The Logical Role of the Four-Valued Bilattice

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Abstract

In his well-known paper "How computer should think" ([Be77b]) Belnap argues that four-valued semantics is a very suitable setting for computerized reasoning. In this paper we vindicate this thesis by showing that the logical *role that the four-valued structure has among Ginsberg's well-known* bilattices *is similar to the role that the two-valued algebra has among Boolean algebras.*

1 Introduction

In [Be77a, Be77b] Belnap introduced a logic for dealing in a useful way with inconsistent and incomplete information. This logic is based on a structure called $FOUR$, which has four truth values: the classical ones, t and f, and two new ones: $\perp$ that intuitively denotes lack of information (no knowledge), and $\top$ that indicates inconsistency ("over"-knowledge). Belnap gave quite convincing arguments why "the way a computer should think" should be based on these four values. In [Gi88] Ginsberg proposed algebraic structures called *bilattices* that naturally generalize Belnap's $FOUR$. The idea is to consider *arbitrary* number of truth values, and to arrange them (as in $FOUR$) in two closely related partial orders, each forming a lattice. The original motivation of Ginsberg for introducing bilattices was to provide a uniform approach for a diversity of applications in computer science. In [AA94, AA96] we presented bilattice-based *logics* and corresponding proof systems. These logics turned out to have many desirable properties (like paraconsistency; see [dC74]). The main goal of the present paper is to show that the *logical* role of $FOUR$ among bilattices is very similar to that the two-valued algebra has among Boolean algebras. Indeed, it turns out that all the natural bilattice-valued logics that we had introduced for various purposes can be characterized using only the four basic values! This does not mean, of course, that bilattices have no value (exactly as the fact, that the logic of Boolean algebras can be characterized in $\{t, f\}$, does not mean that Boolean algebras have no value). It does demonstrate, however, the fundamental role of the four values.

The rest of the paper is organized as follows: In the next section we review the material about bilattices that is relevant to the logical role of $FOUR$. Most of the results of this section have been published before, but some of them are completely new. Section 3 describes the main methods of using bilattices for nonmonotonic and paraconsistent reasoning. Section 4 is the heart of the paper; We show in it that $FOUR$ suffices for all these types of reasoning. Section 5 demonstrates the usefulness of these methods through a practical example of applying them in $FOUR$.

2 Bilattices – background

Bilattice are algebraic structures that were proposed by Ginsberg in [Gi88] as a general framework for applications in AI. Specifically, Ginsberg considered their use in first order theories, truth maintenance systems, and formalisms for default reasoning. The algebraic structure of bilattices has been further investigated by Fitting and Avron [Fi90b, Fi94, Av96]. Fitting has shown that bilattices are very useful tools for providing semantic to logic programs [Fi90a, Fi91, Fi93]. He proposed an extension of Smullyan's tableaux-style proof method to bilattice-valued programs, and showed that this method is sound and complete with respect to a natural generalization of van-Emden and Kowalski's operator (see [Fi90a, Fi91]). Fitting also introduced a multi-valued fixedpoint operator (that generalizes the Gelfond-Lifschitz operator

[GL88]) for providing bilattice-based stable models and well-founded semantics for logic programs (see [Fi93]). A well-founded semantics for logic programs that is based on the bilattice *NINE* (Figure 1) is considered also in [DP95]. Bilattices have also been found useful for nonmonotonic reasoning [AA96], model-based diagnostics [Gi88, AA97a], and reasoning with inconsistent knowledge-bases [Sc96, AA97b].

Definition 2.1 [Gi88] A *bilattice* is a structure $\mathcal{B} = (B, \leq_t, \leq_k, \neg)$ such that B is a nonempty set containing at least two elements; $(B, \leq_t)$, $(B, \leq_k)$ are complete lattices; and $\neg$ is a unary operation on B that has the following properties: (a) if $a \leq_t b$ then $\neg a \geq_t \neg b$, (b) if $a \leq_k b$ then $\neg a \leq_k \neg b$, (c) $\neg\neg a = a$.

In what follows we shall use $\wedge$ and $\vee$ for the meet and join of $\leq_t$, and $\otimes$, $\oplus$ for the meet and join of $\leq_k$. f and t will denote the least and greatest element (respectively) w.r.t. $\leq_t$, and $\bot$ and $\top$ – the least and the greatest element w.r.t. $\leq_k$. It is easy to see that $t, f, \top$, and $\bot$ are all distinct from each other.

Definition 2.2 A bilattice is called *distributive* [Gi88] if all the twelve possible distributive laws concerning $\wedge$, $\vee$, $\otimes$, and $\oplus$ hold. It is called *interlaced* [Fi90a, Fi91] if each one of $\wedge$, $\vee$, $\otimes$, and $\oplus$ is monotonic with respect to both $\leq_t$ and $\leq_k$.

The following subsets of B are used for defining validity of formulae and the associated consequence relations. The elements of these sets are usually called *designated* values:

Definition 2.3 [AA94, AA96]
a) A *bifilter* of a bilattice $\mathcal{B}$ is a nonempty proper subset $\mathcal{F} \subset B$, such that:
$a \wedge b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$,
$a \otimes b \in \mathcal{F}$ iff $a \in \mathcal{F}$ and $b \in \mathcal{F}$.
b) A bifilter $\mathcal{F}$ is called *prime*, if it satisfies also:
$a \vee b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$,
$a \oplus b \in \mathcal{F}$ iff $a \in \mathcal{F}$ or $b \in \mathcal{F}$.

