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Paper: CVA and vulnerable options pricing by correlation expansions.[Link]

A. *Idea and Abstract:*

The paper considers the pricing problem for financial options subject to counter-party credit risk. The impact of a credit event is quantified by the Credit Value Adjustment, which is modeled in a stochastic intensity framework. Thus CVA is represented as the expectation of the derivative's payoff discounted with a rate given by the sum of the risk-free and of the default intensity. Wrong Way Risk is accounted for by considering positive dependence between the exposure and the default event. The calculation maybe tackled by Monte Carlo methods once the dynamics of the stochastic state variables *(risk-free rate and default intensity)* are chosen, but it is computationally very expensive. As an alternative, this paper proposes the correlation expansion method to evaluate CVA with WWR, when the underlying and the intensity dynamics are respectively given by a geometrical Brownian motion and a CIR process.

B. *CVA Evaluation of Vulnerable Options::*

We consider a finite time interval $[0,T]$ and a complete probability space $(\Omega, \mathcal{F}, P)$, endowed with a filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, augmented with the P-null sets and made right continuous. We assume that all the processes have a cádlág version.
The market is described by the interest rate process r_t determining the money market account denoted by $B(t,s) = \mathrm{e}^{\int_t^s r_u du}$ and by a process X_t representing an asset log-price (whose dynamics will be specified later), this process may depend also on other stochastic factors. We assume

- that the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$ is rich enough to support all the aforementioned processes;
- to be in absence of arbitrage;
- that the given probability P is a risk neutral measure, already selected by some criterion.

In this market a defaultable European contingent claim paying $f(X_T)$ at maturity is traded, where f is a function whose regularity properties will be specified later. τ (not necessarily a stopping time w.r.t. the filtration $\mathcal{F}_t$) denotes the default time of the contingent claim and Z_t is an $\mathcal{F}_t$-measurable bounded recovery process.
Evaluating this derivative we need to include the information generated by the default time. We denote by $\mathcal{G}_t$ the progressively enlarged filtration, that makes τ a $\mathcal{G}_t$-stopping time, that is $\mathcal{G}_t = \mathcal{F}_t \vee \sigma(\{\tau \le t\})$.
From now on, we indicate by $H_t = \mathbf{1}_{\{\tau\le t\}}$, the process generating the filtration $\mathcal{H}_t$, so that $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$.
We make the assumption, known as the `H-hypothesis` that

`(H) Every` $\mathcal{F}_t-$ `martingale remains a` $\mathcal{G}_t-$ `martingale.`

Under this assumption, $\mathrm{e}^{X_s}/B(t,s)$ for $s \ge t$ remains a $\mathcal{G}_s$-martingale under the unique extension of the risk neutral probability to the filtration $\mathcal{G}_s$.
In this setting, for any given time $t \in [0,T]$, the price of a defaultable claim, with positive final value $f(X_T)$, default time τ and recovery process Z_t, is given by

$$c^d(t,T) = \mathbf{E}\left[B^{-1}(t,T) f(X_T) 1_{\{\tau>T\}} + B^{-1}(t,\tau) Z_\tau 1_{\{t<\tau\le T\}} \mid \mathcal{G}_t\right] \tag{1}$$

while the corresponding default free value is

$$c(t,T) = \mathbf{E}\left[B^{-1}(t,T) f(X_T) \mid \mathcal{F}_t\right] \tag{2}$$

Correspondingly the CVA, as a function of the running time and of the maturity, is given by

$$CVA(t,T) = \mathbf{E}\left(B^{-1}(t,\tau)Z_\tau 1_{\{t<\tau\leq T\}} \mid \mathcal{G}_t\right) = 1_{\{\tau>t\}}\left[c(t,T) - c^d(t,T)\right] \tag{3}$$

In many situations, investors do not know the default time and they may observe only whether it happened or not. The actual observable quantity is the asset price, therefore the pricing formula (1) in terms of $\mathcal{F}_t$, rather than in terms of $\mathcal{G}_t$.

