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Statistical complexity and disequilibrium

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Abstract

We study the concept of disequilibrium as an essential ingredient of a family of statistical complexity measures. We find that Wootters' objections to the use of Euclidean distances for probability spaces become quite relevant to this endeavor. Replacing the Euclidean distance by the Wootters' one noticeably improves the behavior of the associated statistical complexity measure, as evidenced by its application to the dynamics of the logistic map.

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1. Introduction

Jaynes [1,2] has long ago established the relevance of information theory [3] for theoretical physics. Two essential ingredients of Jaynes' program are (i) Shannon's logarithmic information measure I [4]

$$I = -\sum_{i=1}^{N} p_i \ln[p_i], \tag{1}$$

regarded as the general measure of the uncertainty associated to probabilistic physical processes described by the probability distribution $\{p_i,\ i = 1, \dots, N\}$ and (ii) his celebrated maximum entropy principle (MEP) [1,2]. Traversing a separate track, Kolmogorov and Sinai [5] converted information theory [3] into a powerful tool for the study of dynamical systems. The statistical characterization of *deterministic* sources of *apparent* randomness performed by many authors during the intervening years has shed much light into the intricacies of dynamical behavior by describing the unpredictability of dynamical systems using such tools as metric entropy, Lyapunov exponents, and fractal dimension [6]. It is thus possible to (i) detect the presence and (ii) quantify the degree of the deterministic chaotic behavior [6].

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Now, ascertaining the degree of unpredictability and randomness of a system is not automatically tantamount to adequately grasp the correlational structures that may be present, i.e., to be in a position to capture the relationship between the components of the physical system. These structures strongly influence, of course, the character of the probability distribution that is able to describe the physics one is interested in. Randomness, on the one hand, and structural correlations on the other one, are not totally independent aspects of *this* physics. Certainly, the opposite extremes of (i) perfect order and (ii) maximal randomness possess no structure to speak of [7–9]. In between these two special instances a wide range of possible degrees of physical structure exists, degrees that should be reflected in the features of the underlying probability distribution. One would like that they be adequately captured by some functional $\mathcal{F}[\{p_i\}]$ in the fashion that Shannon's I captures randomness. A suitable candidate to this effect has come to be called the *statistical complexity* (see the enlightening discussion of [10]). $\mathcal{F}[\{p_i\}]$ should, of course, vanish in the two special extreme instances mentioned above.

It is the goal of the present effort that of effecting some in depth analysis of one crucial ingredient shared by a family of statistical complexities that have recently received considerable attention in the pertinent literature. We are referring here to the so-called *disequilibrium* Q, that makes the above functional $\mathcal{F}[\{p_i\}]$ to vanish in the case in which maximal randomness is present. It is clear that I vanishes in the case of perfect order [3].

2. A family of statistical complexity measures

In dealing with the concept of statistical complexity one must start by excluding processes that are certainly not complex, such as those which exhibit periodic motion. White noise random process cannot be assumed to be complex, notwithstanding its irregular and unpredictable character, since it does not contain any nontrivial structure. Statistical complexity (SC) has to do with intrincate structures hidden in the dynamics, emerging from a system which itself is much simpler than its dynamics [11]. SC is characterized by the paradoxical situation of complicated dynamics for simple systems. Of course, if the system itself is already complicated enough and contains many different constituent parts, it is obvious that it may support a rather complicated dynamics, but perhaps without the emergence of nitid and typical characteristic patterns [11].

We will be concerned in this communication with a special SC-family. The members of this family purport to be measures of off-equilibrium "order" in nonequilibrium structures that arise spontaneously in certain situations. As stated above, this type of "order" is not the one associated, for instance, with crystal structures, for which the entropy is very small. Biological life is a typical example of the kind of the "new" order one has in mind here, associated with relatively large entropic values. The associated definitions of *statistical complexity measures* can be divided into three categories [12]. The measure can either (a) grow with increasing disorder (decrease as order increases), (b) be quite small for large amounts of the degree of either order or disorder, with a maximum at some intermediate stage, or (c) grow with increasing order (decrease as disorder increases).