Proposition 2.4 [AA97b] A subset $\mathcal{F}$ of an interlaced bilattice $\mathcal{B}$ is a (prime) bifilter iff it is a (prime) filter relative to $\leq_t$ and $\top \in \mathcal{F}$ (iff it is a (prime) filter relative to $\leq_k$ and $t \in \mathcal{F}$).

From now on (unless otherwise stated) $\mathcal{F}$ will denote a prime bifilter. Obviously, if $a \in \mathcal{F}$ and $b \geq_t a$ or $b \geq_k a$, then $b \in \mathcal{F}$. It immediately follows that $t, \top \in \mathcal{F}$ while $f, \bot \notin \mathcal{F}$.

Example 2.5 Belnap's *FOUR* (Figure 1, left) and Ginsberg's *DEFAULT* (Figure 1, right) are bilattices that contain exactly one bifilter, $\{\top, t\}$, which is prime in both. *NINE* (Figure 1, middle), on the other hand, contains two bifilters: $\{b \mid b \geq_k t\}$ as well as $\{b \mid b \geq_k dt\}$; both are prime.

Definition 2.6 [AA94, AA96] A *logical bilattice* is a pair $(\mathcal{B}, \mathcal{F})$, where $\mathcal{B}$ is a bilattice, and $\mathcal{F}$ is a prime bifilter on $\mathcal{B}$.

The smallest logical bilattice is $(FOUR, \{t, \top\})$; In what follows we shall denote it by $\langle FOUR \rangle$, and write "4" whenever $\langle FOUR \rangle$ should appear as a superscript. Other logical bilattices are, e.g., $(DEFAULT, \{t, \top\})$, $(NINE, \{b \mid b \geq_k t\})$, and $(NINE, \{b \mid b \geq_k dt\})$.

Proposition 2.7 [AA97b] Every distributive bilattice can be turned into a logical bilattice.

Given a logical bilattice $(\mathcal{B}, \mathcal{F})$, the standard notions of valuations, models, etc. are defined in the usual way: A *valuation* ν on B is a function that assigns a truth value from B to each atomic formula. Any valuation is extended to complex formulas in the standard way. ν *satisfies* a formula ψ ($\nu \models \psi$), iff $\nu(\psi) \in \mathcal{F}$. A valuation that satisfies every formula in a given set of formulas, Γ, is said to be a *model* of Γ. The set of the models of Γ is denoted $mod(\Gamma)$.

Note that there are no tautologies in the language of $\{\neg, \vee, \wedge, \otimes, \oplus\}$. In particular, excluded middle is not a valid rule. Hence the definition of the material implication $p \mapsto q$ as $\neg p \vee q$ is not adequate for representing entailments. Instead we use another connective, which does function as an implication:

Definition 2.8 [Av91, AA96] Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. Define: $a \supset b = b$ if $a \in \mathcal{F}$, and $a \supset b = t$ otherwise.

Note that on $\{t, f\}$ the material implication ($\mapsto$) and the new implication ($\supset$) are identical, and both are generalizations of the classical implication. However, Modus Ponens and the deduction theorem for $\supset$ are valid relative to the basic consequence relation (see Definition 3.1 below), while both fail for $\mapsto$.

Using $\supset$, it is possible to define a strong implication and an equivalence operation as follows:[1]

Definition 2.9 [Av91, AA96]
$\psi \to \phi = (\psi \supset \phi) \wedge (\neg\phi \supset \neg\psi)$
$\psi \leftrightarrow \phi = (\psi \to \phi) \wedge (\phi \to \psi)$

[1] See [AA96] for a discussion on the properties of $\to$ and $\leftrightarrow$.

