Paraconsistent Reasoning and Preferential Entailments by Signed Quantified Boolean Formulae

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We introduce a uniform approach of representing a variety of paraconsistent nonmonotonic formalisms by quantified Boolean formulae (QBFs) in the context of multiple-valued logics. We show that this framework provides a useful platform for capturing, in a simple and natural way, a wide range of methods for preferential reasoning. The outcome is a subtle approach to represent the underlying formalisms, which induces a straightforward way to compute the corresponding entailments: By incorporating off-the-shelf QBF solvers it is possible to simulate within our framework various kinds of preferential formalisms, among which are Priest's logic LPm of reasoning with minimal inconsistency, Batens' adaptive logic ACLuNs2, Besnard and Schaub's inference relation $\models_n$, a variety of formula-preferential systems, some bilattice-based preferential relations (e.g., $\models_{\mathcal{I}_1}$ and $\models_{\mathcal{I}_2}$), and consequence relations for reasoning with graded uncertainty, such as the four-valued logic $\models_c^4$.

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1. INTRODUCTION

Preferential reasoning was introduced by McCarthy [1980] and later by Shoham [1988, 1987] as a generalization of the notion of circumscription. This is a common method behind many general patterns of nonmonotonic

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reasoning [Kraus et al. 1990; Lehmann and Magidor 1992; Makinson 1994], and is often used as a technique for defining consequence relations that are *paraconsistent* [da Costa 1974], that is, formalisms in which inconsistent sets of premises do not entail any well-formed formula whatsoever (see, e.g., Arieli [2003], Arieli and Avron [1998], Avron and Lev [2001], Batens [2000], Besnard and Schaub [1996], Carnielli and Marcos [2002], Kifer and Lozinskii [1992], Konieczny and Marquis [2002], Priest [1991], and Schlechta [2000]). The essential idea behind preferential reasoning is that only a "preferred" subset of models of a given theory should be taken into consideration for making inferences from that theory. The relevant models are determined by pre-defined conditions, the satisfaction of which yields the exact kind of preference one wants to work with.

In this article we introduce a uniform setting for representing a variety of preferential paraconsistent consequence relations. This setting is nonclassical in nature, since in the context of classical logic, preferential semantics cannot help to overcome the problem of trivial reasoning with contradictory theories (indeed, if a certain theory has no two-valued models, then it has no preferred models as well). A useful way of reasoning with contradictory classical theories is therefore by embedding them in multiple-valued logics in general, and Belnap's four-valued logic [1977a, 1977b] in particular. The latter is particularly useful for reasoning with uncertainty (see, e.g., Arieli and Avron [1998]) and serves as the underlying multiple-valued semantics in our case as well.

At the computational level, however, implementing paraconsistent reasoning based on four-valued semantics poses important challenges. An effective implementation of theorem provers for one of the existing proof systems for Belnap's logic requires a major effort. In order to handle this problem and provide an efficient way of simulating paraconsistent preferential entailments, we consider the following two-phased approach:

(1) *Encoding of the underlying multivalued logic in terms of classical logic.* For this, we use *signed theories*, apparently first introduced by Besnard and Schaub [1998, 1996] as syntax-independent paraconsistent reasoning systems. These theories are obtained by polynomial-time transformations on the original theories, which implies that preferential four-valued semantics can be effectively implemented by standard theorem proving in two-valued logic (see also Arieli and Denecker [2003]).
(2) *Representing preferences among the models of the theory by quantified Boolean formulae* (QBFs).[1] The use of quantified propositional logic for knowledge representation and reasoning was first proposed by Egly et al. [2000] and then by Besnard et al. [2004, 2003, 2002], who considered the encoding of different forms of nonmonotonic and paraconsistent reasoning by means of QBFs. One important rationale of the QBF approach is that existing solvers can be readily used as back-end tools for implementing the reasoning task at hand.

[1]Thus, preferences are represented by propositional formulae that are extended with quantifiers $\forall, \exists$ over propositional variables; see Section 5.2.

The outcome of our approach is a general represention platform that yields an easy and natural way to handle computational aspects of the underlying consequence relations; by incorporating off-the-shelf computational models for processing QBFs, such as QuBE [Giunchiglia et al. 2001], SEMPROP [Letz 2002], and DECIDE [Rintanen 1999],[2] it is possible to simulate a variety of nonmonotonic and paraconsistent formalisms such as Priest's LPm [1991, 1989], Besnard and Schaub's inference relation $\models_n$ [Besnard and Schaub 1997], various kinds of bilattice-based pointwise-preferential relations [Arieli and Avron 1998, 1996] and formula-preferential relations [Avron and Lev 2001], consequence relations for reasoning with graded uncertainty (such as the four-valued logics $\models_c^4$; see Arieli [2003]), and some adaptive logics (e.g., Batens' ACLuNs2 [2000, 1998, 1989]). The main contribution of this article is, therefore, that it provides a simple, yet general way of representing and reasoning with a variety of many-valued paraconsistent logics (including those simulated in Arieli and Denecker [2003] and Besnard et al. [2003]).

The rest of this article is organized as follows: In the next two sections we set-up our framework. These sections describe, in particular, how to reason with signed formulae in the context of four-valued semantics. Then we show how this framework may be used for simulating paraconsistent nonmonotonic reasoning in the context of two-valued semantics: Section 4 shows how signed formulae may be used in order to simulate a variety of basic (monotonic) consequence relations, and Section 5 shows how preferential derivatives of these consequence relations may be simulated within our framework by incorporating QBFs. In Section 6 we consider some possible extensions of our setting and show how the basic definitions can be generalized accordingly. Finally, in Section 7 we consider some related works and in Section 8 we conclude.[3]

2. FOUR-VALUED SEMANTICS

The formalism that we consider here is based on four-valued semantics and a corresponding four-valued algebraic structure (denoted by $\mathcal{FOUR}$), introduced by Belnap [1977a, 1977b]. This structure is composed of four elements *FOUR* $= \{t, f, \bot, \top\}$, arranged in two lattice structures: one is the standard logical partial order $\leq_t$ which intuitively reflects differences in the "measure of truth" that every value represents. According to this order, f is the minimal element, t maximal, and the other two elements $\bot$ (intuitively representing partial information) and $\top$ (intuitively representing contradictory information) are intermediate values that are incomparable. $(\{t, f, \top, \bot\}, \leq_t)$ is a distributive lattice with an order reversing involution $\neg$ for which $\neg\top = \top$ and $\neg\bot = \bot$. We shall denote the meet and join of this lattice by $\wedge$ and $\vee$, respectively.

The other partial order $\leq_k$ is understood (again, intuitively) as reflecting differences in the amount of *knowledge* or *information* that each truth value exhibits. Again, $(\{t, f, \top, \bot\}, \leq_k)$ is a lattice in which $\bot$ is the minimal element, $\top$ the maximal element, and t, f are incomparable.

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[2]For a list of QBF solvers, see: `http://www.mrg.dist.unige.it/~qube/qbflib/solvers.html`. A comprehensive evaluation of existing QBF solvers appears in Le Berre et al. [2004].

[3]This article is a revised and extended version of Arieli [2004].

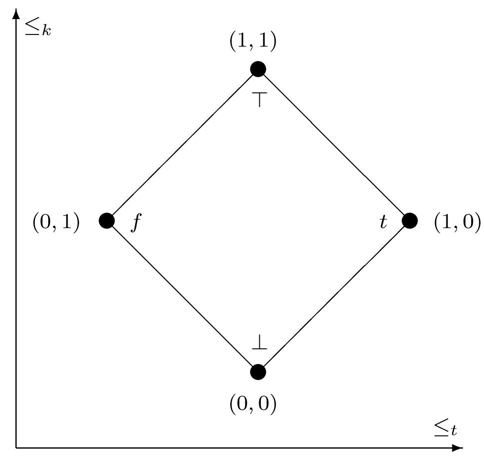

Fig. 1. $\mathcal{FOUR}$.

The elements of $\mathcal{FOUR}$ can be represented by pairs of two-valued components of the lattice $(\{0, 1\}, 0 < 1)$ as follows: $t = (1, 0)$, $f = (0, 1)$, $\top = (1, 1)$, $\bot = (0, 0)$. One way to intuitively understand this representation is that a truth value (x, y) of p corresponds to the amount x of belief in p and the amount y of disbelief in p. The following lemma expresses the partial orders and basic operators of $\mathcal{FOUR}$ in terms of this representation by pairs (see also Figure 1).

LEMMA 2.1 [GINSBERG 1988]. *Let $x, y, x_i, y_i \in \{0, 1\}$ $(i = 1, 2)$. Then:*

(1) $(x_1, y_1) \le_t (x_2, y_2)$ *iff* $x_1 \le x_2$ *and* $y_1 \ge y_2$,
$(x_1, y_1) \le_k (x_2, y_2)$ *iff* $x_1 \le x_2$ *and* $y_1 \le y_2$.

(2) $\neg(x, y) = (y, x)$,
$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2,\ y_1 \wedge y_2)$,
$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2,\ y_1 \vee y_2)$.

The next step in using $\mathcal{FOUR}$ for reasoning is to choose its set of *designated* elements. The obvious choice is $\mathcal{D} = \{t, \top\}$, since both values intuitively represent formulae known to be true. The set $\mathcal{D}$ has the property that $a \wedge b \in \mathcal{D}$ iff both a and b are in $\mathcal{D}$, while $a \vee b \in \mathcal{D}$ iff at least one of a or b is in $\mathcal{D}$, and so $\mathcal{D}$ is a prime filter in $\mathcal{FOUR}$. Note also that $\mathcal{D} = \{(1, x) \mid x \in \{0, 1\}\}$.

From this point on, the various semantic and syntactic notions are defined on $\mathcal{FOUR}$ as natural generalizations of similar classical notions:

—The underlying propositional language consists of an alphabet Σ of propositional variables, propositional constants t and f, and logical symbols $\neg, \wedge, \vee$. We denote elements in Σ by p, q, r, formulae by ψ, ϕ, and sets of formulae by $\mathcal{T}, \mathcal{T}_i$. The set of all atoms occurring in ψ is denoted by $\mathcal{A}(\psi)$. Similarly, $\mathcal{A}(\mathcal{T})$ denotes the union of all the sets $\mathcal{A}(\psi)$ such that $\psi \in \mathcal{T}$.

—A *valuation* ν is a function that assigns a truth value from *FOUR* to each atomic formula, and $\nu(\mathsf{t}) = t$, $\nu(\mathsf{f}) = f$. Any valuation is extended to complex formulae in the obvious way: $\nu(\neg\psi) = \neg\nu(\psi)$, $\nu(\psi \wedge \phi) = \nu(\psi) \wedge \nu(\phi)$, and $\nu(\psi \vee \phi) = \nu(\psi) \vee \nu(\phi)$. We will sometimes write $\psi : b \in \nu$ instead of $\nu(\psi) = b$.

—A valuation ν *satisfies* ψ iff $\nu(\psi) \in \mathcal{D}$. A valuation that satisfies every formula in $\mathcal{T}$ is a *model* of $\mathcal{T}$. The set of models of $\mathcal{T}$ is denoted by *mod*($\mathcal{T}$).

Note that in the four-valued context there are no tautologies in the propositional language defined previously. Thus, for example, excluded middle is not valid, as $\nu(p \vee \neg p) = \bot$ when $\nu(p) = \bot$. This implies that the definition of the material implication $\psi \to \phi$ as $\neg\psi \vee \phi$ is not adequate for representing entailments. Instead, we use here the following connective (see also Arieli and Avron [1998, 1996] and Besnard et al. [2003] and Footnote 5 to follow for further justifications and other applications of this definition).

$$a \supset b = \begin{cases} t & \text{if } a \notin \mathcal{D}, \\ b & \text{otherwise} \end{cases}$$

Note that $a \supset b = a \to b$ when $a, b \in \{t, f\}$, and so the preceding implication is a generalization of the material implication. Like the other binary connectives, we define $\nu(\psi \supset \phi) = \nu(\psi) \supset \nu(\phi)$. The propositional language extended with $\supset$ is denoted by L. As shown in Arieli and Avron [1998], L is functionally complete for $\mathcal{FOUR}$.

LEMMA 2.2. *For* $x_1, x_2, y_1, y_2 \in \{0, 1\}$, $(x_1, y_1) \supset (x_2, y_2) = (\neg x_1 \vee x_2, x_1 \wedge y_2)$.

PROOF. If $x_1 = 1$, then $(x_1, y_1) \in \mathcal{D}$, and so $(x_1, y_1) \supset (x_2, y_2) = (x_2, y_2)$. If $x_1 = 0$, then $(x_1, y_1) \notin \mathcal{D}$, and so $(x_1, y_1) \supset (x_2, y_2) = (1, 0)$. Both of cases are represented by the equation that is specified in the lemma. □

3. SIGNED FORMULAE

It is obvious that the representation of truth values in terms of pairs of two-valued components, considered in the previous section, implies a similar way of representing four-valued valuations; a four-valued valuation ν may be represented in terms of a pair of two-valued components (ν_1, ν_2) by $\nu(p) = (\nu_1(p), \nu_2(p))$. So if, for instance, $\nu(p) = t$, then $\nu_1(p) = 1$ and $\nu_2(p) = 0$. Note also that $\nu = (\nu_1, \nu_2)$ is a four-valued model of $\mathcal{T}$ iff $\nu_1(\psi) = 1$ for every $\psi \in \mathcal{T}$.

Definition 3.1. A *signed alphabet* $\Sigma^{\pm}$ is a set that consists of two symbols p^+, p^- for each atom p of Σ. The language over $\Sigma^{\pm}$ is denoted by $L^{\pm}$. Now:

—The two-valued valuation ν^2 on $\Sigma^{\pm}$ that is *induced by* (or *associated with*) a four-valued valuation $\nu^4 = (\nu_1, \nu_2)$ on Σ, interprets p^+ as $\nu_1(p)$ and p^- as $\nu_2(p)$.

—The four-valued valuation ν^4 on Σ that is *induced by* a two-valued valuation ν^2 on $\Sigma^{\pm}$ is defined, for every atom $p \in \Sigma$, by $\nu^4(p) = (\nu^2(p^+), \nu^2(p^-))$.

In what follows we denote by ν^2 a valuation into $\{0, 1\}$, and by ν^4 a valuation into $\{t, f, \top, \bot\}$.

Definition 3.2. For $p \in \Sigma$ and $\psi, \phi \in L$, define the following formulae in $L^{\pm}$:

$$\begin{array}{ll} \tau_1(p) = p^+ & \tau_2(p) = p^- \\ \tau_1(\neg\psi) = \tau_2(\psi) & \tau_2(\neg\psi) = \tau_1(\psi) \\ \tau_1(\psi \wedge \phi) = \tau_1(\psi) \wedge \tau_1(\phi) & \tau_2(\psi \wedge \phi) = \tau_2(\psi) \vee \tau_2(\phi) \end{array}$$

$$\tau_1(\psi \vee \phi) = \tau_1(\psi) \vee \tau_1(\phi) \qquad \tau_2(\psi \vee \phi) = \tau_2(\psi) \wedge \tau_2(\phi)$$
$$\tau_1(\psi \supset \phi) = \neg\tau_1(\psi) \vee \tau_1(\phi) \qquad \tau_2(\psi \supset \phi) = \tau_1(\psi) \wedge \tau_2(\phi)$$

Given a set $\mathcal{T}$ of formulae in L, we denote $\tau_i(\mathcal{T}) = \{\tau_i(\psi) \mid \psi \in \mathcal{T}\}$ for $i = 1, 2$.

