$, iff it is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ and there is $x \in \overline{\mathcal{D}}$ such that $\tilde{\neg} x \in \mathcal{D}$.*[5]

Non-Deterministic Matrices

The semantics of (standard) matrices is based on the principle of truth-functionality. According to this principle, the truth-value of a complex formula is uniquely determined by the truth-values of its subformulas. However, real-world information is inescapably incomplete, uncertain, vague, imprecise or inconsistent. All these phenomena are related to non-deterministic behavior. The sources of this may vary: partially unknown information, faulty behavior of devices and ambiguity of natural languages are just a few cases in point. Now, truth-functional semantics cannot capture non-deterministic behavior.

A possible solution to this problem is to borrow the idea of *non-deterministic* computations from computability theory and apply it to evaluations of formulas. This leads to the concept of *non-deterministic matrices* (Nmatrices), introduced in (Avron and Lev 2005).[6] Nmatrices are a natural generalization of ordinary matrices, where the truth-value of a formula is chosen non-deterministically from some set of options. They have important applications in reasoning under uncertainty, proof theory, etc. This includes modeling of non-deterministic computations, analysis of non-deterministic behavior of various elements of electrical circuits, handling linguistic ambiguity, and representing incomplete and inconsistent information. For instance, in (Avron and Konikowska 2005) Nmatrices are utilized for knowledge-base integration, and in (Arieli and Zamansky 2009) they are used in the context of distance-based reasoning. In (Avron 2007) and in (Zamansky and Avron 2007) Nmatrices have been used to provide simple and modular non-deterministic semantics for the large family of paraconsistent logics which has been developed by da Costa's

[4] The non-trivial part in this result is that $\vdash_{\mathcal{M}}$ is finitary; It is easy to see that for *every* matrix $\mathcal{M}$ (not necessarily finite), $\vdash_{\mathcal{M}}$ is a structural and consistent tcr.

[5] See also the discussion in (Marcos 2005b).

[6] See (Avron and Zamansky 2010) for a comprehensive survey on Nmatrices.

school, and is known as Logics of Formal Inconsistency (LFIs) (Carnielli, Coniglio, and Marcos 2007).

Below, we recall the basic definitions behind Nmatrices and their logics.

Definition 16 A *non-deterministic matrix* (Nmatrix) for a language $\mathcal{L}$ is a triple $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, where

- $\mathcal{V}$ is a non-empty set (of truth values),
- $\mathcal{D}$ is a non-empty proper subset of $\mathcal{V}$ (the designated elements of $\mathcal{V}$),
- $\mathcal{O}$ includes an n-ary function $\widetilde{\diamond}_{\mathcal{M}} : \mathcal{V}^n \to 2^{\mathcal{V}} \setminus \{\emptyset\}$ for every n-ary connective $\diamond$ of $\mathcal{L}$.

An n-ary connective $\diamond$ is *non-deterministic in* $\mathcal{M}$, if there are some $x_1, \ldots, x_n \in \mathcal{V}$, such that $\tilde{\diamond}(x_1, \ldots, x_n)$ is not a singleton. An Nmatrix $\mathcal{M}$ for $\mathcal{L}$ is called *deterministic* if no connective of $\mathcal{L}$ is non-deterministic in $\mathcal{M}$. Clearly, the matrices considered in the previous section may be associated with corresponding deterministic Nmatrices. We shall say that a matrix $\mathcal{M}$ is *properly* non-deterministic if at least one of the connectives of $\mathcal{L}$ is non-deterministic in $\mathcal{M}$.

Definition 17 Let $\mathcal{M}$ be an Nmatrix for $\mathcal{L}$. An $\mathcal{M}$-*valuation* ν is a function $\nu : \mathcal{W}_{\mathcal{L}} \to \mathcal{V}$ such that for every n-ary connective $\diamond$ of $\mathcal{L}$ and every $\psi_1, \ldots, \psi_n \in \mathcal{W}_{\mathcal{L}}$,

$$\nu(\diamond(\psi_1, \ldots, \psi_n)) \in \tilde{\diamond}(\nu(\psi_1), \ldots, \nu(\psi_n)).$$

As before, we denote the set of all $\mathcal{M}$-valuations by $\Lambda_{\mathcal{M}}$. The notions of a *model* of a formula ψ and of a theory Γ are defined just as in the deterministic case (see Definition 11). Similarly, the relation $\vdash_{\mathcal{M}}$ that is induced by $\mathcal{M}$ is defined exactly as before (see Definition 12).

As in the deterministic case (see Proposition 14), we have the following result:

Proposition 18 (Avron and Lev 2005) *For every propositional language $\mathcal{L}$ and a finite Nmatrix $\mathcal{M}$ for $\mathcal{L}$, $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ is a propositional logic.*

Example 19 Let $\mathcal{M}_2 = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ be an Nmartix for the language $\mathcal{L}_{cl}$ of classical logic, where $\tilde{\neg} f = \{t\}$, $\tilde{\neg} t = \{t, f\}$, and the rest of the connectives are interpreted classically. In (Avron and Lev 2005) it is shown that $\mathbf{L}_{\mathcal{M}_2}$ is the same as the paraconsistent adaptive logic **CLuN** (Batens, De Clercq, and Kurtonina 1999), however it is *not* induced by any finite deterministic matrix. Moreover, as shown in (Avron and Lev 2005), *none* of the two-valued proper Nmatrices can be characterized by a finite (deterministic) matrix. This shows, in particular, that the expressive power of Nmatrices is in general greater than that of ordinary matrices.

The following operation on Nmatrices will be useful in what follows.

