Hypersequential Argumentation Frameworks: An Instantiation in the Modal Logic S5

AnneMarie Borg
Institute of Philosophy II
Ruhr-University Bochum, Germany
annemarie.borg@rub.de

Ofer Arieli
School of Computer Science
The Academic College of Tel-Aviv, Israel
oarieli@mta.ac.il

ABSTRACT

In this paper we introduce *hypersequent-based* frameworks for the modeling of defeasible reasoning by means of logic-based argumentation. These frameworks are an extension of sequent-based argumentation frameworks, in which arguments are represented not only by sequents, but by more general expressions, called *hypersequents*. This generalization allows to incorporate, as the deductive-base of our formalism, some well-studied logics like the modal logic S5, the relevant logic RM, and Gödel–Dummett logic LC, to which no cut-free sequent calculi are known. In this paper we take S5 as the core logic and show that the hypersequent-based argumentation frameworks that are obtained in this case yield a robust defeasible variant of S5 with several desirable properties.

KEYWORDS

Structured argumentation; Dung-style semantics; Nonmonotonic logic; Modal logic; Sequent calculi; Hypersequent calculi

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1 INTRODUCTION

Argumentation theory has been described as "*a core study within artificial intelligence*" [10]. Logical argumentation (sometimes called deductive or structural argumentation) is a branch of argumentation theory in which arguments have a specific structure. This includes rule-based argumentation systems such as ASPIC$^+$ [36], assumption-based argumentation (ABA) systems [13], defeasible logic programming (DeLP) systems [24], and formalisms that are based on Tarskian logics (e.g., [11]), in which classical logic is the deductive base (the so-called *core logic*). The latter were generalized in [4] to *sequent-based argumentation*, where Gentzen's sequents [25], extensively used in proof theory, are incorporated for representing arguments, and attacks are formulated by special inference rules, called *sequent elimination rules*. The result is a generic and modular approach to logical argumentation, in which any logic with a corresponding sound and complete sequent calculus can be used as the underlying core logic.

In this paper, which is a companion paper of [14] (where the core logic is the relevant logic RM), we further extend sequent-based argumentation to *hypersequents* [6, 32, 34]. The latter, which may be regarded as disjunctions of sequents, turned out to be applicable for a large variety of non-classical logics (see [20, 30, 31]), including some well known logics, like S5, RM, and Gödel–Dummett LC, to which no cut-free sequent calculi are known, but all of which do have cut-free hypersequent calculi. Proof systems that admit cut-elimination have multiple proof-theoretic benefits, e.g. they allow for resolution, guarantee the strong normalization property, and imply the subformula property. The latter, meaning that for constructing/proving an argument only its subformulas have to be taken into account, is essential for reducing the proof space when looking for counter-arguments, in which case the cut rule should be avoided.

The usefulness of logical argumentation with hypersequents is demonstrated here on frameworks whose core logic is the well-known modal logic S5. Modal logics have already been used as a descriptive formalism for (abstract) argumentation frameworks (see, for instance, [8, 18, 26, 27]). Here we take S5 and its hypersequential calculus GS5 as 'foundation stones' that allow us to incorporate modalities and advanced proof theoretical techniques in the context of deductive argumentation. As a consequence, argumentation-based methods can be applied to different areas where S5 has been shown useful, like game theory, model checking, temporal reasoning, logics of actions, multi-agent systems and deontic logics for handling conflicting norms.

The rest of the paper is organized as follows. The next two sections contain some preliminary material: first, we review the notions of sequents, hypersequents and the logic S5, and then we recall some basic notions of sequent-based argumentation. In Section 4 we extend sequent-based argumentation frameworks to hypersequential ones, and in Section 5 we consider some properties of these frameworks and their induced entailments, instantiated in S5. In Section 6 we summarize the paper and conclude.

2 PRELIMINARIES

We start by reviewing some general background concerning sequents [25], hypersequents [6], and the modal logic S5.

Throughout the paper we will consider propositional languages, denoted by $\mathcal{L}$. Atomic formulas are denoted by p, q, formulas are denoted by $\gamma, \delta, \phi, \psi$, sets of formulas are denoted by $\mathcal{S}, \mathcal{T}$, and finite sets of formulas are denoted by Γ, Δ, all of which can be primed or indexed.

Definition 2.1. A *logic* for a language $\mathcal{L}$ is a pair $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, satisfying, for every $\mathcal{T}, \mathcal{T}'$ in $\mathcal{L}$, the following properties:

Reflexivity if $\phi \in \mathcal{T}$, then $\mathcal{T} \vdash \phi$;
Monotonicity: if $\mathcal{T}' \vdash \phi$ and $\mathcal{T}' \subseteq \mathcal{T}$, then $\mathcal{T} \vdash \phi$;

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Transitivity (Cut): if $\mathcal{T} \vdash \phi$ and $\mathcal{T}', \phi \vdash \psi$, then $\mathcal{T}, \mathcal{T}' \vdash \psi$.

In addition, a logic is assumed to be *non-trivial* (i.e., there is a non-empty $\mathcal{T}$ and a formula ψ such that $\mathcal{T} \nvdash \psi$), and *structural* (i.e., closed under substitutions: for every substitution θ and every $\mathcal{T}$ and ψ, if $\mathcal{T} \vdash \psi$ then $\{\theta(\varphi) \mid \varphi \in \mathcal{T}\} \vdash \theta(\psi)$).

We suppose that the language $\mathcal{L}$ contains the following connectives:

- $\vdash$-negation $\neg$, satisfying $p \nvdash \neg p$ and $\neg p \nvdash p$, for atomic p, and
- $\vdash$-conjunction $\wedge$, satisfying $\mathcal{S} \vdash \phi \wedge \psi$ iff $\mathcal{S} \vdash \phi$ and $\mathcal{S} \vdash \psi$ for $\mathcal{L}$-formulas ϕ, ψ and a set $\mathcal{S}$ of $\mathcal{L}$-formulas.

In addition, $\mathcal{L}$ may also contain the following connectives:

- $\vdash$-disjunction $\vee$, for which $\mathcal{S}, \phi \vee \psi \vdash \Gamma$ iff $\mathcal{S}, \phi \vdash \Gamma$ and $\mathcal{S}, \psi \vdash \Gamma$,
- $\vdash$-implication $\supset$, for which $\mathcal{S}, \phi \vdash \psi$ iff $\mathcal{S} \vdash \phi \supset \psi$.

In what follows we shall abbreviate $(\phi \supset \psi) \wedge (\psi \supset \phi)$ by $\phi \leftrightarrow \psi$, denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$), the conjunction (respectively, the disjunction) of all the formulas in Γ, and let $\neg\mathcal{S} = \{\neg\phi \mid \phi \in \mathcal{S}\}$.

Definition 2.2. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas.

- $\mathsf{CN}_\mathsf{L}(\mathcal{S})$ denotes the *closure* of $\mathcal{S}$ (so $\mathsf{CN}_\mathsf{L}(\mathcal{S}) = \{\phi \mid \mathcal{S} \vdash \phi\}$).
- $\mathcal{S}$ is *consistent* (for $\vdash$), if there is no finite $\mathcal{S}' \subseteq \mathcal{S}$ such that $\vdash \neg \bigwedge \mathcal{S}'$.[1]

Definition 2.3. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of formulas in $\mathcal{L}$. An *$\mathcal{L}$-sequent* (a *sequent* for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of $\mathcal{L}$-formulas and $\Rightarrow$ is a symbol that does not appear in $\mathcal{L}$.

The formal systems used for the constructions of sequents for a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, are called *sequent calculi* [25]. In what follows we shall assume that a sequent calculus C is sound and complete for its logic (i.e., $\Gamma \Rightarrow \psi$ is provable in C iff $\Gamma \vdash \psi$).

