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# Prioritized Simple Contrapositive Assumption-Based Frameworks

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**Abstract.** Simple contrapositive assumption-based frameworks are a general setting for structured argumentation, providing a robust approach to reasoning with arguments and counter-arguments. In this paper we extend these frameworks with priorities and introduce some new results concerning the semantics of the resulting formalisms.

## 1 INTRODUCTION

Assumption-based frameworks (ABFs) are a well-established form of structured argumentation, aimed at reasoning in the presence of arguments and counter-arguments (see, e.g., [7, 13, 30]). A general form of ABFs called *simple contrapositive assumption-based frameworks* is considered in [22, 23, 24], where it is shown that this family of ABFs has many desirable properties. In this paper we extend simple contrapositive ABFs with priorities, expressing the relative strengths (or reliability) of arguments. While extensions of ABFs with priorities have already been studied in the literature (mainly in the context of $\text{ABA}^+$ frameworks, see e.g. [12, 14]), several new finding on these frameworks are reported, among which are the following:

- Dung-style semantics [17] is considered for prioritized ABFs. It is shown that, like similar cases in other forms of structured argumentation, in many cases the set of naive, stable, and preferred extensions coincide. However, unlike other forms of structured argumentation (including common cases of simple contrapositive ABFs), when priorities are introduced the grounded semantics is not always unique, nor does it necessarily coincide with the well-founded semantics.
- We show that under the reversibility condition (see below), all the extensions of simple contrapositive ABFs are necessarily consistent and are closed under logical consequences (properties that are not assured in general for extensions of prioritized ABFs in particular, and structured argumentation frameworks in general).
- Relations to reasoning with preferred maximally consistent subsets of the premises [8] are shown.
- The impact of the underlying preference setting on the properties of the prioritized ABFs that are induced by them is analyzed in terms of some postulates.
- We define conditions that assure that prioritized ABFs avoid an undesirable property of prioritized systems, known as the *drowning effect*, according to which arguments with lower priorities are excluded in the presence of unrelated arguments with higher priorities.

The outcome of this work is therefore a robust approach of argumentative, preference-based reasoning with defeasible assumptions. This approach is compared in the last section of the paper to other approaches of accommodating priorities in structured argumentation.

## 2 PRELIMINARIES

In what follows we shall denote by $\mathscr{L}$ an arbitrary propositional language. Atomic formulas in $\mathscr{L}$ are denoted by $p,q,r$, compound formulas are denoted by $\psi,\phi,\sigma$, and sets of formulas in $\mathscr{L}$ are denoted by $\Gamma$, $\Delta$, $\Theta$ (possibly primed or indexed). The powerset of $\mathscr{L}$ is denoted by $\wp(\mathscr{L})$.

**Definition 1** A *logic* for a language $\mathscr{L}$ is a pair $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathscr{L}$, that is, a binary relation between sets of formulas and formulas in $\mathscr{L}$, that is reflexive (if $\psi \in \Gamma$ then $\Gamma \vdash \psi$), monotonic (if $\Gamma \vdash \psi$ and $\Gamma \subseteq \Gamma'$ then $\Gamma' \vdash \psi$), and transitive (if $\Gamma \vdash \psi$ and $\Gamma', \psi \vdash \phi$ then $\Gamma, \Gamma' \vdash \phi$).

In addition, we shall assume that $\mathfrak{L}$ is structural (closed under substitutions: if $\Gamma \vdash \psi$ then $\theta(\Gamma) \vdash \theta(\psi)$ for every $\mathscr{L}$-substitution $\theta$) and non-trivial (there are a non-empty set $\Gamma$ and a formula $\psi$ such that $\Gamma \nvdash \psi$).

The $\vdash$-transitive closure of a set $\Gamma$ of $\mathscr{L}$-formulas is $Cn_{\vdash}(\Gamma) = \{\psi \mid \Gamma \vdash \psi\}$. When $\vdash$ is known or arbitrary, we just write $Cn(\Gamma)$.

We shall assume that the language $\mathscr{L}$ contains at least the following connectives and constant:
• a $\vdash$-*negation* $\neg$, satisfying: $p \nvdash \neg p$ and $\neg p \nvdash p$ (for atomic $p$).
• a $\vdash$-*conjunction* $\wedge$, satisfying: $\Gamma \vdash \psi \wedge \phi$ iff $\Gamma \vdash \psi$ and $\Gamma \vdash \phi$.
• a $\vdash$-*disjunction* $\vee$, satisfying: $\Gamma, \phi \vee \psi \vdash \sigma$ iff $\Gamma, \phi \vdash \sigma$ and $\Gamma, \psi \vdash \sigma$.
• a $\vdash$-*implication* $\supset$, satisfying: $\Gamma, \phi \vdash \psi$ iff $\Gamma \vdash \phi \supset \psi$.
• a $\vdash$-*falsity* $\mathsf{F}$, satisfying: $\mathsf{F} \vdash \psi$ for every formula $\psi$.

We abbreviate $\{\neg\gamma \mid \gamma \in \Gamma\}$ by $\neg\Gamma$, and when $\Gamma$ is finite we denote by $\bigwedge\Gamma$ (respectively, by $\bigvee\Gamma$), the conjunction (respectively, the disjunction) of all the formulas in $\Gamma$. We shall say that $\Gamma$ is $\vdash$-*tautological* if $\vdash \Gamma$, and that $\Gamma$ is $\vdash$-*consistent* if $\Gamma \nvdash \mathsf{F}$ (otherwise $\Gamma$ is $\vdash$-inconsistent).

**Definition 2** A logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is *explosive*, if for every $\mathscr{L}$-formula $\psi$ the set $\{\psi, \neg\psi\}$ is $\vdash$-inconsistent.[3] We say that $\mathfrak{L}$ is *contrapositive*, if for every $\Gamma$ and $\psi$ it holds that $\Gamma \vdash \neg\psi$ iff either $\psi = \mathsf{F}$, or for every $\phi \in \Gamma$ we have that $\Gamma \setminus \{\phi\}, \psi \vdash \neg\phi$.

**Note 1** Classical logic ($\mathsf{CL}$), intuitionistic logic, and standard modal logics, are all examples of well-known formalisms which are both explosive and contrapositive. A useful property of an explosive logic $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is that if $S \vdash \psi$ and $S \vdash \neg\psi$, then $S \vdash \phi$.

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[3] That is, $\psi, \neg\psi \vdash \mathsf{F}$. Since $\mathsf{F} \vdash \phi$, by transitivity $\psi, \neg\psi \vdash \phi$. Thus, in explosive logics every formula follows from complementary assumptions.

Next, we define assumption-based argumentation frameworks (ABFs). The following definition generalizes the one from [7].

**Definition 3** An *assumption-based (argumentation) framework* is a tuple $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where:

- $\mathfrak{L} = \langle \mathscr{L}, \vdash \rangle$ is a propositional logic.
- $\Gamma$ (the *strict assumptions*) and $Ab$ (the *candidate/defeasible assumptions*) are distinct (countable) sets of $\mathscr{L}$-formulas, where the former is $\vdash$-consistent and the latter is nonempty.
- $\sim : Ab \to \wp(\mathscr{L})$ is a *contrariness operator*, assigning a finite set of $\mathscr{L}$-formulas to every defeasible assumption in $Ab$, such that for every consistent and non-tautological $\psi \in Ab \setminus \{\mathsf{F}\}$, it holds that $\psi \not\vdash \bigwedge \sim \psi$ and $\bigwedge \sim \psi \not\vdash \psi$.

**Note 2** Unlike the setting of [7], an ABF may be based on *any* Tarskian logic $\mathfrak{L}$. Also, the strict as well as the defeasible assumptions may not be just atomic formulas. We note that the contrariness operator is not a connective of $\mathscr{L}$, as it is restricted to the candidate assumptions.