`Lemma: For any integrable` $\mathcal{G}$`-measurable r.v.` Y`, the following equality holds:`

$$\mathbf{E}\left[\mathbf{1}_{\{\tau>t\}}Y \mid \mathcal{G}_t\right] = P\left(\tau > t \mid \mathcal{G}_t\right)\frac{\mathbf{E}\left[\mathbf{1}_{\{\tau>t\}}Y \mid \mathcal{F}_t\right]}{P\left(\tau > t \mid \mathcal{F}_t\right)} \tag{4}$$

Applying this lemma to the first and the second term of (1) and observing that $1 - H_t = \mathbf{1}_{\{\tau>t\}}$ is $\mathcal{G}_t$-measurable, we obtain

$$\mathbf{E}\left[B^{-1}(t,T)f\left(X_T\right)1_{\{\tau>T\}} \mid \mathcal{G}_t\right] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbf{E}\left[B^{-1}(t,T)f\left(X_T\right)1_{\{\tau>T\}} \mid \mathcal{F}_t\right]}{P\left(\tau > t \mid \mathcal{F}_t\right)} \tag{5}$$

$$\mathbf{E}\left[B^{-1}(t,\tau)Z_\tau 1_{\{t<\tau\leq T\}} \mid \mathcal{G}_t\right] = \mathbf{1}_{\{\tau>t\}}\frac{\mathbf{E}\left[B^{-1}(t,\tau)Z_\tau 1_{\{t<\tau\leq T\}} \mid \mathcal{F}_t\right]}{P\left(\tau > t \mid \mathcal{F}_t\right)} \tag{6}$$

We denote the conditional distribution of the default time τ given $\mathcal{F}_t$ by

$$F_t = P\left(\tau \leq t \mid \mathcal{F}_t\right), \quad \forall t \geq 0,$$

where, for $u \geq t, P\left(\tau \leq u \mid \mathcal{F}_t\right) = \mathbf{E}\left(P\left(\tau \leq u \mid \mathcal{F}_u\right) \mid \mathcal{F}_t\right) = \mathbf{E}\left(F_u \mid \mathcal{F}_t\right)$. If $F_t(\omega) < 1$ for all $t > 0$ (which automatically excludes that $\mathcal{G}_t \equiv \mathcal{F}_t$), we can well define the so called $\mathcal{F}$ - hazard process of τ as

$$\Gamma_t := -\ln\left(1 - F_t\right) \quad \Rightarrow \quad F_t = 1 - \mathrm{e}^{-\Gamma_t} \quad \forall t > 0$$

moreover below we have the $\mathcal{F}$-survival process.

$$S_t := 1 - F_t = \mathrm{e}^{-\Gamma_t} \quad \forall t > 0, \quad S_0 = 1,$$

We assume Γ_t to be differentiable. Its derivative, the intensity process denoted by λ_t, is such that $\Gamma_t = \int_0^t \lambda_u du$. Exploiting (5) and (6) to pass to the $\mathcal{F}_t$ filtration and assuming that $B(t,\cdot)Z.$ is a bounded $\mathcal{F}-$martingale , we may rewrite the pricing formula (1) as obtained by modeling directly the random time τ.

$$c^d(t,T) = 1_{\{\tau>t\}}\mathbf{E}\left[\mathrm{e}^{-\int_t^T(r_s+\lambda_s)ds}f\left(X_T\right) \mid \mathcal{F}_t\right] + 1_{\{\tau>t\}}\mathbf{E}\left[\int_t^T Z_s\lambda_s\mathrm{e}^{-\int_t^s(r_u+\lambda_u)du}ds \mid \mathcal{F}_t\right],$$

This formula can be specialized even further if we assume fractional recovery, $Z_t = Rc(t,T)$ for some $0 \leq R < 1$. Using the Optional Projection Theorem,

$$c^d(t,T) = 1_{\{\tau>t\}}\left[R\mathbf{E}\left[\mathrm{e}^{-\int_t^T r_u du}f\left(X_T\right) \mid \mathcal{F}_t\right] + (1-R)\,\mathbf{E}\left[\mathrm{e}^{-\int_t^T(r_u+\lambda_u)du}f\left(X_T\right) \mid \mathcal{F}_t\right]\right] \tag{7}$$

which can be interpreted as a convex combination of the default free price and the price with default. As a consequence, from (3) we have an expression also for the unilateral CVA as

$$CVA(t,T) = 1_{\{\tau>t\}}(1-R)\mathbf{E}\left[\mathrm{e}^{-\int_t^T r_u du}f\left(X_T\right)\left(1 - \mathrm{e}^{-\int_t^T \lambda_u du}\right) \mid \mathcal{F}_t\right] \tag{8}$$

