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Logical argumentation by dynamic proof systems

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ABSTRACT

In this paper we provide a proof theoretical investigation of logical argumentation, where arguments are represented by sequents, conflicts between arguments are represented by sequent elimination rules, and deductions are made by dynamic proof systems extending standard sequent calculi. The idea is to imitate argumentative movements in which certain claims are introduced or withdrawn in the presence of counter-claims. This is done by a dynamic evaluation of sequences of sequents, in which the latter are considered 'derived' or 'not derived' according to the content of the sequence. We show that decisive conclusions of such a process correspond to well-accepted consequences of the underlying argumentation framework. The outcome is therefore a general and modular proof-theoretical approach for paraconsistent and non-monotonic reasoning with argumentation systems.

1. Introduction

Logical argumentation (sometimes called deductive or structured argumentation) is a logic-based approach for formalizing debates, disagreements, and entailment relations for drawing conclusions from argumentation-based settings. Early works on this subject, in the form of defeasible reasoning, may be traced back to the 1990s (see [36,39,42]). Following Dung's seminal work on semantics for abstract argumentation [19] many follow-up studies have emerged in an attempt to deductively formalize Dung's and related approaches (see, e.g., [1,2,13–15,22,25,32,33,40]). The basic entities in this context are called *arguments*. An argument is a pair of a finite set of formulas (Γ, the support set) and a formula (ψ, the conclusion), expressed in an arbitrary (usually propositional) language, such that the latter follows, according to some underlying logic, from the former. As indicated in [3] and [6], this gives rise to the association of arguments with Gentzen's notion of *sequents* [24], where an argument is expressed by a sequent of the form $\Gamma \Rightarrow \psi$. Accordingly, logical argumentation boils down to the exposition of formalized methods for reasoning with these syntactical objects.

A first step towards a proof theoretical investigation of sequent-based logical argumentation is done in [5,7]. The idea is to consider a generic method of drawing conclusions from a given set of sequent-based arguments, which is tolerant to different logics, languages, and attack relations among the arguments. For this, standard Gentzen-style rules that allow to infer new arguments (sequents) from existing ones are augmented with new rules that allow to *exclude* arguments that were already derived, in the presence of derived, opposing arguments. This gives rise to the notion of *dynamic proofs* (or dynamic derivations), which are intended for explicating actual non-monotonic flavor of reasoning processes in an argumentation framework. The main idea behind these formalisms is that, unlike 'standard' proof methods, an argument can be challenged

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(and possibly withdrawn) by a counter-argument, and so a certain sequent may be considered as not derived at a certain stage of the proof, even if it were considered derived in an earlier stage of the proof. It is only when an argument is 'finally derived' (in the sense that will be explained later on) that it can be safely concluded by the dynamic proof.

In this paper we revise, extend, and improve the work in [5] and [7] (see also Note 4 below).[1] To the best of our knowledge this is the first (archival) paper that provides a comprehensive study on the use of (extended) Gentzen-type sequent calculi for giving semantics to argumentation theory, and for reasoning with argumentation frameworks by dynamic Gentzen-style proof systems. Particular attention is payed to the study of the basic properties and the general theory of these proof systems. Among others, it is shown that despite of their non-monotonic nature, one may still draw solid conclusions from dynamic derivations, which are faithful to the intended semantics of the logical argumentation framework at hand. It is also shown that these derivations preserve some restricted forms of reflexivity, monotonicity, and properly maintain inconsistent information. In particular, this implies that the entailment relations induced by the dynamic proof systems for a large class of frameworks are cumulative in the sense of [23,31]. Finally, like 'standard' proof systems, it is shown that in many cases dynamic proof systems are *proof invariant*, in the sense that if a certain assertion is finally derived by a specific derivation, *any* dynamic derivation can be extended to obtain a new derivation in which that assertion is finally derived.

The rest of this paper is organized as follows. In the next section we recall the notion of sequent-based argumentation and review some related semantic concepts that are used in the context of argumentation frameworks. Then, in Section 3 we define dynamic proofs and consider some of their basic characteristics. Examples of such proofs together with some discussions are given in Section 4, and in Section 5 we consider several properties of the entailment relations that are induced by the dynamic proofs systems. In Section 6 we refer to related work and conclude. Proof invariance and some rationality postulates are shown in the appendices due to the length of their proofs.

2. Preliminaries

We start by reviewing the notion of sequent-based argumentation. First, we recall the more general notion of abstract argumentation frameworks.

2.1. Argumentation frameworks and their semantics

Abstract argumentation frameworks are directed graphs, where the nodes represent (abstract) arguments and the arrows represent attacks between arguments, as defined next.

Definition 1. An *(abstract) argumentation framework* [19] is a pair $\mathcal{AF} = \langle Args, Attack \rangle$, where *Args* is an enumerable set of elements, called *arguments*, and *Attack* is a binary relation on *Args*, whose instances are called *attacks*.

Given an argumentation framework, a key issue in its understanding is to determine what combinations of arguments (called *extensions*) can be collectively accepted from it. For this we recall the notions of *conflict-freeness* and *defense* [19].

Definition 2. Let $\mathcal{AF} = \langle Args, Attack \rangle$ be an argumentation framework, $A \in Args$ an argument, and $\mathcal{E} \subseteq Args$ a set of arguments.

- We say that $\mathcal{E}$ *attacks* A if there is an argument $B \in \mathcal{E}$ that attacks A (i.e., $(B, A) \in Attack$). The set of arguments that are attacked by $\mathcal{E}$ is denoted $\mathcal{E}^+$.
- We say that $\mathcal{E}$ *defends* A if $\mathcal{E}$ attacks every argument that attacks A. We denote by $\mathsf{Def}(\mathcal{E})$ the set of all the elements that are defended by $\mathcal{E}$.
- The set $\mathcal{E}$ is called *conflict-free* if it does not attack any of its elements (i.e., $\mathcal{E}^+ \cap \mathcal{E} = \emptyset$), $\mathcal{E}$ is called *admissible* if it is conflict-free and defends all of its elements (i.e., $\mathcal{E} \subseteq \mathsf{Def}(\mathcal{E})$), and $\mathcal{E}$ is *complete* if it is admissible and contains all the arguments that it defends (i.e., $\mathcal{E} = \mathsf{Def}(\mathcal{E})$).

The requirements defined above express basic properties that every plausible extension of a framework should have. Intuitively, a set of arguments is conflict-free if all of its elements 'can stand together' (since they do not attack each other), and admissibility guarantees that such elements 'can stand on their own', i.e., they are able to respond to any attack by their own attack (see also [9,11]).

Next, we recall some acceptability semantics for an argumentation framework.

Definition 3. Let $\mathcal{AF} = \langle Args, Attack \rangle$ be an argumentation framework.

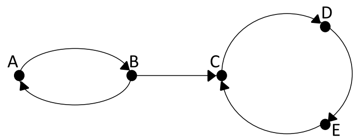
- The minimal complete subset of *Args* is called the *grounded extension* of $\mathcal{AF}$.

[1] In this paper we provide further discussions, examples, and full proofs to the results in [5,7]. Also, Section 4.2 and some parts of Section 5, on the properties of the derivation systems and of the induced entailment relations, have not been presented before.

- A maximal complete subset of *Args* is called a *preferred extension* of $\mathcal{AF}$.
- A complete subset $\mathcal{E}$ of *Args* that attacks every argument in $Args - \mathcal{E}$ is a *stable extension* of $\mathcal{AF}$.

We denote by $\mathsf{Cmpl}(\mathcal{AF})$ (respectively, $\mathsf{Grnd}(\mathcal{AF})$, $\mathsf{Prf}(\mathcal{AF})$, $\mathsf{Stbl}(\mathcal{AF})$) the set of all the complete (respectively, all the grounded, preferred, stable) extensions of $\mathcal{AF}$.[2]

Example 1. Consider the following argumentation framework:



Here $\emptyset$, $\{A\}$, $\{B\}$ and $\{B, D\}$ are admissible sets, and except of $\{B\}$ all of them are also complete. The grounded extension is $\emptyset$, the preferred extensions are $\{A\}$ and $\{B, D\}$, and the stable extension is $\{B, D\}$.

2.2. Sequent-based argumentation frameworks

As indicated previously, in this paper we consider *structured argumentation*, a specific kind of argumentation framework, in which the arguments are not just abstract entities, but represent some assertions (in a given language) that are obtained by a logical correspondence between a set of formulas (the assumptions, or the support set) and a formula (the conclusion). Clearly, this requires the availability of some underlying logic for determining those correspondences. This is what we do next.

In what follows, we shall denote by $\mathcal{L}$ an arbitrary propositional language. Atomic formulas in $\mathcal{L}$ are denoted by p, q, r, arbitrary sets of formulas in $\mathcal{L}$ are denoted by $\mathcal{S}$, $\mathcal{T}$, and *finite* sets of formulas are denoted by Γ, Δ.

Definition 4. A (propositional) *logic* for a language $\mathcal{L}$ is a pair $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, where $\vdash$ is a (Tarskian) consequence relation for $\mathcal{L}$, that is, a binary relation between sets of formulas and formulas in $\mathcal{L}$, satisfying the following conditions:

Reflexivity: $\{\psi\} \vdash \psi$.
Monotonicity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S} \subseteq \mathcal{S}'$, then $\mathcal{S}' \vdash \psi$.
Transitivity: if $\mathcal{S} \vdash \psi$ and $\mathcal{S}', \psi \vdash \phi$ then $\mathcal{S}, \mathcal{S}' \vdash \phi$.

In addition, we shall assume that $\mathfrak{L}$ satisfies the following (standard) conditions:

Structurality: if $\mathcal{S} \vdash \psi$ then $\theta(\mathcal{S}) \vdash \theta(\psi)$ for every $\mathcal{L}$-substitution θ.
Non-triviality: there are a non-empty set $\mathcal{S}$ and a formula ψ such that $\mathcal{S} \nvdash \psi$.
Finiteness: if $\mathcal{S} \vdash \psi$ then there is a *finite* set $\Gamma \subseteq \mathcal{S}$ such that $\Gamma \vdash \psi$.

Structurality assures that inferences are closed under substitutions. Non-triviality excludes trivial logics and (together with structurality) prevents some anomalies, like the inference of an atom q from a distinct atom p. Finiteness is essential for practical reasoning and is satisfied by any logic that has a decent proof system. Its usefulness is demonstrated, e.g., in Note 1 below.

In what follows we shall assume that the language $\mathcal{L}$ contains at least the following connectives:

- a $\vdash$-*negation* $\neg$, satisfying: $p \nvdash \neg p$ and $\neg p \nvdash p$ (for every atomic p), and
- a $\vdash$-*conjunction* $\wedge$, satisfying: $\mathcal{S} \vdash \psi \wedge \phi$ iff $\mathcal{S} \vdash \psi$ and $\mathcal{S} \vdash \phi$.

For a finite set of formulas Γ we denote by $\bigwedge \Gamma$ the conjunction of all the formulas in Γ.

The Tarskian properties impose reasonable constraints on rational reasoners and how they are expected to support their claims (for instance, the arguments in the proofs of most of the papers, including this one, are based on classical logic). Thus, in Section 2.2.1 we use the base (Tarskian) logic at hand for constructing arguments. Then, on top of this, in Sections 2.2.2 and 2.2.3 we define non-monotonic (argumentation-based) entailments that allow to revise conclusions and reason in the presence of conflicts.

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[2] Properties of these extensions can be found in [19]. Further extensions are considered, e.g., in [9–11].

Axioms: $\psi \Rightarrow \psi$

Structural Rules:

Weakening: $\dfrac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}$

Cut: $\dfrac{\Gamma_1 \Rightarrow \Delta_1, \psi \quad \Gamma_2, \psi \Rightarrow \Delta_2}{\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}$

Logical Rules:

$[\wedge\Rightarrow]\ \dfrac{\Gamma, \psi, \varphi \Rightarrow \Delta}{\Gamma, \psi \wedge \varphi \Rightarrow \Delta}$ $\qquad [\Rightarrow\wedge]\ \dfrac{\Gamma \Rightarrow \Delta, \psi \quad \Gamma \Rightarrow \Delta, \varphi}{\Gamma \Rightarrow \Delta, \psi \wedge \varphi}$

$[\vee\Rightarrow]\ \dfrac{\Gamma, \psi \Rightarrow \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \vee \varphi \Rightarrow \Delta}$ $\qquad [\Rightarrow\vee]\ \dfrac{\Gamma \Rightarrow \Delta, \psi, \varphi}{\Gamma \Rightarrow \Delta, \psi \vee \varphi}$

$[\supset\Rightarrow]\ \dfrac{\Gamma \Rightarrow \psi, \Delta \quad \Gamma, \varphi \Rightarrow \Delta}{\Gamma, \psi \supset \varphi \Rightarrow \Delta}$ $\qquad [\Rightarrow\supset]\ \dfrac{\Gamma, \psi \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \psi \supset \varphi, \Delta}$

$[\neg\Rightarrow]\ \dfrac{\Gamma \Rightarrow \Delta, \psi}{\Gamma, \neg\psi \Rightarrow \Delta}$ $\qquad [\Rightarrow\neg]\ \dfrac{\Gamma, \psi \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \neg\psi}$

Fig. 1. The proof system LK.

2.2.1. Arguments as sequents

Logical arguments may be defined in different ways. Following the discussions in [3] and [6], arguments are represented here by the well-known proof-theoretical notion of *sequents* [24].[3]

Definition 5. Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic and $\mathcal{S}$ a set of $\mathcal{L}$-formulas.

- An $\mathcal{L}$*-sequent* (a sequent, for short) is an expression of the form $\Gamma \Rightarrow \Delta$, where Γ and Δ are finite sets of $\mathcal{L}$-formulas, and $\Rightarrow$ is a new symbol (not in the language $\mathcal{L}$).
- An $\mathfrak{L}$*-argument* (an argument, for short) is an $\mathcal{L}$-sequent of the form $\Gamma \Rightarrow \psi$, where $\Gamma \vdash \psi$.
- An $\mathfrak{L}$*-argument* based on $\mathcal{S}$ is an $\mathfrak{L}$-argument $\Gamma \Rightarrow \psi$, where $\Gamma \subseteq \mathcal{S}$. We say that Γ is the *support set* of $\Gamma \Rightarrow \psi$, and that ψ is its *conclusion*. The set of all the $\mathfrak{L}$-arguments that are based on $\mathcal{S}$ is denoted $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$.

An argument $\Gamma \Rightarrow \psi$ is *tautological* (for $\mathfrak{L}$) if $\Gamma = \emptyset$, and it is *contradictory* (for $\mathfrak{L}$) if $\psi \vdash \neg\psi$.

Note 1. Clearly, $\Gamma \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ for some (finite) $\Gamma \subseteq \mathcal{S}$, iff $\mathcal{S} \vdash \psi$.

Proof systems that operate on sequents (and so on arguments) are called *sequent calculi* [24]. The sequent calculi considered here consist of *inference rules* of the form

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \ \ldots \ \Gamma_n \Rightarrow \Delta_n}{\Gamma \Rightarrow \Delta}. \tag{1}$$

In what follows we shall say that the sequents $\Gamma_i \Rightarrow \Delta_i$ $(i = 1, \ldots, n)$ are the *conditions* (or the *prerequisits*) of the rule above, and that $\Gamma \Rightarrow \Delta$ is its *conclusion*.[4]

In the sequel we shall usually assume that the underlying logic has a sound and complete sequent calculus, that is, a sequent-based proof system $\mathfrak{C}$, such that $\Gamma \vdash \psi$ iff the sequent $\Gamma \Rightarrow \psi$ is provable in $\mathfrak{C}$.

Example 2. In this paper we shall mostly use classical logic (CL) for our demonstrations. Gentzen's well-known sequent calculus LK, which is sound and complete for CL, is represented in Fig. 1.

Thus, when CL is the underlying logic, one may derive (e.g., by LK) any argument that corresponds to a classically valid entailment, like CL-tautological arguments obtained by the rule of excluded middle ($\Rightarrow \psi \vee \neg\psi$), arguments that correspond to the Disjunctive Syllogism ($\psi, \neg\psi \vee \phi \Rightarrow \phi$), and so on.

[3] As explained in [3,6], this allows us to consider arguments in an abstract and modular way, where, for instance, the consistency and minimality restrictions that are posed in some other formalisms on the arguments' supports (see, e.g., [13,14]) can be lifted, non-classical logics may serve as the underlying formalisms, and known methodologies and techniques from proof theory may be incorporated.

[4] As usual, axioms are treated as inference rules without conditions, i.e., they are rules of the form $\frac{}{\Gamma \Rightarrow \Delta}$.

2.2.2. Attacks as elimination rules

Different attack relations have been considered in the literature for logical argumentation frameworks (see, e.g., [13,25,36]). In our case, attacks allow for the elimination (or, the discharging) of sequents. We shall denote by $\Gamma \not\Rightarrow \psi$ the elimination of the sequent $\Gamma \Rightarrow \psi$. Alternatively, $\overline{s}$ denotes the elimination of s. Now, a *sequent elimination rule* (or *attack rule*) has a similar form as an inference rule, expect that its conclusion is a renouncing of one of its conditions (for clarity, the renounced sequent will be the last condition of the rule). Thus, elimination rules are of the following form:

$$\frac{\Gamma_1 \Rightarrow \Delta_1 \;\; \dots \;\; \Gamma_n \Rightarrow \Delta_n}{\Gamma_n \not\Rightarrow \Delta_n}. \tag{2}$$

The prerequisites of attack rules usually consist of three ingredients. We shall usually say that the first sequent in the rule's prerequisites is the "attacking" sequent, the last sequent in the rule's prerequisites is the "attacked" sequent, and the other prerequisites are the conditions for the attack. In this view, conclusions of sequent elimination rules are the eliminations of the attacked arguments.

Example 3. Fig. 2 lists some elimination rules in the context of logical argumentation systems (see also [6]). Other attack rules, e.g. for deontic logics and normative reasoning, can be found in [44,45]. Relations between these rules and a study of some of their properties are given in [6] and [25].

2.2.3. Argumentation settings and the induced logical frameworks

We now combine sequents and elimination rules for defining corresponding argumentation frameworks. For this, we need the following definition.

Definition 6. An *argumentation setting* (a setting, for short) is a triple $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$, where $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ is a propositional logic, $\mathfrak{C}$ is a sound and complete sequent calculus for $\mathfrak{L}$, and $\mathfrak{A}$ is a set of attack rules expressed in terms of $\mathcal{L}$-sequents.

Definition 7. Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ be a setting, $\mathcal{S}$ a set of formulas, and θ an $\mathcal{L}$-substitution (i.e., a function representing replacements, in $\mathcal{L}$-formulas, of atomic formulas by $\mathcal{L}$-formulas).

- An inference rule $\mathcal{R}$ of the form of (1) above is *applicable* (for $\mathfrak{S}$, with respect to θ), if for every $1 \le i \le n$, $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ is $\mathfrak{C}$-provable.
- An elimination rule $\mathcal{R}$ of the form of (2) above is $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$*-applicable* (for $\mathfrak{S}$, with respect to θ), if $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ and $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$ are in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and for each $1 < i < n$, $\theta(\Gamma_i) \Rightarrow \theta(\Delta_i)$ is $\mathfrak{C}$-provable.

In the second case above we shall say that $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ $\mathcal{R}$*-attacks* $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$. Note that the attacker and the attacked sequents must be elements of $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ (in order to prevent 'irrelevant attacks' by arguments whose support sets do not belong to $\mathcal{S}$).

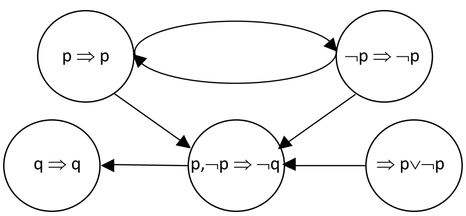
Example 4. Suppose that $\{p, \neg p\} \subseteq \mathcal{S}$. When CL is the underlying logic, the sequents $p \Rightarrow p$ and $\neg p \Rightarrow \neg p$ attack each other according to Undercut (as well as according to other rules in Fig. 2), while the tautological sequent $\Rightarrow \psi \vee \neg\psi$ is not Undercut-attacked by any sequent in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$, since it has an empty support set.

The induced argumentation framework is now defined as follows:

Definition 8. Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ be an argumentation setting and let $\mathcal{S}$ be a set of formulas. The *sequent-based (logical) argumentation framework* for $\mathcal{S}$ (induced by $\mathfrak{S}$) is the argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack} \rangle$, where $(s_1, s_2) \in \mathit{Attack}$ iff there is an $\mathcal{R} \in \mathfrak{A}$ such that s_1 $\mathcal{R}$-attacks s_2.

In what follows, somewhat abusing the notations, we shall sometimes identify *Attack* with $\mathfrak{A}$.

Example 5. Consider the argumentation setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \mathsf{Ucut} \rangle$, based on classical logic CL, its sequent calculus LK (Fig. 1), and Undercut as the sole attack rule (Fig. 2). Let $\mathcal{S}_1 = \{p, \neg p, q\}$. The figure below is part of the graphic representation of $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}_1)$. Here, nodes represent arguments (in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$), and a directed edge from node n_1 to node n_2 indicates that the argument that is represented by node n_1 Ucut-attacks the argument that is represented by node n_2.



Defeat:	[Def]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Compact Defeat:	[C-Def]	$\dfrac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Direct Defeat:	[D-Def]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \phi \quad \Gamma_2, \phi \Rightarrow \psi_2}{\Gamma_2, \phi \not\Rightarrow \psi_2}$
Indirect Defeat:	[I-Def]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Compact Direct Defeat:	[CD-Def]	$\dfrac{\Gamma_1 \Rightarrow \neg \phi \quad \Gamma_2, \phi \Rightarrow \psi_2}{\Gamma_2, \phi \not\Rightarrow \psi_2}$
Compact Indirect Defeat:	[CI-Def]	$\dfrac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Undercut:	[Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Compact Undercut:	[C-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \Gamma_2, \Gamma_2' \Rightarrow \psi_2}{\Gamma_2, \Gamma_2' \not\Rightarrow \psi_2}$
Direct Undercut:	[D-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \gamma_2 \quad \neg \gamma_2 \Rightarrow \psi_1 \quad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2}$
Compact Direct Undercut:	[CD-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \neg \gamma_2 \quad \Gamma_2, \gamma_2 \Rightarrow \psi_2}{\Gamma_2, \gamma_2 \not\Rightarrow \psi_2}$
Canonical Undercut:	[Ca-Ucut]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \bigwedge \Gamma_2 \quad \neg \bigwedge \Gamma_2 \Rightarrow \psi_1 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Rebuttal:	[Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \psi_2 \quad \neg \psi_2 \Rightarrow \psi_1 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Compact Rebuttal 1:	[C-Reb-1]	$\dfrac{\Gamma_1 \Rightarrow \neg \psi_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Compact Rebuttal 2:	[C-Reb-2]	$\dfrac{\Gamma_1 \Rightarrow \psi_2 \quad \Gamma_2 \Rightarrow \neg \psi_2}{\Gamma_2 \not\Rightarrow \neg \psi_2}$
Defeating Rebuttal:	[D-Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \neg \psi_2 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Reductio Defeating Rebuttal:	[RD-Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_2 \Rightarrow \neg \psi_1 \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$
Indirect Rebuttal:	[I-Reb]	$\dfrac{\Gamma_1 \Rightarrow \psi_1 \quad \psi_1 \Rightarrow \varphi \quad \psi_2 \Rightarrow \neg \varphi \quad \Gamma_2 \Rightarrow \psi_2}{\Gamma_2 \not\Rightarrow \psi_2}$

Fig. 2. Sequent elimination rules.

