VOTER MODEL

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Content

Introduction of the Model Used

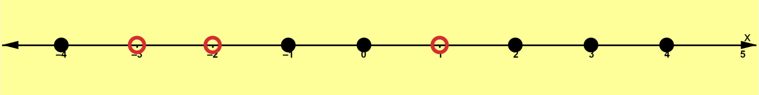
- Why was voter model introduced ?
- Voter model, in which each user holds one of two possible opinions and updates it randomly under the distribution of others beliefs.
- Introduced by,
  - Clifford and Sudbury (1973)- Invasion Process
  - Holley and Liggett (1975)
- Use to describe social dynamics where people are divided between two parties and form their opinion by observing that of others around them.

Clifford and Sudbury

- Paper - A model for spatial conflict.
- Two species compete for territory along their mutual boundary. The species are fairly matched and the result of conflict is the invasion by one of the species of territory held by the other.
- Consider the case where conflict takes place only along the frontier
- The chance of a particular position being overrun depends only on the disposition of the enemy in the immediate neighbourhood.
- We refer to members of the opposing species as black and white cells.
- Two processes are introduced-
  - Swapping Process
  - Invasion Process

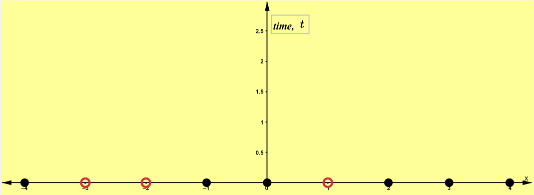
Swapping Process on $\mathbb{R}^1$

- Confrontations are resolved by the exchange of territory
- In the one-dimensional integer lattice case,



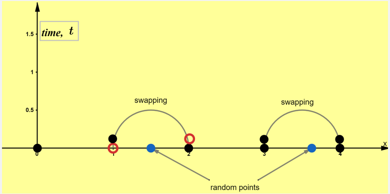
Representation of swapping process

- Consider a plane as follows,



with points thrown down randomly with unit density.

Representation of swapping process



Note

- The probability that adjacent black and white cells swap position in any time interval is the same in both representations.
- With respect to the development of the pattern of colours the processes are stochastically identical.

Lemma

Notation

- $\mathbb{P}_k(n, t)$: The probability that position n at time t is occupied by the cell originally at position k
- $\mathbf{X}_t$ - unit step symmetric random walk in continuous time with rate 2, on integer lattice
- $\mathbb{P}(\mathbf{X}_t = n \mid \mathbf{X}_0 = k) =$Probability that random walk $\mathbf{X}_t$ starting at k, is at n at time t

Lemma

In a swapping process on $\mathbb{Z}$,

$$\mathbb{P}_k(n, t) = \mathbb{P}(\mathbf{X}_t = n \mid \mathbf{X}_0 = k)$$

Theorem 1

Notation

- A = Set of positions initially occupied by black cells
- A_t = Set of positions occupied by black cells at time t
- $\mathbb{P}_A\left(x_0 \in A_{t_0}\right)$ = Probability that at time t_0 the position x_0 is occupied by a black cell

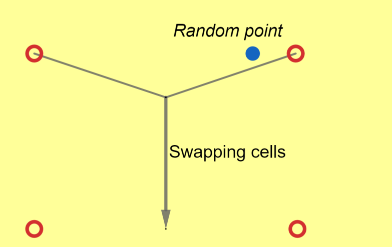
Theorem

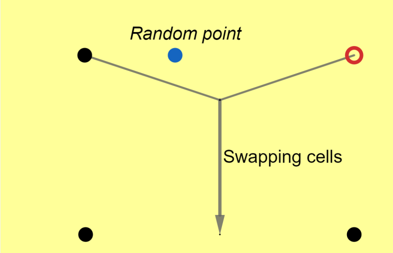
In a swapping process,

$$\mathbb{P}_A\left(x_0 \in A_{t_0}\right) = \mathbb{P}\left(\mathbf{X}_{t_0} \in A \mid \mathbf{X}_0 = x_0\right)$$

Representation of Invasion process

- Either cell may generate a new cell of the same colour
- The neighbour is then eliminated and replaced by the newly born cell.
- The representation of invasion process is essentially the same as the representation of swapping process.
- The differences are
  - The random points have density two
  - Invasion





Theorem 2

- With regard to development of patterns of colours the two processes are equivalent.