2.1 Hypersequents and Their Inference Rules

Unfortunately, ordinary sequent calculi do not satisfactory capture all the interesting logics. For some logics, which have a clear and simple semantics, no cut-free sequent calculus is known. Notable examples are the Gödel–Dummett intermediate logic LC, the relevant logic RM and the modal logic S5. A large range of extensions of Gentzen's original sequent calculi have been introduced in the last decades for providing decent proof systems for different non-classical logics. As we indicated previously, in our context this is very important, e.g., for reducing the proof space in a quest for appropriate arguments and counter-arguments. Here we consider a natural extension of sequent calculi, called *hypersequent calculi*. Hypersequents were independently introduced by Mints [32], Pottinger [34] and Avron [6], nowadays Avron's notation is mostly used (see, e.g., [7]). Intuitively, a hypersequent is a finite multiset (or sequence) of sequents, which is true if and only if at least one of its component sequents is true.

Definition 2.4. An *$\mathcal{L}$-hypersequent* [7] is a finite multiset of sequents of the form $\Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, where $\Gamma_i \Rightarrow \Delta_i$ $(1 \leq i \leq n)$ are $\mathcal{L}$-sequents and $\mid$ is a new symbol, not appearing in $\mathcal{L}$.[2] Each $\Gamma_i \Rightarrow \Delta_i$ is called a *component* of the hypersequent.

Clearly, every ordinary sequent is a hypersequent. We shall denote hypersequents by $\mathcal{G}, \mathcal{H}$, primed or indexed if needed. Given a hypersequent $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$, the *support* of $\mathcal{H}$ is the set $\mathsf{Supp}(\mathcal{H}) = \{\Gamma_1, \ldots, \Gamma_n\}$ and the *consequent* of $\mathcal{H}$ is the formula $\mathsf{Conc}(\mathcal{H}) = \bigvee\Delta_1 \vee \ldots \vee \bigvee\Delta_n$. For a set Λ of hypersequents, we let $\mathsf{Concs}(\Lambda) = \{\mathsf{Conc}(\mathcal{H}) \mid \mathcal{H} \in \Lambda\}$.

Example 2.5. Like in Gentzen's sequent calculi, hypersequent axioms have the form $\psi \Rightarrow \psi$. Consider the right implication rule of Gentzen's calculus LK for classical logic (the rule on the left below). The corresponding hypersequent rule is similar, now with added components (see the rule on the right below):

$$\frac{\Gamma, \phi \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \supset \psi}\ (\Rightarrow\supset) \qquad \frac{\mathcal{G} \mid \Gamma, \phi \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \phi \supset \psi}\ (\Rightarrow\supset)$$

For another example, the left negation rule of LK, presented on the left below, is translated to the hypersequantial rule on the right below:

$$\frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}\ (\neg\Rightarrow) \qquad \frac{\mathcal{G} \mid \Gamma \Rightarrow \Delta, \psi}{\mathcal{G} \mid \Gamma, \neg\psi \Rightarrow \Delta}\ (\neg\Rightarrow)$$

2.2 S5 and the Hypersequent Calculus GS5

Most of the important systems in propositional modal logic (like K, K4, T, and S4) have ordinary, cut-free Gentzen-type formulations. The sequential system for S4, for example, is obtained from that of classical logic by adding to it the following two rules for $\Box$,[3] where $\Box\Gamma$ is a sequence of formulas that begin with $\Box$ (that is, if $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ then $\Box\Gamma = \{\Box\gamma_1, \ldots, \Box\gamma_k\}$):

$$\frac{\Gamma, \phi \Rightarrow \Delta}{\Gamma, \Box\phi \Rightarrow \Delta}\ (\Box\Rightarrow) \qquad \frac{\Box\Gamma \Rightarrow \phi}{\Box\Gamma \Rightarrow \Box\phi}\ (\Rightarrow\Box)$$

The logic S5 is an important modal system for which no such cut-free system is known. In its usual formulation the rule $(\Rightarrow\Box)$ of S4 is strengthened to:

$$\frac{\Box\Gamma \Rightarrow \phi, \Box\Delta}{\Box\Gamma \Rightarrow \Box\phi, \Box\Delta}.$$

It is easy to see, however, that $p \Rightarrow \Box\neg\Box\neg p$ is derivable in this system using a cut on $\Box\neg p$, but it has no cut-free proof.[4]

As shown in [7] and [34], the problem of providing a cut-free formulation for S5 can be a solved with the help of hypersequents. In Figure 1 we recall the hypersequential calculus GS5, introduced in [7]. [5, 6, 7]

[1]Note that if $\mathcal{S}$ is consistent, then so are $\mathsf{CN}_\mathsf{L}(\mathcal{S})$ and $\mathcal{S}'$ for every $\mathcal{S}' \subseteq \mathcal{S}$. If $\mathcal{S}$ is inconsistent, then so is every superset of $\mathcal{S}$.

[2]The common, intuitive interpretation of the sign "|" is disjunction.

[3]For simplicity we deal only with $\Box$, taking $\Diamond$ as a definable connective.

[4]Only *analytic* cut (on subformulas of the proved sequent) suffice for the proof.

[5]The structural rules of GS5 are abbreviated as follows: MS – modularized splitting, IW – internal weakening, EW – external weakening, EC – external contraction.

[6]Other hypersequential calculi for S5 are available, e.g., in [9] and [12].

[7]In the presence of (MS) the rule $(\Rightarrow\Box)$ can be strengthened to the usual rule of S5, in a hypersequential form: $\frac{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\Delta, \phi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\Delta, \Box\phi}$.

Axioms: $\varphi \Rightarrow \varphi$

Logical rules:

$$(\wedge\Rightarrow)\ \frac{\mathcal{G} \mid \Gamma, \varphi, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \wedge \psi \Rightarrow \Delta} \qquad (\vee\Rightarrow)\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta \quad \mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \vee \psi \Rightarrow \Delta} \qquad (\supset\Rightarrow)\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta \quad \mathcal{G} \mid \Gamma, \psi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \varphi \supset \psi \Rightarrow \Delta}$$

$$(\Rightarrow\wedge)\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta \quad \mathcal{G} \mid \Gamma \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \wedge \psi, \Delta} \qquad (\Rightarrow\vee)\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \psi\Delta}{\mathcal{G} | \Gamma \Rightarrow \varphi \vee \psi, \Delta} \qquad (\Rightarrow\supset)\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \psi, \Delta}{\mathcal{G} \mid \Gamma \Rightarrow \varphi \supset \psi, \Delta}$$

$$(\neg\Rightarrow)\ \frac{\mathcal{G} \mid \Gamma \Rightarrow \varphi, \Delta}{\mathcal{G} \mid \Gamma, \neg\varphi \Rightarrow \Delta} \qquad (\Box\Rightarrow)\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma, \Box\varphi \Rightarrow \Delta}$$

$$(\Rightarrow\neg)\ \frac{\mathcal{G} \mid \Gamma, \varphi \Rightarrow \Delta}{\mathcal{G} \mid \Gamma \Rightarrow, \neg\varphi, \Delta} \qquad (\Rightarrow\Box)\ \frac{\mathcal{G} \mid \Box\Gamma \Rightarrow \varphi}{\mathcal{G} \mid \Box\Gamma \Rightarrow \Box\varphi}$$

Structural rules:

$$(\text{Cut})\ \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \varphi, \Delta \quad \mathcal{G} \mid \varphi, \Gamma_2 \Rightarrow \Delta}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta} \qquad (\text{MS})\ \frac{\mathcal{G} \mid \Box\Gamma_1, \Gamma_2 \Rightarrow \Box\Delta_1, \Delta_2}{\mathcal{G} \mid \Box\Gamma_1 \Rightarrow \Box\Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2}$$

$$(\text{IW})\ \frac{\mathcal{G} \mid \Gamma_1 \Rightarrow \Delta_1}{\mathcal{G} \mid \Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2} \qquad (\text{EC})\ \frac{\mathcal{G} \mid s \mid s}{\mathcal{G} \mid s} \qquad (\text{EW})\ \frac{\mathcal{G}}{\mathcal{G} \mid s}$$