Example 3.3. Consider the formula $\psi = \neg(p \vee \neg q) \vee \neg q$. Then,

$$\begin{array}{rclclcl}
\tau_1(\psi) & = & \tau_1(\neg(p \vee \neg q)) \vee \tau_1(\neg q) & = & \tau_2(p \vee \neg q) \vee \tau_2(q) & = \\
& = & (\tau_2(p) \wedge \tau_2(\neg q)) \vee \tau_2(q) & = & (\tau_2(p) \wedge \tau_1(q)) \vee \tau_2(q) & = \\
& = & (p^- \wedge q^+) \vee q^-. & & &
\end{array}$$

We call $\tau_i(\psi)$ $(i = 1, 2)$ the *signed formulae* that are obtained from ψ. Intuitively, $\tau_1(\psi)$ indicates whether ψ should be "at least true" (i.e., it is assigned t or $\top$), and $\tau_2(\psi)$ indicates whether ψ is "at least false". In other words, if $\tau_1(\psi)$ (respectively, $\tau_2(\psi)$) is true in the two-valued context, then ψ (respectively, $\neg\psi$) holds in the four-valued context (compare with Corollaries 3.5 and 3.6).

PROPOSITION 3.4. *Let $\psi \in L$.*

(1) *If ν^4 is induced by ν^2, then $\nu^4(\psi) = (\nu^2(\tau_1(\psi)),\ \nu^2(\tau_2(\psi)))$.*
(2) *If ν^2 is induced by ν^4, then $\nu^4(\psi) = (\nu^2(\tau_1(\psi)),\ \nu^2(\tau_2(\psi)))$.*

PROOF. The proof of both parts is by induction on the structure of ψ. Consider, first, part (1). For $\psi = p$, we have $\nu^4(p) = (\nu^2(p^+), \nu^2(p^-)) = (\nu^2(\tau_1(p)), \nu^2(\tau_2(p)))$. Now, by Lemma 2.1,

- $\psi = \neg\phi$: $\nu^4(\psi) = \nu^4(\neg\phi) = \neg\nu^4(\phi) = \neg\big(\nu^2(\tau_1(\phi)), \nu^2(\tau_2(\phi))\big) =$
$\big(\nu^2(\tau_2(\phi)), \nu^2(\tau_1(\phi))\big) = (\nu^2(\tau_1(\neg\phi)), \nu^2(\tau_2(\neg\phi))\big) =$
$\big(\nu^2(\tau_1(\psi)), \nu^2(\tau_2(\psi))\big)$.
- $\psi = \phi_1 \vee \phi_2$: $\nu^4(\psi) = \nu^4(\phi_1 \vee \phi_2) = \nu^4(\phi_1) \vee \nu^4(\phi_2) =$
$\big(\nu^2(\tau_1(\phi_1)), \nu^2(\tau_2(\phi_1))\big) \vee \big(\nu^2(\tau_1(\phi_2)), \nu^2(\tau_2(\phi_2))\big) =$
$\big(\nu^2(\tau_1(\phi_1)) \vee \nu^2(\tau_1(\phi_2))\ ,\ \nu^2(\tau_2(\phi_1)) \wedge \nu^2(\tau_2(\phi_2))\big) =$
$\big(\nu^2(\tau_1(\phi_1) \vee \tau_1(\phi_2))\ ,\ \nu^2(\tau_2(\phi_1) \wedge \tau_2(\phi_2))\big) =$
$\big(\nu^2(\tau_1(\phi_1 \vee \phi_2))\ ,\ \nu^2(\tau_2(\phi_1 \vee \phi_2))\big) =$
$\big(\nu^2(\tau_1(\psi)), \nu^2(\tau_2(\psi))\big)$.
- $\psi = \phi_1 \wedge \phi_2$: similar (dual) to the case where $\psi = \phi_1 \vee \phi_2$.
- $\psi = \phi_1 \supset \phi_2$: $\nu^4(\psi) = \nu^4(\phi_1 \supset \phi_2) = \nu^4(\phi_1) \supset \nu^4(\phi_2) =$
$\big(\nu^2(\tau_1(\phi_1)), \nu^2(\tau_2(\phi_1))\big) \supset \big(\nu^2(\tau_1(\phi_2)), \nu^2(\tau_2(\phi_2))\big) =$
by Lemma 2.2,
$\big(\neg\nu^2(\tau_1(\phi_1)) \vee \nu^2(\tau_1(\phi_2))\ ,\ \nu^2(\tau_1(\phi_1)) \wedge \nu^2(\tau_2(\phi_2))\big) =$
$\big(\nu^2(\neg\tau_1(\phi_1) \vee \tau_1(\phi_2))\ ,\ \nu^2(\tau_1(\phi_1) \wedge \tau_2(\phi_2))\big) =$
$\big(\nu^2(\tau_1(\phi_1 \supset \phi_2))\ ,\ \nu^2(\tau_2(\phi_1 \supset \phi_2))\big) =$
$\big(\nu^2(\tau_1(\psi))\ ,\ \nu^2(\tau_2(\psi))\big)$.

The proof of part (2) is similar; we show here only the base step. Indeed, for atom p, $\nu^4(p) = (\nu_1(p), \nu_2(p))$. Since ν^2 is induced by ν^4, $\nu^2(p^+) = \nu_1(p)$ and $\nu^2(p^-) = \nu_2(p)$, thus $\nu^4(p) = (\nu^2(p^+), \nu^2(p^-))$. By the definition of τ_1,τ_2, then, $\nu^4(p) = (\nu^2(\tau_1(p)), \nu^2(\tau_2(p)))$. □

COROLLARY 3.5. *If ν^2 is induced by ν^4 or ν^4 is induced by ν^2, then for every $\psi \in L$, $\nu^2(\tau_1(\psi)) = 1$ iff $\nu^4(\psi) \geq_k t$, and $\nu^2(\tau_2(\psi)) = 1$ iff $\nu^4(\psi) \geq_k f$.*

PROOF. $\nu^4(\psi) \geq_k t$ iff $\nu^4(\psi) = (1, x)$ for some $x \in \{0, 1\}$, iff (Proposition 3.4) $\nu^2(\tau_1(\psi)) = 1$. Similarly, $\nu^4(\psi) \geq_k f$ iff $\nu^4(\psi) = (y, 1)$ for some $y \in \{0, 1\}$, iff $\nu^2(\tau_2(\psi)) = 1$. □

The last corollary may be reformulated as follows.

COROLLARY 3.6. *If ν^2 is induced by ν^4 or ν^4 is induced by ν^2, then for every $\psi \in L$, ν^4 satisfies ψ iff ν^2 satisfies $\tau_1(\psi)$, and ν^4 satisfies $\neg\psi$ iff ν^2 satisfies $\tau_2(\psi)$.*

The fact that a formula ψ should have a truth value x may thus be encoded by a signed formula $\mathsf{val}(\psi, x)$ as follows.

Definition 3.7. For $\psi \in L$, define the following signed formulae in $L^{\pm}$:

$$\mathsf{val}(\psi, t) = \tau_1(\psi) \wedge \neg\tau_2(\psi) \qquad \mathsf{val}(\psi, f) = \neg\tau_1(\psi) \wedge \tau_2(\psi)$$
$$\mathsf{val}(\psi, \top) = \tau_1(\psi) \wedge \tau_2(\psi) \qquad \mathsf{val}(\psi, \bot) = \neg\tau_1(\psi) \wedge \neg\tau_2(\psi)$$

Given a set $\mathcal{T}$ of formulae in L, we denote $\mathcal{D}(\mathcal{T}) = \{\mathsf{val}(\psi, t) \vee \mathsf{val}(\psi, \top) \mid \psi \in \mathcal{T}\}$.

PROPOSITION 3.8. *In case that either ν^2 is induced by ν^4 or ν^4 is induced by ν^2, we have that for every $\psi \in L$, $\nu^4(\psi) = x$ iff $\nu^2(\mathsf{val}(\psi, x)) = 1$.*

PROOF. This is another immediate consequence of Proposition 3.4. Consider, for example, $x = \top$. Then $\nu^2(\mathsf{val}(\psi, \top)) = 1$ iff $\nu^2(\tau_1(\psi) \wedge \tau_2(\psi)) = 1$, iff $\nu^2(\tau_1(\psi)) = 1$ and $\nu^2(\tau_2(\psi)) = 1$, iff (Proposition 3.4) $\nu^4(\psi) = \top$. The proof of the other cases is similar. □

The last results may be specified in terms of models of a given theory as follows.

PROPOSITION 3.9. *Let $\mathcal{T}$ be a set of formulae in L.*

(1) *The (two-valued) models of $\tau_1(\mathcal{T})$ are the same as those of $\mathcal{D}(\mathcal{T})$.*
(2) *There is a one-to-one correspondence between the four-valued models of $\mathcal{T}$ and the two-valued models of $\tau_1(\mathcal{T})$:*
    (a) *ν^4 is a model of $\mathcal{T}$ if the two-valued valuation that is associated with ν^4 is a model of $\tau_1(\mathcal{T})$; and*
    (b) *ν^2 is a model of $\tau_1(\mathcal{T})$ if the four-valued valuation that is associated with ν^2 is a model of $\mathcal{T}$.*

    This is also the one-to-one correspondence between the four-valued models of $\mathcal{T}$ and the two-valued models of $\mathcal{D}(\mathcal{T})$.

PROOF. A valuation ν^4 is a model of $\mathcal{T}$ iff for every $\psi \in \mathcal{T}$ $\nu^4(\psi) \in \{t, \top\}$, iff for every $\psi \in \mathcal{T}$ $\nu^4(\psi) \geq_k t$, iff (Corollary 3.5) for every $\psi \in \mathcal{T}$ $\nu^2(\tau_1(\psi)) = 1$, iff ν^2 is a model of $\tau_1(\mathcal{T})$. Similarly, by Proposition 3.8, ν^4 is a model of $\mathcal{T}$ iff for every $\psi \in \mathcal{T}$ $\nu^2(\mathsf{val}(\psi, t)) = 1$ or $\nu^2(\mathsf{val}(\psi, \top)) = 1$, iff ν^2 is a model of $\mathcal{D}(\mathcal{T})$. These arguments easily imply both parts of the proposition. □

4. USING SIGNED FORMULAE TO SIMULATE BASIC ENTAILMENTS

In the next two sections we show how the signed theories introduced previously can be used to simulate paraconsistent reasoning by classical entailments. In this section we show how basic four-valued and three-valued consequence relations can be defined in terms of a classical two-valued entailment of signed theories, and in Section 5 we show that three-valued and four-valued preferential relations can be defined in terms of a classical entailment for the signed theories, augmented with quantified Boolean axioms.

In what follows we denote by $\models^2$ the two-valued classical consequence relation and by $\models^4$ the four-valued counterpart, that is, $\mathcal{T} \models^4 \psi$ if every four-valued model of $\mathcal{T}$ is a four-valued model of ψ. By Proposition 3.9 we immediately have the following theorem.

THEOREM 4.1. *$\mathcal{T} \models^4 \psi$ iff $\tau_1(\mathcal{T}) \models^2 \tau_1(\psi)$ iff $\mathcal{D}(\mathcal{T}) \models^2 \mathcal{D}(\psi)$.*

The preceding theorem implies, in particular, that one can simulate four-valued entailment by two-valued entailment. It follows, therefore, that four-valued reasoning may be implemented by two-valued theorem provers or SAT solvers. Moreover, as $\tau_1(\mathcal{T})$ is obtained from $\mathcal{T}$ in polynomial time, Theorem 4.1 shows that four-valued entailment in the context of Belnap's logic is *polynomially reducible* to classical entailment.[4]

Example 4.2. Let $\mathcal{T}_1 = \{p,\ \neg p,\ q,\ \neg p \vee r,\ \neg q \vee s\}$. Then $\tau_1(\mathcal{T}_1) = \{p^+,\ p^-,\ q^+,\ p^- \vee r^+,\ q^- \vee s^+\}$. In this case, for example, $\tau_1(\mathcal{T}_1) \not\models^2 r^+$ and $\tau_1(\mathcal{T}_1) \not\models^2 s^+$, so indeed $\mathcal{T}_1 \not\models^4 r$ and $\mathcal{T}_1 \not\models^4 s$ (consider, e.g., a valuation that assigns $\top$ to p and q, and f to r and s). Note also that this example shows that $\models^4$ is a paraconsistent consequence relation, since (unlike classical logic), it is not the case that every formula is a $\models^4$-consequence of a classically inconsistent theory.

Consider now $\mathcal{T}_2 = \{p,\ \neg p,\ q,\ p \supset r,\ q \supset s\}$. Here, $\tau_1(\mathcal{T}_2) = \{p^+,\ p^-,\ q^+,\ \neg p^+ \vee r^+,\ \neg q^+ \vee s^+\}$, and this time $\tau_1(\mathcal{T}_2) \models^2 r^+$ and $\tau_1(\mathcal{T}_2) \models^2 s^+$. This corresponds to the fact that $\mathcal{T}_2 \models^4 r$ and $\mathcal{T}_2 \models^4 s$.[5]

It is interesting to note that if the connective $\supset$ does not appear in $\mathcal{T}$, then $\tau_1(\mathcal{T})$ (as well as $\mathcal{D}(\mathcal{T})$) is a *positive theory* (i.e., a theory without negations). In particular then, Theorem 4.1 also implies the following well-known result.

COROLLARY 4.3. *In positive propositional logic (i.e., with respect to the $\{\vee, \wedge\}$-fragment of the language), $\mathcal{T} \models^4 \psi$ iff $\mathcal{T} \models^2 \psi$.*

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[4]See also Arieli and Denecker [2003] for a similar result (for the language without "$\supset$"), obtained by a different transformation.

[5]Note that $\mathcal{T}_2$ is obtained from $\mathcal{T}_1$ by using $\supset$ instead of the material implication $\rightarrow$, so this example demonstrates the fact that in the four-valued setting, Modus Ponens and the Deduction theorem are satisfied by $\supset$ but not by $\rightarrow$. This is another vindication to the claim that in the four-valued setting the former connective is more suitable for representing entailment than the latter.

PROOF. The proof follows from Theorem 4.1 and the fact that in the positive propositional language, $\tau_1(\mathcal{T})$ is the same as $\mathcal{T}$ (using $\Sigma^{\pm}$ instead of Σ). □

Theorem 4.1 also shows that some basic three-valued logics can be simulated in our framework.