Defeasible assertions in an ABF may be attacked in the presence of contrary defeasible information, as defined below.

**Definition 4** Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an ABF, $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$. We say that $\Delta$ *attacks* $\psi$ iff $\Gamma, \Delta \vdash \phi$ for some $\phi \in \sim \psi$. Accordingly, $\Delta$ attacks $\Theta$ if $\Delta$ attacks some $\psi \in \Theta$.

**Example 1** Let $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \emptyset$, and $Ab = \{p, \neg p, q\}$. A corresponding attack diagram is shown in Figure 1.[4]

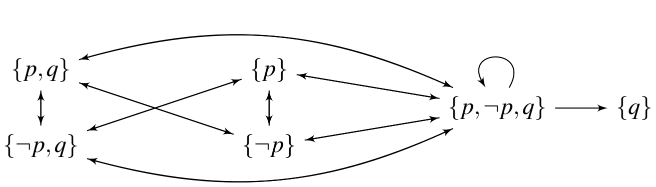


**Figure 1.** An attack diagram for Example 1

Since in classical logic inconsistent premises entail *any* conclusion, the classically inconsistent set $\{p, \neg p, q\}$ attacks all the other sets in the diagram (E.g., $\{p, \neg p, q\}$ attacks $\{q\}$, since $p, \neg p, q \vdash \neg q$).

Definition 4 gives rise to the following adaptation to ABFs of the usual Dung-style semantics [17] for abstract argumentation.

**Definition 5** ([7]) Let $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ be an assumption-based framework, and let $\Delta$ be a set of assumptions. Below, maximum and minimum are taken with respect to set inclusion. We say that:

- $\Delta$ is *closed* (in $\mathbf{ABF}$) if $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$.
- $\Delta$ is *conflict-free* (in $\mathbf{ABF}$) iff there is no $\Delta' \subseteq \Delta$ that attacks some $\psi \in \Delta$.
- $\Delta$ is *naive* (in $\mathbf{ABF}$) iff it is closed and maximally conflict-free.
- $\Delta$ *defends* (in $\mathbf{ABF}$) a set $\Delta' \subseteq Ab$ iff for every closed set $\Theta$ that attacks $\Delta'$ there is $\Delta'' \subseteq \Delta$ that attacks $\Theta$.
- $\Delta$ is *admissible* (in $\mathbf{ABF}$) iff it is closed, conflict-free, and defends every $\Delta' \subseteq \Delta$.
- $\Delta$ is *complete* (in $\mathbf{ABF}$) iff it is admissible and contains every $\Delta' \subseteq Ab$ that it defends.
- $\Delta$ is *well-founded* (in $\mathbf{ABF}$) iff $\Delta = \bigcap \{\Theta \subseteq Ab \mid \Theta \text{ is complete}\}$.[5]
- $\Delta$ is *grounded* (in $\mathbf{ABF}$) iff it is minimally complete.
- $\Delta$ is *preferred* (in $\mathbf{ABF}$) iff it is maximally admissible.
- $\Delta$ is *stable* (in $\mathbf{ABF}$) iff it is closed, conflict-free, and attacks every $\psi \in Ab \setminus \Delta$.

The set of the complete (respectively, the naive, grounded, well-founded, preferred, stable) extensions of $\mathbf{ABF}$ is denoted $\mathsf{Cmp}(\mathbf{ABF})$ (respectively, $\mathsf{Naive}(\mathbf{ABF})$, $\mathsf{Grd}(\mathbf{ABF})$, $\mathsf{WF}(\mathbf{ABF})$, $\mathsf{Prf}(\mathbf{ABF})$, $\mathsf{Stb}(\mathbf{ABF})$). We shall denote by $\mathsf{Sem}(\mathbf{ABF})$ any of the above-mentioned sets. The entailment relations that are induced from an ABF (with respect to a certain semantics) are defined as follows:

**Definition 6** Given an assumption-based framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{WF}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, we denote:[6]

- $\mathbf{ABF} \mid\!\sim^{\cap}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *every* $\Delta \in \mathsf{Sem}(\mathbf{ABF})$.
- $\mathbf{ABF} \mid\!\sim^{\cup}_{\mathsf{Sem}} \psi$ iff $\Gamma, \Delta \vdash \psi$ for *some* $\Delta \in \mathsf{Sem}(\mathbf{ABF})$.

**Example 2** Consider again Example 1 (see also Figure 1). Here, $\mathsf{Naive}(\mathbf{ABF}) = \mathsf{Prf}(\mathbf{ABF}) = \mathsf{Stb}(\mathbf{ABF}) = \{\{p,q\}, \{\neg p, q\}\}$, thus $\mathbf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} q$ for every $* \in \{\cup, \cap\}$ and $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$. Also, $\mathsf{Grd}(\mathbf{ABF}) = \mathsf{WF}(\mathbf{ABF}) = \{\emptyset\}$, thus for $* \in \{\cup, \cap\}$ and $\mathsf{Sem} \in \{\mathsf{Grd}, \mathsf{WF}\}$ we have that $\mathbf{ABF} \mid\!\sim^{*}_{\mathsf{Sem}} \psi$ only if $\psi$ is a $\mathsf{CL}$-tautology.

In the rest of the paper we shall concentrate on the following common family of ABFs (see [22, 23] for a justification of this choice):

**Definition 7** A *simple contrapositive* ABF is an assumption-based framework $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$, where $\mathfrak{L}$ is an explosive and contrapositive logic, and $\sim \psi = \{\neg \psi\}$.

## 3 PRIORITIZED SETTINGS AND pABFS

Sometimes it is useful to extend ABFs with a numeric information for representing preferences among assumptions. This may be done as follows (where lower numbers represent higher preferences):

- An *allocation function* is a total function $g : Ab \to \mathbb{N}$ on the set $Ab$ of candidate assumptions.
- A *numeric aggregation function* $f$ is a total function that maps multisets of non-negative natural numbers into a non-negative real number, such that $\forall x \in \mathbb{N}\ f(\{x\}) = x$. We also assume that an aggregation function is $\subseteq$-coherent in its values, namely, it is either non-decreasing with respect to the subset relation ($f(\Delta') \leq f(\Delta)$ whenever $\Delta' \subseteq \Delta$) or non-increasing with respect to the subset relation ($f(\Delta') \geq f(\Delta)$ whenever $\Delta' \subseteq \Delta$).
- A pair $\mathscr{P} = \langle g, f \rangle$ where $g$ is an allocation function and $f$ is a numerical aggregation function is called a *prioritized setting*.

An allocation function makes preferences among the defeasible information. The sets $Ab_i = \{\psi \in Ab \mid g(\psi) = i\}$ form a partition of $Ab$, which in turn may be viewed as a stratified set. This is sometimes denoted by $Ab = Ab_1 \oplus \ldots \oplus Ab_n$. Aggregation functions are then used for aggregating the preferences. The maximum, minimum, and the summation functions are common aggregation functions.

[4] For reasons that will become apparent in the sequel we include in the diagram only *closed sets* (i.e., only subsets $\Delta \subseteq Ab$ such that $\Delta = Ab \cap Cn_{\vdash}(\Gamma \cup \Delta)$ (see Definition 5). Thus, the set $\{p, \neg p\}$ is omitted from the diagram.

[5] Clearly, the well-founded extension of an ABF is unique.

[6] Unlike standard consequence relations (Definition 1), which are relations between sets of formulas and formulas, the entailments here are relations between ABFs and formulas. This will not cause any confusion.