Figure 1: The hypersequent calculus GS5

Example 2.6. Below we show how $\neg\Box\psi \supset \Box\neg\Box\psi$ (known as Axiom 5 in modal logic) can be proven in GS5:

$$\begin{array}{cl}
\Box\psi \Rightarrow \Box\psi & \\
\hline
\Box\psi, \neg\Box\psi \Rightarrow & (\neg\Rightarrow) \\
\hline
\Box\psi \Rightarrow \mid \neg\Box\psi \Rightarrow & (\text{MS}) \\
\hline
\Rightarrow \neg\Box\psi \mid \neg\Box\psi \Rightarrow & (\Rightarrow\neg) \\
\hline
\Rightarrow \Box\neg\Box\psi \mid \neg\Box\psi \Rightarrow & (\Rightarrow\Box) \\
\hline
\neg\Box\psi \Rightarrow \Box\neg\Box\psi \mid \neg\Box\psi \Rightarrow \Box\neg\Box\psi & (\text{IW} \times 2) \\
\hline
\neg\Box\psi \Rightarrow \Box\neg\Box\psi & (\text{EC}) \\
\hline
\Rightarrow \neg\Box\psi \supset \Box\neg\Box\psi & (\Rightarrow\supset)
\end{array}$$

In [7] it is shown that GS5 admits cut-elimination and that it corresponds to S5 in the following sense:

Theorem 2.7. *Let $\mathcal{H} = \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n$ be a hypersequent, where for each $1 \leq i \leq n$, $\Gamma_i = \{\gamma_1^i, \ldots, \gamma_{m_i}^i\}$ and $\Delta_i = \{\delta_1^i, \ldots, \delta_{l_i}^i\}$. Then $\mathcal{H}$ is derivable in* GS5 *if and only if the following formula $\tau(\mathcal{H})$ is a theorem of* S5*:*[8]

$$\tau(\mathcal{H}) = \Box\left(\neg\gamma_1^1 \vee \ldots \vee \neg\gamma_{m_1}^1 \vee \delta_1^1 \vee \ldots \vee \delta_{l_1}^1\right) \vee \ldots \vee \Box\left(\neg\gamma_1^n \vee \ldots \vee \neg\gamma_{m_n}^n \vee \delta_1^n \vee \ldots \vee \delta_{l_n}^n\right). \tag{1}$$

3 SEQUENT-BASED FRAMEWORKS

In logical argumentation, arguments have a specific structure according to the underlying language (see [11, 33, 35, 38]). A natural representation of such an argument is by a sequent [4] (Definition 2.3).

Definition 3.1. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{S}$ be a set of formulas in $\mathcal{L}$.

- An L-*argument* (an *argument*, for short) is a single-conclusion $\mathcal{L}$-sequent $\Gamma \Rightarrow \psi$,[9] where $\Gamma \vdash \psi$. Γ is called the *support set* of the argument and ψ its *conclusion*.
- An L-*argument based on* $\mathcal{S}$ is an L-argument $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \mathcal{S}$. We denote by $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ the set of all the L-arguments that are based on $\mathcal{S}$.

One of the advantages of sequent-based argumentation is that any logic with a corresponding sound and complete sequent calculus can be used as the core logic. Accordingly, the construction of arguments from simpler arguments is done by the inference rules of the sequent calculus.[10]

Argumentation systems contain also attacks between arguments. In our case, attacks are represented by *sequent elimination rules*. Such a rule consists of an attacking argument (the first condition of the rule), an attacked argument (the last condition of the rule), conditions for the attack (the conditions in between) and a conclusion (the eliminated attacked sequent). The outcome of an application of such a rule is that the attacked sequent is 'eliminated'. The elimination of a sequent $s = \Gamma \Rightarrow \Delta$ is denoted by $\overline{s}$ or $\Gamma \not\Rightarrow \Delta$.

Definition 3.2. A *sequent elimination* (or *attack*) *rule* is a rule $\mathcal{R}$ of the form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \quad \ldots \quad \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n} \quad \mathcal{R} \tag{2}$$

Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic with a sequent calculus C and let $\mathcal{R}$ be an elimination rule as above. If

(1) $\Gamma \Rightarrow \psi \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ is an instance of $\Gamma_1 \Rightarrow \Delta_1$,
(2) $\Gamma' \Rightarrow \psi' \in \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ is an instance of $\Gamma_n \Rightarrow \Delta_n$, and
(3) all the other conditions of $\mathcal{R}$ (i.e., $\Gamma_i \Rightarrow \Delta_i$ for $i = 2, \ldots, n-1$) are provable in C,

we say that the argument $\Gamma \Rightarrow \psi$ $\mathcal{R}$-*attacks* the argument $\Gamma' \Rightarrow \psi'$.

[8] That is, the sequent $\Rightarrow \tau(\mathcal{H})$ is derivable in a standard *ordinary* sequential formulation of S5 (like the one described above).
[9] Set signs in arguments are omitted.
[10] See [4] for further advantages of the sequent-based approach.

Example 3.3. We refer to [4, 39] for a definition of many sequent elimination rules. Below are three of them (assuming that $\Gamma_2 \neq \emptyset$):

$$\text{Defeat (Def):} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \supset \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$$

$$\text{Undercut (Ucut):} \quad \frac{\Gamma_1 \Rightarrow \psi_1 \quad \Rightarrow \psi_1 \leftrightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$$

$$\text{Consistency Undercut (ConUcut):} \quad \frac{\Rightarrow \neg \bigwedge \Gamma \quad \Gamma, \Gamma' \Rightarrow \psi}{\Gamma, \Gamma' \not\Rightarrow \psi}$$

A sequent-based argumentation framework is now defined as follows:

Definition 3.4. A *sequent-based argumentation framework* for a set of formulas $\mathcal{S}$ based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a set ER of sequent elimination rules, is a pair $\mathcal{AF}_{\mathsf{L},\mathsf{ER}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$, where $\mathcal{A} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \times \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ and $(a_1, a_2) \in \mathcal{A}$ iff there is an $\mathcal{R} \in \mathsf{ER}$ such that a_1 $\mathcal{R}$-attacks a_2.

In what follows, to shorten a bit the notations, when the set ER of the elimination rules is known or arbitrary, we shall omit it from the notation of the argumentation framework at hand, and just write $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$.

Example 3.5. Consider the set $\mathcal{S} = \{\Box(p \vee q), \Box(p \vee \neg q), \Box\neg p, r\}$ of S5-formulas. Let $\mathcal{AF}_{\mathsf{S5},\{\mathsf{Def}\}}(\mathcal{S}) = \left\langle \mathrm{Arg}_{\mathsf{S5}}(\mathcal{S}), \mathcal{A} \right\rangle$ be the argumentation framework for $\mathcal{S}$, based on S5 and the attack rule Defeat. The following ordinary sequent arguments are part of $\mathrm{Arg}_{\mathsf{S5}}(\mathcal{S})$:

$a_0 = r \Rightarrow r$
$a_1 = \Box(p \vee q) \Rightarrow \Box(p \vee q)$
$a_2 = \Box(p \vee \neg q) \Rightarrow \Box(p \vee \neg q)$
$a_3 = \Box\neg p \Rightarrow \Box\neg p$
$a_4 = \Box(p \vee \neg q), \Box\neg p \Rightarrow \neg\Box(p \vee q)$
$a_5 = \Box(p \vee q), \Box\neg p \Rightarrow \neg\Box(p \vee \neg q)$
$a_6 = \Box(p \vee q), \Box(p \vee \neg q) \Rightarrow \neg\Box\neg p$

Figure 2 depicts a graphical representation of these arguments and the attacks between them.