Theorem

Starting from the same initial configuration, the probabilities that a certain position will be occupied by a black cell in a swapping process and a fair invasion process are equal.

Basic Voter Model - Overview

- Let us have N agents, each of which can be in one of q states (opinions) $\sigma \in S$.
- Each agent sits on a node of a fully connected network, and they can interact along the edges with their nearest neighbours.
- Some Notations :
  - σ_i = Opinion of the agent i
  - $\Sigma = [\sigma_1, \sigma_2, \cdots, \sigma_N]$, The set of opinions of all the agents

Voter Model (Pictorial Representation)

Figure: Initial Configuration

Voter Model - Scheme

- Any two agents are neighbours, everybody can influence each of its neighbours.
- We choose an agent i *(speaker)* at random out of the N agents.
- Then, choose j *(listener)* randomly among neighbours and set

$$\sigma_j(t+1) = \sigma_i(t).$$

- The process can be summarized as:

$$AB \to AA \quad , \quad BA \to BB.$$

Voter Model (Pictorial Representation)

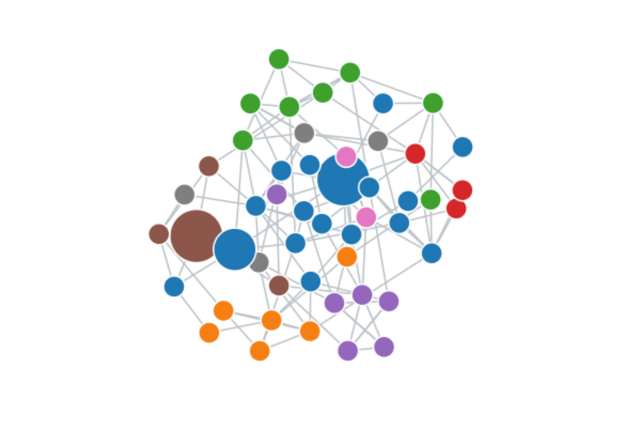


Figure: First Step

Opinion Sets

Definition

$V_i(t) := \{j : j\text{th agent has opinion } i \text{ at time } t\}$.

- $V_i(t)$ may be empty, or may be non-empty but not contain i.
- The number of non empty $V_i(t)$'s for $i = 1, \ldots, N$ can only decrease with time.

Some Basics

- The ultimate behaviour of the system is one of the absorbing states which are for any fixed opinion i,

$$\sigma_j = i \quad \forall 1 \leq j \leq N$$

- The following is a random partition of the set $\{1, 2, \dots, N\}$ for each t,

$$\{V_i(t) : 1 \leq i \leq N\}$$

- An important variable of interest is the consensus time which is time taken to reach one of the absorbing states i,e,

$$T_N^{\text{voter}} := min\{t : V_i(t) = \{1, 2, \dots, N\} \quad \text{for some} \quad 1 \leq i \leq N\}$$

Coalescing MC model

For each agent a set of tokens $C_i(t)$ is assigned varying with time.

- Initialization

$$C_i(0) = \{i\} \quad \forall 1 \leq i \leq N$$

- For i^{th} and j^{th} agent interacting with direction $i \to j$ at time t,

$$C_i(t+1) = \phi, \qquad C_j(t+1) = C_i(t) \cup C_j(t)$$

- Now, the following is also random partition of the set $\{1, 2, \ldots, N\}$ for each t,

$$\{C_i(t) : 1 \leq i \leq N\}$$

- An important variable of interest is coalescence time which is the time taken to reach for some i,

$$T^C := min\{t : C_i(t) = \{1, 2, \ldots, N\} \quad \text{for some} \quad 1 \leq i \leq N\}$$

The duality relationship

For fixed t,

Result

$$\{V_i(t) : 1 \leq i \leq N\} \stackrel{d}{=} \{C_i(t) : 1 \leq i \leq N\}$$

Also,

Result

$$T^C \stackrel{d}{=} T_N^{\text{voter}}$$

- For fixed i, $|V_i(t)|$ changes by ± 1, whereas $|C_i(t)|$ may jump to 0 from a higher value. Hence they are different processes.

