

Chapter 3

Equilibrium points and stability analysis

In this paper we introduce the general class of nonlinear state-space models, of which linear state-space models represent a special case. For these models we introduce the concept of equilibrium point (in case of autonomous state-space models) and of equilibrium point corresponding to a constant input (for general, non-autonomous, state-space models). We also introduce the concepts of stability, attractiveness and asymptotic stability of the equilibria. Finally, we show that the dynamics of these systems in a neighbourhood of an equilibrium condition can be approximated by a linear state-space model and that the analysis of the asymptotic stability of an equilibrium point can often be reduced to the study of the asymptotic stability of the corresponding “linearised” state-space model.

3.1 Nonlinear state-space models

In a general (nonlinear) state-space model three types of variables are involved:

- m scalar input variables, u_1, u_2, \dots, u_m or, equivalently, an m -dimensional input vector

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \in \mathbb{R}^m;$$

- p scalar output variables, y_1, y_2, \dots, y_p or, equivalently, a p -dimensional output vector

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix} \in \mathbb{R}^p;$$

- n scalar state variables, x_1, x_2, \dots, x_n or, equivalently, an n -dimensional state vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n.$$

The describing variables are functions of the (independent) time variable $t \in \mathbb{T}$, that may be either *discrete*, and if so we assume that \mathbb{T} coincides with the integer set \mathbb{Z} or with its subset \mathbb{Z}_+ , or *continuous*, if \mathbb{T} is the real set \mathbb{R} or the nonnegative real half-line \mathbb{R}_+ .

The subset $X \subseteq \mathbb{R}^n$, where the state vector \mathbf{x} takes values, is called *state-space*. The subsets $U \subseteq \mathbb{R}^m$ and $Y \subseteq \mathbb{R}^p$ where the vectors \mathbf{u} and \mathbf{y} take values are the *input alphabet* and the *output alphabet*, or also the input space and the output space.

We will assume that:

1. the *input functions* that act on the system are sequences $\{\mathbf{u}(t)\}_{t \in \mathbb{Z}}$ taking values in (some subset of) \mathbb{R}^m in the discrete-time case, and functions $\mathbf{u}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^m$ at least piece-wise continuous in the continuous-time case;
2. given two instants τ and $T > \tau$, the *system state* $\mathbf{x}(T)$ at time T is uniquely determined by the state $\mathbf{x}(\tau)$ and by the input evolution $\mathbf{u}(\cdot)$ in the time interval $[\tau, T]$;
3. given an instant t , the output value $\mathbf{y}(t)$ at time t depends only on the values at t of the state and of the input.

To describe how the state evolves, we will provide a *state update map*, which allows one to determine how the state variables $x_1(\cdot), x_2(\cdot), \dots, x_n(\cdot)$ evolve over time as a consequence of the applied input variables $u_1(\cdot), u_2(\cdot), \dots, u_m(\cdot)$ and of the initial values assumed by the state variables.

To this end, in the case of a discrete-time system it is sufficient to specify how the state variables update in a single step, by assigning an n -tuple of functions of the following type:

$$\begin{aligned} x_1(t+1) &= f_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ x_2(t+1) &= f_2(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ &\vdots && \vdots && \vdots && \vdots \\ x_n(t+1) &= f_n(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)). \end{aligned} \tag{3.1}$$

Given two time instants τ and T , with $T > \tau$, and upon assuming that the state $\mathbf{x}(\tau)$ and the input function $\mathbf{u}(t)$, $t \in [\tau, T]$, are known, the iterative application of the one-step transition map allows to directly evaluate the state at time T .

For continuous-time systems, the state update map is typically given, in implicit form, by means of a system of first order differential equations

$$\begin{aligned} \dot{x}_1(t) &= f_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ \dot{x}_2(t) &= f_2(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ &\vdots && \vdots && \vdots && \vdots \\ \dot{x}_n(t) &= f_n(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)). \end{aligned} \tag{3.2}$$

In general, the functions f_i are required to be sufficiently regular to guarantee that, for every choice of the “initial condition” $\mathbf{x}(\tau)$ and of the input function $\mathbf{u}(\cdot)$, known from time τ onward, the differential equation (3.2) admits one and only one solution.