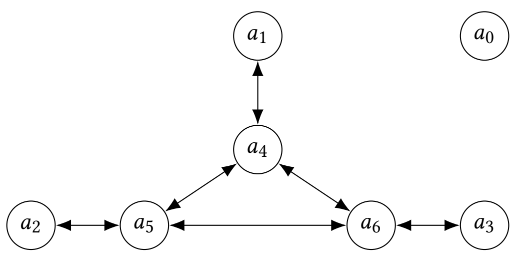

Figure 2: Part of the sequent-based argumentation graph for $\mathcal{S} = \{\Box(p \vee q), \Box(p \vee \neg q), \Box\neg p, r\}$ (Example 3.5)

Given a (sequent-based) argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, we can apply Dung-style semantics [23] to it, to determine what combinations of arguments (called *extensions*) can collectively be accepted from $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$. This is defined next.

Definition 3.6. Let $\mathcal{AF}_{\mathsf{L}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$ be an argumentation framework and let $\mathsf{S} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ be a set of arguments.

- S *attacks* an argument a if there is an $a' \in \mathsf{S}$ such that $(a', a) \in \mathcal{A}$;
- S *defends* an argument a if S attacks every attacker of a;
- S is *conflict-free* if there are no arguments $a_1, a_2 \in \mathsf{S}$ such that $(a_1, a_2) \in \mathcal{A}$;
- S is *admissible* if it is conflict-free and it defends all of its elements.

An admissible set that contains all the arguments that it defends is a *complete extension* of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$. Below are definitions of some particular complete extensions of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$:

- the *grounded extension* of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ is the minimal (with respect to $\subseteq$) complete extension of $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$,[11]
- a *preferred extension* of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ is a maximal (with respect to $\subseteq$) complete extension of $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$,
- a *stable extension* of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ is a complete extension of $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ that attacks every argument not in it.

In what follows we shall refer to either complete (cmp), grounded (gr), preferred (prf) or stable (stb) semantics as *completeness-based semantics*. We denote by $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ the set of all the extensions of $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$ under the semantics $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{gr}, \mathsf{prf}, \mathsf{stb}\}$. The subscript is omitted when this is clear from the context.

Example 3.7. Consider $\mathcal{AF}_{\mathsf{S5},\{\mathsf{Def}\}}(\mathcal{S})$, the argumentation framework from Example 3.5. Note that the argument a_0 is not attacked. Moreover, any attacker of a_0 would have an inconsistent support, and so it is counter-attacked by arguments of the form $\Rightarrow r \vee \neg r$. Therefore, a_0 is in the grounded extension of $\mathcal{AF}_{\mathsf{S5},\{\mathsf{def}\}}(\mathcal{S})$.

Definition 3.8. Given a sequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, the semantics as defined in Def. 3.6 induce corresponding *entailment relations*: $\mathcal{S} \mathrel{|\!\sim}^{\cap}_{\mathsf{sem}} \phi$ ($\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{sem}} \phi$) iff for every (some) extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$, there is an argument $\Gamma \Rightarrow \phi \in \mathcal{E}$ for some $\Gamma \subseteq \mathcal{S}$. Since the grounded extension is unique, $\mathrel{|\!\sim}^{\cap}_{\mathsf{gr}}$ and $\mathrel{|\!\sim}^{\cup}_{\mathsf{gr}}$ coincide, so both of them can be denoted by $\mathrel{|\!\sim}_{\mathsf{gr}}$.

Example 3.9. In Example 3.7 we have that $\mathcal{S} \mathrel{|\!\sim}_{\mathsf{gr}} r$, while $\mathcal{S} \not\mathrel{|\!\sim}_{\mathsf{gr}} \phi$ for any $\phi \in \mathcal{S} \setminus \{r\}$. Thus, for $\mathsf{sem} \in \{\mathsf{cmp}, \mathsf{prf}, \mathsf{stb}\}$, we have that $\mathcal{S} \not\mathrel{|\!\sim}^{\cap}_{\mathsf{sem}} \phi$. On the other hand, for every $\psi \in \mathcal{S}$, it holds that $\mathcal{S} \mathrel{|\!\sim}^{\cup}_{\mathsf{sem}} \psi$.

To define hypersequent-based argumentation frameworks, it is not enough to simply take the hypersequent inference rules to create arguments. A new definition of arguments is required and sequent elimination rules should be turned into hypersequent elimination rules. This is what we will do in the next section.

4 HYPERSEQUENT-BASED FRAMEWORKS

Given a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ with a sound and complete hypersequent calculus H, in what follows an *argument* (or an L-hyperargument) is an $\mathcal{L}$-hypersequent (i.e., whose components are $\mathcal{L}$-sequents) that is provable in H. Since a sequent is a particular case of a hypersequent and hypersequent calculi generalize sequent calculi, arguments in the sense of the previous sections are particular cases of the arguments according to the new definition.

An argument based on a set $\mathcal{S}$ (of formulas in $\mathcal{L}$), is an L-hyperargument $\mathcal{H}$ such that every $\Gamma \in \mathsf{Supp}(\mathcal{H})$ is a subset of

[11] It is well-known [23] that the grounded extension is unique for a given framework.

$\mathcal{S}$. In what follows we shall continue to denote by $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ the set of arguments that are based on $\mathcal{S}$.

As before, arguments are constructed by the inference rules of the hypersequent calculus under consideration. For the attack rules we continue to denote by $\overline{\mathcal{H}}$ the elimination of the hypersequent $\mathcal{H}$. The structure of such rules remains the same as before as well: the first hypersequent in the conditions of the rule is the attacking argument, the last hypersequent is the attacked argument, and the rest of the conditions are to be satisfied for the attack to take place.

Example 4.1. Let $\mathcal{G}, \mathcal{H}$ be two arguments, $\emptyset \neq \Delta_j \in \{\Delta_1, \ldots, \Delta_m\} = \mathrm{Supp}(\mathcal{H})$ and $\Delta'_j \subseteq \Delta_j$. Below are hypersequential counterparts of the elimination rules Defeat, Undercut and Consistency Undercut in Example 3.3.

$$\frac{\mathcal{G} \quad \Rightarrow \mathrm{Conc}(\mathcal{G}) \supset \neg \bigwedge \Delta_j \quad \mathcal{H}}{\overline{\mathcal{H}}} \ \mathrm{Def}_H$$

$$\frac{\mathcal{G} \quad \Rightarrow \mathrm{Conc}(\mathcal{G}) \leftrightarrow \neg \bigwedge \Delta'_j \quad \mathcal{H}}{\overline{\mathcal{H}}} \ \mathrm{Ucut}_H$$

$$\frac{\Rightarrow \neg \bigwedge_{i=1}^{m} \bigwedge \Delta_i \quad \mathcal{H}}{\overline{\mathcal{H}}} \ \mathrm{ConUcut}_H$$

The notion of attack between hypersequents is the same as in Definition 3.2, except that sequents are replaced by hypersequents and the sequent calculus C is replaced by a hypersequent calculus H. Now, a hypersequent-based argumentation framework can be defined in a similar way as that of a sequent-based argumentation framework (cf. Definition 3.4).

Definition 4.2. A *hypersequent-based argumentation framework* for a set of formulas $\mathcal{S}$ based on a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ and a set ER of hypersequent elimination rules, is a pair $\mathcal{AF}_{\mathsf{L},\mathsf{ER}}(\mathcal{S}) = \langle \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}), \mathcal{A} \rangle$, where $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ a set of arguments (i.e., a set of L-hyperarguments), $\mathcal{A} \subseteq \mathrm{Arg}_{\mathsf{L}}(\mathcal{S}) \times \mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$, and $(\mathcal{H}_1, \mathcal{H}_2) \in \mathcal{A}$ iff there is an $\mathcal{R} \in \mathsf{ER}$ such that $\mathcal{H}_1$ $\mathcal{R}$-attacks $\mathcal{H}_2$.