To assure some desirable properties of our setting, we require that the range of the allocation function should be linearly ordered (while in other frameworks, like $\mathrm{ABA}^+$ [14], any preorder is permitted). Yet, the aggregation of the allocations in our case is more general than that of $\mathrm{ABA}^+$, for instance, which allows only comparisons by max-values (called there *weakest link*).

To ease the notations we will sometimes write $f(g(\Delta))$ instead of $f(g(\psi_1),\ldots,g(\psi_n))$ (where $\Delta = \{\psi_1,\ldots,\psi_n\}$). Also, we shall sometimes write $\Delta_1 \preceq_{\mathscr{P}} \Delta_2$, or just $\Delta_1 \preceq_f \Delta_2$ when $g$ is arbitrary, to denote that $f(g(\Delta_1)) \leq f(g(\Delta_2))$. This intuitively indicates that $\Delta_1$ is at least as preferred as $\Delta_2$.

Next, we consider some useful properties of preference settings.

**Definition 8** A prioritized setting $\mathscr{P} = \langle g, f\rangle$ is called:

- *reversible*, if for every nonempty set $\Delta$ and a formula $\phi$, it holds that if $\Delta \succeq_{\mathscr{P}} \phi$, there is a $\delta \in \Delta$ such that $\Delta \cup \{\phi\} \setminus \delta \preceq_{\mathscr{P}} \delta$.
- max-*upper-bounded* (or simply max-*bounded*), if for every set $\Delta$ of formulas, $f(g(\Delta)) \leq \max_{\delta\in\Delta}(f(g(\delta)))$,
- max-*lower-bounded*, if for every set $\Delta$ of formulas, $f(g(\Delta)) \geq \max_{\delta\in\Delta}(f(g(\delta)))$.

**Proposition 1** *A max-bounded prioritized setting is reversible.*[7]

**Example 3** It is easy to see that for every allocation function $g$ the prioritized settings $\mathsf{Min} = \langle g, \min\rangle$ and $\mathsf{Max} = \langle g, \max\rangle$ are max-bounded. Moreover, every prioritized setting with a non-increasing aggregation function is max-bounded. By Proposition 1, these preference settings are also reversible. Also, the prioritized settings $\mathsf{Max} = \langle g, \max\rangle$ and $\mathsf{Sum} = \langle g, \Sigma\rangle$ are max-lower-bounded. Moreover, every prioritized setting with a non-decreasing aggregation function is max-lower-bounded. For an example of a prioritized setting that is max-lower-bounded and reversible yet not max-upper-bounded, consider e.g. $\mathsf{Max}^+ = \langle g, \max^+\rangle$, where $\max^+(\{x\}) = x$ and $\max^+(\Delta) = \max(\Delta)+1$ if $\Delta$ is not singleton.

Prioritized ABFs are defined now as follows:

**Definition 9** A *prioritized assumption-based framework* (pABF, for short) is a pair $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P}\rangle$, where $\mathbf{ABF}$ is an assumption-based argumentation framework and $\mathscr{P}$ is a prioritized setting.

pABFs are similar to (non-prioritized) ABFs, except that the priorities are taken into account when defining attacks (cf. Definition 4). A prioritized ABF is called reversible (respectively, max-bounded, max-lower-bounded), if so is its prioritized setting.

**Definition 10** Let $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P}\rangle$ be a prioritized ABF, where $\mathscr{P} = \langle g, f\rangle$. Let also $\Delta, \Theta \subseteq Ab$, and $\psi \in Ab$. Suppose that $\Delta$ attacks $\psi$ (in the sense of Definition 4). The *$\mathscr{P}$-attacking value* of $\Delta$ on $\psi$ is

$$\mathsf{val}_{f,g}(\Delta,\psi) = \min\{f(g(\Delta')) \mid \Delta' \text{ is a minimal subset of } \Delta \text{ that attacks } \psi\}.$$

We say that $\Delta$ *p-attacks* $\psi$ iff $\Delta$ attacks $\psi$ (in the sense of Definition 4), and $\mathsf{val}_{f,g}(\Delta,\psi) \leq f(g(\psi))$. Again, we say that $\Delta$ p-attacks $\Theta$ if $\Delta$ p-attacks some $\psi \in \Theta$.

**Note 3** A simplified version of the attacks in Definition 10 could be the following: a set $\Delta = \{\psi_1,\ldots,\psi_n\} \subseteq Ab$ p-attacks $\psi$ iff $\Delta$ attacks $\psi$ (in the sense of Definition 4) and $f(g(\Delta)) \leq f(g(\psi))$. However, this alternative definition of p-attacks has some unintuitive consequences. To see this, consider again the ABF of Example 1 with the allocation function $g(p) = 1$, $g(\neg p) = 2$, $g(q) = 3$, and the aggregation function $f = \max$. Note that according to the alternative definition of p-attacks given in this note, $\Theta = \{p, q\}$ does not max-attack $\Delta = \{\neg p\}$, since it has a formula ($q$) which is of a lower preference than the attacked formula in $\Delta$. This seems to be counter-intuitive, since the attack of $\Theta$ on $\Delta$ is 'blocked' by a formula which is 'irrelevant' to the attack.

In contrast to this, $\Theta$ does max-attack $\Delta$ according to Definition 10, as expected, since its attacking value on $\Delta$ is 1, which is smaller than the preference value (2) of the attacked formula in $\Delta$. Indeed, the attacks in Definition 10 take into consideration only the preference values of the formulas that are relevant to the attack. A major advantage of this is considered in Lemma 1 below.

**Lemma 1** *If $\Delta$ p-attacks $\Theta$, so does any superset of $\Delta$.*

**Proof** Suppose that $\Delta$ p-attacks $\Theta$. Then there is a $\psi \in \Theta$ such that $\Delta$ attacks $\psi$ and $\mathsf{val}_{f,g}(\Delta,\psi) \leq f(g(\psi))$. Let now $\Delta \subseteq \Delta'$. By the monotonicity of $\vdash$, $\Delta'$ also attacks $\psi$. Moreover, by Definition 10, $\mathsf{val}_{f,g}(\Delta',\psi) \leq \mathsf{val}_{f,g}(\Delta,\psi)$, and so $\mathsf{val}_{f,g}(\Delta',\psi) \leq f(g(\psi))$. It follows that $\Delta'$ p-attacks $\psi$ and so $\Delta'$ also p-attacks $\Theta$. $\square$

**Lemma 2** *For every $\Delta \subseteq Ab$ and $\psi \in Ab$, $\mathsf{val}_{f,g}(\Delta,\psi) \leq f(g(\Delta))$.*

The semantic notions in Definition 5 are carried on to the prioritized case by replacing attacks with p-attacks. We continue to denote by $\mathsf{Sem}(\mathbf{pABF})$ the extensions of $\mathbf{pABF}$ according to $\mathsf{Sem} \in \{\mathsf{Cmp}, \mathsf{Naive}, \mathsf{Grd}, \mathsf{Prf}, \mathsf{Stb}\}$, and define the entailments $\mid\!\sim^{\cap}_{\mathsf{Sem}}$ and $\mid\!\sim^{\cup}_{\mathsf{Sem}}$ just as in Definition 6, where $\mathbf{pABF}$ replaces $\mathbf{ABF}$.

**Example 4** Consider again the ABF of Example 1 with the allocations $g(p) = 1$, $g(\neg p) = 2$, $g(q) = 3$, and aggregation by $f = \max$. An attack diagram of the prioritized ABF is shown in Figure 2.