2.1. Shiner–Davison–Landsberg complexity measure

Shiner, Davison, and Landsberg (SDL) have recently proposed a *measure of statistical complexity*, based on appropriately defined notions of order and disorder, which has a considerable degree of flexibility in its dependence on these concepts [12]. We will focus attention upon a special choice of the parameters of the SDL statistical complexity measure so that it belongs to the second category mentioned above. The SDL measure is easy to calculate and behaves like an intensive thermodynamical quantity. The possible functional dependencies of the SDL measure encompass those of many earlier definitions of statistical complexity [12].

SDL define, for a given probability distribution $\{p_i\}$ and its associate information measure I [3], an amount of "disorder" H in the fashion $H = I/I_{\max}$, where

$$I_{\max} = I\{\text{uniform probability distribution}\}. \tag{2}$$

Obviously, the associated definition of "order" reads $\Theta = 1 - H$. Both Θ and H lie between 0 and 1. Among the members of the SDL statistical complexity measures (characterized by two parameters α, β) [12] we choose here to work with the $\alpha = \beta = 1$ instance.

One has

$$C^{(\mathrm{SDL})} = H(1-H) = \Theta(1-\Theta). \tag{3}$$

Note that the SDL statistical complexity measure is calculated from the normalized information measure, or entropy. One could raise the objection that $C^{(\mathrm{SDL})}$ is just a simple function of the entropy. As a consequence, it might not contain new information vis-à-vis the measure of order. Such an objection is discussed at length in [13–15].

2.2. *López-Ruiz–Mancini–Calbet statistical complexity measure*

López-Ruiz, Mancini, and Calbet (LMC) have recently proposed a measure of statistical complexity, based on the notion of "disequilibrium" [16–18]. It is rather easy to compute (it is evaluated in terms of common statistical mechanics' concepts). The LMC definition of statistical complexity measure reads

$$C^{(\mathrm{LMC})} = QH, \tag{4}$$

where Q stands for the so-called "disequilibrium" [16] and H has been defined above ($0 \leqslant H \leqslant 1$). Thus, following [16] one defines, in addition to H, the quantity Q as a "distance" in probability space. It measures "how far" $\{p_i\}$ is located, in this space, from the uniform distribution $\{p_e\}$ that characterizes equilibrium in Gibbs' statistical mechanics: $p_e = 1/N$. In his work LMC adopt the definition of Euclidean distance for the evaluation of Q (see Eq. (7)).

It should be noticed that the LMC complexity *is not a trivial function of the entropy*, in the sense that, for a given H-value, there exists a range of complexities between a minimal value $C_{\min}$ and a maximal value $C_{\max}$ [17,18]. Thus, evaluating the complexity provides one with important additional information regarding the peculiarities of a probability distribution.

Landsberg and co-workers showed that their definition of complexity include the López-Ruiz–Mancini–Calbet definition [12]. On the other hand, the LMC complexity definition [16] involves the disequilibrium Q. Alternatively, instead of using the quadratic Euclidean distance between the probability distributions $\{p_i\}$ and $\{p_e\}$, one may take the position that Q is proportional to $S(p|p_e)$, the Kullback–Leibler relative entropy [19]. The Kullback–Leibler measure, for two probability distributions $\{p_i\}$ and $\{q_i\}$, is given by

$$S(p|q) = \sum_i p_i \ln[p_i/q_i], \tag{5}$$

and measures just how similar both probabilities distributions are, taking as a reference the distribution $\{q_i\}$. For $Q \sim S(p|p_e) = (1-H)\ln N$, it easy to see that, starting from the LMC definition we straightforwardly arrive to the SDL one. In summary, the two definitions of statistical complexity that we have here reviewed are intimately linked.

It has been pointed out in [10] that the LMC measure is marred by some troublesome characteristics. For example, it is neither an intensive nor an extensive quantity. Also, it vanishes exponentially in the thermodynamic limit for all one-dimensional, finite-range systems. With reference to the ability of the LMC measure to adequately capture essential dynamical aspects some difficulties have also been encountered in [17]. Our present effort revolves around the idea of improving precisely this aspect of the statistical complexity notion: grasping essential details of the dynamics. To such an end we will analyze one of the two ingredients of the LMC measure: disequilibrium.