Representation of the above model

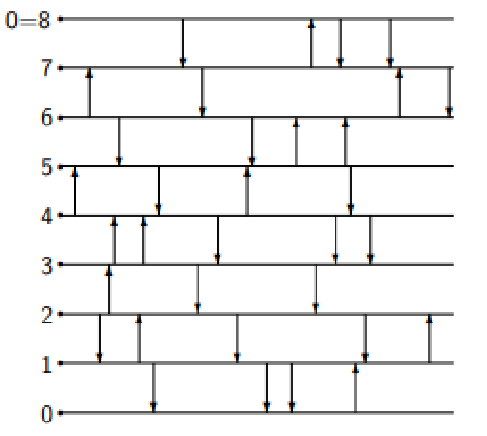


Figure: Moving left to right in time

Representation of the above model

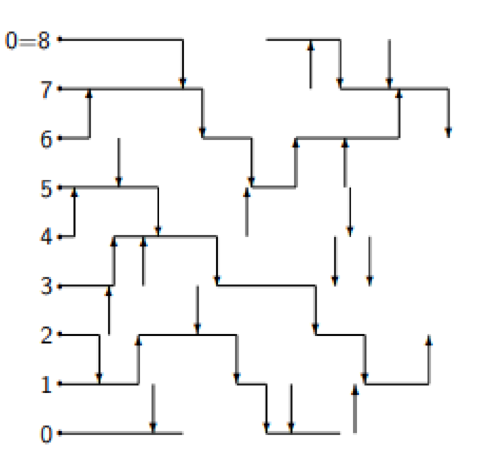


Figure: Left to right and right to Left

Representation of the above model

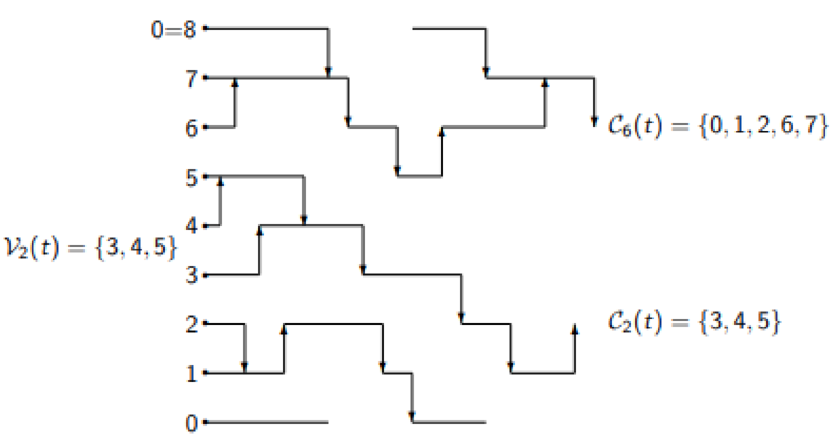


Figure: Duality

Easy Approach for asymptotics

- A slight continuous time variation of the voter model with 2 opinions and N agents is given by the continuous time birth and death chain on $\{1, 2, \ldots, N\}$ with

$$X(t) = \text{number of agents with first opinion at time } t.$$

$$\text{Rates:} \quad \lambda_k = \mu_k = \frac{k(N-k)}{2(N-1)}$$

- Clearly, the absorbing states are 0 and N.