Definition 20 Let $\mathcal{M}_1 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_1 \rangle$ and $\mathcal{M}_2 = \langle \mathcal{V}, \mathcal{D}, \mathcal{O}_2 \rangle$ be Nmatrices for a language $\mathcal{L}$. $\mathcal{M}_1$ is a *simple $\diamond$-refinement* of $\mathcal{M}_2$, if $\tilde{\diamond}_{\mathcal{M}_1}(\overline{x}) \subseteq \tilde{\diamond}_{\mathcal{M}_2}(\overline{x})$ for every n-tuple $\overline{x} \in \mathcal{V}^n$. $\mathcal{M}_1$ is a *simple refinement* of $\mathcal{M}_2$, if $\mathcal{M}_1$ is a simple $\diamond$-refinement of $\mathcal{M}_2$ for every $\diamond$ in $\mathcal{L}$.

Note 21 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for $\mathcal{L}$. A (deterministic) matrix, obtained by choosing one element from each set $\tilde{\diamond}_{\mathcal{M}}(\overline{x})$ (i.e., for every set determined by a connective $\diamond$ in $\mathcal{L}$ and an n-tuple $\overline{x} \in \mathcal{V}^n$), is a simple refinement of $\mathcal{M}$.

Proposition 22 (Avron 2007) *If $\mathcal{M}_1$ is a simple refinement of $\mathcal{M}_2$ then $\vdash_{\mathcal{M}_2} \subseteq \vdash_{\mathcal{M}_1}$.*

Example 23 The two-valued (deterministic) matrix $\mathcal{M}_{cl} = \langle \{t, f\}, \{t\}, \mathcal{O} \rangle$ with ordinary interpretations for the connectives of the standard propositional language $\mathcal{L}_{cl}$, is a simple refinement of the matrix $\mathcal{M}_2$ considered in Example 19. By Proposition 22 and the fact that $\mathbf{L}_{\mathcal{M}_2}$ is paraconsistent while classical logic is not, we have that $\mathbf{L}_{\mathcal{M}_2}$ is strictly weaker than classical logic.

The next proposition is the analogue for the non-deterministic case of Proposition 15.

Proposition 24 *Let $\mathbf{L}_{\mathcal{M}} = \langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$ be a logic induced by an Nmatrix $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ for a language $\mathcal{L}$ with a unary connective $\neg$. Then:*

a) $\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ iff there is an element $x \in \mathcal{D}$ such that $\tilde{\neg} x \cap \overline{\mathcal{D}} \neq \emptyset$.

b) $\neg$ is a weak negation for $\mathbf{L}_{\mathcal{M}}$ iff it is a pre-negation for $\mathbf{L}_{\mathcal{M}}$ and there is $x \in \overline{\mathcal{D}}$ such that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$.

What Three-Valued Non-deterministic Matrices Induce Maximally Paraconsistent Logics?

We now investigate maximal paraconsistency of logics induced by (proper) non-deterministic matrices. In what follows $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ still denotes an Nmatrix for a language $\mathcal{L}$ with a pre-negation $\neg$. We say that $\mathcal{M}$ is (maximally) $\neg$-paraconsistent, if so is $\mathbf{L}_{\mathcal{M}}$.

Proposition 25 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be an Nmatrix for a language $\mathcal{L}$ with a pre-negation $\neg$. Then $\mathcal{M}$ is paraconsistent iff there is some $x \in \mathcal{D}$ such that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$.*

Proof. Suppose that $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$ and let $y \in \tilde{\neg} x \cap \mathcal{D}$. Let $\nu \in \Lambda_{\mathcal{M}}$ such that $\nu(p) = x$, $\nu(\neg p) = y$ and $\nu(q) \in \overline{\mathcal{D}}$. Then ν is an $\mathcal{M}$-model of $\{p, \neg p\}$ but not an $\mathcal{M}$-model of q, hence $p, \neg p \not\vdash_{\mathcal{M}} q$, and so $\mathcal{M}$ is $\neg$-paraconsistent. Conversely, if $\mathcal{M}$ is $\neg$-paraconsistent, then $p, \neg p \not\vdash_{\mathcal{M}} q$ for some $p, q \in \mathcal{A}_{\mathcal{L}}$, and so $mod_{\mathcal{M}}(\{p, \neg p\}) \neq \emptyset$. It follows that there is an $\mathcal{M}$-valuation ν and some $x, y \in \mathcal{D}$ such that $x = \nu(p)$, and $y \in \tilde{\neg} \nu(p)$. Thus, $y \in \mathcal{D} \cap \tilde{\neg} x$, and so $\tilde{\neg} x \cap \mathcal{D} \neq \emptyset$. □

Theorem 26 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a paraconsistent Nmatrix for a language $\mathcal{L}$ with pre-negation $\neg$. If either of the following conditions holds, then $\mathcal{M}$ is* not *maximally $\neg$-paraconsistent:*

a) $\mathcal{D}$ is a singleton.

b) There is $x \in \mathcal{D}$ such that $x \in \tilde{\neg} x$ and $\tilde{\neg} x \cap \overline{\mathcal{D}} \neq \emptyset$.

c) $\mathcal{M}$ is 3-valued proper Nmatrix which is not isomorphic to an Nmatrix in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg} t = \{f\}$, $\tilde{\neg} \top = \{t, f\}$ and $\tilde{\neg} f = \{f\}$ or $\tilde{\neg} f = \{t\}$.

Sketch of proof. The proof of all the three parts is based on the fact that a paraconsistent Nmatrix $\mathcal{M}$ is *non-maximal*, if it is refined by a paraconsistent Nmatrix $\mathcal{M}^*$ (in which $\neg$ is still a pre-negation) and $\vdash_{\mathcal{M}^*} \setminus \vdash_{\mathcal{M}}$ is not empty. Proposition 22 implies that in this case $\mathbf{L}_{\mathcal{M}}$ cannot be maximally paraconsistent. We demonstrate this line of proof by showing Part (b) of the theorem.