Definition 4.4. For a set $\mathcal{T}$ of formulae in L, denote

$$\mathsf{EM}(\mathcal{T}) = \{p \vee \neg p \mid p \in \mathcal{A}(\mathcal{T})\}, \qquad \mathsf{EFQ}(\mathcal{T}) = \{(p \wedge \neg p) \supset \mathsf{f} \mid p \in \mathcal{A}(\mathcal{T})\}.^6$$

COROLLARY 4.5. *Let $\mathcal{T}$ be a set of formulae in L and ψ a formula in L.*

—*Denote by $\models^3_{\mathrm{LP}}$ the entailment relation of Priest's three-valued logic LP* [Priest 1991, 1989]. *Then*

$$\mathcal{T} \models^3_{\mathrm{LP}} \psi \textit{ iff } \tau_1(\mathcal{T} \cup \mathsf{EM}(\mathcal{T})) \models^2 \tau_1(\psi).^7$$

—*Denote by $\models^3_{\mathrm{Kl}}$ the entailment relation of Kleene's three-valued logic* [Kleene 1950]. *Then*

$$\mathcal{T} \models^3_{\mathrm{Kl}} \psi \textit{ iff } \tau_1(\mathcal{T} \cup \mathsf{EFQ}(\mathcal{T})) \models^2 \tau_1(\psi).$$

PROOF. By Theorem 4.1 and the facts that $\mathcal{T} \models^3_{\mathrm{LP}} \psi$ iff $\mathcal{T}, \mathsf{EM}(\mathcal{T}) \models^4 \psi$, and $\mathcal{T} \models^3_{\mathrm{Kl}} \psi$ iff $\mathcal{T}, \mathsf{EFQ}(\mathcal{T}) \models^4 \psi$ (see Arieli and Avron [1998]). □

5. USING SIGNED QBFS TO SIMULATE PREFERENTIAL ENTAILMENTS

5.1 Preferential Reasoning

Consider again the theory $\mathcal{T}_1 = \{p,\ \neg p,\ q,\ \neg p \vee r,\ \neg q \vee s\}$ of Example 4.2. Intuitively, since the information about r is related to inconsistent (thus unreliable) information about p, we have that $\mathcal{T}_1 \not\models^4 r$. However, the fact that $\mathcal{T}_1 \not\models^4 s$ seems more controversial in this case, since the information about q and s is not related to the cause of inconsistency in $\mathcal{T}_1$, and so applying classically valid rules such as the disjunctive syllogism to $\{q, \neg q \vee s\}$ for concluding s from $\mathcal{T}_1$ should be justified here. In terms of Batens [1998, 1989], then, $\models^4$ is not *adaptive*, since it does not presuppose the consistency of all the assertions "unless and until proven otherwise." In other words, although it is possible to distinguish between a consistent fragment and an inconsistent fragment of $\mathcal{T}_1$, it is not the case that assertions that classically follow from the consistent and are not related to the inconsistent fragment are $\models^4$-consequences of $\mathcal{T}_1$. Note further that s is not even a $\models^4$-consequence of the *classically consistent* subtheory $\{q, \neg q \vee s\}$, and so $\models^4$ is strictly weaker than classical logic (see also Arieli and Avron [1998, 1996]). It is well-known that Priest's $\models^3_{\mathrm{LP}}$ (see Corollary 4.5) has the same drawback.

One way to overcome these shortcomings is to refine the underlying consequence relations, and rather than taking into account all models of the premises, to consider only a subset of *preferential models* [McCarthy 1980; Shoham 1988,

[6]EM and EFQ stand for "excluded middle" and "ex falso quodlibet sequitur," respectively. Recall that $\mathcal{A}(\mathcal{T})$ is the set of atoms that appear in the formulae of $\mathcal{T}$.

[7]See Arieli and Denecker [2003] and Besnard et al. [2003, Thm. 2] for other representations of Priest's logic in terms of signed formulae.

Table I. The Four-Valued Models of $\mathcal{T}_1$

No.	p	q	r	s
1	$\top$	t	$\bot$	t
2	$\top$	t	t	t
3	$\top$	t	f	t
4	$\top$	t	$\top$	t
5	$\top$	t	$\bot$	$\top$
6	$\top$	t	t	$\top$
7	$\top$	t	f	$\top$
8	$\top$	t	$\top$	$\top$
9	$\top$	$\top$	$\bot$	$\bot$
10	$\top$	$\top$	$\bot$	t
11	$\top$	$\top$	$\bot$	f
12	$\top$	$\top$	$\bot$	$\top$
13	$\top$	$\top$	t	$\bot$
14	$\top$	$\top$	t	t
15	$\top$	$\top$	t	f
16	$\top$	$\top$	t	$\top$
17	$\top$	$\top$	f	$\bot$
18	$\top$	$\top$	f	t
19	$\top$	$\top$	f	f
20	$\top$	$\top$	f	$\top$
21	$\top$	$\top$	$\top$	$\bot$
22	$\top$	$\top$	$\top$	t
23	$\top$	$\top$	$\top$	f
24	$\top$	$\top$	$\top$	$\top$

1987] as relevant for making inferences. These models are determined according to some preference conditions that can be specified syntactically by a set of (usually second-order) propositions, or by order relations on the space of valuations (see, e.g., Makinson [1994] for a detailed discussion on preferential reasoning). We now introduce a general setting for such order relations.

Definition 5.1. Let ν_1 and ν_2 be two valuations, $\Upsilon \subseteq FOUR$, and Δ be a set of formulae in L. ν_1 is Υ*-preferred* to ν_2 with respect to Δ (denoted $\nu_1 \leq^{\Delta}_{\Upsilon} \nu_2$) if

$$\{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon\} \subseteq \{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon\}.$$

We denote by $\nu_1 <^{\Delta}_{\Upsilon} \nu_2$ that $\nu_1 \leq^{\Delta}_{\Upsilon} \nu_2$ and $\nu_2 \not\leq^{\Delta}_{\Upsilon} \nu_1$.

Definition 5.2. Let $\mathcal{T}, \Delta$ be sets of formulae in L, and $\Upsilon \subseteq FOUR$. A valuation $\nu \in mod(\mathcal{T})$ is a $\leq^{\Delta}_{\Upsilon}$*-minimal model of* $\mathcal{T}$ if there is no $\mu \in mod(\mathcal{T})$ such that $\mu <^{\Delta}_{\Upsilon} \nu$.

Intuitively, Δ represents the "abnormal formulae" (see Batens [1998]) and the purpose is to minimize the Υ-assignments of the elements in Δ. In this respect, Definitions 5.1 and 5.2 may be viewed as a generalization of preferential orders considered elsewhere in the literature: When Υ consists of the designated elements, the order relations of Definition 5.1 are called *formula-preferential orders* [Avron and Lev 2001]. When $\Delta \subseteq \Sigma$, these kinds of orders are called *pointwise-preferential* [Arieli and Avron 1998; Avron and Lev 2001], and their minimal elements are the valuations with minimal sets of atoms[8] that are assigned values in Υ. In the latter case, Υ sometimes consists of "abnormal values" (e.g., $\top$ or $\bot$) that should be assigned to as minimal number of atoms as possible. Note also that in the particular case where $\Delta = \mathcal{T}$ (respectively, where $\Delta = \mathcal{A}(\mathcal{T})$), the purpose is to minimize the Υ-assignments of the (atomic) formulae that appear in (some formulae of) the premises.

Example 5.3. Consider again the set $\mathcal{T}_1 = \{p, \neg p, q, \neg p \vee r, \neg q \vee s\}$ of Example 4.2. The 24 four-valued models of $\mathcal{T}_1$ are listed in Table I.

Anomalous situations such as the ones represented by ν_{21} and ν_{23} (see Table I) are the reason that consequence relations like $\models^4$ are "over-cautious", and causing counterintuitive conclusions, such as $\mathcal{T}_1 \not\models^4 s$, discussed at the beginning of this section. Preferential reasoning avoids these anomalies by taking into account only the most "plausible" models of the premises. In this case, for

[8]Where the minimum is taken with respect to set inclusion.

instance, one could take $\Upsilon = \{\top, \bot\}$ as the set of "abnormal" values (the assignments of which should be minimized), or $\Delta = \{u \wedge \neg u \mid u \in \Sigma\}$ as a set of abnormal formulae. With these choices and in the notations of Table I, the $\leq_{\Upsilon}^{\mathcal{A}(\mathcal{T}_1)}$-minimal models of $\mathcal{T}_1$ are $\nu_2 = \{p:\top, q:t, r:t, s:t\}$ and $\nu_3 = \{p:\top, q:t, r:f, s:t\}$. These are also the $\leq_{\Upsilon}^{\Delta}$-minimal models of $\mathcal{T}_1$, but only ν_2 is a $\leq_{\Upsilon}^{\mathcal{T}_1}$-minimal model of $\mathcal{T}_1$ (note that $\nu_2 <_{\Upsilon}^{\mathcal{T}_1} \nu_3$, since $\nu_3(\neg p \vee r) = \top$ while $\nu_2(\neg p \vee r) = t$).

Definition 5.4. Let $\mathcal{T}, \Delta$ be sets of formulae in L, ψ a formula in L, and $\Upsilon \subseteq$ ***FOUR***. Then $\mathcal{T} \models^4_{(\Upsilon,\Delta)} \psi$ if every $\leq_{\Upsilon}^{\Delta}$-minimal (four-valued) model of $\mathcal{T}$ is a (four-valued) model of ψ.

Example 5.3 (*Continued*). In the notations of Example 5.3,

$$\begin{array}{lll} \mathcal{T}_1 \models^4_{(\Upsilon,\Delta)} s, & \mathcal{T}_1 \models^4_{(\Upsilon,\mathcal{A}(\mathcal{T}_1))} s, & \mathcal{T}_1 \models^4_{(\Upsilon,\mathcal{T}_1)} s, \\ \mathcal{T}_1 \not\models^4_{(\Upsilon,\Delta)} r, & \mathcal{T}_1 \not\models^4_{(\Upsilon,\mathcal{A}(\mathcal{T}_1))} r, & \mathcal{T}_1 \models^4_{(\Upsilon,\mathcal{T}_1)} r. \end{array}$$

It follows that the preferential relations considered here are indeed adaptive, and although $\mathcal{T}_1 \not\models^4 s$, in all of them s is deducible from $\mathcal{T}_1$.

Again, Definition 5.4 includes a variety of preferential consequence relations, many of which correspond to particular cases that were introduced elsewhere. Before considering ways to properly represent these consequence relations and showing how this affects the computational feasibility of the corresponding reasoning process, we list some specific (already investigated) relations among those of Definition 5.4.

Example 5.5. The following consequence relations are representable in terms of Definition 5.4.

(1) Denote by $\models^3_{\text{LPm}}$ the consequence relation of Priest's three-valued logic LPm of minimal inconsistency [Priest 1991, 1989].[9] Then

$$\mathcal{T} \models^3_{\text{LPm}} \psi \text{ iff } \mathcal{T}, \mathsf{EM}(\mathcal{T}) \models^4_{(\{\top\},\Sigma)} \psi$$

or equivalently,

$$\mathcal{T} \models^3_{\text{LPm}} \psi \text{ iff } \mathcal{T}, \mathsf{EM}(\mathcal{T}) \models^4_{(\{\top\},\Delta)} \psi,$$

where $\Delta = \{p \wedge \neg p \mid p \in \mathcal{A}(\mathcal{T})\}$. In fact, if we denote by $\models^3_{(\{\top\},\Sigma)}$ the three-valued counterpart of $\models^4_{(\{\top\},\Sigma)}$,[10] then with the same Δ, we have

$$\mathcal{T} \models^3_{\text{LPm}} \psi \text{ iff } \mathcal{T} \models^3_{(\{\top\},\Sigma)} \psi \text{ iff } \mathcal{T} \models^3_{(\{\top\},\Delta)} \psi$$

The same pointwise consequence relations also simulate Batens' adaptive logic ACLuNs2 [Batens 1998].

(2) Arieli and Avron's pointwise-preferential consequence relation for reasoning with minimal inconsistency $\models^4_{\mathcal{I}_1}$ [Arieli and Avron 1998, 1996] can be represented by the following pointwise consequence relation:

$$\mathcal{T} \models^4_{\mathcal{I}_1} \psi \text{ iff } \mathcal{T} \models^4_{(\{\top\},\Sigma)} \psi$$

---

[9]In Priest [1991, 1989] the language without "$\supset$" is considered, but the results here hold for the extended language as well.

[10]In other words, the same definition, but only with respect to $\{t, f, \top\}$.

Similarly, the consequence relation $\models^4_{\mathcal{I}_2}$ for reasoning with most classical models, introduced in the same papers, can be represented as follows:

$$\mathcal{T} \models^4_{\mathcal{I}_2} \psi \text{ iff } \mathcal{T} \models^4_{(\{\top,\bot\},\Sigma)} \psi$$

(3) Besnard and Schaub's three-valued formula-preferential consequence relation $\models_n$ [Besnard and Schaub 1997] is represented by the following formula-preferential relations:

$$\mathcal{T} \models_n \psi \text{ iff } \mathcal{T}, \mathsf{EM}(\mathcal{T}) \models^4_{(\{\top\},\mathcal{T})} \psi \text{ iff } \mathcal{T} \models^3_{(\{\top\},\mathcal{T})} \psi,$$

where $\models^3_{(\{\top\},\mathcal{T})}$ is the three-valued counterpart (i.e., without $\bot$) of $\models^4_{(\{\top\},\mathcal{T})}$.

(4) Given a set Δ of formulae, denote by $\models^{\mathcal{P}}$ Avron and Lev's Δ-preferential consequence relation that is based on the deterministic four-valued preferential system $\mathcal{P} = (\models^4, \leq^{\Delta}_{\{\top,t\}})$ [Avron and Lev 2001].[11] The intuition here is, again, to consider models of the premises that satisfy a minimal amount of abnormal formulae (in Δ). In our context then, Υ is the set $\mathcal{D} = \{\top, t\}$ of the designated elements in $\mathcal{FOUR}$, and so

$$\mathcal{T} \models^{\mathcal{P}} \psi \text{ iff } \mathcal{T} \models^4_{(\{\top,t\},\Delta)} \psi.$$

(5) Other preferential logics, such as Arieli and Avron's consequence relation $\models^4_k$ for preferring minimal knowledge [Arieli and Avron 1998, 1996] are also representable by the consequence relations of Definition 5.4, using extended languages; see Avron and Lev [2001] for the details.

5.2 QBFs and Signed QBFs

In the following sections we show how the consequence relations that are obtained from Definition 5.4 can be simulated by signed formulae and classical entailment. In order to extend the technique of Section 4 (and the result of Theorem 4.1) to preferential four-valued reasoning, we should express that a given interpretation is minimal with respect to the underlying preference relation, that is, to represent the minimization conditions of Definition 5.2. This is accomplished by introducing (signed) *quantified Boolean formulae* (QBFs) that encode the required axioms.

First, we extend the language L (respectively, $L^{\pm}$) with universal and existential quantifiers $\forall, \exists$ over propositional variables. Denote the extended language by L_{Q} (respectively, $L^{\pm}_{\mathsf{Q}}$). The elements of L_{Q} are called quantified Boolean formulae (QBFs), and the elements of $L^{\pm}_{\mathsf{Q}}$ are called *signed* QBFs. QBFs and signed QBFs are denoted here by the Greek letters Ψ, Φ, and sets of (signed) QBFs are denoted by Γ. Intuitively, the meaning of a QBF of the form $\exists p\, \forall q\, \psi$ is that there exists a truth assignment of p such that for every truth assignment of q, ψ is true. Clearly, every QBF is associated with a logically equivalent propositional formula, thus QBFs can be seen as a conservative extension of classical propositional logic. Next we formalize this intuition.

Consider a QBF Ψ over L_{Q}. An occurrence of an atom p in Ψ is called *free* if it is not in the scope of a quantifier $\mathsf{Q}p$, for $\mathsf{Q} \in \{\forall, \exists\}$. Denote by $\Psi[\phi_1/p_1, \ldots, \phi_n/p_n]$

[11]In Avron and Lev [2001] extensions to nondeterministic matrices are also considered, but we shall not deal with this here.

the uniform substitution of each free occurrence of a variable (atom) p_i in Ψ by a formula ϕ_i, for $i=1,\ldots,n$. Now, the definition of a valuation can be extended to QBFs as follows:

$$\begin{aligned}
&\nu(\neg\psi) = \neg\nu(\psi)\\
&\nu(\psi \circ \phi) = \nu(\psi) \circ \nu(\phi),\ \text{ where } \circ \in \{\wedge, \vee, \supset\}\\
&\nu(\forall p\ \psi) = \nu(\psi[\mathsf{t}/p]) \wedge \nu(\psi[\mathsf{f}/p])\\
&\nu(\exists p\ \psi) = \nu(\psi[\mathsf{t}/p]) \vee \nu(\psi[\mathsf{f}/p])
\end{aligned}$$

As usual, we say that a (two-valued) valuation ν satisfies a QBF Ψ if $\nu(\Psi) = 1$, ν is a *model* of a set Γ of QBFs (denoted $\nu \in mod(\Gamma)$) if ν satisfies every element of Γ, and a QBF Ψ is (classically) entailed *by* Γ (denotated $\Gamma \models^2 \Psi$) if every model of Γ is also a model of Ψ.[12]

5.3 Preferential Reasoning by Signed QBFs

We are now ready to use signed QBFs for representing preferential reasoning. In what follows, $\mathcal{T}$ denotes a *finite* set of formulae in L, and $\mathcal{T}_\wedge$ denotes the conjunction of the elements in $\mathcal{T}$.