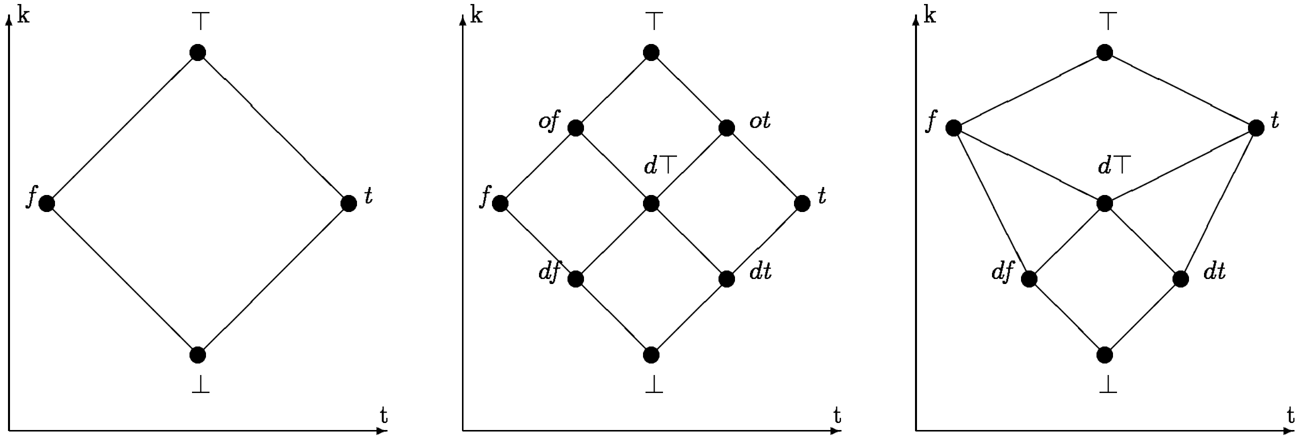

Figure 1: *FOUR*, *NINE*, and *DEFAULT*

Another important property of the language with $\supset$ is given in the next theorem. Its proof will be presented in the full paper.

Theorem 2.10 The language of $\{\neg, \vee, \wedge, \oplus, \otimes, \supset, f\}$ is functionally complete for *FOUR*.

Definition 2.11 Let $(\mathcal{B}_1, \mathcal{F}_1)$ and $(\mathcal{B}_2, \mathcal{F}_2)$ be two logical bilattices. Suppose that b_i is some element of B_i and that ν_i is a valuation on B_i for $i=1,2$.
a) b_1 and b_2 are *similar* if: (i) $b_1 \in \mathcal{F}_1$ iff $b_2 \in \mathcal{F}_2$, and (ii) $\neg b_1 \in \mathcal{F}_1$ iff $\neg b_2 \in \mathcal{F}_2$.
b) ν_1 and ν_2 are similar if for every atomic p, $\nu_1(p)$ and $\nu_2(p)$ are similar.

Note that the two valuations might not be similar even in case they are identical and the underlying bilattice is the same. Consider, e.g., a valuation ν on *NINE* s.t. $\nu(p) = ot$ for some atom p. Then ν for $\mathcal{F} = \{b \mid b \geq_k t\})$ is not similar to ν where the bifilter is $\mathcal{F} = \{b \mid b \geq_k dt\})$.

Proposition 2.12 Let $(\mathcal{B}_1, \mathcal{F}_1)$ and $(\mathcal{B}_2, \mathcal{F}_2)$ be two logical bilattices and suppose that ν_1, ν_2 are two similar valuations on B_1, B_2 (respectively). Then for every formula ψ, $\nu_1(\psi)$ and $\nu_2(\psi)$ are similar.

Proof: By an induction on the structure of ψ (The fact that $\mathcal{F}$ is *prime* is crucial here!). $\Box$

Theorem 2.13 A model of Γ in $\langle FOUR \rangle$ is also a model of Γ in every logical bilattice $(\mathcal{B}, \mathcal{F})$.

Proof: Let $M^{(4)}$ be a model of Γ in *FOUR*, and suppose that $M^{(\mathcal{B},\mathcal{F})}$ is the same valuation defined on some logical bilattice $(\mathcal{B}, \mathcal{F})$. Since every bifilter $\mathcal{F}$ contains $t, \top$ and does not contain $f, \bot$, $M^{(4)}$ and $M^{(\mathcal{B},\mathcal{F})}$ are similar. By Proposition 2.12, $M^{(4)}(\psi)$ and $M^{(\mathcal{B},\mathcal{F})}(\psi)$ are similar for every $\psi \in \Gamma$, and so $M^{(\mathcal{B},\mathcal{F})}$ must be a model of Γ in $(\mathcal{B}, \mathcal{F})$ as well.[2] $\Box$

Notation 2.14 Given a logical bilattice $(\mathcal{B}, \mathcal{F})$. Denote:

$$\mathcal{T}_t^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \in \mathcal{F}, \neg b \notin \mathcal{F}\}$$

$$\mathcal{T}_f^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \notin \mathcal{F}, \neg b \in \mathcal{F}\}$$

$$\mathcal{T}_\top^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \in \mathcal{F}, \neg b \in \mathcal{F}\}$$

$$\mathcal{T}_\bot^{\mathcal{B},\mathcal{F}} = \{b \in B \mid b \notin \mathcal{F}, \neg b \notin \mathcal{F}\}$$

We shall usually omit the superscripts, and just write $\mathcal{T}_t, \mathcal{T}_f, \mathcal{T}_\top, \mathcal{T}_\bot$.[3]

Definition 2.15 Let $(\mathcal{B}, \mathcal{F})$ be a logical bilattice. Define $h : \mathcal{B} \to FOUR$ by $h(b) = x$ iff $b \in \mathcal{T}_x$.

Proposition 2.16
a) h is an homomorphism onto *FOUR*.
b) M is a model of Γ in $(\mathcal{B}, \mathcal{F})$ iff $h \circ M$ is a model of Γ in $\langle FOUR \rangle$.

[2] In the specific case where $(\mathcal{B}, \mathcal{F})$ is interlaced, the last theorem immediately follows from Proposition 3.1 of [Fi91], since it is shown there that *FOUR* is a sub-bilattice of every interlaced bilattice $\mathcal{B}$, so in this case $M^{(4)}(\psi)$ and $M^{(\mathcal{B},\mathcal{F})}(\psi)$ are not only similar, but are actually identical.

