Argumentative Approaches to Reasoning with Consistent Subsets of Premises

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Abstract. It has been shown that entailments based on the maximally consistent subsets (MCS) of a given set of premises can be captured by Dung-style semantics for argumentation frameworks. This paper shows that these links are much tighter and go way beyond simplified forms of reasoning with MCS. Among others, we consider different types of entailments that these kinds of reasoning induce, extend the framework for arbitrary (not necessarily maximal) consistent subsets, and incorporate non-classical logics. The introduction of declarative methods for reasoning with MCS by means of (sequent-based) argumentation frameworks provides, in particular, a better understanding of logic-based argumentation and allows to reevaluate some negative results concerning the latter.

1 Introduction

Reasoning with maximally consistent subsets (MCS) is a common way of maintaining consistency when the set of premises is contradictory. This approach has gained a considerable interest since its introduction by Rescher and Manor [19]. As a result, a number of applications of this approach and its extensions (e.g., [6,9]) were considered for different AI-related areas, such as integration systems [5], belief revision consistency operators [15], and computational linguistics [16].

The relation between MCS-based reasoning and argumentation theory has been already identified in the literature (see, e.g., [1,10,22]). Recently (see [4]), it was shown that sequent-based argumentation frameworks provide a useful platform for representing and reasoning with MCS. In this work we extend the results of [4] to several related formalisms for reasoning with consistent subsets. More specifically, we show that declarative methods based on Dung's semantics for argumentation frameworks [12] can be generalized to more extended settings in which the entailment relations may be moderated, the consistent subsets may not be maximal, and the underlying logic may not be classical logic.

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An important aspect of this work is that our generalizations allow to overcome some shortcoming of reasoning with maximal consistency by argumentation frameworks, reported in [1] (see Sect. 7). We believe that this helps to better understand and evaluate the role of logic-based argumentation systems in properly capturing deductive non-monotonic formalisms.

2 Sequent-Based Argumentation Frameworks

Below, we denote by $\mathcal{L}$ an arbitrary propositional language. Atomic formulas in $\mathcal{L}$ are denoted by p, q, compound formulas are denoted by ψ, ϕ, sets of formulas are denoted by S, T, and *finite* sets of formulas are denoted by Γ, Δ.[1]

Definition 1. A (propositional) *logic* for a language $\mathcal{L}$ is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, that is, a binary relation between sets of formulas and formulas in $\mathcal{L}$, which is reflexive (if $\psi \in \mathsf{S}$ then $\mathsf{S} \vdash \psi$), monotonic (if $\mathsf{S} \vdash \psi$ and $\mathsf{S} \subseteq \mathsf{S}'$, then $\mathsf{S}' \vdash \psi$) and transitive (if $\mathsf{S} \vdash \psi$ and $\mathsf{S}', \psi \vdash \phi$, then $\mathsf{S}, \mathsf{S}' \vdash \phi$).

A logical *argument* is usually regarded as a pair $\langle \Gamma, \psi \rangle$, where Γ is the support set of the argument and ψ is its conclusion (see, [1,7,8,14]). Since we are dealing with arbitrary Tarskian logics, a natural representation of arguments is by the proof theoretical notion of a *sequent* [13] (for a justification of this, see also [3]).

Definition 2. Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic and S a set of $\mathcal{L}$-formulas.

- A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ, Δ are finite sets of $\mathcal{L}$-formulas, and $\Rightarrow$ is a new symbol (not in $\mathcal{L}$).
- An $\mathfrak{L}$-*argument* (or just *argument*) is a sequent $\Gamma \Rightarrow \{\psi\}$, where $\Gamma \vdash \psi$.
- An *argument based on* S is a sequent $\Gamma \Rightarrow \{\psi\}$, for which $\Gamma \subseteq \mathsf{S}$. The set of all the $\mathfrak{L}$-arguments that are based on S is denoted $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$.

In what follows we shall omit the set signs around the premises and conclusions of arguments. We denote: $\mathsf{Prem}(\Gamma \Rightarrow \psi) = \Gamma$ and $\mathsf{Con}(\Gamma \Rightarrow \psi) = \psi$. For a set $\mathcal{S}$ of arguments, $\mathsf{Prem}(\mathcal{S}) = \bigcup\{\mathsf{Prem}(s) \mid s \in \mathcal{S}\}$ and $\mathsf{Con}(\mathcal{S}) = \bigcup\{\mathsf{Con}(s) \mid s \in \mathcal{S}\}$.

