Journal of Logic, Language, and Information 5: 25–63, 1996.

Reasoning with Logical Bilattices

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(Received 31 May 1994; in final form 23 October 1995)

Abstract. The notion of *bilattice* was introduced by Ginsberg, and further examined by Fitting, as a general framework for many applications. In the present paper we develop proof systems, which correspond to bilattices in an essential way. For this goal we introduce the notion of *logical* bilattices. We also show how they can be used for efficient inferences from possibly inconsistent data. For this we incorporate certain ideas of Kifer and Lozinskii, which happen to suit well the context of our work. The outcome are paraconsistent logics with a lot of desirable properties.*

Key words: Substructural logics, relevance logic, many-valued logics, hypersequents

1. Introduction

When using multiple-valued logics, it is usual to order the truth values in a lattice structure. In most cases such a partial order intuitively reflects differences in the "measure of truth" that the lattice elements are supposed to represent. There exist, however, other intuitive criteria of ordering that might be useful. Another reasonable ordering might reflect, for example, differences in the amount of *knowledge* or in the amount of *information* that each of these elements exhibits. Obviously, therefore, there might be cases in which *two* partial orders, each reflecting a different intuitive concept, might be useful. This, for example, has been the case with Belnap's famous four-valued logic (Belnap, 1977a; Belnap, 1977b). Belnap's logic was generalized in Ginsberg (1988), where Ginsberg introduced the notion of *bilattices*, which are algebraic structures that contain two partial orders simultaneously (see Definition 2.1). His motivation was to present a general framework for many applications, like truth maintenance systems and default inferences. The notion was further investigated and applied for logic programming and other purposes by Fitting (1989, 1990a, 1990b, 1991, 1993, 1994).

In all of their applications so far, the role of bilattices was algebraic in nature. In this paper we try to carry bilattices to a new stage in their development by constructing *logics* (i.e.: consequence relations) which are based on bilattices, as well as corresponding proof systems. For this purpose we have found it useful to introduce and investigate the notion of a *logical* bilattice. (All the known bilattices which were actually proposed for applications in the literature fall under this

* A preliminary version of this paper appears in Arieli and Avron (1994).

category). The general logic of these bilattices turned out to have a very nice proof theory. We also show how to use logical bilattices in a more specific way for non-monotonic reasoning and for efficient inferences from inconsistent data (these were, respectively, the original purposes of Belnap and Ginsberg). For this we incorporate certain ideas from Kifer and Lozinskii (1992). We show (so we believe) that bilattices provide a better framework for applying these ideas than the one used in the original paper.

The paper is organized as follows: In the next section we introduce and investigate the notion of logical bilattice. In Section 3 we investigate (from semantical and proof-theoretical points of view) the general logic that is naturally associated with them. This logic is monotonic and paraconsistent. In Section 4 we consider a refined consequence relation which is shown to be *non-monotonic*, and very useful for reasoning in the presence of inconsistency.

2. Logical Bilattices

2.1. Bilattices – General Background

DEFINITION 2.1. A *bilattice* is a structure $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ such that B is a non empty set containing at least two elements; $(B, \leq_t)$, $(B, \leq_k)$ are complete* lattices; and $\neg$ is a unary operation on B that has the following properties:
1. if $a \leq_t b$, then $\neg a \geq_t \neg b$,
2. if $a \leq_k b$, then $\neg a \leq_k \neg b$,
3. $\neg\neg a = a$.

Following Fitting, we shall use $\wedge$ and $\vee$ for the lattice operations which correspond to $\leq_t$, and $\otimes$, $\oplus$ for those that correspond to $\leq_k$. While $\wedge$ and $\vee$ can be associated with their usual intuitive meanings of "and" and "or", one may understand $\otimes$ and $\oplus$ as the "consensus" and the "gullibility" ("accept all") operators, respectively; $p \otimes q$ is the most that p and q can agree on, while $p \oplus q$ accepts the combined knowledge of p with that of q (see also Fitting (1990b, 1994)). A practical application of $\otimes$ and $\oplus$ is provided, for example, in an implementation of a logic programming language designed for distributed knowledge-bases (see Fitting (1991) for more details).

Note that negation is order preserving w.r.t $\leq_k$. This reflects the intuition that $\leq_k$ corrsponds to differences in our *knowledge* about formulae and not to their degrees of truth. Hence, while one expects negation to invert the notion of truth, the role of negation w.r.t. $\leq_k$ is somewhat less transparent: we know no more and no less about $\neg p$ than we know about p (see Ginsberg (1988, p. 269), and Fitting (1990a, p. 239), for further discussion).

* This is Ginsberg's ((1988)) original definition. Some authors have dropped this requirement of completion. We have retained it since we need it in Section 3.5, but apart of that section all our results are valid without this assumption.

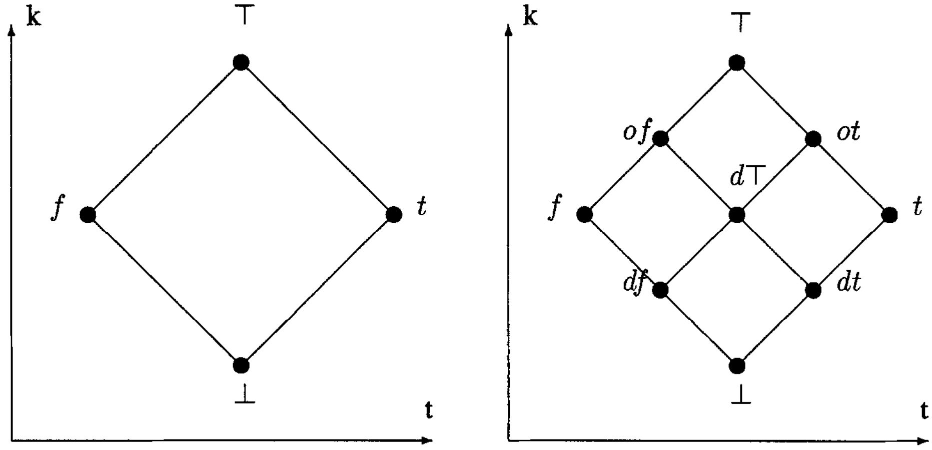

Fig. 1. *FOUR* and *NINE*

We will denote by f and by t the least element and the greatest element (respectively) of B w.r.t $\leq_t$, while $\bot$ and $\top$ will denote the least element and the greatest element of B w.r.t $\leq_k$.* $f, t, \bot$, and $\top$ are all different by Lemma 2.5(a) below, and by the fact that a bilattice contains at least two elements.

DEFINITION 2.2. A bilattice is called *distributive* (Ginsberg, 1988) if all the twelve possible distributive laws concerning $\wedge$, $\vee$, $\otimes$, and $\oplus$ hold.** It is called *interlaced* (Fitting, 1990a) if each one of $\wedge$, $\vee$, $\otimes$, and $\oplus$ is monotonic with respect to both $\leq_t$ and $\leq_k$.

LEMMA 2.3. (Fitting, 1990a) Every distributive bilattice is interlaced.

EXAMPLE 2.4. Figures 1 and 2 contain double Hasse diagrams of some useful bilattices. In these diagrams y is an immediate $\leq_t$-successor of x iff y is on the right side of x, and there is an edge between them; similarly, y is an immediate $\leq_k$-successor of x iff y is above x, and there is an edge between them.

Belnap's *FOUR* (Belnap 1977a, 1977b), drawn in Figure 1, is the smallest bilattice. It easy to check that *FOUR* is distributive. Ginsberg's *DEFAULT* (Figure 2) was introduced in Ginsberg (1988) as a tool for non-monotonic reasoning. The truth values that have a prefix "d" in their names are supposed to represent values of default assumptions (dt = true by default; etc.). It easy to verify that $\neg df = dt$; $\neg dt = df$; $\neg d\top = d\top$. The negations of $\top, t, f, \bot$ are identical to that of *FOUR* (see

* $\bot$ and $\top$ could be thought of as representing no information and inconsistent knowledge, respectively.

** *Infinitary* laws have also been considered in the literature (see, e.g., Fitting, 1993, Definition 3.3). In this paper we do not use such laws. They might be more useful when we enter more deeply to quantification theory in the future.

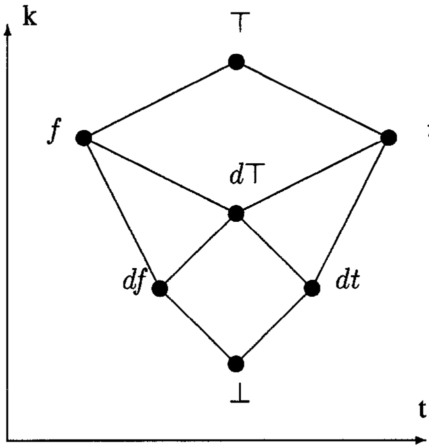

Fig. 2. *DEFAULT*

Lemma 2.5(a) below). This bilattice is not even interlaced;[‡] *NINE* (Figure 1), on the other hand, is distributive, and it contains the default values of *DEFAULT*. In addition, *NINE* has two more truth values, ot and of, where $\neg of = ot$ and $\neg ot = of$.

LEMMA 2.5. Let $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ be a bilattice, and let $a, b \in B$.
a) (Ginsberg, 1988) $\neg(a \wedge b) = \neg a \vee \neg b$; $\neg(a \vee b) = \neg a \wedge \neg b$; $\neg(a \otimes b) = \neg a \otimes \neg b$; $\neg(a \oplus b) = \neg a \oplus \neg b$.
Also, $\neg f = t$; $\neg t = f$; $\neg \bot = \bot$; $\neg \top = \top$.
b) (Fitting, 1991) If $\mathcal{B}$ is interlaced, then: $\bot \wedge \top = f$; $\bot \vee \top = t$; $f \otimes t = \bot$; $f \oplus t = \top$.

DEFINITION 2.6. (Ginsberg, 1988) Let $(L,\leq)$ be a complete lattice. The structure $L \odot L = (L \times L, \leq_t, \leq_k, \neg)$ is defined as follows:
$(y_1, y_2) \geq_t (x_1, x_2)$ iff $y_1 \geq x_1$ and $y_2 \leq x_2$,
$(y_1, y_2) \geq_k (x_1, x_2)$ iff $y_1 \geq x_1$ and $y_2 \geq x_2$,
$\neg(x_1, x_2) = (x_2, x_1)$.

LEMMA 2.7. Let $(L,\leq)$ be a complete lattice. Then:
a) (Fitting, 1990a) $L \odot L$ is an interlaced bilattice.
b) (Ginsberg, 1988) If L is distributive, then so is $L \odot L$.

$L \odot L$ was introduced in Ginsberg (1988), and later examined by Fitting as a general method for constructing bilattices. A truth value $(x, y) \in L \odot L$ may intuitively be understood so that x represents the amount of belief *for* an assertion, and y is the degree of belief *against* it.

EXAMPLE 2.8. Denote the standard two valued structure $\{0,1\}$ by *TWO*. Then *FOUR* is isomorphic to *TWO*$\odot$*TWO*. Similarly, *NINE* is isomorphic to $\{-1, 0, 1\} \odot \{-1, 0, 1\}$.

[‡] For example, $f <_t df$, while $f \otimes d\top = d\top >_t df = df \otimes d\top$.

We conclude this introductory part by considering another bilattice operation, and a corresponding family of bilattices:

DEFINITION 2.9. (Fitting, 1990b) A *conflation*, $-$, is a unary operation on a bilattice B that has the following properties:
1. if $a \leq_k b$ then $-a \geq_k -b$,
2. if $a \leq_t b$ then $-a \leq_t -b$,
3. $--a = a$,
4. $-\neg a = \neg - a$.*

LEMMA 2.10. (Fitting, 1990b) Let $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ be a bilattice, and let $a, b \in B$.
$-(a \wedge b) = -a \wedge -b$; $-(a \vee b) = -a \vee -b$; $-(a \otimes b) = -a \oplus -b$; $-(a \oplus b) = -a \otimes -b$.
Also, $-f = f$; $-t = t$; $-\bot = \top$; $-\top = \bot$.

DEFINITION 2.11. (Fitting, 1994) A bilattice with a conflation is called *classical*, if for every $b \in B$, $b \vee - \neg b = t$.**

EXAMPLE 2.12. *FOUR* is a classical bilattice (where "$-$" is defined according to Lemma 2.10).

Classical bilattices were presented is order to allow analogues of classical tautologies. In particular, in classical bilattices it is really the combination $-\neg$ that plays the role of classical negation.

2.2. Bifilters and Logicality

One of the most important component in a many-valued logic is the subset of the *designated* truth values. This subset is used for defining validity of formulae and a consequence relation. Frequently, in an algebraic treatment of the subject, the set of the designated values forms a filter, or even a prime (ultra-) filter, relative to some natural ordering of the truth values. Natural analogues for bilattices of filters, prime filters, ultrafilters, and set of designated values in general, are the following:

DEFINITION 2.13.
a) A *bifilter* of a bilattice $\mathcal{B} = (B, \leq_t, \leq_k)$ is a nonempty subset $\mathcal{F} \subset B$, $\mathcal{F} \neq B$, such that:
$a \wedge b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$
$a \otimes b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$
b) A bifilter $\mathcal{F}$ is called *prime*, if it satisfies also:

* This requirement is not part of Fitting's original definition. Nevertheless, it is usually assumed when dealing with bilattices that have conflation, and useful for our purposes.
** In the original definition of classical bilattice, Fitting requires that the bilattice would be distributive. This requirement is not essential for the present treatment of such bilattices.

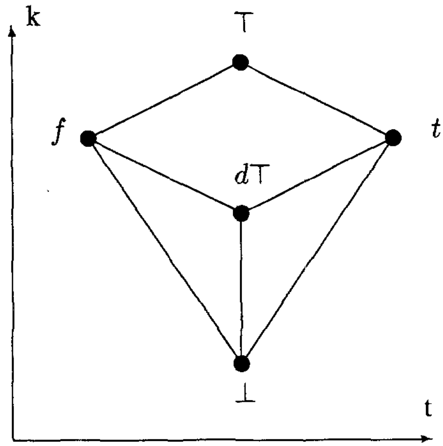

Fig. 3. *FIVE*

$a \vee b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$
$a \oplus b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$
c) Let $\mathcal{B}$ be a bilattice with a conflation. $\mathcal{F}$ is an *ultrabifilter* in $\mathcal{B}$, if it is a prime bifilter, and $b \in \mathcal{F}$ iff $-\neg b \notin \mathcal{F}$.

EXAMPLE 2.14. *FOUR* and *DEFAULT* contain exactly one bifilter: $\{\top, t\}$, which is prime in both, and an ultrabifilter in *FOUR*. $\{\top, t\}$ is also the only bifilter of *FIVE* (Figure 3), but it is *not* prime there: $d\top \vee \bot = t$, while $d\top \notin \mathcal{F}$, and $\bot \notin \mathcal{F}$. *NINE* contains two bifilters: $\{\top, ot, t\}$, as well as $\{\top, ot, t, of, d\top, dt\}$; both are prime, but neither is an ultrabifilter.

PROPOSITION 2.15. (Basic properties of bifilters) Let $\mathcal{F}$ be a bifilter of $\mathcal{B}$; Then:
a) $\mathcal{F}$ is upward-closed w.r.t both $\leq_t$ and $\leq_k$.
b) $t, \top \in \mathcal{F}$, while $f, \bot \notin \mathcal{F}$.
c) In classical bilattices every prime bifilter is also an ultrabifilter.

Proof. Claim (a) follows immediately from the definition of $\mathcal{F}$; the first part of (b) follows from (a), and from the maximality of t and $\top$; the fact that the minimal elements are not in $\mathcal{F}$ follows also from (a), since $\mathcal{F} \neq B$. Finally, part (c) obtains since on the one hand in every classical bilattice $b \vee -\neg b = t \in \mathcal{F}$, and since $\mathcal{F}$ is prime, either $b \in \mathcal{F}$ or $-\neg b \in \mathcal{F}$. On the other hand, $-\neg b \wedge b = -\neg(b \vee -\neg b) = -\neg t = f \notin \mathcal{F}$, therefore $-\neg b \wedge b \notin \mathcal{F}$, and so either $b \notin \mathcal{F}$ or $-\neg b \notin \mathcal{F}$. □

DEFINITION 2.16. A *logical bilattice* is a pair $(\mathcal{B}, \mathcal{F})$, in which $\mathcal{B}$ is a bilattice, and $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

In the next section we shall use logical bilattices for defining logics in a way which is completely analogous to the way Boolean algebras and prime filters are used in classical logic. The role which TWO has among Boolean algebras is taken here by $FOUR$:

THEOREM 2.17. Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. Then there exists a unique homomorphism $h : \mathcal{B} \to FOUR$, such that $h(b) \in \{\top, t\}$ iff $b \in \mathcal{F}$.