Like before, we will omit the subscript ER when the set of elimination rules is known or arbitrary.

Example 4.3. Recall the sequent-based argumentation framework $\mathcal{AF}_{\mathsf{S5},\{\mathrm{Def}\}}(\mathcal{S})$ for the set $\mathcal{S} = \{\Box(p \vee q), \Box(p \vee \neg q), \Box\neg p, r\}$ and the arguments $\{a_0, a_1, a_2, a_3, a_4, a_5, a_6\} \subseteq \mathrm{Arg}_{\mathsf{S5}}(\mathcal{S})$. When generalizing to hypersequents the following three arguments are part of $\mathrm{Arg}_{\mathsf{S5}}(\mathcal{S})$:

$$a_7 = \Box(p \vee \neg q) \Rightarrow \neg\Box(p \vee q) \mid \Box\neg p \Rightarrow \neg\Box(p \vee q)$$
$$a_8 = \Box(p \vee q) \Rightarrow \neg\Box(p \vee \neg q) \mid \Box\neg p \Rightarrow \neg\Box(p \vee \neg q)$$
$$a_9 = \Box(p \vee q) \Rightarrow \neg\Box\neg p \mid \Box(p \vee \neg q) \Rightarrow \neg\Box\neg p$$

Moreover, by using Def_H instead of Def as the attack rule, additional attacks are available as well. For the graphical representation see Figure 3, in which the dashed part represents the ordinary sequent-based framework and the solid part represents the new arguments and attacks in the hypersequent-based framework.[12]

[12] In case of Undercut attacks, the arrows from a_1 to a_4, from a_2 to a_5, and from a_3 to a_6 should be removed.

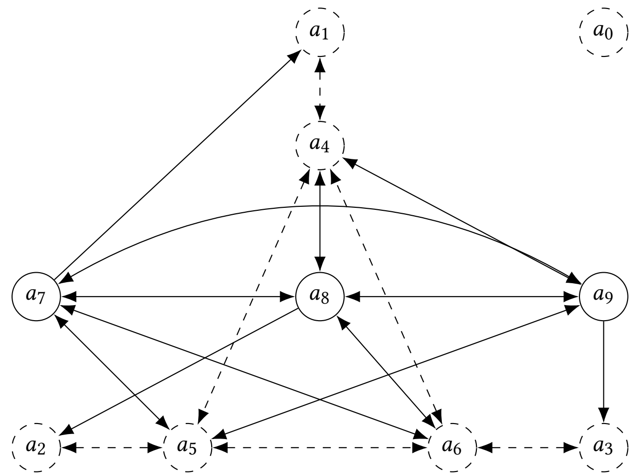

Figure 3: Extension of Figure 2 (the dashed graph) to hypersequents (Example 4.3).

Given a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, Dung-style semantics are defined in an equivalent way to those in Definition 3.6, and so are the corresponding entailment relations (cf. Definition 3.8).

Definition 4.4. Given a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, we denote $\mathcal{S} \mid\!\sim^{\cap}_{H,\mathsf{sem}} \phi$ ($\mathcal{S} \mid\!\sim^{\cup}_{H,\mathsf{sem}} \phi$) iff for every (some) $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ there is an argument $\mathcal{H} \in \mathcal{E}$ such that $\mathrm{Conc}(\mathcal{H}) = \phi$ and $\bigcup \mathrm{Supp}(\mathcal{H}) \subseteq \mathcal{S}$.[13]

5 SOME PROPERTIES OF THE INDUCED ENTAILMENT RELATIONS

In this section we consider some of the properties of the entailment relations $\mid\!\sim^{\cap}_{H,\mathsf{sem}}$ and $\mid\!\sim^{\cup}_{H,\mathsf{sem}}$ defined in Definition 4.4. We start in a general setting, taking a logic $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$, with its hypersequent calculus H, as the core logic of a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, induced by the set of formulas $\mathcal{S}$ and the attack rules $\mathrm{ConUcut}_H$ together with either Ucut_H or Def_H. In what follows, we shall sometimes abbreviate $\mid\!\sim^{\cap}_{H,\mathsf{sem}}$ (for any sem $\in \{\mathsf{cmp}, \mathsf{gr}, \mathsf{prf}, \mathsf{stb}\}$) by $\mid\!\sim^{\cap}$, and $\mid\!\sim^{\cup}_{H,\mathsf{sem}}$ by $\mid\!\sim^{\cup}$.

Proposition 5.1. *If $\mathcal{S}$ is consistent or conflict-free, $\vdash$, $\mid\!\sim^{\cap}$, and $\mid\!\sim^{\cup}$ coincide.*

Proof. It is easy to see that under either of the conditions of the proposition $\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})$ is conflict-free. The proof then follows from the fact that in this case $\mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S})) = \{\mathrm{Arg}_{\mathsf{L}}(\mathcal{S})\}$. □

Lemma 5.2. *Let $\mathcal{H}$ be a hypersequent such that $\bigcup \mathrm{Supp}(\mathcal{H})$ is not consistent, and let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ for some* sem $\in \{\mathsf{cmp}, \mathsf{gr}, \mathsf{prf}, \mathsf{stb}\}$. *In the presence of $\mathrm{ConUcut}_H$, we have that:*

a) $\mathcal{H} \notin \mathcal{E}$, and

b) $\mathcal{E}$ defends arguments from attacks by $\mathcal{H}$.

Proof. Since $\Gamma_{\mathcal{H}} = \bigcup \mathrm{Supp}(\mathcal{H})$ is inconsistent, $\mathcal{G} = \Rightarrow \neg \bigwedge \Gamma_{\mathcal{H}}$ is provable. Note that $\mathcal{G}$ cannot be attacked since it has an empty

[13] Again, the subscript H will be omitted when this is clear from the context.

support, and so $\mathcal{G} \in \mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S})) \subseteq \mathcal{E}$.[14] Moreover, $\mathcal{G}$ ConUcut$_H$-attacks $\mathcal{H}$, thus $\mathcal{H} \notin \mathcal{E}$ (otherwise $\mathcal{E}$ is not conflict free), and $\mathcal{E}$ defends any argument $\mathcal{H}'$ that is attacked by $\mathcal{H}$. □

Proposition 5.3. *We say that $\mathrel{|\!\sim}$ is paraconsistent if it is not trivialized in the presence of inconsistency: for all atoms $p \neq q$ it holds that $p, \neg p \mathrel{|\!\not\sim} q$. Then $\mathrel{|\!\sim}^{\cup}$ and $\mathrel{|\!\sim}^{\cap}$ are paraconsistent.*

Proof. Let $\mathcal{S} = \{p, \neg p\}$. If $\mathcal{S} \mathrel{|\!\sim}^{\cup} q$ then there is some $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ and some $\mathcal{H} \in \mathcal{E}$ such that $\bigcup \mathsf{Supp}(\mathcal{H}) \subseteq \mathcal{S}$ and $\mathsf{Conc}(\mathcal{H}) = q$. Since q does not share any atoms with $\mathcal{S}$, $\mathcal{S} \subseteq \bigcup \mathsf{Supp}(\mathcal{H})$. However, then $\mathsf{Supp}(\mathcal{H})$ is inconsistent and so, by Lemma 5.2, $\mathcal{H} \notin \mathcal{E}$. Thus $\mathcal{S} \mathrel{|\!\not\sim}^{\cup} q$. Since $\mathrel{|\!\sim}^{\cap} \subset \mathrel{|\!\sim}^{\cup}$, we have that $\mathcal{S} \mathrel{|\!\not\sim}^{\cap} q$ as well. □