3. Distances in probability spaces

3.1. *Disequilibrium measure based on the Euclidean distance*

As stated above, the "disequilibrium" Q is the distance between a given probability distribution $\vec{P} = (p_1, \ldots, p_N)$ and the "equilibrium" one $\vec{P}_e = (1/N, \ldots, 1/N)$:

$$Q(\vec{P}) = Q_0 \mathcal{D}(\vec{P}, \vec{P}_e), \tag{6}$$

with $\mathcal{D}$ an appropriate measure of distance. Q_0 is a normalization constant. If $\mathcal{D}$ is the Euclidean norm in $\mathbb{R}^N$ we get the LMC definition, namely,

$$\begin{aligned} Q_{\mathrm{E}}(\vec{P}) &= Q_0^{(\mathrm{E})} \mathcal{D}_{\mathrm{E}}(\vec{P}, \vec{P}_e) = Q_0^{(\mathrm{E})} \|\vec{P} - \vec{P}_e\|_{\mathrm{E}} \\ &= Q_0^{(\mathrm{E})} \sum_{i=1}^{N} \left\{ p_i - \frac{1}{N} \right\}^2, \end{aligned} \tag{7}$$

with $Q_0^{(\mathrm{E})} = N/(N-1)$ so that $0 \leqslant Q_{\mathrm{E}} \leqslant 1$.

This straightforward definition of distance has been criticized by Wootters in an illuminating communication [20]. Essentially, the Euclidean norm ignores the stochastic nature of the vectors $\vec{P}$. Thus, following Wootters' pioneer essay, we redefine the notion of disequilibrium and recast it à la Wooters.

3.2. Disequilibrium measure based on Wootters' ideas

The concept of *statistical distance* originates in a quantum mechanical context. One uses it primarily to distinguish among different preparations of a given quantum state [20], and, more generally, to ascertain to what an extent two such states differ from one another. The concomitant considerations being of an intrinsic statistical nature, the concept can be applied to *any* probabilistic space [20]. The main idea underlying this notion of distance is that of adequately taking into account statistical fluctuations inherent to any finite sample [20]. As a result of the associated statistical errors, the observed frequencies of occurrence of the various possible outcomes typically differ somewhat from the actual probabilities, with the result that two preparations are indistinguishable in a given fixed number of trials if the difference in the actual probabilities is smaller than the size of a typical fluctuation [20].

After these preliminary consideration let us introduce now the notion of *statistical distance* [20] which, we insist, does not coincide with that for the usual Euclidean distance in probability space $\Omega \subset \mathbb{R}^N$. For instance, if $N = 2$, probabilities near $1/2$ are more difficult to distinguish, owing to greater statistical fluctuations, than probabilities near 0 or 1. The *statistical distance* between two points of an N-dimensional probability space is the itstatistical length of the shortest curve connecting the two points in this space. In turn, the statistical length of a curve is the maximum number of mutually distinguishable points along the curve [20].

More precisely, let $\{\xi_i\}$ be the set of occurrence of outcomes (after n trials) associated with the probability set $\{p_i\}$. This set is distributed according to a multinomial distribution, which can be approximated by a Gaussian one when the number of trials is large enough

$$\rho(\vec{\xi}) \propto \exp\left[-\frac{n}{2}\sum_{n=1}^{N}\frac{(\xi_i - p_i)^2}{p_i}\right]. \tag{8}$$

Following Wootters' work, we define first the "region of uncertainty" around the point $\vec{P}$. This zone is comprised of all the points $\vec{\xi}$ such that the exponent in Eq. (8) is in absolute value less than $1/2$. Accordingly, two points $\vec{P}_1$ and $\vec{P}_2 \in \Omega$ will be called "distinguishable in n trials" if their regions of uncertainty *do not overlap*. For n large enough this will be the case if and only if

$$\frac{\sqrt{n}}{2}\left[\sum_{n=1}^{N}\frac{(p_i^{(1)} - p_i^{(2)})^2}{p_i^{(1)}}\right]^{1/2} > 1. \tag{9}$$