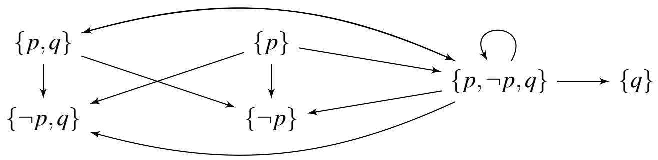


**Figure 2.** An attack diagram for Example 4

Here $\mathsf{Cmp}(\mathbf{pABF}) = \mathsf{Grd}(\mathbf{pABF}) = \mathsf{WF}(\mathbf{pABF}) = \mathsf{Prf}(\mathbf{pABF}) = \mathsf{Stb}(\mathbf{pABF}) = \{\{p,q\}\}$,[8] thus $\mathbf{pABF} \mid\!\sim^{*}_{\mathsf{Sem}} p$ and $\mathbf{pABF} \mid\!\sim^{*}_{\mathsf{Sem}} q$ for every semantics $\mathsf{Sem}$ and every $* \in \{\cup, \cap\}$. Note that in case that the allocation value of $q$ is smaller than those of $p$ and $\neg p$, the set $\{p, \neg p, q\}$ does not attack the sets $\{q\}$ and $\{p,q\}$, in which case the set $\{q\}$ also belongs to $\mathsf{Cmp}(\mathbf{pABF})$. In this case $\mathsf{Grd}(\mathbf{pABF}) = \mathsf{WF}(\mathbf{pABF}) = \{\{q\}\}$, while $\mathsf{Prf}(\mathbf{pABF}) = \mathsf{Stb}(\mathbf{pABF}) = \{\{p,q\}\}$.

In the next sections we consider some properties of prioritized ABFs. In what follows we continue to assume that $\mathbf{ABF}$ in the prioritized framework $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P}\rangle$, is simple contrapositive.[9]

[7] Due to lack of space some proofs are reduced or omitted altogether.

[8] Note that $\{p\}$ is not complete (thus it does not belong to any of the above-mentioned sets), since it defends $q$, which is not a member of this set.

[9] As shown in [22], for simple contrapositive ABFs the closure requirement in Definition 5 is redundant. We shall therefore disregard it in what follows (see also Section 4.3 below).

# 4 ARGUMENTATION-THEORETIC PROPERTIES

First, we check some general properties of the extensions of prioritized ABFs: their inter-relations (Section 4.1) and main characteristics in terms of consistency (Section 4.2) and closure (Section 4.3).

## 4.1 Relations between the Extensions

### 4.1.1 Naive, Preferred and Stable Semantics

In [22, 23] we have shown that in non-prioritized simple contrapositive ABFs the set of naive, preferred and stable extension coincide (that is, if **ABF** is a simple contrapositive ABF without priorities then $\mathsf{Naive}(\mathbf{ABF}) = \mathsf{Prf}(\mathbf{ABF}) = \mathsf{Stb}(\mathbf{ABF})$). As the next examples show, when priorities are involved, this is no longer the case: Example 5 shows a situation in which the naive semantics is different than the preferred and the stable semantics, and Example 6 illustrates a case where the preferred semantics is different than the stable semantics.

**Example 5** In Example 4, the set $\{\neg p, q\}$ is maximally conflict-free (thus naive), but it is not even admissible (not to mention preferred or stable), since it does not defend (any of) its elements.

**Example 6** Consider a prioritized ABF with $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \{\neg(p \wedge q \wedge s)\}$ and $Ab = \{p, q, s, \mathsf{F}\}$, where $g(\psi) = 1$ for every $\psi \in Ab$ and $f = \Sigma$. We define $\Sigma\{\emptyset\} = 0$, thus for every $\psi$ it holds that $\mathsf{val}_{f,g}(\emptyset, \psi) = 0$. Note that $\emptyset$ p-attacks every $\Theta \subseteq Ab$ such that $\mathsf{F} \in \Theta$, and no other subset of $Ab$ attacks another subset of $Ab$. This means that $\{p, q, s\}$ is the only maximally admissible subset of $Ab$, nevertheless it is not closed, since $\mathsf{F} \notin \{p, q, s\}$. If we restrict our attention to maximally admissible *closed* sets, there are three such sets: $\{p, q\}$, $\{s, q\}$ and $\{p, s\}$. However, these sets are not complete since they do not include an unattacked assumption. For example, $\{p, q\}$ does not include $s$, even though $s$ is unattacked.

For similar reasons, neither of these sets are stable, since they do not attack the unattacked element in $Ab$ that is not included in them (e.g., $\{p, q\}$ does not p-attack $s$). Thus, unless further assumptions are posed on the aggregation function (see below), a maximally admissible set might not be complete, preferred extensions might not be stable, and stable extensions might not exist.

Next, we show that, despite of the last example, the sets of preferred and stable extensions still coincide in many prioritized simple contrapositive ABFs (see also Note 5 below).

**Proposition 2** *Let* **pABF** *be a* max*-bounded prioritized ABF and let $\Delta$ be a conflict-free set in Ab. Then $\Delta$ is maximally admissible iff it p-attacks any $A \in Ab \setminus \Delta$.*

**Outline of proof** One direction is clear: if a conflict-free $\Delta$ $p$-attacks any $A \in Ab \setminus \Delta$ it must be maximally admissible. Let now $\Delta$ be a maximally admissible set and suppose towards a contradiction that there is some $\psi \in Ab \setminus \Delta$ s.t. $\Delta$ does not $p$-attack $\psi$. Let $\{\psi_1, \ldots, \psi_n\} = Ab \setminus \Delta$ s.t. $i < j$ if $g(\psi_i) < g(\psi_j)$ (i.e., $\psi_1$ is among the strongest assumptions that are not in $\Delta$, $\psi_2$ has the same properties but has weaker or the same strength as $\psi_1$, and so on). Let $\Delta^\star = \bigcup_{i \geq 0} \Delta_i$, where $\Delta_0 = \Delta$ and for every $0 \leq i \leq n-1$,

$$\Delta_{i+1} = \begin{cases} \Delta_i \cup \{\psi_{i+1}\} & \text{if } \Gamma, \Delta_i \not\vdash \neg\psi_{i+1}, \\ \Delta_i & \text{otherwise.} \end{cases}$$

It can be show that $\Delta^\star$ is an admissible set, and $\Delta \subsetneq \Delta^\star$. This contradicts the maximal admissibility of $\Delta$. ☐

### 4.1.2 Well -Founded and Grounded Semantics

As the next example shows, the grounded semantics is not always unique (unlike, e.g., in abstract argumentation frameworks), and so it does not necessarily coincide with the well-founded semantics (which is unique by its definition).

**Example 7** Consider a pABF with $\mathfrak{L} = \mathsf{CL}$, $\Gamma = \{p \wedge q \supset \neg s, r \supset s, s \supset r\}$, $Ab = \{s, p, q, r\}$, $g(s) = 1$, $g(p) = g(q) = 2$, $g(r) = 3$ and $f = \max$. The p-attack diagram of this pABF is shown in Figure 3.

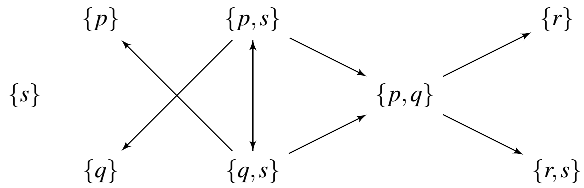


**Figure 3.** An attack diagram for Example 7

Here there is no unique minimal complete extension: $\{s\}$ is not attacked but it is not closed since $\Gamma, \{s\} \vdash r$. Also, $\{r, s\}$ does not defend itself from $\{p, q\}$. This pABF has two minimal complete extensions, $\{p, s, r\}$ and $\{q, s, r\}$, which are also preferred and stable.