Proposition 5.4. *We say that $\mathrel{|\!\sim}$ is non-monotonic if it does not satisfy monotonicity (Definition 2.1), i.e., there are $\mathcal{S}_1, \mathcal{S}_2$ and ϕ such that $\mathcal{S}_1 \mathrel{|\!\sim} \phi$ but $\mathcal{S}_1, \mathcal{S}_2 \mathrel{|\!\not\sim} \phi$. Then $\mathrel{|\!\sim}^{\cap}$ is non-monotonic.*

Proof. Let $\mathcal{S}_1 = \{p\}$ and $\mathcal{S}_2 = \{\neg p\}$. Given a complete (and so conflict-free) extension $\mathcal{E}$ of $\mathcal{S}_1 \cup \mathcal{S}_2$, if a hypersequent $\mathcal{H}$ such that $\mathsf{Conc}(\mathcal{H}) = p$ is in $\mathcal{E}$, there is no hypersequent $\mathcal{H}'$ in $\mathcal{E}$ in which $\mathsf{Conc}(\mathcal{H}) = \neg p$ and vice-versa. It follows that according to skeptical semantics it is not possible to infer both p and $\neg p$ from $\mathcal{S}_1 \cup \mathcal{S}_2$, although by Proposition 5.1 p follows from $\mathcal{S}_1$ and $\neg p$ follows from $\mathcal{S}_2$. □

Next, we consider the rationality postulates introduced in [1, 17] for having a quality measure for the different argumentation frameworks in the literature. The next definitions allow us to adapt those postulates to our notions and notations.

Definition 5.5. Let $\mathsf{L} = \langle \mathcal{L}, \vdash \rangle$ be a logic and let $\mathcal{T}$ be a set of $\mathcal{L}$-formulas. A subset C of $\mathcal{T}$ is a *minimal conflict* of $\mathcal{T}$ (w.r.t. $\vdash$), if C is inconsistent and for any $c \in C$, the set $C \setminus \{c\}$ is consistent. We denote by $\mathsf{Free}(\mathcal{T})$ the set of formulas in $\mathcal{T}$ that are not part of any minimal conflict of $\mathcal{T}$.

Definition 5.6. We say that an argument $\mathcal{H}' = \Gamma'_1 \Rightarrow \phi'_1 \mid \ldots \mid \Gamma'_m \Rightarrow \phi'_m$ is a *sub-argument* of an argument $\mathcal{H} = \Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n$, if for each $i \in \{1, \ldots, m\}$ there is a $j \in \{1, \ldots, n\}$ such that $\Gamma'_i \subseteq \Gamma_j$. The set of sub-arguments of $\mathcal{H}$ is denoted $\mathsf{Sub}(\mathcal{H})$.

In our terms, then, the rationality postulates of [1, 17] are expressible as follows:

Definition 5.7. Given a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{L}}(\mathcal{S})$, the following properties refer to any extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$.[15]

- *closure of extensions*: $\mathsf{Concs}(\mathcal{E}) = \mathsf{CN}_{\mathsf{L}}(\mathsf{Concs}(\mathcal{E}))$.
- *closure under sub-arguments*: if $\mathcal{H} \in \mathcal{E}$ then $\mathsf{Sub}(\mathcal{H}) \subseteq \mathcal{E}$.
- *consistency*: $\mathsf{Concs}(\mathcal{E})$ is consistent.
- *free precedence*: $\mathsf{Arg}_{\mathsf{L}}(\mathsf{Free}(\mathcal{S})) \subseteq \mathcal{E}$.

For the proofs of the rationality postulates, we consider the logic S5 and its corresponding hypersequent calculus GS5 as the core logic of a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{S5}}(\mathcal{S})$, like above, induced by the set of formulas $\mathcal{S}$ and the attack rules

[14] We identify here $\mathsf{Ext}_{\mathsf{grd}}(\mathcal{AF}_{\mathsf{L}}(\mathcal{S}))$ with its single element.

[15] Thus, the postulates are in fact relative to sem-extensions, but in what follows we shall show them for every sem $\in$ {cmp, gr, prf, stb}.

Ucut$_H$ or Def$_H$. For free-precedence we shall also assume ConUcut$_H$. Below are modularized versions of some postulates.

Definition 5.8. Let $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{S5}}(\mathcal{S}))$ and let $\Gamma \subseteq \mathcal{S}$. In what follows we denote:

- $\Gamma_{\Box} = \{\Box\gamma \mid \Box\gamma \in \Gamma\}$
- $\mathcal{E}_{\Box} = \{\mathcal{H} \in \mathcal{E} \mid \mathcal{H} = \Box\Gamma_1 \Rightarrow \Box\Delta_1 \mid \ldots \mid \Box\Gamma_n \Rightarrow \Box\Delta_n\}$.[16]

Definition 5.9. Given a hypersequent-based argumentation framework $\mathcal{AF}_{\mathsf{S5}}(\mathcal{S})$, the following properties refer to any extension $\mathcal{E} \in \mathsf{Ext}_{\mathsf{sem}}(\mathcal{AF}_{\mathsf{S5}}(\mathcal{S}))$.

- *modular closure*: $\Box\mathsf{CN}_{\mathsf{S5}}(\mathsf{Concs}(\mathcal{E}_{\Box})) \subseteq \mathsf{Concs}(\mathcal{E}_{\Box}) \subseteq \mathsf{CN}_{\mathsf{S5}}(\mathsf{Concs}(\mathcal{E}_{\Box}))$.
- *modular consistency*: $\mathsf{Concs}(\mathcal{E}_{\Box})$ is consistent.

Proposition 5.10. *$\mathcal{AF}_{\mathsf{S5}}(\mathcal{S})$ with Def$_H$ or Ucut$_H$ satisfies modular closure of extensions, closure under sub-arguments and modular consistency. In the presence of ConUcut$_H$, it satisfies free precedence.*

Proof. The following facts about derivability in GS5 will be (implicitly) used:

Lemma 5.11.

(1) If $\mathcal{G}_1 \mid \Gamma_1 \Rightarrow \phi_1$ and $\mathcal{G}_2 \mid \Gamma_2, \phi_1 \Rightarrow \phi_2$ are derivable in GS5*, then $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \Gamma_1, \Gamma_2 \Rightarrow \phi_2$ is derivable in* GS5*.*
(2) If $\mathcal{G} \mid \Gamma \Rightarrow \phi \supset \psi, \Delta$ is derivable in GS5 *then so is $\mathcal{G} \mid \Gamma, \phi \Rightarrow \psi, \Delta$.*
(3) If $\mathcal{G} \mid \Delta \Rightarrow \phi$ is derivable in GS5 *then so is $\mathcal{G} \mid \;\Rightarrow \neg\phi \supset \neg \bigwedge \Delta$.*
(4) If $\mathcal{G} \mid \Rightarrow \phi \supset \neg \bigwedge \Gamma'$ is derivable in GS5 *then $\mathcal{G} \mid \Rightarrow \phi \supset \neg \bigwedge \Gamma$ is derivable in* GS5 *for $\Gamma' \subseteq \Gamma$.*
(5) If $\Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n$ is derivable in GS5 *$\Gamma_1, \ldots, \Gamma_n \Rightarrow \phi_1, \ldots, \phi_n$ is also derivable in* GS5*.*
(6) If $\mathcal{G} \mid \Gamma \Rightarrow \Box\phi$ is derivable in GS5*, then so is $\mathcal{G} \mid \Gamma \Rightarrow \phi$.*

Suppose now that $\mathcal{E} \in \mathsf{Ext}_{\mathsf{cmp}}(\mathcal{AF}_{\mathsf{S5}}(\mathcal{S}))$.