Let $\mathcal{M}$ be a paraconsistent Nmatrix, and suppose that there is $x \in \mathcal{D}$ such that $x \in \tilde{\neg}x$ and $\tilde{\neg}x \cap \overline{\mathcal{D}} \neq \emptyset$. By Part (a), if $\mathcal{D} \setminus \{x\} = \emptyset$ then $\mathcal{M}$ cannot be maximal. So let $y \in \mathcal{D} \setminus \{x\}$. Now,

1. if $\tilde{\neg}y \cap \mathcal{D} \neq \emptyset$, we let $\mathcal{M}^*$ be a simple deterministic refinement of $\mathcal{M}$ such that $\tilde{\neg}x \in \overline{\mathcal{D}}$ and $\tilde{\neg}y \in \mathcal{D}$.
2. if $\tilde{\neg}y \cap \overline{\mathcal{D}} \neq \emptyset$, we let $\mathcal{M}^*$ be a simple deterministic refinement of $\mathcal{M}$ such that $\tilde{\neg}x = x$ and $\tilde{\neg}y \in \overline{\mathcal{D}}$.

In both cases $\neg$ is still a pre-negation in $\mathcal{M}^*$, $\mathcal{M}^*$ is $\neg$-paraconsistent, and the logic induced by $\mathcal{M}^*$ extends the logic induced by $\mathcal{M}$ (see Proposition 22). Now, if $\mathcal{V}$ has n elements, then $\mathcal{M}^*$ is an n-valued deterministic matrix, and so $p, \neg p, \neg\neg p, \ldots, \neg^{n-1}p \vdash_{\mathcal{M}^*} \neg^n p$.[7] On the other hand, $p, \neg p, \neg\neg p, \ldots, \neg^{n-1}p \nvdash_{\mathcal{M}} \neg^n p$, since we may take $\nu(p) = \nu(\neg p) = \ldots = \nu(\neg^{n-1}p) = x$, and $\nu(\neg^n p) \in \overline{\mathcal{D}}$. Thus, the logic induced by $\mathcal{M}^*$ properly extends the logic induced by $\mathcal{M}$, and so $\mathcal{M}$ is not maximally paraconsistent. □

Corollary 27 *There is no maximally paraconsistent two-valued Nmatrix.*

Proof. By Part (a) of Theorem 26, since in a two-valued Nmatrix $\mathcal{D}$ is a singleton. □

Corollary 28 *A three-valued proper Nmatrix $\mathcal{M}$ for a language with a* weak *negation $\neg$ can be maximally paraconsistent only if it is isomorphic to an Nmatrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$, in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg}t = \{f\}$, $\tilde{\neg}\top = \{t, f\}$ and $\tilde{\neg}f = \{t\}$.*

Theorem 26 covers all the possibilities of having maximally $\neg$-paraconsistent proper three-valued Nmatrices, except of the following two cases:

a) The interpretation of $\neg$ is given by $\tilde{\neg}t = \{f\}$, $\tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg}f = \{t\}$.

b) The interpretation of $\neg$ is given by $\tilde{\neg}t = \{f\}$, $\tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg}f = \{f\}$.

Next, we consider these cases. First, we suppose that $\neg$ is the only connective of the language. As the next proposition shows, in this case we get maximally paraconsistent logics.

Proposition 29 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a three-valued paraconsistent Nmatrix for a language $\mathcal{L}$ with $\neg$ as the only connective. Suppose that $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, and $\neg$ is interpreted in $\mathcal{M}$ by $\tilde{\neg}t = \{f\}$, $\tilde{\neg}\top = \{t, f\}$, and either $\tilde{\neg}f = \{t\}$ or $\tilde{\neg}f = \{f\}$. Then $\mathcal{M}$ is maximally paraconsistent with respect to $\neg$.*

Proof. We show the case where $\tilde{\neg}f = \{t\}$, leaving the other case to the reader. Let $\mathbf{L}_{\mathcal{M}} = \langle \vdash, \mathcal{L} \rangle$ and let $\vdash'$ be a proper extension of $\vdash$. So there is a finite Γ and a formula ψ s.t. $\Gamma \vdash' \psi$ but $\Gamma \nvdash \psi$. Since $\neg\neg\neg\phi$ is equivalent in $\vdash$ to $\neg\phi$, we may assume that $\Gamma \cup \{\psi\}$ consists only of formulas of the forms p, $\neg p$, or $\neg\neg p$, where p is atomic. Moreover: since Γ cannot contain both $\neg\neg p$ and $\neg p$ (otherwise $\Gamma \vdash \psi$), and $\neg\neg p \vdash p$, we may assume that if $\neg\neg p$ is in Γ then neither p nor $\neg p$ is in Γ.

1. Suppose that $\psi = \neg r$ for atomic r. Then $\neg r \notin \Gamma$. It follows (using weakening if necessary and the fact that $\neg\neg r \vdash r$) that $\Gamma', \neg\neg r \vdash' \neg r$, where r does not occur in Γ' and Γ' has the same properties we assume about Γ. Substituting r for any p such that $\neg\neg p \in \Gamma'$, and q for any other atom occurring in Γ' (and using weakenings if necessary), we get that $q, \neg q, \neg\neg r \vdash' \neg r$. Since $\neg\neg r, \neg r \vdash p$ for any p, we get that $q, \neg q, \neg\neg r \vdash' p$ for any $p, q, r \in \mathcal{A}_{\mathcal{L}}$. Substituting $\neg q$ for r and using the fact that $\neg q \vdash \neg\neg\neg q$, we get that $\neg q, q \vdash' p$ for every $p, q \in \mathcal{A}_{\mathcal{L}}$.
2. Suppose that $\psi = r$ for atomic r. Then neither r nor $\neg\neg r$ is in Γ. Substituting $\neg r$ for r we return to the previous case, and so again $\vdash'$ is not paraconsistent.
3. Suppose that $\psi = \neg\neg r$ for atomic r. Then $\neg\neg r \notin \Gamma$. Since $\neg\neg r, \neg r \vdash q$, also $\neg\neg r, \neg r \vdash' q$, and since $\Gamma \vdash' \neg\neg r$ we get that $\Gamma, \neg r \vdash' q$ for any q that does not occur in Γ and $\neg r$. By substituting $\neg r$ for any p such that $\neg\neg p \in \Gamma$ (such p is necessarily different from r), and r for any atom that is different from q and such that $\neg\neg p$ does not occur in Γ, we get (using weakenings and the fact that $\neg r \vdash \neg\neg\neg r$) that $r, \neg r \vdash' q$. Hence again $\vdash'$ is not paraconsistent. □