Proof. It is immediate that the only function $h : \mathcal{B} \to FOUR$ that satisfies the condition, and is also an homomorphism w.r.t negation, is the following one:

$$h(b) \stackrel{\text{def}}{=} \begin{cases} \top & \text{if } b \in \mathcal{F} \text{ and } \neg b \in \mathcal{F} \\ t & \text{if } b \in \mathcal{F} \text{ and } \neg b \notin \mathcal{F} \\ f & \text{if } b \notin \mathcal{F} \text{ and } \neg b \in \mathcal{F} \\ \bot & \text{if } b \notin \mathcal{F} \text{ and } \neg b \notin \mathcal{F} \end{cases}$$

This entails uniqueness. For existence, note first that h is obviously an homomorphism w.r.t $\neg$. It remains to show that it is also a homomorphism w.r.t $\wedge$, $\vee$, $\otimes$, and $\oplus$.

a) The case of $\wedge$*:*

1. Suppose that $a \wedge b \in \mathcal{F}$ and $\neg(a \wedge b) \in \mathcal{F}$. Then $a \in \mathcal{F}$ and $b \in \mathcal{F}$. In addition, $\neg(a \wedge b) \in \mathcal{F}$, hence $\neg a \vee \neg b \in \mathcal{F}$, and so $\neg a \in \mathcal{F}$ or $\neg b \in \mathcal{F}$ (since $\mathcal{F}$ is prime). It follows that $\{a, \neg a\} \subseteq \mathcal{F}$ or $\{b, \neg b\} \subseteq \mathcal{F}$, hence either $h(a) = \top$ or $h(b) = \top$. Since both $h(a)$ and $h(b)$ are in $\{\top, t\}$, and $\top \wedge \top = \top \wedge t = \top$, it follows that $h(a) \wedge h(b) = \top = h(a \wedge b)$.
2. If $a \wedge b \in \mathcal{F}$ but $\neg(a \wedge b) \notin \mathcal{F}$, then $a \in \mathcal{F}$ and $b \in \mathcal{F}$, but $\neg a \vee \neg b \notin \mathcal{F}$, and so neither $\neg a$ nor $\neg b$ are in $\mathcal{F}$. It follows that $h(a) = h(b) = t$, so this time $h(a) \wedge h(b) = t = h(a \wedge b)$.
3. Suppose that $a \wedge b \notin \mathcal{F}$ and $\neg(a \wedge b) \in \mathcal{F}$. Then either $\neg a \in \mathcal{F}$ or $\neg b \in \mathcal{F}$. Assume, e.g., that $\neg a \in \mathcal{F}$. If $a \notin \mathcal{F}$ then $h(a) = f$ and so $h(a) \wedge h(b) = f = h(a \wedge b)$. If, on the other hand, $a \in \mathcal{F}$, then $h(a) = \top$. In addition $b \notin \mathcal{F}$ (otherwise we would have $a \wedge b \in \mathcal{F}$), and so $h(b) \in \{f, \bot\}$. Since in $FOUR$ $\top \wedge f = \top \wedge \bot = f$, in this case $h(a) \wedge h(b) = f = h(a \wedge b)$.
4. Suppose that $a \wedge b \notin \mathcal{F}$ and $\neg(a \wedge b) \notin \mathcal{F}$. Then $\neg a \notin \mathcal{F}$, $\neg b \notin \mathcal{F}$ and either $a \notin \mathcal{F}$ or $b \notin \mathcal{F}$. It follows that either $h(a) = \bot$ or $h(b) = \bot$. Assume, e.g., the former. Since $\neg b \notin \mathcal{F}$, then $h(b) \in \{t, \bot\}$. But since $\bot \wedge t = \bot \wedge \bot = \bot$, $h(a) \wedge h(b) = \bot = h(a \wedge b)$ in this case.

b) The case of $\vee$*:* Since $a \vee b = \neg(\neg a \wedge \neg b)$, this case follows from the previous one.

c) The case of $\otimes$*:*

1. If $a \otimes b \in \mathcal{F}$ and $\neg(a \otimes b) \in \mathcal{F}$, then since $\neg(a \otimes b) = \neg a \otimes \neg b$, we have that $a, b, \neg a, \neg b \in \mathcal{F}$, hence $h(a) = h(b) = \top$, and so $h(a) \otimes h(b) = \top \otimes \top = \top = h(a \otimes b)$.
2. If $a \otimes b \in \mathcal{F}$ and $\neg(a \otimes b) \notin \mathcal{F}$, then $a \in \mathcal{F}$, $b \in \mathcal{F}$, and either $\neg a \notin \mathcal{F}$ or $\neg b \notin \mathcal{F}$. It follows that both $h(a)$ and $h(b)$ are in $\{\top, t\}$, and at least one of them is t. hence, $h(a) \otimes h(b) = t = h(a \otimes b)$.

3. The case that $a \otimes b \notin \mathcal{F}$ and $\neg(a \otimes b) \in \mathcal{F}$ is similar to the previous one.
4. If $a \otimes b \notin \mathcal{F}$ and $\neg(a \otimes b) \notin \mathcal{F}$ then either $a \notin \mathcal{F}$ or $b \notin \mathcal{F}$, and also either $\neg a \notin \mathcal{F}$ or $\neg b \notin \mathcal{F}$. Assume, e.g., that $a \notin \mathcal{F}$. If also $\neg a \notin \mathcal{F}$, then $h(a) = \bot$, and so $h(a) \otimes h(b) = \bot = h(a \otimes b)$. If, on the other hand, $\neg a \in \mathcal{F}$, then $\neg b \notin \mathcal{F}$, and so we get that $h(a) = f$, and $h(b) \in \{t, \bot\}$. Since in *FOUR* $f \otimes t = f \otimes \bot = \bot$, we have again that $h(a) \otimes h(b) = \bot = h(a \otimes b)$.

d) The case of $\oplus$:

1. Assume that $a \oplus b \in \mathcal{F}$ and $\neg(a \oplus b) \in \mathcal{F}$. Then $a \in \mathcal{F}$ or $b \in \mathcal{F}$. Assume, e.g., that $a \in \mathcal{F}$; then $h(a) \in \{\top, t\}$. If in addition $\neg a \in \mathcal{F}$, then $h(a) = \top$, and so $h(a) \oplus h(b) = \top = h(a \oplus b)$. Otherwise, $\neg b \in \mathcal{F}$, and so $h(b) \in \{\top, f\}$. Since in *FOUR*, $\top \oplus \top = \top \oplus t = \top \oplus f = t \oplus f = \top$, we have that $h(a) \oplus h(b) = \top = h(a \oplus b)$.
2. If $a \oplus b \in \mathcal{F}$ and $\neg(a \oplus b) \notin \mathcal{F}$, then $a \in \mathcal{F}$ or $b \in \mathcal{F}$, and neither $\neg a$ nor $\neg b$ are in $\mathcal{F}$. It follows that $h(a)$, $h(b)$ are both in $\{t, \bot\}$, and at least on of then is t. Hence, $h(a) \oplus h(b) = t = h(a \oplus b)$.
3. The case that $a \oplus b \notin \mathcal{F}$ and $\neg(a \oplus b) \in \mathcal{F}$ is similar to the previous one.
4. If $a \oplus b \notin \mathcal{F}$ and $\neg(a \oplus b) \notin \mathcal{F}$, then $a, \neg a, b, \neg b$ are all not in $\mathcal{F}$, and so $h(a) = h(b) = \bot$. It follows that $h(a) \oplus h(b) = \bot = h(a \oplus b)$. □

Note: For Boolean algebras we have, in fact, a weaker theorem: given x from a Boolean algebra B, and a filter F $\subseteq$ B s.t. $x \notin$ F, we have an homomorphism h_x: B $\to$ *TWO* w.r.t $\neg, \wedge, \vee$ s.t. $h_x(x) \notin \mathrm{F}(TWO)$, and $h_x(y) \in \mathrm{F}(TWO)$ for every $y \in$ F. In our case, the same h is good for all x. On the other hand, in Boolean algebras we have the property that if $x, y \in$ B and $x \neq y$, then there is an homomorphism h: B $\to$ *TWO* which separates them. This further implies that equalities which hold in *TWO* are valid in any Boolean algebra. Logical bilattices and *FOUR*, in contrast, do not enjoy this property. Thus, the distributive law $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ is valid in *FOUR*, but *not* in every logical bilattice in general (take, e.g., *DEFAULT*).

DEFINITION 2.18. An *ultralogical bilattice* is a pair $(\mathcal{B}, \mathcal{F})$, where $\mathcal{B}$ is a bilattice with a conflation, and $\mathcal{F}$ is an ultrabifilter on $\mathcal{B}$.

As it follows from Proposition 2.15(c), ultralogical bilattices are natural extensions of Fitting's notion of classical bilattices. Also, they have several similar properties to those of logical bilattices. The next proposition is one such an instance (cf. Theorem 2.17):

PROPOSITION 2.19. Let $(\mathcal{B}, \mathcal{F})$ be an ultralogical bilattice. Then there exists a unique homomorphism $h : \mathcal{B} \to FOUR$, such that $h(b) \in \{\top, t\}$ iff $b \in \mathcal{F}$.

Proof. Similar to that of Theorem 2.17. The only extra thing that we need to check is the case of conflation. Again, we shall examine the four possible cases:

1. $h(b) = \top \Rightarrow b \in \mathcal{F}, \neg b \in \mathcal{F} \Rightarrow -\neg b \notin \mathcal{F}, -\neg\neg b \notin \mathcal{F} \Rightarrow \neg - b \notin \mathcal{F}, -b \notin \mathcal{F} \Rightarrow$ $h(-b) = \bot = -h(b)$.

2. $h(b)=t \Rightarrow b\in\mathcal{F}$, $\neg b\notin\mathcal{F} \Rightarrow -\neg b\notin\mathcal{F}$, $-\neg\neg b\in\mathcal{F} \Rightarrow \neg-b\notin\mathcal{F}$, $-b\in\mathcal{F} \Rightarrow$ $h(-b)=t=-h(b)$.
3. $h(b)=f \Rightarrow b\notin\mathcal{F}$, $\neg b\in\mathcal{F} \Rightarrow -\neg b\in\mathcal{F}$, $-\neg\neg b\notin\mathcal{F} \Rightarrow \neg-b\in\mathcal{F}$, $-b\notin\mathcal{F} \Rightarrow$ $h(-b)=f=-h(b)$.
4. $h(b)=\bot \Rightarrow b\notin\mathcal{F}$, $\neg b\notin\mathcal{F} \Rightarrow -\neg b\in\mathcal{F}$, $-\neg\neg b\in\mathcal{F} \Rightarrow \neg-b\in\mathcal{F}$, $-b\in\mathcal{F} \Rightarrow$ $h(-b)=\top=-h(b)$. □

Since ultralogical bilattices seems to be quite rare,* we shall concentrate in what follows on logical bilattices.

Next we discuss the existence of bifilters and prime bifilters, concentrating on an important special case:

DEFINITION 2.20. Let $\mathcal{B}$ be a bilattice. Define:
- $\mathcal{D}_k(\mathcal{B}) \stackrel{\text{def}}{=} \{\, x \mid x \geq_k t \,\}$ (*designated* values of $\mathcal{B}$ w.r.t $\leq_k$)
- $\mathcal{D}_t(\mathcal{B}) \stackrel{\text{def}}{=} \{\, x \mid x \geq_t \top \,\}$ (*designated* values of $\mathcal{B}$ w.r.t $\leq_t$)

Intuitively, each element of $\mathcal{D}_k(\mathcal{B})$ represents a truth value which is known to be "at least true" (Belnap, 1977b, p. 36). Hence it seems that $\mathcal{D}_k(\mathcal{B})$ is a particularly natural candidate to play the role of the set of the designated values of $\mathcal{B}$.

EXAMPLE 2.21.
a) $\mathcal{D}_k(FOUR) = \mathcal{D}_t(FOUR) = \{\top, t\}$.
b) $\mathcal{D}_k(FIVE) = \mathcal{D}_t(FIVE) = \{\top, t\}$.
c) $\mathcal{D}_k(DEFAULT) = \mathcal{D}_t(DEFAULT) = \{\top, t\}$.
d) $\mathcal{D}_k(NINE) = \mathcal{D}_t(NINE) = \{\top, ot, t\}$.
e) $\mathcal{D}_k(L\odot L) = \mathcal{D}_t(L\odot L) = \{\, (sup(L), x) \mid x \in L \,\}$.

PROPOSITION 2.22. (Basic properties of $\mathcal{D}_k(\mathcal{B})$ and $\mathcal{D}_t(\mathcal{B})$)
a) $t, \top \in \mathcal{D}_k(\mathcal{B})$, while $f, \bot \notin \mathcal{D}_k(\mathcal{B})$. The same is true for $\mathcal{D}_t(\mathcal{B})$.
b) $\mathcal{D}_k(\mathcal{B}) \cup \mathcal{D}_t(\mathcal{B}) \subseteq \mathcal{F}$.

Proof. The first part concerning $\mathcal{D}_k(\mathcal{B})$ of (a) is obvious. To see that $f \notin \mathcal{D}_k(\mathcal{B})$, assume the countrary. Then $f \geq_k t$ and so also $\neg f \geq_k \neg t$, which means that $t \geq_k f$, hence $f=t$. This entails that $\mathcal{B}$ contains just one element, but this contradicts the definition of a bilattice. An even simpler argumenet holds for $\bot$. Claim (b) follows immediately from Proposition 2.15. □

PROPOSITION 2.23. If $\mathcal{D}_k(\mathcal{B})=\mathcal{D}_t(\mathcal{B})$, then $\mathcal{D}_k(\mathcal{B})$ is the smallest bifilter (i.e: it is contained in any other bifilter).

Proof. For every $a, b\in B$, $a\wedge b\in\mathcal{D}_t(\mathcal{B})$ iff $a\wedge b\geq_t\top$, iff $a\geq_t\top$ and $b\geq_t\top$, iff $a\in\mathcal{D}_t(\mathcal{B})$ and $b\in\mathcal{D}_t(\mathcal{B})$. Similarly, $a\otimes b\in\mathcal{D}_k(\mathcal{B})$ iff $a\in\mathcal{D}_k(\mathcal{B})$ and $b\in\mathcal{D}_k(\mathcal{B})$. Hence, if $\mathcal{D}_k(\mathcal{B})=\mathcal{D}_t(\mathcal{B})$ then $\mathcal{D}_k(\mathcal{B})$ is a bifilter of $\mathcal{B}$. That $\mathcal{D}_k(\mathcal{B})$ is *the smallest* bifilter in this case follows from Proposition 2.22(b). □

* Even *NINE* with either one of its two prime bifilters is not ultralogical bilattice.

PROPOSITION 2.25. Let $\mathcal{B}$ be an interlaced bilattice. Then:
a) $\mathcal{D}_k(\mathcal{B}) = \mathcal{D}_t(\mathcal{B})$.
b) $\{b, \neg b\} \subseteq \mathcal{D}(\mathcal{B})$ iff $b = \top$.

Proof. Suppose that $\mathcal{B}$ is interlaced. Then:
a) $b \geq_t \top \Rightarrow b \wedge \top = \top \Rightarrow b \wedge \top \geq_k t \Rightarrow b \vee (b \wedge \top) \geq_k b \vee t \Rightarrow b \geq_k t$. Similarly, $b \geq_k t \Rightarrow b \otimes t = t \Rightarrow b \otimes t \geq_t \top \Rightarrow b \oplus (b \otimes t) \geq_t b \oplus \top \Rightarrow b \geq_t \top$. Hence $\mathcal{D}_k(\mathcal{B}) = \mathcal{D}_t(\mathcal{B})$.
b) If $b = \top$, then $b = \neg b = \top \geq_k t$, hence $\{b, \neg b\} \in \mathcal{D}_k(\mathcal{B})$. The other direction: if $\{b, \neg b\} \in \mathcal{D}_k(\mathcal{B})$, then $b \geq_k t$ and $\neg b \geq_k t$, hance $b \geq_k t$ and $b = \neg\neg b \geq_k \neg t = f$, and so $b \geq_k t \oplus f = \top$ (see Lemma 2.5(b)). But $\top$ is the greatest element w.r.t $\leq_k$, hence $b = \top$. □

COROLLARY 2.26. For every interlaced bilattice $\mathcal{B}$, $\langle \mathcal{B} \rangle$ is defined (In particular, $\langle L \odot L \rangle$ is defined for every complete lattice L).

Proof. Follows from section (a) of the last proposition, and from Proposition 2.23. □

From the last corollary it follows that if $\mathcal{B}$ is interlaced, then $\langle \mathcal{B} \rangle$ is a logical bilattice iff $\mathcal{D}(\mathcal{B})$ is prime. In fact, $\langle \mathcal{B} \rangle$ is logical bilattice in all the examples which were actually used in the literature for constructive purposes. This is true even for $\langle DEFAULT \rangle$, although it is not interlaced. $\langle FIVE \rangle$, in contrast, is not a logical bilattice.