We shall use standard *sequent calculi* [13] for constructing arguments from simpler arguments. This is done by *inference rules* of the following form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ \ldots \ \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}. \tag{1}$$

We shall say that the sequents $\Gamma_i \Rightarrow \Delta_i$ ($i = 1, \ldots, n$) are the *conditions* (or the *prerequisites*) of the rule in (1) and that $\Gamma \Rightarrow \Delta$ is its *conclusion*.

Attack rules in our case allow for the elimination (discharging) of sequents. We shall denote by $\Gamma \not\Rightarrow \psi$ the elimination of the sequent $\Gamma \Rightarrow \psi$. Alternatively, $\overline{s}$

[1] Thus, unlike Γ, Δ, when S, T are assumed to be finite, this will be indicated explicitly.

denotes the elimination of s. Now, a *sequent elimination rule* (or an *attack rule*) is a rule of the following form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ \ldots \ \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n}. \tag{2}$$

The prerequisites of attack rules usually consist of three ingredients. The first sequent in the rule's prerequisites is the "attacking" sequent, the last sequent in the rule's prerequisites is the "attacked" sequent, and the other prerequisites are the conditions for the attack. Conclusions of elimination rules are the eliminations of the attacked arguments.

Example 1. The following rule is known as Undercut (abbreviation: Ucut):

$$\frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2' \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}.$$

This rule intuitively reflects the idea that an argument attacks another argument when the conclusion of the former contradicts some premises of the latter. We refer to [7,8,14,17] for other attack rules and to [3] for their representations by sequents. Further elimination rules for normative reasoning and deontic logics can be found in [20].

Given two arguments s_1 and s_2 in $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ and an elimination rule $\mathcal{R}$, we say that s_1 *$\mathcal{R}$-attacks* s_2 if s_1 is in the form of the attacker of $\mathcal{R}$, s_2 is in the form of the attacked sequent of $\mathcal{R}$, and all the conditions in $\mathcal{R}$ hold (i.e., are provable by the underlying sequent calculus).

Example 2. Let $\mathsf{S} = \{p, \neg p, q\}$ and denote classical logic by CL. Then $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ are both in $\mathsf{Arg}_{\mathsf{CL}}(\mathsf{S})$ and each one Undercut-attacks the other one.

Our setting induces an argumentation framework in the sense of Dung [12]:

Definition 3. Given a set S of $\mathcal{L}$-formulas, a *sequent-based argumentation framework* for S (induced by a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, a sequent calculus $\mathfrak{C}$ for $\mathfrak{L}$, and a set $\mathfrak{A}$ of attack rules) is the pair $\mathcal{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack} \rangle$, where $(s_1, s_2) \in \mathit{Attack}$ iff s_1 $\mathcal{R}$-attacks s_2 for some $\mathcal{R} \in \mathfrak{A}$.[2]

Following Dung [12], to define the sets of arguments (called *extensions*), the elements of which can collectively be accepted from a given sequent-based framework $\mathcal{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack} \rangle$, we first extend the notion of attack to sets of arguments. A set $\mathcal{S} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ *attacks* an argument t if there is an argument $s \in \mathcal{S}$ that attacks t (i.e., $(s, t) \in \mathit{Attack}$). The set of arguments that are attacked by $\mathcal{S}$ is denoted $\mathcal{S}^+$. We say that $\mathcal{S}$ *defends* s if $\mathcal{S}$ attacks every argument t that attacks s.

Now, $\mathcal{S}$ is called *conflict-free* (in $\mathcal{AF}(\mathsf{S})$) if it does not attack any of its elements (i.e., $\mathcal{S}^+ \cap \mathcal{S} = \emptyset$), $\mathcal{S}$ is an *admissible extension* of $\mathcal{AF}(\mathsf{S})$ if it is

[2] Somewhat abusing the notations, we shall sometimes identify *Attack* with $\mathfrak{A}$.

conflict-free and defends all of its elements, and $\mathcal{S}$ is a *complete extension* of $\mathcal{AF}(\mathsf{S})$ if it is admissible and contains all the arguments that it defends.