Note that each of the Nmatrices considered in Proposition 29 induces the same logic as its refinement by a matrix $\mathcal{M}'$, in which $\tilde{\neg}_{\mathcal{M}'}\top = \{t\}$. This shows that the maximally paraconsistent logics that are obtained in these cases can be induced by *deterministic matrices* (cf. Theorem 33).

Next, we assume that in addition to $\neg$ there is another connective with a corresponding properly non-deterministic interpretation. In this case maximal paraconsistency cannot be achieved.

Proposition 30 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a three-valued paraconsistent Nmatrix for a language $\mathcal{L}$ that includes the connectives $\neg$ and $\diamond$. Suppose that $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\diamond$ has a properly non-deterministic interpretation in $\mathcal{M}$, and $\neg$ is interpreted in $\mathcal{M}$ by $\tilde{\neg}t = \{f\}, \tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg}f = \{f\}$ or $\tilde{\neg}f = \{t\}$. Then $\mathcal{M}$ is not maximally paraconsistent with respect to $\neg$.*

Proof. Let $\mathcal{M}'$ be a simple refinement of $\mathcal{M}$ that is the same as $\mathcal{M}$ except that $\tilde{\neg}\top = \{t\}$. Then $\mathcal{M}'$ is still $\neg$-paraconsistent and $\neg$ is still a pre-negation for $\mathbf{L}_{\mathcal{M}'}$. However, by Part (c) of Theorem 26, $\mathcal{M}'$ is not maximally paraconsistent. As $\vdash_{\mathcal{M}} \subseteq \vdash_{\mathcal{M}'}$ (Proposition 22), $\mathcal{M}$ cannot be maximally paraconsistent either. □

It remains to check the case where all the connectives of the language, except for $\neg$ (which is interpreted by $\tilde{\neg}\top =$

[7] See the proof of Theorem 3.4 in (Avron and Lev 2005). Here, as usual, $\neg^k\psi$ abbreviates the formula that consists of k occurrences of $\neg$ followed by ψ.

$\{t, f\}$, $\tilde{\neg}t = \{f\}$, and $\tilde{\neg}f = \{t\}$ or $\tilde{\neg}f = \{f\}$), have deterministic interpretations. As Proposition 31 and 32 show, in this case we don't have a unique answer to the question whether the underlying Nmatrix is maximally paraconsistent.

Proposition 31 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a three-valued paraconsistent Nmatrix for a language $\mathcal{L}$. Suppose that $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\neg$ is interpreted in $\mathcal{M}$ by $\tilde{\neg}t = \{f\}, \tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg}f = \{f\}$ or $\tilde{\neg}f = \{t\}$, and all the other connectives in $\mathcal{L}$ have deterministic interpretations. If all the complex formulas can get only values in $\{t, f\}$, then $\mathcal{M}$ is maximally paraconsistent under the conditions defined in Theorem 35 below.*

Proof. Let $\mathcal{M}'$ be a simple refinement of $\mathcal{M}$ in which $\tilde{\neg}\top = \{t\}$. For every $\nu \in \Lambda_{\mathcal{M}}$ consider a valuation $\nu' \in \Lambda_{\mathcal{M}'}$ that is the same as ν for every atom p, except that $\nu'(p) = t$ if $\nu(p) = \top$. By the fact that $\nu(\psi) \in \{t, f\}$ for every complex formula ψ, it is easy to see that for every formula ϕ, $\nu(\phi)$ is designated iff $\nu'(\phi)$ is designated. It follows that the logics induced by $\mathcal{M}$ and by $\mathcal{M}'$ are the same, and so, by Theorem 35 below, $\mathcal{M}$ is maximally paraconsistent. $\square$

Proposition 32 *Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a three-valued paraconsistent Nmatrix for a language $\mathcal{L}$. Suppose that $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\neg$ is interpreted in $\mathcal{M}$ by $\tilde{\neg}t = \{f\}, \tilde{\neg}\top = \{t, f\}$, and $\tilde{\neg}f = \{f\}$ or $\tilde{\neg}f = \{t\}$, and all the other connectives in $\mathcal{L}$ have deterministic interpretations. If there is a connective $\diamond$ which is $\top$-free (i.e., $\tilde{\diamond}(\top, \ldots, \top) = \{\top\}$), then $\mathcal{M}$ is not maximally paraconsistent.*

Proof. Let $\mathcal{M}'$ be a simple refinement of $\mathcal{M}$, in which $\tilde{\neg}_{\mathcal{M}'}\top = \{t\}$. Then $\mathcal{M}'$ is paraconsistent and $\neg$ is still a pre-negation for $\mathbf{L}_{\mathcal{M}'}$. Now, $p_1, \neg p_1, \ldots, p_n, \neg p_n \nvdash_{\mathcal{M}} \tilde{\neg}\tilde{\diamond}(p_1, \ldots, p_n)$ (consider $\nu \in \Lambda_{\mathcal{M}}$ where for all $1 \leq i \leq n$ $\nu(p_i) = \top$, $\nu(\neg p_i) = t$, and $\nu(\neg\diamond(p_1, \ldots, p_n)) = f$). On the other hand, $p_1, \neg p_1, \ldots, p_n, \neg p_n \vdash_{\mathcal{M}'} \tilde{\neg}\tilde{\diamond}(p_1, \ldots, p_n)$ (if ν' is an $\mathcal{M}'$-model of the left-hand side then $\nu'(p_i) = \top$ for every p_i, and as $\diamond$ is $\top$-free, $\nu'(\diamond(p_1, \ldots, p_n)) = \top$ as well. Thus, $\nu'(\neg\diamond(p_1, \ldots, p_n)) = t$). It follows that $\mathcal{M}$ is not maximally paraconsistent. $\square$

We conclude this section by showing that for characterizing three-valued maximally paraconsistent logics it is enough to consider only deterministic matrices.