Definition 5.6. For $\Upsilon = \{x_1,\ldots,x_n\} \subseteq \mathit{FOUR}$, denote $\Upsilon(\psi) = \mathsf{val}(\psi, x_1) \vee \ldots \vee \mathsf{val}(\psi, x_n)$.

Note 5.7. By Proposition 3.8, if ν^2 is induced by ν^4 or ν^4 is induced by ν^2, then $\nu^4(\psi) \in \Upsilon$ iff $\nu^2(\Upsilon(\psi)) = 1$.

Definition 5.8. $\mathcal{A}^{\pm}(\mathcal{T}) = \{p^+ \mid p \in \mathcal{A}(\mathcal{T})\} \cup \{p^- \mid p \in \mathcal{A}(\mathcal{T})\}$.

PROPOSITION 5.9. *Let $\Delta = \{\psi_1,\ldots,\psi_k\}$ and $\mathcal{T}$ be finite sets of formulae in L, and let $\mathcal{A}^{\pm}(\mathcal{T} \cup \Delta) = \{p_1,\ldots,p_n\}$. Then ν^4 is a $\leq^{\Delta}_{\Upsilon}$-minimal model of $\mathcal{T}$ iff the two-valued valuation ν^2 that is associated with ν^4 is a model of $\tau_1(\mathcal{T})$ and the following signed QBF, denoted* $\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T})$*,*

$$\forall q_1,\ldots,q_n \Bigg(\tau_1(\mathcal{T}_\wedge)[q_1/p_1,\ldots,\ q_n/p_n] \to \left(\bigwedge_{i=1}^{k} (\Upsilon(\psi_i)[q_1/p_1,\ldots,\ q_n/p_n] \to \Upsilon(\psi_i) \right) \to \bigwedge_{i=1}^{k} (\Upsilon(\psi_i) \to \Upsilon(\psi_i)[q_1/p_1,\ldots,\ q_n/p_n])) \Bigg).$$

PROOF. By Proposition 3.9-(2), ν^4 is a model of $\mathcal{T}$ iff ν^2 is a model of $\tau_1(\mathcal{T})$. It remains to show, then, that the fact that ν^2 satisfies $\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T})$ is a necessary and sufficient condition for assuring that ν^4 is $\leq^{\Delta}_{\Upsilon}$-minimal among the models of $\mathcal{T}$. For this, denote by $(\{r_1,\ldots,r_n\}:\mu_1, \{s_1,\ldots,s_m\}:\mu_2)$ a valuation that interprets the symbols in $\{r_1,\ldots,r_n\}$ according to μ_1 and the symbols in $\{s_1,\ldots,s_m\}$ according to μ_2. Now, suppose that μ_1^4 and μ_2^4 are two models of $\mathcal{T}$. By Proposition 3.9-(2) and Note 5.7, $\mu_1^4 \leq^{\Delta}_{\Upsilon} \mu_2^4$ iff $(\{r_1,\ldots,r_n\}:\mu_1^2, \{s_1,\ldots,s_n\}:\mu_2^2)$ satisfies

[12] We refer to Besnard et al. [2004] for a detailed description of quantified Boolean formulae, including some historical remarks and relevant complexity issues.

$\bigwedge_{i=1}^{k}(\Upsilon(\psi_i)[r_1/p_1, \ldots, r_n/p_n] \rightarrow \Upsilon(\psi_i)[s_1/p_1, \ldots, s_n/p_n])$, where μ_1^2 and μ_2^2 are the two-valued interpretations that are induced by (the four-valued interpretations) μ_1^4 and μ_2^4, respectively. It follows, then, that ν^2 satisfies $\mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T})$ iff for every valuation μ^2 that satisfies $\tau_1(\mathcal{T}_\wedge)$ (i.e., $\mu^2 \in mod(\tau_1(\mathcal{T}))$, if $(\{p_1, \ldots, p_n\} : \nu^2, \{q_1, \ldots, q_n\} : \mu^2)$ satisfies $\bigwedge_{i=1}^{k}(\Upsilon(\psi_i)[q_1/p_1, \ldots, q_n/p_n] \rightarrow \Upsilon(\psi_i))$, then the same valuation also satisfies $\bigwedge_{i=1}^{k}(\Upsilon(\psi_i) \rightarrow \Upsilon(\psi_i)[q_1/p_1, \ldots, q_n/p_n])$. Thus ν^2 satisfies $\mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T})$ iff for every model μ^4 of $\mathcal{T}$ for which $\mu^4 \leq_\Upsilon^\Delta \nu^4$, it is also true that $\nu^4 \leq_\Upsilon^\Delta \mu^4$. This means that ν^2 satisfies $\mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T})$ iff there is no model μ^4 of $\mathcal{T}$ such that $\mu^4 <_\Upsilon^\Delta \nu^4$, and so ν^2 satisfies $\mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T})$ iff ν^4 is indeed $\leq_\Upsilon^\Delta$-minimal among the models of $\mathcal{T}$, as required. $\square$

Note 5.10. As the QBF $\mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T})$ of the last proposition expresses minimization, it is clearly related to the notion of circumscription [McCarthy 1980], in which second-order formulae are usually used for expressing minimization. Indeed, an alternative way of regarding QBFs is as a particular class of second-order languages where predicates are restricted to arity 0. We refer to Section 7, and in particular to Corollary 7.4, where we compare our approach to related works that use circumscription as the primary methods for encoding (multivalued and preferential) entailments.

Proposition 5.9 immediately implies the following theorem and corollary, applied to finite sets $\mathcal{T}$, Δ of formulae in L.

THEOREM 5.11. *$\mathcal{T} \models^4_{(\Upsilon,\Delta)} \psi$ iff $\tau_1(\mathcal{T}), \mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T}) \models^2 \tau_1(\psi)$.*

COROLLARY 5.12. *$\mathcal{T} \models^4_{(\Upsilon,\Delta)} \psi$ iff $\tau_1(\mathcal{T}_\wedge) \wedge \mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T}) \rightarrow \tau_1(\psi)$ is valid in the two-valued semantics.*

Example 5.13. Let $\mathcal{T} = \{p_1, \neg p_1, p_2\}$, $\Upsilon = \{\top\}$, and $\Delta = \mathcal{A}(\mathcal{T}) = \{p_1, p_2\}$. In this case, for every $p \in \Sigma$, $\Upsilon(p) = p^+ \wedge p^-$. Thus

$$\begin{aligned}\mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T}) \;=\; & \forall q_1^+ q_1^- q_2^+ q_2^- \Big(q_1^+ \wedge q_1^- \wedge q_2^+ \rightarrow \\ & \Big(\big((q_1^+ \wedge q_1^-) \rightarrow (p_1^+ \wedge p_1^-)\big) \wedge \big((q_2^+ \wedge q_2^-) \rightarrow (p_2^+ \wedge p_2^-)\big) \rightarrow \\ & \big((p_1^+ \wedge p_1^-) \rightarrow (q_1^+ \wedge q_1^-)\big) \wedge \big((p_2^+ \wedge p_2^-) \rightarrow (q_2^+ \wedge q_2^-)\big)\Big)\Big).\end{aligned}$$

Both $\nu_1 = \{p_1^+ : t,\ p_1^- : t,\ p_2^+ : t,\ p_2^- : t\}$ and $\nu_2 = \{p_1^+ : t,\ p_1^- : t,\ p_2^+ : t,\ p_2^- : f\}$ satisfy $\tau_1(\mathcal{T}) = \{p_1^+, p_1^-, p_2^+\}$, but only ν_2 also satisfies $\mathsf{Min}(\leq_\Upsilon^\Delta, \mathcal{T})$. The four-valued valuation that is associated with ν_2 is $\{p_1 : \top, p_2 : t\}$, and this indeed is the only $\leq_\Upsilon^\Delta$-minimal model of $\mathcal{T}$. Thus, for instance, $\mathcal{T} \not\models^4_{(\Upsilon,\Delta)} \neg p_2$.

Example 5.14. By Theorem 5.11, it is now possible to simulate the consequence relations of Example 5.5 by classical entailment. Indeed, if $\mathcal{T}, \Delta$ are finite sets of formulae in L, then

$\mathcal{T} \models^3_{\mathrm{LPm}} \psi$ iff $\tau_1(\mathcal{T} \cup \mathsf{EM}(\mathcal{T}))$, $\mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\{\top\}}, \mathcal{T}) \models^2 \tau_1(\psi)$. Similarly for $\models^4_{\mathcal{I}_1}$.

$\mathcal{T} \models^4_{\mathcal{I}_2} \psi$ iff $\tau_1(\mathcal{T})$, $\mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\{\top,\bot\}}, \mathcal{T}) \models^2 \tau_1(\psi)$.

$\mathcal{T} \models_n \psi$ iff $\tau_1(\mathcal{T} \cup \mathsf{EM}(\mathcal{T}))$, $\mathsf{Min}(\leq^{\mathcal{T}}_{\{\top\}}, \mathcal{T}) \models^2 \tau_1(\psi)$.

$\mathcal{T} \models^{\mathcal{P}} \psi$ where $\mathcal{P} = (\models^4, \leq^\Delta_{\{\top,t\}})$ iff $\tau_1(\mathcal{T})$, $\mathsf{Min}(\leq^\Delta_{\{\top,t\}}, \mathcal{T}) \models^2 \tau_1(\psi)$.

5.4 Complexity

The representation theorems by signed formulae (Theorems 4.1 and 5.11) allow, in particular, to derive complexity results for the corresponding consequence relations. For instance, Theorem 4.1 and Corollary 4.5 show that the entailment problems for $\models^4$, $\models^3_{\mathrm{KL}}$, and $\models^3_{\mathrm{LP}}$ can be reduced (using a polynomial-time transformation) to the problem of entailment in classical logic, which implies that the corresponding decision problems are in coNP (moreover, this fact, together with polynomial-time reductions from SAT, show the well-known result that these problems are actually coNP-complete; see, e.g., Cadoli and Schaerf [1996], Costa-Marquis and Marquis [2002], and Konieczny and Marquis [2002]).

By Theorem 5.11 one can derive complexity results for the preferential versions of the aforementioned consequence relations (see also Costa-Marquis and Marquis [2004, 2002] and Konieczny and Marquis [2002] for related results about the computational complexity of some of these relations).

PROPOSITION 5.15. *The decision problems for $\models^4_{(\Upsilon,\Delta)}$ and $\models^3_{(\Upsilon,\Delta)}$ are in Π^P_2.*

PROOF. By Theorem 5.11, entailment of a theory $\mathcal{T}$ with respect to $\models^4_{(\Upsilon,\Delta)}$ is equivalent to classical entailment checking with respect to $\tau_1(\mathcal{T}) \cup \{\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T})\}$. Thus, this decision problem can be encoded by QBFs in prenex normal form with exactly one quantifier alternation (and requires a polynomial-time transformation). The proposition is obtained now by the following well-known result.

PROPOSITION 5.16 [WRATHALL 1976]. *Given a propositional formula ψ whose atoms are partitioned into $i \geq 1$ sets $\{p^1_1, \dots, p^1_{m_1}\}, \dots, \{p^i_1, \dots, p^i_{m_i}\}$, deciding if*

$$\exists p^1_1, \dots, \exists p^1_{m_1}, \forall p^2_1, \dots, \forall p^2_{m_2}, \dots, \mathsf{Q} p^i_1, \dots, \mathsf{Q} p^i_{m_i} \psi$$

is true (where $\mathsf{Q} = \exists$ if i is odd and $\mathsf{Q} = \forall$ if i is even), is Σ^P_i-complete. Dually, deciding whether

$$\forall p^1_1, \dots, \forall p^1_{m_1}, \exists p^2_1, \dots, \exists p^2_{m_2}, \dots, \mathsf{Q} p^i_1, \dots, \mathsf{Q} p^i_{m_i} \psi$$

is true (where $\mathsf{Q} = \forall$ if i is odd and $\mathsf{Q} = \exists$ if i is even) is Π^P_i-complete.

For the three-valued case, note that $\mathcal{T}$ has to be extended with the set $\mathsf{EM}(\mathcal{T})$ or $\mathsf{EFQ}(\mathcal{T})$ that forces three-valued assignments, but this does not change the complexity of the corresponding decision problems. □

Note that under the usual assumptions of complexity theory, the complexity bound specified in the last corollary is *not* strict for *every* consequence relation of the form $\models^4_{(\Upsilon,\Delta)}$ and $\models^3_{(\Upsilon,\Delta)}$. For instance, when $\Delta = \emptyset$, consequence relations of the form $\models^4_{(\Upsilon,\Delta)}$ are the same as $\models^4$, thus the corresponding decision problems are (as noted before) already in coNP (i.e., in Π^P_1). However, as shown in Costa-Marquis and Marquis [2004, 2002], the decision problems for many other consequence relations of the form $\models^4_{(\Upsilon,\Delta)}$ and $\models^3_{(\Upsilon,\Delta)}$ are hard for Π^P_2. We refer to Costa-Marquis and Marquis [2004] for a detailed proof of the Π^P_2-hardness of $\models^3_{\mathrm{LPm}}$, even in case that all the formulae are in conjunctive normal form. This also implies the hardness of $\models^4_{\mathcal{I}_1}$ and $\models^4_{\mathcal{I}_2}$, because of the following result.

PROPOSITION 5.17 [ARIELI AND AVRON 1998]. *Let $\mathcal{T}$ be a set of formulae and ψ a formula in the language of* $\{\neg, \wedge, \vee, \mathsf{t}, \mathsf{f}\}$. *Then $\mathcal{T} \models^{3}_{\mathrm{LPm}} \psi$ iff $\mathcal{T} \models^{4}_{\mathcal{I}_2} \psi$. If ψ is a formula in CNF, none of its conjuncts a tautology, then we have in addition that $\mathcal{T} \models^{3}_{\mathrm{LPm}} \psi$ iff $\mathcal{T} \models^{4}_{\mathcal{I}_1} \psi$.*

It follows, then, that the evaluation of the resulting QBFs for the consequence relations considered previously resides in the same complexity class as the decision of the original problems!

6. GENERALIZATIONS

6.1 Reasoning with Graded Abnormality

The consequence relation $\models^{4}_{(\Upsilon,\Delta)}$ of Definition 5.4 can be generalized in several ways to capture other consequence relations considered in the literature. In this section we demonstrate one possible extension and show how to simulate, by signed QBFs and classical entailment, preferential reasoning with different levels of uncertainty.

Definition 6.1. A partial order $\prec$ on a set $\mathcal{S}$ is called *modular* if $y \prec x_2$ for every $x_1, x_2, y \in \mathcal{S}$ such that $x_1 \not\prec x_2$, $x_2 \not\prec x_1$, and $y \prec x_1$.