Result for Birth and Death processes

$$T^{\text{hit}}_{0,N} := \min\{t : X(t) = 0 \text{ or } N\}$$

- By general birth-and-death formulas,

$$\mathbb{E}_k T^{\text{hit}}_{0,N} = \frac{2(N-1)}{N}\left(k\left(h_{N-1} - h_{k+1}\right) + (N-k)\left(h_{N-1} - h_{N-k+1}\right)\right)$$

where $h_m := \sum_{i=1}^{m} 1/i$.
- This is maximized by $k = \lfloor N/2 \rfloor$, and

$$\max \mathbb{E}_k T^{\text{hit}}_{0,N} \sim (2\log 2)N.$$

Framing the inequality

1. For the true voter model with N different opinions, assign two dummy opinions to k and $N - k$ agents respectively.
2. The process is continued with the changes in opinions carried out in both the original and dummy opinion.
3. The number of agents having first dummy opinion follow the 2 opinion model.
4. The original opinion being same for any two agents always implies same dummy opinion but not vice versa.
5. This implies the dummy opinions become all same if all the original opinions become same.

Weak Asymptotics

1. 1^{st} Inequality

$$\mathbb{P}_k \left(T^{\text{hit}}_{0,N} > t \right) \leq \mathbb{P} \left(T^{\text{voter}}_N > t \right)$$

2. 2^{nd} Inequality

$$\mathbb{P}_k \left(T^{\text{hit}}_{0,N} > t \right) \geq \frac{2k(N-k-1)}{N(N-1)} \cdot \mathbb{P} \left(T^{\text{voter}}_N > t \right)$$

With $k = \lfloor N/2 \rfloor$ implies

$$\mathbb{E} T^{\text{voter}}_N \leq (4 \log 2 + o(1)) N$$

Occupation Numbers

Definition

Occupation numbers are defined as

$$N_j = \sum_{i=1}^{N} \mathbf{1}(\sigma_i = j) = |V_i|$$

or equivalently the densities

$$n_j = N_j/N,$$

for each opinion $j \in \{1, 2, \cdots, N\}$.

Events

- When there are 2 opinions it is equivalent to renaming, $\sigma = \pm 1$. The state is described by one dynamical variable only, which is called *magnetisation.*

$$m = \frac{N_+ - N_-}{N}$$

- The probabilities of the three possible events at a step are

$$\mathbb{P}\left\{m \to m + \tfrac{2}{N}\right\} = \frac{(N_+)\times(N_-)}{N\times(N-1)} = \frac{1}{4}\left(1 - m^2\right)\left(\frac{N}{N-1}\right)$$
$$\mathbb{P}\left\{m \to m - \tfrac{2}{N}\right\} = \frac{(N_-)\times(N_+)}{N\times(N-1)} = \frac{1}{4}\left(1 - m^2\right)\left(\frac{N}{N-1}\right)$$
$$\mathbb{P}\{m \to m\} = \frac{(N^2{}_+)+(N^2{}_-)}{N\times(N-1)} = \left(\frac{1}{2}\left(1 + m^2\right) - \frac{1}{N}\right)\left(\frac{N}{N-1}\right)$$