We conclude this section with a definition of some useful properties that a sequent-based argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack \rangle$ may have. Some of these properties will be needed in the sequel for considering (and proving) different characteristics of dynamic derivations and the entailment relations induced by them (see, for instance, Definition 20, Proposition 6, and its proof in Appendix A.

Definition 9. Let $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack \rangle$ be a (sequent-based) argumentation framework, $\sim$ an equivalence relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, and $\equiv$ an equivalence relation on $\mathcal{L}$. We define the following properties of $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$:

- *Preservation of tautological arguments*: $\{\Rightarrow \psi \mid \Rightarrow \psi \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) - \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})^{+}$.
- *Irreflexivity*: *Attack* is irreflexive.
- *Acyclicity*: There is no odd-length cycle of attacks in *Attack*: if $(s_1, s_2), (s_2, s_3), \ldots, (s_{2n}, s_{2n+1}) \in Attack$ for some $n \in \mathbb{N}$, then $(s_{2n+1}, s_1) \notin Attack$.
- *Non-transitivity*: *Attack* is non-transitive.

- *Symmetry*: *Attack* is symmetric.
- *Symmetry modulo* $\sim$: If $(s_1, s_2) \in Attack$ then $(s_2', s_1') \in Attack$ for some $s_1' \sim s_1$ and $s_2' \sim s_2$.
- *Left invariance modulo* $\sim$: If $(s_1, s_2) \in Attack$ then $(s_1', s_2) \in Attack$ for every $s_1' \sim s_1$.
- *Right invariance modulo* $\sim$: If $(s_1, s_2) \in Attack$ then $(s_1, s_2') \in Attack$ for every $s_2' \sim s_2$.
- *Support invariance modulo* $\equiv$: If $(\Gamma_1 \Rightarrow \psi_1, \Gamma_2 \Rightarrow \psi_2) \in Attack$, then
  a) $(\Gamma_1' \Rightarrow \psi_1, \Gamma_2 \Rightarrow \psi_2) \in Attack$ for every $\Gamma_1' \subseteq \mathcal{S}$ such that $\bigwedge \Gamma_1' \equiv \bigwedge \Gamma_1$, and
  b) $(\Gamma_1 \Rightarrow \psi_1, \Gamma_2' \Rightarrow \psi_2) \in Attack$ for every $\Gamma_2' \subseteq \mathcal{S}$ such that $\bigwedge \Gamma_2' \equiv \bigwedge \Gamma_2$.
- *Conclusion invariance modulo* $\equiv$: If $(\Gamma_1 \Rightarrow \psi_1, \Gamma_2 \Rightarrow \psi_2) \in Attack$, then
  a) $(\Gamma_1 \Rightarrow \psi_1', \Gamma_2 \Rightarrow \psi_2) \in Attack$ for every $\psi_1' \equiv \psi_1$, and
  b) $(\Gamma_1 \Rightarrow \psi_1, \Gamma_2 \Rightarrow \psi_2') \in Attack$ for every $\psi_2' \equiv \psi_2$.

Note 2. Concerning the properties in Definition 9, we note that:

1. Preservation of tautological arguments means that tautological arguments are not attacked; Acyclicity generalizes Irreflexivity to any cycle of attacks whose length is odd; Non-transitivity assures that no sequent attacks and defends another sequent at the same time; Symmetry modulo $\sim$ is weaker than symmetry, as it allows to exchange the attacking and the attacked arguments only by some of their $\sim$-equivalent arguments.
2. When $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ is left [right] invariant modulo $\sim$ we shall say that $\sim$ is *left* [*right*] *congruent* on $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$.[5] Indeed, this notion can be expressed in terms of invariance relations (see [34]), where for every $s \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $R_s = \{t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid (t, s) \in Attack\}$, $\sim$ is a right congruence on $(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack)$ iff $\sim$ is invariant on $\{R_s \mid s \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\}$.
3. A common way to define $\equiv$ is by logical equivalence. The relation $\sim$ may be defined in some cases by logical equivalence on the arguments' support sets (see, e.g., the next example and Example 15).

Example 6. Let $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ be an argumentation framework induced by a setting $\mathfrak{S}$ in which $\mathfrak{A}$ consists of any of the attack rules in Fig. 2, except the rebuttals and the direct attacks. Let $\sim$ be an equivalence relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, defined by $\Gamma_1 \Rightarrow \psi_1 \sim \Gamma_2 \Rightarrow \psi_2$ iff $\Gamma_1 = \Gamma_2$. Then $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ is right invariant modulo $\sim$ (alternatively, $\sim$ is a right congruence on $(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack)$).

The above properties of argumentation frameworks may be attached to specific attack rules as follows.

Definition 10. An elimination rule $\mathcal{R}$ is *irreflexive* (respectively: *tautology preserving*, *acyclic*, *non-transitive*, *symmetric*, etc.) with respect to a logic $\mathfrak{L} = \langle \mathcal{L}, \vdash\rangle$, if for every set $\mathcal{S}$ of $\mathcal{L}$-formulas and for every calculus $\mathfrak{C}$ which is sound and complete for $\mathfrak{L}$, the argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ for $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \{\mathcal{R}\}\rangle$ is irreflexive (respectively: *tautology preserving*, *acyclic*, *non-transitive*, *symmetric*, etc.).

3. Dynamic proofs

We now consider the notions of proofs (or derivations) for argumentation settings. Such proofs are meant to indicate what assertions may be concluded from a given argumentation framework. Intuitively, the idea is that, given an argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ where $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}\rangle$, dynamic proofs are sequences of tuples constructed by applications of inference rules from $\mathfrak{C}$ and eliminating rules from $\mathfrak{A}$. This allows to provide derivations for sequents of the form $\Gamma \Rightarrow \psi$ for some $\Gamma \subseteq \mathcal{S}$, in which case ψ is regarded an argumentative consequence of $\mathcal{S}$ (with respect to the setting $\mathfrak{S}$).

In what follows we fix a given setting $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}\rangle$, thus the underlying logic, a Gentzen-type proof system for it, and the elimination rules, are pre-determined.

Definition 11. A (proof) *tuple* (also called *derivation step* or *proof step*) is a quadruple $\langle i, s, \mathsf{J}, \mathsf{A}\rangle$, where i (the tuple's index) is a natural number, s (the tuple's sequent) is either a sequent or an eliminated sequent, J (the tuple's justification) is a string, and A (the tuple's attacker) is the emptyset or a singleton of a sequent.[6]

As noted previously, unlike 'standard' Gentzen-type systems, dynamic proofs may consist not only of applications of rules for introducing new sequents, but also applications of rules for eliminating sequents. This is defined next.

Definition 12. Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}\rangle$ be a setting and $\mathcal{S}$ a set of formulas in $\mathcal{L}$. A *simple* (dynamic) *derivation* (with respect to $\mathfrak{S}$ and $\mathcal{S}$) is a finite sequence $\mathcal{D} = \langle T_1, \dots T_m\rangle$ of proof tuples, where each $T_i \in \mathcal{D}$ is of one of the following forms:

[5] In the terminology of Yeh ([47, p. 10]) this means that $\sim$ is a *weak congruence of type III* on the graph $(\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack^T)$, where $Attack^T = \{(s, t) \mid (t, s) \in Attack\}$.

[6] In what follows we shall sometimes omit the last component of a tuple in case that it is the emptyset, and omit the set signs (the parentheses) in case that it is a singleton.

- $T_i = \langle i, \theta(\Gamma) \Rightarrow \theta(\Delta), \mathsf{J}, \emptyset\rangle$, where there is an inference rule $\mathcal{R} \in \mathfrak{C}$ of the form of (1) above that is applicable for some $\mathcal{L}$-substitution θ, and for every $1 \le k \le n$ there is a proof tuple $\langle i_k, s_k, \mathsf{J}_k, \emptyset\rangle$ in which $i_k < i$ and s_k is the sequent $\theta(\Gamma_k) \Rightarrow \theta(\Delta_k)$. In this case, J = "$\mathcal{R}; i_1, \dots, i_n$". In what follows we shall call T_i an *introducing tuple*.
- $T_i = \langle i, \theta(\Gamma_n) \not\Rightarrow \theta(\Delta_n), \mathsf{J}, \theta(\Gamma_1) \Rightarrow \theta(\Delta_1)\rangle$, where there is an elimination rule $\mathcal{R} \in \mathfrak{A}$ of the form of (2) above that is $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$-applicable for some $\mathcal{L}$-substitution θ,[7] and for every $1 \le k \le n$ there is a proof tuple $\langle i_k, s_k, \mathsf{J}_k, \emptyset\rangle$, in which $i_k < i$ and $s_k = \theta(\Gamma_k) \Rightarrow \theta(\Delta_k)$. In this case, J ="$\mathcal{R}; i_1, \dots, i_n$". In what follows we shall call T_i an *eliminating tuple*.

In the sequel we shall sometimes identify introducing tuples with their derived sequents and eliminating tuples with their eliminated sequents.

Example 7. Consider again the argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}_1)$ from Example 5, in which $\mathfrak{S} = \langle \mathsf{CL}, LK, \mathsf{Ucut}\rangle$ and $\mathcal{S}_1 = \{p, \neg p, q\}$. Below is a simple derivation with respect to $\mathfrak{S}$ and $\mathcal{S}_1$. To simplify the reading, in this and other derivations in the rest of the paper we shall sometimes use abbreviations or omit some details, e.g. the tuple signs in proof steps.

1.	$p \Rightarrow p$	Axiom	
2.	$\Rightarrow p, \neg p$	$[\Rightarrow \neg]$, 1	
3.	$\Rightarrow p \vee \neg p$	$[\Rightarrow \vee]$, 2	
4.	$p \vee \neg p \Rightarrow \neg(p \wedge \neg p)$	...	
5.	$\neg(p \wedge \neg p) \Rightarrow p \vee \neg p$	...	
6.	$q \Rightarrow q$	Axiom	
7.	$\neg p \Rightarrow \neg p$	Axiom	
8.	$p \not\Rightarrow p$	Ucut, 7, 7, 7, 1	$\neg p \Rightarrow \neg p$
9.	$p \Rightarrow \neg\neg p$	...	
10.	$\neg\neg p \Rightarrow p$	...	
11.	$\neg p \not\Rightarrow \neg p$	Ucut, 1, 9, 10, 7	$p \Rightarrow p$

Note that in this derivation Tuples 8 and 11 are eliminating while the other tuples are introducing.

Given a simple derivation $\mathcal{D}$, we shall denote by $\mathsf{Top}(\mathcal{D})$ the tuple with the highest index in $\mathcal{D}$ and by $\mathsf{Tail}(\mathcal{D})$ the simple derivation $\mathcal{D}$ without $\mathsf{Top}(\mathcal{D})$. Also, we shall denote by $\mathcal{D}' = \mathcal{D} \oplus \langle T_1, \dots, T_n\rangle$ the simple derivation whose prefix is $\mathcal{D}$ and whose suffix is $\langle T_1, \dots, T_n\rangle$ (thus, for instance, when $n = 1$ we have that $T = \mathsf{Top}(\mathcal{D} \oplus T)$ and $\mathcal{D} = \mathsf{Tail}(\mathcal{D} \oplus T)$). We call $\mathcal{D}'$ the *extension* of $\mathcal{D}$ by $\langle T_1, \dots, T_n\rangle$.

To indicate that the validity of a derived sequent (in a simple derivation) is in question due to attacks on it, we need the following evaluation process.

Definition 13. Given a simple derivation $\mathcal{D}$, the iterative top-down function $\mathsf{Evaluate}(\mathcal{D})$ given in Fig. 3, computes the following three sets: $\mathsf{Elim}(\mathcal{D})$ – the sequents that (at least once in $\mathcal{D}$) are attacked by an attacker which is not already attacked, $\mathsf{Attack}(\mathcal{D})$ – the sequents that attack a sequent in $\mathsf{Elim}(\mathcal{D})$, and $\mathsf{Accept}(\mathcal{D})$ – the derived sequents in $\mathcal{D}$ that are not in $\mathsf{Elim}(\mathcal{D})$.

Example 8. Consider the simple derivation in Example 7.

- After Step 6 of the derivation we have that $q \Rightarrow q$ and $p \Rightarrow p$ are in Derived and no sequent is in Elim, thus both of these sequents are also in Accept.
- After Step 8 of the derivation we still have that $q \Rightarrow q \in \mathsf{Accept}$, and now also $\neg p \Rightarrow \neg p \in \mathsf{Accept}$. However, $p \Rightarrow p$ is in Elim (since it is attacked by $\neg p \Rightarrow \neg p \in \mathsf{Attack}$).
- After Step 11 of the derivation $q \Rightarrow q$ remains in Accept, while the statuses of the other two sequents are reversed: $p \Rightarrow p \in \mathsf{Accept}$ while $\neg p \Rightarrow \neg p \in \mathsf{Elim}$.

[7] Remember that this means, in particular, that the attacking sequent $\theta(\Gamma_1) \Rightarrow \theta(\Delta_1)$ and the attacked sequent $\theta(\Gamma_n) \Rightarrow \theta(\Delta_n)$ are both in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. This prevents situations in which, e.g., $\neg p \Rightarrow \neg p$ attacks $p \Rightarrow p$, although $\mathcal{S} = \{p\}$.

```
function Evaluate(𝒟)  /* 𝒟 – a simple derivation */
Attack := ∅;  Elim := ∅;  Derived := ∅;
while (𝒟 is not empty) do {

    if (Top(𝒟) = ⟨i, s, J, ∅⟩) then  /* Top(𝒟) is an introducing tuple */
       Derived := Derived ∪ {s};
    if (Top(𝒟) = ⟨i, s̄, J, r⟩) then  /* Top(𝒟) is an attacking tuple */
       if (r ∉ Elim) then Elim := Elim ∪ {s} and Attack := Attack ∪ {r};
    𝒟 := Tail(𝒟); }

Accept := Derived − Elim;
return (Attack, Elim, Accept)
```

Fig. 3. Evaluation of a simple derivation.

Using the sets in the last definition (and algorithm), we can now specify a condition for the coherence of a derivation. Intuitively, it assures that eliminating tuples represent 'firm' attacks: there is no sequent in the underlying derivation that eliminates another sequent, and later on is eliminated itself.

Definition 14. A simple derivation $\mathcal{D}$ is *coherent*, if $\mathsf{Attack}(\mathcal{D}) \cap \mathsf{Elim}(\mathcal{D}) = \emptyset$.

Example 9. The simple derivation in Example 7 is coherent.

Next, we show that the evaluation process (in Fig. 3) for a derivation $\mathcal{D}$ is adequate in terms of the argumentation framework that is induced by $\mathcal{D}$.

Definition 15. Let $\mathcal{D}$ be a simple derivation. The *sequent-based argumentation framework that is induced by* $\mathcal{D}$ is the graph $\mathcal{AF}(\mathcal{D}) = \langle Derived(\mathcal{D}), Attack(\mathcal{D})\rangle$, where $s \in Derived(\mathcal{D})$ if there is an introducing tuple $\langle i, s, \mathsf{J}, \emptyset\rangle$ in $\mathcal{D}$, and $(r, s) \in Attack(\mathcal{D})$ if there is an eliminating tuple $\langle i, \overline{s}, \mathsf{J}, r\rangle$ in $\mathcal{D}$.[8]

Proposition 1. *For every simple derivation $\mathcal{D}$ the set $\mathsf{Accept}(\mathcal{D})$ is conflict-free in $\mathcal{AF}(\mathcal{D})$. If $\mathcal{D}$ is coherent, $\mathsf{Accept}(\mathcal{D})$ is a stable extension of $\mathcal{AF}(\mathcal{D})$.*

Proof. If $\mathsf{Accept}(\mathcal{D})$ is not conflict-free in $\mathcal{AF}(\mathcal{D})$ then there are $s, t \in \mathsf{Accept}(\mathcal{D})$ such that $\langle i, \overline{s}, \mathsf{J}, t\rangle \in \mathcal{D}$ for some $i \in \mathbb{N}$ and some justification J. Since t is accepted, it is not eliminated, and so by the evaluation algorithm s is eliminated, in a contradiction to the assumption that s is also accepted.

Suppose now that $\mathcal{D}$ is coherent. We have to show that $\mathsf{Accept}(\mathcal{D})$ is admissible, complete, and stable (that is, $\mathsf{Accept}(\mathcal{D}) \cup \mathsf{Accept}(\mathcal{D})^+ = \mathsf{Derived}(\mathcal{D})$).

(I) $\mathsf{Accept}(\mathcal{D})$ **is admissible:** Suppose that there is some $s \in \mathsf{Accept}(\mathcal{D})$ that is attacked by t, i.e., there is $T_i = \langle i, \overline{s}, \mathsf{J}, t\rangle \in \mathcal{D}$. Since $s \notin \mathsf{Elim}(\mathcal{D})$ (because it is accepted), t must be in $\mathsf{Elim}(\mathcal{D})$. This means that there is some $j > i$ such that $T_j = \langle j, \overline{t}, \mathsf{J}, r\rangle \in \mathcal{D}$ for some $r \notin \mathsf{Elim}(\mathcal{D})$. It follows that $r \in \mathsf{Attack}(\mathcal{D})$ and since $\mathcal{D}$ is coherent, r is not eliminated later on (i.e., in the remaining j iterations of the evaluation algorithm). Thus, $r \in \mathsf{Accept}(\mathcal{D})$, which means that the attacker (t) of s is attacked by an element (r) of $\mathsf{Accept}(\mathcal{D})$. Thus $s \in \mathsf{Def}(\mathsf{Accept}(\mathcal{D}))$, and so $\mathsf{Accept}(\mathcal{D}) \subseteq \mathsf{Def}(\mathsf{Accept}(\mathcal{D}))$.

(II) $\mathsf{Accept}(\mathcal{D})$ **is complete:** Suppose that $s \in \mathsf{Def}(\mathsf{Accept}(\mathcal{D}))$. Then for every proof tuple $T_i = \langle i, \overline{s}, \mathsf{J}, t\rangle \in \mathcal{D}$ there is a proof tuple $T_j = \langle j, \overline{t}, \mathsf{J}, r\rangle \in \mathcal{D}$ and $r \in \mathsf{Accept}(\mathcal{D})$. Now,
- If $j > i$ then since $r \notin \mathsf{Elim}(\mathcal{D})$ we have that $t \in \mathsf{Elim}(\mathcal{D})$, and so s is not eliminated.
- If $i > j$ then either $t \in \mathsf{Elim}(\mathcal{D})$ and again s is not eliminated, or $t \notin \mathsf{Elim}(\mathcal{D})$, thus $t \in \mathsf{Attack}(\mathcal{D})$ (because of T_i) and $t \in \mathsf{Elim}(\mathcal{D})$ (because of T_j), and so $\mathcal{D}$ is not coherent, in contradiction to our assumption.

By the two items above, if s is attacked in $\mathcal{D}$, its attacker must be in $\mathsf{Elim}(\mathcal{D})$, and so $s \in \mathsf{Accept}(\mathcal{D})$. Thus, $\mathsf{Def}(\mathsf{Accept}(\mathcal{D})) \subseteq \mathsf{Accept}(\mathcal{D})$, and by the admissibility of $\mathsf{Accept}(\mathcal{D})$, $\mathsf{Def}(\mathsf{Accept}(\mathcal{D})) = \mathsf{Accept}(\mathcal{D})$.

(III) $\mathsf{Accept}(\mathcal{D})$ **is stable:** In other words, $\mathsf{Accept}(\mathcal{D}) \cup \mathsf{Accept}(\mathcal{D})^+ = \mathsf{Derived}(\mathcal{D})$. Since $\mathsf{Derived}(\mathcal{D}) = \mathsf{Accept}(\mathcal{D}) \cup \mathsf{Elim}(\mathcal{D})$, it is enough to show that $\mathsf{Accept}(\mathcal{D})^+$ coincides with $\mathsf{Elim}(\mathcal{D})$. Indeed,

[8] Note that while *Derived*($\mathcal{D}$) is the same as the set $\mathsf{Derived}(\mathcal{D})$ produced by the function $\mathsf{Evaluate}(\mathcal{D})$ (in Fig. 3), *Attack*($\mathcal{D}$) is *not* the same as the set $\mathsf{Attack}(\mathcal{D})$ produced by that function, since here just the existence of an eliminating tuple merits a directed edge from the attacker to the attacked sequent, no matter whether the attacker is counter-attacked.