We next provide a sufficient and necessary conditions for $\mathcal{D}(\mathcal{B})$ to be prime in a particularly important case. It will follow that logical bilattices are very common, and easily constructed:

PROPOSITION 2.27. If L is a complete lattice, then $\langle L \odot L \rangle$ is a logical bilattice iff $\sup(L)$ is join irreducible (i.e.: if $a \vee b = \sup(L)$, then $a = \sup(L)$ or $b = \sup(L)$).

Proof. Denote the suprimum of L by ∇_L. Then:
($\Leftarrow$) Assume that ∇_L is join irreducible. Since $L \odot L$ is interlaced, then by Corollary 2.26, $\mathcal{D}(L \odot L)$ is a bifilter. It remains to show that it is also a *prime* bifilter. Indeed, $(x_1, x_2) \vee (y_1, y_2) \in \mathcal{D}(L \odot L)$ iff $(x_1 \vee_L y_1, x_2 \wedge_L y_2) \in \mathcal{D}(L \odot L)$ iff $(x_1 \vee_L y_1) = \nabla_L$ (see Example 2.21(e)), iff $x_1 = \nabla_L$ or $y_1 = \nabla_L$, iff $(x_1, x_2) \in \mathcal{D}(L \odot L)$ or $(y_1, y_2) \in \mathcal{D}(L \odot L)$. The proof in the case of $\oplus$ is similar.
($\Rightarrow$) Assume that $L \odot L$ is prime, and that $x \vee y = \nabla_L$ for $x, y \in L$. Take arbitrary $z \in L$. Then, $(x, z) \vee (y, z) = (x \vee y, z) = (\nabla_L, z) \in \mathcal{D}(L \odot L)$, hence $(x, z) \in \mathcal{D}(L \odot L)$ or $(y, z) \in \mathcal{D}(L \odot L)$. It follows that $x = \nabla_L$ or $y = \nabla_L$ (by Example 2.21(e) again). □

COROLLARY 2.28.
a) $\langle FOUR \rangle$ ($\equiv \langle \{0,1\} \odot \{0,1\} \rangle$) and $\langle NINE \rangle$ ($\equiv \langle \{-1,0,1\} \odot \{-1,0,1\} \rangle$) are both logical bilattices.

b) More generally, if L is a chain, or if $\sup(L)$ has a unique predecessor, then $\langle L \odot L\rangle$ is a logical bilattice.

3. The Basic Logic of Logical Bilattices

3.1. Syntax and Semantics

We shall first treat the propositional case.

DEFINITION 3.1.
a) The language BL (Bilattice-based Language) is the standard propositional language over $\{\wedge, \vee, \neg, \otimes, \oplus\}$.
b) BL^- is BL together with a unary connective, $-$, for conflation.
c) $BL(4)$ $(BL^-(4))$ is BL (BL^-) enriched with the propositional constants $\{f, t, \bot, \top\}$.
d) Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. $BL(\mathcal{B})$ is BL enriched with a propositional constant for each element in B. We shall usually employ the same symbol and name for each $b \in B$ and its corresponding propositional constant.

Given a bilattice $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$, perhaps with conflation, the semantic notion of a valuation in B for sentences in $BL(\mathcal{B})$ is defined in the obvious way. The associated logics are also defined in the most natural way:

DEFINITION 3.2.
a) Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. $\Gamma \models_{BL(\mathcal{B},\mathcal{F})} \Delta$ (where Γ, Δ are finite sets of formulae in $BL(\mathcal{B})$) iff for every valuation ν such that $\nu(\psi) \in \mathcal{F}$ for every $\psi \in \Gamma$, there exists some $\phi \in \Delta$ such that $\nu(\phi) \in \mathcal{F}$ as well.
b) Suppose that all the sentences in $\Gamma \cup \Delta$ are in the language BL (resp. in $BL(4)$). Then $\Gamma \models_{BL} \Delta$ (resp. $\Gamma \models_{BL(4)} \Delta$), iff $\Gamma \models_{BL(\mathcal{B},\mathcal{F})} \Delta$ for every $(\mathcal{B}, \mathcal{F})$.

Two important properties of $\models_{BL}$ are given in the following proposition:

PROPOSITION 3.3.
a) $\models_{BL}$ has no tautologies.
b) $\models_{BL}$ is paraconsistent: $p, \neg p \not\models_{BL} q$.
Proof.
a) Let ψ be any sentence in BL, and suppose that ν is a valuation (in $FOUR$, say) that assigns all the propositional variables in ψ the value $\bot$. Then $\nu(\psi) = \bot$ as well, so ψ is not valid.
b) Set, e.g., $\nu(p) = \top$ and $\nu(q) = f$. □

Note that the first part of the last proposition fails in $BL(4)$, since both t and $\top$ are valid.

Our next theorem is an easy consequence of Theorem 2.17. It shows that in order to check consequence in any logical bilattice, it is sufficient to check it in $\langle FOUR\rangle$.

THEOREM 3.4. Let Γ and Δ be finite sets of formulae in BL (in $BL(4)$). Then $\Gamma \models_{BL} \Delta$ ($\Gamma \models_{BL(4)} \Delta$) iff $\Gamma \models_{\langle FOUR \rangle} \Delta$.*

Proof. One direction is trivial. For the other, suppose that for some logical bilattice $(\mathcal{B}, \mathcal{F})$, $\Gamma \not\models_{BL(\mathcal{B},\mathcal{F})} \Delta$, where Γ, Δ in $BL(4)$. Let ν be an assignment in $\mathcal{B}$ such that $\nu(\psi) \in \mathcal{F}$ if $\psi \in \Gamma$, and $\nu(\psi) \notin \mathcal{F}$ if $\psi \in \Delta$. Then $h \circ \nu$, where h is the homomorphism defined in Theorem 2.17, is easily seen to be a valuation in $FOUR$ with the same properties, hence $\Gamma \not\models_{\langle FOUR \rangle} \Delta$. □

The next proposition, which provides a semi-CNF for formulae, will be needed later.

PROPOSITION 3.5. Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. For every sentence ψ in $BL(\mathcal{B})$ one can construct a sentence ψ', so that ψ' is a $\wedge$-conjunction of $\vee$-disjunction of literals, and for every ν over $\mathcal{B}$, $\nu(\psi) \in \mathcal{F}$ iff $\nu(\psi') \in \mathcal{F}$. If ψ is in $BL(4)$ then the same ψ' is good for every logical bilattice $(\mathcal{B}, \mathcal{F})$.

Proof. From the properties of negation it is obvious that for every sentence ψ we can find a sentence ψ' in a negation normal form (i.e. in ψ' the negation precedes only propositional variables), s.t. $\nu(\psi) = \nu(\psi')$ for every valuation ν. It suffices, therefore, to prove the proposition for sentences in a negation normal form. This is done by an induction on the number of operations in ψ (negation excluded): the case where ψ is literal is obvious. If $\psi = \psi_1 \wedge \psi_2$ or $\psi = \psi_1 \otimes \psi_2$, take $\psi' = \psi_1' \wedge \psi_2'$. Then for every ν, $\nu(\psi) \in \mathcal{F}$ iff $\nu(\psi_1) \in \mathcal{F}$ and $\nu(\psi_2) \in \mathcal{F}$, iff $\nu(\psi_1') \in \mathcal{F}$ and $\nu(\psi_2') \in \mathcal{F}$, iff $\nu(\psi_1' \wedge \psi_2') \in \mathcal{F}$, iff $\nu(\psi') \in \mathcal{F}$. Finally, suppose that $\psi = \psi_1 \vee \psi_2$ or $\psi = \psi_1 \oplus \psi_2$. Let $\psi_1' = \psi'^1_1 \wedge \psi'^2_1 \wedge \ldots \wedge \psi'^n_1$ and $\psi'_2 = \psi'^1_2 \wedge \psi'^2_2 \wedge \ldots \wedge \psi'^m_2$ (where ψ'^j_i are $\vee$-disjunction of literals). Let $\psi' = \bigwedge_{1 \leq i \leq n, 1 \leq j \leq m} (\psi'^i_1 \vee \psi'^j_2)$. Assume that $\nu(\psi) \in \mathcal{F}$. Then either $\nu(\psi_1) \in \mathcal{F}$ or $\nu(\psi_2) \in \mathcal{F}$. Assume, e.g., the former. Then $\nu(\psi'^i_1) \in \mathcal{F}$ for every $1 \leq i \leq n$, hence $\nu(\psi'^i_1 \vee \psi'^j_2) \in \mathcal{F}$ for every i, j, and so $\nu(\psi') \in \mathcal{F}$. For the converse, assume that $\nu(\psi) \notin \mathcal{F}$. Then both $\nu(\psi_1)$ and $\nu(\psi_2)$ are not in $\mathcal{F}$, hence $\nu(\psi'^i_1)$ and $\nu(\psi'^j_2)$ are not in $\mathcal{F}$ for some i, j, and so $\nu(\psi'^i_1 \vee \psi'^j_2) \notin \mathcal{F}$. It follows that $\nu(\psi') \notin \mathcal{F}$. □

Notes:

1. ψ and ψ' above are *not* equivalent, i.e: there may be some valuation ν, s.t. $\nu(\psi) \neq \nu(\psi')$. All the proposition claims is that ψ and ψ' are true with respect to the same valuations.**
2. We could, of course, use $\otimes$ and $\oplus$ (or $\otimes$ and $\vee$, etc.) instead of $\wedge$ and $\vee$, without any change in the proof.

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* There is a related, weaker theorem (10.5) in Fitting (1994).

** The situation is in some sense analogous to that of Skolemizing and satisfiability in first order classical logic; The Skolemized version of a sentence is satisfiable iff the original sentence is satisfiable, but the two sentences are *not* equivalent.

3.2. Proof Theory

Since $\models_{BL}$ does not have valid formulae, it cannot have a Hilbert-type representation. However, there is a nice Gentzen-type formulation, which we shall call *GBL* (*GBL*(4)):

The System *GBL*:

Axioms:

$$\Gamma, \psi \Rightarrow \psi, \Delta$$

Rules: Exchange, Contraction, and the following logical rules:

$$[\wedge\Rightarrow]\ \frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \wedge \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \wedge \phi}\ [\Rightarrow\wedge]$$

$$[\neg\wedge\Rightarrow]\ \frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \wedge \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \wedge \phi)}\ [\Rightarrow\neg\wedge]$$

$$[\vee\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \vee \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \vee \phi}\ [\Rightarrow\vee]$$

$$[\neg\vee\Rightarrow]\ \frac{\Gamma, \neg\psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \vee \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \vee \phi)}\ [\Rightarrow\neg\vee]$$

$$[\otimes\Rightarrow]\ \frac{\Gamma, \psi, \phi \Rightarrow \Delta}{\Gamma, \psi \otimes \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \phi}{\Gamma \Rightarrow \Delta, \psi \otimes \phi}\ [\Rightarrow\otimes]$$

$$[\neg\otimes\Rightarrow]\ \frac{\Gamma, \neg\psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \otimes \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi \quad \Gamma \Rightarrow \Delta, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \otimes \phi)}\ [\Rightarrow\neg\otimes]$$

$$[\oplus\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \oplus \phi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi, \phi}{\Gamma \Rightarrow \Delta, \psi \oplus \phi}\ [\Rightarrow\oplus]$$

$$[\neg\oplus\Rightarrow]\ \frac{\Gamma, \neg\psi \Rightarrow \Delta \quad \Gamma, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \oplus \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \neg\psi, \neg\phi}{\Gamma \Rightarrow \Delta, \neg(\psi \oplus \phi)}\ [\Rightarrow\neg\oplus]$$

$$[\neg\neg\Rightarrow]\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma, \neg\neg\psi \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \neg\neg\psi}\ [\Rightarrow\neg\neg]$$

In *GBL*(4) the following axioms are also included:

$$\Gamma, \neg t \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, t$$

$$\Gamma, f \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \neg f$$

$$\Gamma, \bot \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \top$$

$$\Gamma, \neg\bot \Rightarrow \Delta \qquad \Gamma \Rightarrow \Delta, \neg\top$$

The positive rules for $\wedge$ and $\otimes$ are identical. Both behave as classical conjunction. The difference is with respect to the negations of $p \wedge q$ and $p \otimes q$. Unlike the conjunction of classical logic, the negation of $p \otimes q$ is equivalent to $\neg p \otimes \neg q$. This follows from the fact that $p \leq_k q$ iff $\neg p \leq_k \neg q$. The difference between $\vee$ and $\oplus$ is similar.

DEFINITION 3.6. Δ *follows* from Γ in *GBL* (notation: $\Gamma \vdash_{GBL} \Delta$) if $\Gamma \Rightarrow \Delta$ is provable in *GBL*.

THEOREM 3.7.
a) (*Soundness and Completeness*) $\Gamma \models_{BL} \Delta$ iff $\Gamma \vdash_{GBL} \Delta$.
b) (*Cut Elimination*) If $\Gamma_1 \vdash_{GBL} \Delta_1, \psi$ and $\Gamma_2, \psi \vdash_{GBL} \Delta_2$, then $\Gamma_1, \Gamma_2 \vdash_{GBL} \Delta_1, \Delta_2$.

Proof. The soundness part is easy, and is left to the reader. We prove completeness and cut-elimination together by showing that if $\Gamma \Rightarrow \Delta$ has no cut-free proof then $\Gamma \not\models_{BL} \Delta$. The proof is by an induction on the complexity of the sequent $\Gamma \Rightarrow \Delta$:

• The base step: Suppose that $\Gamma \Rightarrow \Delta$ consists only of literals. If Γ and Δ have a literal in common then $\Gamma \Rightarrow \Delta$ is obviously valid (and is provable without cut), while if Γ and Δ have no literal in common, then consider the following assignment ν in *FOUR*:

$$\nu(p) \stackrel{\text{def}}{=} \begin{cases} \top & \text{if both } p \text{ and } \neg p \text{ are in } \Gamma \\ \bot & \text{if both } p \text{ and } \neg p \text{ are in } \Delta \\ t & \text{if } (p \in \Gamma \text{ and } \neg p \notin \Gamma) \text{ or } (p \notin \Delta \text{ and } \neg p \in \Delta) \\ f & \text{if } (p \notin \Gamma \text{ and } \neg p \in \Gamma) \text{ or } (p \in \Delta \text{ and } \neg p \notin \Delta) \end{cases}$$

Obviously, this is a well defined valuation, which gives all the literals in Γ values in $\{\top, t\}$, and all the literals in Δ values in $\{\bot, f\}$. Hence ν refutes $\Gamma \Rightarrow \Delta$ in $\langle FOUR \rangle$. Hence, $\Gamma \not\models_{BL} \Delta$.

• The induction step: The crucial observation is that all the rules of the system *GBL* are reversible, both semantically and proof-theoretically (a direct demonstration in the proof-theoretical case requires cuts). There are many cases to consider here. We shall treat in detail only the case in which a sentence of the form $\psi \wedge \phi$ is in $\Gamma \cup \Delta$. Before doing so we note that the case in which a sentence of the form $\neg\psi$ belongs to $\Gamma \cup \Delta$ should be split into the subcases $\psi = \neg\phi$, $\psi = \phi_1 \wedge \phi_2$, etc. (The case in which $\psi = \neg p$ where p is atomic was already taken care of in the base step).
(*i*) Suppose that $\psi \wedge \phi \in \Gamma$, i.e.: $\Gamma = \Gamma', \psi \wedge \phi$. Consider the sequent $\Gamma', \psi, \phi \Rightarrow \Delta$.

By induction hypothesis, either $\Gamma', \psi, \phi \Rightarrow \Delta$ is provable without a cut (and then $\Gamma', \psi \wedge \phi \Rightarrow \Delta$ is provable without cut, using $[\wedge \Rightarrow]$), or else there is a valuation that refutes $\Gamma', \psi, \phi \Rightarrow \Delta$. In the latter case the same valuation refutes $\Gamma', \psi \wedge \phi \Rightarrow \Delta$ as well.

