Linear Algebra
A gentle introduction

Linear Algebra has become as basic and as applicable as calculus, and fortunately it is easier.

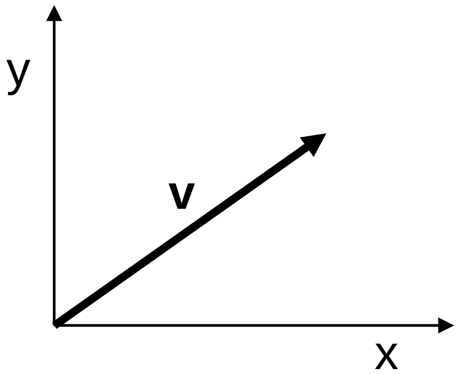
--Gilbert Strang, MIT

What is a Vector ?

- Think of a vector as a *directed line segment in N-dimensions*! (has "length" and "direction")

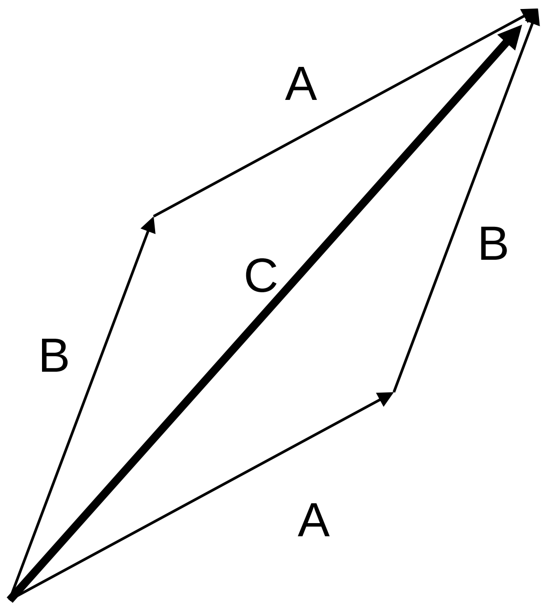
- Basic idea: convert geometry in higher dimensions into algebra!
  - Once you define a "nice" *basis* along each dimension: x-, y-, z-axis …
  - Vector becomes a N x 1 matrix!
  - $\mathbf{v} = [a \ \ b \ \ c]^T$
  - Geometry starts to become linear algebra on vectors like **v**!

$$\vec{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$



Vector Addition: A+B

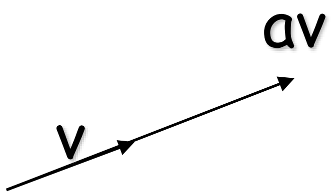
$$\mathbf{A+B} = (x_1, x_2) + (y_1, y_2) = (x_1 + y_1, x_2 + y_2)$$



A+B = C
(use the head-to-tail method to combine vectors)

Scalar Product: $a\mathbf{v}$

$$a\mathbf{v} = a(x_1, x_2) = (ax_1, ax_2)$$



Change only the length ("scaling"), but keep *direction fixed*.

Sneak peek: matrix operation (**Av**) can change *length*, *direction and also dimensionality*!

Vectors: Dot Product

$$A \cdot B = A^T B \; = \begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} d \\ e \\ f \end{bmatrix} = ad + be + cf$$