Consider now a smooth curve parameterized by t that connects the two points ($\vec{P}(0) = \vec{P}_1$ and $\vec{P}(1) = \vec{P}_2$). Let us denote it by $\vec{P}(t) \in \Omega$ with $0 \leqslant t \leqslant 1$. Its statistical length (in probability space) [20], according to the criterion given by Eq. (9) is [20]

$$\mathcal{L} = \frac{1}{2}\int_0^1 dt \left\{\sum_{n=1}^{N}\frac{1}{p_i(t)}\left[\frac{dp_i(t)}{dt}\right]^2\right\}^{1/2}, \tag{10}$$

that, with the change of variables $x_i = (p_i)^{1/2}$, becomes simply

$$\mathcal{L} = \int_0^1 dt \left[\sum_{n=1}^{N}\left(\frac{dx_i}{dt}\right)^2\right]^{1/2}. \tag{11}$$

This is the usual Euclidean length of the curve in x-space. The requirement that the curve $\vec{P}(t) \in \Omega$ is given by $1 = \sum_{i=1}^{N} p_i(t) = \sum_{i=1}^{N} x_i^2(t)$. Thus, the curve $\vec{X}(t)$ must lie on the unit sphere in the x-space. The statistical distance between $\vec{P}_1$ and $\vec{P}_2$ is therefore the shortest distance, on the unit sphere, between the unit vectors $\vec{X}_1$ and $\vec{X}_2$. This shortest distance is equal to the angle between the above two vectors and is given by [20]

$$\mathcal{D}_W(\vec{P}_1, \vec{P}_2) = \cos^{-1}\left\{\sum_{n=1}^{N}\left[p_i^{(1)}\right]^{1/2}\left[p_i^{(2)}\right]^{1/2}\right\}, \tag{12}$$

that vanishes when the two probability distributions coincide, and attains its maximum value ($\pi/2$) when the two vectors are orthogonal, i.e., when each outcome which has a positive probability according to one distribution has zero probability according to the other. As emphasized by Wootters [20], this is the most natural notion of distance on probability space, since it takes into account the actual difficulty of distinguishing different probabilistic experiments.

Inspired by Wootters [20] we thus rewrite the disequilibrium Q on the basis of a statistical distance specifically designed for a probability space Ω. Using the above results the Wootters' disequilibrium reads

$$Q_{\mathrm{W}}(\vec{P}) = Q_0^{(\mathrm{W})}\mathcal{D}_{\mathrm{W}}(\vec{P}, \vec{P}_e) = Q_0^{(\mathrm{W})} \cos^{-1}\left\{ \sum_{i=1}^{N} [p_i]^{1/2} \left[\frac{1}{N}\right]^{1/2} \right\}, \qquad (13)$$

with $Q_0^{(\mathrm{W})} = 1/\cos^{-1}\{[1/N]^{1/2}\}$ and $0 \leqslant Q_{\mathrm{W}} \leqslant 1$.

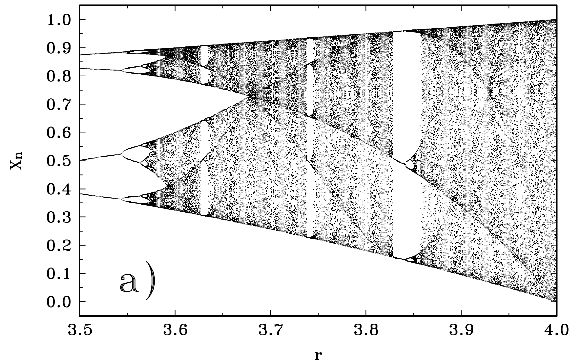
4. Application

We will compare the Wootters' SC ($C_{\mathrm{W}}^{(\mathrm{LMC})} = Q_{\mathrm{W}} H$) to the LMC one ($C_{\mathrm{E}}^{(\mathrm{LMC})} = Q_{\mathrm{E}} H$) with reference to the logistic map, in order to determine whether the former allows for some more detailed grasping of the dynamics than the latter. We deal with the map $F : x_n \to x_{n+1}$, described by the ecologically motivated, dissipative system described by the first order difference equation

$$x_{n+1} = r x_n (1 - x_n) \quad (0 \leqslant x_n \leqslant 1,\ 0 < r \leqslant 4). \qquad (14)$$

The dynamical behavior is controlled by r. Fig. 1(a) shows the well-known bifurcation diagram for the logistic map for $3.5 \leqslant r \leqslant 4.0$. In Fig. 1(b) the corresponding Lyapunov exponent is shown. For $r > r_\infty$ the orbit-diagram reveals an "strange" mixture of order and chaos. The onset of chaos is apparent near $r_\infty \approx 3.569946$, where λ first becomes positive, reflecting the exponential divergence of trajectories. For $r > r_\infty$ the Lyapunov exponent increases globally, except for the dips one sees in the windows of periodic behavior.