Simulating magnetisation

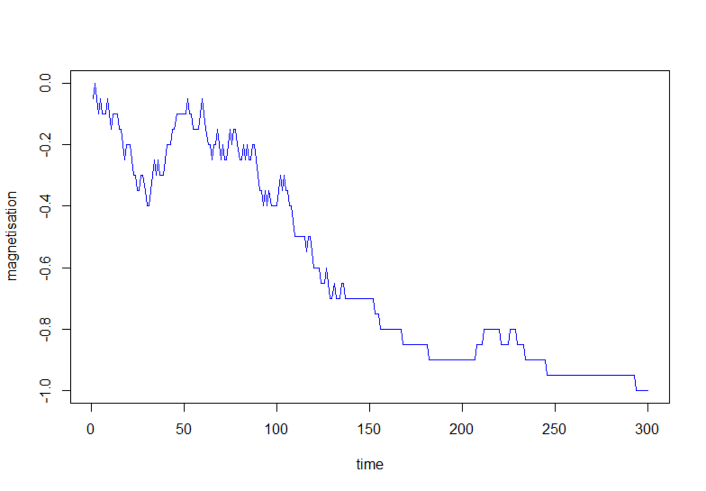


Figure: Model with $N = 40$

Simulating magnetisation

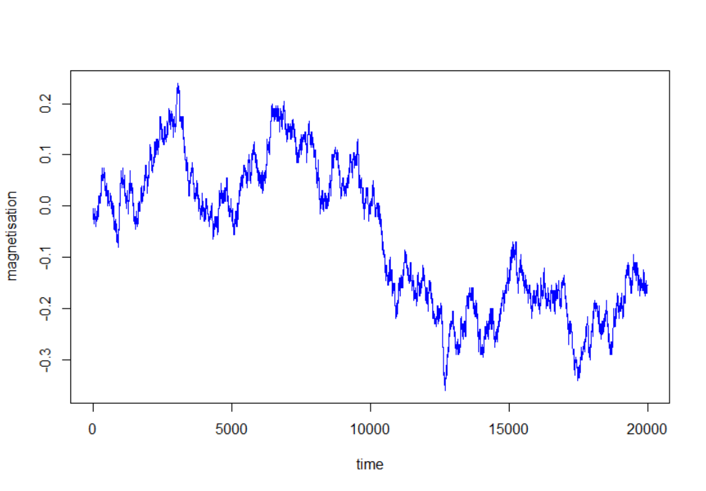


Figure: Model with $N = 4000$

Partial Differential Equation

The probability density $P_m(m, \tau)$ for $m(t)$ evolves according to the partial differential equation where time is re scaled as $N \to \infty$ by τN^2

$$\frac{\partial}{\partial \tau} P_m(m, \tau) = \frac{\partial^2}{\partial m^2} \left[\left(1 - m^2\right) P_m(m, \tau) \right] \qquad \text{where } \tau = \frac{t}{N^2}$$

What if q is large?

- The distribution of occupation numbers say, $D(n)$:

$$D(n) = \frac{N}{q} \sum_{\sigma=1}^{q} \delta\left(n - n_\sigma\right) \quad ; \delta(x) = 1 \text{ if } \quad x = 0$$

- When $N \to \infty$ and $q \to \infty$ and substituting the variable $x = 2n - 1$ we arrive at the equation

$$\frac{\partial}{\partial \tau} P_n(x, \tau) = \frac{\partial^2}{\partial x^2} \left[\left(1 - x^2\right) P_n(x, \tau) \right] \qquad P_n(n) \approx D(n)$$

Eigen Vectors

Linear Operator Form

$\frac{\partial}{\partial \tau} P(x,\tau) = \mathcal{L} P(x,\tau)$ where the linear operator $\mathcal{L}$ acts as $(\mathcal{L} f)(x) = \frac{\partial^2}{\partial x^2} \left[\left(1 - x^2\right) f(x) \right]$.

Equation 2

$$\left(1 - x^2\right) \Phi_c''(x) - 4x\Phi_c'(x) + (c-2)\Phi_c(x) = 0$$

where $\phi_c(x)$ is the eigen vector for c.

Expansion

$$P(x,\tau) = \sum_c A_c e^{-c\tau} \Phi_c(x)$$

where A_c comes from the initial conditions.

Solutions for the equations

$c = 0$

Eigenvectors corresponding to eigenvalue $c = 0$,

$$\Phi_{01} = \delta(x-1) \quad , \quad \Phi_{02} = \delta(x+1)$$

$c \neq 0$

$c \neq 0$, the solution is an ordinary function of x plus a pair of δ -functions,

$$\Phi_c = \phi_{c+}\delta(x-1) + \phi_{c-}\delta(x+1) + \phi_c(x)\theta(x-1)\theta(x+1)$$

ϕ_{c+}, ϕ_{c-} are real numbers and $\phi_c(x)$ is a double differentiable function.