- EXERCISE 3.1.1 [SEMIGROUP PROPERTY] The state update map exhibits the “composition” or “semigroup” property. Consider three time instants $t_1 \leq t_2 \leq t_3$. Given the initial state $\mathbf{x}(t_1)$ and the input signal $\mathbf{u}(\cdot)$ from t_1 onward, the state of the system at the final time instant t_3 can be obtained either
 - by applying the state transition map to the whole interval $[t_1, t_3]$,
 - or
 - by applying it first to the interval $[t_1, t_2]$, to determine the state $\mathbf{x}(t_2)$, and subsequently, upon having assumed $\mathbf{x}(t_2)$ as initial state, by applying it to the interval $[t_2, t_3]$.

Verify that the two procedures lead to the same value for the state at time t_3 .

The model description is completed by the *output map*, that determines by means of a static relationship the values of the outputs y_1, y_2, \dots, y_p at time t as functions of the state variables and input values at the same time. To this end, a p -tuple of functions

$$\begin{aligned} y_1(t) &= h_1(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ y_2(t) &= h_2(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)), \\ &\vdots && \vdots && \vdots && \vdots \\ y_p(t) &= h_p(x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \end{aligned} \tag{3.3}$$

is given.

Example 3.1.1 [DYNAMICS OF THE SUGAR AND INSULIN CONCENTRATIONS IN THE BLOOD]. The mathematical models of the relationship between the concentrations of insulin and sugar in the blood provide a good framework to study the effects of different treatment options in diabetes. Limiting ourselves to an extremely simplified model, we denote by $x_1(t)$ and $x_2(t)$ the concentrations of insulin and sugar in the blood of an individual, and with $u_1(t)dt$ and $u_2(t)dt$ the food and insulin intakes in the interval dt .

To build up the model, we need to translate into mathematical terms the following biochemical phenomena:

- in a long period of fasting with no insulin intake, i.e., $u_1(t) = u_2(t) = 0$, the system reaches an equilibrium condition in which the insulin concentration in the blood is zero ($x_2(t) = 0$), and the concentration of sugar is constant ($x_1(t) = c$);
- if the concentration $x_1(t)$ of the sugar exceeds the level c in a time interval dt , the pancreas secretes insulin in the blood elevating the concentration of insulin by $a_{21}(x_1(t) - c)dt$; if $x_1(t) < c$ the pancreas does not secrete insulin;
- in dt the insulin concentration in the blood, $x_2(t)$, undergoes (in the absence of external intake or of pancreas secretion) a decrease of $a_{22}x_2(t)dt$;
- in dt the concentration of insulin can grow as a result of an external intake equal to $u_2(t)dt$.

As a consequence, the dynamics of insulin concentration can be represented by the equations

$$\begin{cases} \dot{x}_2(t) = a_{21}(x_1(t) - c) - a_{22}x_2(t) + b_2u_2(t), & \text{if } x_1(t) > c, \\ \dot{x}_2(t) = -a_{22}x_2(t) + b_2u_2(t), & \text{if } x_1(t) \leq c, \end{cases} \tag{3.4}$$

where the constants a_{21}, a_{22}, b_2 are characteristic of the specific individual.

For the sugar dynamics one has to take into account the following facts:

- the presence of insulin in the blood stimulates the metabolism of sugar, which leads to reduce the sugar concentration $x_1(t)$ in the blood, with a decrease in dt which is proportional to the product of the concentrations $x_1(t)x_2(t)$;
- if $x_1(t) < c$, the liver releases in the blood, in the interval dt , an amount of sugar that is proportional to $c - x_1(t)$; if $x_1(t) > c$, the liver does not release sugar;
- the sugar concentration can increase as a result of sugar intake through food, in an amount that is proportional to $u_1(t)$ (assuming that the diet remains homogeneous over time).

We can then write the equations

$$\begin{cases} \dot{x}_1(t) = -a_{12}x_1(t)x_2(t) + b_1u_1(t), & \text{if } x_1(t) > c, \\ \dot{x}_1(t) = a_{11}(c - x_1(t)) - a_{12}x_1(t)x_2(t) + b_1u_1(t), & \text{if } x_1(t) \leq c. \end{cases} \quad (3.5)$$

Also in this case the constants a_{11}, a_{12}, b_1 are peculiar of the specific individual.