- To see that $\mathsf{Elim}(\mathcal{D}) \subseteq \mathsf{Accept}(\mathcal{D})^+$, let $s \in \mathsf{Elim}(\mathcal{D})$. Hence, there is an attacking tuple $T = \langle i, \bar{s}, \mathsf{J}, t\rangle \in \mathcal{D}$ and when the algorithm reaches T, we have that $t \notin \mathsf{Elim}(\mathcal{D})$. Thus $t \in \mathsf{Attack}(\mathcal{D})$, and since $\mathcal{D}$ is coherent, $t \notin \mathsf{Elim}(\mathcal{D})$ also when the algorithm terminates. It follows that $t \in \mathsf{Accept}(\mathcal{D})$, and so $s \in \mathsf{Accept}(\mathcal{D})^+$.
- To see that $\mathsf{Accept}(\mathcal{D})^+ \subseteq \mathsf{Elim}(\mathcal{D})$, let $s \in \mathsf{Accept}(\mathcal{D})^+$. Then there is a tuple $T = \langle i, \bar{s}, \mathsf{J}, t\rangle \in \mathcal{D}$ such that $t \in \mathsf{Accept}(\mathcal{D})$. Since $\mathcal{D}$ is coherent, at the end of the execution of the algorithm $t \notin \mathsf{Elim}(\mathcal{D})$. Thus, since $\mathsf{Elim}(\mathcal{D})$ grows monotonically during the execution, in particular $t \notin \mathsf{Elim}(\mathcal{D})$ when the algorithm reaches the tuple T. It follows that $s \in \mathsf{Elim}(\mathcal{D})$. □

Interestingly, the following proposition also holds:

Proposition 2. *Let $\mathcal{D}$ be a simple derivation. If $\mathcal{E}$ is a stable extension of $\mathcal{AF}(\mathcal{D})$ then there is a coherent simple derivation $\mathcal{D}'$ such that $\mathcal{AF}(\mathcal{D}') = \mathcal{AF}(\mathcal{D})$ and $\mathcal{E} = \mathsf{Accept}(\mathcal{D}')$.*

Proof. Let $\mathcal{D}$ be a simple derivation and $\mathcal{E}$ a stable extension of the sequent-based argumentation framework $\mathcal{AF}(\mathcal{D}) = \langle \mathit{Derived}(\mathcal{D}), \mathit{Attack}(\mathcal{D})\rangle$ that is induced by $\mathcal{D}$. Consider a simple derivation $\mathcal{D}'$ which is a concatenation of the following sequences $\mathcal{D}'_1 \oplus \mathcal{D}'_2 \oplus \mathcal{D}'_3$, where $\mathcal{D}'_1$ contains the tuples introducing the sequents in $\mathit{Derived}(\mathcal{D})$, $\mathcal{D}'_3$ consists of tuples of the form $\langle i, \bar{s}, \mathsf{J}, t\rangle$ where $t \in \mathcal{E}$ and $s \in \mathcal{E}^+$, and $\mathcal{D}'_2$ consists of the attacking tuples for the other elements in $\mathit{Attack}(\mathcal{D})$ (the order of the elements in $\mathcal{D}'_2$ and in $\mathcal{D}'_3$ may be arbitrary, and some of these sequences may be empty for some $\mathcal{D}'$). Now, by the definition of $\mathcal{D}'$, clearly $\mathcal{AF}(\mathcal{D}') = \mathcal{AF}(\mathcal{D})$. Also, since $\mathcal{E}$ is stable, $\mathcal{E}^+ = \mathit{Derived}(\mathcal{D}) - \mathcal{E} = \mathit{Derived}(\mathcal{D}') - \mathcal{E}$, and so when the algorithm completes its pass over $\mathcal{D}'_3$ it holds that $\mathsf{Attack} = \mathcal{E}$ and $\mathsf{Elim} = \mathit{Derived}(\mathcal{D}') - \mathcal{E}$. Clearly, the other tuples will not affect these sets, thus $\mathcal{D}'$ is coherent (since $\mathsf{Accept}(\mathcal{D}') \cap \mathsf{Elim}(\mathcal{D}') = \emptyset$) and $\mathsf{Accept}(\mathcal{D}') = \mathit{Derived}(\mathcal{D}) - \mathsf{Elim}(\mathcal{D}) = \mathcal{E}$. □

Together, Propositions 1 and 2 show a correspondence between the accepted sets of coherent simple derivations and the stable models of the sequent-based argumentation frameworks that are induced by those derivations.

Now we are ready to define derivations in a dynamic proof system.

Definition 16. Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}\rangle$ be an argumentation setting and let $\mathcal{S}$ be a set of formulas in $\mathcal{L}$. A *(dynamic) derivation* (for $\mathfrak{S}$, based on $\mathcal{S}$) is a simple derivation $\mathcal{D}$ (with respect to $\mathfrak{S}$ and $\mathcal{S}$) of one of the following forms:

a) $\mathcal{D} = \langle T\rangle$, where $T = \langle 1, s, \mathsf{J}, \emptyset\rangle$ is a proof tuple.
b) $\mathcal{D}$ is an extension of a dynamic derivation by a sequence $\langle T_1, \dots, T_n\rangle$ of introducing tuples (of the form $\langle i, s, \mathsf{J}, \emptyset\rangle$), whose derived sequents (the s's) are not in $\mathsf{Elim}(\mathcal{D})$.
c) $\mathcal{D}$ is an extension of a dynamic derivation by a sequence $\langle T_1, \dots, T_n\rangle$ of eliminating tuples (of the form $\langle i, \bar{s}, \mathsf{J}, r\rangle$), such that:
   (i) $\mathcal{D}$ is coherent: $\mathsf{Attack}(\mathcal{D}) \cap \mathsf{Elim}(\mathcal{D}) = \emptyset$, and
   (ii) the new attacking sequents (the r's) are not $\mathfrak{A}$-attacked by sequents in $\mathsf{Accept}(\mathcal{D}) \cap \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, where the attack is based on prerequisit conditions in $\mathcal{D}$.

Note 3. Conditions (i) and (ii) of Definition 16(c) assure that the attacks of the derivation are 'sound': by coherence neither of the attacking sequents of the additional elimination tuples is in $\mathsf{Elim}(\mathcal{D})$, and by Condition (ii) they are not attacked by an accepted $\mathcal{S}$-based sequent. As we show below (see Footnote 13), these two conditions are not dependent.

Dynamic derivations are therefore simple derivations that are progressed (i.e., extended) in a restricted manner. Accordingly, after each extension the status of the derived sequents is updated. Thus, derived sequents may be eliminated ("marked as unreliable") in light of new proof tuples, but also the other way around is possible: an eliminated sequent may be 'restored' if its attacking tuple is counter-attacked by a new eliminating tuple. It follows that previously derived data may not be derived anymore (and vice-versa) until and unless new derived information revises the state of affairs.

Example 10. It is easy to verify that the simple derivation given in Example 7 satisfies the conditions in Definition 16, and so it is also a dynamic derivation. Example 8 demonstrates the dynamic nature of this derivation. For instance, although the sequent $\neg p \Rightarrow \neg p$ is derived in Step 7 of the derivation, it is eliminated in Step 11 of the derivation as a consequence of an Undercut attack, initiated by $p \Rightarrow p$.

Further examples of dynamic derivations are considered in Section 4.

The following property of dynamic derivations immediately follows from their definition.

Proposition 3. *Every dynamic derivation is coherent.*

The next definition, of the outcomes of a dynamic derivation, states that we can safely (or 'finally') derive a derived sequent only when we are sure that there is no scenario in which it will be eliminated in some extension of the derivation.

Definition 17. Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ be a setting and let $\mathcal{S}$ be a set of formulas in $\mathcal{L}$. A sequent s is *finally derived* (or safely derived) in a dynamic derivation $\mathcal{D}$ (for $\mathfrak{S}$, based on $\mathcal{S}$), if $s \in \mathsf{Accept}(\mathcal{D})$, and $\mathcal{D}$ cannot be extended to a dynamic derivation $\mathcal{D}'$ (for $\mathfrak{S}$, based on $\mathcal{S}$) such that $s \in \mathsf{Elim}(\mathcal{D}')$.

A few notes are in order here. The concept of final derivability resembles a similar concept used in the context of adaptive logics for representing an irreversible acceptance of formulas by a derivation process (see, e.g., [12,43]). The non-monotonic nature of the formalisms in both cases dictates the introduction of external considerations (i.e., which are not expressible in the proofs themselves, see the examples in the next section) for assuring that derived data will not be refuted during the progressing of the derivation. It follows that the two kinds of derivations (the standard one and the final one) are inherently different: derived sequents may be eliminated (as a consequence of an application of eliminating tuples in which derived sequents are attacked), while finally derived sequents are non-eliminated derived objects. It follows that final derivability, unlike standard derivability, is monotonic in the length of the dynamic derivations. Indeed,

Proposition 4. *If s is finally derived in $\mathcal{D}$ then it is finally derived in any extension of $\mathcal{D}$.*

Proof. Suppose that s is finally derived in $\mathcal{D}$ but it is not finally derived in some extension $\mathcal{D}'$ of $\mathcal{D}$. This means that there is some extension $\mathcal{D}''$ of $\mathcal{D}'$ in which $s \in \mathsf{Elim}(\mathcal{D}'')$. Since $\mathcal{D}''$ is also an extension of $\mathcal{D}$, we get a contradiction to the final derivability of s in $\mathcal{D}$. $\square$

Another notable difference between ordinary and final derivability is related to their consistency. Consider the argumentation system in Example 5 with $\mathcal{S}_1 = \{p, \neg p, q\}$. As shown in Example 7, it might happen that the set of sequents derived from this framework contain contradictory conclusions (see, e.g., Tuples 1 and 7 in the dynamic derivation of Example 7). This cannot happen as far as final derivation is concerned. In fact, as shown in [4], the set of conclusions of the sequents that are finally derived in the setting $\mathfrak{S}$ of Example 5 from a finite set $\mathcal{S}$ of formulas, is equal to the transitive closure of the intersection of all the maximally consistent subsets of $\mathcal{S}$ (see [4] for the exact details and definitions). We refer to the next section for a further discussion on this distinction (e.g., Proposition 8) and other general properties of final derivability.

The induced entailment is now defined as follows:

Definition 18. Given an argumentation setting $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ and a set $\mathcal{S}$ of formulas, we denote by $\mathcal{S} \mid\!\sim_{\mathfrak{S}} \psi$ that there is an $\mathcal{S}$-based dynamic derivation for $\mathfrak{S}$, in which $\Gamma \Rightarrow \psi$ is finally derived for some finite $\Gamma \subseteq \mathcal{S}$.

When the underlying argumentation setting is clear from the context we shall sometimes abbreviate $\mid\!\sim_{\mathfrak{S}}$ by $\mid\!\sim$. Some basic properties of this entailment are considered in Section 5.

4. Examples and discussion

To get some more insight on dynamic proofs and their constructions we first examine in this section a few particular derivations and then consider some general properties of dynamic proofs.

4.1. Some simple case studies

Example 11. Consider again the simple derivation of Example 7, for the setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \mathsf{Ucut} \rangle$, based on the set of formulas $\mathcal{S}_1 = \{p, \neg p, q\}$. By Example 10 this is a dynamic derivation.

Note that after Step 6 of the derivation, the sequent $q \Rightarrow q$ is finally derived. Indeed, the only sequents in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}_1)$ that can potentially attack $q \Rightarrow q$ are of the form $p, \neg p \Rightarrow \psi$ or $p, \neg p, q \Rightarrow \psi$, where ψ is logically equivalent to $\neg q$. However, those sequents are counter-attacked by $\Rightarrow p \vee \neg p$ (which is derived in Tuple 3), using the justifications in Tuples 4 and 5.[9] Hence, this derivation cannot be extended to a derivation in which $q \Rightarrow q$ is eliminated, and so $\mathcal{S}_1 \mid\!\sim q$.

The situation is completely different as far as $p \Rightarrow p$ is concerned. This is due to the fact that although this sequent is derived by Tuple 1, after Step 8 of the derivation $p \Rightarrow p$ is eliminated, and after the extension with Tuples 9–11 $\neg p \Rightarrow \neg p$ is eliminated due to the Ucut-attack on it by the eliminating tuple 11. At this point of the derivation, $p \Rightarrow p$ is not eliminated anymore (see also Example 8). Nevertheless, $p \Rightarrow p$ can be re-attacked by the sequent $\neg p \Rightarrow \neg p$,[10] thus reproducing $p \not\Rightarrow p$, and so forth. As a consequence, neither of $p \Rightarrow p$ nor $\neg p \Rightarrow \neg p$ is finally derived by the derivation of Example 7. In an analogous way any dynamic derivation based on $\mathcal{S}_1$ can always be extended in such a way that all the sequents in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1)$ whose conclusion is p (respectively, $\neg p$) are eliminated, and so $\mathcal{S}_1 \not\mid\!\sim p$ (respectively, $\mathcal{S}_1 \not\mid\!\sim \neg p$).

[9] In is important to note that Ucut-attackers of $q \Rightarrow q$ like $p, \neg p, q \Rightarrow \neg q$ may still be derived in an extension of $\mathcal{D}$, however, they cannot be used for eliminating $q \Rightarrow q$. Any attempt to introduce an eliminating tuple with $q \not\Rightarrow q$ will fail due to Condition (ii) in Definition 16(c) because, as noted above, the attacker of $q \Rightarrow q$ is counter-attacked by the sequent $\Rightarrow p \vee \neg p$ in Tuple 3 of $\mathcal{D}$.

[10] Alternatively, $p \Rightarrow p$ may be re-attacked by any sequent of the form $\neg p \Rightarrow \psi$, where ψ is equivalent to $\neg p$ (for instance, $\psi = \neg^{2n+1} p$, where $\neg^n p$ denotes the atom p preceded by n negations).

This state of affairs is intuitively justified by the fact that while q is not related to the inconsistency in $\mathcal{S}_1$ and so it may safely follow from $\mathcal{S}_1$, the information in $\mathcal{S}_1$ about p is contradictory, and so neither p nor $\neg p$ may be safely inferred from $\mathcal{S}_1$.

Example 12. Let us consider the following variation of Example 7. The underlying setting is the same as in that example: $\mathfrak{S} = \langle \mathsf{CL}, LK, \mathsf{Ucut} \rangle$, but now we take the conjunction of p and q: $\mathcal{S}'_1 = \{p \wedge q, \neg p\}$. Again, although both of $p \wedge q \Rightarrow p$ and $\neg p \Rightarrow \neg p$ are LK-derivable, neither p nor $\neg p$ follows according to $\mathfrak{S}$ from $\mathcal{S}'_1$, because, e.g., the first sequent Ucut-attacks the other sequent and is Ucut-attacked by the sequent $\neg p \Rightarrow \neg(p \wedge q)$ (the details are quite similar to those in Examples 7 and 11). This time, however, q is *not* $\mathfrak{S}$-derivable from $\mathcal{S}'_1$, because both the sequents $p \wedge q \Rightarrow q$ and $\neg p, p \wedge q \Rightarrow q$ are also Ucut-attacked by the LK-derivable sequent $\neg p \Rightarrow \neg(p \wedge q)$ and cannot be permanently defended by sequents in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}'_1)$.[11]

This example shows in particular that $\mid\!\sim_{\mathfrak{S}}$ is sensitive to the syntactic form of the premises: although $\mathcal{S}_1$ and $\mathcal{S}'_1$ are CL-equivalent, their $\mathfrak{S}$-conclusions are not the same. In our case this may be intuitively justified by the fact that in $\mathcal{S}'_1$, unlike in $\mathcal{S}_1$, q is not neutral with respect to the inconsistency of the set of premises and it is 'linked' to p by the conjunction (as is also reflected by the above Ucut-attack on $p \wedge q$). Indeed, syntax sensitivity is not unusual in non-monotonic reasoning and this what one expects when, e.g., maximally consistent subsets of premises are taken into account (see [41]), or when inconsistency measurements are incorporated (see [27]).[12]

Example 13. Consider a logic with a negation $\neg$ (i.e., $p \nvdash \neg p$ and $\neg p \nvdash p$), which doesn't respect the introduction of double-negation (i.e., $p \nvdash \neg\neg p$), and suppose that Direct Defeat (D-Def; See Fig. 2) is the only attack rule. Let $\mathcal{S}_2 = \{p, \neg p, \neg\neg p, \neg\neg\neg p, \neg\neg\neg\neg p\}$. We write s_i ($i \in \mathbb{N}$) for the sequent $\neg^i p \Rightarrow \neg^i p$ (where $\neg^i p$ is the formula in which p is preceded by i negations. In particular, $s_0 = p \Rightarrow p$). Note that by reflexivity s_i is provable in any complete calculus for the base logic. Now, consider the following sequence $\mathcal{D}$ of proof tuples:

1.	s_0	Axiom	
2.	s_1	Axiom	
3.	s_2	Axiom	
4.	$\overline{s_1}$	D-Def, 3, 3, 2	s_2
5.	s_3	Axiom	
6.	$\overline{s_0}$	D-Def, 2, 2, 1	s_1
7.	$\overline{s_2}$	D-Def, 5, 5, 3	s_3
8.	s_4	Axiom	

It is easy to verify that $\mathcal{D}$ is a valid dynamic derivation. Extending it only with the tuple

9.	$\overline{s_3}$	D-Def, 8, 8, 5	s_4

yields a simple derivation $\mathcal{D}'$, in which the attacker (s_4) is not counter-attacked by an accepted sequent, yet $\mathcal{D}'$ is not coherent since $s_1 \in \mathsf{Attack}(\mathcal{D}') \cap \mathsf{Elim}(\mathcal{D}')$.[13] Note, however, that $\mathcal{D}$ may be extended to a coherent derivation containing Tuple 9, provided that the latter is introduced together with the following eliminating tuple:

10.	$\overline{s_1}$	D-Def, 3, 3, 2	s_2

Indeed, the extension of $\mathcal{D}$ with the sequence $\langle T_9, T_{10} \rangle$ *is* a valid dynamic derivation. This demonstrates the need in Definition 16 to allow the addition of more than one elimination tuple at a time.[14]

Let us now check what can be finally derived from $\mathcal{S}_2$. First, the sequent s_4 is attacked according to D-Def only by sequents whose right-hand side is $\neg^5 p$, but since double-negation introduction does not hold, such sequents cannot be in

[11] Note that the $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}'_1)$-sequent $p \wedge q \Rightarrow p$ does not prevent the Ucut-attack on $p \wedge q \Rightarrow q$ by the $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S}'_1)$-sequent $\neg p \Rightarrow \neg(p \wedge q)$, because the latter attacks both of them. This situation is different from the one in Example 7 (and Example 11), where $\Rightarrow p \vee \neg p$ 'blocks' any potential Ucut-attack on $q \Rightarrow q$, since in Example 7 $\Rightarrow p \vee \neg p$ couldn't be counter Ucut-attacked.

[12] Syntax dependency ceases to hold when $\mathcal{S}_1$ (or $\mathcal{S}'_1$) is consistent. This follows from Proposition 10 below.

[13] This shows, in particular, that the two conditions in Definition 16(c) are not dependent.

[14] This example also demonstrates the fact that while introducing tuples need to be produced only once in a derivation, elimination tuples may be repeated several times (modulo their index), as in the case of Tuples 4 and 10 in the last derivation.

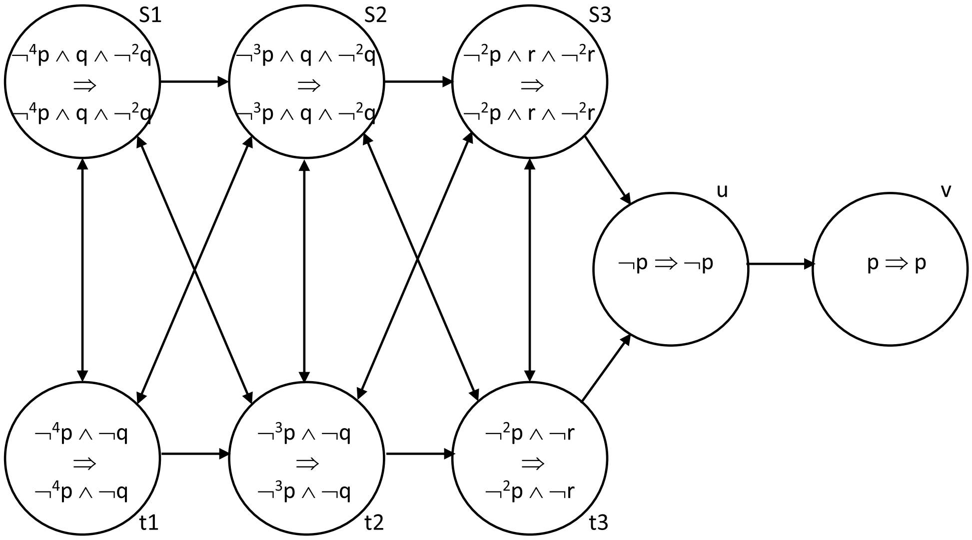


Fig. 4. The argumentation framework of Example 14.

$\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_2)$. It follows that s_4 is finally derived by the above derivation, and so $\mathcal{S}_2 \mid\!\sim \neg^4 p$. Also, s_3 cannot be finally derived, since any derivation in which it is derived can be extended by a tuple of the form $\langle i, \overline{s_3}, \text{D-Def}, s_4\rangle$, which causes the elimination of s_3. Thus $\mathcal{S}_2 \not\mid\!\sim \neg^3 p$. In turn, since the attacker (s_3) of s_2 is eliminated and cannot be recovered, s_2 is finally derived, thus $\mathcal{S}_2 \mid\!\sim \neg\neg p$. Similar considerations show that in this case $\mathcal{S}_2 \not\mid\!\sim \neg p$ and that $\mathcal{S}_2 \mid\!\sim p$.

Note 4. The last example emphasizes the basic difference between the derivation process introduced here and the one considered in [5]. While the process in [5] allows to reintroduce sequents irrespective of whether they are attacked, here the way sequents can be introduced in a proof is restricted and it depends on the already introduced elimination sequents. Thus, e.g., while according to the approach in [5] the sequent $\neg p \Rightarrow \neg p$ may be reintroduced in an extension of the dynamic derivation of Example 13, this is not possible according to the present formalism. Hence, according to [5] only s_4 is finally derivable in Example 13, while in our case both s_2 and s_0 are also finally derivable, although they are attacked at a certain point. This allows for a better 'diffusion of attacks' and it is in line with standard extensions of the corresponding argumentation frameworks (see [19]): although s_2 is attacked by s_3, that attack is counter-attacked by s_4, and so s_2 is 'defended' or 'reinstated' by s_4 (see also Proposition 1).

Example 14. Consider again the setting of Example 13. Fig. 4 represents the relevant arguments with their notations (for instance, t_1 denotes the argument $\neg^4 p \wedge \neg q \Rightarrow \neg^4 p \wedge \neg q$), and the corresponding attacks among them. Consider now the following derivation (we use below the arguments' names as specified in Fig. 4, and instead of deriving the justifications of the attacks, we just mention them in the justification components of the attacking tuples):

1.	s_1	Axiom	
2.	t_1	Axiom	
3.	t_2	Axiom	
4.	s_3	Axiom	
5.	u	Axiom	
6.	v	Axiom	
7.	$\overline{t_1}$	D-Def (since $\neg^4 p \wedge q \wedge \neg^2 q \Rightarrow \neg^2 q$)	s_1
8.	$\overline{t_2}$	D-Def (since $\neg^4 p \wedge q \wedge \neg^2 q \Rightarrow \neg^4 p$)	s_1
9.	$\overline{u}$	D-Def (since $\neg^2 p \wedge r \wedge \neg^2 r \Rightarrow \neg^2 p$)	s_3

Note that in this derivation v is finally derived. Indeed, its only attacker, u, is eliminated. Any attempt to re-accept u (in order to initiate an attack on v) by attacking u's attacker (s_3) in some extension of this derivation will necessarily fail, since the defenders of u (namely s_2 and t_2) are attacked by the accepted sequent s_1 (and all the attackers of the latter are also eliminated and cannot be re-accepted).