[3] These four items could serve as a tableaux signs, comparable to the usual T and F. Alternatively, one may use signs corresponding to: 'is in $\mathcal{F}$', 'is not in $\mathcal{F}$', 'negation is in $\mathcal{F}$', and 'negation is not in $\mathcal{F}$'. This gives simple tableaux rules which correspond to the Gentzen system presented in [AA94, AA96].

3 Reasoning with logical bilattices

3.1 The basic consequence relation

We start with the simplest consequence relation which naturally corresponds to logical bilattices.

Definition 3.1 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice, and suppose that Γ, Δ are two sets of formulae. $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ if every model of Γ is a model of some formula in Δ.

$\models^{\mathcal{B},\mathcal{F}}$ is a consequence relation in the standard sense of Scott. In [AA94, AA96] it is shown that this relation is sound and complete w.r.t. a certain cut-free Gentzen-type system, and that it is also monotonic, compact, and paraconsistent. The main drawbacks of $\models^{\mathcal{B},\mathcal{F}}$ are that it is strictly weaker than classical logic even for consistent theories, and that it *always* invalidates some intuitively justified inference rules, like the Disjunctive Syllogism.

In the next subsections we consider two possibilities of refining $\models^{\mathcal{B},\mathcal{F}}$. The main theme is to restrict the set of models we take into account, using some *preference* criteria. This is the idea behind the notion of a *preferential logic* considered in [Sh87, Sh88]. This approach has recently received a considerable attention (see, e.g., [Ma89, KLM90, Pr91, LM92, KL92, Ma94, Sc97]).

3.2 The logics $\models_k^{\mathcal{B},\mathcal{F}}$

A natural approach for reducing the set of models which are used for drawing conclusions is to consider only the k-minimal ones. The idea behind this approach is that one should not assume anything that is not *really* known.

Definition 3.2 Let M_1, M_2 be two four-valued valuations, and Γ – a set of formulae.
a) M_1 is *k-smaller* than M_2 ($M_1 \leq_k M_2$) if for every atomic p, $M_1(p) \leq_k M_2(p)$.
b) M is a *k-minimal model* of Γ if M is a $\leq_k$-minimal element of $mod(\Gamma)$.

Definition 3.3 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice, and suppose that Γ, Δ are two sets of formulae. $\Gamma \models_k^{\mathcal{B},\mathcal{F}} \Delta$ if every k-minimal model of Γ is a model of some formula in Δ.

In [AA97b] it is shown, among other things, that $\models_k^{\mathcal{B},\mathcal{F}}$ is nonmonotonic, paraconsistent, a plausibility logic [Le92], and is identical to $\models^{\mathcal{B},\mathcal{F}}$ in the language without $\supset$.

3.3 The logics $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$

The motivation behind the last family of bilattice-based logics that we consider here is perhaps the closest in spirit to the original idea of Da-Costa [dC74] on paraconsistent reasoning: We allow a nontrivial reasoning in the presence of inconsistency, while still trying to minimize the amount of contradictions. This approach reflects the intuition that while one has to deal with conflicts in a nontrivial way, contradictory data corresponds to inadequate information about the real world, and therefore should be minimized.

Definition 3.4 [AA94, AA96] Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice. A subset $\mathcal{I}$ of B (the carrier of $\mathcal{B}$) is called an *inconsistency set* in $(\mathcal{B},\mathcal{F})$, if it has the following properties: **(a)** $b \in \mathcal{I}$ iff $\neg b \in \mathcal{I}$, **(b)** $\mathcal{F} \cap \mathcal{I} = \mathcal{T}_\top$.

Lemma 3.5 Let $\mathcal{I}$ be an inconsistency set. Then: **(a)** $\mathcal{T}_\top \subseteq \mathcal{I} \subseteq \mathcal{T}_\top \cup \mathcal{T}_\perp$, **(b)** $\top \in \mathcal{I}$ and $t, f \notin \mathcal{I}$.

Example 3.6 $\mathcal{T}_\top$ and $\mathcal{T}_\top \cup \mathcal{T}_\perp$ are respectively the minimal and maximal inconsistency sets in every logical bilattice. In $\langle FOUR \rangle$ the former set is $\mathcal{I}_1 = \{\top\}$ and the latter is $\mathcal{I}_2 = \{\top, \perp\}$.

Notation 3.7 $I(\nu,\mathcal{I}) = \{p \mid p$ is atomic and $\nu(p) \in \mathcal{I}\}$. Intuitively, $I(\nu,\mathcal{I})$ is the set of the inconsistent assignments of a valuation ν w.r.t. an inconsistency set $\mathcal{I}$.

Definition 3.8 Let Γ be a set of formulae, and M, N – models of Γ.
a) M is *more consistent* than N w.r.t. $\mathcal{I}$ ($M >_{\mathcal{I}} N$) if $I(M,\mathcal{I}) \subset I(N,\mathcal{I})$.
b) M is a *most consistent* model of Γ w.r.t. $\mathcal{I}$ ($\mathcal{I}$-mcm, in short), if there is no other model of Γ which is more consistent than M. The set of all the $\mathcal{I}$-mcms of Γ is denoted $mcm(\Gamma,\mathcal{I})$.