**Note 4** In [22, 23] we have shown that in the non-prioritized case, when $\mathsf{F} \in Ab$, the grounded and the well-founded semantics coincide and are unique. As Example 6 shows, in prioritized ABFs this is no longer the case. (Example 6 also shows that in general, Dung fundamental lemma, stating that if $\Delta$ is admissible and defends $\psi$ then $\Delta \cup \{\psi\}$ is also admissible, does not hold for prioritized ABFs).

## 4.2 Consistency of Extensions

Extensions of reversible prioritized ABFs are consistent:

**Proposition 3** *Let* $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ *be a reversible prioritized ABF, Then* **pABF** *satisfies the consistency postulate in [11]: There is no conflict-free set $\Delta \subseteq Ab$ such that $\Gamma, \Delta \vdash \neg\psi$ for some $\psi \in \Delta$.*

**Proof** Suppose that $\Delta \subseteq Ab$ is conflict-free in **pABF** and that $\Gamma \cup \Delta' \vdash \neg\phi$ for some $\{\phi\} \cup \Delta' \subseteq \Delta$. This means that $\mathsf{val}_{f,g}(\Delta', \phi) > f(g(\phi))$ (otherwise, $\mathsf{val}_{f,g}(\Delta', \phi) \leqslant f(g(\phi))$ and so $\Delta'$ p-attacks $\phi$, thus $\Delta$ cannot be conflict-free). Since $\mathscr{P}$ is reversible, there is a $\psi \in \Delta'$ s.t. $f(g(\Delta' \cup \{\phi\} \setminus \{\psi\})) \leqslant f(g(\psi))$, and since $\mathsf{val}_{f,g}(\Delta' \cup \{\phi\} \setminus \{\psi\}, \psi) \leqslant f(g(\Delta' \cup \{\phi\} \setminus \{\psi\}))$ (Lemma 2), we have that $\mathsf{val}_{f,g}(\Delta' \cup \{\phi\} \setminus \{\psi\}, \psi) \leqslant f(g(\psi))$. Now, since $\mathfrak{L}$ is contrapositive, $\Gamma' \cup \Delta' \cup \{\phi\} \setminus \{\psi\} \vdash \neg\psi$. Consequently, $\Delta' \cup \{\phi\} \setminus \{\psi\} \subset \Delta$ p-attacks $\psi \in \Delta$, contradicting the assumption that $\Delta$ is conflict-free. ☐

The next example shows that the reversibility requirement from the aggregation function in Proposition 3 is indeed necessary.

**Example 8** Consider a variation of Example 6 where $\mathsf{F}$ is removed from $Ab$, namely: $\Gamma = \{\neg(p \wedge q \wedge s)\}$, $Ab = \{p, q, s\}$, $g(\delta) = 1$ for every $\delta \in Ab$, and $f = \Sigma$. Clearly, $\mathscr{P} = \langle g, f \rangle$ is not reversible (for instance, $\{p, q\} \succeq_{\mathscr{P}} s$, yet neither $\{p, s\} \preceq_{\mathscr{P}} q$ nor $\{q, s\} \preceq_{\mathscr{P}} p$). Also, similar considerations as in Example 6 show that there is no p-attack in this example. Thus, there is one maximally admissible set: $Ab$. However, this set is not consistent. Thus, consistency can be violated when $f$ is not reversible.

## 4.3 Closure of Extensions

Next, we consider the closure requirement from extensions (see Definition 5). As Example 8 shows, this requirement is, in general *not* redundant in prioritized ABFs. However, as we show below, under the assumption that the aggregation function is reversible, the closure requirement may be lifted. This result generalizes similar results shown in [22, 23] for simple contrapositive ABFs without priorities.

**Proposition 4** *Let* $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF.*
• *If* $\mathbf{pABF}$ *is reversible, the closure requirement is redundant in the definition of stable extensions (Definition 5): Any conflict-free* $\Delta \subseteq Ab$ *that p-attacks every* $A \in Ab \setminus \Delta$ *is closed.*
• *If* $\mathbf{pABF}$ *is* max*-bounded, the closure requirement is redundant in the definition of preferred extensions (Definition 5): Any* $\Delta \subseteq Ab$ *that is conflict free and maximally admissible is closed.*

**Proof** For the first item, suppose that $\Delta$ p-attacks every $\psi \in Ab \setminus \Delta$, yet $\Gamma, \Delta \vdash \phi$ for some $\phi \in Ab \setminus \Delta$. Since $\Delta$ p-attacks $\phi$, it holds that $\Gamma, \Delta \vdash \neg\phi$. Thus, by Note 1, we have that $\Gamma, \Delta \vdash \mathsf{F}$. By Proposition 3, this is a contradiction to the assumption that $\Delta \subseteq Ab$ is conflict-free.

For the second item, suppose that $\Delta \subseteq Ab$ is conflict free and maximally admissible. By proposition 2, $\Delta$ attacks every $A \in Ab \setminus \Delta$. By the first item of the proposition, this means that $\Delta$ is closed. □

**Note 5** By Propositions 4 and 1, Proposition 2 may be restated as follows: The stable extensions and the preferred extensions of a max-bounded prioritized ABF coincide.

# 5 REPRESENTATION OF PREFERRED MAXIMALLY CONSISTENT SUBSETS

The relation between prioritized argumentation frameworks and reasoning with prioritized maximally consistent subsets of the premises has been investigated in several different contexts (see, e.g. [4, 5, 28] and [21, Chapter 7]). In this section we show that under certain assumptions, prioritized assumption-based argumentation can represent Brewka's preferred sub-theories [8], as defined next.

**Definition 11** Let $Ab = Ab_1 \oplus \ldots \oplus Ab_n$ (recall Section 3), and let $\Delta, \Theta \subseteq Ab$. We say that $\Delta$ is *preferred* over $\Theta$ (with respect to $\mathscr{P}$), denoted $\Delta \sqsubset_{\mathscr{P}} \Theta$ (or just $\Delta \sqsubset \Theta$ when $\mathscr{P}$ is known or arbitrary), iff there is an $1 \leq i \leq n$ such that $Ab_j \cap \Delta = Ab_j \cap \Theta$ for every $1 \leq j < i$, and $Ab_i \cap \Delta \supsetneq Ab_i \cap \Theta$.

Thus, $\Delta$ is preferred over $\Theta$ when both sets have the same $i-1$ stratifications with the $g$-most preferred formulas, and the $i$-th stratification of $\Delta$ properly contains that of $\Theta$. This is a kind of lexicographic preference in term of the $g$-values. In turn, this preference can be posed on the maximally consistent subsets of $Ab$:

**Definition 12** Let $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ be a prioritized ABF

- $\Delta \subseteq Ab$ is a *maximally consistent set* (MCS) in $\mathbf{ABF}$, if (a) $\Gamma, \Delta \not\vdash \mathsf{F}$ and (b) $\Gamma, \Delta' \vdash \mathsf{F}$ for every $\Delta \subsetneq \Delta' \subseteq Ab$.[10] The set of the maximally consistent sets in $\mathbf{ABF}$ is denoted $\mathsf{MCS}(\mathbf{ABF})$.
- $\Delta \subseteq Ab$ is a *preferred maximally consistent set* (pMCS) in $\mathbf{pABF}$, if $\Delta \in \mathsf{MCS}(\mathbf{ABF})$ and there is no $\Theta \in \mathsf{MCS}(\mathbf{ABF})$ such that $\Theta \sqsubset \Delta$. The set of the prioritized maximally consistent sets in $\mathbf{ABF}$ is denoted $\mathsf{MCS}_{\sqsubset}(\mathbf{pABF})$.

[10] In what follows, (a) is called the consistency condition and (b) is the maximality condition.