Theorem 33 *Let $\mathcal{M}$ be an n-valued maximally paraconsistent Nmatrix. Then there is a (deterministic) matrix $\mathcal{M}^*$ which induces the same (maximally paraconsistent) logic.*

Proof. From Part (a) of Theorem 26 it follows that $\mathcal{D}$ has at least two elements. From this fact, together with Propositions 24 and 25, it follows that there are two different elements t and $\top$ in $\mathcal{D}$ and an element $f \in \overline{\mathcal{D}}$ such that $f \in \tilde{\neg}t$, while $\tilde{\neg}\top \cap \mathcal{D} \neq \emptyset$ (note that it is possible that also $\tilde{\neg}t \cap \mathcal{D} \neq \emptyset$, or that $\tilde{\neg}\top \cap \overline{\mathcal{D}} \neq \emptyset$, or that $\mathcal{D}$ contains other elements besides t and $\top$). Let $\mathcal{M}^*$ be any matrix which is a simple refinement of $\mathcal{M}$ for which $\tilde{\neg}_{\mathcal{M}^*}t = f$, and $\tilde{\neg}_{\mathcal{M}^*}\top \in \tilde{\neg}_{\mathcal{M}^*}\top \cap \mathcal{D}$. Then, by Proposition 22, the logic of $\mathcal{M}^*$ extends that of $\mathcal{M}$, and it is paraconsistent with respect to $\neg$ (which is still pre-negation in $\mathcal{M}^*$). Since $\mathcal{M}$ is maximally paraconsistent, this implies that $\vdash_{\mathcal{M}} = \vdash_{\mathcal{M}^*}$. $\square$

All Natural Three-Valued Paraconsistent Logics Induced by Deterministic Matrices are Maximal

By Theorem 33, it is enough to consider the maximal paraconsistency of logics induced by deterministic three-valued matrices. In this section we show that all natural paraconsistent logics of this kind are in fact maximal (in the strong sense).

Again, a matrix $\mathcal{M}$ is (maximally) $\neg$-paraconsistent iff so is $\mathbf{L}_{\mathcal{M}}$.

Proposition 34 *A three-valued matrix $\mathcal{M}$ with a pre-negation $\neg$ is $\neg$-paraconsistent iff it is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$, $\tilde{\neg}t = f$, and $\tilde{\neg}\top \neq f$.*

Proof. Suppose that $\mathcal{M}$ is isomorphic to a matrix $\langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ satisfying the conditions in the proposition. Since $\tilde{\neg}t = f$, by Item (a) in Proposition 15, $\neg$ is a pre-negation. Also, $\nu = \{p : \top, q : f\}$ is an $\mathcal{M}$-model of $\{p, \neg p\}$ that does not $\mathcal{M}$-satisfy q, thus $p, \neg p \nvdash_{\mathcal{M}} q$, and so $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent.

For the converse, suppose that $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent. Since $\neg$ is a pre-negation for $\mathbf{L}_{\mathcal{M}}$, by Item (a) in Proposition 15 again, there is an element in $\mathcal{D}$, denote it t, such that $\tilde{\neg}t \notin \mathcal{D}$. So let $f \in \overline{\mathcal{D}}$ such that $\tilde{\neg}t = f$. Also, since $\mathbf{L}_{\mathcal{M}}$ is $\neg$-paraconsistent, we have that $p, \neg p \nvdash_{\mathcal{M}} q$ for some $p, q \in \mathcal{A}_{\mathcal{L}}$, and so $mod_{\mathcal{M}}(\{p, \neg p\}) \neq \emptyset$. In this case t cannot be the only designated element, since otherwise for $\nu \in mod_{\mathcal{M}}(\{p, \neg p\})$ necessarily $\nu(p) = t$. But $\nu(\neg p) = \tilde{\neg}t = f \notin \mathcal{D}$, and so $\nu \notin mod_{\mathcal{M}}(\{p, \neg p\})$. It follows that $\mathcal{V} = \{t, \top, f\}$, where $\top \in \mathcal{D}$, and f is the only non-designated element. Also, by the discussion above, for $\nu \in mod_{\mathcal{M}}(\{p, \neg p\})$ necessarily $\nu(p) = \top$. This implies that $\nu(\neg p) = \tilde{\neg}\top \in \mathcal{D}$, and so $\tilde{\neg}\top \neq f$. $\square$