4.2. Useful properties of dynamic derivations

In what follows we give some general observations regarding dynamic proofs and their properties. We show these properties for argumentation frameworks that satisfy a common normalization property, called weak symmetry. For this, we first need the following definition (and notations).

Definition 19. Given a sequent-based argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}\rangle$, induced by a setting $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}\rangle$, we consider the following sets:

- $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})) = \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) - \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})^+$
- $\mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S}) = \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) - \Big(\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})) \cup \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+\Big)$
- $\mathit{Attack}^* = \mathit{Attack} \cap \Big(\mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S}) \times \mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S})\Big)$

Thus, the set $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ consists of the non-attacked arguments in $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$. The argumentation framework $\mathcal{AF}^*_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}^*\rangle$ is the sub-graph of $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ that excludes the arguments in $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})) \cup \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ (and so $\mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S})$ is $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ without these arguments). We call $\mathcal{AF}^*_{\mathfrak{S}}(\mathcal{S})$ the *inner framework* of $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$.

Definition 20. Let $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}\rangle$ be a (sequent-based) argumentation framework and let $\sim$ be a right congruent relation on $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$.[15] We say that $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ is $\sim$*-weakly symmetric*, if it's inner framework, $\mathcal{AF}^*_{\mathfrak{S}}(\mathcal{S})$, is irreflexive and symmetric modulo $\sim$.

As the next example and proposition show, $\sim$-weakly symmetric frameworks are quite common.

Example 15. The argumentation framework that is induced by the setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \mathsf{Ucut}\rangle$, considered in Example 5, where $\sim$ is defined by $\Gamma_1 \Rightarrow \psi_1 \sim \Gamma_2 \Rightarrow \psi_2$ iff $\Gamma_1 = \Gamma_2$, and the argumentation frameworks $\mathcal{AF}_{\mathfrak{S}}$ considered in Example 6, are $\sim$-weakly symmetric.

The framework of the last example are particular cases of the following class of frameworks

Definition 21. A setting $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}\rangle$ is called *SAC* (support attacking, contrapositive), if the following conditions are satisfied:

1. The sequent calculus $\mathfrak{C}$ of the core logic $\mathfrak{L}$ admits *Contraposition*:
   If the sequent $\Delta \Rightarrow \neg \bigwedge \Theta$ is $\mathfrak{C}$-derivable, then for every $\Theta' \subseteq \Theta$ and $\Delta' \subseteq \Delta$, the sequent $(\Delta - \Delta') \cup \Theta' \Rightarrow \neg \bigwedge (\Theta - \Theta') \cup \Delta'$ is also $\mathfrak{C}$-derivable.
2. The set $\mathfrak{A}$ consists of any attack rules in Fig. 2 except rebuttals and direct attacks. (Note that the common property of these rulers is that the attack is on the support set of the attacked sequent.)

An argumentation framework induced by a SAC setting is called SAC framework.

Next we show that the fact that the argumentation frameworks considered in Example 15 are $\sim$-weakly symmetric, is not a coincidence.

Proposition 5. *Every SAC argumentation framework is $\sim$-weakly symmetric, where $\sim$ is defined by: $\Gamma_1 \Rightarrow \psi_1 \sim \Gamma_2 \Rightarrow \psi_2$ iff $\Gamma_1 = \Gamma_2$.*

Proof. We show the proposition for Indirect Defeat.

For symmetry, suppose that there are $\Delta \Rightarrow \phi'$ and $\Theta \Rightarrow \psi'$ in $\mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S})$ such that $(\Delta \Rightarrow \phi', \Theta \Rightarrow \psi') \in \mathit{Attack}$. Hence $\phi' \Rightarrow \neg \bigwedge \Theta'$ is $\mathfrak{C}$-derivable for some $\Theta' \subseteq \Theta$. Thus, by contraposition, $\Theta' \Rightarrow \neg\phi'$ and $\neg\phi' \Rightarrow \neg \bigwedge \Delta$ are also $\mathfrak{C}$-derivable,[16] and so by monotonicity and cut, also $\Theta \Rightarrow \neg \bigwedge \Delta$ is in $\mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S})$. It follows that $(\Theta \Rightarrow \neg \bigwedge \Delta, \Delta \Rightarrow \phi) \in \mathit{Attack}$.

For irreflexivity, suppose that there are ψ and ψ' such that $(\Delta \Rightarrow \psi, \Delta \Rightarrow \psi') \in \mathit{Attack}$. Thus $\psi \Rightarrow \neg \bigwedge \Delta'$ is $\mathfrak{C}$-derivable for some $\Delta' \subseteq \Delta$. By cut, $\Delta \Rightarrow \neg \bigwedge \Delta'$ is $\mathfrak{C}$-derivable, and so, by contraposition, $\Rightarrow \neg \bigwedge \Delta$ is in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Now, since $(\Rightarrow \neg \bigwedge \Delta, \Delta \Rightarrow \phi) \in \mathit{Attack}$ and $\Rightarrow \neg \bigwedge \Delta \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$, it follows that $\Delta \Rightarrow \phi \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. In particular, $(\Delta \Rightarrow \phi, \Delta \Rightarrow \phi) \notin \mathit{Attack}^*$. □

[15] Recall by Item 2 of Note 2 that this means that for every $s, s', t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, if $s \sim s'$ then $(t, s) \in \mathit{Attack}$ implies that $(t, s') \in \mathit{Attack}$ as well.

[16] Note that $\Delta \neq \emptyset$ since $\Delta \Rightarrow \phi \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$.

In what follows we show some interesting and useful properties of dynamic derivations for weakly symmetric frameworks. The first property is that final derivability is invariant of particular patterns of dynamic derivations.

Proposition 6. *Let $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ be a $\sim$-weakly symmetric argumentation framework for some right congruent relation $\sim$ on $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$. If a sequent s is finally derived in a dynamic derivation for $\mathfrak{S}$ that is based on $\mathcal{S}$, then every dynamic derivation (for $\mathfrak{S}$ that is based on $\mathcal{S}$) can be extended to a dynamic derivation (for $\mathfrak{S}$ that is based on $\mathcal{S}$) in which s is finally derived.*

Proof. See Appendix A. □

Example 16. By Example 15, final derivability for the setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \mathsf{Ucut}\rangle$ (or for any SAC framework) is invariant of particular patterns of dynamic derivations.

An interesting corollary of (the proof of) Proposition 6 is that for proving entailment of the form $\mathcal{S} \mid\!\sim \psi$, induced by weakly symmetric frameworks, dynamic derivations with only introducing tuples suffice. Eliminating tuples are required for providing constructive refutations (that is, they are needed for showing that $\mathcal{S} \not\mid\!\sim \psi$).

Proposition 7. *Let $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ be a $\sim$-weakly symmetric argumentation framework for some right congruent relation $\sim$ on $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$. A sequent is finally derived in a derivation (for $\mathfrak{S}$ that is based on $\mathcal{S}$) iff it is finally derived in a derivation (for $\mathfrak{S}$ that is based on $\mathcal{S}$) that consists only of introducing tuples.*

Proof. One direction is trivial. For the other direction, suppose that s is finally derived in a dynamic derivation $\mathcal{D}$. By Lemma 4 in the proof of Proposition 6 (see Appendix A), all the attackers of s are attacked by elements in $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})) \cap \overline{\mathcal{D}}$. Let $\{t_1, \dots, t_n\}$ be the set of these attackers and let $\{s_1, \dots, s_n\} \subseteq \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ be sequents such that $(s_i, t_i) \in Attack$ (for $i = 1, \dots, n$). Let now $\mathcal{D}'$ be the dynamic derivation in which s is derived, and for each $i = 1, \dots, n$, t_i and s_i are derived. Note that $\mathcal{D}'$ is coherent since it doesn't contain any elimination tuples. Furthermore,

1. all the attackers of s are present in $\mathcal{D}'$, and
2. according to Condition c-(ii) in Definition 16, no elimination tuple of the form $(k, \bar{s}, J, t_i)$ can be added in any extension $\mathcal{D}''$ of $\mathcal{D}'$, because $(s_i, t_i) \in Attack$, and since $s_i \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$, necessarily $s \in \mathsf{Accept}(\mathcal{D}'')$.

By the two facts above, s is finally derived in $\mathcal{D}'$ (in a derivation consisting only of introducing tuples). □

Note 5. Some remarks concerning the implications of Proposition 7 are in order here.

1. Clearly, Proposition 7 is useful since it allows to considerably reduce the search space of proofs of finally derived sequents in weakly symmetric frameworks.
2. Although elimination rules are not required for final derivations in weakly symmetric frameworks, in general the presence of such rules may enable the final derivation of certain arguments that could not be finally derived otherwise. In other words, if $\mathfrak{S}_1 = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}_1\rangle$ and $\mathfrak{S}_2 = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}_2\rangle$ are two settings (and even SAC settings) such that $\mathfrak{A}_1 \subset \mathfrak{A}_2$, it is *not* necessarily the case that $\mid\!\sim_{\mathfrak{S}_2} \subseteq \mid\!\sim_{\mathfrak{S}_1}$. This is demonstrated in the next example.

Example 17. Let Weak Rebuttal (WReb) be the same attack rule as Rebuttal (Fig. 2), but with the requirement that the support set of the attacked sequent should not be empty ($\Gamma_2 \neq \emptyset$).[17] Consider now the setting $\mathfrak{S}_1 = \langle \mathsf{CL}, LK, \{\mathsf{WReb}\}\rangle$ and $\mathcal{S} = \{p, \neg p, q\}$. Then $\mathcal{S} \not\mid\!\sim_{\mathfrak{S}_1} q$, since every sequent of the form $\Gamma \Rightarrow q$ in $\mathsf{Arg}_{\mathsf{CL}}(\mathcal{S})$ is attacked by $p, \neg p \Rightarrow \neg q$. Now, if we add Undercut to the attack rules of the setting, i.e., if $\mathfrak{S}_2 = \langle \mathsf{CL}, LK, \{\mathsf{WReb,Ucut}\}\rangle$, we have that this time $\mathcal{S} \mid\!\sim_{\mathfrak{S}_2} q$, since $\Rightarrow \neg(p \wedge \neg p)$ attacks $p, \neg p \Rightarrow \neg q$ and is not attacked itself (due to the restriction of WReb).

Another interesting property of dynamic proof systems (obtained by SAC and some other settings) is that no complementary conclusions are allowed by them even when the set of premises is contradictory. This is shown next.

Proposition 8. *Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A}\rangle$ be an argumentation setting and let $\mathcal{S}$ be a set of formulas in the underlying language. If $\mathfrak{S}$ is either a SAC setting or is a $\sim$-weakly symmetric setting in which $\mathfrak{A}$ contains one of the rebuttal attacks in Fig. 2, then there is no formula ψ for which both $\mathcal{S} \mid\!\sim_{\mathfrak{S}} \psi$ and $\mathcal{S} \mid\!\sim_{\mathfrak{S}} \neg\psi$.*

Proof. Suppose for a contradiction that $\mathcal{S} \mid\!\sim_{\mathfrak{S}} \psi$ and $\mathcal{S} \mid\!\sim_{\mathfrak{S}} \neg\psi$. Then, there is a dynamic derivation $\mathcal{D}$ in which a sequent $s = \Gamma \Rightarrow \psi$ is finally derived for some $\Gamma \subseteq \mathcal{S}$, and there is a dynamic derivation $\mathcal{D}'$ in which $s' = \Gamma' \Rightarrow \neg\psi$ is finally derived for some $\Gamma' \subseteq \mathcal{S}$. By Lemma 4 in the proof of Proposition 6 (see Appendix A),

[17] In particular, WReb is tautology preserving in the sense of Definition 10.

(†) all attackers of s and s' are attacked by elements in $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ and $s, s' \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$.

Now, if there is a rebuttal attack in $\mathfrak{A}$ we have $(s', s) \in Attack$, in contradiction to (†). Alternatively, if $\mathfrak{S}$ is a SAC setting, $\mathfrak{C}$ admits of contraposition, thus by cut $\Gamma \Rightarrow \neg \bigwedge \Gamma'$ attacks s'. Hence, $\Gamma \Rightarrow \neg \bigwedge \Gamma' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$, and so there is a $t = \Theta \Rightarrow \neg \bigwedge \Gamma'' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ that attacks $\Gamma \Rightarrow \neg \bigwedge \Gamma'$. Note that $\Theta = \emptyset$ since otherwise $\Gamma, \Gamma' \Rightarrow \neg \bigwedge \Theta$ attacks t (by contraposition and monotonicity we have $\Gamma, \Gamma' \vdash \neg \bigwedge \Theta$). In this case t also attacks s, which is a contradiction to (†). □

5. Some properties of $\mid\!\sim$

In this section we consider some properties of the entailment relations that are induced by dynamic proof systems according to Definition 18.

5.1. Relations between $\mid\!\sim$ and $\vdash$

We start with some results concerning the relations between the base consequence relation and the entailments induced by the corresponding argumentation setting. In these propositions we refer to an entailment $\mid\!\sim$ that is induced by an argumentation setting $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ with a base logic $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$.

First, we show that Tarskian consequence relations may be viewed as particular $\mathfrak{S}$-entailments:

Proposition 9. *If $\mathfrak{A} = \emptyset$ then $\mid\!\sim$ and $\vdash$ coincide.*

Proof. If there are no attack rules, dynamic derivations are in fact standard $\mathfrak{C}$-proofs in which every derived sequent is finally derived. Thus, $\mathcal{S} \mid\!\sim_{\mathfrak{S}} \psi$ iff there is a derivation of $\Gamma \Rightarrow \psi$ in $\mathfrak{C}$ for some finite $\Gamma \subseteq \mathcal{S}$. Since $\mathfrak{C}$ is sound and complete for $\mathfrak{L}$, the latter is a necessary and sufficient condition for $\mathcal{S} \vdash \psi$. □

Another case where $\mid\!\sim$ and $\vdash$ correlate is the following:

Proposition 10. *If $\mathcal{S}$ is conflict-free with respect to $\mathfrak{S}$ (that is, there are no $\mathfrak{A}$-attacks between the elements in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$) then $\mathcal{S} \mid\!\sim \psi$ iff $\mathcal{S} \vdash \psi$.*

Proof. If there are no attacks between arguments in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ then no attack rule in $\mathfrak{A}$ is applicable, and so the proof is similar to that of Proposition 9. □

In general, however, $\mid\!\sim$ is weaker than $\vdash$.

Proposition 11. *If $\mathcal{S} \mid\!\sim \psi$ then $\mathcal{S} \vdash \psi$.*

Proof. If $\mathcal{S} \mid\!\sim \psi$ then there is an $\mathcal{S}$-based dynamic derivation for $\mathfrak{S}$, in which $\Gamma \Rightarrow \psi$ is finally derived for some finite $\Gamma \subseteq \mathcal{S}$. In particular, there is a proof in $\mathfrak{C}$ for $\Gamma \Rightarrow \psi$. Since $\mathfrak{C}$ is complete for $\mathfrak{L}$, this implies that $\Gamma \vdash \psi$, and by the monotonicity of $\mathfrak{L}$ we have that $\mathcal{S} \vdash \psi$. □

The converse of Proposition 11 holds for $\vdash$-theorems and rules that preserve tautological arguments (see Definition 10).

Proposition 12. *If $\mathfrak{A}$ consists only of tautology preserving rules, then $\vdash \psi$ implies that $\mid\!\sim \psi$.*

Proof. If $\vdash \psi$ then the sequent $\Rightarrow \psi$ is provable in $\mathfrak{C}$. Since there are only tautology-preserving rules in $\mathfrak{A}$, this sequent cannot be attacked, and so any $\mathfrak{C}$-proof of $\Rightarrow \psi$ is also a dynamic derivation for $\mathfrak{S}$, in which $\Rightarrow \psi$ is finally derived. □

Corollary 1. *If $\mathfrak{A}$ consists only of theorem-preserving rules, then*

1. *$\vdash \psi$ iff $\mid\!\sim \psi$, and*
2. *$\mathfrak{C}$ is sound and weakly complete for $\mid\!\sim$ (that is, $\mid\!\sim \psi$ iff $\Rightarrow \psi$ is $\mathfrak{C}$-derivable).*

Proof. By Propositions 11 and 12. □

Finally, the next proposition shows that for a large class of argumentation settings the corresponding entailment is closed under the consequence relation of the core logic.

Proposition 13. *If $\mid\!\sim$ is induced by a SAC setting (Definition 21) then for every finite set $\mathcal{S}$ of formulas, $\mathcal{S} \mid\!\sim \phi$ iff $\{\psi \mid \mathcal{S} \mid\!\sim \psi\} \vdash \phi$.*

Proof. See Appendix B. □

Note 6. A strengthening of the base logic may not have the same effect on the induced entailment: If $\mathfrak{S}_1 = \langle \mathfrak{L}_1, \mathfrak{C}, \mathfrak{A} \rangle$ and $\mathfrak{S}_2 = \langle \mathfrak{L}_2, \mathfrak{C}, \mathfrak{A} \rangle$ are two argumentation settings where $\mathfrak{L}_1 = \langle \mathcal{L}, \vdash_1 \rangle$ and $\mathfrak{L}_2 = \langle \mathcal{L}, \vdash_2 \rangle$ are two logics (for the same language) such that $\vdash_1 \subset \vdash_2$, it is *not* necessarily the case that $\mathrel{|\!\sim}_{\mathfrak{S}_1} \subseteq \mathrel{|\!\sim}_{\mathfrak{S}_2}$. To see this, consider e.g. $\mathcal{S} = \{p, \neg p\}$, the logic $\mathfrak{L}_1$ induced by LK without the negation introduction rule ($[\Rightarrow \neg]$, see Fig. 1), and Rebuttal as the sole attack rule. Then, for instance, $\mathcal{S} \mathrel{|\!\sim}_{\mathfrak{S}_1} \neg p$ (since $\neg p \Rightarrow \neg p$ cannot be attacked by Rebuttal in $\mathfrak{L}_1$), but when one switches to classical logic, neither p nor $\neg p$ is finally derivable, because the strengthening of the core logic yields additional attacks.

5.2. Cautious reflexivity

As the examples in Section 4 show, in general $\mathrel{|\!\sim}$ is *not* reflexive: a formula ψ does not necessarily follow from $\mathcal{S}$ even if $\psi \in \mathcal{S}$. Yet, the next proposition and corollary show that $\mathrel{|\!\sim}$ is *cautiously reflexive*.

Proposition 14. *If $\mathcal{S}$ is conflict-free then $\mathcal{S} \mathrel{|\!\sim} \psi$ for all $\psi \in \mathcal{S}$.*

Proof. This is a direct corollary of Proposition 10 and the fact that $\mathcal{S} \vdash \psi$ for every $\psi \in \mathcal{S}$ (since $\vdash$ is reflexive). □

Corollary 2.

1. *For every formula ψ such that $\{\psi\}$ is conflict-free in $\mathfrak{S}$, we have that $\psi \mathrel{|\!\sim} \psi$.*
2. *For every atom p it holds that $p \mathrel{|\!\sim} p$.*

Note 7. The condition in the last proposition and corollary is indeed required. For instance, if $\mathrel{|\!\sim}$ is the entailment relation that is induced by $\mathfrak{S} = \langle \mathsf{CL}, LK, \{\mathsf{Ucut}\} \rangle$ (Example 7) then $p \wedge \neg p \mathrel{|\!\not\sim} p \wedge \neg p$.

5.3. Restricted monotonicity

Clearly, $\mathrel{|\!\sim}$ is not monotonic. For instance, by Corollary 2 $p \mathrel{|\!\sim} p$, while Example 7 shows a case in which $p, \neg p, q \mathrel{|\!\not\sim} p$. Like reflexivity, monotonicity can be guaranteed in particular cases. For instance, as Proposition 15 below shows, when adding unrelated information to a SAC framework, this information should not disturb previous inferences. For this proposition we first define in precise terms what 'unrelated information' means and then recall the known notion of uniformity.

Definition 22. Let $\mathcal{S}$ be a set of formulas and ψ a formula in a language $\mathcal{L}$. We denote by $\mathsf{Atoms}(\mathcal{S})$ the set of atomic formulas that appear (in some subformula of a formula) in $\mathcal{S}$. We say that $\mathcal{S}$ is *relevant* to ψ, if $\mathsf{Atoms}(\mathcal{S}) \cap \mathsf{Atoms}(\{\psi\}) = \emptyset$ implies that $\mathcal{S} = \emptyset$. A nonempty set $\mathcal{S}$ is *irrelevant* to a (nonempty) set $\mathcal{T}$ if $\mathcal{S}$ is not relevant to any formula in $\mathcal{T}$, i.e.: $\mathsf{Atoms}(\mathcal{S}) \cap \mathsf{Atoms}(\mathcal{T}) = \emptyset$.

Definition 23. Let $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$ be a propositional logic. A set $\mathcal{S}$ of formulas (in $\mathcal{L}$) is called $\vdash$-*consistent*, if there exists a formula ψ (in $\mathcal{L}$) such that $\mathcal{S} \nvdash \psi$. We say that $\mathfrak{L}$ is *uniform*, if $\mathcal{S}_1 \vdash \psi$ when $\mathcal{S}_1, \mathcal{S}_2 \vdash \psi$ and $\mathcal{S}_2$ is $\vdash$-consistent and irrelevant to $\mathcal{S}_1 \cup \{\psi\}$.

Note 8. By Łos-Suzsko Theorem [30], a finitary propositional logic $\langle \mathcal{L}, \vdash \rangle$ is uniform iff it has a single characteristic matrix (see also [46]). Thus, classical logic as well as many other logics are uniform.

Proposition 15. *Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathcal{A} \rangle$ be a SAC settings whose base logic $\mathfrak{L}$ is uniform and let $\mathrel{|\!\sim}$ be the induced entailment. Suppose that $\mathcal{S}_1 \cup \mathcal{S}_2$ is a finite set of formulas in the underlying language. If $\mathcal{S}_1 \cup \{\phi\}$ is irrelevant to $\mathcal{S}_2$ then $\mathcal{S}_1 \mathrel{|\!\sim} \phi$ iff $\mathcal{S}_1, \mathcal{S}_2 \mathrel{|\!\sim} \phi$.*

A few notes concerning the last proposition are in order prior to its proof. Proposition 15 shows that entailments induced by SAC settings satisfy the property of *non-interference* [17] with respect to finite sets of assumptions. One direction of it (if $\mathcal{S}_1 \mathrel{|\!\sim} \phi$ then $\mathcal{S}_1, \mathcal{S}_2 \mathrel{|\!\sim} \phi$) is a restricted form of monotonicity to irrelevant premise sets. Note that non-interference is more than a bi-directional version of uniformity for non-monotonic entailments (or of the *basic relevance criterion* [8]), since here $\mathcal{S}_2$ may not be consistent.