Definition 3.9 $\Gamma \models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}} \Delta$ if every $\mathcal{I}$-mcm of Γ is a model of some formula of Δ.

Note: Several relations similar to $\models_{\mathcal{I}}^{\mathcal{B},\mathcal{F}}$ are considered in the literature. Perhaps the closest in spirit is Priest's LPm ([Pr89, Pr91]). In our terms, Priest considers the inconsistency set $\mathcal{I} = \mathcal{T}_\top$. In the 3-valued case (which is the case Priest considers) this is the only inconsistency set, and it consists only of $\top$. In the general (multi-valued) case there are many others.

Kifer and Lozinskii ([KL92]) also propose a similar relation (denoted there $\models\!\approx_\Delta$, where Δ stands for the values that are considered as representing inconsistent knowledge). This relation is considered in the framework of annotated logics ([Su90a, Su90b]). See [AA96] for a discussion on the similarities and the differences between $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ and $\approx_\Delta$.

Some of the basic properties of $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}}$ are considered in [AA96, AA97b]. We just note here that this relation is not only nonmonotonic, paraconsistent, and a plausibility logic [Le92], but in case that $\mathcal{T}_\perp \subseteq \mathcal{I}$ (i.e. $\mathcal{I} = \mathcal{T}_\perp \cup \mathcal{T}_\top$) it also behaves like classical logic when dealing with consistent theories.

4 Characterizations in terms of $\langle FOUR \rangle$

In this section we give several theorems that show that all the families of bilattice-based logics that we have considered here can be defined in terms of $\langle FOUR\rangle$. Again, we start with the basic consequence relation:

Theorem 4.1 [AA96] $\Gamma \models^{\mathcal{B},\mathcal{F}} \Delta$ iff $\Gamma \models^4 \Delta$.

Proof: The "only if" part follows from Theorem 2.13 and Proposition 2.12. The converse follows from Propositions 2.12 and 2.16. □

Theorem 4.2 Let $(\mathcal{B},\mathcal{F})$ be a logical bilattice s.t. $\inf_k \mathcal{F} \in \mathcal{F}$.[4] Then $\Gamma \models^{\mathcal{B},\mathcal{F}}_k \Delta$ iff $\Gamma \models^4_k \Delta$.

Outline of proof: We shall use the following two lemmas:

Lemma 4.2-A: $\forall x \in \{t, f, \top, \perp\}$ $\inf_k \mathcal{T}^{\mathcal{B},\mathcal{F}}_x \in \mathcal{T}^{\mathcal{B},\mathcal{F}}_x$. Moreover: $\inf_k \mathcal{T}^{\mathcal{B},\mathcal{F}}_\perp = \perp$, $\inf_k \mathcal{T}^{\mathcal{B},\mathcal{F}}_t = \inf_k \mathcal{F} = \min_k \mathcal{F}$, $\inf_k \mathcal{T}^{\mathcal{B},\mathcal{F}}_f = \neg \inf_k \mathcal{F} = \neg \min_k \mathcal{F}$, and $\inf_k \mathcal{T}^{\mathcal{B},\mathcal{F}}_\top = \min_k \mathcal{F} \oplus \neg \min_k \mathcal{F}$.

Lemma 4.2-B: Suppose that M is a k-minimal model of Γ in $(\mathcal{B},\mathcal{F})$, and let $h : \mathcal{B} \to FOUR$ be the homomorphism defined in 2.15. Then $h \circ M$ is a k-minimal model of Γ in $\langle FOUR\rangle$.

The "if" direction of Theorem 4.2 now easily follows from Lemma 4.2-B: Suppose that for some logical bilattice $(\mathcal{B},\mathcal{F})$, $\Gamma \not\models^{\mathcal{B},\mathcal{F}}_k \Delta$. Let M be a k-minimal model of Γ s.t. $M(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. By Lemma 4.2-B $h \circ M$ is a k-minimal model of Γ in $\langle FOUR\rangle$ which is similar to M. Therefore $(h \circ M)(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$, and so $\Gamma \not\models^4_k \Delta$.
The other direction: Suppose that $\Gamma \not\models^4_k \Delta$. Then there is a k-minimal model M of Γ in $\langle FOUR\rangle$ s.t. $M(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. Define a valuation M' on B as follows: $M'(p) = \inf_k \mathcal{T}_{M(p)}$ (p atomic). By Proposition 2.12 and Lemma 4.2-A, $h \circ M' = M$. Hence (by Proposition 2.16) M' is a model of Γ, and $M'(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. Moreover, M' is a k-minimal model of Γ, and so $\Gamma \not\models^{\mathcal{B},\mathcal{F}}_k \Delta$. Indeed, if N is another model of Γ s.t. $N <_k M'$, then $h \circ N \leq_k h \circ M' = M$. Also, there is p s.t. $N(p) <_k M'(p)$ and so $N(p) \notin \mathcal{T}_{M(p)}$. Hence $h(N(p)) \neq M(p)$, and so actually $h \circ N <_k M$. Since $h \circ N$ is a model of Γ in $\langle FOUR\rangle$ (because $N \in mod(\Gamma)$), M is not k-minimal – a contradiction. □

Theorem 4.3 For every logical bilattice $(\mathcal{B},\mathcal{F})$ and an inconsistency set $\mathcal{I}$ there is an inconsistency set $\mathcal{J}$ in $\langle FOUR\rangle$ s.t. $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$ iff $\Gamma \models^4_{\mathcal{J}} \Delta$.