Theorem 35 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ with a pre-negation $\neg$. Suppose that there is a formula $\Psi(p, q)$ in $\mathcal{L}$ such that for all $\nu \in \Lambda_{\mathcal{M}}$ $\nu(\Psi) = t$ if either $\nu(p) \neq \top$ or $\nu(q) \neq \top$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. Let $\langle \mathcal{L}, \vdash \rangle$ be a (finitary) propositional logic that is strictly stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$. Then there is a finite theory Γ and a formula ψ in $\mathcal{L}$, such that $\Gamma \vdash \psi$ but $\Gamma \nvdash_{\mathcal{M}} \psi$. In particular, there is a valuation $\nu \in mod_{\mathcal{M}}(\Gamma)$ such that $\nu(\psi) = f$. Consider the substitution θ, defined for every $p \in \mathsf{Atoms}(\Gamma \cup \{\psi\})$ by

$$\theta(p) = \begin{cases} q_0 & \text{if } \nu(p) = t, \\ \neg q_0 & \text{if } \nu(p) = f, \\ p_0 & \text{if } \nu(p) = \top, \end{cases}$$

where p_0 and q_0 are two different atoms in $\mathcal{L}$. Note that $\theta(\Gamma)$ and $\theta(\psi)$ contain (at most) the variables p_0, q_0, and that for every valuation $\mu \in \Lambda_{\mathcal{M}}$ where $\mu(p_0) = \top$ and $\mu(q_0) = t$ it holds that $\mu(\theta(\phi)) = \nu(\phi)$ for every formula ϕ such that $\mathsf{Atoms}(\{\phi\}) \subseteq \mathsf{Atoms}(\Gamma \cup \{\psi\})$. Thus,

($\star$) *any* $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$, $\mu(q_0) = t$ is an $\mathcal{M}$-model of $\theta(\Gamma)$ that does not $\mathcal{M}$-satisfy $\theta(\psi)$.

Now, consider the following two cases:

Case I. There is a formula $\phi(p,q)$ such that for every $\mu \in \Lambda_{\mathcal{M}}$, $\mu(\phi) \neq \top$ if $\mu(p) = \mu(q) = \top$.

In this case, let $\mathsf{tt} = \Psi(q_0, \phi(p_0, q_0))$. Note that $\mu(\mathsf{tt}) = t$ for every $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$. Now, as $\vdash$ is structural, $\Gamma \vdash \psi$ implies that

$$\theta(\Gamma)\,[\mathsf{tt}/q_0] \vdash \theta(\psi)\,[\mathsf{tt}/q_0]. \tag{1}$$

Also, by the property of tt and by $(\star)$, any $\mu \in \Lambda_{\mathcal{M}}$ for which $\mu(p_0) = \top$ is a model of $\theta(\Gamma)\,[\mathsf{tt}/q_0]$ but does not $\mathcal{M}$-satisfy $\theta(\psi)\,[\mathsf{tt}/q_0]$. Thus,

- $p_0, \neg p_0 \vdash_{\mathcal{M}} \theta(\gamma)\,[\mathsf{tt}/q_0]$ for every $\gamma \in \Gamma$. As $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, this implies that

$$p_0, \neg p_0 \vdash \theta(\gamma)\,[\mathsf{tt}/q_0] \text{ for every } \gamma \in \Gamma. \tag{2}$$

- The set $\{p_0,\ \neg p_0,\ \theta(\psi)[\mathsf{tt}/q_0]\}$ is not $\mathcal{M}$-satisfiable, thus $p_0, \neg p_0, \theta(\psi)\,[\mathsf{tt}/q_0] \vdash_{\mathcal{M}} q_0$. Again, as $\langle \mathcal{L}, \vdash \rangle$ is stronger than $\langle \mathcal{L}, \vdash_{\mathcal{M}} \rangle$, we have that

$$p_0,\ \neg p_0,\ \theta(\psi)\,[\mathsf{tt}/q_0] \vdash q_0. \tag{3}$$

By (1)–(3) and by Lemma 4, $p_0, \neg p_0 \vdash q_0$, thus $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent.

Case II. For every formula ϕ in p, q and for every $\mu \in \Lambda_{\mathcal{M}}$, if $\mu(p) = \mu(q) = \top$ then $\mu(\phi) = \top$.

Again, as $\vdash$ is structural, and since $\Gamma \vdash \psi$,

$$\theta(\Gamma)\,[\Psi(q_0,q_0)/q_0] \vdash \theta(\psi)\,[\Psi(q_0,q_0)/q_0]. \tag{4}$$

In addition, $(\star)$ above entails that any valuation $\mu \in \Lambda_{\mathcal{M}}$ such that $\mu(p_0) = \top$ and $\mu(q_0) \in \{t, f\}$ is a model of $\theta(\Gamma)\,[\Psi(q_0,q_0)/q_0]$ which is not a model of $\theta(\psi)\,[\Psi(q_0,q_0)/q_0]$. Thus, the only $\mathcal{M}$-model of $\{p_0, \neg p_0, \theta(\psi)\,[\Psi(q_0,q_0)/q_0]\}$ is the one in which both of p_0 and q_0 are assigned the value $\top$. It follows that $p_0, \neg p_0, \theta(\psi)\,[\Psi(q_0,q_0)/q_0] \vdash_{\mathcal{M}} q_0$. Thus,

$$p_0, \neg p_0, \theta(\psi)\,[\Psi(q_0,q_0)/q_0] \vdash q_0. \tag{5}$$

By using $(\star)$ again (for $\mu(q_0) \in \{t, f\}$) and the condition of case II (for $\mu(q_0) = \top$), we have:

$$p_0, \neg p_0 \vdash \theta(\gamma)\,[\Psi(q_0,q_0)/q_0] \text{ for every } \gamma \in \Gamma. \tag{6}$$

Again, by (4)–(6) above and by Lemma 4, we have that $p_0, \neg p_0 \vdash q_0$, and so $\langle \mathcal{L}, \vdash \rangle$ is not $\neg$-paraconsistent in this case either. $\square$

Note 36

1. The requirement on the underlying language, stated in Theorem 35, is very minor, and all the interesting three-valued logics that we are aware of meet it (see Example 40 below).
2. Once a three-valued paraconsistent logic $\mathbf{L}$ satisfies the condition of Theorem 35, not only is it maximally paraconsistent, but so must be also any three-valued extension of it which is obtained by enriching the languages of $\mathbf{L}$ with extra three-valued connectives.

   Below are two particular cases of Theorem 35.