Proof of Proposition 15. We show the claim for the case that Undercut is an attack rule. We will indicate in the proof (see Footnote 18) what changes are necessary for Canonical Undercut. The cases of Defeat attacks and other forms of Undercut are shown in a very similar way and are left to the reader.

In what follows we will denote by $\overline{\mathcal{D}}$ the set of all the sequents that occur in the proof tuples of a (simple) derivation $\mathcal{D}$.

Suppose that $\mathcal{S}_1 \mid\!\sim \phi$. Then, there is a dynamic derivation $\mathcal{D}_1$, in which some $\Gamma \Rightarrow \phi$ is finally derived where $\Gamma \subseteq \mathcal{S}_1$. By Proposition 7 we can assume that $\mathcal{D}_1$ does not contain any eliminating tuples. Let $\mathcal{D}_1^\star$ be the extension of $\mathcal{D}_1$ in which for each $\Omega \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$ which is inconsistent, $\Rightarrow \neg \bigwedge \Omega$ is added to $\mathcal{D}_1$. Note that since $\mathfrak{S}$ is a SAC setting, for every inconsistent Ω, it holds that $\vdash \neg \bigwedge \Omega$ and so $\Rightarrow \neg \bigwedge \Omega$ is derivable in $\mathfrak{C}$. Also, since $\mathcal{S}_1 \cup \mathcal{S}_2$ is finite, there are only finitely many such sequents.

Assume now for a contradiction that $\mathcal{S}_1, \mathcal{S}_2 \not\mid\!\sim \phi$. Note that $\mathcal{D}_1^\star$ is also a dynamic derivation based on $\mathcal{S}_1 \cup \mathcal{S}_2$. Let $\mathcal{D}_1^\perp$ be the extension of $\mathcal{D}_1^\star$ by the eliminating tuples $\langle k, \Omega \not\Rightarrow \delta, J_k, \Rightarrow \neg \bigwedge \Omega\rangle$ for all the inconsistent subsets $\Omega \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$. By our assumption, there is an extension $\mathcal{D}_2$ of $\mathcal{D}_1^\perp$, in which $\Gamma \Rightarrow \phi$ is eliminated by some eliminating tuple $T_k = \langle k, \Gamma \not\Rightarrow \phi, J_k, \Theta \Rightarrow \psi\rangle$. Thus, $\psi \Rightarrow \neg \bigwedge \Gamma'$ is derivable in $\mathfrak{C}$ for some non-empty $\Gamma' \subseteq \Gamma$.

By cut, $\Theta \Rightarrow \neg \bigwedge \Gamma'$ is also derivable in $\mathfrak{C}$. Note that Θ is consistent (otherwise it would be attacked by $\Rightarrow \neg \bigwedge \Theta'$ for some $\Theta' \subseteq \Theta$ in $\mathcal{D}_1^\star$, in which case T_k cannot be added to the proof in view of Definition 16 c.ii). Thus, $\Theta \cap \mathcal{S}_2$ is consistent as well. Thus, by the uniformity of $\mathfrak{L}$, $\Theta - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Gamma'$ is derivable in $\mathfrak{C}$.

Note that, again in view of the uniformity of $\mathfrak{L}$,

(†) for every attacker $\Theta' \Rightarrow \psi' \in \mathsf{Accept}(\mathcal{D}_2) \cap \mathsf{Arg}(\mathcal{S}_1 \cup \mathcal{S}_2)$ of some $\Omega \Rightarrow \delta \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1)$ there is a $\Theta' - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Omega'$ (for some $\Omega' \subseteq \Omega$) that also attacks $\Omega \Rightarrow \delta$.

Let $(\Theta'_1 - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Omega'_1,\ \Omega_1 \Rightarrow \delta_1), \ldots, (\Theta'_n - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Omega'_n,\ \Omega_n \Rightarrow \delta_n) \in \mathit{Attack}$ be a list of all such pairs (including $(\Theta - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Gamma',\ \Gamma \Rightarrow \phi)$).

We now show that $\Gamma \Rightarrow \phi$ is not finally derived in $\mathcal{D}_1$, which is a contradiction. For this, we extend $\mathcal{D}_1$ to $\mathcal{D}'_1$ with a proof of the sequents $\Theta'_i - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Omega'_i$ for every $i = 1, \ldots, n$. Subsequently we also add, for every $i = 1, \ldots, n$ for which $\Omega_i \Rightarrow \delta_i \in \overline{\mathcal{D}'_1}$, an eliminating tuple for $\Omega_i \not\Rightarrow \delta_i$, based on the attack by $\Theta'_i - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Omega'_i$.

We now show that $\mathcal{D}'_1$ is coherent. For this we have to show that there are no $1 \leq i, j \leq n$ for which $\Theta'_i - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Omega'_i$ is $\Omega_j \Rightarrow \delta_j$. Assume for a contradiction that there are such i and j. But this implies that Θ'_i is inconsistent, which contradicts that $\Theta'_i \Rightarrow \psi'_i \in \mathsf{Accept}(\mathcal{D}_2)$, since $\Rightarrow \neg \bigwedge \Theta'_i \in \overline{\mathcal{D}_1^\perp}$.

Note also that the extension of $\mathcal{D}_1$ by the eliminating tuples is in accordance with Definition 16 c.ii. To show this suppose some $\Theta'_i - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Omega'_i$ is attacked by some $\Lambda \Rightarrow \sigma \in \overline{\mathcal{D}'_1} \cap \mathsf{Arg}(\mathcal{S}_1)$. Then $\Lambda \Rightarrow \sigma$ also attacks $\Theta'_i \Rightarrow \psi'_i \in \mathsf{Accept}(\mathcal{D}_2)$, and so $\Lambda \Rightarrow \sigma \in \mathsf{Elim}(\mathcal{D}_2)$. Thus, there is an attacker $\Theta'_j \Rightarrow \psi'_j \in \mathsf{Accept}(\mathcal{D}_2)$ of $\Lambda \Rightarrow \sigma$. In view of (†), then, $\langle k, \Lambda \not\Rightarrow \sigma, J_k, \Theta'_j - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Lambda\rangle$ was one of the eliminating tuples added to $\mathcal{D}'_1$, and hence $\Lambda \Rightarrow \sigma \in \mathsf{Elim}(\mathcal{D}'_1)$.

The considerations above show that $\mathcal{D}'_1$ is a valid dynamic derivation. This derivation is based on $\mathcal{S}_1$, extends $\mathcal{D}_1$, and in which $\Gamma \Rightarrow \phi \in \mathsf{Elim}(\mathcal{D}'_1)$. This is a contradiction to the final derivability of $\Gamma \Rightarrow \phi$.

Suppose now that $\mathcal{S}_1, \mathcal{S}_2 \mid\!\sim \phi$. Again, by Proposition 7 there is a dynamic derivation $\mathcal{D}$ that is based on $\mathcal{S}_1 \cup \mathcal{S}_2$, in which some $\Gamma \Rightarrow \phi$ is finally derived for $\Gamma \subseteq \mathcal{S}_1 \cup \mathcal{S}_2$, and which doesn't contain any eliminating tuples. Note that Γ is consistent since otherwise we can extend $\mathcal{D}$ with $\Rightarrow \neg \bigwedge \Gamma$ and subsequently eliminate $\Gamma \Rightarrow \phi$. So $\Gamma \cap \mathcal{S}_2$ is consistent as well. Hence, by the uniformity of $\mathfrak{L}$, also $\Gamma - \mathcal{S}_2 \vdash \phi$.

We now extend $\mathcal{D}$ to $\mathcal{D}'$ (with line numbers $l+1, \ldots, l'$) by deriving $\Omega - \mathcal{S}_2 \Rightarrow \psi$ for each $\Omega \Rightarrow \psi \in \overline{\mathcal{D}}$ for which $\Omega - \mathcal{S}_2 \vdash \psi$. Thus, $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ is in $\mathsf{Accept}(\mathcal{D}')$. Note that in $\mathcal{D}'$ we have that

(†) for every attacker $s \in \overline{\mathcal{D}'}$ of some $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1)$ there is a $s' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1) \cap \overline{\mathcal{D}'}$ that attacks t.

This is warranted in view of the uniformity of $\mathfrak{L}$: if some $\Omega \Rightarrow \psi \in \overline{\mathcal{D}'}$ attacks some $\Theta \Rightarrow \delta \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1)$ then $\psi \vdash \neg \bigwedge \Theta'$ for some $\Theta' \subseteq \Theta \subseteq \mathcal{S}_1$, and by cut and uniformity, $\Omega - \mathcal{S}_2 \Rightarrow \neg \bigwedge \Theta' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1)$ attacks $\Theta \Rightarrow \delta$. (If Θ is inconsistent, $\Rightarrow \neg \bigwedge \Theta \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1)$ attacks $\Theta \Rightarrow \delta$.)

Let $\mathsf{Atoms}(\mathcal{S}_2) = \{p_1, \ldots, p_n\}$ and let $\{p'_1, \ldots, p'_n\}$ be a set of atoms for which $(\mathsf{Atoms}(\mathcal{S}_1) \cup \mathsf{Atoms}(\mathcal{S}_2)) \cap \{p'_1, \ldots, p'_n\} = \emptyset$. We denote by $\mathcal{D}'[p_1, \ldots, p_n/p'_1, \ldots, p'_n]$ the result of simultaneously replacing all occurrences of p_i by p'_i for $i = 1, \ldots, n$ in each $t \in \overline{\mathcal{D}'}$. Let now $\mathcal{D}''$ be the result of concatenating $\mathcal{D}'$ and $\mathcal{D}'[p_1, \ldots, p_n/p'_1, \ldots, p'_n]$ (after changing the line numbers $1, \ldots, l'$ of the tuples in $\mathcal{D}'[p_1, \ldots, p_n/p'_1, \ldots, p'_n]$ to $l'+1, \ldots, l'+l'+1$, respectively). Note that $\mathcal{D}''$ is still a dynamic derivation that is based on $\mathcal{S}_1 \cup \mathcal{S}_2$, and which extends $\mathcal{D}'$. Thus, $\Gamma \Rightarrow \phi$ is finally derived in $\mathcal{D}''$.

Note that $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ is also finally derived in $\mathcal{D}''$, since every attacker of $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ is also an attacker of $\Gamma \Rightarrow \phi$, and the latter is finally derived in $\mathcal{D}''$.[18] Since $\mathcal{D}''$ doesn't contain eliminating tuples, $\mathcal{D}''$ is also a dynamic derivation based on $\mathcal{S}_1$.

Assume now for a contradiction that $\mathcal{S}_1 \not\mid\!\sim \phi$. Then $\mathcal{D}''$ can be extended to $\mathcal{D}'''$ in such a way that $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ is eliminated. Let $\mathcal{D}^\star$ be the result of replacing each proof tuple $\langle l, s, J, t\rangle$ which was added in $\mathcal{D}'''$ to $\mathcal{D}''$ by $\langle l, s[p_1, \ldots, p_n/p'_1, \ldots, p'_n], J', t[p_1, \ldots, p_n/p'_1, \ldots, p'_n]\rangle$, where $s[p_1, \ldots, p_n/p'_1, \ldots, p'_n]$ and $t[p_1, \ldots, p_n/p'_1, \ldots, p'_n]$ are the results of simultaneously replacing each p_i with p'_i (for $i = 1, \ldots, n$) in s and t, and J' is the result of replacing each line

[18] Note that that $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ is finally derived in $\mathcal{D}''$ also when Canonical Undercut is incorporated. To see this, assume for a contradiction that $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ is *not* finally derived in $\mathcal{D}''$ in this case. Then there is an extension $\mathcal{D}'''$ of $\mathcal{D}''$ in which $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ is eliminated due to some eliminating tuple $\langle k, \Gamma - \mathcal{S}_2 \not\Rightarrow \phi, \Theta \Rightarrow \neg \bigwedge (\Gamma - \mathcal{S}_2)\rangle$. However, for every canonical undercutter $\Theta \Rightarrow \neg \bigwedge (\Gamma - \mathcal{S}_2)$ of $\Gamma - \mathcal{S}_2 \Rightarrow \phi$, the sequent $\Theta \Rightarrow \neg \bigwedge \Gamma$ is also $\mathfrak{C}$-derivable (indeed, by contraposition and monotonicity, $\Theta \vdash \neg \bigwedge \Gamma$ since $\Theta \vdash \neg \bigwedge (\Gamma - \mathcal{S}_2)$) and this sequent is a canonical undercutter of $\Gamma \Rightarrow \phi$. Thus, we can further extend $\mathcal{D}'''$ by deriving $\langle k', \Gamma \not\Rightarrow \phi, \Theta \Rightarrow \neg \bigwedge \Gamma\rangle$ so that $\Gamma \Rightarrow \phi$ is eliminated. This contradicts that $\Gamma \Rightarrow \phi$ is finally derived in $\mathcal{D}''$.

number in J which refers to a line in $\mathcal{D}'$ with the corresponding line number in $l'+1,\ldots,l'+l'+1$. Since $\mathcal{D}'''$ is a dynamic derivation based on $\mathcal{S}_1$ it is easy to see that also $\mathcal{D}^\star$ is a dynamic derivation based on $\mathcal{S}_1$. Note that:

$$(\ddagger) \quad \text{every } \Theta \Rightarrow \psi \in \overline{\mathcal{D}^\star} \text{ for which } \Theta \cap \mathcal{S}_2 \neq \emptyset \text{ is in } \overline{\mathcal{D}'}.$$

Let now $\mathcal{D}^\ddagger$ be the result of adding after each eliminating tuple $\langle k, \Theta \not\Rightarrow \psi, J, t\rangle$ in $\mathcal{D}^\star$ eliminating tuples $\langle k', \Theta' \not\Rightarrow \psi', J', t\rangle$ for every $\Theta' \Rightarrow \psi' \in \overline{\mathcal{D}'}$ for which $\Theta \subset \Theta' \subseteq (\mathcal{S}_1 \cup \mathcal{S}_2)$ and $\Theta' \cap \mathcal{S}_2 \neq \emptyset$ (where the line numbers and the justifications are adjusted accordingly). Thus,

$$(\star) \quad \text{whenever } \Theta \Rightarrow \psi \in \mathsf{Elim}(\mathcal{D}^\ddagger), \text{ also } \Theta' \Rightarrow \psi' \in \mathsf{Elim}(\mathcal{D}^\ddagger).$$

Note that $\mathcal{D}^\ddagger$ is also a coherent simple derivation based on $\mathcal{S}_1 \cup \mathcal{S}_2$, since $\mathcal{D}^\star$ is coherent, $\mathsf{Attack}(\mathcal{D}^\ddagger) = \mathsf{Attack}(\mathcal{D}^\star) \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1)$, and $\mathsf{Elim}(\mathcal{D}^\ddagger) - \mathsf{Elim}(\mathcal{D}^\star) \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1 \cup \mathcal{S}_2) - \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}_1)$. By (†), (‡), and ($\star$), Requirement (c.ii) of Definition 16 also holds for $\mathcal{D}^\ddagger$. To see this let $\Theta \Rightarrow \psi \in \mathsf{Attack}(\mathcal{D}^\ddagger)$ and suppose that $t = \Omega \Rightarrow \sigma \in \overline{\mathcal{D}^\ddagger}$ attacks $\Theta \Rightarrow \psi$. If $t \in \mathsf{Arg}(\mathcal{S}_1)$ we know that $t \in \mathsf{Elim}(\mathcal{D}^\star) \subseteq \mathsf{Elim}(\mathcal{D}^\ddagger)$ since $\mathcal{D}^\star$ is coherent. Suppose then that $t \in \mathsf{Arg}(\mathcal{S}_1 \cup \mathcal{S}_2) - \mathsf{Arg}(\mathcal{S}_1)$. By (‡), $t \in \overline{\mathcal{D}'}$. Then, by (†) $t' = \Omega - \mathcal{S}_2 \Rightarrow \sigma \in \mathcal{D}''$ also attacks $\Theta \Rightarrow \psi$. Since $\mathcal{D}^\star$ is a dynamic derivation, $t' \in \mathsf{Elim}(\mathcal{D}^\star) \subseteq \mathsf{Elim}(\mathcal{D}^\ddagger)$, and by ($\star$), $t \in \mathsf{Elim}(\mathcal{D}^\ddagger)$.

We have shown that $\mathcal{D}^\ddagger$ is also a dynamic derivation based on $\mathcal{S}_1 \cup \mathcal{S}_2$. However, since $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ is eliminated in $\mathcal{D}^\ddagger$, this contradicts the final derivability of $\Gamma - \mathcal{S}_2 \Rightarrow \phi$ in $\mathcal{D}''$. □

Note 9. The assumption that the sets of premises are finite is necessary for Proposition 15. Indeed, let $\mathfrak{L} = \langle \mathcal{L}, \vdash\rangle$ be a logic with a $\vdash$-conjunction $\wedge$ and a $\vdash$-negation $\neg$, satisfying the following rules:

$$\frac{\Gamma \Rightarrow \neg\psi, \neg\phi}{\Gamma \Rightarrow \neg(\psi \wedge \phi)}, \qquad \frac{\Gamma \Rightarrow \neg(\psi \wedge \phi)}{\Gamma \Rightarrow \neg\psi, \neg\phi}. \tag{3}$$

We denote by $\mid\!\sim$ the entailment relation (according to Definition 18) for an argumentation setting whose base logic is $\mathfrak{L}$ and whose sole attack rule is Canonical Undercut.

Let $\mathcal{S}_1 = \{p_1, \neg p_1, \neg\neg p_1\}$ and $\mathcal{S}_2 = \{p_i \mid i \geq 2\}$. Clearly, $\mathcal{S}_2$ is $\vdash$-consistent and irrelevant to $\mathcal{S}_1$. Yet, we show that: (a) $\mathcal{S}_1 \mid\!\sim p_1$ while: (b) $\mathcal{S}_1, \mathcal{S}_2 \not\mid\!\sim p_1$.

A derivation for proving Claim (a) may be the following:

1.	$p_1 \Rightarrow p_1$	Axiom	
2.	$\neg p_1 \Rightarrow \neg p_1$	Axiom	
3.	$\neg\neg p_1 \Rightarrow \neg\neg p_1$	Axiom	
4.	$\neg p_1 \not\Rightarrow \neg p_1$	Can-Ucut, 3, 3, 3, 2	$\neg\neg p_1 \Rightarrow \neg\neg p_1$

The only potential attacker of Tuple 1 is Tuple 2, but the latter is eliminated and there is no way to attack its attacker, Tuple 3. Thus $p_1 \Rightarrow p_1$ is finally derived here.

For Claim (b), note that once $\mathcal{S}_2$ is available, we can for instance extend the previous derivation by:

5.	$\neg p_1, p_2 \Rightarrow \neg p_1$	Weakening, 2	
6.	$p_1 \not\Rightarrow p_1$	Can-Ucut, 5, 2, 2, 1	$\neg p_1, p_2 \Rightarrow \neg p_1$[19]

Tuple 1 can still be defended by eliminating $\neg p_1, p_2 \Rightarrow \neg p_1$ (using the rules (3) above), but then it may be re-attacked, e.g., by $\neg p_1, p_3 \Rightarrow \neg p_1$ (a weakening of Tuple 2), and so on. It follows that Tuple 1 is never finally derived. A similar argument applies to other ways of deriving p_1, such as by using $p_1, p_2 \Rightarrow p_1$.

5.4. Cumulativity

In the previous sections we have shown that although $\mid\!\sim$ is not a Tarskian consequence relation (being, e.g., non-monotonic), it still satisfies some cautious versions of the properties listed in Definition 4. This suggests that entailments induced by argumentation settings may satisfy the rationality postulates for non-monotonic reasoning (NMR), introduced by Makinson and Gärdenfors [23,31], and by Kraus, Lehmann, and Magidor [29]. In this section we consider these postulates.

Definition 24. An entailment $\mid\!\sim$ is called $\vdash$-*cumulative* if the following conditions are satisfied:

[19] Note that with Ucut the extension of Lines 1–5 by Line 6 would not be allowed, since Tuple 5 is Ucut-attacked by Tuple 3.

Cautious Reflexivity (CR): If $\psi \Rightarrow \psi$ is not contradictory[20] then $\psi \mid\!\sim \psi$.
Cautious Monotonicity (CM): If $\mathcal{S} \mid\!\sim \phi$ and $\mathcal{S} \mid\!\sim \psi$ then $\mathcal{S}, \phi \mid\!\sim \psi$.
Cautious Cut (CC): If $\mathcal{S} \mid\!\sim \phi$ and $\mathcal{S}, \phi \mid\!\sim \psi$ then $\mathcal{S} \mid\!\sim \psi$.[21]
Left Logical Equivalence (LLE): If $\phi \vdash \psi$ and $\psi \vdash \phi$ then $\mathcal{S}, \phi \mid\!\sim \rho$ iff $\mathcal{S}, \psi \mid\!\sim \rho$.
Right Weakening (RW): If $\phi \vdash \psi$ and $\mathcal{S} \mid\!\sim \phi$ then $\mathcal{S} \mid\!\sim \psi$.

A cumulative entailment is called *preferential* if it satisfies the following condition:

Disjunction Property (OR): If $\mathcal{S}, \phi \mid\!\sim \rho$ and $\mathcal{S}, \psi \mid\!\sim \rho$ then $\mathcal{S}, \phi \vee \psi \mid\!\sim \rho$.

In what follows we shall restrict ourselves to cases where the set $\mathcal{S}$ in the rules above is finite.

Proposition 16. *Every entailment $\mid\!\sim$ that is induced by a SAC setting $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ where $\mathfrak{L} = \langle \mathcal{L}, \vdash \rangle$, is $\vdash$-cumulative.*

Proof. Since $\psi \Rightarrow \psi$ is not contradictory, $\{\psi\}$ is conflict free, and since $\psi \vdash \psi$, by Proposition 10, $\psi \mid\!\sim \psi$ as well. This shows (CR). For (RW) assume that $\mathcal{S} \mid\!\sim \phi$. By Proposition 13, $\{\rho \mid \mathcal{S} \mid\!\sim \rho\} \vdash \phi$, and if $\phi \vdash \psi$, by cut we get that $\{\rho \mid \mathcal{S} \mid\!\sim \rho\} \vdash \psi$. By Proposition 13 again, $\mathcal{S} \mid\!\sim \psi$.