The minimal complete extension of $\mathcal{AF}(\mathsf{S})$ is called the *grounded extension* of $\mathcal{AF}(\mathsf{S})$, and a maximal complete extension $\mathcal{AF}(\mathsf{S})$ is called a *preferred extension* of $\mathcal{AF}(\mathsf{S})$. A complete extension $\mathcal{AF}(\mathsf{S})$ is called a *stable extension* of $\mathcal{AF}(\mathsf{S})$ if $\mathcal{S} \cup \mathcal{S}^{+} = \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$. We write $\mathsf{Adm}(\mathcal{AF}(\mathsf{S}))$ [respectively: $\mathsf{Cmp}(\mathcal{AF}(\mathsf{S}))$, $\mathsf{Prf}(\mathcal{AF}(\mathsf{S}))$, $\mathsf{Stb}(\mathcal{AF}(\mathsf{S}))$] for the set of all the admissible [respectively: complete, preferred, stable] extensions of $\mathcal{AF}(\mathsf{S})$. Similarly, $\mathsf{Grd}(\mathcal{AF}(\mathsf{S}))$ denotes the unique grounded extension of $\mathcal{AF}(\mathsf{S})$.

Example 3. Figure 1 depicts part of an argumentation framework for the set $\mathsf{S} = \{p, \neg p, q\}$, based on classical logic, where Undercut is the single attack rule.

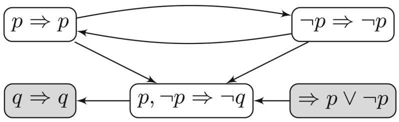

Fig. 1. (Part of the) argumentation framework for Example 3

Note that the gray-colored rightmost node is non-attacked since it has an empty support set. That node counter attacks any attacker of the other gray-colored node, whose sequent is $q \Rightarrow q$, because any argument in $\mathsf{Arg}_{\mathsf{CL}}(\mathsf{S})$ whose conclusion is logically equivalent to $\neg q$ must contain both p and $\neg p$ in its support set. It follows that the gray-colored nodes of the figure are in the grounded extension (and so in every complete extension) of $\mathsf{Arg}_{\mathsf{CL}}(\mathsf{S})$.

3 Reasoning with Maximally Consistent Subsets

As indicated previously, our primary goal in this work is to provide argumentative approaches for reasoning with inconsistent premises by their (maximally) consistent subsets. This may be represented as follows:

Definition 4. Let S be a set of formulas. We denote by $\mathrm{Cn}(\mathsf{S})$ the transitive closure of S with respect to classical logic and by $\mathsf{MCS}(\mathsf{S})$ the set of all the maximally consistent subsets of S (where maximality is taken with respect to the subset relation). We denote:

- $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$ iff $\psi \in \mathrm{Cn}(\bigcap \mathsf{MCS}(\mathsf{S}))$.
- $\mathsf{S} \mid\!\sim_{\cup\mathsf{mcs}} \psi$ iff $\psi \in \bigcup_{\mathsf{T} \in \mathsf{MCS}(\mathsf{S})} \mathrm{Cn}(\mathsf{T})$.

Example 4. Consider the theory $\mathsf{S} = \{p, \neg p, q\}$. Since $\mathsf{MCS}(\mathsf{S}) = \{\{p, q\}, \{\neg p, q\}\}$, every formula in S follows according to $\mid\!\sim_{\cup\mathsf{mcs}}$ from S, but (unlike classical logic!) $\mathsf{S} \not\mid\!\sim_{\cup\mathsf{mcs}} r$ and $\mathsf{S} \not\mid\!\sim_{\cup\mathsf{mcs}} p \wedge \neg p$. Note that $\mid\!\sim_{\mathsf{mcs}}$ is more cautious than $\mid\!\sim_{\cup\mathsf{mcs}}$, and it does not allow to infer p nor $\neg p$ from S. Still, we have, e.g., that $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} q$, since $q \in \bigcap \mathsf{MCS}(\mathsf{S})$.

The entailments $\mid\!\sim_{\mathsf{mcs}}$ and $\mid\!\sim_{\cup\mathsf{mcs}}$ are sometimes called "free" and "existential", respectively. Next, we recall the two argumentation-based approaches, introduced in [4], for computing these entailments.

Definition 5. Let $\mathcal{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}\rangle$. We denote $\mathsf{S} \mid\!\sim_{\mathsf{gr}} \psi$ if there is an $s \in \mathsf{Grd}(\mathcal{AF}(\mathsf{S}))$ such that $\mathsf{Con}(s) = \psi$. The entailments $\mid\!\sim_{\cap\mathsf{prf}}$, $\mid\!\sim_{\cup\mathsf{prf}}$, $\mid\!\sim_{\cap\mathsf{stb}}$ and $\mid\!\sim_{\cup\mathsf{stb}}$ are defined similarly, where $\mathsf{Grd}(\mathcal{AF}(\mathsf{S}))$ is replaced, respectively, by $\bigcap \mathsf{Prf}(\mathcal{AF}(\mathsf{S}))$, $\bigcup \mathsf{Prf}(\mathcal{AF}(\mathsf{S}))$, $\bigcap \mathsf{Stb}(\mathcal{AF}(\mathsf{S}))$, and $\bigcup \mathsf{Stb}(\mathcal{AF}(\mathsf{S}))$.