Outline of proof: In the course of this proof we shall use the following conventions: whenever ν is a function from the atomic formulae to $\{t, f, \top, \perp\}$, ν^4 denotes its expansion to complex formulae in $FOUR$, and ν^B denotes the corresponding valuation on B.[5] Also, $h : \mathcal{B} \to FOUR$ denotes the homomorphism onto $FOUR$, defined in 2.15.

Lemma 4.3-A: $\nu^4 = h \circ \nu^B$.

Corollary 4.3-B: ν^B is a model of Γ in $(\mathcal{B},\mathcal{F})$ iff ν^4 is a model of Γ in $\langle FOUR\rangle$.

The rest of the proof is divided into two cases that correspond to the two possibilities of defining an inconsistency set in $\langle FOUR\rangle$:

- *Case* A: $\mathcal{T}_\perp \subseteq \mathcal{I}$,
- *Case* B: $\mathcal{T}_\perp \setminus \mathcal{I} \neq \emptyset$.

For each case define a corresponding inconsistency set in $\langle FOUR\rangle$. In case A let $\mathcal{J} = \mathcal{I}_2 = \{\top, \perp\}$, and in case B let $\mathcal{J} = \mathcal{I}_1 = \{\top\}$.

Lemma 4.3-C: In case A, $M \in mcm(\Gamma, \mathcal{I})$ in $(\mathcal{B},\mathcal{F})$ iff $h \circ M \in mcm(\Gamma, \mathcal{I}_2)$ in $\langle FOUR\rangle$.

Corollary 4.3-D: In case A, $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$ iff $\Gamma \models^4_{\mathcal{I}_2} \Delta$.
Proof: Suppose that $\Gamma \not\models^4_{\mathcal{I}_2} \Delta$. Then there is an assignment ν on $FOUR$ s.t. ν^4 is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR\rangle$ that is not a model of any $\delta \in \Delta$. By Lemma 4.3-A, $\nu^4 = h \circ \nu^B$ and by 4.3-B, 4.3-C, ν^B is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$ s.t. $\nu^B(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. Hence $\Gamma \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$. For the converse, assume that M is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$ which is not a model of any formula in Δ. Then, by 4.3-B and 4.3-C, $h \circ M$ is an $\mathcal{I}_2$-mcm of Γ in $\langle FOUR\rangle$, and $h \circ M(\delta) \in \{f, \perp\}$ for every $\delta \in \Delta$. Therefore $\Gamma \not\models^4_{\mathcal{I}_2} \Delta$.

[4] This is clearly the case whenever $\mathcal{B}$ is finite. It can also be shown that if $\mathcal{B}$ is interlaced then $\inf_k \mathcal{F} \in \mathcal{F}$ iff $\inf_t \mathcal{F} \in \mathcal{F}$. Moreover, in this case $\inf_t \mathcal{F} = \inf_k \mathcal{F} \wedge \top$ while $\inf_k \mathcal{F} = \inf_t \mathcal{F} \otimes t$.

[5] Note that although $\nu^4(p) = \nu^B(p)$ when p is atomic, this might not be the case in general, unless $\mathcal{B}$ is interlaced.

Let us turn now to case B, in which there is some $\alpha \in \mathcal{T}_\perp \setminus \mathcal{I}$. Suppose that M is a model of Γ in $(\mathcal{B},\mathcal{F})$. Consider the valuation M_α, defined for every atomic formula p as follows:

$$M_\alpha(p) = \begin{cases} \alpha & \text{if } M(p) \in \mathcal{T}_\perp \cap \mathcal{I} \\ M(p) & \text{otherwise} \end{cases}$$

Since obviously $h \circ M = h \circ M_\alpha$, then in particular:

$$(*)\ I(h \circ M, \mathcal{I}_1) = I(h \circ M_\alpha, \mathcal{I}_1)$$

Lemma 4.3-E: If $M \in mcm(\Gamma, \mathcal{I})$ then $M = M_\alpha$. If $M = M_\alpha$ then $I(M, \mathcal{I}) = I(h \circ M, \mathcal{I}_1)$.

Lemma 4.3-F: In case B, if $M \in mcm(\Gamma, \mathcal{I})$ in $(\mathcal{B},\mathcal{F})$ then $h \circ M \in mcm(\Gamma, \mathcal{I}_1)$ in $\langle FOUR \rangle$.

Corollary 4.3-G: In case B, $\Gamma \models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$ iff $\Gamma \models^4_{\mathcal{I}_1} \Delta$.