Solution for the equations

Equation 2 (With Conditions)

$$\left(1 - x^2\right) \phi_c''(x) - 4x\phi_e'(x) + (c - 2)\phi_c(x) = 0$$

accompanied by conditions:

$$\lim_{x \to +1} \phi_c(x) = -\frac{c}{2}\phi_{c+}$$

$$\lim_{x \to -1} \phi_c(x) = -\frac{c}{2}\phi_{c-}$$

The general solution of the equation exhibits behaviour $\phi_c(x) \sim (1 \mp x)^\alpha$ at $x \to \pm 1$, where either $\alpha = 0$ or $\alpha = -1$.

Solving equation 3

The coefficients in the solution of expansion:$P(x,\tau) = \sum_c A_c e^{-c\tau} \Phi_c(x)$ with initial conditions $P(x,0) = P_0(x)$. The coefficients come out to be:

Initial Configuration

$$A_c = \frac{\int P_0(x)\psi_c(x)\mathrm{d}x}{\int \phi_c(x)\psi_c(x)\mathrm{d}x}$$

Where, the set of left eigen vectors of the operator $\mathcal{L}$ is given by:

$$\left(1 - x^2\right) \psi_c''(x) + c\psi_c(x) = 0$$

Probabilities

Let us denote $P_{\rm st}(\tau)$ is the probability density for ending at time τ in the stationary frozen configuration with all agents in the same state.

The probability that the stationary configuration was not reached before time τ :

$$P^{>}_{st}(\tau) \equiv \int_{\tau}^{\infty} P_{\rm st}\left(\tau'\right) \mathrm{d}\tau' = 1 - \lim_{\varepsilon \to 0^{+}} \left(\int_{-1-\varepsilon}^{-1+\varepsilon} + \int_{1-\varepsilon}^{1+\varepsilon} \right) P(x,\tau) \mathrm{d}x$$

Explicit Form

$$P^{>}_{st}(\tau) = \sum_{c>0} 2A_c \frac{\phi_c(-1) + \phi_c(1)}{c} e^{-c\tau}$$

Asymptotics

The distribution of waiting times will have an exponential tail

$$P_{st}^{>}(\tau) \sim e^{-2\tau}, \tau \to \infty$$

From initial condition $P_0(x) = \delta(x - x_0)$ we can also compute the pre factor in the leading term for large τ.

Distribution

$$P_{\mathrm{st}}^{>}(\tau) \simeq \frac{6}{4}\left(1 - x_0^2\right) e^{-2\tau}, \tau \to \infty$$

Voter Model Visualization

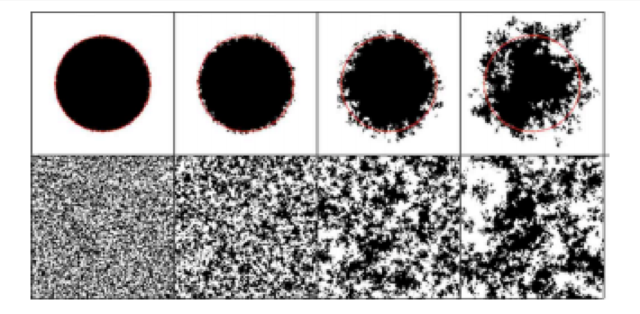


Figure: Evolution of a two-dimensional voter model starting from a droplet or a fully disordered configuration.

Stationary State Probability

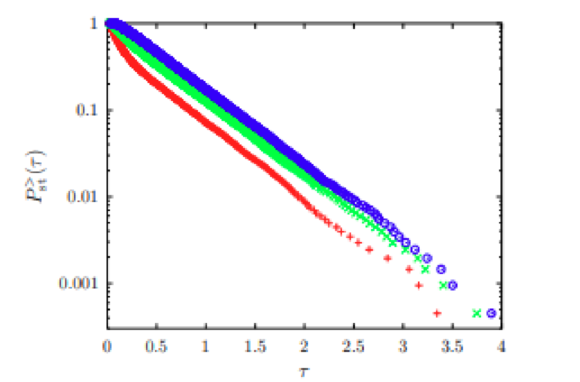


Figure: Probability of reaching the stationary state in time larger than τ , q = 2, N = 2000. The values of initial fraction p of opinions +1 are 0.1 (+) 0.2 ($\times$) and 0.7 (.)