Let us revisit now the LMC treatment of the logistic map [16,17]. Following [16], for each parameter value, r, the dynamics of the logistic map was reduced to a binary sequence (0 if $x \leqslant 1/2$; 1 if $x > 1/2$) and binary strings of length 12 were considered as states of



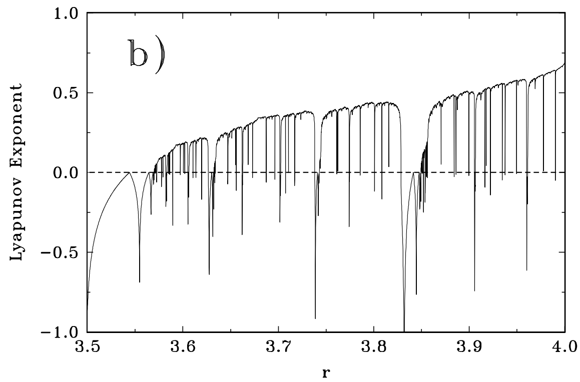


Fig. 1. (a) Orbit diagram for the logistic map as function of parameter r. (b) Lyapunov exponent for the logistic map as function of parameter r.

the system. The concomitant probabilities are assigned according to the frequency of occurrence after running over at least 2^{22} iterations.

LMC pay particular attention to the vicinity of $r = 3.8284$, where a transition from chaos to a 3-periodic orbit (via intermittence) takes place It is our intention that of performing here a more general analysis. As a consequence, we study the whole region $r_\infty \cong 3.57 < r < 4.0$, and effect a comparison between the descriptions provided by the original LMC statistical complexity measure, see Fig. 2 and the one we are advancing here, whose results are displayed in Fig. 3. Remember that our proposal is to replace Euclidean disequilibrium by the Wootters' one.

We observe, in both instances (Euclidean and Wootter's), an abrupt complexity growth around $r = 3.56$, with maxima in the vicinity of $r > r_\infty \cong 3.57$. After we pass this point, the two measures behave in quite different fashions. Notwithstanding the fact that both measures almost vanish within the large stability

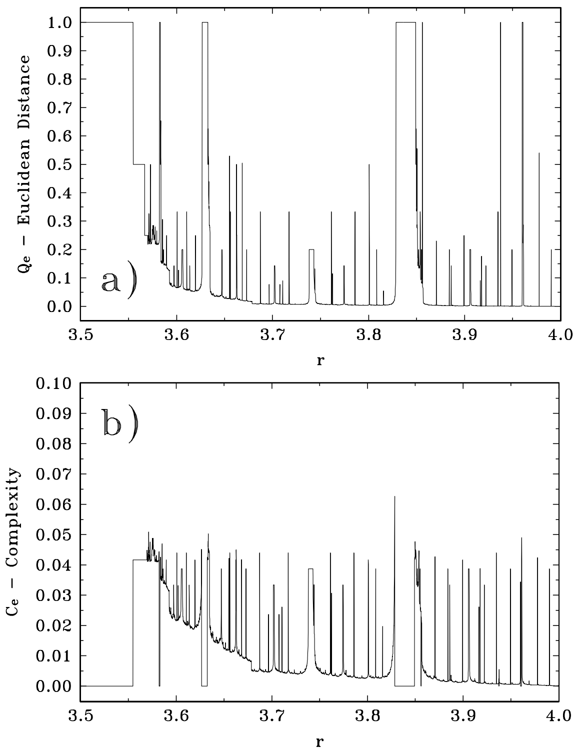


Fig. 2. (a) Euclidean disequilibrium (Q_{E}) and (b) LMC complexity—Euclidean ($C_{\mathrm{E}}^{(\mathrm{LMC})}$) evaluated for the logistic map (binary sequence) as function of parameter r.