Think of the dot product as a matrix multiplication

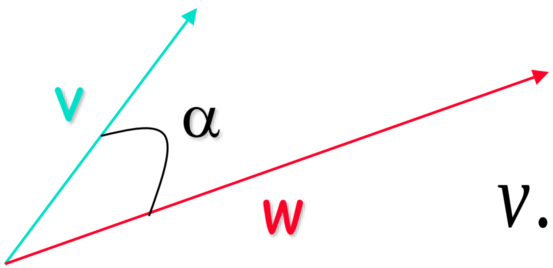
$$\|A\|^2 = A^T A = aa + bb + cc$$

The magnitude is the dot product of a vector with itself

$$A \cdot B = \|A\| \; \|B\| \cos(\theta)$$

The dot product is also related to the angle between the two vectors

Inner (dot) Product: v.w or $\mathbf{w}^T\mathbf{v}$



$$v.w = (x_1, x_2).(y_1, y_2) = x_1 y_1 + x_2 . y_2$$

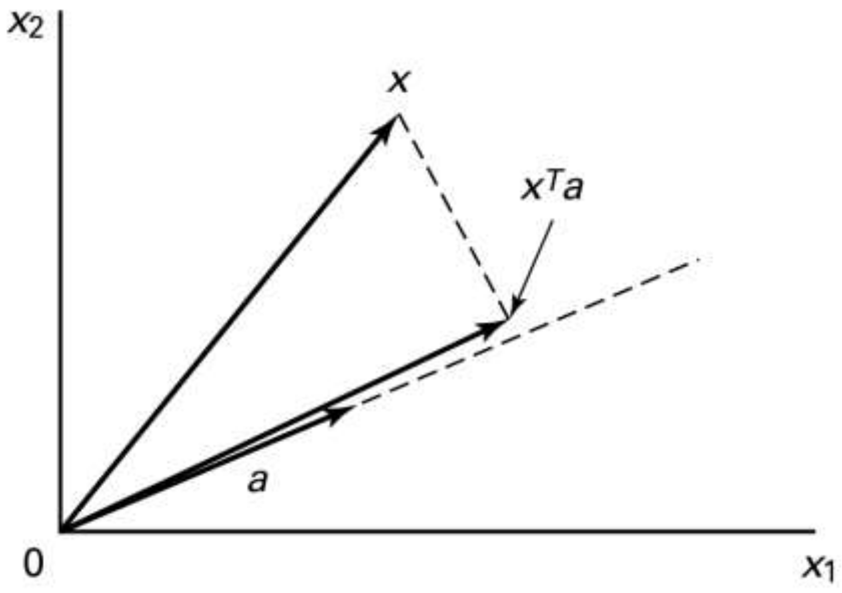
The inner product is a **SCALAR!**

$$v.w = (x_1, x_2).(y_1, y_2) = \| v \| \cdot \| w \| \cos \alpha$$

$$v.w = 0 \Leftrightarrow v \perp w$$

If vectors **v, w** are "columns", then dot product is $\mathbf{w}^T\mathbf{v}$

Projection: Using Inner Products (I)

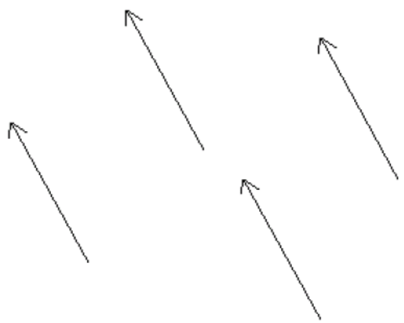


Projection of x along the direction **a** (||**a**|| = 1).

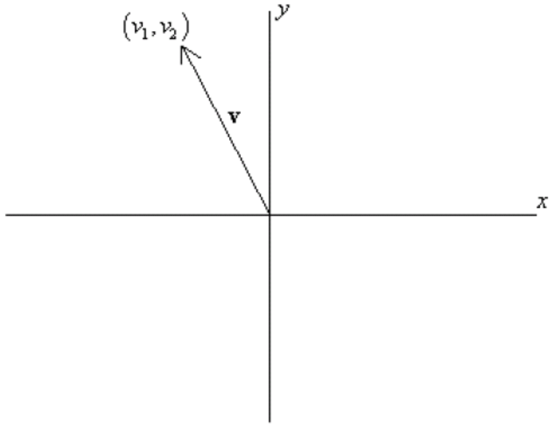
$\mathbf{p} = \mathbf{a}\,(a^T x)$
$\|a\| = a^T a = 1$

Bases & Orthonormal Bases

- Basis (or axes): frame of reference



vs



Basis: a space is totally defined by a set of vectors – any point is a *linear combination* of the basis

Ortho-Normal: orthogonal + normal

[**<u>Sneak peek</u>**:

<u>Orthogonal</u>: dot product is zero

<u>Normal</u>: magnitude is one]

$$x = [1 \quad 0 \quad 0]^T \qquad x \cdot y = 0$$

$$y = [0 \quad 1 \quad 0]^T \qquad x \cdot z = 0$$

$$z = [0 \quad 0 \quad 1]^T \qquad y \cdot z = 0$$

What is a Matrix?

- A matrix is a set of elements, organized into rows and columns

$$\begin{array}{cc} & \text{rows} \longrightarrow \\ \text{columns} \Big\downarrow & \begin{bmatrix} a & b \\ c & d \end{bmatrix} \end{array}$$

Basic Matrix Operations

- Addition, Subtraction, Multiplication: creating new matrices (or functions)

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a+e & b+f \\ c+g & d+h \end{bmatrix}$$

Just add elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a-e & b-f \\ c-g & d-h \end{bmatrix}$$

Just subtract elements

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

Multiply each row by each column

Matrix Times Matrix

$$\mathbf{L} = \mathbf{M} \cdot \mathbf{N}$$

$$\begin{bmatrix} l_{11} & l_{12} & l_{13} \\ l_{21} & l_{22} & l_{23} \\ l_{31} & l_{32} & l_{33} \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \cdot \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}$$

$$l_{12} = m_{11}n_{12} + m_{12}n_{22} + m_{13}n_{32}$$

Multiplication

- Is AB = BA? Maybe, but maybe not!