(*ii*) Suppose that $\psi \wedge \phi \in \Delta$, i.e.: $\Delta = \Delta', \psi \wedge \phi$. Consider the sequents $\Gamma \Rightarrow \Delta', \psi$ and $\Gamma \Rightarrow \Delta', \phi$. Again, either both have cut-free proofs, and then $\Gamma \Rightarrow \Delta', \psi \wedge \phi$ also has a proof without a cut (using $[\Rightarrow \wedge]$), or there is an assignment that refutes either sequent, and the same assignment refutes $\Gamma \Rightarrow \Delta', \psi \wedge \phi$ as well. □

Notes:

1. It is obvious from the proof that we can delete contraction from the list of the rules, and restrict the axioms to the case that Γ, Δ, ψ, and ϕ contains only literals.
2. The $\{\wedge, \vee, \neg\}$ fragment of *GBL* was called "the basic $\{\wedge, \vee, \neg\}$ system" in Avron (1991a), and was introduced there following a different motivation. It had generally been known as the system of "first degree entailments" in relevance logic (see Anderson, 1975; Dunn, 1986), since it is well known that $\psi_1, \ldots, \psi_n \Rightarrow \phi_1, \ldots, \phi_m$ is provable in it, iff $\psi_1 \wedge \ldots \wedge \psi_n \to \phi_1 \vee \ldots \vee \phi_m$ is provable in the system R (or E) of Anderson and Belnap, iff $\nu(\psi_1 \wedge \ldots \wedge \psi_n) \leq_t \nu(\phi_1 \vee \ldots \vee \phi_m)$ for every valuation ν in *FOUR*. It is not difficult to show that this fragment of *GBL* is valid in any distributive lattice with an involution ("valid" – in the sense that $\psi_1, \ldots, \psi_n \Rightarrow \phi_1, \ldots, \phi_m$ is provable in *GBL* if $\nu(\psi_1) \wedge \ldots \wedge \nu(\psi_n) \leq_t \nu(\phi_1) \vee \ldots \vee \nu(\phi_m)$ for every valuation ν). Hence we have an alternative soundness and completeness theorem relative to these structures.
3. In Avron (1991b) it is shown that if we add $\Gamma, \neg\psi, \psi \Rightarrow \Delta$ as an axiom to the $\{\wedge, \vee, \neg\}$ (or $\{\wedge, \vee, \neg, f, t\}$) fragment of *GBL*, we get a sound and complete system for Kleene 3-valued logic, while if we add $\Gamma \Rightarrow \Delta, \psi, \neg\psi$ we get one of the basic three-valued paraconsistent logics (Also known as basic J_3 – see, e.g., Chapter IX of Epstein (1990) as well as d'Ottaviano and da-Costa (1970), d'Ottaviano (1985), Avron (1986) and Rozoner (1989)). By adding both axioms, we get classical logic.
4. In order to add a conflation to *GBS* one needs to expand it with additional rules for the left and right combination of $-$ with $\wedge, \vee, \otimes, \oplus$ and $-$ (10 new rules altogether). These rules are the duals of the corresponding rules of negation. For example,

$$[-\wedge\Rightarrow]\ \frac{\Gamma, -\psi, -\phi, \Rightarrow \Delta}{\Gamma, -(\psi \wedge \phi) \Rightarrow \Delta} \qquad [-\otimes\Rightarrow]\ \frac{\Gamma, -\psi \Rightarrow \Delta \quad \Gamma, -\phi \Rightarrow \Delta}{\Gamma, -(\psi \otimes \phi) \Rightarrow \Delta}$$

In addition, one should add four more rules for the combination of negation and conflation:

$$[-\neg\Rightarrow]\ \frac{\Gamma, \Rightarrow \Delta, \psi}{\Gamma, -\neg\psi \Rightarrow \Delta} \qquad [\Rightarrow -\neg]\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\neg\psi}$$

$$[\neg-\Rightarrow]\ \frac{\Gamma, \Rightarrow \Delta, \psi}{\Gamma, \neg-\psi \Rightarrow \Delta} \qquad [\Rightarrow -\neg]\ \frac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg-\psi}$$

Using Theorem 2.19 it is straightforward to extend the proof of Theorem 3.7 to the case of ultralogical bilattices and the resulting systems. Note that in the presence of conflation we do have provable sequents of the form $\Gamma \Rightarrow$ and $\Rightarrow \Delta$.

5. In order to get a sound and complete system for $BL(\mathcal{B})$ for any logical bilattice $\mathcal{B}$, we have to add axioms to *GBL* for every $b \in B$, according to the homomorphism h of Theorem 2.17. For example, if for some $b \in B$ $h(b) = t$, then we add $\Gamma \Rightarrow \Delta, b$ and $\Gamma, \neg b \Rightarrow \Delta$.

For the single-conclusioned fragment of $\models_{BL}$ we have a stronger result:

DEFINITION 3.8. GBL_I (*Intuitionistic GBL*) is the system obtained from *GBL* by allowing a sequent to have exactly *one* formula to the r.h.s of $\Rightarrow$, and by replacing the rules which have more than one formula on their r.h.s (or empty r.h.s) by the corresponding intuitionistic rules. $GBL_I(4)$ is defined similarly.*

For example, in GBL_I, $[\Rightarrow\vee]$ is replaced with the following two rules:

$$\frac{\Gamma \Rightarrow \psi}{\Gamma \Rightarrow \psi \vee \phi} \qquad \frac{\Gamma \Rightarrow \phi}{\Gamma \Rightarrow \psi \vee \phi}$$

In case of $BL(4)$, all the axioms of the form $b \Rightarrow$ (where $b \in \{f, \neg t, \bot, \neg\bot\}$) are replaced by $b \Rightarrow \psi$ for arbitrary ψ.

THEOREM 3.9. $\Gamma \models_{BL} \psi$ iff $\Gamma \vdash_{GBL_I} \psi$. A similar result holds for $BL(4)$.

Proof. We start with two lemmas:

LEMMA 3.9a: Suppose that $\vdash_{GBL} \Gamma \Rightarrow \Delta$, where Δ is not empty, and Γ consists only of literals. Then $\vdash_{GBL_I} \Gamma \Rightarrow \psi$ for some ψ in Δ (note that if Δ is empty, then $\vdash_{GBL_I} \Gamma \Rightarrow \psi$ for *every* ψ).

Proof of Lemma 3.9a: By an easy induction on the length of a cut-free proof of $\Gamma \Rightarrow \Delta$ in *GBL*: It is trivial in the case where $\Gamma \Rightarrow \Delta$ is an axiom. For the induction step we use the fact that since Γ consists of literals, all the rules employed are r.h.s rules. We will prove the case of the rules for $\vee$ as an example:

• Suppose that $\Delta = \Delta', \phi\vee\tau$ and $\Gamma \Rightarrow \Delta$ was inferred from $\Gamma \Rightarrow \Delta', \phi, \tau$. By induction hypothesis either $\vdash_{GBL_I} \Gamma \Rightarrow \phi$, or $\vdash_{GBL_I} \Gamma \Rightarrow \tau$, or $\vdash_{GBL_I} \Gamma \Rightarrow \psi$, for some $\psi \in \Delta'$. In the third case we are done, while in the first two we infer $\vdash_{GBL_I} \Gamma \Rightarrow \phi\vee\tau$

* Note that $\neg\neg\psi \Rightarrow \psi$ obtains in both new systems, so the analogy with intuitionistic logic is not perfect.

using the intuitionistic rules for introduction of $\vee$.

• Suppose that $\Delta = \Delta', \neg(\phi \vee \tau)$ and $\Gamma \Rightarrow \Delta$ was inferred from $\Gamma \Rightarrow \Delta', \neg\phi$ and $\Gamma \Rightarrow \Delta', \neg\tau$. By induction hypothesis either $\vdash_{GBL_I} \Gamma \Rightarrow \psi$, for some $\psi \in \Delta'$, in which case we are done, or both $\vdash_{GBL_I} \Gamma \Rightarrow \neg\phi$ and $\vdash_{GBL_I} \Gamma \Rightarrow \neg\tau$. In this case, $\Gamma \Rightarrow \neg(\phi \vee \tau)$ follows immediately by $[\Rightarrow \neg\vee]$.

LEMMA 3.9b: For every Γ there exist sets Γ_i $(i = 1 \dots n)$ s.t:

1. For every i, Γ_i consists of literals.
2. For every Δ, $\vdash_{GBL} \Gamma \Rightarrow \Delta$ iff for every i, $\vdash_{GBL} \Gamma_i \Rightarrow \Delta$.
3. For every Δ there is a cut-free proof of $\Gamma \Rightarrow \Delta$ from $\Gamma_i \Rightarrow \Delta$ $(i = 1 \dots n)$, where Δ is the r.h.s of all the sequents involved, and the only rules used are l.h.s rules.

Proof of Lemma 3.9b: By induction on the complexity of Γ, using the fact that all the l.h.s rules of *GBL* are reversible, and their active formulae belong to the l.h.s of the premises.

Proof of Theorem 3.9: Assume that $\vdash_{GBL} \Gamma \Rightarrow \psi$. Then $\vdash_{GBL} \Gamma_i \Rightarrow \psi$ for the Γ_i's given in Lemma 3.9b. Lemma 3.9a implies, then, that $\vdash_{GBL_I} \Gamma_i \Rightarrow \psi$ $(i = 1 \dots n)$. The third property of $\Gamma_1, \dots \Gamma_n$ in Lemma 3.9b implies that $\vdash_{GBL_I} \Gamma \Rightarrow \psi$, since GBL_I and *GBL* have the same l.h.s rules. □

Notice that the last theorem is still true if we add $\Gamma, \psi, \neg\psi \Rightarrow \Delta$ to the axioms of *GBL*, and $\Gamma, \psi, \neg\psi \Rightarrow \phi$ to the axioms of GBL_I. In contrast, the theorem fails if we add $\Gamma \Rightarrow \Delta, \psi, \neg\psi$ as an axiom, or the classical introduction rules of $\neg$, or implication with the classical rules. That is why classical logic is not a conservative extension of intuitionistic logic. This is also the reason why the theorem fails for the conservative extension of *GBL* with the implication we introduce in the forthcoming sections.

We end this subsection with two other fundamental properties of $\models_{BL}$:

THEOREM 3.10. (*Monotonicity and Compactness*) Let Γ, Δ be arbitrary sets of formulae in *BL* (possibly infinite). Define $\Gamma \models_{BL} \Delta$ exactly as in the finite case. Then $\Gamma \models_{BL} \Delta$ iff there exist *finite* sets Γ', Δ' such that $\Gamma' \subseteq \Gamma$, $\Delta' \subseteq \Delta$, and $\Gamma' \models_{BL} \Delta'$ (iff $\vdash_{GBL} \Gamma' \Rightarrow \Delta'$).

Proof. Suppose that Γ, Δ are sets for which no such Γ', Δ' exist. Construct a refuting ν in *FOUR* as follows: first, extend the pair (Γ, Δ) to a maximal pair (Γ^*, Δ^*) with the same property. Then, for any ψ, either $\psi \in \Gamma^*$ or $\psi \in \Delta^*$ (Otherwise, $(\Gamma^* \cup \{\psi\}, \Delta^*)$ and $(\Gamma^*, \Delta^* \cup \{\psi\})$ do not have the property, and so there are finite $\Gamma' \subseteq \Gamma^*$, and $\Delta' \subseteq \Delta^*$ such that $\Gamma', \psi \models_{BL} \Delta'$ and there are finite $\Gamma'' \subseteq \Gamma^*$, and $\Delta'' \subseteq \Delta^*$ such that $\Gamma'' \models_{BL} \psi, \Delta''$. It follows that $\Gamma' \cup \Gamma'' \models_{BL} \Delta' \cup \Delta''$, contradicting the definition of (Γ^*, Δ^*)).

Define ν from the set of all sentences to *FOUR* as follows:

$$\nu(\psi) \stackrel{\text{def}}{=} \begin{cases} \top & \text{if } \psi \in \Gamma^* \text{ and } \neg\psi \in \Gamma^* \\ t & \text{if } \psi \in \Gamma^* \text{ and } \neg\psi \in \Delta^* \\ f & \text{if } \psi \in \Delta^* \text{ and } \neg\psi \in \Gamma^* \\ \bot & \text{if } \psi \in \Delta^* \text{ and } \neg\psi \in \Delta^* \end{cases}$$

Obviously, $\nu(\psi) \in \mathcal{D}(FOUR)$ for all $\psi \in \Gamma^*$, while $\nu(\psi) \notin \mathcal{D}(FOUR)$ if $\psi \in \Delta^*$. It remains to show that ν is indeed a valuation (i.e. it respects the operations). We will prove the case of $\wedge$, leaving the other cases to the reader. For this, we first note the following facts:

1. If $\psi \in \Delta^*$ or $\phi \in \Delta^*$, then $\psi \wedge \phi \in \Delta^*$
   (Since $\psi \wedge \phi \models_{BL} \psi$ and $\psi \wedge \phi \models_{BL} \phi$, $\psi \wedge \phi$ cannot be in Γ^*)
2. If $\psi \in \Gamma^*$, then $\psi \wedge \phi \in \Gamma^*$ $(\in \Delta^*)$ iff $\phi \in \Gamma^*$ $(\in \Delta^*)$. Similarly, If $\phi \in \Gamma^*$, then $\psi \wedge \phi \in \Gamma^*$ $(\in \Delta^*)$ iff $\psi \in \Gamma^*$ $(\in \Delta^*)$.
   (Suppose that $\psi \in \Gamma^*$. If also $\phi \in \Gamma^*$, then $\psi \wedge \phi$ cannot be in Δ^*, since $\psi, \phi \models_{BL} \psi \wedge \phi$, So $\psi \wedge \phi \in \Gamma^*$ as well. If, on the other hand, $\phi \in \Delta^*$, then also $\psi \wedge \phi \in \Delta^*$, by (1)).
3. If $\neg\psi \in \Gamma^*$ or $\neg\phi \in \Gamma^*$, then $\neg(\psi \wedge \phi) \in \Gamma^*$ (similar to (1)).
4. If $\neg\psi \in \Delta^*$ then $\neg(\psi \wedge \phi) \in \Delta^*$ iff $\neg\phi \in \Delta^*$ (similar to (2)).

Using (1)–(4), it is straightforward to check that $\nu(\psi \wedge \phi) = \nu(\psi) \wedge \nu(\phi)$ for every ψ, ϕ. For example, if $\nu(\psi) = f$ then $\psi \in \Delta^*$ and $\neg\psi \in \Gamma^*$, thus, by (1) and (3), $\psi \wedge \phi \in \Delta^*$ and $\neg(\psi \wedge \phi) \in \Gamma^*$. Hence $\nu(\psi \wedge \phi) = f = \nu(\psi) \wedge \nu(\phi)$ in this case. The other cases are handled similarly. □

THEOREM 3.11. *(Interpolation)* Suppose that $\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2$ is provable in $BL(4)$. Then there exists a sentence ψ such that both $\Gamma_1 \Rightarrow \Delta_1, \psi$ and $\psi, \Gamma_2 \Rightarrow \Delta_2$ are provable in $BL(4)$, and ψ contains only atomic formulae which are common to $\Gamma_1 \Rightarrow \Delta_1$ and to $\Gamma_2 \Rightarrow \Delta_2$. In particular, $\psi \Rightarrow \phi$ iff ψ and ϕ have an interpolant.

Proof. By Maehera's method (see Taukeuti, 1975, Chapter 1). □

3.3. The Symmetric Consequence Relation

The consequence relation, $\models_{BL}$, as defined above, meets the symmetry conditions for $\neg, \wedge, \vee$ as defined in Avron (1991b). It follows from the discussion there that it is possible to define an associated symmetric consequence relation, $\models^s_{BL}$, for which Proposition 3.13 below will be valid:

DEFINITION 3.12. The *symmetric version, $\models^s_{BL}$, of $\models_{BL}$*, is defined as follows: $\psi_1, \ldots, \psi_n \models^s_{BL} \phi_1, \ldots, \phi_m$ if:
a) for every $1 \leq j \leq m$, $\psi_1, \ldots, \psi_n, \neg\phi_1, \ldots, \neg\phi_{j-1}, \neg\phi_{j+1}, \ldots, \neg\phi_m \models_{BL} \phi_j$,
b) for every $1 \leq i \leq n$, $\psi_1, \ldots, \psi_{i-1}, \psi_{i+1}, \ldots, \psi_n, \neg\phi_1, \ldots, \neg\phi_m \models_{BL} \neg\psi_i$.