And finally, under independence between λ_t and (X_t, r_t), the second term in (7) simplifies further to

$$\mathbf{E}\left[\mathrm{e}^{-\int_t^T (r_s+\lambda_s)ds} f\left(X_T\right) \mid \mathcal{F}_t\right] = \mathbf{E}\left[\mathrm{e}^{-\int_t^T r_s ds} f\left(X_T\right) \mid \mathcal{F}_t\right] \mathbf{E}\left[\mathrm{e}^{-\int_t^T \lambda_s ds} \mid \mathcal{F}_t\right] \tag{9}$$

Correspondingly, we get a similar factorization for the CVA

$$\begin{aligned} CVA(t,T) &= 1_{\{\tau>t\}}(1-R)\mathbf{E}\left[\mathrm{e}^{-\int_t^T r_u du} f\left(X_T\right) \mid \mathcal{F}_t\right] \mathbf{E}\left[\left(1-\mathrm{e}^{-\int_t^T \lambda_u du}\right) \mid \mathcal{F}_t\right] \\ &= 1_{\{\tau>t\}}(1-R)c(t,T)\frac{P\left(t<\tau\leq T \mid \mathcal{F}_t\right)}{P\left(\tau\geq t \mid \mathcal{F}_t\right)} \end{aligned}$$

In this case, the two factors are respectively the price of a European derivative and the price of a bond. Thus we may arrive at explicit formulas whenever the models for X and λ are appropriately chosen.

B. The Model

We assume that in the given probability space, the following diffusion dynamics are satisfied

$$\begin{aligned} X_s &= x + \int_t^s \left(r_u - \tfrac{\sigma^2}{2}\right) du + \sigma\left(B_s - B_t\right), \quad x\in\mathbb{R} \\ \lambda_s &= \lambda + \int_t^s \gamma\left(\theta-\lambda_u\right) du + \eta\int_t^s \sqrt{\lambda_u} dY_u, \quad \lambda>0 \\ r_s &= r + \int_t^s k\left(\mu - r_u\right) du + \nu\left(W_s - W_t\right), \quad r>0, \end{aligned}$$

where the parameters are such that $k,\theta,\eta,\sigma,\mu>0,\gamma,\nu\geq 0, 2k\theta>\eta^2$ and B,Y,W are correlated Brownian motions with a given correlation matrix. To simplify calculations, we assume independence between the interest rate and default intensity, i.e. between Y and W; with this choice we may represent the triple B,Z,W as

$$B_t = \rho B_t^1 + \delta B_t^2 + \sqrt{1-\rho^2-\delta^2} B_t^3, \quad Y_t = B_t^1, \quad W_t = B_t^2;$$

where (B^1, B^2, B^3) is a 3 -dimensional Brownian motion and $\delta^2+\rho^2\leq 1$. Under independence we have an explicit expression of the factor $E\left[\mathrm{e}^{-\int_t^T \lambda_s ds} \mid \mathcal{F}_t\right]$ in (9), being the bond price with a CIR interest rate. The problem is then reduced to computing the other factor representing the price of the European derivative.

C. Correlation Expansion:

For simplicity, we assume $R=0$ and the short rate to be constant, $r_t\equiv r$ for all $t\in[0,T]$. The model, which we write in flow notation, is hence reduced to

$$\begin{cases} X_s^{t,x,\lambda} = x + \left(r-\frac{\sigma^2}{2}\right)(s-t) + \sigma\left[\rho\left(B_s^1 - B_t^1\right) + \sqrt{1-\rho^2}\left(B_s^2 - B_t^2\right)\right] \\ \lambda_s^{t,\lambda} = \lambda + \int_t^s \gamma\left(\theta - \lambda_u^{t,\lambda}\right) du + \int_t^s \eta\sqrt{\lambda_u^{t,\lambda}} dB_u^1 \end{cases}$$