Relations between prioritized argumentation frameworks and reasoning with prioritized maximally consistent subsets of the premises are shown in the next two lemmas:

**Lemma 3** *Let* $\mathbf{pABF}$ *be a* max*-lower-bounded and reversible pABF, and let* $\Delta$ *be a stable extension of* $\mathbf{pABF}$. *Then* $\Delta \in \mathsf{MCS}_{\sqsubset}(\mathbf{pABF})$

**Lemma 4** *Let* $\mathbf{pABF}$ *be a* max*-bounded pABF, and let* $\Delta \in \mathsf{MCS}_{\sqsubset}(\mathbf{pABF})$. *Then* $\Delta$ *is a stable extension of* $\mathbf{pABF}$.

When the aggregation is by the maximum function, the two lemmas above give a full characterization of the preferred and the stable semantics in terms of preferred maximally consistent sets.

**Proposition 5** *Let* $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ *be a prioritized ABF in which* $\mathscr{P} = \langle g, \max \rangle$ *for some allocation function g. Then* $\mathsf{Prf}(\mathbf{pABF}) = \mathsf{Stb}(\mathbf{pABF}) = \mathsf{MCS}_{\sqsubset}(\mathbf{pABF})$.

**Proof** Since $\mathscr{P} = \langle g, \max \rangle$ is max-bounded, by Proposition 2, $\mathsf{Prf}(\mathbf{pABF}) = \mathsf{Stb}(\mathbf{pABF})$. Since it is also max-lower-bounded, by Lemmas 3 and 4, $\mathsf{Stb}(\mathbf{pABF}) = \mathsf{MCS}_{\sqsubset}(\mathbf{pABF})$. Altogether, we get the proposition. □

# 6 PREFERENCE HANDLING PROPERTIES

## 6.1 Preference-Related Postulates

In this section we consider a series of postulates that are concerned with the handling of preferences in prioritized ABFs. In particular, we show how the properties of the preference setting affect the properties of the resulting prioritized ABF (under the preferred and stable semantics).

### *6.1.1 Degenerated Preferences*

We start with two postulates that relate extensions of prioritized ABFs and extensions of their non-prioritized fragments. The first one (introduced in [1, 10]) refers to situations in which the preference setting is degenerated.

**Empty Preferences (for** $\mathsf{Sem}$**):** If $\mathscr{P}$ is a degenerated preference setting (i.e., if $g$ is a uniform allocation function), then $\mathsf{Sem}(\mathbf{pABF}) = \mathsf{Sem}(\mathbf{ABF})$.

Empty preferences is satisfied by every prioritized ABF in which the aggregation function is invariant to multiple occurrences, namely: if $S$ is a set and $S'$ is a multiset with the same elements as $S$ (so $S'$ may have multiple instances of the same element in $S$ but not new elements not in $S$), then $f(S) = f(S')$. This is the case, e.g., when $f = \min$ or $f = \max$.

**Proposition 6** *Let* $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ *be a pABF with a preference setting* $\mathscr{P} = \langle g, f \rangle$. *If f is invariant to multiple occurrences then* $\mathbf{pABF}$ *satisfies the empty preferences postulate for every* $\mathsf{Sem}$.

### *6.1.2 Preferences as Criteria for Selecting Extensions*

The next property also relates the extensions of a prioritized ABF to the extension of its ABF. This postulate is taken from [29]. Intuitively, it may be understood by the fact that priorities allow to select the 'best' extensions according to some preference criteria, in the sense that any extension of a pABF is an extension of the corresponding ABF.[11]

[11] In a way, this resembles what is called in [2] *refining argumentation frameworks by preferences*, where priorities are used for selecting extensions rather than for defining attacks.

**Extensions Selection (for** $\mathsf{Sem}$**):** If $\mathscr{E} \in \mathsf{Sem}(\mathbf{pABF})$ then $\mathscr{E} \in \mathsf{Sem}(\mathbf{ABF})$.

**Proposition 7** *Let* $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ *be a reversing pABF. Then* $\mathbf{pABF}$ *satisfies the extensions selection postulate for every* $\mathsf{Sem} \in \{\mathsf{Naive}, \mathsf{Prf}, \mathsf{Stb}\}$.

### 6.1.3 Conflict Preservation

The next postulate is considered, e.g., in [1, 3, 26].

**Conflict Preservation (for** $\mathsf{Sem}$**):** If $\mathscr{E} \in \mathsf{Sem}(\mathbf{pABF})$ and $\Delta$ p-attacks $\Theta$, either $\Delta \notin \mathscr{E}$ or $\Theta \notin \mathscr{E}$.

Conflict preservation follows in our case from the fact that every $\mathscr{E} \in \mathsf{Sem}(\mathbf{pABF})$ is conflict-free. This property is not so obvious in other formalisms in which attacks are sometimes discarded due to preference over arguments (see [12] for some examples).

### 6.1.4 Inclusion of the Most Preferred Assumptions

The next principle is concerned with the inclusion in extensions of the 'strongest' assumptions (see [3, 12]).

**Preferred Assumptions (for** $\mathsf{Sem}$**):** $\mathsf{Min}_g(Ab) = \{\psi \in Ab \mid \neg\exists \phi \in Ab$ such that $g(\phi) < g(\psi)\} \subseteq \mathscr{E}$ for every $\mathscr{E} \in \mathsf{Sem}(\mathbf{pABF})$.

Clearly, the principle above cannot hold in our setting unless $\mathsf{Min}_g(Ab)$ itself is $\vdash$-consistent (otherwise $\mathscr{E}$ is not conflict free). A sufficient condition for assuring this principle for stable semantics in max-lower-bounded and reversible pABFs is given next:

**Proposition 8** *Let* $\mathbf{pABF}$ *be a* max*-lower-bounded and reversible pABF. If* $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\sqsubset_{\mathscr{P}}}(\mathbf{pABF})$ *then* $\mathbf{pABF}$ *satisfies the principle of preferred arguments for the stable semantics.*

**Proof** Let $\mathscr{E} \in \mathsf{Stb}(\mathbf{pABF})$. By Lemma 3, $\mathscr{E} \in \mathsf{MCS}_{\sqsubset_{\mathscr{P}}}(\mathbf{pABF})$. Since $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\sqsubset_{\mathscr{P}}}(\mathbf{pABF})$, we get $\mathsf{Min}_g(Ab) \subseteq \mathscr{E}$. $\square$

Note that, by Proposition 5, when $\mathscr{P} = \langle g, \max \rangle$, the condition that $\mathsf{Min}_g(Ab) \subseteq \bigcap \mathsf{MCS}_{\sqsubset_{\mathscr{P}}}(\mathbf{pABF})$ is also necessary for assuring the satisfaction of the preferred argument postulate for stable and preferred semantics.

### 6.1.5 Brewka-Eiter Principle

The next postulate in taken from [9]. It says that given $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$, if $\Delta$ and $\Theta$ are two extensions of $\mathbf{ABF}$, each one contains a single element that is not in the other one, and the element in $\Theta \setminus \Delta$ is preferred over the element in $\Delta \setminus \Theta$, then $\Delta$ is not a p-extension of $\mathbf{pABF}$ (see also [12]).

**BE Principle (for** $\mathsf{Sem}$**):** If $\Delta = \Lambda \cup \{\phi\} \in \mathsf{Sem}(\mathbf{ABF})$ and $\Theta = \Lambda \cup \{\psi\} \in \mathsf{Sem}(\mathbf{ABF})$ for some $\phi, \psi \notin \Lambda$, and if $g(\psi) < g(\phi)$, then $\Delta \notin \mathsf{Sem}(\mathbf{pABF})$.