PROPOSITION 3.13. $\models^s_{BL}$ has the following properties:

a) $\models^s_{BL}$ is a consequence relation in the extended sense of Avron (1991a, 1991b). In other words: $\psi \models^s_{BL} \psi$ for every formula ψ, and if $\Gamma_1 \models^s_{BL} \Delta_1, \psi$ and $\Gamma_2, \psi \models^s_{BL} \Delta_2$ (where Γ_1, Γ_2, Δ_1 and Δ_2 are *multisets* of formulae) then $\Gamma_1, \Gamma_2 \models^s_{BL} \Delta_1, \Delta_2$.

b) If $\Gamma \models^s_{BL} \psi$, then $\Gamma \models_{BL} \psi$.

c) $\neg$ is an internal negation with respect to $\models^s_{BL}$, i.e.: $\Gamma \models^s_{BL} \psi, \Delta$ iff $\Gamma, \neg\psi \models^s_{BL} \Delta$, and $\Gamma, \psi \models^s_{BL} \Delta$ iff $\Gamma \models^s_{BL} \neg\psi, \Delta$.

d) $\models^s_{BL}$ is the maximal single-conclusioned consequence relation having properties (a)–(c).

e) $\wedge$ and $\vee$ are, respectively, combining conjunction and disjunction for $\models^s_{BL}$: $\Gamma \models^s_{BL} \psi \wedge \phi, \Delta$ iff $\Gamma \models^s_{BL} \psi, \Delta$ and $\Gamma \models^s_{BL} \phi, \Delta$. Similarly, $\Gamma, \psi \vee \phi \models^s_{BL} \Delta$ iff $\Gamma, \psi \models^s_{BL} \Delta$ and $\Gamma, \phi \models^s_{BL} \Delta$.

f) $\models_{BL}$ and $\models^s_{BL}$ have the same logical theorems. In other words, for any ψ, $\models_{BL} \psi$ iff $\models^s_{BL} \psi$.*

g) From $\psi \models^s_{BL} \phi$ and $\phi \models^s_{BL} \psi$ it follows that $\Theta(\psi) \models^s_{BL} \Theta(\phi)$ and $\Theta(\phi) \models^s_{BL} \Theta(\psi)$ for every scheme Θ (The proof is by induction on the complexity of Θ).

Notes:

1. Similar symmetric versions, with similar properties, can be given, of course, to the other consequence relations defined in the previous section.
2. The converse of property (b) above does not hold (unless Γ is empty, as in property (f)). Thus, $p, q \models_{BL} p$ but $p, q \not\models^s_{BL} p$ (which shows also that $\models^s_{BL}$ is *non-monotonic*). Hence the single-conclusioned fragment of $\models^s_{BL}$ is strictly weaker then that of $\models_{BL}$. Thus, $\models^s_{BL}$ can be used to express stronger connections than those allowed by $\models_{BL}$.
3. Both weakening and contraction fail for $\models^s_{BL}$. We have already seen an example for the failure of weakening. As for contraction, we note that $\models^s_{BL} \neg\psi \vee \psi, \neg\psi \vee \psi$, but $\not\models^s_{BL} \neg\psi \vee \psi$.** This demonstrates great similarity with linear logic (Girard, 1987). In fact, $\neg$ behaves exactly as linear negation, while $\wedge$ and $\vee$ corresponds to the "additives" of linear logic. In the next subsection we will introduce connectives which correspond to the "multiplicatives" of linear logic as well. On the other hand, there is nothing in linear logic which corresponds to either $\otimes$ or $\oplus$.‡

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* For the case of $\models_{BL}$, but not $\models_{BL(4)}$, this holds in fact vacuously. The situation is different, though, for the stronger language introduced below.

** This can directly be seen from the definition of $\models^s_{BL}$. It can also be inferred from 3.13(b), using only the fact that $\not\models_{BL} \neg\psi \vee \psi$.

‡ Clearly not the connectives which have the same notations in Girard (1987)!

4. Property (g) above fails for $\models_{BL}$. Thus, $p \vee q \models_{BL} p \oplus q$, and $p \oplus q \models_{BL} p \vee q$, but $\neg(p \oplus q) \not\models_{BL} \neg(p \vee q)$. Moreover, for the implication $\supset$ we introduce in the next section, we have that $p \supset p \models_{BL} q \supset q$ and $q \supset q \models_{BL} p \supset p$, while $\neg(p \supset p) \not\models_{BL} \neg(q \supset q)$. For the fragment of $\{\neg, \wedge, \vee\}$ we do have (g) as an *admissible* rule. In other words, if ψ and ϕ are in this fragment, and it is *actually* the case that $\psi \models_{BL} \phi$ and $\phi \models_{BL} \psi$, then $\Theta(\psi) \models_{BL} \Theta(\phi)$. This follows (using induction on the complexity of Θ) from the fact that for such ψ and ϕ, if $\psi \models_{BL} \phi$ then $\neg\phi \models_{BL} \neg\psi$. However, this rule is not *derivable*: from $\psi \Rightarrow \phi$ and $\phi \Rightarrow \psi$ one cannot *infer* in *GBL* $\neg\psi \Rightarrow \neg\phi$.

3.4. Implication Connectives

3.4.1. *Weak Implication*

As we have noted, $\models_{BL}$ and $\models^{s}_{BL}$ correspond to different degrees of entailment between premises and conclusions. Being consequence relations they can be used, however, only as separated frameworks for making conclusions. It would be much more convenient to be able to treat them within one framework. For this we need appropriate implication *connectives*, which would correspond to those consequence relations. In general, the existence of an appropriate implication connective is a major requirement for a logic. First of all, it allows us to reduce questions of deducibility to questions of theoremhood, and to express the various consequence relations among sentences by other sentences of the language. Moreover, higher order rules (like: "if ψ entails ϕ then not-ϕ entails that not-ψ") can be expressed only if we have a corresponding implication in our disposal. If more than one consequence relation is relevant, the use of corresponding implication connectives allow us also to express higher-order connections among those relations.

Unfortunately, the language *BL*, rich as it is, lacks an appropriate general implication connectives (this is clear from the fact that it has no tautologies). We can try to use $\neg\psi \vee \phi$ as expressing implication of ϕ by ψ (henceforth we shall use $\leadsto$ for this connective), but this is not adequate, since both modus ponens and the deduction theorem fail for this connective. The natural thing to do, therefore, is to enrich the language of *BL* so that this problem will be eliminated. Again, Avron (1991b) provides a clue how to get implication connectives that correspond to both $\models_{BL}$ and $\models^{s}_{BL}$, by adding only *one* connective. What we need is an internal implication, $\supset$, for $\models_{BL}$, which satisfies the *symmetry conditions* for implication:

- $\Gamma, \psi \models_{BL} \phi, \Delta$ iff $\Gamma \models_{BL} \psi \supset \phi, \Delta$.
- If $\Gamma, \psi, \neg\phi \models_{BL} \Delta$ then $\Gamma, \neg(\psi \supset \phi) \models_{BL} \Delta$.
- If $\Gamma \models_{BL} \psi, \Delta$ and $\Gamma \models_{BL} \neg\phi, \Delta$, then $\Gamma \models_{BL} \neg(\psi \supset \phi), \Delta$.

These conditions can easily be translated into rules of a sequential calculus. Therefore, it is easier to start by extending the language and the proof system, then to look for an appropriate semantics.

DEFINITION 3.14.
a) $BL_{\supset}, BL_{\supset}(4), BL_{\supset}(\mathcal{B})$ are the extensions of the various languages defined above with the connective $\supset$.
b) $GBL_{\supset}$ ($GBL_{\supset}(4)$) is obtained from GBL ($GBL(4)$) by the additions of the following rules:

$$[\supset\Rightarrow]\ \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \phi \Rightarrow \Delta}{\Gamma, \psi \supset \phi \Rightarrow \Delta} \qquad \frac{\Gamma, \psi \Rightarrow \phi, \Delta}{\Gamma \Rightarrow \psi \supset \phi, \Delta}\ [\Rightarrow\supset]$$

$$[\neg\supset\Rightarrow]\ \frac{\Gamma, \psi, \neg\phi \Rightarrow \Delta}{\Gamma, \neg(\psi \supset \phi) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma \Rightarrow \neg\phi, \Delta}{\Gamma \Rightarrow \neg(\psi \supset \phi), \Delta}\ [\Rightarrow\neg\supset]$$

c) $\Gamma \vdash_{GBL_{\supset}} \Delta$ iff $\Gamma \Rightarrow \Delta$ is provable in $GBL_{\supset}$.

We turn now to the semantics of $\supset$:

DEFINITION 3.15.
a) Given a logical bilatice $(\mathcal{B}, \mathcal{F})$, the operation $\supset$ is defined as follows:*

$$a \supset b \stackrel{\text{def}}{=} \begin{cases} b & \text{if } a \in \mathcal{F} \\ t & \text{if } a \notin \mathcal{F} \end{cases}$$

b) Using part (a), the consequence relation $\models_{BL}$ ($\models_{BL(4)}$) is extended to $\models_{BL_{\supset}}$ ($\models_{BL_{\supset}(4)}$) in the language $BL_{\supset}$ ($BL_{\supset}(4)$) in the obvious way.

PROPOSITION 3.16. Both modus ponens and the deduction theorem are valid for $\supset$ in $\models_{BL_{\supset}}$ ($\models_{BL_{\supset}(4)}$).
Proof. Easy, and is left to the reader. □

PROPOSITION 3.17. Theorem 2.17 is still valid when $\supset$ is allowed: if $(\mathcal{B}, \mathcal{F})$ is a logical bilattice, then there exists a unique homomorphism (relative to $\neg$, $\wedge$, $\vee$, $\otimes$, $\oplus$, and $\supset$) $h : \mathcal{B} \to FOUR$, s.t. $h(b) \in \{\top, t\}$ iff $b \in \mathcal{F}$.
Proof. Almost identical to that of Theorem 2.17. We only have to check that h as defined there is an homomorphism w.r.t $\supset$ also. Well, if $a \in \mathcal{F}$, then $a \supset b = b$, so $h(a \supset b) = h(b) = h(a) \supset h(b)$, since $h(a) \in \{\top, t\}$ when $a \in \mathcal{F}$. On the other hand, if $a \notin \mathcal{F}$, then $a \supset b = t$ and so $h(a \supset b) = h(t) = t$. But since in this case $h(a) \in \{\bot, f\}$, then $h(a) \supset h(b)$ is also t, no matter what $h(b)$ is. □

PROPOSITION 3.18. Let Γ and Δ be finite sets of formulae in BL (in $BL(4)$). Then $\Gamma \models_{BL_{\supset}} \Delta$ ($\Gamma \models_{BL_{\supset}(4)} \Delta$) iff $\Gamma \models_{\langle FOUR \rangle} \Delta$.
Proof. Identical to the proof of Theorem 3.4, using the previous proposition instead of Theorem 2.17. □

* Note that unlike the operations we delt with so far, $\supset$ is defined *only* for *logical* bilattices.

Note: In contrast to Theorem 3.4, Proposition 3.5 fails for $BL_{\supset}$. Thus, $p \supset p$ is always valid (i.e.: always has a value in $\mathcal{F}$), while the language of $\{\neg, \wedge, \vee, \otimes, \oplus\}$ contains no such formula. The same argument shows also the following proposition:

PROPOSITION 3.19. *$BL_{\supset}$ is a proper extension of BL.*

THEOREM 3.20.
a) (*Soundness, Completeness*) $\Gamma \models_{BL_{\supset}} \Delta$ iff $\Gamma \vdash_{GBL_{\supset}} \Delta$ (similarly for $GBL_{\supset}(4)$).*
b) The *Cut Elimination Theorem* is valid for $GBL_{\supset}$ and for $GBL_{\supset}(4)$.

Proof. Soundness is easy, and is again left to the reader. The combined proof of completeness and cut-elimination is identical to that in the case of $\models_{BL}$ (Theorem 3.7). We only have to check that all the rules of $\supset$ are again reversible, both proof theoretically and semantically. We do this here for the case of $[\neg\supset\Rightarrow]$: First, we observe that it is easy to show that $\psi, \neg\phi \Rightarrow \neg(\psi \supset \phi)$ is provable, and so it is valid (by soundness). Hence, if $\Gamma, \neg(\psi \supset \phi) \Rightarrow \Delta$ is valid (provable), then a cut (which is a valid rule) with $\psi, \neg\phi \Rightarrow \neg(\psi \supset \phi)$ gives that $\Gamma, \psi, \neg\phi \Rightarrow \Delta$ is also valid (provable). □

COROLLARY 3.21.
a) *$GBL_{\supset}$ is a conservative extension of GBL.*
b) *$GBL_{\supset}$ is still paraconsistent.*
c) The $\{\wedge, \vee, \supset\}$-part of $\models_{BL_{\supset}}$ is identical of that of classical logic.

Proof.
a) This is direct implication of cut-elimination. It also a corollary of the soundness and completeness results for both.
b) We still have that $p, \neg p \not\models_{GBL_{\supset}} q$.
c) The $\{\wedge, \vee, \supset\}$-part of $GBL_{\supset}$ is identical to that of the usual Gentzen-type system of classical logic. By cut-elimination, this part of the system is complete for the corresponding fragment of $\models_{BL_{\supset}}$.** □

Other properties of $\models_{BL}$ which can be generalized to $\models_{BL_{\supset}}$ are compactness and interpolation:

THEOREM 3.22. *$\models_{BL_{\supset}}$ enjoys compactness, monotonicity, and interpolation.*

Proof. Identical to these of Theorems 3.10 and 3.11. The only necessary addition to the proof of 3.10 is showing that ν as defined there is a valuation also with respect to $\supset$ (i.e.: $\nu(\psi \supset \phi) = \nu(\psi) \supset \nu(\phi)$). We leave this to the reader (compare to the proof of Proposition 3.17). □

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* It is not difficult to check that in *FOUR* our definition of $\supset$ is the *only* possible definition for which this is true.

** From part (c) of the corollary it follows that the critical connective of *GBL* is *negation*.

On the other hand, Theorem 3.9 *cannot* be extended to $\models_{BL_\supset}$. This is obvious from the fact that the $\{\wedge, \vee, \supset\}$-fragment of $GBL_\supset$ is identical to the classical one, and so it is strictly stronger than its intuitionistic version (Thus, $(\psi \supset \phi) \supset \psi \vdash_{GBL_\supset} \psi$, but $(\psi \supset \phi) \supset \psi \nvdash_{GBL_I} \psi$).

We have already noted that unlike $\models_{BL}$, $\models_{BL_\supset}$ does have valid formulae. This fact, together with the existence of an internal implication, indicate that for $\models_{BL_\supset}$ it might be possible to provide a sound and complete Hilbert-type representation. This indeed is the case:[‡]

The System *HBL*

Defined Connective:

$$\psi \equiv \phi \stackrel{\text{def}}{=} (\psi \supset \phi) \wedge (\phi \supset \psi)$$

Inference Rule:

$$\frac{\psi \quad \psi \supset \phi}{\phi}$$

Axioms:

$[\supset 1]$	$\psi \supset \phi \supset \psi$
$[\supset 2]$	$(\psi \supset \phi \supset \tau) \supset (\psi \supset \phi) \supset (\psi \supset \tau)$
$[\supset 3]$	$((\psi \supset \phi) \supset \psi) \supset \psi$
$[\wedge \supset]$	$\psi \wedge \phi \supset \psi \qquad \psi \wedge \phi \supset \phi$
$[\supset \wedge]$	$\psi \supset \phi \supset \psi \wedge \phi$
$[\otimes \supset]$	$\psi \otimes \phi \supset \psi \qquad \psi \otimes \phi \supset \phi$
$[\supset \otimes]$	$\psi \supset \phi \supset \psi \otimes \phi$
$[\supset \vee]$	$\psi \supset \psi \vee \phi \qquad \phi \supset \psi \vee \phi$
$[\vee \supset]$	$(\psi \supset \tau) \supset (\phi \supset \tau) \supset (\psi \vee \phi \supset \tau)$
$[\supset \oplus]$	$\psi \supset \psi \oplus \phi \qquad \phi \supset \psi \oplus \phi$
$[\oplus \supset]$	$(\psi \supset \tau) \supset (\phi \supset \tau) \supset (\psi \oplus \phi \supset \tau)$
$[\neg \wedge]$	$\neg(\psi \wedge \phi) \equiv \neg\psi \vee \neg\phi$
$[\neg \vee]$	$\neg(\psi \vee \phi) \equiv \neg\psi \wedge \neg\phi$

[‡] In the formulae below the association of nested implications should be taken to the right.

$[\neg\otimes]\quad \neg(\psi \otimes \phi) \equiv \neg\psi \otimes \neg\phi$

$[\neg\oplus]\quad \neg(\psi \oplus \phi) \equiv \neg\psi \oplus \neg\phi$

$[\neg\supset]\quad \neg(\psi \supset \phi) \equiv \psi \wedge \neg\phi$

$[\neg\neg]\quad \neg\neg\psi \equiv \psi$

Note: Again we note the critical role of negation in this system.

THEOREM 3.23. $GBL_{\supset}$ and *HBL* are equivalent. In particular:
a) $\psi_1, \ldots, \psi_n \vdash_{GBL_{\supset}} \phi_1, \ldots, \phi_m$ iff $\vdash_{HBL} \psi_1 \wedge, \ldots, \wedge\psi_n \supset \phi_1 \vee, \ldots, \vee\phi_m$ (or just $\phi_1 \vee, \ldots, \vee\phi_m$ in case that $n=0$).
b) Let Γ be any set of sentences, and ψ – a sentence. Then $\Gamma \vdash_{HBL} \psi$ iff every valuation ν in *FOUR*, which gives all the sentences in Γ designated values, does the same to ψ.