Results on Complete Graphs

1. Master Equation for the probability density $P(\mu, t')$

$$\frac{\partial P(\mu, t')}{\partial t'} = \frac{\partial^2}{\partial \mu^2}\left[\left(1-\mu^2\right) P(\mu, t')\right]$$

2. The natural scaling of time with the number of sites is taken as

$$t' = \frac{t}{N^2}.$$

Probabilities on Complete Graphs

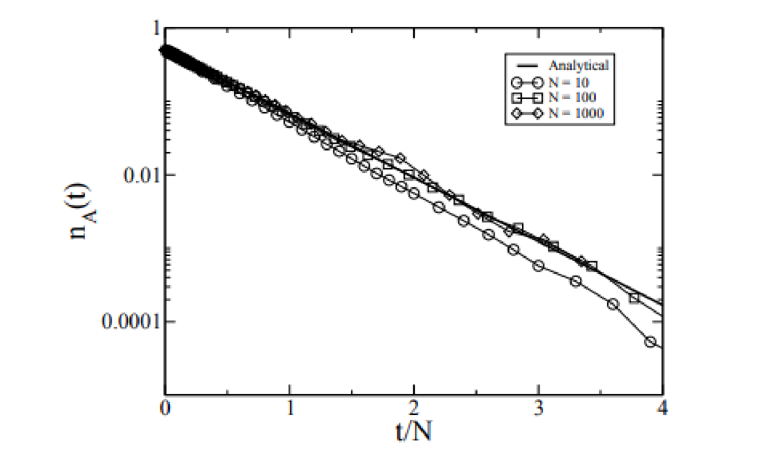


Figure: Fraction of active edges $n_A(t)$ for voter dynamics on a complete graph

Results:

By standard methods for the Fokker-Planck equation, for large $t' = \frac{t}{N^2}$, the fraction of edges connecting nodes with opposite values of the variable (active edges) is

$$n_A(t') = \frac{\left(1 - m_0^2\right)}{2} e^{-2t'}$$

where m_0 is the initial magnetization.

A Basic Implementation of the Voter Model

Each node ("voter") takes one of the finite discrete states (say, black , white, red and blue—one could view it as a political opinion). The voter model considers only one opinion transfer event at a time between a pair of connected nodes that are randomly chosen from the network. There are three minor variations of how those nodes are chosen:

- Original ("pull") version: First, a "listener" node is randomly chosen from the network, and then a "speaker" node is randomly chosen from the listener's neighbors.
- Reversed ("push") version: First, a "speaker" node is randomly chosen from the network, and then a "listener" node is randomly chosen from the speaker's neighbors.

Representation of the above model

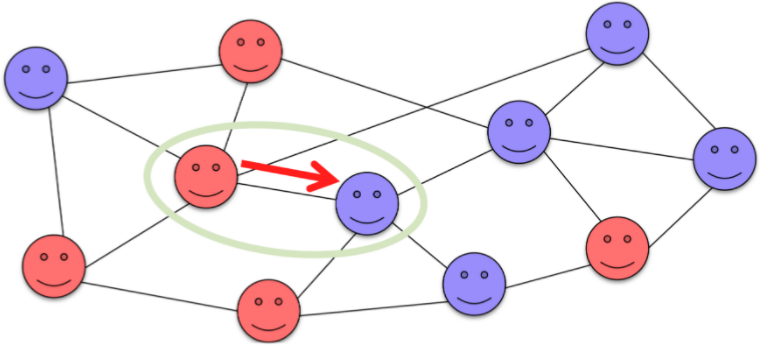


Figure: Each time a pair of connected nodes are randomly selected (light green circle), the state of the speaker node (left) is copied to the listener node (right).