The proofs of the three other conditions are longer and rather technical, so we moved them to Appendix B. Note that LLE is shown there for a larger class of entailment relations than that of the proposition. □

Note 10. The following example shows that $\mid\!\sim$ is *not* preferential even for SAC settings. Consider the setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \mathsf{Ucut} \rangle$ and the set $\mathcal{S} = \{p \wedge \neg q, p \wedge \neg r\}$. We show that $\mathcal{S}, q \mid\!\sim p$ and $\mathcal{S}, r \mid\!\sim p$, but $\mathcal{S}, q \vee r \not\mid\!\sim p$. Indeed, here is a dynamic derivation that is based on $\mathcal{S} \cup \{r\}$ and in which $p \wedge \neg q \Rightarrow p$ is finally derived (thus $\mathcal{S}, r \mid\!\sim p$).[22]

1.	$p \wedge \neg q \Rightarrow p \wedge \neg q$	Ax
2.	$p \wedge \neg q \Rightarrow p$	LK
3.	$\Rightarrow \neg((p \wedge \neg r) \wedge r)$	LK
4.	$\Rightarrow \neg((p \wedge \neg r) \wedge r \wedge (p \wedge \neg q))$	LK
5.	$p \wedge \neg r, r \Rightarrow \neg(p \wedge \neg q)$	LK
6.	$p \wedge \neg r, r, p \wedge \neg q \Rightarrow \neg(p \wedge \neg q)$	LK
7.	$\neg((p \wedge \neg r) \wedge r) \Rightarrow \neg((p \wedge \neg r) \wedge r)$	Ax
8.	$p \wedge \neg r, r \not\Rightarrow \neg(p \wedge \neg q)$	Def, 3, 7, 7, 5
9.	$\neg((p \wedge \neg r) \wedge r \wedge (p \wedge \neg q)) \Rightarrow \neg((p \wedge \neg r) \wedge r \wedge (p \wedge \neg q))$	Ax
10.	$p \wedge \neg r, r, p \wedge \neg q \not\Rightarrow \neg(p \wedge \neg q)$	Def, 4, 9, 9, 6

Here, not only that $p \wedge \neg q \Rightarrow p$ is derived in Tuple 2 (based on Tuple 1), it is also finally derived, because the (non-attacked) sequents in Tuples 3 and 4 block (i.e., counter Ucut-attack) any attempt to Ucut-attack $p \wedge \neg q \Rightarrow p$. Indeed, the potential attacker of Tuple 5 is counter-attacked in Tuple 8 and the potential attacker of Tuple 6 is counter-attacked in Tuple 10.

Similar considerations show that $\mathcal{S}, q \mid\!\sim p$. However, $\mathcal{S}, q \vee r \not\mid\!\sim p$. The reason is the following: Given the premise set $\mathcal{S} \cup \{r \vee q\}$ there are essentially two ways to conclude p: either by deriving $s_1 = p \wedge \neg q \Rightarrow p$ or by deriving $s_2 = p \wedge \neg r \Rightarrow p$. In contrast to the proof above, we can construct non-tautological attackers to each of these two sequents, namely: $p \wedge \neg q, r \vee q \Rightarrow \neg(p \wedge \neg r)$ and $p \wedge \neg r, r \vee q \Rightarrow \neg(p \wedge \neg q)$. This makes it impossible to finally derive neither s_1 nor s_2. It follows that the OR postulate does not hold in this case, and so $\mid\!\sim$ is not a preferential entailment.

5.5. Paraconsistency and crash resistance

We conclude this section by considering two properties of $\mid\!\sim$ that are related to the handling of inconsistent information. The first one, called $\neg$-paraconsistency [18], may be inherited from the base logic $\vdash$:

Proposition 17. *If $\vdash$ is $\neg$-paraconsistent (that is, there are atoms p, q such that $p, \neg p \nvdash q$) then so is $\mid\!\sim$.*

[20] Recall Definition 5.
[21] The combination of CM and CC is sometimes called cumulativity [23,31]: If $\mathcal{S} \mid\!\sim \phi$ then $\mathcal{S} \mid\!\sim \psi$ iff $\mathcal{S}, \phi \mid\!\sim \psi$.
[22] The fourth components of the tuples as well as some derivation steps are omitted (in which case LK is mentioned in the justification part).

Proof. Since $\vdash$ is paraconsistent, $p, \neg p \nvdash q$. Thus, by Proposition 11, $p, \neg p \not\mathrel{|\!\sim} q$. $\square$

Proposition 15 allows us to show another interesting property regarding inconsistency maintenance: *crash resistance* [17] of dynamic proofs systems for SAC settings.

Definition 25. Let $\mathcal{S}$ be a set of $\mathcal{L}$-formulas such that $\mathsf{Atoms}(\mathcal{S}) \subsetneq \mathsf{Atoms}(\mathcal{L})$ (that is, there is at least one atomic formula that is not mentioned in $\mathcal{S}$), and let $\mathrel{|\!\sim}$ be an entailment relation on $\mathcal{L}$.

- We say that $\mathcal{S}$ is *contaminating for* $\mathrel{|\!\sim}$, if for every set $\mathcal{T}$ of $\mathcal{L}$-formulas that is irrelevant to $\mathcal{S}$ (in the sense of Definition 22), and for every $\mathcal{L}$-formula ψ, it holds that $\mathcal{S} \mathrel{|\!\sim} \psi$ iff $\mathcal{S}, \mathcal{T} \mathrel{|\!\sim} \psi$.
- We say that $\mathrel{|\!\sim}$ is *crash-resistant*, if there is no set of $\mathcal{L}$-formulas that is contaminating for $\mathrel{|\!\sim}$.

Proposition 18. *Let $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ be a SAC setting whose base logic $\mathfrak{L}$ is uniform. Then $\mathrel{|\!\sim}_{\mathfrak{S}}$ is crash resistant.*

Proof. If $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \mathfrak{A} \rangle$ is a SAC setting whose base logic $\mathfrak{L}$ is uniform then by Proposition 15 there is no set of $\mathcal{L}$-formulas that is contaminating for $\mathrel{|\!\sim}_{\mathfrak{S}}$, and so the latter is crash-resistant. $\square$

6. Conclusion

Several different approaches to logical argumentation have been introduced in the literature. This includes formalisms that are based on classical logic [13,14], defeasible reasoning [26,35,37,42] the ASPIC$^+$ system [33,40], assumption-based argumentation [20], default logic [38], situation calculus [16], and so forth. Like ASPIC$^+$, our approach provides a very flexible environment for logical argumentation, as it leaves open the choices of the underlying language, the core logic, and the adequate calculus. This flexibility carries on to the representation of arguments that avoids the minimality and consistency constraints posed on the premises of arguments in, e.g., [13].

The rationality behind our approach is to synthesize proof theoretical and argumentation-based methods.[23] On one hand arguments need to be logically justified and constructed by means of formal proofs, but on the other hand the existence of such proofs is not sufficient for accepting the underlying arguments in the presence of counter-arguments. In a sense, then, dynamic derivations resemble dialogues or disputes that are meant to resolve disagreements, and in which conclusions are accepted upon the inability to provide counter-arguments. The main contribution of this paper is therefore the providing of non-monotonic extensions for Gentzen-style proof systems in terms of argumentation-based considerations. In particular, standard argumentation semantics are related to proof theoretic aspects of reasoning.

Our proof format is closely related to dynamic proofs in adaptive logics [12,43], which so far have used Hilbert-style proofs. Our choice of using Gentzen-type systems is motivated by the understanding that in a Hilbert systems finding a derivation for a formula may be a tricky business, as one has to guess which axiom and which inference rule to use without systematic reliance on the syntax of the given formula. In sequent calculi there is less of this kind of guesswork, and derivations are largely syntax-directed. Another major difference from the original proof methods used in the context of adaptive logic is that the progressing of dynamic derivations is uni-directional: the statuses of tuples (derived, accepted, attacked, etc) are determined only in view of tuples that are derived later in the proof, in contrast to bi-directional marking in dynamic proofs of adaptive logic, where a proof line may be marked in view of a previously derived line. By this, we keep the resemblances to standard proof procedures that are progressing in one direction only.

In future work we plan to incorporate more expressive languages involving priorities and, in the longer run, first-order arguments. Also, we hope to develop mechanisms for automatically detecting finally derived sequents in particular scenarios. Another issue to be investigated is considered in the following note.

Note 11. Given a sequent calculus $\mathcal{C}$, the induced consequence relation we have used in the paper for the dynamic derivations is defined by $\mathcal{T} \vdash_{\mathcal{C}} \psi$ if there exists a finite $\Gamma \subseteq \mathcal{T}$ such that $\Gamma \Rightarrow \psi$ is derivable in $\mathcal{C}$ (Recall that we used here only sound and complete calculi $\mathcal{C}$, that is, for which $\vdash_{\mathcal{C}}$ coincides with the consequence relations $\vdash$ of the base logic). Yet, there is another common way to define $\mathcal{C}$-based consequence relations: $\mathcal{T} \Vdash_{\mathcal{C}} \psi$ if the sequent $\Rightarrow \psi$ follows from the set of sequents $\{\Rightarrow \phi \mid \psi \in \mathcal{T}\}$. Note that even in cases where $\vdash_{\mathcal{C}}$ coincides with $\Vdash_{\mathcal{C}}$ (e.g., when $\mathcal{C} = LK$), still the induced argumentation-based entailment relations (and so the dynamic proof systems) may be different. To see this consider, for instance, the setting $\mathfrak{S} = \langle \mathsf{CL}, LK, \mathsf{Ucut} \rangle$ of Example 7, and define $\mathcal{S} \mathrel{||\!\sim}_{\mathfrak{S}} \psi$ if $\Rightarrow \psi$ is finally derived in a dynamic derivation that may contain introducing tuples for sequents of the form $\Rightarrow \phi$ for some $\phi \in \mathcal{S}$. Clearly, $\mathrel{||\!\sim}_{\mathfrak{S}}$ is the counterpart of $\mathrel{|\!\sim}_{\mathfrak{S}}$ (Definition 18), where $\Vdash_{\mathcal{C}}$, instead of $\vdash_{\mathcal{C}}$, is the underlying relation for $\mathcal{C}$-proofs. Now, $\mathrel{||\!\sim}_{\mathfrak{S}}$ is reflexive (since $\Rightarrow \phi$ for any $\phi \in \mathcal{S}$ cannot be attacked by Ucut), while $\mathrel{|\!\sim}_{\mathfrak{S}}$ is not reflexive (recall, e.g., Example 11). On the other hand, $\mathrel{|\!\sim}_{\mathfrak{S}}$ has the

[23] The incorporation of proof theoretical methods in general, and sequent calculi in particular, in the context of argument-based dialectical processes may be traced back (at least) to Dunne and Bench-Capon's 2003 paper [21]. Their motivation for considering sequent calculi in this context is quite different, though: the complexity of proofs in a cut-free Gentzen-type sequent calculus is analyzed in order to obtain some results concerning the computational complexity of argument games.

property of Proposition 8, while $|\!\!\sim_{\mathfrak{S}}$ does not have it (Indeed, by reflexivity, when $\mathcal{S} = \{p, \neg p\}$ we have that $\mathcal{S} \,|\!\!\sim_{\mathfrak{S}} p$ and $\mathcal{S} \,|\!\!\sim_{\mathfrak{S}} \neg p$). A detailed analysis of the argumentation-based entailments of the form $|\!\!\sim_{\mathfrak{S}}$ and their dynamic proof systems is beyond the scope of this paper, and it is left for future work.

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Appendix A. Proof invariance under weak symmetry

The purpose of this appendix is to show that (final) provability in dynamic systems for frameworks that are weakly symmetric is proof invariant: if a certain argument s is finally derivable, then any dynamic derivation for the same setting can be extended in such a way that s will be finally derived in that derivation (see Proposition 6 and Proposition 23 below). For the proof, we first consider some notations and lemmas.

Given a sequent-based (logical) argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}\rangle$ for a set of formulas $\mathcal{S}$ (induced by setting $\mathfrak{S}$), we define a corresponding framework, for the same setting and set of formulas, but whose arguments are equivalence classes of arguments in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. This is defined next.

Definition 26. Let $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack}\rangle$ be a sequent-based argumentation framework and let $\sim$ be an equivalence relation on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$.

- The $\sim$-*equivalence class* of $s \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ is the set $[s] = \{s' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) \mid s' \sim s\}$. The corresponding set of arguments is the quotient set of $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, namely: $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim\, = \{[s] \mid s \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})\}$.
- We write $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})$ for the quotient graph[24] of $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$, i.e.: $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim, \mathit{Attack}/\!\sim\rangle$, where $\mathit{Attack}/\!\sim\, = \{([s],[t]) \mid (s,t) \in \mathit{Attack}\}$. In what follows we shall refer to $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})$ as the *quotient framework* of $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$.

In terms of the definition above we recall that $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ is $\sim$-weakly symmetric (Definition 20) iff $\sim$ is a right congruence[25] on $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ and $(\mathit{Attack}/\!\sim)^*$ is symmetric and irreflexive, where:

- $\mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S})/\!\sim\; = \; \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim\; - \Big(\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})) \;\cup\; (\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})))^+\Big)$,[26]
- $(\mathit{Attack}/\!\sim)^* = \mathit{Attack}/\!\sim\; \cap \Big(\mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S})/\!\sim\; \times\; \mathsf{Arg}^*_{\mathfrak{L}}(\mathcal{S})/\!\sim\Big)$.

Using the notations above, we can show proof invariance (under $\sim$-weak symmetry). First, since the proof in what follows is rather long and technical, we give an outline of it.

Outline of the proof. Suppose that $\mathcal{D}$ is a dynamic derivation in which s is finally derived from $\mathcal{S}$ and that $\mathcal{D}'$ is an arbitrary dynamic derivation (for the same setting) which is also based on $\mathcal{S}$.

1. In view of Proposition 19 below we can extend $\mathcal{D}'$ to $\mathcal{D}''$ in such a way that every $t \in \overline{\mathcal{D}}$ is added, where $\overline{\mathcal{D}}$ denotes the set of all sequents that occur in the derivation $\mathcal{D}$.
2. In Lemma 4 we then show that every attacker in $\mathsf{Arg}_{\mathfrak{S}}(\mathcal{S})$ of s is in $\overline{\mathcal{D}} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ and that $s \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. Hence, every attacker $t \in \mathsf{Arg}_{\mathfrak{S}}(\mathcal{S})$ of s is in $\overline{\mathcal{D}''} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$.
3. Let $\mathsf{R}_{\overline{\mathcal{D}''}}$ be a set that contains for each $t \in \mathcal{D}'' \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ a $t' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ for which $(t', t) \in \mathit{Attack}$.
4. In Proposition 21 below we show that there is a stable extension $\mathcal{E}$ of $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!(\mathcal{D}''/\!\sim \cup\, \mathsf{R}_{\overline{\mathcal{D}''}}/\!\sim)$, where the latter is the restriction of $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})$ to the arguments in $\overline{\mathcal{D}''}/\!\sim \cup\, \mathsf{R}_{\overline{\mathcal{D}''}}/\!\sim$, such that $[s] \in \mathcal{E}$.
5. In Lemma 2 we then show that $\mathcal{D}''$ can be extended to $\mathcal{D}^*$ in such a way that $\mathsf{R}_{\overline{\mathcal{D}''}} \subseteq \overline{\mathcal{D}^*}$ and $s \in \mathsf{Accepted}(\mathcal{D}^*)$. Since all attackers of s are attacked by elements in $\mathsf{R}_{\overline{\mathcal{D}''}} \subseteq \overline{\mathcal{D}^*} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$, s is finally derived in $\mathcal{D}^*$.
6. Since $\mathcal{D}^*$ is an extension of $\mathcal{D}'$ and $\mathcal{D}'$ is a dynamic derivation based on $\mathcal{S}$, this suffices to prove proof invariance.

We now provide the details of the proof outlined above.

Proposition 19. *Let $\mathcal{D}$ be a dynamic derivation for $\mathfrak{S}$, and denote by $\overline{\mathcal{D}}$ the set of sequents that appear in the derivation $\mathcal{D}$. For every $s \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) - \overline{\mathcal{D}}$ there is an extension $\mathcal{D}'$ of $\mathcal{D}$ such that $s \in \mathsf{Accept}(\mathcal{D}')$.*

[24] See, e.g., [28, Definition 2.8].
[25] Recall Item 2 in Note 2.
[26] Recall the definition of Root in Definition 19.

Proof. Since $s \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, there is a proof of s in the underlying calculus $\mathfrak{C}$. We therefore extend $\mathcal{D}$ to $\mathcal{D}'$ by adding each sequent t in such a proof which is not already in $\overline{\mathcal{D}}$. Clearly, $\mathsf{Elim}(\mathcal{D}) = \mathsf{Elim}(\mathcal{D}')$ and $\mathsf{Attack}(\mathcal{D}) = \mathsf{Attack}(\mathcal{D}')$, hence $\mathcal{D}'$ is coherent, and $s \in \mathsf{Accept}(\mathcal{D}')$. $\square$

Next we extend $\mathcal{D}'$ of the last proposition in such a way that s is finally derived in the derivation that is obtained. For this we first relate in the next proposition an argumentation framework with its quotient argumentation framework (for a given equivalence relation $\sim$).

Proposition 20. *$\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ be an argumentation framework and $\sim$ a right congruence on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$.*

1. *$s \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ iff $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))$.*
2. *$s \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ iff $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))^+$.*
3. *Let $\bigcup \mathcal{E} = \bigcup\{[s] \mid [s] \in \mathcal{E}\}$. If $\mathcal{E} \in \mathsf{Stbl}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))$ then $\bigcup \mathcal{E} \in \mathsf{Stbl}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$.*

Proof.
1. ($\Rightarrow$) Suppose that $[s] \notin \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))$. Then there is a $[t] \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim$ for which $([t],[s]) \in Attack/\!\sim$. Thus, there are $t' \in [t]$ and $s' \in [s]$ such that $(t', s') \in Attack$. Since $\sim$ is a right congruence and $s \sim s'$ also $(t', s) \in Attack$. Hence, $s \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$.
($\Leftarrow$) Suppose that $s \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$. Then there is a $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ for which $(t, s) \in Attack$. Hence, $([t],[s]) \in Attack/\!\sim$, and so $[s] \notin \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))$.
2. ($\Rightarrow$) Suppose that $s \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. Then, there is a $t \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ for which $(t, s) \in Attack$. Hence, $([t],[s]) \in Attack/\!\sim$. Since by Item 1 $[t] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))$, we have that $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))^+$.
($\Leftarrow$) Suppose that $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))^+$. Then there is a $[t] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))$ for which $([t],[s]) \in Attack/\!\sim$. Thus, there are $t' \in [t]$ and $s' \in [s]$ for which $(t', s') \in Attack$. Since $\sim$ is a right congruence and $s \sim s'$ also $(t', s) \in Attack$. By Item 1 again, $t' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$, and so $s \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$.
3. Suppose that $\mathcal{E} \in \mathsf{Stbl}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))$, and let $s, t \in \bigcup \mathcal{E}$. Then $(s, t) \notin Attack$, otherwise $([s],[t]) \in Attack/\!\sim$, in which case $\mathcal{E}$ is not conflict-free (and so not stable). Thus $\bigcup \mathcal{E}$ is conflict-free. Let now $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) - \bigcup \mathcal{E}$. Then $[t] \in (\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim) - \mathcal{E}$. Since $\mathcal{E}$ is stable, there is a $[s] \in \mathcal{E}$ such that $([s],[t]) \in Attack/\!\sim$. Thus, there is are $s' \in [s]$ and $t' \in [t]$ such that $(s', t') \in Attack$. Since $\sim$ is a right congruence, also $(s', t) \in Attack$. In particular, there is a $s' \in \bigcup \mathcal{E}$ such that $(s', t) \in Attack$. It follows that $\bigcup \mathcal{E}$ is a stable extension of $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$. $\square$

Definition 27. Let $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ and $\mathcal{A} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. We denote by $\mathsf{R}_{\mathcal{A}}$ an arbitrary hitting set of $\big\{\{s \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})) \mid (s,t) \in Attack\} \mid t \in \mathcal{A}\big\}$.[27]

Intuitively, $\mathsf{R}_{\mathcal{A}}$ consists of elements in $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ that attack elements in $\mathcal{A}$ (at least one attacker for each element in $\mathcal{A}$).

Note 12. In case that the set $\mathcal{A}$ in Definition 27 is finite, there is a finite set $\mathsf{R}_{\mathcal{A}}$.

Definition 28. Given $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ and $\mathcal{A} \subseteq \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, we denote by $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) \downarrow \mathcal{A}$ the framework that is obtained by restricting $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ to the elements in $\mathcal{A}$. That is: $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) \downarrow \mathcal{A} = \langle \mathcal{A},\ Attack \cap (\mathcal{A} \times \mathcal{A})\rangle$.

Note 13. In terms of Definition 28, the inner framework of $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ (considered in the paragraph below Definition 19) is $\mathcal{AF}^{*}_{\mathfrak{S}}(\mathcal{S}) = \mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) \downarrow \mathsf{Arg}^{*}_{\mathfrak{L}}(\mathcal{S})$.

In what follows we suppose that $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ is a $\sim$-weakly symmetric framework, where $\sim$ is a right congruence on $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})$ is the corresponding quotient framework.

The next proposition shows that for any subset of arguments $\mathcal{A} \subseteq \mathsf{Arg}_{\mathfrak{S}}(\mathcal{S})$ that contains for each $t \in \mathcal{A} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ an $s_t \in \mathcal{A} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ for which $(s_t, t) \in Attack$ and for each $s \in \mathcal{A} - \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}\!\downarrow\!\mathcal{A})$, there is a stable extension $\mathcal{E}$ of $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!(\mathcal{A}/\!\sim)$ such that $[s] \in \mathcal{E}$.

Proposition 21. *Let $\mathcal{A} \subseteq \mathsf{Arg}_{\mathfrak{S}}(\mathcal{S})$ and let $\mathsf{R}_{\mathcal{A}}$ be any hitting set as considered in Definition 27. We denote: $\mathcal{A}_{\sim} =_{\mathsf{df}} \{[s] \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim \mid s \in \mathcal{A} \cup \mathsf{R}_{\mathcal{A}}\}$. Then:*

1. *if $[s] \in \mathcal{A}_{\sim} - \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{A}_{\sim})^+$, $[s]$ is an element of some stable extension $\mathcal{E}$ of $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{A}_{\sim}$ for which $\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{A}_{\sim}) \subseteq \mathcal{E}$,*
2. *$\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{A}_{\sim})^+ \subseteq \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))^+$,*
3. *$\mathsf{Stbl}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{A}_{\sim}) \neq \emptyset$.*

[27] Recall that Δ is a *hitting set* of a set of sets $\Xi = \{\Delta_i \mid i \in I\}$ iff for all $i \in I$, $\Delta \cap \Delta_i \neq \emptyset$.