Proof. It is possible to prove (a) purely proof theoretically. This is easy but tedious (the well-known fact that every $\{\wedge, \vee, \supset\}$-classical tautology is provable from the corresponding fragment of *HBL* can shorten things a lot, though). Part (b) follows then from the completeness and the compactness of $GBL_{\supset}$. Alternatively, one can prove (b) first (and then (a) is an immediate corollary). For this, assume that $\Gamma \not\vdash_{HBL} \psi$. Extend Γ to a maximal theory Γ^*, such that $\Gamma^* \not\vdash_{HBL} \psi$. By the deduction theorem for $\supset$ (which obviously obtains here), and from the maximality of Γ^*, $\Gamma^* \not\vdash_{HBL} \phi$ iff $\Gamma^* \vdash_{HBL} \phi \supset \psi$. Hence, if τ is any sentence, then if $\Gamma^* \not\vdash_{HBL} \psi \supset \tau$, then $\Gamma^* \vdash_{HBL} (\psi \supset \tau) \supset \psi$ and so $\Gamma^* \vdash_{HBL} \psi$ by $[\supset 3]$; a contradiction. It follows that $\Gamma^* \vdash_{HBL} \psi \supset \tau$ for every τ, and so for every ϕ and τ:

(*) if $\Gamma^* \not\vdash_{HBL} \phi$ then $\Gamma^* \vdash_{HBL} \phi \supset \tau$.

Define now a valuation ν as follows:

$$\nu(\phi) \stackrel{\text{def}}{=} \begin{cases} \top & \text{if } \Gamma^* \vdash_{HBL} \phi \text{ and } \Gamma^* \vdash_{HBL} \neg\phi \\ \bot & \text{if } \Gamma^* \not\vdash_{HBL} \phi \text{ and } \Gamma^* \not\vdash_{HBL} \neg\phi \\ t & \text{if } \Gamma^* \vdash_{HBL} \phi \text{ and } \Gamma^* \not\vdash_{HBL} \neg\phi \\ f & \text{if } \Gamma^* \not\vdash_{HBL} \phi \text{ and } \Gamma^* \vdash_{HBL} \neg\phi \end{cases}$$

Obviously, $\nu(\phi)$ is designated whenever $\Gamma^* \vdash_{HBL} \phi$, while $\nu(\psi)$ is not. It remains to show that ν is actually a valuation. We shall show that $\nu(\phi \supset \tau) = \nu(\phi) \supset \nu(\tau)$, and that $\nu(\phi \vee \tau) = \nu(\phi) \vee \nu(\tau)$, leaving the other cases for the reader.

To show that $\nu(\phi \vee \tau) = \nu(\phi) \vee \nu(\tau)$, we note first that axioms $[\supset\vee]$ and $[\vee\supset]$, together with the above characterization (*) of the non-theorems of Γ^*, imply that $\Gamma^* \vdash_{HBL} \phi \vee \tau$ iff either $\Gamma^* \vdash_{HBL} \phi$, or $\Gamma^* \vdash_{HBL} \tau$. Axiom $[\neg\vee]$, on the other hand, entails that $\Gamma^* \vdash_{HBL} \neg(\phi \vee \tau)$ iff both $\Gamma^* \vdash_{HBL} \neg\phi$, and $\Gamma^* \vdash_{HBL} \neg\tau$. From these facts the desired equation easily follows.

In showing that $\nu(\phi \supset \tau) = \nu(\phi) \supset \nu(\tau)$, we distinguish between two cases:
case 1: $\nu(\phi) \in \{f, \bot\}$. This means, on the one hand, that $\nu(\phi) \supset \nu(\tau) = t$. On the

other hand, it is equivalent to $\Gamma^* \nvdash_{HBL} \phi$. By (*) above, and by axiom $[\neg \supset]$ this entails that $\Gamma^* \vdash_{HBL} \phi \supset \tau$ but $\Gamma^* \nvdash_{HBL} \neg(\phi \supset \tau)$. Hence $\nu(\phi \supset \tau) = t = \nu(\phi) \supset \nu(\tau)$. *case 2:* $\nu(\phi) \in \{t, \top\}$. Then $\nu(\phi) \supset \nu(\tau) = \nu(\tau)$. In addition, it means that $\Gamma^* \vdash_{HBL} \phi$, and so (by axioms $[\supset 1]$ and $[\neg \supset]$), $\Gamma^* \vdash_{HBL} \phi \supset \tau$ iff $\Gamma^* \vdash_{HBL} \tau$, and $\Gamma^* \vdash_{HBL} \neg(\phi \supset \tau)$ iff $\Gamma^* \vdash_{HBL} \neg\tau$. It follows that $\nu(\phi \supset \tau) = \nu(\tau)$ too. □

COROLLARY 3.24. *HBL* is well-axiomatized: a complete and sound axiomatization of every fragment of $\models_{BL_\supset}$, which includes $\supset$, is given by the axioms of *HBL* which mention only the connectives of that fragment.

Proof. The above proof shows, as it is, the completeness of the axioms which mention only $\{\vee, \supset, \neg\}$ for the corresponding fragment. All the other cases in which $\neg$ is included are similar. If $\neg$ is not included, then the system is identical to the system for positive classical logic, which is known to have this property.* □

Note: The $\{\neg, \wedge, \vee, \supset\}$-fragment of $GBL_\supset$ and *HBL* were called in Avron (1991b) the "basic systems". Again, it is shown there that by adding $\Gamma \Rightarrow \Delta, \psi, \neg\psi$ to $GBL_\supset$, and either $\neg\psi \vee \psi$ or $(\psi \supset \phi) \supset (\neg\psi \supset \phi) \supset \phi$ to *HBL*, we get complete proof systems for the full three-valued logic of $\{t, f, \bot\}$. This logic is an extension of Kleene three-valued logic, which is equivalent to the logic of LPF (Barringer, 1984; Jones, 1986). If, on the other hand, we add $\Gamma, \psi, \neg\psi \Rightarrow \Delta$ to $GBL_\supset$ and $\neg\psi \supset (\psi \supset \phi)$ to *HBL*, we get complete proof systems for the three-valued logic of $\{t, f, \top\}$ (also known as J_3 – see note (3) after Theorem 3.7).

3.4.2. *Strong Implication*

The implication connective $\supset$ has two drawbacks: the main one is that even in case $\psi \supset \phi$ and $\phi \supset \psi$ are both valid, ψ and ϕ might not be equivalent (in the sense that one can be substituted for the other in any context). For example, if $\psi = \neg(\tau \supset \rho)$ and $\phi = \tau \wedge \neg\rho$, then both $\psi \supset \phi$ and $\phi \supset \psi$ are valid, but $\neg\psi \supset \neg\phi$ is not. The second disadvantage is that $\psi \supset \phi$ may be true, its conclusion false, without this entailing that the premise is also false (for example: $\bot \supset f = t$).

This drawbacks of $\supset$ are, in fact, drawbacks of $\models_{BL_\supset}$, the consequence relation on which it is based. What we can do, however, using the general theory developed in Avron (1991b), is to define in $\models_{BL_\supset}$ an implication connective, which corresponds to $\models^s_{BL_\supset}$ and does not suffer from these disadvantages.

DEFINITION 3.25. (strong implication)**

- $\psi \to \phi \stackrel{\text{def}}{=} (\psi \supset \phi) \wedge (\neg\phi \supset \neg\psi)$
- $\psi \leftrightarrow \phi \stackrel{\text{def}}{=} (\psi \to \phi) \wedge (\phi \to \psi)$

* Without $\neg$ there is no difference between $\wedge$ and $\otimes$, and no difference between $\vee$ and $\oplus$.

** In this definition too, the role of negation is critical.

PROPOSITION 3.26. $\models^s_{BL_\supset}$ has all the properties stated for $\models^s_{BL}$ in Proposition 3.13. In addition, $\rightarrow$ is an internal implication for it: $\Gamma, \psi \models^s_{BL_\supset} \phi$ iff $\Gamma \models^s_{BL_\supset} \psi \rightarrow \phi$ (in particular, $\psi, \psi \rightarrow \phi \models^s_{BL_\supset} \phi$).

Proof. These are all immediate consequences of the general theory in Avron (1991b), and the fact that $\neg, \wedge$ and $\supset$ satisfy in $\models_{BL_\supset}$ their corresponding symmetry conditions as defined there (basically this means that the relevant rules of $GBL_\supset$ are valid). □

PROPOSITION 3.27. Let ψ, ϕ, τ be formulae in $BL_\supset$, and ν – any evaluation in $FOUR$. Then:

a) $\nu(\psi \rightarrow \phi) \in \mathcal{D}(FOUR)$, iff $\nu(\psi) \leq_t \nu(\phi)$.

b) $\nu(\psi \leftrightarrow \phi) \in \mathcal{D}(FOUR)$, iff $\nu(\psi) = \nu(\phi)$.

Proof. Left to the reader. □

COROLLARY 3.28. $\psi \leftrightarrow \phi \models_{BL_\supset} \Theta(\psi) \leftrightarrow \Theta(\phi)$ for every scheme Θ. In other words, $\leftrightarrow$ is a *congruence* connective.

Proof. Immediate from part (b) of the last proposition, and from the fact that $\models_{BL_\supset}$ is the same as $\models_{\langle FOUR \rangle}$. □

Proposition 3.27 provides us with an easy method of checking validity or invalidity of sentences containing $\rightarrow$. Using this method it is straightforward to check the next two propositions:

PROPOSITION 3.29. The following are valid in $\models_{BL_\supset}$ ($\models_{BL_\supset(4)}$):

$$\psi \rightarrow \psi$$

$$(\psi \rightarrow \phi) \rightarrow (\phi \rightarrow \tau) \rightarrow (\psi \rightarrow \tau)$$

$$(\psi \rightarrow \phi \rightarrow \tau) \rightarrow \phi \rightarrow \psi \rightarrow \tau$$

$$(\psi \rightarrow \phi) \rightarrow \psi \rightarrow \psi \rightarrow \phi$$

$$\psi \wedge \phi \rightarrow \psi \quad , \quad \psi \wedge \phi \rightarrow \phi$$

$$(\psi \rightarrow \phi) \wedge (\psi \rightarrow \tau) \rightarrow \psi \rightarrow \phi \wedge \tau$$

$$\psi \otimes \phi \rightarrow \psi \quad , \quad \psi \otimes \phi \rightarrow \phi$$

$$(\psi \rightarrow \phi) \otimes (\psi \rightarrow \tau) \rightarrow \psi \rightarrow \phi \otimes \tau$$

$$\psi \rightarrow \psi \vee \phi \quad , \quad \phi \rightarrow \psi \vee \phi$$

$$(\psi \rightarrow \tau) \vee (\phi \rightarrow \tau) \rightarrow \psi \vee \phi \rightarrow \tau$$

$$\psi \rightarrow \psi \oplus \phi \quad , \quad \phi \rightarrow \psi \oplus \phi$$

$$(\psi \to \tau) \oplus (\phi \to \tau) \to \psi \oplus \phi \to \tau$$

$$\psi \leftrightarrow \neg\neg\psi$$

$$(\psi \to \phi) \leftrightarrow (\neg\phi \to \neg\psi)$$

$$\psi \wedge (\phi \vee \tau) \leftrightarrow (\psi \wedge \phi) \vee (\psi \wedge \tau)$$

$$\psi \otimes (\phi \oplus \tau) \leftrightarrow (\psi \otimes \phi) \oplus (\psi \otimes \tau)$$

$$\neg(\psi \wedge \phi) \leftrightarrow \neg\psi \vee \neg\phi$$

$$\neg(\psi \vee \phi) \leftrightarrow \neg\psi \wedge \neg\phi$$

$$\neg(\psi \otimes \phi) \leftrightarrow \neg\psi \otimes \neg\phi$$

$$\neg(\psi \oplus \phi) \leftrightarrow \neg\psi \oplus \neg\phi$$

PROPOSITION 3.30. The following are *not valid* in $\models_{BL_\supset}$ ($\models_{BL_\supset(4)}$):

$$\psi \to \phi \to \psi$$

$$(\psi \to \psi \to \phi) \to \psi \to \phi$$

$$\neg\psi \to \psi \to \phi$$

$$\psi \to \phi \to \psi \wedge \phi$$

$$\psi \to \phi \to \psi \otimes \phi$$

Notes:

1. If we compare the list above with the usual formal system for the relevance logic R (Anderson and Belnap, 1975; Dunn, 1986), we see that the only axiom of R which is not valid for this interpretation of $\to$ is the contraction axiom: $(\psi \to \psi \to \phi) \to \ \psi \to \phi$. It is worth noting that the omission of this axiom is also the main difference between the linear logic of Girard (see Girard, 1987) and the usual relevance logics. In fact, the last two propositions are true for linear logic as well (with the exception of the converse of contraction, the distributive schemes, and the parts concerning $\otimes$ and $\oplus$, of course), if we interprate $\neg$ and $\to$ as linear negation and implication (respectively), and $\wedge, \vee$ as the "additives". Note, however, that the "mix" (or "mingle") axiom $\psi \to \psi \to \psi$ *is* valid.
2. On $\{t, f, \bot\}$, $\to$ is exactly Lukasiewicz implication (Lukasiewicz, 1967; Urquhart, 1984), while on $\{t, f, \top\}$ it is Sobocinski implication (Sobocinski, 1952), which is the implication of RM_3 - the strongest logic in the family of relevance logics.

3. By using $\rightarrow$, we can sometimes translate "annotated atomic formulae" from Subrahmanian's annotated logic (see Subrahmanian, 1990a, 1990b; da-Costa et al., 1990; Kifer and Lozinskii, 1992; Kifer and Subrahmanian, 1992): The translation of $\psi : b$ to $BL(4)$ when $b \in FOUR$, and when the partial order in the (semi)lattice is $\leq_t$, is simply $b \rightarrow \psi$.

PROPOSITION 3.31. $\models_{BL_{\supset}} (\psi \supset \phi) \leftrightarrow \phi \vee (\psi \rightarrow (\psi \rightarrow \phi))$
Proof. This can easily be checked in $FOUR$. □

The last proposition means that it is possible to choose $\rightarrow$ rather than $\supset$ as the primitive implication of the language. We prefer the latter, though, since the intuitive meaning of both is then clearer. Also, the corresponding proof systems are much simpler if we follow this choice. Using $\rightarrow$, on the other hand, is more convenient for relating our logic to other known logics, as we have just seen.

Our next proposition brings us back to the relations between our logic and relevance logic:

PROPOSITION 3.32. Let ψ and ϕ be in the language of $\{\neg, \wedge, \vee\}$; then the following assertions are equivalent:
a) $\psi \models_{BL} \phi$
b) $\psi \models^{s}_{BL} \phi$
c) $\models_{BL} \psi \supset \phi$
d) $\models_{BL} \psi \rightarrow \phi$
e) $\models_{R} \psi \rightarrow \phi$
Proof. That $\psi \models_{BL} \phi$ iff $\models_R \psi \rightarrow \phi$ was noted already after Theorem 3.7. That $\models_{BL} \psi \supset \phi$ iff $\psi \models_{BL} \phi$ is an instance of the deduction theorem for $\supset$. Similarly, the equivalence of $\models_{BL} \psi \rightarrow \phi$ and $\psi \models^{s}_{BL} \phi$ follows from the deduction theorem for $\rightarrow$ relative to $\models^{s}_{BL}$, and the fact that $\models_{BL} \psi$ iff $\models^{s}_{BL} \psi$. Finally, $\models_{BL} \psi \rightarrow \phi$ iff $\nu(\psi) \leq_t \nu(\phi)$ for every valuation ν in $FOUR$, and it is well known (see Anderson and Belnap, 1975; Dunn, 1986) that if ψ and ϕ are in the $\{\neg, \wedge, \vee\}$-language, then $\vdash_R \psi \rightarrow \phi$ under exactly the same circumstances. □

We end this subsection with a short demonstration of the potential use of $\models_{BL_{\supset}}$ as well as of its various implication connectives. Recall that we are using $\rightsquigarrow$ to denote the implication of the classical calculus (i.e: $\psi \rightsquigarrow \phi = \neg\psi \vee \phi$).