Closure under sub-arguments: We show the claim for Def$_H$ (the case for Ucut$_H$ is left to the reader). Assume that $\mathcal{H} \in \mathcal{E}$ and let $\mathcal{G} \in \mathsf{Sub}(\mathcal{H})$. We shall show that $\mathcal{G} \in \mathcal{E}$ as well. Indeed, if $\mathcal{G}'$ attacks $\mathcal{G}$, then $\Rightarrow \mathsf{Conc}(\mathcal{G}') \supset \neg \bigwedge \Gamma_i^{\mathcal{G}}$ is derivable in GS5 for some $\Gamma_i^{\mathcal{G}} \in \mathsf{Supp}(\mathcal{G})$. But $\mathcal{G} \in \mathsf{Sub}(\mathcal{H})$, thus, $\Gamma_i^{\mathcal{G}} \subseteq \Gamma_j^{\mathcal{H}} \in \mathsf{Supp}(\mathcal{H})$ and by Lemma 5.11.4, $\Rightarrow \mathsf{Conc}(\mathcal{G}') \supset \neg \bigwedge \Gamma_j^{\mathcal{H}}$ is derivable in GS5. Hence, $\mathcal{G}'$ attacks $\mathcal{H}$ as well. However, since $\mathcal{H} \in \mathcal{E}$, $\mathcal{E}$ defends $\mathcal{H}$ and hence $\mathcal{E}$ also defends $\mathcal{G}$. Since $\mathcal{E}$ is a completeness-based extension, $\mathcal{G} \in \mathcal{E}$.

Modular closure of extensions: Clearly (see Definition 2.2), it holds that $\mathsf{Concs}(\mathcal{E}_{\Box}) \subseteq \mathsf{CN}_{\mathsf{S5}}(\mathsf{Concs}(\mathcal{E}_{\Box}))$. Suppose now that $\phi \in \mathsf{CN}_{\mathsf{S5}}(\mathsf{Concs}(\mathcal{E}_{\Box}))$. Then there are arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{E}$ of the form $\mathcal{H}_i = \Box\Gamma_1^i \Rightarrow \Box\psi_1^i \mid \ldots \mid \Box\Gamma_{m_i}^i \Rightarrow \Box\psi_{m_i}^i$ with $\phi_i = \Box\psi_1^i \vee \ldots \vee \Box\psi_{m_i}^i$, and $\phi_1, \ldots, \phi_n \vdash_{\mathsf{S5}} \phi$. It can be shown that the sequent $\{\Box\Gamma_j^k\}_{k=1,\ldots,n, j=1,\ldots,m_k} \Rightarrow \phi_1 \wedge \ldots \wedge \phi_n$ is derivable in GS5, so by transitivity the sequent $\{\Box\Gamma_j^k\}_{k=1,\ldots,n, j=1,\ldots,m_k} \Rightarrow \phi$ is also derivable in GS5. By $(\Rightarrow\Box)$ we derive $\{\Box\Gamma_j^k\}_{k=1,\ldots,n, j=1,\ldots,m_k} \Rightarrow \Box\phi$, and so by (MS) the hypersequent $\mathcal{H} = \Box\Gamma_1^1 \Rightarrow \Box\phi \mid \ldots \mid \Box\Gamma_{m_1}^1 \Rightarrow \Box\phi \mid \ldots \mid \Box\Gamma_1^n \Rightarrow \Box\phi \mid \ldots \mid \Box\Gamma_{m_n}^n \Rightarrow \Box\phi$ is provable

[16] Recall that $\Box\Gamma = \{\Box\gamma \mid \gamma \in \Gamma\}$.

in GS5. For both attack rules Def_H and Ucut_H, any attacker of $\mathcal{H}$ is an attacker of one of the arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n$, so $\mathcal{E}$ defends $\mathcal{H}$ and since $\mathcal{E}$ is complete, $\mathcal{H} \in \mathcal{E}$. Thus $\mathcal{H} \in \mathcal{E}_{\Box}$, and so $\Box\phi \in \mathrm{Concs}(\mathcal{E}_{\Box})$.

Modular consistency: Assume, towards a contradiction, that $\mathrm{Concs}(\mathcal{E}_{\Box})$ is not consistent. Then there are arguments $\mathcal{H}_1, \ldots, \mathcal{H}_n \in \mathcal{E}_{\Box}$ of the form $\mathcal{H}_i = \Box\Gamma_1^i \Rightarrow \Box\psi_1^i \mid \ldots \mid \Box\Gamma_{m_i}^i \Rightarrow \Box\psi_{m_i}^i$ with $\phi_i = \Box\psi_1^i \vee \ldots \vee \Box\psi_{m_i}^i$ and $C = \{\phi_1, \ldots, \phi_n\}$ is an inconsistent set. Since C is inconsistent, $\Rightarrow \neg \bigwedge_{j=1}^n \phi_j$ is derivable. Let $\psi = \phi_1 \wedge \ldots \wedge \phi_n$. By Lemma 5.11.5 and ($\Rightarrow\vee$) it follows that $\mathcal{H}_i' = \Box\Gamma_1^i, \ldots, \Box\Gamma_{m_i}^i \Rightarrow \phi_i$ are derivable, for all $i \in \{1, \ldots, n\}$. By monotonicity and ($\Rightarrow\wedge$) on $\mathcal{H}_i'$ ($i = 1, \ldots, n$), we derive $\Box\Gamma_1^1, \ldots, \Box\Gamma_{m_1}^1, \ldots, \Box\Gamma_1^n, \ldots, \Box\Gamma_{m_n}^n \Rightarrow \psi$. By applying ($\wedge\Rightarrow$), ($\neg\Rightarrow$) and ($\Rightarrow\neg$) it follows that $\neg\psi, \Box\Gamma_1^1, \ldots, \Box\Gamma_{m_1}^1, \ldots, \Box\Gamma_{k-1}^j, \Box\Gamma_{k+1}^j, \ldots, \Box\Gamma_1^n, \ldots, \Box\Gamma_{m_n}^n \Rightarrow \neg \bigwedge \Box\Gamma_k^j$ is derivable for any $j \in \{1, \ldots, n\}$ and $k \in \{1, \ldots, m_j\}$. So from now on we shall assume that $\Box\Gamma_k^j$ is absent from the left-hand side of the sequents, for a particular k and j. By cut (Lemma 5.11.1) with $\Rightarrow \neg\psi$, we get $\Box\Gamma_1^1, \ldots, \Box\Gamma_{m_1}^1, \ldots, \Box\Gamma_1^n, \ldots, \Box\Gamma_{m_n}^n \Rightarrow \neg \bigwedge \Box\Gamma_k^j$ and by applying ($\Rightarrow\Box$), $\Box\Gamma_1^1, \ldots, \Box\Gamma_{m_1}^1, \ldots, \Box\Gamma_1^n, \ldots, \Box\Gamma_{m_n}^n \Rightarrow \Box\neg \bigwedge \Box\Gamma_k^j$. Now, by modalized splitting (*MS*), $\mathcal{G}' = \Box\Gamma_1^1 \Rightarrow \Box\neg \bigwedge \Box\Gamma_k^j \mid \ldots \mid \Box\Gamma_{m_1}^1 \Rightarrow \Box\neg \bigwedge \Box\Gamma_k^j \mid \ldots \mid \Box\Gamma_1^n \Rightarrow \Box\neg \bigwedge \Box\Gamma_k^j \mid \ldots \mid \Box\Gamma_{m_n}^n \Rightarrow \Box\neg \bigwedge \Box\Gamma_k^j$ is derivable in GS5. By Lemma 5.11.6 it follows that $\mathcal{G} = \Box\Gamma_1^1 \Rightarrow \neg \bigwedge \Box\Gamma_k^j \mid \ldots \mid \Box\Gamma_{m_1}^1 \Rightarrow \neg \bigwedge \Box\Gamma_k^j \mid \ldots \mid \Box\Gamma_1^n \Rightarrow \neg \bigwedge \Box\Gamma_k^j \mid \ldots \mid \Box\Gamma_{m_n}^n \Rightarrow \neg \bigwedge \Box\Gamma_k^j$ is derivable in GS5. Note that any attacker of $\mathcal{G}$ is an attacker of one of the $\mathcal{H}_1, \ldots, \mathcal{H}_n$, therefore $\mathcal{G} \in \mathcal{E}$. However, $\mathcal{G}$ attacks $\mathcal{H}_j$, a contradiction with the assumption that $\mathcal{E}$ is conflict-free.