Representation of the above model

Figure: Iterations=128

Representation of the above model

Figure: Iterations=384

Representation of the above model

Figure: Iterations=685

Representation of the above model

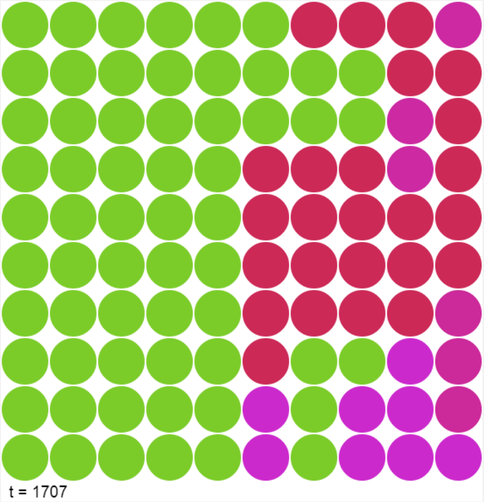


Figure: Iterations=1707

Representation of the above model

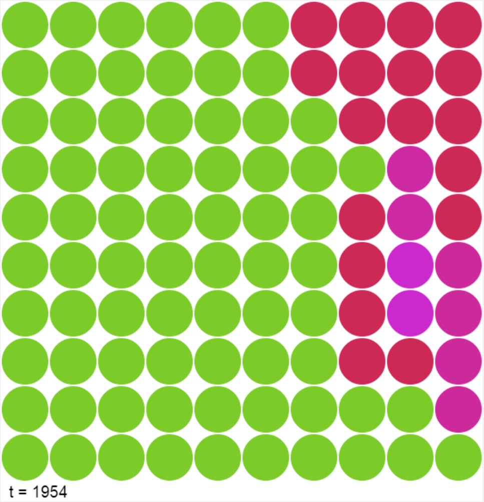


Figure: Iterations=1954

Representation of the above model

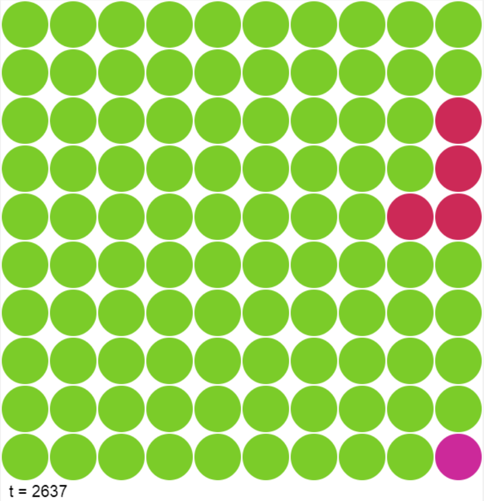


Figure: Iterations=2637

Representation of the above model

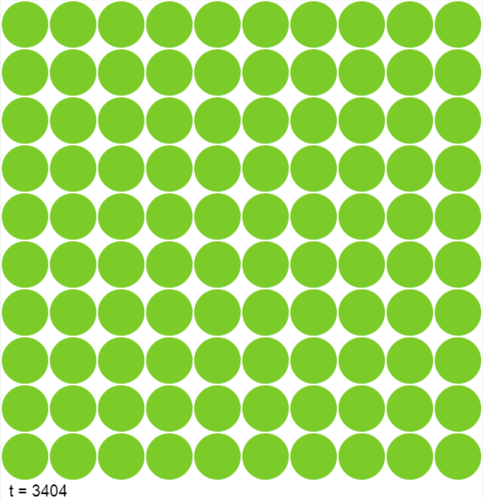


Figure: Iterations=3404

Results on Complete Graphs

1. Random fluctuations bring eventually all surviving runs to the fully ordered absorbing state; however, as long as the runs survive they do not order on average.
2. The decay of $n_A(t)$ is just a consequence of the decay of $P_{st}^{>}(t)$.

THANK YOU!