The two-dimensional diffusion $\mathbf{U}_t^{t,x,\lambda,\rho} := \left(X_s^{t,x,\lambda}, \lambda_s^{t,\lambda}\right)$ is a Markov process since the coefficients,

$$\mu(x,\lambda) := \begin{pmatrix} r-\frac{\sigma^2}{2} \\ \gamma(\theta-\lambda) \end{pmatrix}, \quad \text{and} \quad \Sigma(x,\lambda) := \begin{pmatrix} \sigma\rho & \sigma\sqrt{1-\rho^2} \\ \eta\sqrt{\lambda} & 0 \end{pmatrix}$$

are deterministic. This implies that the price $c^d(t,T)$ of any European defaultable derivative with payoff $F\left(\mathbf{U}_T^{t,x,\lambda,\rho}\right)$ will be a deterministic function $u(\cdot)$ of all the initial data, that is

$$u(x,\lambda,t,T;\rho) = \mathrm{e}^{-r(T-t)}\mathbf{E}\left(\mathrm{e}^{-\int_t^T \lambda_s^{t,\lambda} ds} F\left(\mathbf{U}_T^{t,x,\lambda,\rho}\right)\right).$$

By the Feymann-Kac formulas, $u(x, \lambda, t, T; \rho)$ solves the parabolic PDE

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{L}^\rho u = 0 \\ u(x, \lambda, T, T; \rho) = (\mathrm{e}^x - K)^+, \end{cases}$$

where we denoted $\mathcal{L}^\rho = \mathcal{L}^0 + \rho \mathcal{A}$, with

$$\mathcal{L}^0 := \frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \frac{\eta^2 \lambda}{2}\frac{\partial^2}{\partial \lambda^2} + \left(r - \frac{\sigma^2}{2}\right)\frac{\partial}{\partial x} + \gamma(\theta - \lambda)\frac{\partial}{\partial \lambda} - r - \lambda$$
$$\mathcal{A} := \eta\sigma\sqrt{\lambda}\frac{\partial^2}{\partial x \partial \lambda}.$$

By the Markov property and Feymann-Kac formulas, $g_0(x, \lambda, t, T)$ we have the following representation

$$g_0(x, \lambda, t, T) = \mathrm{e}^{-r(T-t)}\mathbf{E}\left(\mathrm{e}^{-\int_t^T \lambda_s^{t,\lambda} ds}\left(\mathrm{e}^{X_T^{t,x,\lambda}} - K\right)^+\right) = \mathbf{E}\left(\mathrm{e}^{-\int_t^T \lambda_s^{t,\lambda} ds}\right)\mathrm{e}^{-r(T-t)}\mathbf{E}\left(\left(\mathrm{e}^{X_T^{t,x}} - K\right)^+\right)$$

D. *CVA and the change of measure approach:*

Using Girsanov's theorem, in the stochastic-intensity default setup the starting point is the following formula for the time-zero CVA of portfolio price process V_t :

$$CVA(0, T) = -(1 - R)\int_0^T \mathbf{E}\left[\frac{V_t^+}{B(0, t)}\zeta_t\right] dG(t) \tag{10}$$

where $\mathbf{E}[.]$ is the expectation under the risk-neutral measure. The EPE (expected positive exposure) under WWR is the function

$$EPE(t) = \mathbf{E}\left[\frac{V_t^+}{B(0, t)}\zeta_t\right].$$

Girsanov's theorem is used to factorize the EPE. By defining an equivalent martingale measure $Q^{C^{\mathcal{F},t}} \sim Q$ and the following as:

$$Z_s^t := \frac{dQ^{C^{\mathcal{F},t}}}{dQ} = \frac{M_s^t}{M_0^t}, \qquad \text{where } M_s^t = \mathbf{E}\left[\frac{1}{B(0, t)}\lambda_t S_t \mid \mathcal{F}_s\right], s \in [0, t],$$

$$\mathbf{E}\left[\frac{V_t^+}{B(0, t)}\zeta_t\right] = \mathbf{E}^{C^{\mathcal{F},t}}\left[V_t^+\right]\mathbf{E}\left[\frac{\zeta_t}{B(0, t)}\right]$$

