# 1 Proof by Induction

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DONE

1) Base case
2) Assume n=k
3) Prove n=k+1
4) Conclusion

## Divisibility

> Prove $2^{n+2}+3^{2n+1}$ divisible by 7 for all $n \geq 1$

Base case n=1

$2^{(1)+2}+3^{2(1)+1} = 35$

Assume true for n=k

$2^{k+2}+3^{2k+1} = 7a$

$\Rightarrow 2^{k+2} = 7a - 3^{2k+1}$

Prove true for n=k+1

$2^{k+3}+3^{2(k+1)+1} = 2^{k+3}+3^{2k+3}$

Use assumption

$= 2(7a - 3^{2k+1}) + 3^{2k+3}$

$= 7(2a) - 2(3^{2k+1}) + 9(3^{2k+1})$

$= 7(2a + 3^{2k+1})$

$\therefore$ divisible by 7

Conclusion

If n=k is true, then n=k+1 is true. As true for n=1, $\therefore 2^{n+2}+3^{2n+1}$ is divisible by 7 for all values $n \geq 2, n \in \mathbb{Z}$ Q.E.D //

## Sums of Series

> Prove $\sum_{r=1}^{n} r^2 = \frac{n}{6}(n+1)(2n+1)$

Base case n=1

LHS: $\sum_{1}^{1} 1^2 = 1$ RHS: $\frac{1}{6}(2)(3) = 1$

LHS = RHS $\therefore$ true for base case

Assume true for n=k

$\sum_{1}^{k} r^2 = \frac{k}{6}(k+1)(2k+1)$

Prove true for n=k+1

RHS: $\frac{k+1}{6}(k+2)(2k+3)$

LHS: $\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k+1)^2$

Use assumption

$= \frac{k}{6}(k+1)(2k+1) + (k+1)^2$

$= \frac{k+1}{6}(k(2k+1) + 6k + 6)$

$= \frac{k+1}{6}(2k^2 + 7k + 6)$

$= \frac{k+1}{6}(2k+3)(k+2)$

RHS = LHS $\therefore$ true for n=k+1

Conclusion

If true for n=k, then true for n=k+1. Shown true for n=1 hence true for all $n \geq 1, n \in \mathbb{Z}$ Q.E.D //

## Inequalities

> Prove $P(n): 2^n > n+4$ for $n \geq 3$

Base case n=3

LHS: $2^3 = 8$ RHS: $3+4 = 7$

$\therefore$ LHS > RHS true for n=3

Assume true for n=k

$2^k > k+4$

Prove true for n=k+1

LHS: $2^{k+1}$ RHS: $k+5$

$2^{k+1} = 2(2^k) > 2(k+4)$

Use assumption

$\therefore 2^{k+1} > 2k+8 > k+5$
LHS RHS

$\therefore 2^{k+1} > k+5$

$\therefore$ LHS > RHS true for n=k+1

Conclusion

If true for n=k, then true for n=k+1. As true for n=3, then true for $n \geq 3, n \in \mathbb{Z}$ Q.E.D //

## Matrices

> $A = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$, Prove $A^n = \begin{pmatrix} 1 & 2^{n+1}-2 \\ 0 & 2^n \end{pmatrix}$

Base case n=1

$A^1 = \begin{pmatrix} 1 & 2^{1+1}-2 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

Assume true for n=k

$A^k = \begin{pmatrix} 1 & 2^{k+1}-2 \\ 0 & 2^k \end{pmatrix}$

Prove true for n=k+1

$A^{k+1} = A^k A$

Use assumption

$= \begin{pmatrix} 1 & 2^{k+1}-2 \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2+2(2^{k+1}-2) \\ 0 & 2^k \times 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 2+2^{k+2}-4 \\ 0 & 2^{k+1} \end{pmatrix}$

$= \begin{pmatrix} 1 & 2^{k+2}-2 \\ 0 & 2^{k+1} \end{pmatrix}$

If true for n=k, then true for n=k+1. As true for n=1, then true for $n \geq 1, n \in \mathbb{Z}$ Q.E.D //

## Sequences

> $u_1 = -1$, $u_{n+1} = \frac{3u_n}{5u_n+6}$, for $n \geq 1$

Prove $u_n = \frac{3}{2^n-5}$

Base case n=1

LHS: $u_1 = -1$ from start

RHS: $\frac{3}{2-5} = -1$ $\therefore$ RHS = LHS

Assumption n=k

$u_k = \frac{3}{2^k-5}$

Prove true for n=k+1

RHS: $\frac{3}{2^{k+1}-5}$

LHS: $u_{k+1} = \frac{3u_k}{5u_k+6}$

Use assumption

$= 3\left(\frac{3}{2^k-5}\right) \div \left(5\left(\frac{3}{2^k-5}\right)+6\right)$

$= \left(\frac{9}{\cancel{2^k-5}}\right)\left(\frac{\cancel{2^k-5}}{15+6(2^k-5)}\right)$

$= \frac{3}{2^{k+1}-5}$

Conclusion

If true for n=k, then true for n=k+1. Shown true for n=1, $\therefore$ true for all $n \geq 1$, $n \in \mathbb{Z}$ Q.E.D //

De Moivres Theorem

$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$