Proof. *Item 1.* Let $\{[s_1], [s_2], \ldots\}$ be an enumeration of $\mathcal{A}_\sim - \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)^+$ such that $s_1 = s$. We consider the set $\mathcal{E} = \bigcup_{i \ge 1} \mathcal{E}_i$, where $\mathcal{E}_1 = \{[s_1]\}$, and for every $i \ge 1$

$$\mathcal{E}_{i+1} = \begin{cases} \mathcal{E}_i \cup \{[s_{i+1}]\} & \forall [s'] \in \mathcal{E}_i, ([s'], [s_{i+1}]) \notin (Attack/\!\sim) \cap (\mathcal{A}_\sim \times \mathcal{A}_\sim), \\ \mathcal{E}_i & \text{otherwise.} \end{cases}$$

We claim that $\mathcal{E}$ is a stable extension of $\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim$, containing the argument $[s]$.

Note first that $\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim) \subseteq \mathcal{E}$. To see this, let $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)$. Then $[s] = [s_i]$ for some $i \ge 1$. Clearly, $[s_i] \in \mathcal{E}_i$ since there is no $[t] \in \mathcal{A}_\sim$ for which $([t], [s_i]) \in (Attack/\!\sim)$.

To show that $\mathcal{E}$ is stable, we first show that $\mathcal{E}$ is conflict-free. For this we show the following lemma:

Lemma 1. *Let* $Attack^{\sim} = \big((Attack/\!\sim) \cap (\mathcal{A}_\sim)^2\big) - \big(\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim) \cup (\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim))^+\big)^2$. *Then* $Attack^{\sim} \subseteq (Attack/\!\sim)^* \cap (\mathcal{A}_\sim)^2$.

To see this note that (†) $\mathcal{A}_\sim \cap \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S})) \subseteq \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)$ and thus also (‡) $\mathcal{A}_\sim \cap \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}))^+ \subseteq (\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim))^+$. We have: $(Attack/\!\sim)^* \cap (\mathcal{A}_\sim)^2 = ((Attack/\!\sim) \cap (\mathcal{A}_\sim)^2) - \big((\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S})) \cap \mathcal{A}_\sim) \cup ((\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S})))^+ \cap \mathcal{A}_\sim)\big)^2$. By (†) and (‡), $(Attack/\!\sim)^* \cap (\mathcal{A}_\sim)^2 \supseteq ((Attack/\!\sim) \cap (\mathcal{A}_\sim)^2) - \big(\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim) \cup (\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S})){\downarrow}\mathcal{A}_\sim)^+\big)^2 = Attack^{\sim}$. □

From this, the definition of $Attack^{\sim}$, and since $\mathcal{AF}_{\mathfrak{G}}(\mathcal{S})$ is $\sim$-weakly symmetric, it immediately follows that $Attack^{\sim}$ is both symmetric and irreflexive.

Now, to prove the conflict-freeness of $\mathcal{E}$, let $[s_i], [s_j] \in \mathcal{E}$ and assume for a contradiction that $([s_i], [s_j]) \in ((Attack/\!\sim) \cap \mathcal{A}_\sim)^2$. By the construction, $i > j$. Since $\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)^+ \cap \mathcal{E} = \emptyset$, $([s_i], [s_j]) \in Attack^{\sim}$. By the symmetry of $Attack^{\sim}$ also $([s_j], [s_i]) \in Attack^{\sim}$, which is a contradiction, since then $[s_i] \notin \mathcal{E}$ by the construction of $\mathcal{E}_i$. Altogether, this shows that $\mathcal{E}$ is conflict-free.

Suppose that $[t] \in \mathcal{A}_\sim - \mathcal{E}$. In case that $[t] \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)^+$, we have $[t] \in \mathcal{E}^+$, since $\mathsf{Root}(\mathcal{AF}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim) \subseteq \mathcal{E}$. Otherwise, $[t] = [s_j]$ for some $j \ge 1$. Hence, there is a $[s_i] \in \mathcal{E}$ with $1 \le i < j$ such that $([s_i, s_j]) \in Attack/\!\sim$. Again, $[t] \in \mathcal{E}^+$. Altogether, this shows that $\mathcal{E}$ is a stable extension of $\mathcal{AF}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim$.

Item 2. Suppose that $[t] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)^+$ and assume for a contradiction that $[t] \notin \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}))^+$. Thus, there is an $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)$, for which $([s], [t]) \in Attack/\!\sim$. By the supposition, $[s] \notin \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}))$. Assume now that $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}))^+$. Thus, there is a $s' \in [s] \cap \mathcal{A}$. By Proposition 20 (item 2), $s' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{G}}(\mathcal{S}))^+$. Hence, there is a $t_{s'} \in \mathsf{R}_{\mathcal{A}}$ for which $(t_{s'}, s') \in Attack$. Thus, $([t_{s'}], [s]) \in Attack/\!\sim$ and hence, $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)^+$. This contradicts that $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)$. Thus, $[s] \notin \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}))^+$.

Since $\{[s], [t]\} \cap (\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}) \cup \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}))^+)) = \emptyset$ and since $(Attack/\!\sim)^*$ is symmetric, also $([t], [s]) \in Attack/\!\sim$. But then $[s] \notin \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)$, a contradiction. Hence, $[t] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}))^+$.

Item 3. In case that $\mathcal{A}_\sim = \emptyset$ the claim is trivial. Suppose that $\mathcal{A}_\sim \neq \emptyset$. If $\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim) \neq \emptyset$, then $\mathcal{A}_\sim - \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)^+ \neq \emptyset$. Otherwise, also $\mathcal{A}_\sim - \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim)^+ = \mathcal{A}_\sim \neq \emptyset$. By Item 1, $\mathcal{AF}_{\mathfrak{G}}(\mathcal{S}){\downarrow}\mathcal{A}_\sim$ has a stable model. □

Proposition 22. *Every argumentation framework $\mathcal{AF}_{\mathfrak{G}}(\mathcal{S})$ that is $\sim$-weakly symmetric (where $\sim$ is some right congruence on the set of arguments) has a stable model.*

Proof. By Item 3 of Proposition 21 (where $\mathcal{A} = \mathsf{Arg}_{\mathfrak{G}}(\mathcal{S})$), we have $\mathsf{Stbl}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S})) \neq \emptyset$. Thus, by Item 3 of Proposition 20, $\mathsf{Stbl}(\mathcal{AF}_{\mathfrak{G}}(\mathcal{S})) \neq \emptyset$. □

Recall that proof invariance basically means that if a sequent s is finally derived in a certain derivation, then every derivation in the same setting and which is based on the same formulas, can be extended in such a way that s will be finally derived in the extended derivation. The next three lemmas show how such an extension may be constructed.

Lemma 2. *Let $\mathcal{D}$ be a dynamic derivation for $\mathfrak{G}$ that is based on $\mathcal{S}$, let $\mathsf{R}_{\overline{\mathcal{D}}}$ be a finite hitting set as in Definition 27 and Note 12,*[28] *let $\mathcal{D}_\sim = \{[s] \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim \mid s \in \overline{\mathcal{D}} \cup \mathsf{R}_{\overline{\mathcal{D}}}\}$, and let $\mathcal{E} \in \mathsf{Stbl}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}) \downarrow \mathcal{D}_\sim)$.*[29] *We denote: $\overline{\mathcal{D}}[\mathcal{E}] = \{s \in \overline{\mathcal{D}} \mid [s] \in \mathcal{E}\}$. There is an extension $\mathcal{D}'$ of $\mathcal{D}$ such that*

1. $\overline{\mathcal{D}}[\mathcal{E}] = \mathsf{Accept}(\mathcal{D}')$,

[28] Recall that $\overline{\mathcal{D}}$ denotes the set of all sequents that occur in the derivation $\mathcal{D}$.

[29] By Proposition 21 (Item 3), $\mathsf{Stbl}(\mathcal{AF}^{\sim}_{\mathfrak{G}}(\mathcal{S}) \downarrow \mathcal{D}_\sim) \neq \emptyset$.

2. $\mathsf{Elim}(\mathcal{D}') = \overline{\mathcal{D}} - \overline{\mathcal{D}}[\mathcal{E}]$, *and*
3. *for every* $s \in \overline{\mathcal{D}} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ *there is a* $t_s \in \overline{\mathcal{D}'} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ *such that* $\mathcal{D}'$ *contains an eliminating tuple in which* t_s *attacks* s, *and* $s \in \mathsf{Elim}(\mathcal{D}')$.

Proof. Let $\mathcal{E} = \{[s_1], \ldots, [s_n]\}$ be a stable extension of $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}) \downarrow \mathcal{D}_{\sim}$. We construct the derivation $\mathcal{D}'$ as follows:

- Suppose that $s \in (\overline{\mathcal{D}} - \overline{\mathcal{D}}[\mathcal{E}]) - \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ attacks some $t \in \overline{\mathcal{D}}[\mathcal{E}]$. Then $([s],[t]) \in Attack/\!\sim$. Hence, there is a $[t'] \in \mathcal{E}$ such that $([t'],[s]) \in Attack/\!\sim$. Thus, there are $s' \in [s]$ and $t_s \in [t']$ such that $(t_s, s') \in Attack$. Since $\sim$ is a right congruence, $(t_s, s) \in Attack$. Since $t_s \sim t'$, $[t_s] = [t'] \in \mathcal{E}$.
- Suppose that $s \in (\overline{\mathcal{D}} - \overline{\mathcal{D}}[\mathcal{E}]) \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. Hence, (†) there is a $t_s \in \mathsf{R}_{\overline{\mathcal{D}}}$ for which $(t_s, s) \in Attack$. By Proposition 20 (item 1) $[t_s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))$ and thus $[t_s] \in \mathcal{E}$.
- Thus, for each $s \in \overline{\mathcal{D}} - \overline{\mathcal{D}}[\mathcal{E}]$, we have found a $t_s \in \bigcup \mathcal{E}$ such that $(t_s, s) \in Attack$. We introduce each such t_s (which is not already in $\overline{\mathcal{D}}$), denote the resulting derivation by $\mathcal{D}''$, and suppose that the indices of the new tuples are $l+1, \ldots, l+m''$.
- For every $s \in \overline{\mathcal{D}} - \overline{\mathcal{D}}[\mathcal{E}]$ there is now a sequent $t_s \in \overline{\mathcal{D}''}$ such that $(t_s, s) \in Attack$, where the attack is obtained by an application of some elimination rule $\mathcal{R}$ under some condition. As in Proposition 19, we add to $\mathcal{D}''$ proofs of all the conditions of these rules, which are not already occurring in $\mathcal{D}''$. Let the indices of the new tuples be $l+m''+1, \ldots, l+m'''$ and let the resulting derivation be denoted by $\mathcal{D}'''$.
- Now, (‡) we add elimination tuples in which t_s attacks s for every $t_s \in \overline{\mathcal{D}''}$ and every $s \in \overline{\mathcal{D}} - \overline{\mathcal{D}}[\mathcal{E}]$, for which $(t_s, s) \in Attack$. We denote the resulting derivation by $\mathcal{D}'$ and suppose that the indices of the new tuples are $l+m'''+1, \ldots, l+m'$.

It is not difficult to verify that the resulting derivation is valid. Coherence follows since sequents are added to Elim only until the top-down evaluation algorithm of Fig. 3 reaches line $l+m'''+1$. The reason for this is that all other elimination tuples $\tau = \langle l', \bar{s}, \ldots, t\rangle$ in $\mathcal{D}'$ in which t attacks s and $l' < l+m'''+1$, are such that either $t \in \overline{\mathcal{D}} - \overline{\mathcal{D}}[\mathcal{E}]$ or $t \in \overline{\mathcal{D}}[\mathcal{E}]$. In the former case, t has already been added to Elim (in one of the lines $l+m'''+1, \ldots, l+m'$) and so no new element is added to Elim when τ is reached in the algorithm. In the latter case the attacked sequent (s) is already in Elim, since its attacker t_s already attacked it in one of the lines $l+m'''+1, \ldots, l+m'$. Hence, $\mathsf{Attack}(\mathcal{D}') \subseteq \bigcup \mathcal{E}$. Since $\bigcup \mathcal{E}$ is conflict-free, $\mathsf{Attack}(\mathcal{D}') \cap \mathsf{Elim}(\mathcal{D}') \subseteq \bigcup \mathcal{E} \cap \mathsf{Elim}(\mathcal{D}') = \emptyset$, which shows that $\mathcal{D}'$ is coherent, that $\overline{\mathcal{D}}[\mathcal{E}] = \mathsf{Accept}(\mathcal{D}')$, and that $\mathsf{Elim}(\mathcal{D}') = \overline{\mathcal{D}} - \overline{\mathcal{D}}[\mathcal{E}]$. The last item of the proposition immediately follows from the construction (by (†) and (‡)). □

Lemma 3. *Let* $\mathcal{D}$ *be a dynamic derivation for* $\mathfrak{S}$ *that is based on* $\mathcal{S}$ *and let* $s \in \overline{\mathcal{D}}$ *be a sequent such that for all* $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ *for which* $(t, s) \in Attack$, *it holds that* $t \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. *If there is a sequent* $t \notin \overline{\mathcal{D}}$ *such that* $(t, s) \in Attack$, *then there is an extension* $\mathcal{D}'$ *of* $\mathcal{D}$ *such that* $s \in \mathsf{Elim}(\mathcal{D}')$ *and* $t \in \mathsf{Accept}(\mathcal{D}')$.

Proof. We construct $\mathcal{D}'$ as follows: Let $\mathcal{E}$ be a stable extension of $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}) \downarrow D_{\sim}$ where $\mathcal{D}_{\sim}$ is defined as in Lemma 2. (Again, by Item 3 of Proposition 21 such a stable extension exists). We extend $\mathcal{D}$ to $\mathcal{D}''$ as in Lemma 2. Suppose that the indices of the new tuples are $l+1, \ldots, l+n$. By Item 2 of Lemma 2, $t \notin \mathsf{Elim}(\mathcal{D}'')$, since $t \notin \overline{\mathcal{D}}$.

We now add the proof of t (as in Proposition 19) and an eliminating tuple in which t attacks s, resulting in a derivation $\mathcal{D}'$. To see that $\mathcal{D}'$ is a valid derivation we have to show coherence. For this we first show that (†) all sequents s' which are eliminated by s in $\mathcal{D}''$ are also eliminated by some $t_{s'} \in \mathsf{R}_{\overline{\mathcal{D}}} \subseteq \overline{\mathcal{D}''} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ in one of the lines $l+1, \ldots, l+n$. With (†) we know that $\mathsf{Elim}(\mathcal{D}') = \mathsf{Elim}(\mathcal{D}'') \cup \{s\}$ and $\mathsf{Attack}(\mathcal{D}') = (\mathsf{Attackers}(\mathcal{D}'') \cup \{t\}) - \{s\}$. Since $t \notin \mathsf{Elim}(\mathcal{D}')$ the coherence of $\mathcal{D}'$ follows from the coherence of $\mathcal{D}''$.

To show (†), suppose that s' is a sequent eliminated by s in $\mathcal{D}''$. Then $(s, s') \in Attack$ and hence $s' \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$. Since $(t, s') \in Attack$, $s \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$. Also, since every attacker of s is attacked by an element in $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$, $s \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. Assume for a contradiction that $s' \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. By Proposition 20, $([s],[s']) \in (Attack/\!\sim)^\star$. Then, $([s'],[s]) \in Attack/\!\sim$ by the symmetry of $(Attack/\!\sim)^\star$. Hence, for some $s'' \in [s']$, $(s'', s) \in Attack$ since $\sim$ is a right congruence. So, s'' is attacked by some sequent in $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ and by Proposition 20 so is s'. This contradicts the assumption that $s' \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. Thus, $s' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ and by the definition of $\mathsf{R}_{\mathcal{A}}$ there is a $t_{s'} \in \mathsf{R}_{\mathcal{A}}$ for which $(t_{s'}, s') \in Attack$. □

Lemma 4. *Let* $\mathcal{D}$ *be a dynamic derivation for* $\mathfrak{S}$ *that is based on* $\mathcal{S}$. *If* s *is finally derived in* $\mathcal{D}$ *then (i) all the attackers* $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ *of* s *are in* $\overline{\mathcal{D}} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ *and (ii)* $s \notin \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$.

Proof. Assume for a contradiction that there is a sequent $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}) - \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$ such that $(t, s) \in Attack$. We distinguish two cases, (a) $t \in \overline{\mathcal{D}}$ and (b) $t \notin \overline{\mathcal{D}}$. In case (a) let $\mathcal{D}^* = \mathcal{D}$. In case (b) we extend $\mathcal{D}$ to $\mathcal{D}^*$ with a derivation of t (as in Proposition 19). By Item 2 of Proposition 20, $[t] \in (\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim) - (\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))^+)$. Where $\mathsf{R}_{\overline{\mathcal{D}^*}}$ is a finite hitting set as in Definition 27 and Note 12, let $\mathcal{D}^*_{\sim} = \{[s] \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim \mid s \in \mathcal{D}^* \cup \mathsf{R}_{\overline{\mathcal{D}^*}}\}$. By Item 2 of Proposition 21, $[t] \in \mathcal{D}^*_{\sim} - \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}) \downarrow \mathcal{D}^*_{\sim})^+$. By Item 1 of Proposition 21, $[t]$ is an element of some stable extension of $\mathcal{E}$ of $\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}) \downarrow (\mathcal{D}^*_{\sim})$. By Lemma 2, there is an extension $\mathcal{D}'$ of $\mathcal{D}^*$ such that $t \in \mathsf{Accept}(\mathcal{D}')$ and $s \in \mathsf{Elim}(\mathcal{D}')$, which is a contradiction to the assumption that s is finally derived in $\mathcal{D}$. Hence, (†) any attacker t of s is in $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$.

Finally, by (†) and Lemma 3, every attacker t of s is in $\overline{\mathcal{D}}$. This completes the proof for (i). Now, (ii) follows immediately in view of (i). □

Now we are ready to show the main result of the appendix (which is a slight reformulation of Proposition 6):

Proposition 23. *Let $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), Attack\rangle$ be a $\sim$-weakly symmetric framework. Let $\mathcal{D}$ and $\mathcal{D}'$ be two dynamic derivations for $\mathfrak{S}$ that are based on $\mathcal{S}$. If s is finally derived in $\mathcal{D}$, there is an extension of $\mathcal{D}'$ such that s is also finally derived in that extension.*

Proof. Let $\mathcal{D}$, $\mathcal{D}'$ and s be as in the proposition. We construct the required extension of $\mathcal{D}'$ as follows.

- Let $\mathcal{D}''$ be the result of extending $\mathcal{D}'$ by adding to the latter (the proofs of) every $t \in \overline{\mathcal{D}} - \overline{\mathcal{D}'}$ (as in Proposition 19).
- By Lemma 4, all the attackers $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ of s are in $\overline{\mathcal{D}} \cap \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$. Thus, by Item 1 of Proposition 20, (†) every attacker $[t] \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim$ of $[s]$ is in $\mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))^+ \cap \overline{\mathcal{D}}/\!\sim$.
- Let $\mathcal{D}''_{\sim} = \{[t] \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})/\!\sim \mid s \in \overline{\mathcal{D}''} \cup \mathsf{R}_{\overline{\mathcal{D}''}}\}$, where $\mathsf{R}_{\overline{\mathcal{D}''}}$ is a finite hitting set as in Definition 27 and Note 12. We now show that $[s] \in \mathcal{D}''_{\sim} - \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{D}''_{\sim})^+$. Assume for a contradiction that $[s] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{D}''_{\sim})^+$. Thus, there is a $[t] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{D}''_{\sim})$ such that $([t],[s]) \in Attack/\!\sim$. By (†), also $[t] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S}))^+ \cap \overline{\mathcal{D}}/\!\sim$. This implies that there are $t' \in \mathsf{R}_{\overline{\mathcal{D}''}}$ and $t'' \in [t]$ for which $(t', t'') \in Attack$. Thus, $([t'],[t]) \in Attack/\!\sim$ which contradicts $[t] \in \mathsf{Root}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{D}''_{\sim})$.
- By Item 1 of Proposition 21, there is a $\mathcal{E} \in \mathsf{Stbl}(\mathcal{AF}^{\sim}_{\mathfrak{S}}(\mathcal{S})\!\downarrow\!\mathcal{D}''_{\sim})$ for which $[s] \in \mathcal{E}$.
- By Lemma 2, there is an extension $\mathcal{D}^*$ of $\mathcal{D}''$ (based on the hitting set $\mathsf{R}_{\overline{\mathcal{D}''}}$) for which $D''[\mathcal{E}] = \mathsf{Accepted}(\mathcal{D}^*)$ and hence $s \in \mathsf{Accepted}(\mathcal{D}^*)$.
- We have that s is finally derived in $\mathcal{D}^*$ since all of its attackers t are eliminated in $\mathcal{D}^*$ by some $t_s \in \mathsf{R}_{\overline{\mathcal{D}''}} \subseteq \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})) \cap \mathcal{D}^*$ which cannot itself be eliminated in any extension of the proof. □

Appendix B. Rationality postulates for non-monotonic reasoning

In this appendix we prove some NMR postulates satisfied by $\mid\!\sim$, as claimed in Propositions 13 and 16.

Closure with respect to the base logic

First, we show Proposition 13: If $\mid\!\sim$ is induced by a SAC setting (Definition 21) then for every finite set $\mathcal{S}$ of formulas, $\mathcal{S} \mid\!\sim \phi$ iff $\{\psi \mid \mathcal{S} \mid\!\sim \psi\} \vdash \phi$.

For this, in what follows we assume that $\mathcal{S}$ is finite, and show the claim for Indirect Defeat attacks.