Modular orders will be used here for grading uncertainty. As shown in Lehmann and Magidor [1992], $\prec$ is a modular order on $\mathcal{S}$ iff there is a total order $<$ on a set $\mathcal{S}'$ and a function $g : \mathcal{S} \to \mathcal{S}'$ such that $x_1 \prec x_2$ iff $g(x_1) < g(x_2)$. For a modular order $\prec$ on *FOUR* then, there is a partition $\Upsilon_1 \ldots \Upsilon_m$ of *FOUR* such that $x \prec y$ iff $x \in \Upsilon_i$, $y \in \Upsilon_j$, and $1 \leq i < j \leq m$.

Let $\mathcal{T}$ and Δ be sets of formulae in L. Let $\prec$ be a modular order on *FOUR* and $\nu, \mu \in mod(\mathcal{T})$. Denote $\nu \prec^{\Delta} \mu$, if there is a $\psi \in \Delta$ such that $\nu(\psi) \prec \mu(\psi)$ and for every $\phi \in \Delta$, either $\nu(\phi) \prec \mu(\phi)$, or $\nu(\phi)$ and $\mu(\phi)$ are $\prec$-incomparable.

A valuation $\nu \in mod(\mathcal{T})$ is a $\prec^{\Delta}$-minimal model of $\mathcal{T}$ if there is no $\mu \in mod(\mathcal{T})$ such that $\mu \prec^{\Delta} \nu$. Denote $\mathcal{T} \models^{4}_{(\prec,\Delta)} \psi$ if every $\prec^{\Delta}$-minimal model of $\mathcal{T}$ is a model of ψ.

Example 6.2. The consequence relation $\models^{4}_{c_3}$, introduced in Arieli [2003], is one example of formalisms for reasoning with graded uncertainty. In this case Δ consists of the atomic formulae, and the preferred (i.e., the $\prec^{\Delta}$-minimal) models are determined according to a modular order $\prec_{c_3}$ on *FOUR* that has three "uncertainty levels." t, f are the $\prec_{c_3}$-minimal elements, $\bot$ is the $\prec_{c_3}$-intermediate value, and $\top$ is the $\prec_{c_3}$-maximal (i.e., the most abnormal) one. Clearly then, $\models^{4}_{c_3}$ is a particular case of $\models^{4}_{(\prec,\Delta)}$: for every set $\mathcal{T}$ of formulae and a formula ψ in L, we have that $\mathcal{T} \models^{4}_{c_3} \psi$ iff $\mathcal{T} \models^{4}_{(\prec_{c_3},\mathcal{A}(\mathcal{T}))} \psi$.

Consider, for instance, $\mathcal{T} = \{\neg q,\ (p \supset q) \vee (\neg q \supset \neg p),\ (\neg p \supset q) \vee (\neg q \supset p)\}$. This theory has three $\prec_{c_3}$-minimal models: $\nu_1 = \{p : \bot,\ q : f\}$, $\nu_2 = \{p : t,\ q : \top\}$, and $\nu_3 = \{p : f,\ q : \top\}$. Therefore, for example, $\mathcal{T} \models^{4}_{(\prec_{c_3},\mathcal{A}(\mathcal{T}))} p \supset q$ and $\mathcal{T} \not\models^{4}_{(\prec_{c_3},\mathcal{A}(\mathcal{T}))} q \supset p$.

In order to simulate consequence relations such as $\models^{4}_{c_3}$ in our framework, it is necessary to extend Definition 5.1. In particular, Υ should be partitioned according to the underlying preference order.

Definition 6.3. Let ν_1, ν_2 be two valuations, Δ a set of formulae, and $\vec{\Upsilon} = \vec{\Upsilon}_{\prec} = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m\}$—a partition of *FOUR*. Denote $\nu_1 \leq^{\Delta}_{\vec{\Upsilon}} \nu_2$ if the following conditions are satisfied:

$$\{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon_1\} \subseteq \{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon_1\}$$
$$\{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon_2\} \subseteq \{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon_1 \cup \Upsilon_2\}$$
$$\ldots$$
$$\{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon_{m-1}\} \subseteq \{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon_1 \cup \ldots \cup \Upsilon_{m-1}\}$$

Denote by $\nu_1 <^{\Delta}_{\vec{\Upsilon}} \nu_2$ that $\nu_1 \leq^{\Delta}_{\vec{\Upsilon}} \nu_2$ and $\nu_2 \not\leq^{\Delta}_{\vec{\Upsilon}} \nu_1$. $\nu_1 \in mod(\mathcal{T})$ is a $\leq^{\Delta}_{\vec{\Upsilon}}$-*minimal model* of $\mathcal{T}$ if there is no $\nu_2 \in mod(\mathcal{T})$ such that $\nu_2 <^{\Delta}_{\vec{\Upsilon}} \nu_1$.

Preferential reasoning with graded uncertainty can now be defined in our context as follows.

Definition 6.4. Denote $\mathcal{T} \models^4_{(\vec{\Upsilon},\Delta)} \psi$, if any $\leq^{\Delta}_{\vec{\Upsilon}}$-minimal model of $\mathcal{T}$ is a model of ψ.

The following proposition follows from the fact that $\nu_1 \prec^{\Delta} \nu_2$ iff $\nu_1 <^{\Delta}_{\vec{\Upsilon}} \nu_2$.

PROPOSITION 6.5. *$\mathcal{T} \models^4_{(\prec,\Delta)} \psi$ iff $\mathcal{T} \models^4_{(\vec{\Upsilon},\Delta)} \psi$.*

Example 6.6. Consider again the consequence relation of Example 6.2. Let $\Upsilon_1 = \{t, f\}$, $\Upsilon_2 = \{\perp\}$, $\Upsilon_3 = \{\top\}$, and $\vec{\Upsilon} = \{\Upsilon_1, \Upsilon_2, \Upsilon_3\}$. Then:

(1) for every $\nu_1, \nu_2 \in mod(\mathcal{T})$, $\nu_1 \preceq_{c_3} \nu_2$ iff $\nu_1 \leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}} \nu_2$; and so
(2) $\mathcal{T} \models^4_{c_3} \psi$ iff $\mathcal{T} \models^4_{(\prec_{c_3},\mathcal{A}(\mathcal{T}))} \psi$ iff $\mathcal{T} \models^4_{(\vec{\Upsilon},\mathcal{A}(\mathcal{T}))} \psi$.

Note that, in fact, the definition of preferential reasoning with graded uncertainty (Definition 6.4) is a conservative extension of the definition of consequence relations for preferential reasoning (Definition 5.4).

PROPOSITION 6.7. *For every consequence relation of the form $\models^4_{(\Upsilon,\Delta)}$, there is a partition $\vec{\Upsilon} = \{\Upsilon_1, \Upsilon_2\}$ of FOUR such that $\models^4_{(\Upsilon,\Delta)}$ and $\models^4_{(\vec{\Upsilon},\Delta)}$ coincide.*

PROOF. Given $\Upsilon \subseteq$ *FOUR*, let $\vec{\Upsilon} = \{FOUR - \Upsilon, \Upsilon\}$. By Definitions 5.1 and 6.3, for every two valuations ν_1 and ν_2, it holds that $\nu_1 <^{\Delta}_{\Upsilon} \nu_2$ iff $\{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon\} \subseteq \{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon\}$, iff $\{\psi \in \Delta \mid \nu_2(\psi) \in FOUR - \Upsilon\} \subseteq \{\psi \in \Delta \mid \nu_1(\psi) \in FOUR - \Upsilon\}$, iff $\nu_1 <^{\Delta}_{\vec{\Upsilon}} \nu_2$. Thus by Definitions 5.4 and 6.4, $\models^4_{(\Upsilon,\Delta)}$ and $\models^4_{(\vec{\Upsilon},\Delta)}$ are identical. □

The converse of the last proposition is not true; there are consequence relations of the form $\models^4_{(\vec{\Upsilon},\Delta)}$ that are not the same as any consequence relation of the form $\models^4_{(\Upsilon,\Delta)}$. The consequence relation of Examples 6.2 and 6.6 is one example of this (in Arieli [2003], it is shown that this consequence relation is different than any other four-valued consequence relation that is defined by a pointwise-preferential order on *FOUR*). It follows, therefore, that consequence relations of the form $\models^4_{(\vec{\Upsilon},\Delta)}$ may be divided into three nonempty classes of paraconsistent entailments:

(1) When $\vec{\Upsilon}$ is the degenerated partition (i.e., $\vec{\Upsilon} = \{FOUR\}$), $\models^4_{(\vec{\Upsilon},\Delta)}$ is the same as the monotonic consequence relation $\models^4$;
(2) two-leveled partitions of *FOUR* induce the consequence relations of Section 5; and
(3) other consequence relations of the form $\models^4_{(\vec{\Upsilon},\Delta)}$ have more than two levels of uncertainty.

The next step is to simulate reasoning with graded uncertainty by (signed) QBFs. For this, $\leq^{\Delta}_{\vec{\Upsilon}}$-minimal valuations should be described by an appropriate signed QBF. In the remainder of this section, for the sake of keeping this QBF as simple as possible, we consider only the case where $\Delta = \mathcal{A}(\mathcal{T})$. However, similar results can be obtained for *any* finite set Δ of formulae in L (as in Proposition 5.9).

Proposition 6.8. *Let $\mathcal{T}$ be a finite set of formulae in L, and let $\mathcal{A}^{\pm}(\mathcal{T}) = \{p_1, \ldots, p_n\}$. Then ν^4 is a $\leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}}$-minimal model of $\mathcal{T}$, where $\vec{\Upsilon} = \{\Upsilon_1, \Upsilon_2, \ldots, \Upsilon_m\}$ is a partition of FOUR, iff the two-valued valuation ν^2 that is associated with ν^4 is a model of $\tau_1(\mathcal{T})$ and the following signed QBF, denoted* $\mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}}, \mathcal{T})$

$$\forall q_1, \ldots, q_n \Bigg(\tau_1(\mathcal{T}_\wedge)[q_1/p_1, \ldots, q_n/p_n] \longrightarrow \Bigg(\bigwedge_{i=1}^{n} \Bigg((\Upsilon_1(p_i) \to \Upsilon_1(q_i)) \wedge \ldots \wedge \Bigg(\Upsilon_{m-1}(p_i) \to \bigvee_{j=1}^{m-1} \Upsilon_j(q_i) \Bigg) \Bigg) \longrightarrow \bigwedge_{i=1}^{n} \Bigg((\Upsilon_1(q_i) \to \Upsilon_1(p_i)) \wedge \ldots \wedge \Bigg(\Upsilon_{m-1}(q_i) \to \bigvee_{j=1}^{m-1} \Upsilon_j(p_i) \Bigg) \Bigg) \Bigg) \Bigg).$$

Proof. Again, we denote by $(\{r_1, \ldots, r_n\} : \mu_1, \{s_1, \ldots, s_m\} : \mu_2)$ a valuation that interprets the symbols in $\{r_1, \ldots, r_n\}$ according to μ_1 and the symbols in $\{s_1, \ldots, s_m\}$ according to μ_2. Now, the proof is similar to that of Proposition 5.9, using the fact that $\mu_1^4 \leq^{\Delta}_{\vec{\Upsilon}} \mu_2^4$ iff the valuation $(\{r_1, \ldots, r_n\} : \mu_1^2, \{s_1, \ldots, s_n\} : \mu_2^2)$ satisfies

$$\bigwedge_{i=1}^{n} \Bigg((\Upsilon_1(s_i) \to \Upsilon_1(r_i)) \wedge \ldots \wedge \Bigg(\Upsilon_{m-1}(s_i) \to \bigvee_{j=1}^{m-1} \Upsilon_j(r_i) \Bigg) \Bigg),$$

where μ_1^2 and μ_2^2 are the two-valued interpretations that are induced by μ_1^4 and μ_2^4, respectively. □

By Proposition 6.8, for a finite set of formulae $\mathcal{T}$ and a formula ψ in L, we have the following.

Proposition 6.9. $\mathcal{T} \models^4_{(\vec{\Upsilon},\mathcal{A}(\mathcal{T}))} \psi$ *iff* $\tau_1(\mathcal{T}), \mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}}, \mathcal{T}) \models^2 \tau_1(\psi)$.

Corollary 6.10. $\mathcal{T} \models^4_{(\prec,\mathcal{A}(\mathcal{T}))} \psi$ *iff* $\tau_1(\mathcal{T}), \mathsf{Min}(\leq^{\mathcal{A}(\mathcal{T})}_{\vec{\Upsilon}}, \mathcal{T}) \models^2 \tau_1(\psi)$.

Proof. By propositions 6.5 and 6.9. □

The last results imply that the decision problem for consequence relations with graded abnormality remains in the second level of the polynomial hierarchy (compare with Proposition 5.15).

PROPOSITION 6.11. *The decision problem for* $\models^4_{(\vec{\Upsilon},\mathcal{A}(\mathcal{T}))}$ *is in* Π^P_2.

PROOF. Similar to that of Proposition 5.15, using Proposition 6.9. □

Again, as noted in Section 5.4, Π^P_2 is not a strict bound for every decision problem of entailments of the form $\models^4_{(\vec{\Upsilon},\mathcal{A}(\mathcal{T}))}$ (e.g., when $\vec{\Upsilon} = \{FOUR\}$, $\models^4_{(\vec{\Upsilon},\mathcal{A}(\mathcal{T}))}$ is equivalent to $\models^4$, the decision problem of which is in Π^P_1). However, as noted in the paragraph following the proof of Proposition 5.15, many decision problems for consequence relations of the form $\models^4_{(\Upsilon,\mathcal{A}(\mathcal{T}))}$ (which are by Proposition 6.7 also of the form $\models^4_{(\vec{\Upsilon},\mathcal{A}(\mathcal{T}))}$) *are* hard for Π^P_2. Next we give one more example.

PROPOSITION 6.12. *The decision problem for* $\models^4_{c_3}$ *is* Π^P_2*-complete.*

PROOF. In Section 5.4 we have shown that the decision problem for $\models^4_{\mathcal{I}_2}$ is hard for Π^P_2, even when all the formulae are in CNF. The claim now follows from Example 6.6 and Proposition 6.11 (which imply that $\models^4_{c_3}$ is in Π^P_2), together with the fact, shown in Arieli [2003], that in the language of $\{\neg, \wedge, \vee, \mathsf{t}, \mathsf{f}\}$ it holds that $\mathcal{T} \models^4_{c_3} \psi$ iff $\mathcal{T} \models^4_{\mathcal{I}_2} \psi$ (which implies that $\models^4_{c_3}$ is hard for Π^P_2). □

Other generalizations of Definition 5.1 could be useful as well. For instance, the set Δ may contain formulae with different levels of abnormality, in which case it should be graded. Again, it is possible to simulate reasoning with corresponding consequence relations by signed QBFs, just as described before for cases in which Υ is graded.

6.2 Beyond Four-Valued Semantics

In this section we show that by using the same techniques as those introduced earlier, it is possible to perform preferential reasoning in general multiple-valued logics (having *arbitrarily many* truth values). For instance, default reasoning in the context of nine-valued semantics [Arieli and Avron 1996] (see Figure 2) may be simulated though either Kleene's three-valued logic [Kleene 1950] or Priest's three-valued logic LP [Priest 1991, 1989] (see Example 6.22 to follow). Essentially, we do so by repeating the same process as described before, using signed formulae for the basic entailment, and quantified Boolean formulae for expressing the preferential semantics. Next are the details.