The measure $Q^{C^{\mathcal{F},t}}$ is called wrong-way measure and it is associated to the measure $C^{\mathcal{F},t} = B(0, \cdot)M_\cdot^t$. The dynamics of V_t under the measure $Q^{C^{\mathcal{F},t}}$ by assuming a continuous dynamic for V_t under Q described by a SDE is as follows, the change of measure results in a drift adjustment; the risk free rate being constant implies that $\mathbf{E}\left[B(0, t)^{-1}\zeta_t\right] = -\mathrm{e}^{-rt}$. The explicit expression of the new drift is

$$\theta_t^s \equiv \theta_t^s(\lambda_t) = \rho\eta\sqrt{\lambda_t}\left(\frac{A^\lambda(s, t)B_t^\lambda(s, t)}{A^\lambda(s, t)B_t^\lambda(s, t)\lambda_t - A_t^\lambda(s, t)} - B^\lambda(s, t)\right),$$

The expression $EPE(t) = -\mathrm{e}^{-rt}\mathbf{E}^{C^{-o*}}[c(t, T)]$ can be evaluated analytically leading to

$$E^{C^{\mathcal{F},t}}\left[\frac{c(t, T)}{B(0, t)}\right] \approx \mathrm{e}^{x_0 + \sigma\Theta_t}N\left(\frac{\hat{\alpha}(t) + \beta(t)\sigma\sqrt{t}}{\sqrt{1 + \beta^2(t)}}\right) - \mathrm{e}^{\kappa - rT}N\left(\frac{\hat{\alpha}(t) - \sigma\sqrt{T - t}}{\sqrt{1 + \beta^2(t)}}\right) \tag{11}$$

where

$$\Theta(t) = \int_0^t \theta(u,t)du, \quad \theta(u,t) = \theta_u^t(\lambda(u)), \quad \hat{\alpha}(t) = \alpha(t) + \frac{\Theta_t}{\sqrt{T-t}}$$

$$\alpha(t) = \frac{1}{\sigma\sqrt{T-t}}\left(x_0 - \kappa + \left(r + \frac{\sigma^2}{2}\right)T - \sigma^2 t\right), \quad \beta(t) = \sqrt{\frac{t}{T-t}}$$

Two deterministic proxies $\lambda(t)$ are considered: $\mathbf{E}[\lambda_t]$ and $\mathbf{E}^{C^{\mathcal{F},t}}[\lambda_t]$. The first is analytically known, the second requires an approximation step. Inserting (11) in (10) a numerical integration procedure gives the CVA under WWR.

1 Extension Work and Results

The model described earlier is simulated with $S_0 = 100, \sigma = 1, \rho = 0.3, T = 1, \lambda = 0.04, \gamma = 0.2, K = 100$