Proof: If $\Gamma \not\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \Delta$ then there exists an $\mathcal{I}$-mcm M of Γ s.t. $M(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. By Lemma 4.3-F, $h \circ M$ is an $\mathcal{I}_1$-mcm of Γ in $\langle FOUR \rangle$ and $(h \circ M)(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. Therefore $\Gamma \not\models^4_{\mathcal{I}_1} \Delta$. For the converse, assume that $\Gamma \not\models^4_{\mathcal{I}_1} \Delta$. Suppose that ν is an assignment on $\{t, f, \top, \perp\}$ s.t. ν^4 is an $\mathcal{I}_1$-mcm of Γ in $\langle FOUR \rangle$ and $\nu^4(\delta) \notin \{t, \top\}$ for every $\delta \in \Delta$. By Lemma 4.3-A $\nu^4 = h \circ \nu^B$. By Corollary 4.3-B ν^B is a model of Γ in $(\mathcal{B},\mathcal{F})$ s.t. $\nu^B(\delta) \notin \mathcal{F}$ for every $\delta \in \Delta$. But for every $\psi \in \Gamma$, $\nu^B(\psi) \in \mathcal{F}$ iff $\nu^B_\alpha(\psi) \in \mathcal{F}$, and so ν^B_α is also a model of Γ in $(\mathcal{B},\mathcal{F})$ and $\forall \delta \in \Delta\ \nu^B_\alpha(\delta) \notin \mathcal{F}$. It is left to show, then, that ν^B_α is an $\mathcal{I}$-mcm of Γ in $(\mathcal{B},\mathcal{F})$. Suppose otherwise. Then there is an $\mathcal{I}$-mcm M of Γ, s.t. $M >^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \nu^B_\alpha$. Since $(\nu^B_\alpha)_\alpha = \nu^B_\alpha$ and (by Corollary 4.3-E) $M = M_\alpha$, we have:

$$\begin{array}{lll} I(h \circ M, \mathcal{I}_1) & = I(M, \mathcal{I}) & \text{by Lemma4.3-E} \\ & \subset I(\nu^B_\alpha, \mathcal{I}) & \text{by the choice of } M \\ & = I(h \circ \nu^B_\alpha, \mathcal{I}_1) & \text{by Lemma4.3-E} \\ & = I(h \circ \nu^B_\alpha, \mathcal{I}_1) & \text{by } (*) \\ & = I(h \circ \nu^B, \mathcal{I}_1). & \end{array}$$

Therefore $h \circ M >^4_{\mathcal{I}_1} h \circ \nu^B = \nu^4$. Since $h \circ M$ is a model of Γ (because M is), this is a contradiction.
This concludes the proof of Corollary 4.3-G and Theorem 4.3. $\Box$

Corollary 4.4 Let $(\mathcal{B},\mathcal{F})$ and $\mathcal{I}$ be some logical bilattice and an inconsistency set in it. Then:
(a) If $\mathcal{T}^{\mathcal{B},\mathcal{F}}_\perp \not\subset \mathcal{I}$ then $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \equiv \models^4_{\mathcal{I}_1}$,
(b) If $\mathcal{T}^{\mathcal{B},\mathcal{F}}_\perp \subset \mathcal{I}$ then $\models^{\mathcal{B},\mathcal{F}}_{\mathcal{I}} \equiv \models^4_{\mathcal{I}_2}$.

Proof: Easily follows from the proof of Theorem 4.3. $\Box$

5 An example of reasoning in $\langle FOUR \rangle$: A diagnosis

Figure 2 depicts a circuit that consists of six components: two and-gates $A1$ and $A2$, two xor-gates $X1$ and $X2$, and two or-gates $O1$ and $O2$. It shows also the results of an experiment which was done with this circuit. According to this experiment the circuit is faulty; the values of the output lines of $X2$ and $O1$ are not the expected ones. The third output line (that of $O2$) does have the expected value, although one of its inputs is not known. Our goal is to find some minimal set of components the collective failure of which can explain the observed malfunction.
A description of this circuit, together with the results of the experiment, is given in Figure 3.[6] [7] Here, every non-grounded formula represents its set of ground instances. $\Box\psi$ abbreviates the formula $\psi \wedge (\neg\psi \supset f)$. Its intuitive meaning is that ψ is "absolutely true", i.e. ψ is known to be true, while its negation can *never* be valid. Since we know in advance the values of three input wires and of all the output wires, as well as the kind of each gate in the system, we attached this certainty operator ($\Box$) to the corresponding predicates. The correct behavior of each gate, on the other hand, is only a default assumption, therefore the predicate ok is not preceded by the $\Box$-operator. Note that the resulting knowledge-base is classically inconsistent.

Denote the knowledge-base that describes this circuit by Γ. In $\langle FOUR \rangle$, Γ has 232 models, but just three k-minimal ones (which in this case are also the $\mathcal{I}_1$-mcms of Γ).The table of Figure 4 lists these models. We have omitted from it predicates that have the same value in all the models of Γ, and any predicate that always has the same value as some other predicate that already appears in the table.

From the table it follows that:

$$\begin{array}{rl} \Gamma \models^4_k & \neg\mathtt{ok}(X1) \ \vee \\ & (\neg\mathtt{ok}(X2) \ \wedge \ \neg\mathtt{ok}(A2)) \ \vee \\ & (\neg\mathtt{ok}(X2) \ \wedge \ \neg\mathtt{ok}(O1)) \end{array}$$

This exactly corresponds to the diagnoses for the possible causes of the malfunction of a similar (but simpler) circuit in [Re87, Example 2.2] and [Gi88, Sections 15,16].