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & \ldots \\ \ldots & \ldots \end{bmatrix} \qquad \begin{bmatrix} e & f \\ g & h \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ea+fc & \ldots \\ \ldots & \ldots \end{bmatrix}$$

- Matrix multiplication AB: apply transformation B first, and then again transform using A!
- Heads up: multiplication is NOT commutative!

- **<u>Note</u>**: If A and B both represent either pure "*<u>rotation</u>*" or "*<u>scaling</u>*" they can be interchanged (i.e. AB = BA)

Matrix operating on vectors

- Matrix is like a <u>function</u> that <u>transforms the vectors on a plane</u>
- Matrix operating on a general point => transforms x- and y-components
- *System of linear equations*: matrix is just the bunch of coeffs !

- x' = ax + by
- y' = cx + dy

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x' \\ y' \end{bmatrix}$$

Direction Vector Dot Matrix

$$\mathbf{v}' = \mathbf{M}\cdot\mathbf{v} = \begin{bmatrix} a_x & b_x & c_x & d_x \\ a_y & b_y & c_y & d_y \\ a_z & b_z & c_z & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} v_x \\ v_y \\ v_z \\ 1 \end{bmatrix}$$

$$v'_x = v_x a_x + v_y b_x + v_z c_x$$

$$v'_y = v_x a_y + v_y b_y + v_z c_y$$

$$v'_z = v_x a_z + v_y b_z + v_z c_z$$

$$\mathbf{v}' = v_x\mathbf{a} + v_y\mathbf{b} + v_z\mathbf{c}$$

Inverse of a Matrix

- Identity matrix:
  AI = A
- Inverse exists only for square matrices that are non-singular
  - Maps N-d space to another N-d space bijectively
- Some matrices have an inverse, such that:
  $AA^{-1} = I$
- Inversion is tricky:
  $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$

  Derived from non-commutativity property

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Determinant of a Matrix

- Used for inversion
- If det(A) = 0, then A has no inverse

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \det(A) = ad - bc$$

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

http://www.euclideanspace.com/maths/algebra/matrix/functions/inverse/threeD/index.htm

Transpose of a Matrix

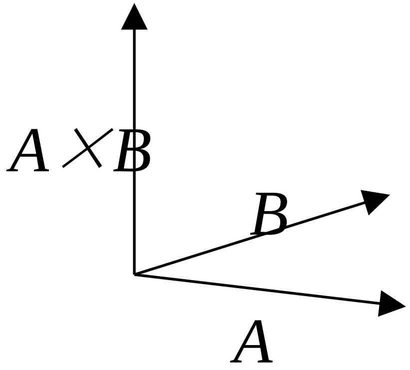
- Written A^T (transpose of A)

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad A^T = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$$

- Keep the diagonal but reflect all other elements about the diagonal
  $a_{ij} = a_{ji}$ where i is the row and j the column
  in this example, elements c and b were exchanged
- For orthonormal matrices $A^{-1} = A^T$

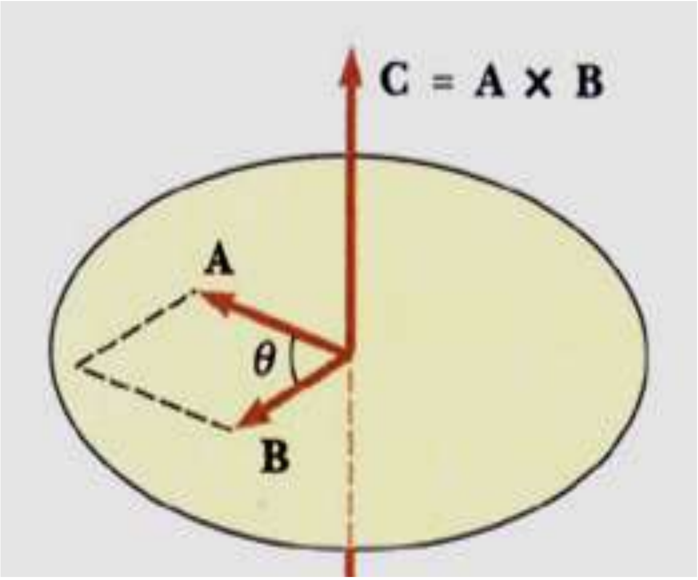
Vectors: Cross Product

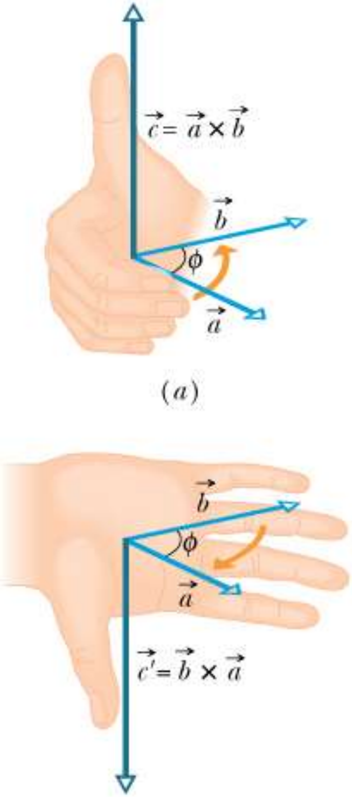
- The cross product of vectors A and B is a vector C which is perpendicular to A and B
- The magnitude of C is proportional to the sin of the angle between A and B
- The direction of C follows the **right hand rule** if we are working in a right-handed coordinate system