Lemma 5. *If $s = \Delta \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ and for every $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ for which $(t, s) \in Attack$ it holds that $t \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))^+$, then s is finally derivable and $\mathcal{S} \mid\!\sim \phi$.*

Proof. We construct a proof in which s is finally derived as follows. We start with a derivation of s. Let now $\{t_1, \dots, t_n\}$ be the set of all attackers of s in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Since $\mathcal{S}$ is finite, this set is finite as well. We now derive each of these attackers. By our supposition, for each t_i there is an $s_i \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ for which $(s_i, t_i) \in Attack$. We now derive each s_i. Finally, we add $\overline{t_i}$ for each $i \in I$ with the attacker s_i. Let the resulting dynamic derivation be $\mathcal{D}$. Note that $\mathcal{D}$ is coherent since $\mathsf{Attack}(\mathcal{D}) = \{s_1, \dots, s_n\}$, $\mathsf{Elim}(\mathcal{D}) = \{t_1, \dots, t_n\}$ and $\{s_1, \dots, s_n\} \cap \{t_1, \dots, t_n\} = \emptyset$. Now, s is finally derived in $\mathcal{D}$ since every attacker t_i of s is eliminated by an argument without attackers. □

Lemma 6. *If $s = \Delta \Rightarrow \phi \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ and there is a $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ for which $(s, t) \in Attack$, then $\Delta = \emptyset$.*

Proof. Assume for a contradiction that $\Delta \neq \emptyset$ and let $t = \Lambda \Rightarrow \psi$. Then $\phi \Rightarrow \neg \bigwedge \Lambda'$ is derivable for some $\Lambda' \subseteq \Lambda$. By Contraposition and Cut, $\Lambda' \Rightarrow \neg \bigwedge \Delta \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, thus t attacks s. This contradicts the assumption that $s \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$. □

Lemma 7. *If $\Delta \Rightarrow \phi$ and $\Delta' \Rightarrow \phi'$ are finally derivable from $\mathcal{S}$, also $\Gamma \Rightarrow \psi$ is finally derivable from $\mathcal{S}$ for any Γ for which $\Gamma \vdash \psi$ and $\Gamma \subseteq \Delta \cup \Delta'$.*

Proof. By Lemma 5, we need to show that every attacker $t \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ of $s = \Gamma \Rightarrow \psi$ is attacked by some $s_t \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$. Suppose that s is attacked by some $\Lambda \Rightarrow \sigma \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Then $\sigma \Rightarrow \neg \bigwedge \Gamma'$ is derivable for some $\Gamma' \subseteq \Gamma$ and thus, where $\Delta'' = \Delta \cap \Gamma'$ and $\Delta''' \subseteq \Delta' \cap \Gamma'$, $\sigma \Rightarrow \neg \bigwedge(\Delta'' \cup \Delta''')$ is derivable. By Contraposition and Cut, $\Lambda, \Delta'' \Rightarrow \neg \bigwedge \Delta''' \in \mathsf{Arg}_{\mathfrak{S}}(\mathcal{S})$ attacks $\Delta' \Rightarrow \phi'$. By Lemma 4, $\Lambda, \Delta'' \Rightarrow \neg \bigwedge \Delta'''$ is attacked by some $\Omega \Rightarrow \neg \bigwedge(\Lambda' \cup \Delta'''') \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ where $\Lambda' \subseteq \Lambda$ and $\Delta'''' \subseteq \Delta''$. By Lemma 6, $\Omega = \emptyset$ (note for this that $\Omega \Rightarrow \neg \bigwedge(\Lambda' \cup \Delta'''')$ is attacked by $\Lambda', \Delta'''' \Rightarrow \neg \bigwedge \Omega$). Hence, $\emptyset \Rightarrow \neg \bigwedge(\Lambda' \cup \Delta'''') \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ and by contraposition we obtain $\Lambda' \Rightarrow \neg \bigwedge \Delta''''$ which attacks $\Delta \Rightarrow \phi$. Again, by Lemmas 4 and 6, $\Lambda' \Rightarrow \neg \bigwedge \Delta''''$ is attacked by some $\emptyset \Rightarrow \neg \bigwedge \Lambda'' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ where $\Lambda'' \subseteq \Lambda'$. Thus, $\emptyset \Rightarrow \neg \bigwedge \Lambda''$ attacks $\Lambda \Rightarrow \sigma$. □

Closure: $\mathcal{S} \mid\!\sim \phi$ iff $\{\psi \mid \mathcal{S} \mid\!\sim \psi\} \vdash \phi$.

Proof. The left-to-right direction is trivial. Suppose that $\{\psi \mid \mathcal{S} \mid\!\sim \psi\} \vdash \phi$. Since $\mathfrak{L}$ is finitary, there are $\phi_1, \ldots, \phi_n$ for which $\mathcal{S} \mid\!\sim \phi_1, \ldots, \mathcal{S} \mid\!\sim \phi_n$ and $\phi_1, \ldots, \phi_n \vdash \phi$. Hence, there are $\Delta_1 \Rightarrow \phi_1, \ldots, \Delta_n \Rightarrow \phi_n$ that are finally derivable from $\mathcal{S}$. Note that $s = \Delta_1, \ldots, \Delta_n \Rightarrow \phi \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. In view of Lemma 7, s is finally derivable. Thus, $\mathcal{S} \mid\!\sim \phi$. □

Cautious monotonicity and cautious cut

Next, we consider cautious versions of monotonicity and cut.

Cautious Cut (CC): If $\mathcal{S} \mid\!\sim \phi$ and $\mathcal{S}, \phi \mid\!\sim \psi$ then $\mathcal{S} \mid\!\sim \psi$.

Proof. Suppose that $\mathcal{S} \mid\!\sim \phi$ and $\mathcal{S}, \phi \mid\!\sim \psi$. Then there are sequents $\Delta \Rightarrow \phi$ and $\Gamma \Rightarrow \psi$ there are finally derivable, respectively, from $\mathcal{S}$ and from $\mathcal{S} \cup \{\phi\}$.

Suppose first that $\phi \notin \Gamma$ and hence $\Gamma \subseteq \mathcal{S}$. Suppose also that a sequent $\Lambda \Rightarrow \neg \bigwedge \Gamma' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ attacks $\Gamma \Rightarrow \psi$ where $\Gamma' \subseteq \Gamma$. By Lemmas 4 and 6, $\Lambda \Rightarrow \neg \bigwedge \Gamma'$ is attacked by some tautological sequent $\Rightarrow \neg \bigwedge \Lambda' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S} \cup \{\phi\}))$ where $\Lambda' \subseteq \Lambda$. Note that $\Rightarrow \neg \bigwedge \Lambda' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$. Thus, by Lemma 5, $\Gamma \Rightarrow \psi$ is finally derivable from $\mathcal{S}$, and so $\mathcal{S} \mid\!\sim \psi$.

Suppose now that $\phi \in \Gamma$. By Cut, $\Delta, \Gamma \setminus \{\phi\} \Rightarrow \psi$ is derivable, and so it is in $\mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. Suppose that $\Lambda \Rightarrow \neg \bigwedge(\Delta' \cup \Gamma') \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$ attacks $\Delta, \Gamma \setminus \{\phi\} \Rightarrow \psi$, where $\Delta' \subseteq \Delta$ and $\Gamma' \subseteq \Gamma \setminus \{\phi\}$. By Contraposition, $\Lambda, \Gamma' \Rightarrow \neg \bigwedge \Delta' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, and it attacks $\Delta \Rightarrow \phi$. By Lemmas 4 and 6 some tautological sequent $\Rightarrow \neg \bigwedge(\Lambda' \cup \Gamma'') \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ attacks $\Lambda, \Gamma' \Rightarrow \neg \bigwedge \Delta'$, where $\Lambda' \subseteq \Lambda$ and $\Gamma'' \subseteq \Gamma'$. By Contraposition, $\Lambda' \Rightarrow \neg \bigwedge \Gamma'' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, and it attacks $\Gamma \Rightarrow \psi$ in $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S} \cup \{\phi\})$. By Lemmas 4 and 6, there is a tautological sequent $\Rightarrow \neg \bigwedge \Lambda'' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S} \cup \{\phi\}))$ that attacks $\Lambda' \Rightarrow \neg \bigwedge \Gamma''$ for some $\Lambda'' \subseteq \Lambda'$. Note that $\Rightarrow \neg \bigwedge \Lambda'' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ and it also attacks $\Lambda \Rightarrow \neg \bigwedge(\Delta' \cup \Gamma')$. By Lemma 5, $\Delta, \Gamma \setminus \{\phi\} \Rightarrow \psi$ is finally derivable from $\mathcal{S}$, and so $\mathcal{S} \mid\!\sim \psi$ in this case as well. □

Cautious Monotonicity (CM): If $\mathcal{S} \mid\!\sim \phi$ and $\mathcal{S} \mid\!\sim \psi$ then $\mathcal{S}, \phi \mid\!\sim \psi$.

Proof. Suppose that $\mathcal{S} \mid\!\sim \phi$ and $\mathcal{S} \mid\!\sim \psi$. Hence, there is a sequent $\Gamma \Rightarrow \psi$ that is finally derivable from $\mathcal{S}$ for some $\Gamma \subseteq \mathcal{S}$. Suppose that some sequent $\Lambda \Rightarrow \neg \bigwedge \Gamma' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S} \cup \{\phi\})$ attacks $\Gamma \Rightarrow \psi$. We now distinguish the two cases $\phi \in \Lambda$ and $\phi \notin \Lambda$ and show that in each case there is a $\Rightarrow \neg \bigwedge \Lambda' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S} \cup (\phi))$ for some $\Lambda' \subseteq \Lambda$ that attacks $\Lambda \Rightarrow \neg \bigwedge \Gamma'$. Thus, by Lemma 5, $\Gamma \Rightarrow \psi$ is finally derivable from $\mathcal{S} \cup \{\phi\}$. Hence $\mathcal{S}, \phi \mid\!\sim \psi$.

If $\phi \notin \Lambda$, by Lemmas 4 and 6, $\Lambda \Rightarrow \neg \bigwedge \Gamma'$ is attacked by some tautological sequent $\Rightarrow \neg \bigwedge \Lambda' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$. Note that $\Rightarrow \neg \bigwedge \Lambda' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S} \cup \{\phi\}))$ as well.

Suppose now that $\phi \in \Lambda$. We know that there is a $\Delta \Rightarrow \phi$ that is finally derivable from $\mathcal{S}$. By Cut, $\Lambda \setminus \{\phi\}, \Delta \Rightarrow \neg \bigwedge \Gamma' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$. By Lemmas 4 and 6, there is a tautological sequent $\Rightarrow \neg \bigwedge(\Lambda' \cup \Delta') \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ that attacks $\Lambda \setminus \{\phi\}, \Delta \Rightarrow \neg \bigwedge \Gamma'$ where $\Lambda' \subseteq \Lambda \setminus \{\phi\}$ and $\Delta' \subseteq \Delta$. By Contraposition, $\Lambda' \Rightarrow \neg \bigwedge \Delta' \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, and it attacks $\Delta \Rightarrow \phi$. By Lemmas 4 and 6 again, there is a tautological sequent $\Rightarrow \neg \bigwedge \Lambda'' \in \mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}))$ for some $\Lambda'' \subseteq \Lambda'$ that attacks $\Lambda' \Rightarrow \neg \bigwedge \Delta'$ and thus also attacks $\Lambda \Rightarrow \neg \bigwedge \Gamma'$. Note that $\Rightarrow \neg \bigwedge \Lambda''$ is also in $\mathsf{Root}(\mathcal{AF}_{\mathfrak{S}}(\mathcal{S} \cup \{\phi\}))$. □

Left logical equivalence

Let $\sigma[\phi/\psi]$ be the result of uniformly substituting subformulas ϕ in σ with ψ. Let $\Delta[\phi/\psi] = \{\delta[\phi/\psi] \mid \delta \in \Delta\}$, $(\Gamma \Rightarrow \Delta)[\phi/\psi] = \Gamma[\phi/\psi] \Rightarrow \Delta[\phi/\psi]$ and $(\Gamma \not\Rightarrow \Delta)[\phi/\psi] = \Gamma[\phi/\psi] \not\Rightarrow \Delta[\phi/\psi]$. Finally, given a set Ξ of sequents and eliminated sequents, we denote $\Xi[\phi/\psi] = \{s[\phi/\psi] \mid s \in \Xi\} \cup \{\overline{s}[\phi/\psi] \mid \overline{s} \in \Xi\}$.

Definition 29. A (sequent-based) argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S}) = \langle \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S}), \mathit{Attack} \rangle$ is *invariant under replacement of $\vdash$-equivalents*, if for every ψ and ϕ such that $\phi \vdash \psi$ and $\psi \vdash \phi$, and for every $s_1, s_2 \in \mathsf{Arg}_{\mathfrak{L}}(\mathcal{S})$, if $(s_1, s_2) \in \mathit{Attack}$ then also $(s_1[\phi/\psi], s_2[\phi/\psi]) \in \mathit{Attack}$.

An elimination rule $\mathcal{R}$ is *invariant under replacement of $\vdash$-equivalents*, if for every set $\mathcal{S}$ of $\mathcal{L}$-formulas and for every calculus $\mathfrak{C}$ which is sound and complete for $\mathfrak{L}$, the argumentation framework $\mathcal{AF}_{\mathfrak{S}}(\mathcal{S})$ for $\mathfrak{S} = \langle \mathfrak{L}, \mathfrak{C}, \{\mathcal{R}\} \rangle$ is invariant under replacement of $\vdash$-equivalents.[30]

In the remainder of this section we suppose that $\mathcal{R}$ is invariant under replacement of equivalents.

Definition 30. A dynamic derivation $\mathcal{D}$ can be written as a sequence $\langle \mathcal{D}_1, \ldots, \mathcal{D}_n \rangle$ of sequences $\mathcal{D}_i$ of tuples, where for each odd index i the sequence $\mathcal{D}_i$ consists only of introducing tuples and for each even index i the sequence $\mathcal{D}_i$ consists only of eliminating tuples (see also the paragraph that proceeds Note 3). We write $\mathcal{D}[\phi/\psi]$ for the derivation represented by the sequence $\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_n \rangle$, where $\mathcal{D}'_i$ $(1 \leq i \leq n)$ is defined as follows:

[30] Cf. Definitions 9 and 10.

- if i is an odd index and $\mathcal{D}_i = \langle\langle l_1, s_1, \mathsf{J}_1, t_1\rangle, \ldots, \langle l_m, s_m, \mathsf{J}_m, t_m\rangle\rangle$, then $\mathcal{D}'_i$ is the result of sequentially adding to $\mathcal{D}'_{i-1}$ the derivations of $s_i[\phi/\psi]$ $(i = 1, \ldots, m)$,[31] possibly by using previously derived sequents in $\mathcal{D}'_i$ and $\mathcal{D}'_j$ for $j < i$.
- if i is an odd index and $\mathcal{D}_i = \langle\langle l_1, \overline{s_1}, \mathsf{J}_1, t_1\rangle, \ldots, \langle l_m, \overline{s_m}, \mathsf{J}_m, t_m\rangle\rangle$, then $\mathcal{D}'_i$ is the result of adding to $\mathcal{D}'_{i-1}$ the sequence $\langle\langle l'_1, \overline{s_1}[\phi/\psi], \mathsf{J}'_1, t_1[\phi/\psi]\rangle, \ldots, \langle l'_m, \overline{s_m}[\phi/\psi], \mathsf{J}'_m, t_m[\phi/\psi]\rangle\rangle$, where $l'_1 = 1 + |\bigcup_{1 \le j < i} \mathcal{D}'_j|$ and $l'_k = j'_1 + k - 1$ for each $1 < k \le m$, and for each $1 \le j \le m$, J_j is the adjusted justification in terms of tuple indexes.

Lemma 8. *Suppose that $\phi \vdash \psi$ and $\psi \vdash \phi$, and let $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n\rangle$ be a dynamic proof based on $\mathcal{S} \cup \{\phi\}$. Then $\mathcal{D}[\phi/\psi] = \langle \mathcal{D}'_1, \ldots, \mathcal{D}'_n\rangle$ is a dynamic proof based on $\mathcal{S} \cup \{\psi\}$, such that for each $1 \le i \le n$, $\mathsf{Elim}(\mathcal{D}[\phi/\psi]) = \mathsf{Elim}(D)[\phi/\psi]$ and $\mathsf{Attack}(\mathcal{D}[\phi/\psi]) = \mathsf{Attack}(\mathcal{D})[\phi/\psi]$.*

Proof. We show this inductively for each $i \ge 1$.

- The case $i = 1$ is trivial: $\mathsf{Elim}(\mathcal{D}_1) = \mathsf{Elim}(\mathcal{D}_1)[\phi/\psi] = \mathsf{Elim}(\mathcal{D}'_1) = \emptyset$. Similarly, $\mathsf{Attack}(\mathcal{D}_1) = \mathsf{Attack}(\mathcal{D}_1)[\phi/\psi] = \mathsf{Attack}(\mathcal{D}'_1) = \emptyset$. Moreover, $\mathcal{D}'_1$ is coherent since so is $\mathcal{D}_1$.
- Induction step: The inductive hypothesis is that $\mathsf{Elim}(\langle \mathcal{D}_1, \ldots, \mathcal{D}_i\rangle)[\phi/\psi] = \mathsf{Elim}(\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_i\rangle)$, that $\mathsf{Attack}(\langle \mathcal{D}_1, \ldots, \mathcal{D}_i\rangle)[\phi/\psi] = \mathsf{Attack}(\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_i\rangle$, and that $\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_i\rangle$ is coherent. We consider two cases:
  - If $i+1$ is odd, then by the induction hypothesis, $\mathsf{Elim}(\langle \mathcal{D}_1, \ldots, \mathcal{D}_{i+1}\rangle) = \mathsf{Elim}(\langle \mathcal{D}_1, \ldots, \mathcal{D}_i\rangle = \mathsf{Elim}(\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_i\rangle = \mathsf{Elim}(\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_{i+1}\rangle)$ and $\mathsf{Attack}(\langle \mathcal{D}_1, \ldots, \mathcal{D}_{i+1}\rangle) = \mathsf{Attack}(\langle \mathcal{D}_1, \ldots, \mathcal{D}_i\rangle) = \mathsf{Attack}(\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_i\rangle = \mathsf{Attack}(\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_{i+1}\rangle)$. The coherence of $\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_{i+1}\rangle$ follows from the coherence of $\langle \mathcal{D}_1, \ldots, \mathcal{D}_{i+1}\rangle$.
  - If $i+1$ is even, then the fact that $\mathsf{Elim}(\langle \mathcal{D}_1, \ldots, \mathcal{D}_{i+1}\rangle)[\phi/\psi] = \mathsf{Elim}(\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_{i+1}\rangle)$ and that $\mathsf{Attack}(\langle \mathcal{D}_1, \ldots, \mathcal{D}_{i+1}\rangle)[\phi/\psi] = \mathsf{Attack}(\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_{i+1}\rangle)$ follows from Item 2 in Definition 30 and the inductive hypothesis. The coherence of $\langle \mathcal{D}'_1, \ldots, \mathcal{D}'_{i+1}\rangle$ follows then from the coherence of $\langle \mathcal{D}_1, \ldots, \mathcal{D}_{i+1}\rangle$. □

Lemma 9. *Suppose that $\phi \vdash \psi$ and $\psi \vdash \phi$ and let $\mathcal{D}$ be a dynamic derivation based on $\mathcal{S} \cup \{\phi\}$. If s is finally derived from in $\mathcal{D}$ (based on $\mathcal{S} \cup \{\phi\}$) then $s[\phi/\psi]$ is finally derived in $\mathcal{D}[\phi/\psi]$ (based on $\mathcal{S} \cup \{\psi\}$).*

Proof. Let $\mathcal{D} = \langle \mathcal{D}_1, \ldots, \mathcal{D}_n\rangle$ and let $s \in \overline{\mathcal{D}}$. In the following we suppose that n is even (for the other case the proof can easily be adjusted). Suppose that $s[\phi/\psi]$ is not finally derived from $\mathcal{S} \cup \{\psi\}$ in $\mathcal{D}[\phi/\psi]$. Then there is an extension $\mathcal{D}^\star = \langle \mathcal{D}'_1, \ldots, \mathcal{D}'_n, \mathcal{D}^\star_1, \ldots, \mathcal{D}^\star_m\rangle$ of $\mathcal{D}[\phi/\psi]$, in which $s[\phi/\psi]$ is eliminated. We now show that then s is not finally derived in $\mathcal{D}$ from $\mathcal{S} \cup \{\phi\}$. For this we extend $\mathcal{D}$ in the following way:

- For each odd i for which $\mathcal{D}^\star_i = \langle\langle l_1, s_1, \mathsf{J}_1\rangle, \ldots, \langle l_m, s_m, \mathsf{J}_m\rangle\rangle$, we let $\mathcal{D}^\dagger_i$ be the result of adding to $\mathcal{D}_{i+1}$ derivations of $s_1[\psi/\phi], \ldots s_m[\psi/\phi]$ (without introducing sequents that were previously used in $\overline{\mathcal{D}}$ or in some $\overline{\mathcal{D}^\dagger_j}$ for $j \le i$).
- For each even i for which $\mathcal{D}^\star_i = \langle\langle l_1, \overline{s_1}, \mathsf{J}_1, t_1\rangle, \ldots, \langle l_m, \overline{s_m}, \mathsf{J}_m, t_m\rangle\rangle$, we let

  $$\mathcal{D}^\dagger_i = \langle\langle l'_1, \overline{s_1}[\psi/\phi], \mathsf{J}_1, t_1[\psi/\phi]\rangle, \ldots, \langle l'_m, \overline{s_m}[\psi/\phi], \mathsf{J}'_m, t_m[\psi/\phi]\rangle\rangle,$$

  where J_k (for $1 \le k \le m$) is the adjusted justification J_k with respect to the tuple indexes.

Denote the resulting proof by $\mathcal{D}^\dagger$. In a similar way as in Lemma 8 it can be shown that $\mathsf{Elim}(\mathcal{D}^\dagger) = \mathsf{Elim}(\mathcal{D}^\star)[\psi/\phi]$ and $\mathsf{Attack}(\mathcal{D}^\dagger) = \mathsf{Attack}(\mathcal{D}^\star)[\psi/\phi]$. From this it immediately follows that $\mathcal{D}^\dagger$ is coherent and that $s \in \mathsf{Elim}(\mathcal{D}^\dagger)$. □

Left Logical Equivalence (LLE): If $\phi \vdash \psi$ and $\psi \vdash \phi$ then $\mathcal{S}, \phi \mid\!\sim \sigma$ iff $\mathcal{S}, \psi \mid\!\sim \sigma$.

Proof. Suppose that $\mathcal{S}, \phi \mid\!\sim \sigma$. Then there is a dynamic derivation $\mathcal{D}$ in which some sequent $s = \Delta \Rightarrow \sigma$ is finally derived from $\mathcal{S} \cup \{\phi\}$. By Lemma 9, $s[\phi/\psi]$ is finally derived in $\mathcal{D}[\phi/\psi]$ from $\mathcal{S} \cup \{\psi\}$. Thus $\mathcal{S}, \psi \mid\!\sim \sigma$. The other direction is analogous. □

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