6.2.1 *The Lattice-Valued Setting.* Consider a complete lattice $\mathfrak{L} = (\mathcal{L}, \leq_{\mathcal{L}})$ with a negation operator $\neg$.[13] Let $\mathcal{D}_{\mathfrak{L}}$ be the set of designated elements of $\mathcal{L}$. As usual in multiple-valued logics (and as we did in the case of $\mathcal{FOUR}$), we require that $\mathcal{D}_{\mathfrak{L}}$ be a *filter* in $\mathfrak{L}$, namely, it is a nonempty proper subset of $\mathcal{L}$ such that for every $x, y \in \mathcal{L}$, $x \wedge y \in \mathcal{D}_{\mathfrak{L}}$ iff $x \in \mathcal{D}_{\mathfrak{L}}$ and $y \in \mathcal{D}_{\mathfrak{L}}$. If, in addition, for every $x, y \in \mathcal{L}$, $x \vee y \in \mathcal{D}_{\mathfrak{L}}$ iff $x \in \mathcal{D}_{\mathfrak{L}}$ or $y \in \mathcal{D}_{\mathfrak{L}}$, then $\mathcal{D}_{\mathfrak{L}}$ is a *prime* filter in $\mathcal{L}$. Note that by its definition $\mathcal{D}_{\mathfrak{L}}$ is $\leq_{\mathcal{L}}$-upwards closed, and so $\max(\mathcal{L}) \in \mathcal{D}_{\mathfrak{L}}$ while $\min(\mathcal{L}) \notin \mathcal{D}_{\mathfrak{L}}$.

[13]In other words, for every $x, y \in \mathcal{L}$, $x \leq_{\mathcal{L}} y$ iff $\neg y \leq_{\mathcal{L}} \neg x$, and for every $x \in \mathcal{L}$, $\neg\neg x = x$.

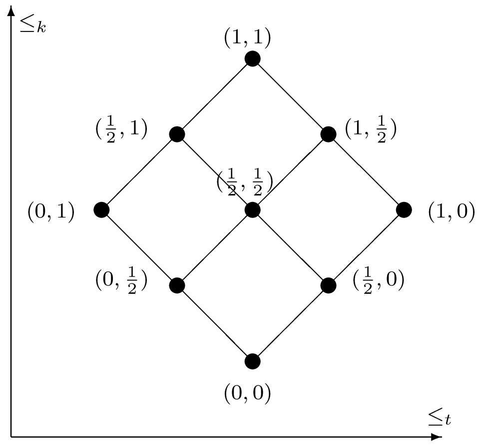

Fig. 2. $\mathcal{THREE} \odot \mathcal{THREE}$.

Now, the connectives of the language L are defined by the corresponding operators on $\mathcal{L}$: conjunction, disjunction, and negation correspond, respectively, to the meet, join, and negation operators on $\mathcal{L}$; the definition of the implication connective $\supset$ depends, as in the case of $\mathcal{FOUR}$, on the choice of designated elements: $x \supset y = \max(\mathcal{L})$ if $x \notin \mathcal{D}_{\mathfrak{L}}$, otherwise $x \supset y = y$.

The other semantical notions and corresponding consequence relations are now defined in the obvious way:

—A (multiple-valued) valuation ν is a function that assigns an element of $\mathcal{L}$ to each atomic formula in Σ. A valuation is extended to complex formulae in the standard way: $\nu(\neg\psi) = \neg\nu(\psi)$, $\nu(\psi \wedge \phi) = \nu(\psi) \wedge \nu(\phi)$, $\nu(\psi \vee \phi) = \nu(\psi) \vee \nu(\phi)$, and $\nu(\psi \supset \phi) = \nu(\psi) \supset \nu(\phi)$.

—A valuation ν satisfies a formula ψ if $\nu(\psi) \in \mathcal{D}_{\mathfrak{L}}$.

—A valuation ν is a model of $\mathcal{T}$ if it satisfies every formula in $\mathcal{T}$. We shall continue to denote by $mod(\mathcal{T})$ the set of the models of $\mathcal{T}$.

A natural definition for a consequence relation with respect to this $\mathcal{L}$-valued semantics is the following.

Definition 6.13. Let $\mathfrak{L}$ be a lattice and $\mathcal{D}_{\mathfrak{L}}$ a prime filter in it. For a set $\mathcal{T}$ of formulae and a formula ψ, denote $\mathcal{T} \models^{\mathfrak{L},\mathcal{D}_{\mathfrak{L}}} \psi$ if every model of $\mathcal{T}$ is a model of ψ.

Common examples of logics that are obtained from the preceding definitions are classical logic where $\mathfrak{L}$ is the two-valued lattice $(\{t, f\}, f <_{\mathcal{L}} t)$ and $\mathcal{D}_{\mathfrak{L}} = \{t\}$, Kleene's three-valued logic [Kleene 1950], where $\mathfrak{L}$ is the three-valued lattice $(\{t, f, \bot\}, f \leq_{\mathcal{L}} \bot \leq_{\mathcal{L}} t)$ and $\mathcal{D}_{\mathfrak{L}} = \{t\}$, Priest's three-valued logic [Priest 1989], where $\mathfrak{L}$ is the three-valued lattice $(\{t, f, \top\}, f \leq_{\mathcal{L}} \top \leq_{\mathcal{L}} t)$ and $\mathcal{D}_{\mathfrak{L}} = \{t, \top\}$, and Belnap's four-valued logic [Belnap 1977b] defined in Section 2. Note that the former two logics are not paraconsistent while the latter two are.

6.2.2 *The Bilattice-Valued Setting.* In previous sections we have reduced four-valued paraconsistent reasoning to classical entailment by "splitting" syntactical and semantical objects (e.g., the alphabet, truth values, and truth assignments) to their positive and negative counterparts. The same idea can be applied here, this time in the $\mathfrak{L}$-valued setting. This leads us to the following construct.

Definition 6.14 [*Ginsberg* 1988]. Let $\mathfrak{L} = (\mathcal{L}, \leq_{\mathcal{L}})$ be a complete lattice. The structure $\mathfrak{L} \odot \mathfrak{L} = (\mathcal{L} \times \mathcal{L}, \leq_t, \leq_k, \neg)$ consists of pairs of elements from $\mathcal{L}$ that are arranged in two complete lattice structures as follows:

— $(\mathcal{L} \times \mathcal{L}, \leq_t)$, where $(x_1, y_1) \leq_t (x_2, y_2)$ iff $x_1 \leq_{\mathcal{L}} x_2$ and $y_1 \geq_{\mathcal{L}} y_2$; and
— $(\mathcal{L} \times \mathcal{L}, \leq_k)$, where $(x_1, y_1) \leq_k (x_2, y_2)$ iff $x_1 \leq_{\mathcal{L}} x_2$ and $y_1 \leq_{\mathcal{L}} y_2$.

The unary operation $\neg$ is defined on $\mathcal{L} \times \mathcal{L}$ by $\neg(x, y) = (y, x)$.

Figures 1 and 2 show two structures that are particular cases of Definition 6.14 where $\mathfrak{L}$ is a chain of two and three elements, respectively. The structure that $\mathfrak{L} \odot \mathfrak{L}$ forms is called a *bilattice* [Fitting 1990; Ginsberg 1988], denoting that the set of truth values is simultaneously arranged in two (related) lattice structures. As in the four-valued case, a truth value $(x, y) \in \mathfrak{L} \odot \mathfrak{L}$ may intuitively be understood so that x represents the amount of evidence *for* an assertion, while y represents the amount of evidence *against* it. It is easy to verify that the basic $\leq_t$-operators are defined in the same way as in $\mathcal{FOUR}$, namely, that for every $x_1, x_2, y_1, y_2 \in \mathcal{L}$,

$$(x_1, y_1) \vee (x_2, y_2) = (x_1 \vee x_2,\ y_1 \wedge y_2),$$
$$(x_1, y_1) \wedge (x_2, y_2) = (x_1 \wedge x_2,\ y_1 \vee y_2).[14]$$

Also, the $\leq_k$-minimal element of $\mathcal{L} \odot \mathcal{L}$ is $(\min(\mathcal{L}), \min(\mathcal{L}))$, the $\leq_k$-maximal one is $(\max(\mathcal{L}), \max(\mathcal{L}))$, the $\leq_t$-minimal element is $(\min(\mathcal{L}), \max(\mathcal{L}))$, and the $\leq_t$-maximal one is $(\max(\mathcal{L}), \min(\mathcal{L}))$.

Clearly, the structure of Definition 6.14 is a natural extension of Belnap's four-valued structure. As before, we denote by $\mathcal{D}_{\mathfrak{L}\odot\mathfrak{L}}$ a (fixed) set of designated values of $\mathfrak{L} \odot \mathfrak{L}$, which is a filter of $\mathcal{L} \times \mathcal{L}$. The other semantical notions are defined accordingly.

Given a bilattice $\mathfrak{L} \odot \mathfrak{L}$ and a (prime) filter $\mathcal{D}_{\mathfrak{L}\odot\mathfrak{L}}$ of it, one may define a corresponding consequence relation in a way similar to the lattice-valued case.

Definition 6.15. $\mathcal{T} \models^{\mathfrak{L}\odot\mathfrak{L},\, \mathcal{D}_{\mathfrak{L}\odot\mathfrak{L}}} \psi$ if every $\mathcal{L} \times \mathcal{L}$-valued model of $\mathcal{T}$ (with respect to $\mathcal{D}_{\mathfrak{L}\odot\mathfrak{L}}$) is also a model of ψ.

The preferential derivative of $\models^{\mathfrak{L}\odot\mathfrak{L},\, \mathcal{D}_{\mathfrak{L}\odot\mathfrak{L}}}$ is defined as follows (compare with Definition 5.4).

Definition 6.16. Let Δ be a set of formulae in L, and $\Upsilon \subseteq \mathcal{L} \times \mathcal{L}$.

[14] It worth noting that the $\leq_k$-join $\oplus$ and the $\leq_k$-meet $\otimes$ in $\mathfrak{L} \odot \mathfrak{L}$ have similar definitions. For every $x_1, x_2, y_1, y_2 \in \mathcal{L}$, $(x_1, y_1) \oplus (x_2, y_2) = (x_1 \vee x_2,\ y_1 \vee y_2)$ and $(x_1, y_1) \otimes (x_2, y_2) = (x_1 \wedge x_2,\ y_1 \wedge y_2)$. These operators, nevertheless, will not play an important role in what follows.

(1) Define a relation $\leq_{\Upsilon}^{\Delta}$ on $\mathcal{L}\times\mathcal{L}$-valued valuations as in Definitions 5.1 and 5.2:
   (a) For two valuations ν_1 and ν_2 into $\mathcal{L}\times\mathcal{L}$, we denote $\nu_1 \leq_{\Upsilon}^{\Delta} \nu_2$ (ν_1 is Υ-preferred to ν_2 with respect to Δ) if $\{\psi \in \Delta \mid \nu_1(\psi) \in \Upsilon\} \subseteq \{\psi \in \Delta \mid \nu_2(\psi) \in \Upsilon\}$.
   We write $\nu_1 <_{\Upsilon}^{\Delta} \nu_2$ if $\nu_1 \leq_{\Upsilon}^{\Delta} \nu_2$ and $\nu_2 \not\leq_{\Upsilon}^{\Delta} \nu_1$.
   (b) A valuation $\nu \in mod(\mathcal{T})$ is a $\leq_{\Upsilon}^{\Delta}$-minimal model of $\mathcal{T}$ if there is no model μ of $\mathcal{T}$ such that $\mu <_{\Upsilon}^{\Delta} \nu$.

(2) $\mathcal{T} \models_{(\Upsilon,\Delta)}^{\mathfrak{L}\odot\mathfrak{L},\mathcal{D}_{\mathfrak{L}\odot\mathfrak{L}}} \psi$ if every $\leq_{\Upsilon}^{\Delta}$-minimal ($\mathcal{L}\times\mathcal{L}$-valued) model of $\mathcal{T}$ (with respect to $\mathcal{D}_{\mathfrak{L}\odot\mathfrak{L}}$) is also a model of ψ.

Example 6.17. Consider again $\mathcal{T}_1$ of Example 5.3 and the bilattice $\mathcal{THREE} \odot \mathcal{THREE}$ (Figure 2) with $\mathcal{D}_{\mathcal{THREE}\odot\mathcal{THREE}} = \{(x, y) \mid (x, y) \geq_k (1, 0)\}$. Suppose that one wants to consider only those models of $\mathcal{T}_1$ that are "as consistent as possible" (i.e., the models of $\mathcal{T}_1$ that assign either true $(1, 0)$ or false $(0, 1)$ to a maximal amount of atomic formulae that appear in the premise formulae). In this case, $\Upsilon = \mathcal{THREE} \times \mathcal{THREE} - \{(1, 0), (0, 1)\}$ and $\Delta = \mathcal{A}(\mathcal{T}_1)$. Now, abbreviate by 9 the pair $(\mathcal{THREE} \odot \mathcal{THREE}, \mathcal{D}_{\mathcal{THREE}\odot\mathcal{THREE}})$. Then, for instance,

$$\mathcal{T}_1 \models^9_{(\Upsilon,\Delta)} p, \quad \mathcal{T}_1 \models^9_{(\Upsilon,\Delta)} q, \quad \mathcal{T}_1 \models^9_{(\Upsilon,\Delta)} s, \quad \mathcal{T}_1 \not\models^9_{(\Upsilon,\Delta)} r,$$
$$\mathcal{T}_1 \models^9_{(\Upsilon,\Delta)} \neg p, \quad \mathcal{T}_1 \not\models^9_{(\Upsilon,\Delta)} \neg q, \quad \mathcal{T}_1 \not\models^9_{(\Upsilon,\Delta)} \neg s, \quad \mathcal{T}_1 \not\models^9_{(\Upsilon,\Delta)} \neg r.$$

6.2.3 *Relating the Two Settings.* In what follows, we fix some complete lattice $\mathfrak{L} = (\mathcal{L}, \leq_L)$ with a prime filter $\mathcal{D}_{\mathfrak{L}}$ and consider the associated bilattice $\mathfrak{L} \odot \mathfrak{L}$ with a set of designated elements $\mathcal{D} = \mathcal{D}_{\mathcal{L}} \times \mathcal{L}$ of $\mathcal{L} \times \mathcal{L}$. Note that $\mathcal{D}$ is a legitimate choice of a set of designated elements due to the following result.

PROPOSITION 6.18 [ARIELI AND AVRON 2000, PROP. 14]. *$\mathcal{D}_{\mathfrak{L}}$ is a (prime) filter in $\mathfrak{L}$ iff $\mathcal{D}_{\mathcal{L}} \times \mathcal{L}$ is a (prime) filter in $\mathcal{L} \times \mathcal{L}$.*

We again start with the basic consequence relations of these settings and then turn to the preferential cases.

THEOREM 6.19. $\mathcal{T} \models^{\mathfrak{L}\odot\mathfrak{L},\mathcal{D}} \psi$ iff $\tau_1(\mathcal{T}) \models^{\mathfrak{L},\mathcal{D}_{\mathfrak{L}}} \tau_1(\psi)$.

PROOF (OUTLINE). Essentially, we repeat the process described in Section 3, and extend the corresponding results to the (bi)lattice-valued case.