Free precedence: Suppose that $\mathrm{ConUcut}_H$ is one of the attack rules. It can easily be shown that any of the considered attack rules is *conflict-dependent* [1], namely: if $\mathcal{H}, \mathcal{G} \in \mathrm{Arg}_{\mathrm{S5}}(\mathcal{S})$ and $\mathcal{H}$ attacks $\mathcal{G}$, then $\bigcup \mathrm{Supp}(\mathcal{H}) \cup \bigcup \mathrm{Supp}(\mathcal{G})$ is inconsistent. Suppose now that $\mathcal{H} = \Gamma_1 \Rightarrow \phi_1 \mid \ldots \mid \Gamma_n \Rightarrow \phi_n \in \mathrm{Arg}_{\mathrm{S5}}(\mathsf{Free}(\mathcal{S}))$. Thus $\bigcup \mathrm{Supp}(\mathcal{H}) \subseteq \mathsf{Free}(\mathcal{S})$. To see that $\mathcal{H} \in \mathcal{E}$ assume that $\mathcal{G} = \Delta_1 \Rightarrow \psi_1 \mid \ldots \mid \Delta_m \Rightarrow \psi_m \in \mathrm{Arg}_{\mathrm{S5}}(\mathcal{S})$ attacks $\mathcal{H}$. Since the attack rules are conflict-dependent, there is a minimal conflict $C \subseteq \bigcup \mathrm{Supp}(\mathcal{H}) \cup \bigcup \mathrm{Supp}(\mathcal{G})$. But $\bigcup \mathrm{Supp}(\mathcal{H}) \subseteq \mathsf{Free}(\mathcal{S})$, thus $C \subseteq \bigcup \mathrm{Supp}(\mathcal{G})$, and so $\mathrm{Supp}(\mathcal{G})$ is not consistent. By Lemma 5.2(b), then, $\mathcal{H}$ is defended by $\mathcal{E}$. Since $\mathcal{G}$ is arbitrary and $\mathcal{E}$ is complete, $\mathcal{H} \in \mathcal{E}$. Thus $\mathrm{Arg}_{\mathrm{S5}}(\mathsf{Free}(\mathcal{S})) \subseteq \mathcal{E}$. ◻

We conclude this section by a short remark on the consistency postulate. As noted e.g. in [2, 19], in logical argumentation the deductive closure of extensions according to Dung-style semantics (Definition 3.6) may not be consistent in some cases. As the next example shows, this phenomenon carries on to hypersequential frameworks, so the consistency postulate may be violated in our case as well.

Example 5.12. Consider the set $\mathcal{S}' = \{p \vee q, p \vee \neg q, \neg p, r\}$, obtained from the set $\mathcal{S}$ of Example 3.5 by omitting the appearances of the modal operator $\Box$. The arguments of Example 3.5, now without the $\Box$'s, belong to $\mathcal{AF}_{\mathrm{S5}}(\mathcal{S}')$. These arguments, and the attacks between them, can be represented as in Figure 2. Note that these arguments, viewed as hypersequents, cannot be split up, since (*MS*) is applicable only to formulas with the $\Box$-operator. It follows that $\mathcal{E}' = \{a_1', a_2', a_3'\}$ (where a_i' denotes the argument a_i from Example 3.5 without the $\Box$-operator) is admissible in $\mathcal{AF}_{\mathrm{S5}}(\mathcal{S}')$, but $\mathrm{Conc}(\mathcal{E}')$ is not consistent.

The last example demonstrates the necessity to trade the consistency postulate for its modularized version (see Proposition 5.10). Indeed, when switching to hypersequential frameworks with modular operators, consistency *can be* guaranteed for arguments in the form of the elements of $\mathcal{E}_{\Box}$ (see Definition 5.8). This is intuitively explained by the introduction of new arguments (and so new attacks), which are obtained by the modularized splitting rule (*MS*), as is illustrated in the next example.

Example 5.13. Unlike the situation described in Example 5.12, in the hypersequent-based argumentation framework of Example 4.3 (Figure 2) $\mathrm{S} = \{a_1, a_2, a_3\}$ cannot be part of a complete extension of $\mathcal{AF}_{\mathrm{S5}}(\mathcal{S})$. Indeed, a complete extension $\mathcal{E}$, such that $\mathrm{S} \subseteq \mathcal{E}$, has to defend a_2 from the attack by a_8. In order to do so, $\mathcal{E}$ must be extended with an argument like a_4, a_6, a_7 or a_9, but then the new extension is not conflict-free anymore.

6 CONCLUSION AND FURTHER WORK

In this paper we have generalized sequent-based argumentation to hypersequents. Hypersequential argumentation, like sequential argumentation, allows for a great flexibility in defining arguments and attack rules, and avoids some limitations of other approaches to deductive logical argumentation, like the requirement that the arguments' support sets should be consistent and minimal, or the adherence to classical logic as the sole core logic (see, e.g., [2, 11]).

Hypersequential frameworks are particularly useful for logics that lack a decent sequent calculus and so cannot be successfully applied for sequent-based argumentation. Such a logic is the modal logic S5, which lacks a cut-free Gentzen-type proof system. It was shown that hypersequential frameworks that are based on this logic together with some standard attack rules (formalized for hypersequential arguments) have some interesting properties, among others modular closure and modular consistency, which are otherwise not available. This opens up possibilities to integrate further modal logics and extended frameworks to capture, for instance, deontic and temporal logics.

Another interesting application of our setting is within the context of agent-based modeling with structured arguments. Indeed, modal logics serve as a foundation for different multi-agent systems (see [40]), and the use of modal logics together with structured argumentation in multi-agent systems was already investigated in the literature. For instance, in [28] beliefs or information states of an agent are represented in a modal language, and their interactions are studied from the point of view of argumentation theory, by means of dialogues, to obtain further information about arguments and attacks. A multi-agent setting where agents reason about knowledge and/or beliefs of other agents, represented as modality-based arguments, can also be found e.g. in [37]. A recent modeling of a multi-agent system based on abstract argumentation theory can be found in [16], where agents represent scientists that have to explore an argumentative landscape, consisting of several scientific theories, and the theory that has the most acceptable arguments is determined by Dung's semantics. Our setting is particularly useful for a similar purpose, since S5 is considered an important epistemic logic [22], (see [21] for a recent overview of the research in this direction), and so S5 is a basis of both single and multi-agent logical systems. In that respect, we intend to extend the dynamic proof theory introduced in [5] from sequent-based frameworks to hypersequent-based ones, so that the S5-based hypersequential frameworks presented here will allow an agent to automatically reason about knowledge, beliefs, obligations, and arguments, depending on the interpretation of the modalities.

Another interesting direction for future work is to further increase the expressive power of the frameworks by adding preferences among the arguments. Note that to some extent this may be done already in the current setting. Indeed, when e.g $\mathcal{S} = \{\Box p, \neg p\}$ is taken as (part of) the set of premises, the argument $\Box p \Rightarrow \Box p$ defeats $\neg p \Rightarrow \neg p$ (this is so since $\Box p \supset p$ is an S5-axiom and double-negation elimination holds in S5), but not vice versa. For a more fine-grained approach one would probably need to incorporate a priority function for expressing preferences among arguments , or introduce a preference criterion, based on the number of modalities in a formula, as is done in other contexts of prioritized or probabilistic argumentation frameworks (see, e.g., [3, 29] for some discussions and further references). In both cases, the attack rules, like those in Example 4.1, would remain the same, but their application will be determined by the relative strengths of the attacking and the attacked arguments (that is, the attacking argument should not be weaker than the attacked argument [3, 15]).

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