Proposition 1. [4] *Let* S *be a set of formulas and* ψ *a formula. Consider the sequent-based argumentation framework* $\mathcal{AF}(\mathsf{S})$ *for* S*, induced by classical logic, Gentzen's sequent calculus LK for it* [13]*, and Undercut (Example 1) as the sole attack rule. Then:*

1. $\mathsf{S} \mid\!\sim_{\mathsf{gr}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$.
2. $\mathsf{S} \mid\!\sim_{\cup\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cup\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cup\mathsf{mcs}} \psi$.

Example 5. Let $\mathsf{S} = \{p, \neg p, q\}$. By the discussion in Examples 3 and 4, $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} q$, $\mathsf{S} \mid\!\sim_{\mathsf{gr}} q$, $\mathsf{S} \mid\!\sim_{\cap\mathsf{prf}} q$, and $\mathsf{S} \mid\!\sim_{\cap\mathsf{stb}} q$. By Proposition 1, this is not a coincidence.

4 Generalization I: More Moderated Entailments

Let $\mathsf{S}' = \{p \wedge q, \neg p \wedge q\}$. Here, $\bigcap \mathsf{MCS}(\mathsf{S}') = \emptyset$, and so only tautological formulas follow according to $\mid\!\sim_{\mathsf{mcs}}$ from S'. Yet, one may argue that in this case formulas in $\mathrm{Cn}(\{q\})$ should also follow from S', since they follow according to classical logic from every set in $\mathsf{MCS}(\mathsf{S}')$. This gives rise to the following variation of $\mid\!\sim_{\mathsf{mcs}}$.

Definition 6. Given a set S of formulas and a formula ψ, we denote by $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$ that $\psi \in \bigcap_{\mathsf{T} \in \mathsf{MCS}(\mathsf{S})} \mathrm{Cn}(\mathsf{T})$.

Note 1. Clearly, if $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \psi$ then $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$. However, as noted in the discussion before Definition 6, the converse does not hold. Indeed, $\mathsf{S}' \mid\!\sim_{\cap\mathsf{mcs}} q$ while $\mathsf{S}' \not\mid\!\sim_{\mathsf{mcs}} q$.

For characterizing $\mid\!\sim_{\cap\mathsf{mcs}}$ in terms of Dung-style semantics we need to revise the set of arguments as follows: Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic, then

$$\mathsf{Arg}^{\star}_{\mathfrak{L}}(\mathsf{S}) = \{\psi \Rightarrow \phi \mid \psi \vdash \phi \text{ and } \psi = \bigvee_{1 \leq i \leq n} \bigwedge \Gamma_i, \text{ where } \forall_i \Gamma_i \subseteq \mathsf{S}\}.$$

Note that the definition of $\mathsf{Arg}^{\star}_{\mathfrak{L}}(\mathsf{S})$ resembles that of $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S})$ using a different form of support sets. Intuitively, this is explained by the need to provide in the support set different alternatives for deriving the conclusion of the argument, according to the more moderated entailment $\mid\!\sim_{\cap\mathsf{mcs}}$.

Example 6. Consider again the set $\mathsf{S}' = \{p \wedge q, \neg p \wedge q\}$. Then, e.g., $p \wedge q \Rightarrow q$, $\neg p \wedge q \Rightarrow q$ and $(p \wedge q) \vee (\neg p \wedge q) \Rightarrow q$ are all in $\mathsf{Arg}^{\star}_{\mathfrak{L}}(\mathsf{S}')$. Note that while the first two sequents are also in $\mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}')$, the last one is not.

Now, for $\mathsf{sem} \in \{\mathsf{grd}, \cap\mathsf{prf}, \cap\mathsf{stb}\}$, we define $\mid\!\sim^{\star}_{\mathsf{sem}}$ just as $\mid\!\sim_{\mathsf{sem}}$ (Definition 5), where $\mathcal{AF}(\mathsf{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}\rangle$ is substituted by $\mathcal{AF}^{\star}(\mathsf{S}) = \langle \mathsf{Arg}^{\star}_{\mathfrak{L}}(\mathsf{S}), \mathit{Attack}\rangle$. Like Proposition 1, these Dung-style semantics may be used for characterizing the MCS-based entailments under consideration.