Definition 37 Let $\mathcal{M} = \langle \mathcal{V}, \mathcal{D}, \mathcal{O} \rangle$ be a matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$. Then $\neg$ is *an extension in* $\mathbf{L}_{\mathcal{M}}$ *of classical negation*, if there are $t \in \mathcal{D}$ and $f \in \overline{\mathcal{D}}$, such that $\tilde{\neg} t = f$ and $\tilde{\neg} f = t$.

Clearly, an extension in $\mathbf{L}_{\mathcal{M}}$ of classical negation is a weak negation for $\mathbf{L}_{\mathcal{M}}$. Moreover, by Proposition 34, when $\mathcal{M}$ is a three-valued paraconsistent matrix, the only extensions of classical negation are Kleene's negation (in which $\neg\top = \top$) and Sette's negation (in which $\neg\top = t$); See also Example 40 below.

Corollary 38 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$ that extends classical negation and a binary connective $+$ such that for every $x \in \mathcal{V}$, $x + t = t + x = t$. Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. By Theorem 35, where $\Psi(p,q) = (p{+}\neg p){+}(q{+}\neg q)$. $\square$

Corollary 39 *Let $\mathcal{M}$ be a three-valued paraconsistent matrix for a language $\mathcal{L}$ that includes a unary connective $\neg$ that extends classical negation and a propositional constant* f *(for which $\nu(\mathsf{f}) = f$ for all $\nu \in \Lambda_{\mathcal{M}}$). Then $\mathcal{M}$ is maximally $\neg$-paraconsistent for $\mathcal{L}$.*

Proof. By Theorem 35, where $\Psi(p,q) = \neg\mathsf{f}$. $\square$

Example 40 Theorem 35 and Corollaries 38, 39 imply that all of the following well-known three-valued logics are maximally paraconsistent for their languages:

- Sette's logic P_1 (Sette 1973), induced from the matrix $\mathsf{P}_1 = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\rightarrow}, \tilde{\neg}\} \rangle$ is maximally paraconsistent for the language of $\{\neg, \vee, \wedge, \rightarrow\}$ with the following interpretations of its connectives:

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	t
$\top$	t	t	t

$\tilde{\wedge}$	t	f	$\top$
t	t	f	t
f	f	f	f
$\top$	t	f	t

$\tilde{\rightarrow}$	t	f	$\top$
t	t	f	t
f	t	t	t
$\top$	t	f	t

$\tilde{\neg}$	
t	f
f	t
$\top$	t

- Priest's LP (Priest 1989), induced from the matrix $\mathsf{LP} = \langle \{t, f, \top\}, \{t, \top\}, \{\tilde{\vee}, \tilde{\wedge}, \tilde{\neg}\} \rangle$ is maximally paraconsistent for the language of $\{\neg, \vee, \wedge\}$ with the following standard interpretations (Kleene 1950):

$\tilde{\vee}$	t	f	$\top$
t	t	t	t
f	t	f	$\top$
$\top$	t	$\top$	$\top$

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$\top$
f	f	f	f
$\top$	$\top$	f	$\top$

$\tilde{\neg}$	
t	f
f	t
$\top$	$\top$

- Sobociński's three-valued logic S_3, induced by the matrix $\mathcal{S}$ considered in Example 9, is maximally paraconsistent, as the connective $+$, defined by $x + y = \neg x \rightarrow y$, meets the condition of Theorem 35.
- Let $\mathbf{L}$ be a logic that is obtained from either of P_1, LP or S_3 by enriching its language with extra three-valued connectives. Then $\mathbf{L}$ is also a maximally paraconsistent logic. This includes the following logics:

1. PAC (Batens 1980; Avron 1991), extending LP by an implication connective $\supset$, defined by: $x \supset y = y$ if $x \in \{t, \top\}$, otherwise $x \supset y = t$.
2. J_3 (D'Ottaviano 1985), obtained from PAC by adding the propositional constant f.
3. The logic of the *maximally monotonic* language in (Avron 1999) that consists of the connectives of LP and two propositional constants f and T, where the latter is defined by $\nu(\mathsf{T}) = \top$ for every $\nu \in \Lambda_{\mathcal{M}}$.
4. The logic of the *functionally complete* language in (Avron 1999), consisting of the connectives of PAC and the two propositional connectives considered in the previous item.
5. The semi-relevant logic SRM_3, that can be obtained from Sobocinski's three-valued matrix $\mathcal{S}$ by the addition of the standard three-valued interpretations for $\wedge$ and $\vee$, as in LP.

- All the three-valued logics of formal inconsistency (LFIs) that are shown in (Carnielli, Coniglio, and Marcos 2007; Marcos 2010) to be maximally paraconsistent (in the weak sense) with respect to classical logic, are also covered by Theorem 35. More specifically, these logics are induced by three-valued matrices for the language of $\{\neg, \circ, \vee, \wedge, \to\}$, in which $\mathcal{V} = \{t, \top, f\}$, $\mathcal{D} = \{t, \top\}$ and the interpretations of the connectives are as follows (below, we denote by '$x \wr y$' that x and y are two optional values):

$\tilde{\wedge}$	t	f	$\top$
t	t	f	$t \wr \top$
f	f	f	f
$\top$	$t \wr \top$	f	$t \wr \top$

$\tilde{\vee}$	t	f	$\top$
t	t	t	$t \wr \top$
f	t	f	$t \wr \top$
$\top$	$t \wr \top$	$t \wr \top$	$t \wr \top$

$\tilde{\to}$	t	f	$\top$
t	t	f	$t \wr \top$
f	t	t	$t \wr \top$
$\top$	$t \wr \top$	f	$t \wr \top$

	$\tilde{\neg}$	$\tilde{\circ}$
t	f	t
f	t	t
$\top$	$t \wr \top$	f

Thus, there are 2 interpretations for $\neg$, 2^3 interpretations for $\wedge$, 2^5 interpretations for $\vee$, and 2^4 interpretations for $\to$, altogether 2^{13} (8192) distinct logics.