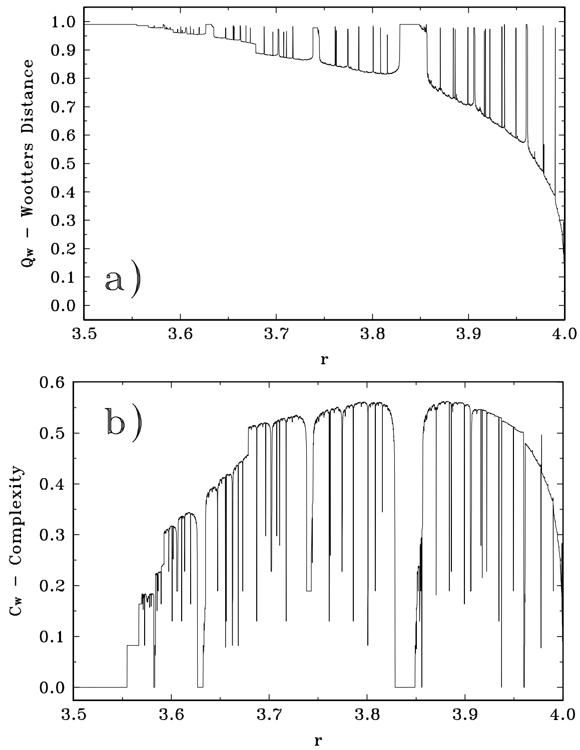


Fig. 3. (a) Wootters disequilibrium (Q_{W}) and (b) LMC complexity—Wootters ($C_{\mathrm{W}}^{(\mathrm{LMC})}$) evaluated for the logistic map (binary sequence) as function of parameter r.

windows, in the inter-windows region they noticeably differ.

In the latter zone, the original LMC measure globally decreases, reaching almost null values. Consider, for example, $r \in [3.58 < r < 3.62]$ in Fig. 2. The many peaks indicate a local complexity growth. Comparison with the orbit-diagram (see Fig. 1(a)) indicates that the peaks coincide with the periodic windows. The conclusion is inescapable. The original LMC measure regards this periodic motion as having a more complex character than that pertaining to the neighboring chaotic zone. Notice instead that the Wootters' measure does grow in the inter-windows region and rapidly falls within the periodic windows (see Fig. 3).

The arbiter that passes judgment on the controversy between the two measures is the behavior of the Lyapunov exponent (see Fig. 1(b)). The λ (and, as a result, the associated degree of chaoticity), grow with r, reaching a maximum at $r = 4$. One would expect that a sensible statistical complexity measure should accompany such a growth.

In other words, a reasonable statistical complexity measure should take very small values at $r = r_\infty$ and then grow together with the degree of chaoticity till, the chaoticity becoming too large, the complexity falls again. When chaos totally prevails, the statistical complexity should vanish. Our à la Wootters modified measure behaves precisely in this fashion, as one clearly appreciates in Fig. 3.

5. Conclusions

The notion of statistical complexity advanced by López-Ruiz, Mancini, and Calbet (LMC) [16] constitutes an important step towards building up an armory of measures that are both easily to compute and to intuitively grasp. It exhibits, nonetheless, some difficulties that have been nitidly pointed out in, for instance, Refs. [10,17]. As a further step in the direction inaugurated by López-Ruiz, Mancini, and Calbet, we have here called attention to the mind-opening study of Wootters' [20] regarding the notion of distance in probability spaces.

Following the suggestions of [20], we have revisited the LMC statistical complexity measure and replaced their definition by one that incorporates, instead of the Euclidean disequilibrium, the Wootters' one. Application of the ensuing statistical complexity measure to the logistic map shows that important improvements are thereby achieved. The new measure does behave in a manner compatible with that of the Lyapunov exponents, while the LMC one does not. We recognize the fact that not all the objections of Ref. [10] have been here overcome. However, it is our contention that the LMC path is a valid one, worth pursuing into some more depth.

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