Proposition 2. *Let* S *be a finite set of formulas and* ψ *a formula. Consider the sequent-based argumentation framework* $\mathcal{AF}^{\star}(\mathsf{S})$ *for* S*, induced by classical logic, Gentzen's sequent calculus LK for it, and Undercut as the sole attack rule. Then:* $\mathsf{S} \mid\!\sim^{\star}_{\mathsf{gr}} \psi$ *iff* $\mathsf{S} \mid\!\sim^{\star}_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mid\!\sim^{\star}_{\cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \psi$*.*

Outline of proof. Given a finite set S of formulas, we let: $\mathsf{S}^{\wedge} = \{\bigwedge\Gamma \mid \Gamma \subseteq \mathsf{S}\}$ and $\mathsf{S}^{\star} = \{\Psi_1 \vee \ldots \vee \Psi_n \mid \Psi_1, \ldots, \Psi_n \in \mathsf{S}^{\wedge}\}$. Then, for every $\mathsf{sem} \in \{\mathsf{grd}, \cap\mathsf{prf}, \cap\mathsf{stb}\}$ it can be shown that $\mathsf{S} \mid\!\sim_{\cap\mathsf{mcs}} \phi$ iff $\mathsf{S}^{\star} \mid\!\sim_{\cap\mathsf{mcs}} \phi$ iff $\mathsf{S}^{\star} \mid\!\sim_{\mathsf{mcs}} \phi$ iff $\mathsf{S}^{\star} \mid\!\sim_{\mathsf{sem}} \phi$ iff $\mathsf{S} \mid\!\sim^{\star}_{\mathsf{sem}} \phi$, and so the proposition is obtained. □

5 Generalization II: Lifting Subset Maximality

Next, we consider the following strengthening, by Benferhat, Dubois and Prade [6], of the entailment relation from Definition 4.

Definition 7. Given a set S of propositions and a formula ϕ, we denote by $\mathsf{S} \mid\mid\!\sim_{\mathsf{mcs}} \phi$ that: (1) $\mathsf{T} \vdash_{\mathsf{CL}} \phi$ for some consistent subset T of S, and (2) There is no consistent subset T' of S such that $\mathsf{T}' \vdash_{\mathsf{CL}} \neg\phi$.

To see how the entailment relation of the last definition is represented in sequent-based argumentation frameworks, let us denote by $\mid\mid\!\sim_{\mathsf{gr}}$ the entailment that is defined like $\mid\!\sim_{\mathsf{gr}}$ (Definition 5), except that instead of Undercut the attack relations are the following[3]:

Consistency Undercut (ConUcut): $$\frac{\Rightarrow \neg\bigwedge\Gamma_2' \qquad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$$

Defeating Rebuttal (DefReb): $$\frac{\Gamma_1 \Rightarrow \psi_1 \qquad \Rightarrow \psi_1 \supset \neg\psi_2 \qquad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$$

Again, $\mid\mid\!\sim_{\mathsf{mcs}}$-entailments are characterized by Dung's semantics as follows:

Proposition 3. *For a finite set* S *of formulas and a formula* ψ*, we have that* $\mathsf{S} \mid\mid\!\sim_{\mathsf{mcs}} \psi$ *iff* $\mathsf{S} \mid\mid\!\sim_{\mathsf{gr}} \psi$*.*

Proof. If $\mathsf{S} \mid\mid\!\sim_{\mathsf{mcs}} \psi$ then $\mathsf{T} \vdash_{\mathsf{CL}} \psi$ for some $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ and there is no $\mathsf{T}' \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T}' \vdash_{\mathsf{CL}} \neg\psi$. Thus, $s = \Delta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$ for some finite $\Delta \subseteq \mathsf{T}$. Since Δ is consistent, s is not ConUcut-attacked. To see that s is defended from any DefReb-attack, suppose that $\Gamma' \Rightarrow \neg\psi \in \mathsf{Arg}(\mathsf{S})$. Then $\Gamma' \vdash_{\mathsf{CL}} \neg\psi$, thus Γ' is an inconsistent finite subset of S. It follows that $\Rightarrow \neg\bigwedge\Gamma' \in \mathsf{Arg}(\mathsf{S})$. Clearly,

[3] To prevent attacks on tautologies, in Defeating Rebuttal we assume that $\Gamma_2 \neq \emptyset$.

$\Rightarrow \neg\bigwedge\Gamma' \in \mathsf{Arg}(\mathsf{S}) \setminus \mathsf{Arg}(\mathsf{S})^+$, and so indeed any DefReb-attacker of s is counter-ConUcut-attacked by an argument in $\mathsf{Arg}(\mathsf{S})$ (which itself is not attacked), thus s is defended. It follows, then, that $s \in \mathsf{Grd}(\mathcal{AF}(\mathsf{S}))$.