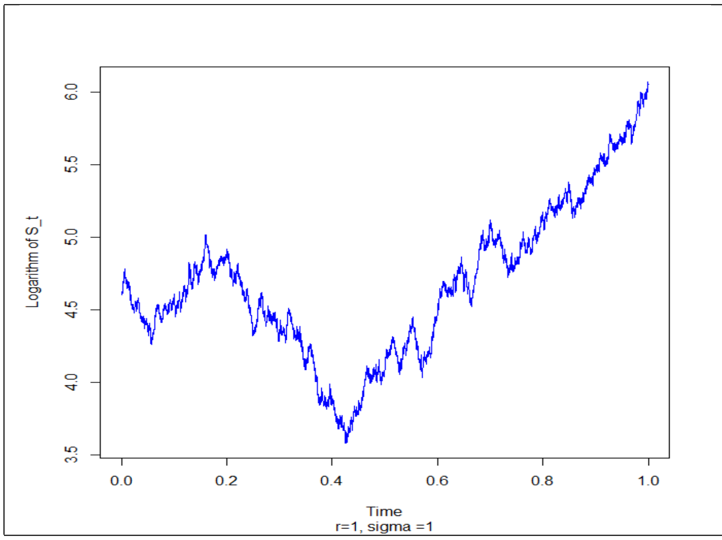
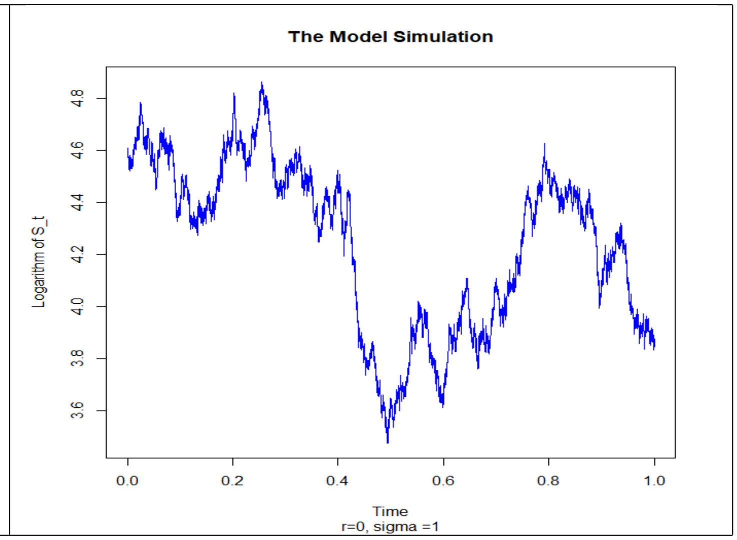
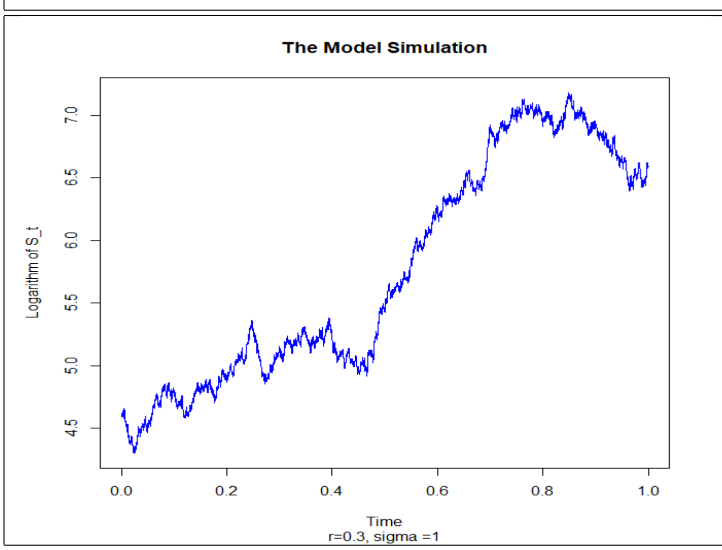
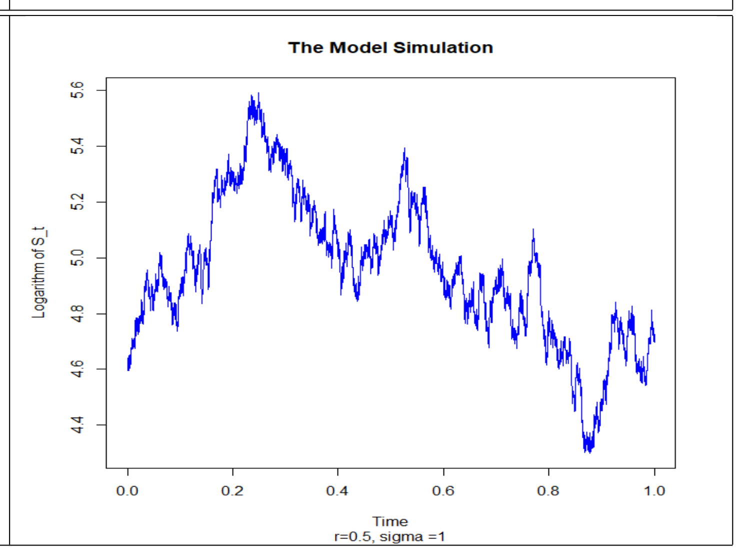
Plot 1	Plot 2
	
Plot 3	**Plot 4**
	

Table 1: Simulation Results for the Logarithm of the Stock Price

```
Description of The Plots
```

- The CVA is calculated for fixed parameters at different values of the volatility η and the results are noted as follows. The correlation expansion method was used to obtain the plots which is computationally efficient as compared to Monte Carlo methods.