$$\|A \times B\| = \|A\| \, \|B\| \sin(\theta)$$

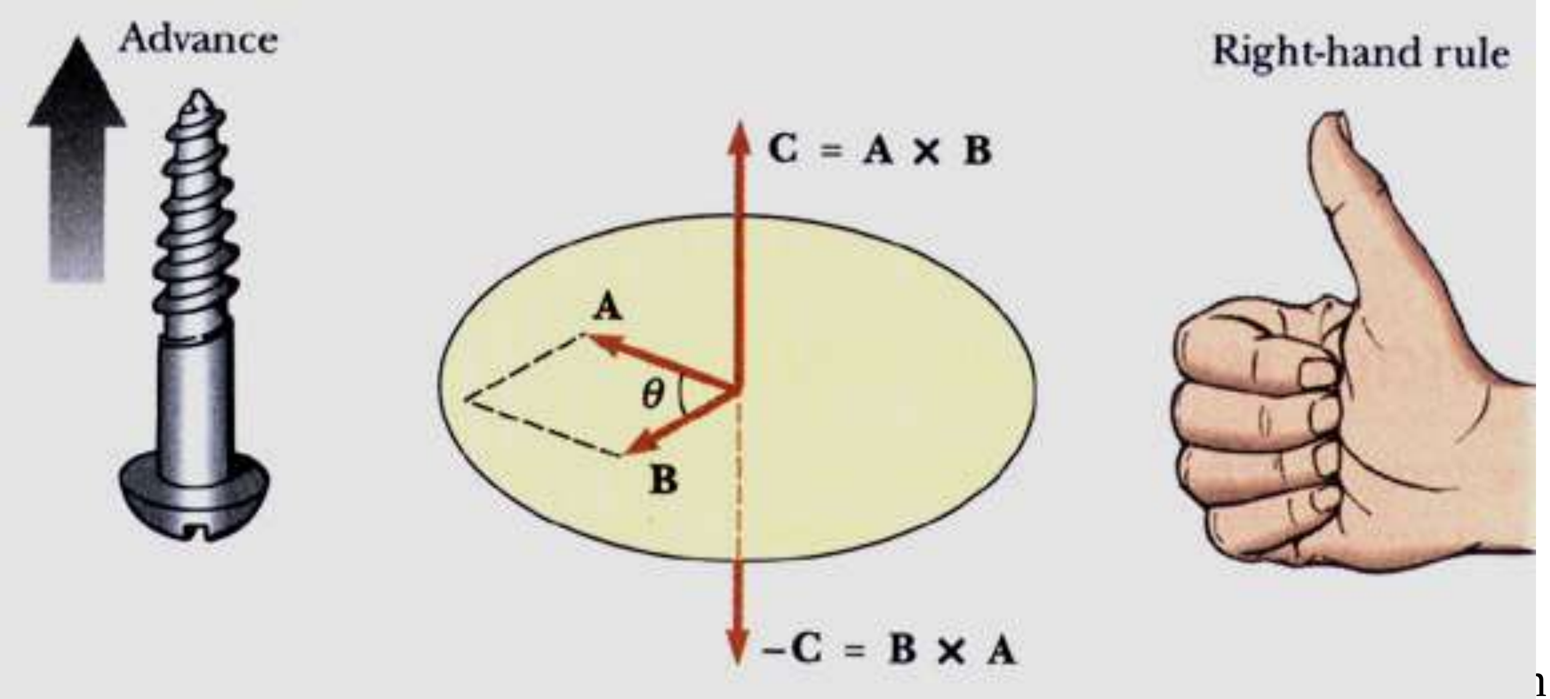
MAGNITUDE OF THE CROSS PRODUCT





DIRECTION OF THE CROSS PRODUCT

- The right hand rule determines the direction of the cross product



For more details

- Prof. Gilbert Strang's course videos:
- http://ocw.mit.edu/OcwWeb/Mathematics/18-06Spring-2005/VideoLectures/index.htm

- Esp. the lectures on eigenvalues/eigenvectors, singular value decomposition & applications of both. (second half of course)

- Online Linear Algebra Tutorials:
- http://tutorial.math.lamar.edu/AllBrowsers/2318/2318.asp