Suppose now that $\mathsf{S} \mid\!\not\sim_{\mathsf{mcs}} \psi$. This means that either there is no $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T} \vdash_{\mathsf{CL}} \psi$, or otherwise there is a set $\mathsf{T} \in \mathsf{MCS}(\mathsf{S})$ such that $\mathsf{T} \vdash_{\mathsf{CL}} \neg\psi$. In the first case the only sequents s such that $\mathsf{Prem}(s) \subseteq \mathsf{S}$ and $\mathsf{Cons}(s) = \psi$ are those for which $\Rightarrow \neg\bigwedge\Gamma$ is provable in LK, where $\Gamma \subseteq \mathsf{Prem}(s)$. Hence, all of these sequents are not members of any admissible extension of $\mathcal{AF}(\mathsf{S})$. In the second case we can construct an admissible extension $\mathcal{E}$ such that $s \in \mathcal{E}^+$ for any $s = \Delta \Rightarrow \psi \in \mathsf{Arg}(\mathsf{S})$ by letting $\mathcal{E} = \mathsf{Arg}(\mathsf{T})$. It is easy to verify that $\mathcal{E} \in \mathsf{Adm}(\mathcal{AF}(\mathsf{S}))$, thus $s \notin \mathsf{Grd}(\mathcal{AF}(\mathsf{S}))$. □

6 Generalization III: Beyond Classical Logic

In this section we consider base logics that may not be classical. In this context we also introduce generalized definitions for consistency.

Extending the setting to arbitrary propositional Tarskian logics (Definition 1) is straightforward, as the sequent-based frameworks described in the second section may be based on any such logic. The extended settings allow to introduce more expressive arguments (involving, for instance, modal operators) or exclude unwanted arguments (like $\neg\neg\psi \Rightarrow \psi$, which is unacceptable by intuitionists).

Example 7. Let $\mathsf{S} = \{p, q, \neg(p \wedge q)\}$. When classical logic is the base logic each pair of assertions in S initiates an Undercut-attack on the sequent corresponding to the third assertion. For instance, $p, \neg(p \wedge q) \Rightarrow \neg q$ Ucut-attacks $q \Rightarrow q$.

Suppose now that the base logic is Priest's 3-valued paraconsistent logic LP [18]. This time, while $\neg(p \wedge q) \Rightarrow \neg(p \wedge q)$ is still attacked (by $p, q \Rightarrow p \wedge q$), the sequents $p \Rightarrow p$ and $q \Rightarrow q$ are not attacked by Ucut, since in LP sequents of the form $p, \neg(p \wedge q) \Rightarrow \neg q$ are *not* derivable.

For extending the condition of consistency we introduce the following notion:

Definition 8. Let $\varrho(\mathcal{L})$ be the set of the finite sets of the formulas in $\mathcal{L}$. A function $g : \varrho(\mathcal{L}) \to \mathcal{L}$ is called *cautiously* $\vdash$*-reversing* if the following two properties are satisfied:

$\vdash$-monotonicity: If $\Gamma \vdash g(\Delta)$ then $\Gamma \vdash g(\Delta \cup \Delta')$.

$\vdash$-reversibility: If $\Gamma, \Sigma \vdash g(\Sigma \cup \Delta)$ then $\Gamma \vdash g(\Sigma \cup \Delta)$.

Example 8. Let $g(\Gamma) = \bigvee_{\psi\in\Gamma} \neg\psi$. It can be shown that g is cautiously reversing with respect to different many-valued logics, among which are Priest's LP (mentioned above), Post's many-valued systems with a single designated element, and Łukasiewicz m-valued logics $Ł_m$, where the truth values are linearly ordered and no more than the top $\frac{m}{2}$-ones are designated (see [21, pages 252 and 260]).