[6] We use here the material implication $\mapsto$ for standard implications. This allows us a comparison with other treatments. One should consider, however, replacing it by $\supset$. This will be done elsewhere.

[7] To avoid overloading, we use $+$ (rather than $\oplus$) for the xor operation.

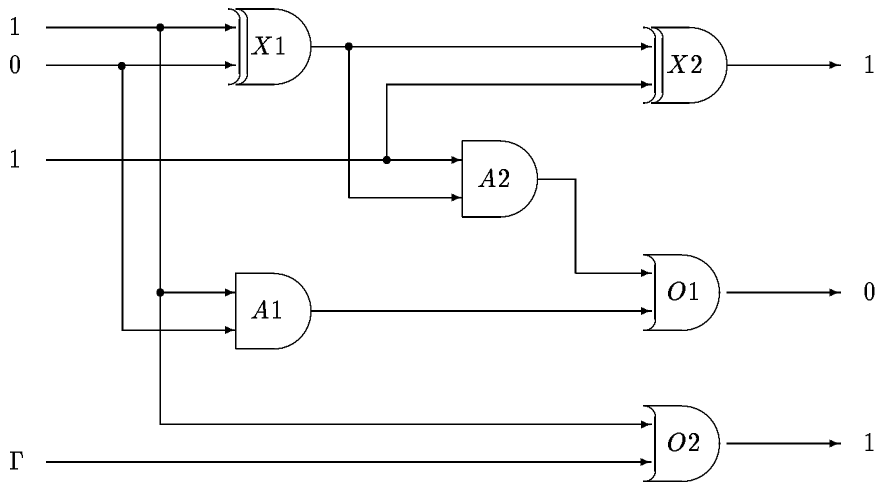

Figure 2: A faulty circuit

$\texttt{andG}(x) \wedge \texttt{ok}(x) \mapsto (\texttt{out}(x) \leftrightarrow (\texttt{in1}(x) \wedge \texttt{in2}(x)))$,
$\texttt{xorG}(x) \wedge \texttt{ok}(x) \mapsto (\texttt{out}(x) \leftrightarrow (\texttt{in1}(x) + \texttt{in2}(x)))$,
$\texttt{orG}(x) \wedge \texttt{ok}(x) \mapsto (\texttt{out}(x) \leftrightarrow (\texttt{in1}(x) \vee \texttt{in2}(x)))$,
$\neg(\texttt{andG}(x) \wedge \texttt{orG}(x)) \wedge \neg(\texttt{andG}(x) \wedge \texttt{xorG}(x)) \wedge \neg(\texttt{xorG}(x) \wedge \texttt{orG}(x))$,
$\texttt{in1}(X1) \leftrightarrow \texttt{in1}(A1)$, $\texttt{in1}(X1) \leftrightarrow \texttt{in1}(O2)$, $\texttt{in2}(X1) \leftrightarrow \texttt{in2}(A1)$,
$\texttt{in1}(A2) \leftrightarrow \texttt{in2}(X2)$, $\texttt{out}(X1) \leftrightarrow \texttt{in2}(A2)$, $\texttt{out}(X1) \leftrightarrow \texttt{in1}(X2)$,
$\texttt{out}(A1) \leftrightarrow \texttt{in2}(O1)$, $\texttt{out}(A2) \leftrightarrow \texttt{in1}(O1)$,
$\Box\texttt{andG}(A1)$, $\Box\texttt{andG}(A2)$, $\Box\texttt{xorG}(X1)$, $\Box\texttt{xorG}(X2)$, $\Box\texttt{orG}(O1)$, $\Box\texttt{orG}(O2)$,
$\Box\texttt{in1}(X1)$, $\Box\neg\texttt{in2}(X1)$, $\Box\texttt{in1}(A2)$, $\Box\texttt{out}(X2)$, $\Box\neg\texttt{out}(O1)$, $\Box\texttt{out}(O2)$,
$\texttt{ok}(A1)$, $\texttt{ok}(A2)$, $\texttt{ok}(X1)$, $\texttt{ok}(X2)$, $\texttt{ok}(O1)$, $\texttt{ok}(O2)$.

Figure 3: The circuit description

Model No.	in2 $A2$	in1 $O1$	in2 $O1$	in2 $O2$	ok $A1$	ok $A2$	ok $X1$	ok $X2$	ok $O1$	ok $O2$
$M1$	f	f	f	$\bot$	t	t	$\top$	t	t	t
$M2$	t	f	f	$\bot$	t	$\top$	t	$\top$	t	t
$M3$	t	t	f	$\bot$	t	t	t	$\top$	$\top$	t

Figure 4: The k-minimal models of Γ

6 Conclusion

Bilattices are algebraic structures that have been shown useful in several areas of computer science. The smallest non-degenerated bilattice, $\langle FOUR \rangle$, consists of four elements. The goal of this work has been to show that the logical role of $\langle FOUR \rangle$ among (logical) bilattices is similar to that the two-valued (classical) lattice has among Boolean algebras. As such, $\langle FOUR \rangle$ provides a useful framework for capturing some standard non-monotonic methods and paraconsistent techniques. The outcome is, so we believe, a vindication of Belnap's thesis that "the way a computer should think" should be based on the four basic values.

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