This principle doesn't hold for prioritized ABFs in general, as demonstrated by the following example:

**Example 9** Consider again Example 8 (i.e., where $\Gamma = \{\neg(p \wedge q \wedge s)\}$ and $Ab = \{p, q, s\}$), but this time with $g(p) = 1$, $g(q) = 2$, $g(s) = 3$ and $f = \min$. It can be verified that $\mathsf{Stb}(\mathbf{pABF}) = \{\{p, q\}, \{p, s\}\}$ and $\mathsf{Stb}(\mathbf{ABF}) = \{\{p, q\}, \{p, s\}, \{q, s\}\}$. This constitutes a violation of the BE-principle, since $\{p, q\}, \{p, s\} \in \mathsf{Stb}(\mathbf{ABF})$ and $g(q) < g(s)$, yet $\{p, s\} \in \mathsf{Stb}(\mathbf{pABF})$.

**Proposition 9** *Let* $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ *be a reversing pABF that is max-lower-bounded. Then* $\mathbf{pABF}$ *satisfies the BE-principle for the stable semantics.*

**Proof** Let $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ be as in the proposition. Let $\Delta, \Theta \in \mathsf{Stb}(\mathbf{ABF})$ and $\Lambda \cup \{\phi, \psi\} \subseteq Ab$ s.t. $\phi, \psi \notin \Lambda$ and $\Delta = \Lambda \cup \{\phi\}$ and $\Theta = \Lambda \cup \{\psi\}$ and $g(\psi) < g(\phi)$. Since $\Delta, \Theta \in \mathsf{Stb}(\mathbf{ABF})$, by [22, Theorem 1], $\Delta, \Theta \in \mathsf{MCS}(\mathbf{ABF})$. Also, $\Theta \sqsubset \Delta$ (recall Definition 11), and so $\Delta \notin \mathsf{MCS}_{\sqsubset}(\mathbf{pABF})$. By Lemma 3, $\Delta \notin \mathsf{Stb}(\mathbf{pABF})$. $\square$

**Note 6** Let $\mathbf{pABF}$ be a reversing pABF that is max-lower-bounded. If $\mathbf{pABF}$ is also max-upper-bound (and so necessarily $f = \max$), we have by Proposition 2 that $\mathsf{Prf}(\mathbf{pABF}) = \mathsf{Stb}(\mathbf{pABF})$, and so in this case the BE-principle holds for the preferred semantics as well.

### 6.1.6 Principle of Tolerance

The last postulate that we consider is the following:

**Tolerance (for** $\mathsf{Sem}$**):** If $\mathsf{Sem}(\mathbf{ABF}) \neq \emptyset$ then $\mathsf{Sem}(\mathbf{pABF}) \neq \emptyset$.

The principle of tolerance for complete and preferred semantics is clear from the fact that $\mathbf{pABF}$ is in particular an argumentation framework, and so $\mathsf{Cmp}(\mathbf{pABF})$ and $\mathsf{Prf}(\mathbf{pABF})$ are not empty. This principle for stable semantics holds for max-bounded pABFs by Proposition 2, and for max-lower-bounded and reversible pABF by Lemma 4. (As noted in [12], when the prioritized assumption-based framework $\mathrm{ABA}^+$ is concerned (see [14]), the principle of tolerance does not hold for the stable semantics).

## 6.2 Avoidance of the Drowning Effect

A desirable property of prioritized information systems in general, and pABFs in particular, is that their conclusions shouldn't be altered when assumptions with a lower priority are added. In this section we consider this property in our context.

**Definition 13** Let $\mathbf{pABF}' = \langle \mathbf{ABF}', \mathscr{P}' \rangle$ be a prioritized ABF that is obtained from $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P} \rangle$ by adding to $\mathbf{ABF}$ some defeasible assumptions whose priorities are lower than those in $Ab$, namely:
• $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim \rangle$ and $\mathbf{ABF}' = \langle \mathfrak{L}, \Gamma, Ab \cup Ab', \sim \rangle$ for $Ab' \neq \emptyset$,
• if $\mathscr{P} = \langle g, f \rangle$ then $\mathscr{P}' = \langle g', f \rangle$, where $g'(\psi) > \max\{g(\varphi) \mid \varphi \in Ab\}$ if $\psi \in Ab'$ and $g'(\psi) = g(\psi)$ otherwise.
In this case we say that $\mathbf{pABF}'$ is *an extension of* $\mathbf{pABF}$ *by least-preferred assumptions*.

Below, we use the notations of Definition 13 and assume that $\mathbf{pABF}'$ is an extension of $\mathbf{pABF}$ by least-preferred assumptions (in particular, $Ab$ is extended by lower-prioritized assumptions in $Ab'$).

**Lemma 5** *If* $\Delta$ *p-attacks* $\psi$ *in* $\mathbf{pABF}$ *then* $\Delta$ *p-attacks* $\psi$ *in* $\mathbf{pABF}'$.

**Lemma 6** *If* $f$ *is non-decreasing and* $\Delta$ *p-attacks* $\psi \in Ab$ *in* $\mathbf{pABF}'$ *then* $\Delta \cap Ab$ *p-attacks* $\psi$ *in* $\mathbf{pABF}$.

**Note 7** The requirement in Lemma 6 that $f$ is non-decreasing is indeed necessary. To see this, consider $\mathbf{pABF}$ with $\Gamma = \emptyset$, $Ab = \{p\}$ and $g(p) = 1$, and $\mathbf{pABF}'$ with $\Gamma = \emptyset$, $Ab = \{p\}$, $Ab' = \{\neg p\}$ and $g'(p) = 1$, $g'(\neg p) = 2$, where in both cases $f = \min$. Clearly, $\mathbf{pABF}'$ extended $\mathbf{pABF}$ with the least-preferred assumption $\neg p$. Moreover, $\Delta = \{p, \neg p\}$ p-attacks $p$ in $\mathbf{pABF}'$, but $\Delta \cap Ab = \{p\}$ does not p-attacks $p$ in $\mathbf{pABF}'$, simply because $p \not\vdash \neg p$.

**Corollary 1** *Let $f$ be a non-decreasing aggregation and let $\psi \in Ab$. Then $\Delta$ p-attacks $\psi$ in $\mathbf{pABF}'$ iff $\Delta \cap Ab$ p-attacks $\psi$ in $\mathbf{pABF}$.*

**Proof** One direction is Lemma 6. For the other direction, suppose that $\Delta \cap Ab$ p-attacks $\psi$ in $\mathbf{pABF}$. By Lemma 5, $\Delta \cap Ab$ p-attacks $\psi$ also in $\mathbf{pABF}'$, and by Lemma 1, $\Delta$ p-attacks $\psi$ in $\mathbf{pABF}'$. □

Lemmas 5 and 6 imply that:

**Lemma 7** *Let $f$ be a non-decreasing aggregation function. If $\Delta \in \mathsf{Cmp}(\mathbf{pABF}')$ then $\Delta' = \Delta \cap Ab \in \mathsf{Cmp}(\mathbf{pABF})$.*

The next property assures that conclusions of a pABF are preserved under extensions of the pABF by least-preferred assumptions.

**Definition 14** An aggregation function $f$ (and so every priority setting $\mathscr{P} = \langle g, f\rangle$ that is obtained from it) *avoids the drowning effect* with respect to $\mid\!\sim$, if for every $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P}\rangle$ with $\mathscr{P} = \langle g, f\rangle$, and for every extension $\mathbf{pABF}'$ of $\mathbf{pABF}$ by least-preferred assumptions, $\mathbf{pABF} \mid\!\sim \psi$ implies that $\mathbf{pABF}' \mid\!\sim \psi$ (for every formula $\psi$).