- The next part shows the variation of the CVA with different values of θ. The variation is observed at two different values of volatility η and the results are as follows.

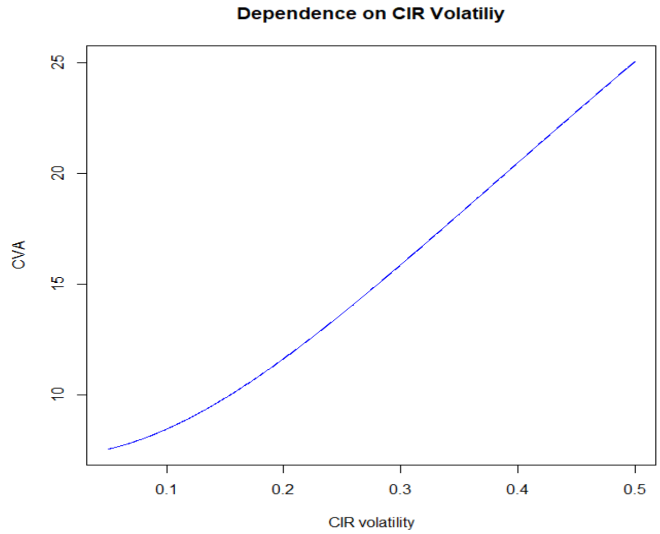


Table 2: Dependence on CIR Volatility

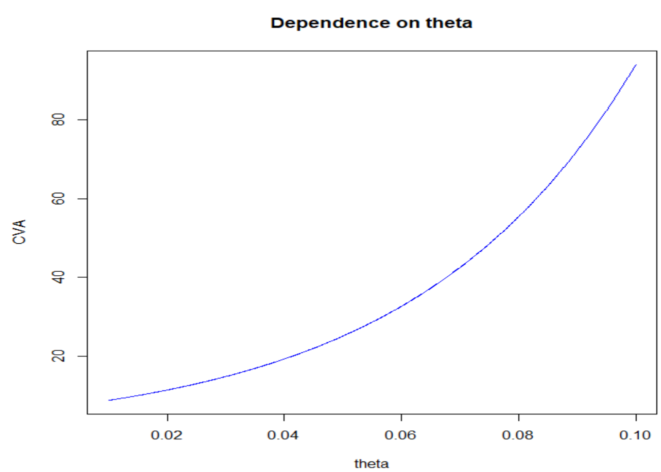
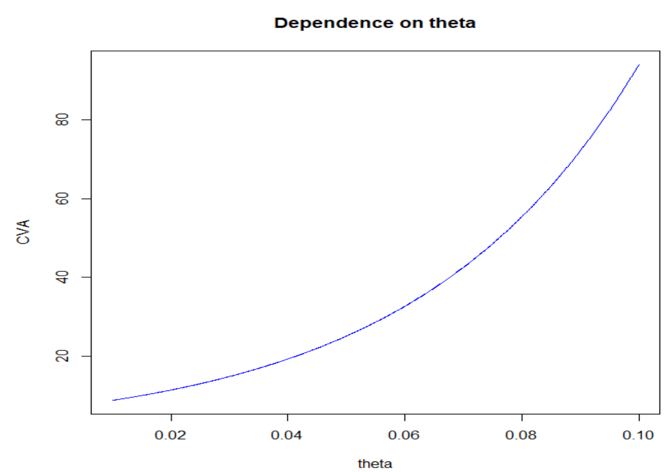
$\eta = 0.3$	$\eta = 0.5$
	

Table 3: Dependence of CVA on θ

2 Comments

- The Credit Value Adjustment is a important feature in the financial field to understand counter party credit risk.
- The model used consists of an intensity process that is correlated with the stock price.
- The CVA is approximated effectively with the correlation expansion method which is faster than general Monte Carlo and Finite difference methods.