As in Definition 3.1, we define a signed alphabet $\Sigma^{\pm}$ for a given alphabet Σ. Then, $\mathcal{L}$-valuations ($\nu^{\mathcal{L}}$) and $\mathcal{L}\times\mathcal{L}$-valuations ($\nu^{\mathcal{L}\times\mathcal{L}}$) are associated with each other, exactly like the two- and four-valued case (see again Definition 3.1). By induction on the structure of the formulae in the language L, one can show that if $\nu^{\mathcal{L}}$ is induced by $\nu^{\mathcal{L}\times\mathcal{L}}$ or $\nu^{\mathcal{L}\times\mathcal{L}}$ is induced by $\nu^{\mathcal{L}}$, then for every formula ψ,

$$\nu^{\mathcal{L}\times\mathcal{L}}(\psi) = (\nu^{\mathcal{L}}(\tau_1(\psi)),\ \nu^{\mathcal{L}}(\tau_2(\psi))),$$

where the τ_i transformations ($i = 1, 2$) are the same as those of Definition 3.2 (see the proof of Proposition 3.4). By the last equations then,

$$\nu^{\mathcal{L}\times\mathcal{L}}(\psi) \in \mathcal{D} = \mathcal{D}_{\mathcal{L}} \times \mathcal{L} \ \text{ iff } \ \nu^{\mathcal{L}}(\tau_1(\psi)) \in \mathcal{D}_{\mathcal{L}},$$
$$\nu^{\mathcal{L}\times\mathcal{L}}(\neg\psi) \in \mathcal{D} \ \text{ iff } \ \nu^{\mathcal{L}}(\tau_2(\psi)) \in \mathcal{D}_{\mathcal{L}},$$

which means that $\nu^{\mathcal{L}\times\mathcal{L}}$ satisfies ψ iff $\nu^{\mathcal{L}}$ satisfies $\tau_1(\psi)$, and $\nu^{\mathcal{L}\times\mathcal{L}}$ satisfies $\neg\psi$ iff $\nu^{\mathcal{L}}$ satisfies $\tau_2(\psi)$ (compare with Corollary 3.6). It follows that there is a one-to-one correspondence between the $\mathcal{L}\times\mathcal{L}$-valued models of $\mathcal{T}$ (with respect to $\mathcal{D}$) and the $\mathcal{L}$-valued models of $\tau_1(\mathcal{T})$ (with respect to $\mathcal{D}_{\mathcal{L}}$), and so we are done. □

We turn now to the preferential case. Again, paraconsistent preferential reasoning in $\mathfrak{L}\odot\mathfrak{L}$ may be simulated in $\mathfrak{L}$ by signed QBFs in a way which is similar to the one described in Section 5.3. The only difference is that unlike the four-valued case, it is not possible to represent every element in $\mathcal{L}\times\mathcal{L}$ by a formula in L (i.e., it is not necessarily possible to define formulae val, like those of Definition 3.7, that satisfy the property of Proposition 3.8), so it is not always possible to define formulae $\Upsilon(\psi)$ that hold iff the truth value of ψ is in Υ (see Note 5.7). We therefore have to impose the following restriction on the choice of Υ.

Definition 6.20. Given a bilattice $\mathfrak{L}\odot\mathfrak{L}$ and filter $\mathcal{F}$ in $\mathcal{L}\times\mathcal{L}$, consider the following partition of $\mathcal{L}\times\mathcal{L}$:

$$\mathfrak{T}_t = \{x \in \mathcal{L}\times\mathcal{L} \mid x \in \mathcal{F}, \neg x \notin \mathcal{F}\} \qquad \mathfrak{T}_f = \{x \in \mathcal{L}\times\mathcal{L} \mid x \notin \mathcal{F}, \neg x \in \mathcal{F}\}$$
$$\mathfrak{T}_\top = \{x \in \mathcal{L}\times\mathcal{L} \mid x \in \mathcal{F}, \neg x \in \mathcal{F}\} \qquad \mathfrak{T}_\bot = \{x \in \mathcal{L}\times\mathcal{L} \mid x \notin \mathcal{F}, \neg x \notin \mathcal{F}\}$$

A set $\Upsilon \subseteq \mathcal{L}\times\mathcal{L}$ is called *perceptive* if for every $c \in \{t, f, \top, \bot\}$, either $\mathfrak{T}_c \subseteq \Upsilon$ or $\Upsilon \cap \mathfrak{T}_c = \emptyset$.

Consider again the signed QBF $\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T})$ introduced in Proposition 5.9 for a theory $\mathcal{T}$ and fixed sets Δ, Υ. The analog of Theorem 5.11 is now the following.

THEOREM 6.21. *Let $\mathcal{T},\Delta$ be finite sets of formulae in L, and Υ a nonempty perceptive subset of $\mathcal{L}\times\mathcal{L}$. Then $\mathcal{T} \models^{\mathfrak{L}\odot\mathfrak{L},\mathcal{D}}_{(\Upsilon,\Delta)} \psi$ iff $\tau_1(\mathcal{T}), \mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T}) \models^{\mathfrak{L},\mathcal{D}_{\mathfrak{L}}} \tau_1(\psi)$.*

PROOF (OUTLINE). In the proof of Theorem 6.19 we have shown that if $\nu^{\mathcal{L}}$ is induced by $\nu^{\mathcal{L}\times\mathcal{L}}$ or $\nu^{\mathcal{L}\times\mathcal{L}}$ is induced by $\nu^{\mathcal{L}}$, then for every formula ψ, $\nu^{\mathcal{L}\times\mathcal{L}}$ satisfies ψ iff $\nu^{\mathcal{L}}$ satisfies $\tau_1(\psi)$, and $\nu^{\mathcal{L}\times\mathcal{L}}$ satisfies $\neg\psi$ iff $\nu^{\mathcal{L}}$ satisfies $\tau_2(\psi)$. In the notations of Definition 3.7 then, for every formulae ψ and every $c \in \{t, f, \top, \bot\}$, $\nu^{\mathcal{L}\times\mathcal{L}}(\psi) \in \mathfrak{T}_c$ iff $\nu^{\mathcal{L}}$ satisfies $\mathsf{val}(\psi, c)$ (the proof of this is similar to that of Proposition 3.8, using the fact that $\mathcal{D}_{\mathfrak{L}}$ is a filter in $\mathfrak{L}$). Now, Υ is a nonempty perceptive subset of $\mathcal{L}\times\mathcal{L}$, so there are elements $c_1, \ldots, c_k \in \{t, f, \top, \bot\}$ such that $\Upsilon = \mathfrak{T}_{c_1} \cup \ldots \cup \mathfrak{T}_{c_k}$. Let $\Upsilon(\psi) = \mathsf{val}(\psi, c_1) \vee \ldots \vee \mathsf{val}(\psi, c_k)$. We have that $\nu^{\mathcal{L}\times\mathcal{L}}(\psi) \in \Upsilon$, iff $\nu^{\mathcal{L}\times\mathcal{L}}(\psi) \in \mathfrak{T}_c$ for some $c \in \{c_1, \ldots, c_k\}$, iff $\nu^{\mathcal{L}}(\mathsf{val}(\psi, c)) \in \mathcal{D}_{\mathfrak{L}}$ for the same $c \in \{c_1, \ldots, c_k\}$, iff $\nu^{\mathcal{L}}$ satisfies $\Upsilon(\psi)$ (compare with Note 5.7).[15] This implies that $\nu^{\mathcal{L}\times\mathcal{L}}$ is a $\leq^{\Delta}_{\Upsilon}$-minimal model of $\mathcal{T}$ iff $\nu^{\mathcal{L}}$ is a model of $\tau_1(\mathcal{T})$ and $\mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T})$ (the proof of this is completely analogous to that of Proposition 5.9, using the fact that we have shown that for a perceptive set Υ, Proposition 3.9-(2) and Note 5.7 are both valid in the $\mathcal{L}\times\mathcal{L}$-case as well). □

Example 6.22. Consider again the bilattice $\mathcal{THREE}\odot\mathcal{THREE}$ of Figure 2. This structure may be used, for example, for default reasoning (where the values $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$ stand for "true by default" and "false by default," respectively, see e.g., Arieli and Avron [1998, 1996]). This structure has two sets that are

[15]Here we are using the fact that $\mathcal{D}_{\mathfrak{L}}$ is a *prime* filter in $\mathfrak{L}$.

prime filters with respect to both $\leq_t$ and $\leq_k$, namely, $\mathcal{D}_1 = \{(x_1, x_2) \mid (x_1, x_2) \geq_k (1, 0)\}$ and $\mathcal{D}_2 = \{(x_1, x_2) \mid (x_1, x_2) \geq_k (\frac{1}{2}, 0)\}$. These sets are therefore natural candidates for being the set of the designated elements. Note moreover, that $\mathcal{D}_1 = \{1\} \times \{0, \frac{1}{2}, 1\}$ and $\mathcal{D}_2 = \{1, \frac{1}{2}\} \times \{0, \frac{1}{2}, 1\}$. By Theorems 6.19 and 6.21 then, (preferential) reasoning with $\mathcal{THREE} \odot \mathcal{THREE}$ and $\mathcal{D}_1$ can be simulated in $\mathcal{THREE}$ with $\mathcal{D}_{\mathcal{THREE}} = \{1\}$ (a setting that corresponds to Kleene's three-valued logic), while (preferential) reasoning with $\mathcal{THREE} \odot \mathcal{THREE}$ and $\mathcal{D}_2$ can be simulated in $\mathcal{THREE}$ with $\mathcal{D}_{\mathcal{THREE}} = \{1, \frac{1}{2}\}$ (a setting that corresponds to Priest's three-valued logic LP).

7. RELATED WORKS

A key issue in our approach is the encoding of many-valued logics in terms of two-valued classical logic by means of signed formulae. Such formulae were introduced by Besnard and Schaub [1998, 1996] as syntax-independent paraconsistent reasoning systems. A similar idea is used in Arieli and Denecker [2003, 2002], and Besnard et al. [2003], where different transformations of formulae to signed formulae are considered for reducing multiple-valued entailments to classical entailments. We find the τ-transformations used here (Definition 3.2) somewhat more natural and general for conversions to signed theories, as three-valued semantics is inherent in the formalism of Besnard et al. [2003] (this is implied by Definition 1 in that paper), and the language considered in Arieli and Denecker [2003, 2002] is not functionally complete.

The use of QBF axiomatic theories for representing preferred models is another fundamental ingredient of our approach. Using QBFs for knowledge representation and reasoning is proposed by Egly et al. [2000], who showed that several major problems from propositional nonmonotonic reasoning can be translated into QBFs and resolved by the QBF-based system QUIP. Quantified propositional logic for paraconsistent reasoning has been independently considered in the preliminary version of this article [Arieli 2004] and by Besnard et al., who defined a variety of paraconsistent consequence relations by means of default logic [Besnard et al. 2002] and showed how to circumscribe inconsistent theories in the context of three-valued logics [Besnard et al. 2003] (see also Besnard et al. [2004] for a survey on these and related methods). Although the QBF axiomatic theories in Besnard et al. [2003] are different than ours, in both cases the formulae are obtained from the original theories by an effective and efficient (polynomial-time constructible) encoding. Moreover, when the complexity bounds given in Propositions 5.15 and 6.11 for the underlying consequence relations are strict, determining the validity of the QBFs resulting from the transformations is not computationally harder than checking inferences with respect to those consequence relations.[16]

The main difference between the approach in this article and that of Besnard et al. [2003] concerns the representation level. Whereas the encoding used by Besnard et al. [2003] is sensitive to the underlying three-valued semantic and the interrelations among its elements, here the role of truth values

[16]For instances of those relations that have lower complexity than Π_2^P, efficient reductions to classical logic are more appropriate; see, for example, Arieli and Denecker [2003, 2002].

is somewhat more transparent. As a consequence, our formalism can be generalized relatively easily, as can be verified by comparing the computation of the basic four-valued entailment (Theorem 4.1) to that of lattice-valued entailment (Theorem 6.19), or by relating the simulation of preferential entailments (Theorem 5.11) to the reasoning process with their graded extensions (Proposition 6.9). Similar generalizations are less natural in the case of the formalism of Besnard et al. [2003]. For instance, switching from the axiomatization of Priest's three-valued logic $\models^3_{\mathrm{LP}}$ (which is the basic three-valued entailment considered in Besnard et al. [2003]) to that of Kleene's three-valued logic $\models^3_{\mathrm{Kl}}$ requires, according to the approach of Besnard et al. [2003], a modification in the parametrized translation to signed formulae, while in our case the same transformation is appropriate for both logics (Corollary 4.5), as well as for their generalizations to logics with more truth values and/or with preferential semantics. Now, as the consequence relations considered in Besnard et al. [2003] are particular cases of those conveyed by Definition 5.4, our formalism may be viewed as an alternative approach for simulating the three-valued consequence relations considered in Besnard et al. [2003] which is tolerant to refinements of the preference criteria at hand, and offers a simple way of generalizing the underlying semantics to algebraic structures with arbitrarily many truth values.

Another approach for reducing (multiple-valued) preferential reasoning to (higher-order) classical propositional logic is considered in Arieli and Denecker [2003]. This approach expresses preferences in terms of second-order formulae, so (instead of QBF solvers) algorithms for processing circumscriptive theories (i.e., reducing second-order formulae to their first-order equivalents) are needed in order to implement preferential reasoning. In the propositional case, a circumscriptive encoding is given in Egly et al. [2000] (see also Lifschitz [1985]). As both methods have the same origin (i.e., second-order languages; see Note 5.10), this duality is quite natural. Next we formulate the exact relation between the present approach and the circumscription-based approach of Arieli and Denecker [2003] with respect to the propositional fragment of L (i.e., the language without "$\supset$").[17]

Define the *scope* of a negation operator $\neg$ in the formula $\neg\psi$ as the set of all occurrences of propositional symbols in ψ. An occurrence of p in a formula ψ is *positive* if it appears in the scope of an even number of negation operators in ψ, otherwise it is a *negative* occurrence.

Definition 7.1 [*Arieli and Denecker* 2003]. Let ψ be a formula in Σ. Denote by $\overline{\psi}$ the formula in $\Sigma^{\pm}$, obtained from ψ by substituting every positive occurrence in ψ of an atomic formula p by p^+ and replacing every negative occurrence in ψ of an atomic formula p by $\neg p^-$. Given a theory $\mathcal{T}$, the set $\{\overline{\psi} \mid \psi \in \mathcal{T}\}$ is denoted by $\overline{\mathcal{T}}$.

Example 7.2. Consider again the formula $\psi = \neg(p \vee \neg q) \vee \neg q$ of Example 3.3. The first occurrence of q in ψ is positive, and the second occurrence of q as well as the (single) occurrence of p in ψ are negative. Thus, the signed

[17]This fragment is the language considered in Arieli and Denecker [2003].

formula that is obtained from ψ is $\overline{\psi} = \neg(\neg p^- \vee \neg q^+) \vee \neg\neg q^-$. Note that $\overline{\psi}$ is logically equivalent to $\tau_1(\psi)$ (see Example 3.3). As the following proposition shows, this equivalence is not accidental.

PROPOSITION 7.3. *If $\nu^4(\psi) = x$ and ν^2 is the two-valued valuation that is induced by ν^4, then $\nu^2(\overline{\psi}) = 1$ iff $\nu^2(\tau_1(\psi)) = 1$ (iff $x \geq_k t$).*

PROOF. Let $\nu^4(\psi) = (\nu_1(\psi), \nu_2(\psi))$ and let ν^2 be the two-valued valuation that is induced by ν^4. In Arieli and Denecker [2003, Prop. 2.4] it is shown that $\nu^2(\overline{\psi}) = \nu_1(\psi)$. The claim now follows from item (2) of Proposition 3.4. □

COROLLARY 7.4. *Let $\mathcal{T}$ be a finite theory in the propositional fragment of L. Then:*

(1) *The two-valued models of $\overline{\mathcal{T}}$ are the same as the two-valued models of $\tau_1(\mathcal{T})$;*
(2) *$\mathcal{T} \models^4 \psi$ iff $\overline{\mathcal{T}} \models^2 \overline{\psi}$ (iff $\tau_1(\mathcal{T}) \models^2 \tau_1(\psi)$); and*
(3) *$\mathcal{T} \models^4_{(\Upsilon,\Delta)} \psi$ iff $\overline{\mathcal{T}}, \mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T}) \models^2 \overline{\psi}$ (iff $\tau_1(\mathcal{T}), \mathsf{Min}(\leq^{\Delta}_{\Upsilon}, \mathcal{T}) \models^2 \tau_1(\psi)$).*

PROOF. Part (1) follows from Proposition 7.3, part (2) follows from part (1) and Theorem 4.1, and part (3) follows from part (1) and Theorem 5.11. □

It follows, then, that our embedding in four-valued semantics captures the reductions to classical entailments considered in Arieli and Denecker [2003], but in a more general context. In particular, our approach extends that of Arieli and Denecker [2003] in the sense that the language L considered here (unlike the $\leq_k$-monotonic language used in Arieli and Denecker [2003]) is functionally complete for $\mathcal{FOUR}$.[18] Also, we simulate a wider range of preferential logics and provide a natural approach to reasoning with graded abnormality.