Frequently, the conditions of the attack rules considered e.g. in [7,8,14,17] are violated in logics that do not respect (at least one of) the standard negation rules of LK (i.e., if $\Gamma \Rightarrow \psi, \Delta$ then $\Gamma, \neg\psi \Rightarrow \Delta$ and if $\Gamma, \psi \Rightarrow \Delta$ then $\Gamma \Rightarrow \neg\psi, \Delta$), in which cases alternative negation rules often operate on one side of the sequents. One way to reflect this in our case is to consider confluence of premises in the attacking and the attacked sequents:

Definition 9. Let g be a cautiously $\vdash$-reversing function. For $\Gamma_1' \cup \Gamma_2' \neq \emptyset$ and $\gamma = g(\Gamma_1' \cup \Gamma_2')$, we define:

$$\text{Confluent } g\text{-Undercut:} \quad \frac{\Gamma_1, \Gamma_1' \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \gamma \quad \gamma \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}.$$

We denote by $\mid\!\sim^g_{\mathsf{gr}}$, $\mid\!\sim^g_{\cap\mathsf{prf}}$, $\mid\!\sim^g_{\cap\mathsf{stb}}$, $\mid\!\sim^g_{\cup\mathsf{prf}}$ and $\mid\!\sim^g_{\cup\mathsf{stb}}$, the counterparts, for a logic $\mathfrak{L}$, of the entailments in Definition 5, where the attack relation of the underlying sequent-based argumentation framework is Confluent g-Undercut.

Next, we characterize the above entailments in terms of the following generalizations of maximally consistent subsets, based on $\vdash$-reversing functions.

Definition 10. Let $g : \wp(\mathcal{L}) \to \mathcal{L}$, and $\Sigma_1, \Sigma_2 \in \wp(\mathcal{L})$.

- Σ_1, Σ_2 are *g-reversible*, if $\Sigma_1 \vdash g(\Sigma_2)$ or $\Sigma_2 \vdash g(\Sigma_1)$.
- Σ_1, Σ_2 are *g-coherent*, if there are no subsets Σ_1' and Σ_2' of Σ_1 and Σ_2 that are g-reversible.
- $\Sigma \in \wp(\mathcal{L})$ is *g-coherent* if so are every $\Sigma_1, \Sigma_2 \subseteq \Sigma$.
- A g-coherent set Σ is *maximal*, if none of its proper supersets is g-coherent. We denote by $\mathsf{MAX}_g(\mathsf{S})$ the set of the maximally g-coherent subsets of S.

Note 2. If $\Sigma_1, \Sigma_2 \in \wp(\mathcal{L})$ are g-reversible, then (without loss of generality) $\Sigma_1 \vdash g(\Sigma_2)$. By monotonicity, $\Sigma_1 \vdash g(\Sigma_1 \cup \Sigma_2)$ and so by $\vdash$-reversibility, $\vdash g(\Sigma_1 \cup \Sigma_2)$. It follows that g-coherence makes sense only for logics that have tautologies (like CL and LP). In the rest of this section we thus restrict ourselves to such logics.

Example 9. Consider the logic LP and the cautiously reversing function $g(\Gamma) = \bigvee_{\psi \in \Gamma} \neg\psi$, considered in Example 8. For $\mathsf{S} = \{p, q, \neg(p \wedge q)\}$ (Example 7) we have that $\mathsf{MAX}_g(\mathsf{S}) = \{\{p, q\}, \{p, \neg(p \wedge q)\}, \{q, \neg(p \wedge q)\}\}$.

Note that while the elements in $\mathsf{MAX}_g(\mathsf{S})$ are the same as the maximally consistent subsets of S with respect to classical logic, the setting in Example 9 is different from the one that is based on classical logic and Undercut. Indeed,

1. Since LP is weaker than CL, the extensions of the current framework are $\subseteq$-smaller than those of the CL-based framework. For instance, we have that $p, \neg(p \wedge q) \Rightarrow \neg q \in \mathsf{Arg}_{\mathsf{CL}}(\mathsf{S})$ while $p, \neg(p \wedge q) \Rightarrow \neg q \notin \mathsf{Arg}_{\mathsf{LP}}(\mathsf{S})$.
2. The use of Undercut instead of Confluent g-Undercut is not appropriate for LP, since the extensions for the framework with Undercut (unlike those of the framework with Confluent g-Undercut) are not closed under LP-inferences.

Definition 11. Let g be a $\vdash$-reversing function and S a set of formulas, and let $\mathrm{Cn}_{\vdash}(\mathsf{S})$ be the transitive closure of S with respect to $\vdash$. We denote:

- $\mathsf{S} \mid\!\sim_{\mathsf{MAX}_g} \psi$ iff $\psi \in \mathrm{Cn}_{\vdash}(\bigcap \mathsf{MAX}_g(\mathsf{S}))$.
- $\mathsf{S} \mid\!\sim_{\cup\mathsf{MAX}_g} \psi$ iff $\psi \in \bigcup_{\mathsf{T} \in \mathsf{MAX}_g(\mathsf{S})} \mathrm{Cn}_{\vdash}(\mathsf{T})$.

Next we show that the correspondence between Dung's semantics and MCS-reasoning carries on to non-classical logics (proof is omitted due to lack of space).