3 R Code and Numerical Results

```r
######################
#The Model Simulation#
######################


#Fixing the parameters#

r=0.3
x=log(100)
sigma=1
rho=0.3
T=1
N=10000
lambda=0.04
gamma=0.2
theta=0.05
eta=0.5
beta=sqrt((gamma^2)+(eta^2))
K=100
script_k=log(K)
time_points=seq(from = 0, to = T, length.out = N)
#The SMB simulations
B1=rep(0,times=N)
B2=rep(0,times=N)
for (i in 1:N)
{
  t=time_points[i]
  if(t>0)
  {
    B1[i]=B1[(i-1)]+rnorm(1,mean = 0,sd=sqrt(t-time_points[(i-1)]))
    B2[i]=B2[(i-1)]+rnorm(1,mean = 0,sd=sqrt(t-time_points[(i-1)]))
  }
}
B=(rho*B1)+(sqrt(1-rho^2)*B2)

#The path#

X=x+(r-((sigma^2)/2))*time_points+sigma*B
#Plotting the path
plot(time_points,X,xlab = "Time",ylab = "Logarithm of S_t",col="blue",type = "l",sub =
    "r=0.7, sigma =1")
title("The Model Simulation")


############################
#Dependence on CIR volatility#
############################


eta_values=seq(from=0.05, to=0.5, length.out = 1000)
g_0=NULL
for (eta in eta_values)
{
  beta=sqrt((gamma^2)+(eta^2))
```

```
  B_1_func<-function(tau)
  {
    val=(2*gamma*theta/(eta^2))*log((2*beta*exp((gamma+beta)*tau/2))/(beta-gamma+((
   gamma+beta)*exp(beta*tau))))
    return(val)
  }
  B_2_func<-function(tau)
  {
    val=(2*(exp(beta*tau)-1))/(beta-gamma+((gamma+beta)*exp(beta*tau)))
    return(val)
  }
  t=0
  first_term=exp(-B_1_func(T-t)-B_2_func(T-t))
  d_1=(x-script_k+(r+((sigma^2)/2))*(T-t))/(sigma*sqrt(T-t))
  d_2=(x-script_k+(r-((sigma^2)/2))*(T-t))/(sigma*sqrt(T-t))
  second_term=pnorm(d_1,mean = 0,sd=1)*exp(x)
  third_term=pnorm(d_2,mean = 0,sd=1)*K*exp(-r*(T-t))
  val=first_term*(second_term-third_term)
  g_0=c(g_0,val)
}

plot(eta_values,g_0,pch=16,type = "l",col="blue",xlab = "CIR volatility",ylab = "CVA")
title("Dependence on CIR Volatiliy")


#####################
#Dependence on theta#
#####################


beta=0.5
beta=sqrt((gamma^2)+(eta^2))
theta_values=seq(from=0.01, to=0.1, length.out = 1000)
g_0=NULL
for (theta in theta_values)
{
  B_1_func<-function(tau)
  {
    val=(2*gamma*theta/(eta^2))*log((2*beta*exp((gamma+beta)*tau/2))/(beta-gamma+((
   gamma+beta)*exp(beta*tau))))
    return(val)
  }
  B_2_func<-function(tau)
  {
    val=(2*(exp(beta*tau)-1))/(beta-gamma+((gamma+beta)*exp(beta*tau)))
    return(val)
  }
  t=0
  first_term=exp(-B_1_func(T-t)-B_2_func(T-t))
  d_1=(x-script_k+(r+((sigma^2)/2))*(T-t))/(sigma*sqrt(T-t))
  d_2=(x-script_k+(r-((sigma^2)/2))*(T-t))/(sigma*sqrt(T-t))
  second_term=pnorm(d_1,mean = 0,sd=1)*exp(x)
  third_term=pnorm(d_2,mean = 0,sd=1)*K*exp(-r*(T-t))
  val=first_term*(second_term-third_term)
  g_0=c(g_0,val)
}

plot(theta_values,g_0,pch=16,type = "l",col="blue",xlab = "theta",ylab = "CVA")

title("Dependence on theta")
```