EXAMPLE 3.33. Consider the following knowledge-base:

$bird(tweety) \rightsquigarrow fly(tweety)$

$penguin(tweety) \supset bird(tweety)$

$penguin(tweety) \rightarrow \neg fly(tweety)$

$bird(tweety)$

Note that we are using different implication connectives according to the strength we attach to each entailment: Penguins *never* fly. This is a characteristic feature of penguins, and there are no exceptions to that, hence we use the strongest implication ($\rightarrow$) in the third assertion in order to express this fact. The second assertion states that every penguin is a bird. Again, there are no exceptions to that fact. Still, penguins are not *typical* birds, thus they should not inherit all the properties we expect birds to have. The use of a weaker implication ($\supset$) forces us, indeed, to infer that something is a bird whenever we know that it is a penguin, but it does not forces us to infer that it has every property of a bird. Finally, the first assertion states only a default feature of birds, hence we attach the weakest implication ($\leadsto$) to it. Indeed, since from ψ and $\psi \leadsto \phi$ we cannot infer ϕ (by $\models_{BL}$) without more information, the first assertion does not cause automatic inference of flying abilities just from the fact that something is a bird. It does give, however, strong connection between the two facts.

The above knowledge-base does not allow us to infer whether tweety is a penguin or not (as it should be), and if it can fly or not (which is less satisfactory; we shall return to it in the next section). However, if we add to the knowledge-base an extra assumption, $penguin(tweety)$, we can infer $\neg fly(tweety)$ but we still *cannot* infer $fly(tweety)$, as should be expected.

3.5. Adding Quantifiers

So far we have concentrated on propositional languages and systems. The justification for this is that the main ideas and innovations are all on this level. Extending our notions and results to first order languages can be done in a rather standard way. We can take $\forall$, for example, as a generalization of $\wedge$. Having then an appropriate structure D, and an assignment ν of values to variables and truth values to atomic formula, we let $\nu(\forall x\psi(x))$ be $inf_{\leq_t}\{\nu(\psi(d) \mid d \in D\}$. Here we are using, of course, the fact that we assume $\mathcal{B}$ to be a *complete* lattice relative to $\leq_t$. The corresponding Gentzen-type rules are then:

$$[\forall\Rightarrow]\ \frac{\Gamma, \psi(s) \Rightarrow \Delta}{\Gamma, \forall x\psi(x) \Rightarrow \Delta} \qquad \frac{\Gamma \Rightarrow \psi(y), \Delta}{\Gamma \Rightarrow \forall x\psi(x), \Delta}\ [\Rightarrow\forall]$$

$$[\neg\forall\Rightarrow]\ \frac{\Gamma, \neg\psi(y) \Rightarrow \Delta}{\Gamma, \neg\forall x\psi(x) \Rightarrow \Delta} \quad \frac{\Gamma \Rightarrow \neg\psi(s), \Delta}{\Gamma \Rightarrow \neg\forall x\psi(x), \Delta}\ [\Rightarrow\neg\forall]$$

In these rules we assume, as usual, that the variable y does not appear free in Γ or in Δ. Corresponding soundness and completeness as well as cut elimination theorems can be proved relative to *FOUR* with no great difficulties. We omit here the details. We just note that one can introduce also, in the obvious way, quantifiers which correspond to $\otimes$ and $\oplus$.

4. A More Subtle Consequence Relation

$\models_{BL}$ should be taken as a first approximation of what can be safely inferred when we have a classically inconsistent knowledge-base; this safety is its main advantage. The disadvantage is that $\models_{BL}$ is somewhat "over cautious". Thus, in Example 3.33 we would have liked to be able to infer $fly(tweety)$ from the original knowledge-base, before the new information, $penguin(tweety)$, is added to it. We cannot do this, of course, since $\models_{BL}$ is monotonic.

There is more than one way of introducing other consequence relations, which are less cautious, and enjoy non-monotonicity; we present here one example. The idea is taken from a paper of Kifer and Lozinskii (1992). Their idea, basically, is to order models of a given knowledge-base in a way that somehow reflects their degree of consistency, and then take into account only the models which are maximal w.r.t this order. The main difference is that they were using just ordinary (semi)lattices, in which the partial order relation corresponds, intuitively, to our $\leq_k$. Hence, no direct interpretation of the standard logical connectives ($\wedge, \vee$) was available to them. They were forced, therefore, to use an unnatural language, in which the *atomic* formulae are of the form $p : b$ (where p is an atomic formula of the basic language, and b – a value from the semilattice). $\psi : b$ is meaningless, however, for nonatomic ψ. The use of bilattices allows us to give the standard logical language a direct interpretation, and so gives a meaning to every annotated formula. On the other hand, by using $\mathcal{F}$ we can dispense with annotated formulae altogether, as we do below.*

DEFINITION 4.1. Let $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ be a logical bilattice. A subset $\mathcal{I}$ of B is called an *inconsistency set*, if it has the following properties:
a) $b \in \mathcal{I}$ iff $\neg b \in \mathcal{I}$.
b) $b \in \mathcal{F} \cap \mathcal{I}$ iff $b \in \mathcal{F}$ and $\neg b \in \mathcal{F}$.**

Notes:
1. From (b), always $\top \in \mathcal{I}$. Also, from (b), $t \notin \mathcal{I}$, and so, from (a), $f \notin \mathcal{I}$.
2. As for $\bot$, both $\mathcal{I} \cup \{\bot\}$ and $\mathcal{I} \setminus \{\bot\}$ are inconsistency sets in case $\mathcal{I}$ is. Now, on one hand, in every bilattice, $\neg\bot = \bot$ (Proposition 2.5), so $\bot$ has some features that may be associated with inconsistent elements. On the other hand, $\bot$ intuitively reflects no knowledge at all about the assertions it represents; in particular, one might not take such assertions to be inconsistent. We shall usually prefer, therefore, to take $\bot$ as consistent (see also the note after Proposition 4.13).

EXAMPLE 4.2. The following are all inconsistency sets:
a) $\mathcal{I}_1 = \{b \mid b \in \mathcal{F} \text{ and } \neg b \in \mathcal{F}\}$.

* Despite the fact that this method of using "annotated" atomic formulae is quite common, it is still artificial from a logical point of view, since semantic notions interfere within the syntax.

** In Kifer and Lozinskii (1992) the inconsistent values are defined quite differently; see there for the details.

b) $\mathcal{I}_2 = \{b \mid b = \neg b\}$.
c) $\mathcal{I}_3 = \{b \mid b = \neg b\ ,\ b \neq \bot\}$.
$\mathcal{I}_1$ is the minimal possible inconsistency set in every in every $(\mathcal{B}, \mathcal{F})$. In case that $\mathcal{B}$ in interlaced, and $\mathcal{F} = \mathcal{D}(\mathcal{B})$, $\mathcal{I}_1$ is just $\{\top\}$ (see Proposition 2.25). $\mathcal{I}_2$ and $\mathcal{I}_3$ are always inconsistency sets in case $\mathcal{B}$ is interlaced, and $\mathcal{F} = \mathcal{D}(\mathcal{B})$. There are, however, other cases in which they are inconsistency sets, for example in *DEFAULT*.

We fix henceforth some logical bilattice $(\mathcal{B}, \mathcal{F})$, and an inconsistency subset $\mathcal{I}$ of it. Unless otherwise stated, all the definitions below will be relative to $(\mathcal{B}, \mathcal{F})$ and $\mathcal{I}$. We will refer to the members of $\mathcal{I}$ (the members of $\mathcal{B} \setminus \mathcal{I}$) as the inconsistent (consistent) truth values of $\mathcal{B}$.

NOTATION 4.3.
a) $\mathcal{A}(\Gamma)$ denotes the set of the atomic formulae that appear in some formula of Γ.
b) For a valuation M of Γ, denote: $I_M(\Gamma) = \{p \in \mathcal{A}(\Gamma) \mid M(p) \in \mathcal{I}\}$.

DEFINITION 4.4. Let Γ and Δ be two sets of formulae, and M, N – models of Γ.
a) M is *more consistent* model of Γ than N, if the set of the atomic formulae in $\mathcal{A}(\Gamma)$ that are assigned under M values from $\mathcal{I}$, is properly contained in the corresponding set of N (i.e: $I_M(\Gamma) \subset I_N(\Gamma)$).
b) M is a *most consistent* model of Γ (mcm, in short), if there is no other model of Γ which is more consistent than M.
c) $\Gamma \models_{con} \Delta$ if every mcm of Γ is a model of some formula of Δ.

EXAMPLE 4.5. Let us return to the knowledge-base KB of Example 3.33. Take $\mathcal{F} = \{t, \top\}$ and $\mathcal{B}$ – any bilattice in which this $\mathcal{F}$ is a prime bifilter (e.g: *FOUR*, *DEFAULT*). Let $\mathcal{I}$ be any inconsistency set in $\mathcal{B}$ (obviously, $\mathcal{F} \cap \mathcal{I} = \{\top\}$). Relative to $(\mathcal{B}, \mathcal{F})$ and $\mathcal{I}$, this knowledge-base has exactly one mcm, and it takes values in $\{t, f\}$. Hence, if ψ is in the language $\{\neg, \wedge, \vee, \supset\}$, then $KB \models_{con} \psi$ iff ψ follows classically from KB. Thus (unlike in the case of $\models_{BL}$!):

$$KB \models_{con} bird(tweety),\ KB \models_{con} \neg penguin(tweety),\ KB \models_{con} fly(tweety),$$
$$KB \not\models_{con} \neg bird(tweety),\ KB \not\models_{con} penguin(tweety),\ KB \not\models_{con} \neg fly(tweety).$$

Now, consider again what happens when we add $penguin(tweety)$ to KB: The new knowledge-base, KB', has two mcms, M_1 and M_2, where:

$$M_1(bird(tweety)) = t,\ M_1(penguin(tweety)) = \top,\ M_1(fly(tweety)) = \top,$$
$$M_2(bird(tweety)) = \top,\ M_2(penguin(tweety)) = t,\ M_2(fly(tweety)) = f.$$

This time, therefore,

$$KB' \models_{con} bird(tweety),\ KB' \models_{con} penguin(tweety),\ KB' \models_{con} \neg fly(tweety),$$

$KB' \not\models_{con} \neg bird(tweety)$, $KB' \not\models_{con} \neg penguin(tweety)$, $KB' \not\models_{con} fly(tweety)$.

It follows that $\models_{con}$ is a non-monotonic consequence relation, which seems to behave according to our expectations. □

Some important properties of $\models_{con}$ are summarized below:

PROPOSITION 4.6. If $\Gamma \models_{BL} \Delta$ then $\Gamma \models_{con} \Delta$.

Proof. If every model of Γ satisfies some formula of Δ, then obviously every mcm of Γ does so. □

PROPOSITION 4.7. $\models_{con}$ is non-monotonic.

Proof. Consider, e.g., $\Gamma = \{p, \neg p \vee q\}$. In every mcm, M, p and q must have consistent values (since the valuation that assigns t to each one of them, is an mcm of Γ). Also, $M(p) \in \mathcal{F}$, since M is a model of Γ. If $M(\neg p) \in \mathcal{F}$ also, then $M(p) \in \mathcal{F} \cap \mathcal{I}$ (from Definition 4.1(b)), so $M(p)$ is inconsistent. Hence $M(\neg p) \notin \mathcal{F}$. But $M(\neg p \vee q) \in \mathcal{F}$, hence $M(q) \in \mathcal{F}$. So, $\Gamma \models_{con} q$ in every $(\mathcal{B}, \mathcal{F})$ and $\mathcal{I}$. Obviously, however, $\Gamma, \neg p \not\models_{con} q$ (take, e.g., M s.t. $M(p) = \top$, and $M(q) = f$). □

PROPOSITION 4.8. $\models_{con}$ is paraconsistent:
$p, \neg p \not\models_{con} q$, and even $p \vee q, \neg(p \vee q) \not\models_{con} q$.

Proof. Consider any valuation that assigns p the value $\top$, and assigns q the value f. □

PROPOSITION 4.9. If Γ and ψ are in the language of $\{\neg, \wedge, \vee, \supset, f, t\}$, and $\Gamma \models_{con} \psi$, then ψ classically follows from Γ.

Proof. The crucial property of the language here is that if all the atomic formulae get values in $\{f, t\}$, then so does any formula in the language. Now, if Γ is classically consistent, then it has a model in $\{t, f\}$, and so all its mcms assign the members of $\mathcal{A}(\Gamma)$ consistent values. Hence, if $\Gamma \models_{con} \psi$, then every model of Γ that assigns the members of $\mathcal{A}(\Gamma)$ consistent values, is a model of ψ. In particular, every model of Γ that assigns the members of $\mathcal{A}(\{\Gamma, \psi\})$ classical values (i.e.: $\{t, f\}$), is a model of ψ, and so ψ follows classically from Γ. If Γ is classically inconsistent, then any ϕ follows from it classically (in particular ψ). □.

A partial converse for *consistent* theories is given in the next proposition:

PROPOSITION 4.10. Let Γ be a classically consistent set in the language of $\{\neg, \wedge, \vee, f, t\}$, and let ψ be a sentence in the same language, which classically follows from Γ. Then there exist sentences ϕ and τ, such that:

1) ψ is classically equivalent to ϕ,
2) τ is a tautology,
3) $\psi \models_{BL(4)} \phi \wedge \tau$ and $\phi \wedge \tau \models_{BL(4)} \psi$,
4) $\Gamma \models_{con} \phi$.

Proof. Let ψ' be a sentence like in Proposition 3.5. ψ' can be written in the form $\phi \wedge \tau$, where τ is the conjunction of all the conjuncts in ψ' which are tautologies (i.e.: contains some atomic formula and its negation as disjuncts), and ϕ is the conjunction of the other conjuncts of ψ' (if either set of conjuncts is empty, we take it to be t). ϕ and τ obviously satisfy properties (2) and (3). Since classical logic is an extension of $\models_{BL(4)}$ w.r.t. the language under consideration, ψ is classically equivalent to $\phi \wedge \tau$, and so to ϕ (since τ is a tautology). It remains to prove (4). It is easy to see that $\Gamma \models_{con} \phi_1 \wedge \ldots \wedge \phi_n$ iff $\Gamma \models_{con} \phi_i$ for every $i = 1 \ldots n$. Hence, (4) follows from the following lemma:

LEMMA 4.11. Let Γ be a classically consistent set in the language of $\{\neg, \wedge, \vee, f, t\}$, and ψ – a clause that does not contain any pair of an atomic formula and its negation. If ψ follows classically from Γ, then $\Gamma \models_{con} \psi$.

Proof. We will show that if $\Gamma \not\models_{con} \psi$, then there is a classical model of Γ, which is not a model of ψ. Indeed, let M be an mcm of Γ s.t. $M(\psi) \notin \mathcal{F}$. Consider the valuation M', defined as follows:

$$M'(p) \stackrel{\text{def}}{=} \begin{cases} t & \text{if } M(p) \in \mathcal{F}, \text{ and } p \in \mathcal{A}(\Gamma, \psi). \\ f & \text{if } M(\neg p) \in \mathcal{F}, \text{ and } p \in \mathcal{A}(\Gamma, \psi). \\ t & \text{if } M(p) \notin \mathcal{F},\ M(\neg p) \notin \mathcal{F}, \text{ and } \neg p \text{ appears as a literal in } \psi. \\ f & \text{if } M(p) \notin \mathcal{F},\ M(\neg p) \notin \mathcal{F}, \text{ and } p \text{ appears as a literal in } \psi. \\ t & \text{otherwise} \end{cases}$$

Exactly as in the proof of Proposition 4.9, the fact that Γ is classically consistent entails that $M(p)$ is consistent for every p in $\mathcal{A}(\Gamma)$. Hence there cannot be any p in $\mathcal{A}(\Gamma)$ s.t. both $M(p)$ and $M(\neg p)$ are in $\mathcal{F}$ (otherwise, from (b) in Definition 4.1, $M(p) \in \mathcal{I}$). On the other hand, if $p \in \mathcal{A}(\psi)$ then either p or $\neg p$ is a disjunct of ψ. Since $M(\psi) \notin \mathcal{F}$, this implies that either $M(p) \notin \mathcal{F}$, or $M(\neg p) \notin \mathcal{F}$. These two facts and our explicit assumption on ψ imply that M' above is well defined. Obviously, M' is a classical valuation. Now, by Proposition 3.5, there is a set of clauses Γ', s.t. $\mathcal{A}(\Gamma) = \mathcal{A}(\Gamma')$, every model of Γ is also a model of Γ', and vice-versa. Since M is a model of Γ, it is also a model of Γ'. Hence, for every clause $\phi \in \Gamma'$ with literals l_i $(i = 1 \ldots n)$, there is at least one literal, l_i, s.t. $M(l_i) \in \mathcal{F}$. From the definition of M', $M'(l_i) \in \mathcal{F}$ as well, thus M' is a model of Γ'. Hence M' is a model of Γ as well. On the other hand, $M'(\psi) = f$, since for every literal l_i that appears in ψ, $M'(l_i) = f$. Indeed, without a loss of generality, suppose that $l_i = \neg p$. Since $M(\psi) \notin \mathcal{F}$, also $M(\neg p) \notin \mathcal{F}$. If $M(p) \in \mathcal{F}$, then $M'(p) = t$, and so $M'(l_i) = M'(\neg p) = \neg M'(p) = \neg t = f$. If $M(p) \notin \mathcal{F}$, then since $\neg p$ appears as a literal in ψ, $M'(p) = t$ in this case as well, and again $M'(l_i) = f$. M' is, therefore, a classical model of Γ, which is not a model of ψ. Hence ψ does not follow classically from Γ. □

Note: The crucial Lemma 4.11 does not hold under stronger assumptions:
a) If we allow the appearance of $\supset$ in Γ, then consider $\langle FOUR \rangle$ with $\mathcal{I} = \{\top\}$,

and $\Gamma=\{p\supset q, p\supset\neg q\}$, $\psi=\neg p$. ψ follows classically from Γ, but the valuation M, where $M(p)=\bot$, and $M(q)=t$, is an example of an mcm of Γ, which is not a model of ψ.
b) If ψ contains a literal and its negation, then consider again $\langle FOUR\rangle$ with $\mathcal{I}=\{\top\}$. This time, $p\vee\neg p$ follows classically from q, but $q\not\models_{con} p\vee\neg p$ (consider, e.g., $M(q)=t, M(p)=\bot$).*

As we have already shown, $\models_{con}$ is non-monotonic. We next show that in addition it satisfies some properties that one might like a non-monotonic logic to have:

DEFINITION 4.12. (Lehmann, 1992): A *plausibility logic* is a logic that satisfies the following conditions (for finite Γ, Δ):

Inclusion: $\Gamma, \psi \Rightarrow \psi$.