Proposition 4. *Let g be a $\vdash$-reversing function and S a set of formulas. Then:*

1. $\mathsf{S} \mid\!\sim^{g}_{\mathsf{gr}} \psi$ *iff* $\mathsf{S} \mid\!\sim^{g}_{\cap\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mid\!\sim^{g}_{\cap\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\mathsf{MAX}_g} \psi$.
2. $\mathsf{S} \mid\!\sim^{g}_{\cup\mathsf{prf}} \psi$ *iff* $\mathsf{S} \mid\!\sim^{g}_{\cup\mathsf{stb}} \psi$ *iff* $\mathsf{S} \mid\!\sim_{\cup\mathsf{MAX}_g} \psi$.

7 Discussion, In View of Related Work

In this paper we have introduced a series of generalizations of the work in [4], concerning the relations between Dung-style semantics for argumentation frameworks. The relations between these two formalisms have already been investigated in [10] and then in [1,22]. Our approach extends these works in several ways, the most significant ones are the following:

1. In [10,22] the base logic is classical logic. Here (as well as in [1], which continues the work in [10]), *any* propositional language and Tarskian logic is supported. This allows, for instance, to include modal operators in arguments and use paraconsistent logics [11] as the underlying platform for reasoning.
2. According to [1,22] (following [7]), the support of an argument s must be a *consistent and* $\subseteq$*-minimal* set of formulas that entails the conclusion.
   In our setting the argument's support may be *any* finite set that logically implies the argument's conclusion (see [2,3] for a justification of this).
3. The intended semantics in [1] is captured by the entailment $\mid\!\sim_{\cap\mathsf{mcs}}$ in Definition 6, which is only one way of reasoning with MCS. In this paper we also provide argumentation inspired characterizations of other entailments, such as $\mid\!\sim_{\mathsf{mcs}}$ (Definition 4) and $\|\!\sim_{\mathsf{mcs}}$ (Definition 7).

Interestingly, the study of reasoning with maximal consistency by deductive argumentation has led the authors of [1] to conclude that according to Dung's setting, maximal conflict-free sets of arguments (forming what is known as 'naive semantics') are sufficient in order to derive reasonable conclusions and so "the different acceptability semantics defined in the literature are not necessary, and the notion of defense is useless". In view of this statement we note that the removal of the restrictions on the notion of arguments, as well as the introduction of new types of consistency-based entailments, allow us to overcome the shortcoming identified in [1]. Indeed, as the next example shows, in our setting argumentation-based MCS-reasoning does not collapse to naive semantics.

Example 10. Let us consider a sequent-based argumentation framework for $\mathsf{S} = \{p \wedge \neg p\}$, based on classical logic, in which Undercut is the single attack rule.

Let $\mathcal{E} = \{p \wedge \neg p \Rightarrow \psi \mid \psi$ is not a classical logic tautology$\}$. This set is maximal conflict-free. Indeed, the only way to undercut the arguments in $\mathcal{E}$ is by producing an argument of the form $\Gamma \Rightarrow \phi$ where ϕ is logically equivalent to $\neg(p \wedge \neg p)$, which means that ϕ is a classical logic tautology. However, these attacking arguments are excluded from $\mathcal{E}$ and so $\mathcal{E}$ is conflict-free. Moreover, the only arguments from $\mathsf{Arg}_{\mathsf{CL}}(\mathsf{S})$ that were excluded from $\mathcal{E}$ are those that have tautologies of classical logic as conclusions. This implies that $\mathcal{E}$ is maximal in the property of being conflict-free. Now, $\mathsf{S} \mid\!\sim_{\mathsf{mcs}} \neg(p \wedge \neg p)$ while with naive semantics $\neg(p \wedge \neg p)$ doesn't follow from S, since $\mathcal{E}$ is a maximal conflict-free set that does not entail $\neg(p \wedge \neg p)$.

The more lenient view of arguments in our setting (Item 2 above) not only enables the last example, but also warrants a large variety of attack rules which can be applied to arguments in the form of sequents. This is demonstrated, for instance, by the results and the attack rules in [20], which somewhat challenge the observation in [1] that "the notion of inconsistency [...] should be captured by a symmetric attack relation". Furthermore, the results given in this paper stand against the conclusion in [1], that "Dung's framework seems problematic when applied over deductive logical formalisms". We thus believe that, beyond our primary goal of demonstrating the strong ties between reasoning with maximal consistency and argumentation theory, this paper calls for a reevaluation of some negative conclusions in the literature concerning logic-based argumentation.

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