- More generally, *any* three-valued paraconsistent logic with a language that includes the unary operators $\neg$ ('negation') and $\circ$ ('consistency'), where $\neg$ is interpreted as in J_3 or P_1 and $\circ$ is interpreted as in the previous item, by $\tilde{\circ}t = t$, $\tilde{\circ}f = t$, and $\tilde{\circ}\top = f$, is maximally paraconsistent in the strong sense, because $\nu(\circ \circ \psi) = t$ for every valuation ν and formula ψ. This includes, e.g., logics like J_3 and P_1, since $\circ$ with the above interpretation is definable in them.

Summary and Open Questions

In this paper, we have shown that all natural three-valued deterministic logics are maximal in the strong sense, while (almost) no three-valued non-deterministic logic is. An interesting related result is that maximal paraconsistency of finite matrices can be fully characterized by n-valued *deterministic* matrices. The questions whether *all* the three-valued paraconsistent logics induced by deterministic matrices are maximal, and is every maximally paraconsistent n-valued Nmatrix reducible to a three-valued matrix, remain open.

Acknowledgement

We thank João Marcos for his helpful comments. This research was supported by The Israel Science Foundation (grant No 809-06).

References

Arieli, O., and Zamansky, A. 2009. Distance-based non-determinisitc semantics for reasoning with uncertainty. *Logic Journal of the IGPL* 17(4):325–350.

Avron, A., and Konikowska, B. 2005. Multi-valued calculi for logics based on non-determinism. *Logic Journal of the IGPL* 13(4):365–387.

Avron, A., and Lev, I. 2005. Non-deterministic multi-valued structures. *Logic and Computation* 15:241–261.

Avron, A., and Zamansky, A. 2010. Non-deterministic semantics for logical systems – A survey. In Gabbay, D., and Guenthner, F., eds., *Handbook of Philosophical Logic*. Kluwer. To appear.

Avron, A. 1984. Relevant entailment – semantics and formal systems. *Journal of Symbolic Logic* 49:334–342.

Avron, A. 1991. Natural 3-valued logics: Characterization and proof theory. *Journal of Symbolic Logic* 56(1):276–294.

Avron, A. 1999. On the expressive power of three-valued and four-valued languages. *Logic and Computation* 9(6):977–994.

Avron, A. 2007. Non-deterministic semantics for logics with a consistency operator. *Journal of Approximate Reasoning* 45:271–287.

Batens, D.; De Clercq, K.; and Kurtonina, N. 1999. Embedding and interpolation for some paralogics. The propositional case. *Reports on Mathematical Logic* 33:29–44.

Batens, D. 1980. Paraconsistent extensional propositional logics. *Logique et Analyse* 90/91:195–234.

Carnielli, W.; Coniglio, M.; and Marcos, J. 2007. Logics of formal inconsistency. In Gabbay, D., and Guenthner, F., eds., *Handbook of Philosophical Logic*, volume 14. Springer. 1–93.

Carnielli, W.; Marcos, J.; and de Amo, S. 2000. Formal inconsistency and evolutionary databases. *Logic and logical philosophy* 8:115–152.

da Costa, N. C. 1974. On the theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic* 15:497–510.

Delgrande, J., and Mylopoulos, J. 1985. Knowledge representation: Features of knowledge. In *Advanced Course: Fundamentals of Artificial Intelligence*, LNCS 232, 3–36. Springer.

D'Ottaviano, I. M. L. 1985. The completeness and compactness of a three-valued first-order logic. *Revista Colombiana de Matematicas* XIX(1–2):31–42.

Gabbay, D., and Wansing, H. 1999. *What is Negation?*, volume 13 of *Applied Logic Series*. Springer.

Gottwald, S. 2001. *A Treatise on Many-Valued Logics, Studies in Logic and Computation*, volume 9. Research Studies Press, Baldock.

Jaśkowski, S. 1999. On the discussive conjunction in the propositional calculus for inconsistent deductive systems. *Logic and Logical Philosophy* 7:57–59. Translation of the original paper from 1949.

Karpenko, A. 2000. A maximal paraconsistent logic: The combination of two three-valued isomorphs of classical propositional logic. In *Frontiers of Paraconsistent Logic*. Research Studies Press. 181–187.

Kleene, S. C. 1950. *Introduction to Metamathematics*. Van Nostrand.

Marcos, J. 2005a. On a problem of da Costa. In Sica, G., ed., *Essays on the Foundations of Mathematics and Logic*, volume 2. Polimetrica. 39–55.

Marcos, J. 2005b. On negation: Pure local rules. *Journa of Applied Logic* 3(1):185–219.

Marcos, J. 2010. 8K solutions and semi-solutions to a problem of da Costa. Submitted.

Priest, G. 1989. Reasoning about truth. *Artificial Intelligence* 39:231–244.

Sette, A. M. 1973. On propositional calculus P_1. *Mathematica Japonica* 16:173–180.

Shoesmith, D. J., and Smiley, T. J. 1971. Deducibility and many-valuedness. *J. Symbolic Logic* 36:610–622.

Shoesmith, D. J., and Smiley, T. J. 1978. *Multiple Conclusion Logic*. Cambridge University Press.

Sobociński, B. 1952. Axiomatization of a partial system of three-value calculus of propositions. *Journal of Computing Systems* 1:23–55.

Urquhart, A. 2001. Many-valued logic. In Gabbay, D., and Guenthner, F., eds., *Handbook of Philosophical Logic*, volume II. Kluwer. 249–295.

Zamansky, A., and Avron, A. 2007. Effective non-deterministic semantics for first-order LFIs. *Multiple-Valued Logic and Soft Computing* 13:335–372.