Right Monotonicity: If $\Gamma\Rightarrow\Delta$, then $\Gamma\Rightarrow\psi, \Delta$.

Cautious Left Monotonicity: If $\Gamma\Rightarrow\psi$ and $\Gamma\Rightarrow\Delta$, then $\Gamma,\psi\Rightarrow\Delta$.**

Cautious Cut: If $\Gamma,\psi_1,\ldots,\psi_n\Rightarrow\Delta$ and $\Gamma\Rightarrow\psi_i,\Delta$ for $i=1\ldots n$, then $\Gamma\Rightarrow\Delta$.

PROPOSITION 4.13. $\models_{con}$ satisfies Inclusion, Right Monotonicity, and Cautious Left Monotonicity. $\models_{con}$ also satisfies Cautious Cut iff there exists $\beta\in\mathcal{B}$ s.t. $\beta\notin\mathcal{I}\cup\mathcal{F}\cup\{b\mid\neg b\in\mathcal{F}\}$, and the language is $BL(4)$ (Hence $\models_{con}$ is a plausibility logic under these conditions).‡

Proof. Inclusion and Right Monotonicity follow immediately from the definition of $\models_{con}$.

Proof of Cautious Left Monotonicity: Assume that $\Gamma\models_{con}\psi$, $\Gamma\models_{con}\Delta$, and let M be any mcm of $\{\Gamma,\psi\}$. We will show that M is also a mcm of Γ. Since $\Gamma\models_{con}\Delta$, this will imply that M satisfies some formula in Δ, and so $\Gamma,\psi\models_{con}\Delta$. Now, M is certainly a model of Γ. Assume that it is *not* an mcm of Γ. Then there is a model of Γ that is strictly more consistent than M. Since Γ is finite, there is an mcm N of Γ, which is strictly more consistent than M; and so $I_N(\Gamma)\subset I_M(\Gamma)$. Consider the valuation N' that is defined as follows: $N'(p)=N(p)$ for every $p\in\mathcal{A}(\Gamma)$ and $N'(p)=b$ otherwise, where b is any consistent truth value. Obviously, N' is an mcm of Γ. Since $\Gamma\models_{con}\psi$, N' is a model of $\{\Gamma,\psi\}$. Now, $I_{N'}(\Gamma,\psi)=I_{N'}(\Gamma)=I_N(\Gamma)\subset I_M(\Gamma)\subseteq I_M(\Gamma,\psi)$. Hence N' is a model of $\{\Gamma,\psi\}$, which is more consistent than M. This contradicts the fact that M is an mcm of $\{\Gamma,\psi\}$.

* One can replace here $\{q\}$ by $\{q, q\vee p\}$, if one wishes $\mathcal{A}(\psi)$ to be a subset of $\mathcal{A}(\Gamma)$.

** This rule was first proposed in Gabbay (1985).

‡ In Proposition 4.10 of Arieli and Avron (1994) the bilattice under consideration should have been interlaced, and $\beta=\bot$ (these assumptions were used there for the proof of the Cautious Cut). Here we prove the proposition for *any* logical bilattice, and for β as defined above, which may be different from $\bot$.

Proof of Cautious Cut under the specified conditions: Assume that $\Gamma, \psi_1, \ldots, \psi_n \models_{con} \Delta$ and $\Gamma \models_{con} \psi_i, \Delta$ for $i=1 \ldots n$. Let M be an mcm of Γ. We will show that M is a model of some formula of Δ. For this, define another valuation, M', by:

$$M'(p) \stackrel{\text{def}}{=} \begin{cases} M(p) & \text{if } p \in \mathcal{A}(\Gamma) \\ \beta & \text{otherwise} \end{cases}$$

Obviously, $M'(\phi) = M(\phi)$ for every ϕ s.t. $\mathcal{A}(\phi) \subseteq \mathcal{A}(\Gamma)$. Hence M' is also an mcm of Γ. Thus, M' is either a model of some $\phi \in \Delta$, or M' is a model of $\psi_1, \ldots, \psi_n$. Since $M'(p) \in \mathcal{I}$ implies that $p \in \mathcal{A}(\Gamma)$, and since M' is an mcm of Γ, M' is necessarily an mcm of $\{\Gamma, \psi_1, \ldots, \psi_n\}$ in the second case. Hence, again, M' is a model of some $\phi \in \Delta$. It follows that in either cases $M'(\phi) \in \mathcal{F}$ for some $\phi \in \Delta$. It remains to show that $M(\phi) \in \mathcal{F}$ whenever $M'(\phi) \in \mathcal{F}$. Indeed, by Proposition 3.5 there exists a formula ϕ', which is a conjunction of disjunctions of literals, s.t. for every valuation ν, $\nu(\phi) \in \mathcal{F}$ iff $\nu(\phi') \in \mathcal{F}$. If $M'(\phi) \in \mathcal{F}$, then $M'(\phi') \in \mathcal{F}$ also, so $M'(D) \in \mathcal{F}$ for every conjunct D of ϕ'. Now, $M'(D) \in \mathcal{F}$ iff there is a literal $l \in D$ s.t. $M'(l) \in \mathcal{F}$. But since l is a literal, it is obvious that $M'(l) \in \mathcal{F}$ only if $M'(l) \neq \beta$ and $M'(\neg l) \neq \beta$, so $M(l) = M'(l)$. Hence $M(l) \in \mathcal{F}$ as well. It follows that $M(D) \in \mathcal{F}$ also, and so $M(\phi') \in \mathcal{F}$, implying that $M(\phi) \in \mathcal{F}$.

To show the necessity of the conditions we note that:

1) If $\supset$ is in the language, then for every $\mathcal{B}$, $\mathcal{F}$, and $\mathcal{I}$: $q \models_{con} q \vee p$, and $q, q \vee p \models_{con} (p \supset \neg q) \vee (\neg p \supset \neg q)$, but $q \not\models_{con} (p \supset \neg q) \vee (\neg p \supset \neg q)$ (take a valuation M, s.t. $M(q) = t$ and $M(p) = \top$).

2) If $\mathcal{B} = \mathcal{I} \cup \mathcal{F} \cup \{b \mid \neg b \in \mathcal{F}\}$, then $q \models_{con} q \vee p$ and $q, q \vee p \models_{con} p \vee \neg p$, but $q \not\models_{con} p \vee \neg p$ (consider M, s.t.$M(q) = t$ and $M(p) = \bot$). □

Note: If $\bot \notin \mathcal{I}$ (see note 2 after Definition 4.1) then the condition for Cautious Cut is satisfied for $\beta = \bot$.

The crucial point in the counterexamples given in the last proof, is that the cut formula contain atomic formula that does not appear in $\mathcal{A}(\Gamma)$. In fact, it is easy to show that otherwise the rule is valid with no extra assumption:

DEFINITION 4.14. (*Analytic Cautious Cut*) If $\Gamma, \psi_1, \ldots, \psi_n \models_{con} \Delta$ and $\Gamma \models_{con} \psi_i, \Delta$ for $i=1 \ldots n$, and if $\mathcal{A}(\{\psi_1, \ldots, \psi_n\}) \subseteq \mathcal{A}(\Gamma)$, then $\Gamma \models_{con} \Delta$.

PROPOSITION 4.15. Analytic Cautious Cut is valid rule for $\models_{con}$.

Proof. Let M be any mcm of Γ. We will show that M is a model of some formula in Δ. If not, then M is a model of ψ_i $(i = 1 \ldots n)$, since $\Gamma \models_{con} \psi_i, \Delta$. Hence M is a model of $\{\Gamma, \psi_1, \ldots \psi_n\}$. It is obviously an mcm of this set, since any model which is more consistent than M w.r.t $\{\Gamma, \psi_1, \ldots \psi_n\}$, is also a more consistent model than M w.r.t Γ (using the fact that $\mathcal{A}(\{\psi_1, \ldots, \psi_n\}) \subseteq \mathcal{A}(\Gamma)$). Since $\Gamma, \psi_1, \ldots, \psi_n \models_{con} \Delta$, M *is* a model of some formula of Δ after all. □

PROPOSITION 4.16. All the rules of *GBL* are valid for $\models_{con}$.

Proof. The validity of Exchange and Contraction is immediate from the definition of $\models_{con}$. The introduction rules on the right, as well as their inverses, are valid for exactly the same reasons that they are valid in $\models_{BL}$. The rules $[\wedge \Rightarrow]$ and $[\otimes \Rightarrow]$ are valid, since the models of $\{\Gamma, \psi, \phi\}$, $\{\Gamma, \psi \wedge \phi\}$, and $\{\Gamma, \psi \otimes \phi\}$, are the same, hence the mcms of these sets are also the same. Similar argument works for $[\neg\neg \Rightarrow]$. The rules $[\vee \Rightarrow]$ and $[\oplus \Rightarrow]$ are proved in Lehmann (1992) to be valid in every plausibility logic, which satisfies $[\Rightarrow \vee]$, $[\Rightarrow \oplus]$, and their converses. The proof there does not use in fact the full power of Cautious Cut, but only that of Analytic Cautious Cut. For the reader convenience, we repeat the arguments, adjusted to our logic, for the case of $[\oplus \Rightarrow]$:

(1)	$\Gamma, \psi \Rightarrow \psi, \phi$	Inclusion and Right Monotonicity.
(2)	$\Gamma, \psi \Rightarrow \psi \oplus \phi$	(1), $[\Rightarrow \oplus]$.
(3)	$\Gamma, \psi \Rightarrow \Delta$	Hypothesis.
(4)	$\Gamma, \psi, \psi \oplus \phi \Rightarrow \Delta$	(2), (3), Left Cautious Monotonicity.
(5)	$\Gamma, \psi, \psi \oplus \phi \Rightarrow \phi, \Delta$	(4), Right Monotonicity.
(6)	$\Gamma, \psi \oplus \phi \Rightarrow \psi \oplus \phi$	Inclusion.
(7)	$\Gamma, \psi \oplus \phi \Rightarrow \psi, \phi$	(6), Inverse rule of $[\Rightarrow \oplus]$.
(8)	$\Gamma, \psi \oplus \phi \Rightarrow \psi, \phi, \Delta$	(7), Right Monotonicity.
(9)	$\Gamma, \psi \oplus \phi \Rightarrow \phi, \Delta$	(5), (8), Analytic Cautious Cut.
(10)	$\Gamma, \phi, \psi \oplus \phi \Rightarrow \Delta$	Proved like (4), exchanging the roles of ψ and ϕ.
(11)	$\Gamma, \psi \oplus \phi \Rightarrow \Delta$	(9), (10), Analytic Cautious Cut.

Finally, $[\neg\wedge \Rightarrow]$, $[\neg\vee \Rightarrow]$, $[\neg\otimes \Rightarrow]$, and $[\neg\oplus \Rightarrow]$ all follow from Lemma 2.5(a), together with the previous observations. □

Some other nice properties that are true in every plausibility logic which satisfies $[\Rightarrow \vee]$, $[\Rightarrow \oplus]$, and their converses, are listed in the next proposition (see Lehmann, 1992):

PROPOSITION 4.17. Let Γ, Δ be sets of formulae, and ψ, ϕ, τ – formulae in *BL*. Then:

Left Equivalence: $$\frac{\Gamma, \psi \models_{con} \phi \quad \Gamma, \phi \models_{con} \psi \quad \Gamma, \psi \models_{con} \Delta}{\Gamma, \phi \models_{con} \Delta}$$

Right equivalence: $$\frac{\Gamma, \psi \models_{con} \phi \quad \Gamma, \phi \models_{con} \psi \quad \Gamma \models_{con} \psi, \Delta}{\Gamma \models_{con} \phi, \Delta}$$

Loop: $$\frac{\psi \models_{con} \phi \quad \phi \models_{con} \tau \quad \tau \models_{con} \psi}{\psi \models_{con} \tau}$$

$$\frac{\psi \models_{con} \tau \quad \psi \vee \phi \models_{con} \psi}{\psi \vee \phi \models_{con} \tau} \qquad \frac{\psi \models_{con} \tau \quad \psi \oplus \phi \models_{con} \psi}{\psi \oplus \phi \models_{con} \tau}$$

$$\frac{\phi \vee \tau \models_{con} \phi \quad \psi \vee \phi \models_{con} \psi}{\psi \vee \tau \models_{con} \psi} \qquad \frac{\phi \oplus \tau \models_{con} \phi \quad \psi \oplus \phi \models_{con} \psi}{\psi \oplus \tau \models_{con} \psi}$$

As we have shown, $\models_{con}$ has a lot of desirable properties. We should mention, however, that $\models_{con}$ is *not* closed under substitutions. In other words: it is sensitive to the choice of the atomic formulae. Thus, although $\neg p, p \vee q \models_{con} q$, when p and q are *atomic*, it is not true in general that $\neg\psi, \psi \vee \phi \models_{con} \phi$ (take, e.g., $\mathcal{B} = FOUR$, $\psi = \neg(\neg p \wedge p)$, and $\phi = q$). This, however, is unavoidable when one wants to achieve both Lemma 4.11 and Proposition 4.8 above.

5. Conclusion and Further Work

Bilattices have had an extensive use in several areas, most notably in logic programming, but their role so far was mainly semantic in nature. We develop a real notion of logic based on bilattices, giving associated consequence relations and corresponding proof systems. These consequence relations are strongly related to non-monotonic reasoning, and especially to reasoning in the presence of inconsistent data.

This, however, is not the end of the work. The basic languages mentioned here are, as their name suggests, only basic. It seems that additional connectives are required in order to get more expressive languages. Such languages should be able to describe more precisely the specific bilattice under consideration. One would like, for example, to express in a knowledge-base over *DEFAULT* that a certain formula is considered to be true by default, or that the result of $f \otimes t$ should be considered as $d\top$ rather than $\bot$. This can be achieved, e.g., by defining a connective that reflects equivalences in formula assignments, or by defining some kind of analogue to the ":"-connective of annotated logic. The guard connective, investigated in Fitting (1994), might also be considered.

The consequence relations are also a matter for further examination. As we have shown (Theorem 3.4), the basic consequence relation, $\models_{BL}$, is no more than the logic of *FOUR*. Nevertheless, it is obviously desirable to take advantage of the availability of other values in the bilattice under consideration, for example the default values $\{df, dt\}$ of *DEFAULT*. Considering $\models_{con}$ was a first step, since we take into account not just the designated elements of the bilattice, but also those that were considered as inconsistent. For $\models_{con}$ *FOUR* is no longer a single representative of all the logical bilattices. For example, by taking $\mathcal{B}$ to be *FOUR* with the inconsistency set $\mathcal{I} = \{b \in \mathcal{B} \mid b = \neg b\}$, we have that $q,\ p \supset \neg q \models_{con} \neg p$, and $p \vee q \models_{con} \neg p \vee p$, while if we take $\mathcal{B}$ to be *DEFAULT* with the same definition of inconsistency set, these consequences are no longer valid. $\models_{con}$ seems to be, however, somewhat too crude, since it treats uniformly the whole set of atoms that are assigned inconsistent values under a given valuation. As a result, the preferences among the valuations are due to "global" considerations rather than pointwise ones. A future work should seek for a refinment of this relation, which might as well reflect the specific structure of the bilattice (especially its partial orders).

Another natural issue for a further research is to investigate how the resulting logics are affected by the choice of the bilattice under consideration, the truth values that are taken to be designated, and the choice of the inconsistency subsets.

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