8. CONCLUSION

Motivated by the need to find practical and effective methods for paraconsistent reasoning, signed formulae serve here as a representative platform that can be used by a variety of off-the-shelf QBF solvers for drawing plausible conclusions from possibly inconsistent and/or incomplete theories.

Benchmark studies on the performance of QBF solvers in comparison to that of neighboring tools (e.g., the answer set programming solvers dlv [Eiter et al. 1998] and smodels [Niemelä and Simons 1996], and the SAT solver zChaff [Moskewicz et al. 2001]) have shown that in some cases, the performance of the QBF solvers is below that of other solvers (see, e.g., Arieli et al. [2004] for a comparative study in the context of database repair, and Egly et al. [2000] for some benchmark results for abduction problems and graph problems). Still, the performance of many QBF solvers is continuously improving as the underlying techniques become more sophisticated. This includes adaptations of the Davis-Putnam-Logemann-Loveland (DPLL) algorithm for propositional logic [Davis et al. 1962; Davis and Putnam 1960] to quantified propositional logic [Cadoli et al. 2002, 1998], extensions of resolution-based reasoning [Kleine-Büning et al. 1995], employment of lookback techniques [Letz 2002], and

[18] See Arieli and Avron [1998] for a proof of the functional completeness of L.

incorporation of new solving paradigms based on symbolic reasoning strategies [Benedetti 2005]. Different methods for converting formulae to prenex CNF and specifying them in Dimacs or Rintanen format (which is the conventional form that QBF solvers accept) also help to reduce the evaluation time of different solvers (see Egly et al. [2004]). Moreover, QBF solvers provide an easy way of obtaining a prototypical implementation for many different problems, and this kind of implementation is particularly suitable for our purpose, since quantified Boolean formulae offer a natural way of expressing minimization (which is in many cases a major consideration in preferential reasoning). We believe, therefore, that QBF solvers are becoming powerful mechanisms that provide robust ways of computing preferential entailments in general, and paraconsistent reasoning in particular.

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REFERENCES

ARIELI, O. 2004. Paraconsistent preferential reasoning by signed quantified Boolean formulae. In *Proceedings of the 16th European Conference on Artificial Intelligence* (ECAI), R. López de Mántaras and L. Saitta, Eds. IOS Press, Amsterdam, The Netherlands, 773–777.

ARIELI, O. 2003. Reasoning with different levels of uncertainty. *Appl. Non-Classical Logics 13*, 3-4, 317–343.

ARIELI, O. AND AVRON, A. 2000. Bilattices and paraconsistency. In *Frontiers of Paraconsistent Logic*, D. Batens et al., Eds. Studies in Logic and Computation, vol. 8. Research Studies Press, Taunton, UK, 11–27.

ARIELI, O. AND AVRON, A. 1998. The value of the four values. *Artif. Intell. 102*, 1, 97–141.

ARIELI, O. AND AVRON, A. 1996. Reasoning with logical bilattices. *J. Logic Lang. Info. 5*, 1, 25–63.

ARIELI, O. AND DENECKER, M. 2003. Reducing preferential paraconsistent reasoning to classical entailment. *Logic Comput. 13*, 4, 557–580.

ARIELI, O. AND DENECKER, M. 2002. Modeling paraconsistent reasoning by classical logic. In *Proceedings of the 2nd Symposium on Foundations of Information and Knowledge Systems* (FoIKS), T. Eiter and K. D. Schewe, Eds. Lecture Notes in Computer Science, vol. 2284, Springer-Verlag, 1–14.

ARIELI, O., DENECKER, M., VAN NUFFELEN, B., AND BRUYNOOGHE, M. 2004. Database repair by signed formulae. In *Proceedings of the 3rd Symposium on Foundations of Information and Knowledge Systems* (FoIKS), D. Seipel and J. M. Turull Torres, Eds. Lecture Notes in Computer Science, vol. 2942, Springer-Verlag, 14–30.

AVRON, A. AND LEV, I. 2001. A formula-preferential base for paraconsistent and plausible non-monotonic reasoning. In *Proceedings of the Joint KRR-IJCAI Workshop on Inconsistency in Data and Knowledge*. 60–70.

BATENS, D. 1989. Dynamic dialectical logics. In *Paraconsistent Logic. Essay on the Inconsistent*, G. Priest et al., Eds. Philosophia Verlag, 187–217.

BATENS, D. 1998. Inconsistency-adaptive logics. In *Logic at Work*, E. Orlowska, Ed. Physica Verlag, 445–472.

BATENS, D. 2000. A survey on inconsistency-adaptive logics. In *Frontiers of Paraconsistent Logic*, D. Batens et al., Eds. Studies in Logic and Computation, vol. 8. Research Studies Press, Taunton, UK, 69–73.

BELNAP, N. D. 1977a. How a computer should think. In *Contemporary Aspects of Philosophy*, G. Ryle, Ed. Oriel Press, 30–56.

BELNAP, N. D. 1977b. A useful four-valued logic. In *Modern Uses of Multiple-Valued Logics*, J. M. Dunn and G. Epstein, Eds. Reidel, 7–37.

BENEDETTI, M. 2005. Evaluating QBFs via symbolic skolemization. In *Proceedings of the 11th International Conference on Logic Programming, Artificial Intelligence and Reasoning* (LPAR), F. Baader and A. Voronkov, Eds. Lecture Notes in Artificial Intelligence, vol. 3452, Springer-Verlag, 285–300.

BESNARD, P. AND SCHAUB, T. 1996. A simple signed system for paraconsistent reasoning. In *Proceedings of the 5th European Conference on Logics in Artificial Intelligence* (JELIA), J. Alferes et al., Eds. Lecture Notes in Artificial Intelligence, vol. 1126, Springer-Verlag, 404–416.

BESNARD, P. AND SCHAUB, T. 1997. Circumscribing inconsistency. In *Proceedings of the 15th International Joint Conference on Artificial Intelligence* (IJCAI). Morgan Kaufmann, San Francisco, CA, 150–155.

BESNARD, P. AND SCHAUB, T. 1998. Signed systems for paraconsistent reasoning. *Automat. Reason. 20*, 1, 191–213.

BESNARD, P., SCHAUB, T., TOMPITS, H., AND WOLTRAN, S. 2002. Paraconsistent reasoning via quantified Boolean formulas I: Axiomatizing signed systems. In *Proceedings of the 8th European Conference on Logics in Artificial Intelligence* (JELIA), S. Flesca et al., Eds. Lecture Notes in Artificial Intelligence, vol. 2424, Springer-Verlag, 320–331.

BESNARD, P., SCHAUB, T., TOMPITS, H., AND WOLTRAN, S. 2003. Paraconsistent reasoning via quantified Boolean formulas II: Circumscribing inconsistent theories. In *Proceedings of the 7th European Conference on Symbolic and Quantitative Approaches to Reasoning with Uncertainty* (ECSQARU), T. D. Nielsen and N. L. Zhang, Eds. Lecture Notes in Artificial Intelligence, vol. 2711, Springer-Verlag, 528–539.

BESNARD, P., SCHAUB, T., TOMPITS, H., AND WOLTRAN, S. 2004. Representing paraconsistent reasoning via quantified propositional logic. In *Inconsistency Tolerance*, L. Bertossi et al., Eds. Lecture Notes in Computer Science, vol. 3300, Springer-Verlag, 84–118.

CADOLI, M., GIOVANARDI, A., AND SCHAERF, M. 1998. An algorithm to evaluate quantified Boolean formulae. In *Proceedings of the 15th National Conference on Artificial Intelligence* (AAAI). AAAI Press, 262–267.

CADOLI, M. AND SCHAERF, M. 1996. On the complexity of entailment in propositional multivalued logics. *Ann. Math. Artif. Intell. 18*, 29–50.

CADOLI, M., SCHAERF, M., GIOVANARDI, A., AND GIOVANARDI, M. 2002. An algorithm to evaluate quantified Boolean formulae and its experimental evaluation. *Automat. Reason. 28*, 2, 101–142.

CARNIELLI, W. A. AND MARCOS, J. 2002. A taxonomy of C-systems. In *Paraconsistency: The Logical Way to the Inconsistent*, W. A. Carnielli et al., Eds. Lecture Notes in Pure and Applied Mathematics, vol. 228. Marcel Dekker, New York, 1–94.

COSTA-MARQUIS, S. AND MARQUIS, P. 2004. On the complexity of paraconsistent inference relations. In *Inconsistency Tolerance*, L. Bertossi et al., Eds. Lecture Notes in Computer Science, vol. 3300, Springer-Verlag, 151–190.

COSTA-MARQUIS, S. AND MARQUIS, P. 2002. Complexity results for paraconsistent inference relations. In *Proceedings of the International Conference on Principles of Knowledge Representation and Reasoning* (KR). Morgan Kaufmann, San Francisco, CA, 61–72.

DA COSTA, N. C. A. 1974. On the theory of inconsistent formal systems. *Notre Dame J. Formal Logic 15*, 497–510.

DAVIS, M., LOGEMANN, G., AND LOVELAND, D. 1962. A machine program for theorem proving. *Commun. ACM 7*, 5, 394–397.

DAVIS, M. AND PUTNAM, H. 1960. A computing procedure for quantification theory. *J. ACM 7*, 3, 201–215.

EGLY, U., EITER, T., TOMPITS, H., AND WOLTRAN, S. 2000. Solving advanced reasoning tasks using quantified Boolean formulas. In *Proceedings of the 16th National Conference on Artificial Intteligence* (AAAI). AAAI Press, 417–422.

EGLY, U., SEIDL, M., TOMPITS, H., WOLTRAN, S., AND ZOLDA, M. 2004. Comparing different prenexing strategies for quantified Boolean formulas. In *Proceedings of the 6th International Conference on Theory and Applications of Satisfiability* (SAT), E. Guinchiglia and A. Tacchella, Eds. Lecture Notes in Computer Science, vol. 2919, Springer-Verlag, 214–228.

EITER, T., LEONE, N., MATEIS, C., PFEIFER, G., AND SCARCELLO, F. 1998. The KR system dlv: Progress report, comparisons and benchmarks. In *Proceedings of the 6th International Conference on*

Principles of Knowledge Representation and Reasoning (KR). Morgan Kaufmann, San Fracisco, CA, 406–417.

FITTING, M. 1990. Kleene's logic, generalized. *Logic Comput. 1*, 797–810.

GINSBERG, M. L. 1988. Multi-Valued logics: A uniform approach to reasoning in AI. *Comput. Intell. 4*, 256–316.

GIUNCHIGLIA, E., NARIZZANO, M., AND TACCHELLA, A. 2001. QuBE: A system for deciding quantified Boolean formulas satisfiability. In *Proceedings of the 1st International Conference on Automated Reasoning* (IJCAR), R. Gor et al., Eds. Lecture Notes in Computer Science, vol. 2083, Springer-Verlag, 364–369.

KIFER, M. AND LOZINSKII, E. L. 1992. A logic for reasoning with inconsistency. *Auto. Reason. 9*, 2, 179–215.

KLEENE, S. C. 1950. *Introduction to Metamathematics*. Van Nostrand, New York.

KLEINE-BÜNING, H., KARPINSKI, M., AND FÖGEL, A. 1995. Resolution for quantified Boolean formulas. *Inf. Comput. 177*, 1, 12–18.

KONIECZNY, S. AND MARQUIS, P. 2002. Three-Valued logics for inconsistency handling. In *Proceedings of the 8th European Conference on Logics in Artificial Intelligence* (JELIA). Lecture Notes in Artificial Intelligence, vol. 2424, Springer-Verlag, 332–344.

KRAUS, S., LEHMANN, D., AND MAGIDOR, M. 1990. Nonmonotonic reasoning, preferential models and cumulative logics. *Artif. Intel. 44*, 1–2, 167–207.

LE BERRE, D., SIMON, L., AND TACCHELLA, A. 2004. Challenges in the QBF arena: The SAT'03 evaluation of QBF solvers. In *Proceedings of the 6th International Conference on Theory and Applications of Satisfiability* (SAT), E. Guinchiglia and A. Tacchella, Eds. Lecture Notes in Computer Science, vol. 2919, Springer-Verlag, 468–485.

LEHMANN, D. AND MAGIDOR, M. 1992. What does a conditional knowledge base entail? *Artif. Intel. 55*, 1–60.

LETZ, R. 2002. Lemma and model caching in decision procedures for quantified Boolean formulas. In *Proceedings of the International Conference on Automated Reasoning with Analytic Tableaux and Related Methods* (TABLEAUX), U. Egly and G. C. Fermüler, Eds. Lecture Notes in Artificial Intelligence, vol. 2381, Springer-Verlag, 160–175.

LIFSCHITZ, V. 1985. Computing circumscription. In *Proceedings of the 9th International Joint Conference on Artificial Intelligence* (IJCAI). Morgan Kaufmann, San Francisco, CA, 121–127.

MAKINSON, D. 1994. General patterns in nonmonotonic reasoning. In *Handbook of Logic in Artificial Intelligence and Logic Programming*, D. Gabbay et al., Eds. Oxford Science Publications, 35–110.

MCCARTHY, J. 1980. Circumscription—A form of non monotonic reasoning. *Artif. Intel. 13*, 1–2, 27–39.

MOSKEWICZ, M., MADIGAN, C., ZHAO, Y., ZHANG, L., AND MALIK, S. 2001. Chaff: Engineering an efficient SAT solver. In *Proceedings of the 39th Design Automation Conference* (DAC). ACM, New York, 530–535.

NIEMELÄ, I. AND SIMONS, P. 1996. Efficient implementation of the well-founded and stable model semantics. In *Proceedings of the Joint International Conference and Symposium on Logic Programming* (JICSLP). MIT Press, Cambridge, MA, 289–303.

PRIEST, G. 1991. Minimally inconsistent LP. *Studia Logica 50*, 321–331.

PRIEST, G. 1989. Reasoning about truth. *Artif. Intel. 39*, 231–244.

RINTANEN, J. T. 1999. Improvements of the evaluation of quantified Boolean formulae. In *Proceedings of the 16th International Joint Conference on Artificial Intelligence* (IJCAI). Morgan Kaufmann, San Francisco, CA, 1192–1197.

SCHLECHTA, K. 2000. Unrestricted preferential structures. *Logic Comput. 10*, 4, 573–581.

SHOHAM, Y. 1987. A semantical approach to nonmonotonic logics. In *Readings in Non-Monotonic Reasoning*, M. L. Ginsberg, Ed. Morgan Kauffman, Los Altos, CA, 227–249.

SHOHAM, Y. 1988. *Reasoning About Change: Time and Causation from the Standpoint of Artificial Intelligence*. MIT Press, Cambridge, MA.

WRATHALL, C. 1976. Complete sets and the polynomial-time hierarchy. *Theoret. Comput. Sci. 3*, 1, 23–33.

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