**Proposition 10** *Any non-decreasing aggregation function avoids the drowning effect with respect to $\mid\!\sim^{\cap}_{\mathsf{Cmp}}$.*

**Proof** Let $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P}\rangle$ be a prioritized ABF with $\mathbf{ABF} = \langle \mathfrak{L}, \Gamma, Ab, \sim\rangle$ and $\mathscr{P} = \langle g, f\rangle$ where $f$ is non-decreasing, and let $\mathbf{pABF}'$ be an extension of $\mathbf{pABF}$ by least-preferred assumptions. Suppose for a contradiction that for some formula $\psi$ it holds that $\mathbf{pABF} \mid\!\sim^{\cap}_{\mathsf{Cmp}} \psi$ but $\mathbf{pABF}' \not\mid\!\sim^{\cap}_{\mathsf{Cmp}} \psi$. The latter means that there is some complete extension $\Delta$ of $\mathbf{pABF}'$ for which $\Delta \nvdash_{\mathfrak{L}} \psi$. By the monotonicity of $\vdash_{\mathfrak{L}}$ it holds that $\Delta' \nvdash_{\mathfrak{L}} \psi$ for every $\Delta' \subseteq \Delta$. In particular, $\Delta' \nvdash_{\mathfrak{L}} \psi$ when $\Delta' = \Delta \cap Ab$. But by Lemma 7, $\Delta'$ is a complete extension of $\mathbf{pABF}$, in a contradiction to $\mathbf{pABF} \mid\!\sim^{\cap}_{\mathsf{Cmp}} \psi$. □

**Corollary 2** *Let $\mathbf{pABF} = \langle \mathbf{ABF}, \mathscr{P}\rangle$ be a prioritized ABF. If $\mathscr{P} = \langle g, \max\rangle$ or $\mathscr{P} = \langle g, \Sigma\rangle$, then for every extension $\mathbf{pABF}'$ of $\mathbf{pABF}$ by least-preferred assumptions and for every formula $\psi$, we have that $\mathbf{pABF} \mid\!\sim^{\cap}_{\mathsf{Cmp}} \psi$ implies that $\mathbf{pABF}' \mid\!\sim^{\cap}_{\mathsf{Cmp}} \psi$.*

**Proof** By Proposition 10, since both the maximum function and the summation function are non-decreasing. □

**Note 8** Since $\mid\!\sim^{\cap}_{\mathsf{Cmp}}$ is non-monotonic, extending a prioritized ABF with extra assumptions does not guarantee the preservation of its conclusions. Indeed, consider for instance the prioritized framework $\mathbf{pABF}_1$, based on $\mathsf{CL}$, with $\Gamma = \emptyset$, $Ab = \{\neg p, q\}$, the allocation function $g(\neg p) = 2$, $g(q) = 3$, and the aggregation function $f = \max$. Clearly, $\mathbf{pABF}_1 \mid\!\sim^{\cap}_{\mathsf{Cmp}} \neg p$. Now, let $\mathbf{pABF}_2$ be a pABF that is obtained by adding to $Ab$ of $\mathbf{pABF}_1$ the assumption $p$ with $g(p) = 1$. This is the pABF considered in Example 4 (see also Figure 2), and as it is shown there, $\mathbf{pABF}_2 \not\mid\!\sim^{\cap}_{\mathsf{Cmp}} \neg p$ (in fact, even $\mathbf{pABF}_2 \not\mid\!\sim^{\cup}_{\mathsf{Cmp}} \neg p$).

**Note 9** Results similar to those of Proposition 10 and Corollary 2 hold also with respect to preferred and stable semantics. We omit them due to lack of space.

## 7 SUMMARY IN VIEW OF RELATED WORK

Simple contrapositive assumption-based argumentation frameworks are useful structures for accommodating logical argumentation. In this paper we extended these frameworks with information about the relative strength of their arguments. Table 1 summarizes some of the results for stable semantics.

| Property of the pABF | Conditions on the preference setting |
|---|---|
| Consistency | Reversible |
| Closure | Reversible |
| $\mathsf{Stb} \subseteq \mathsf{MCS}_{\sqsubset}$ | Max-lower-bounded |
| $\mathsf{Stb} \supseteq \mathsf{MCS}_{\sqsubset}$ | Max-upper-bounded |
| Empty preferences | Invariance of multiple-occurrence |
| Extension selection | Reversible |
| Conflict preservation | – |
| Preferred assumptions | Reversible & Max-lower-bounded |
| Brewka-Eiter postulate | Reversible & Max-lower-bounded |
| Tolerance | Reversible & Max-lower-bounded, or Max-upper-bounded |
| No drowning effect | Non-decreasing |

**Table 1.** Summary of the results for the stable semantics

Priorities have been integrated in all the major approaches to structured argumentation frameworks, including the assumption-based argumentation formalism $\mathrm{ABA}^+$ (see [12, 14]), $\mathrm{ASPIC}^+$ systems [27, 28], sequent-based argumentation frameworks [5, 6], and dialectical argumentation frameworks [15, 16].

Apart from $\mathrm{ABA}^+$, the incorporation of priorities in all of these settings is similar: for the attack to take place the attacking argument should be at least as preferred as the attacked argument. The $\mathrm{ABA}^+$ system, in contrast, is based on the idea of *reverse defeats*: A set of assumptions $\Delta$ reverse defeats a set of assumptions $\Theta$ if and only if either $\Delta$ attacks $\Theta$ and $\Delta$ is not less preferred than $\Theta$, or $\Theta$ attacks $\Delta$ and $\Theta$ is (strictly) less preferred than $\Delta$.[12] The use of reverse defeats is required for avoiding some violations of rationality postulates such as consistency (see [14] for more details). However, in [21, Chapter 7] it is shown that such reverse defeats are actually superfluous when assuming that the inference relation is closed under contraposition (as in our case), and when using the max-attacks. In Proposition 3 we generalized this result and showed that contraposition together with reversibility of the preference function is sufficient to guarantee consistency (and thus reverse defeats are superfluous).

Another difference between the present work and the one in [14] is that the latter concentrates on the weakest link principle (i.e, max-attacks) for comparing arguments, while we do not confine ourselves to a particular preference setting.

In [27], Modgil and Prakken show that for $\mathrm{ASPIC}^+$-based frameworks including priorities, the preferred and stable extensions coincide and correspond to the set of preferred sub-theories of the set of premises (see [8] and Definition 12). Similar results for sequent-based argumentation frameworks are shown in [6] (see also [4]). However, we note that both in $\mathrm{ASPIC}^+$ and in sequent-based frameworks, a finite set of (defeasible) assumptions gives rise to an infinite set of arguments. The fact that for ABFs the size of an argumentation graph is bounded by the size of the power-set of defeasible assumptions is a benefit of (prioritized) ABFs.

For ASPIC-like systems, Dung and his co-authors have made axiomatic studies on the inferential behaviour of preference-based structured argumentation (see, for instance, [18, 19, 20]). Their study is mainly concerned with arguments constructed on the basis of defeasible rules, and as such the focus of their work is different then the research done in this paper.

In dialectical argumentation frameworks the problem of having infinite number of arguments out of a finite set of assumptions is avoided by what is called in [15, 16] depth-bounded logics. Dialectical argumentation using depth-bounded logics can capture preferred sub-theories and bring about finite argumentation frameworks, given

12 See [25] for the use of similar principles in the context of abstract argumentation frameworks.

a finite set of defeasible assumptions. However, [15, 16] only study preferred sub-theories based on classical logic (and max-attacks), whereas we show that preferred-subtheories based on *any* contrapositive Tarskian logic (and any reversible and max-bounded preference settings) can be represented by simple contrapositive pABFs.

Future work includes, among others, a study of the non-monotonic properties of the entailment relations that are induced by pABFs, and investigations of translation methods from and to formalisms of incorporating preferences in related areas (e.g., logic programming).

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