Let us introduce the function

$$g(y) := \begin{cases} y, & \text{if } y > 0, \\ 0, & \text{if } y \leq 0, \end{cases}$$

and let us combine (3.4) and (3.5), thus obtaining the overall model for the dynamics of sugar and insulin:

$$\begin{cases} \dot{x}_1(t) = a_{11}g(c - x_1(t)) - a_{12}x_1(t)x_2(t) + b_1u_1(t), \\ \dot{x}_2(t) = a_{21}g(x_1(t) - c) - a_{22}x_2(t) + b_2u_2(t). \end{cases} \quad (3.6)$$

System (3.6) consists of non-linear equations, due to the presence of the term x_1x_2 and of the function $g(\cdot)$, and it provides a state-space model for the phenomenon we are studying. In that model $u_1(\cdot)$ and $u_2(\cdot)$ are regarded as inputs, and the vector with entries $x_1(\cdot)$ and $x_2(\cdot)$ (which exhibits the separation property, being the solution of (3.6)) as state vector. As an output one can assume one of the state variables (possibly both of them).

The equations of a discrete-time or a continuous-time system are often written in compact form as:

$$\begin{cases} \mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)) \end{cases} \quad (3.7)$$

and

$$\begin{cases} \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{h}(\mathbf{x}(t), \mathbf{u}(t)), \end{cases} \quad (3.8)$$

where \mathbf{f} and \mathbf{h} denote vectors with entries f_1, \dots, f_n and h_1, \dots, h_p , respectively.

Both in the discrete and in the continuous cases, it is clear that the state update map and the input/output relationship of the system are *causal*, by this meaning that the state and the output at a generic instant t do not depend on the values of the input at time instants subsequent to t .

When the variable \mathbf{y} in the output equation only depends on the state \mathbf{x} and not on the input \mathbf{u} , the dynamical system is called *strictly proper*. In this case, as the output depends on the input only through the state equation, the output value $\mathbf{y}(t)$ at time t is independent of the input value at time t , $\mathbf{u}(t)$.

Systems described by (3.7) and (3.8) enjoy additional properties: they are *time-invariant*, by this meaning invariant with respect to a shift of the time variable.

To formalise this concept, it is convenient to introduce the time shift operator σ^Δ , where $\Delta \in \mathbb{T}$ is an integer or a real quantity, depending on whether we consider discrete-time or continuous-time signals. It associates with a generic function $\mathbf{u}(\cdot)$ (defined on \mathbb{T}) the function $\sigma^\Delta \mathbf{u}(\cdot)$, “shifted” by Δ with respect to $\mathbf{u}(\cdot)$, and hence defined at every time $t \in \mathbb{T}$, as

$$(\sigma^\Delta \mathbf{u})(t) = \mathbf{u}(t + \Delta).$$

Therefore we will say that the system is time-invariant if for every choice of

an initial time τ and a final one $t \geq \tau$,

an initial state $\mathbf{x}(\tau) = \mathbf{x}_0$,

an input function $\mathbf{u}(\cdot)$,

the subsequent values of the state and the output at time t coincide with those that the state and the output would take at time $t - \Delta$ when assuming

as initial time $\tau - \Delta$ and as final time $t - \Delta$,

as initial state $\mathbf{x}(\tau - \Delta) = \mathbf{x}_0$, and

as input function $\sigma^\Delta \mathbf{u}(\cdot)$.

It is immediate to verify that the state update map of the discrete-time system (3.7) enjoys the time-invariance property: it is sufficient to apply $t - \tau$ times the one step update equation to the two intervals $[\tau, t]$ and $[\tau - \Delta, t - \Delta]$. In the continuous-time case (3.7), verifying this property is slightly more elaborate.

- EXERCISE 3.1.2 Verify that the system described by the differential equation $\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t))$ is time-invariant.

Solution. Assume that the function $\mathbf{x}(\cdot)$ satisfies the differential equation with $\mathbf{x}(\tau) = \mathbf{x}_0$ and assume

$$\bar{\mathbf{x}}(\cdot) = (\sigma^\Delta \mathbf{x})(\cdot), \quad \bar{\mathbf{u}}(\cdot) = (\sigma^\Delta \mathbf{u})(\cdot).$$

Then $\bar{\mathbf{x}}(\cdot)$ satisfies the differential equation

$$\frac{d\bar{\mathbf{x}}(t - \Delta)}{d(t - \Delta)} = \frac{d\bar{\mathbf{x}}(t - \Delta)}{dt} = \frac{d\mathbf{x}(t)}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) = \mathbf{f}(\bar{\mathbf{x}}(t - \Delta), \bar{\mathbf{u}}(t - \Delta))$$

namely, upon assuming $t' = t - \Delta$,

$$\frac{d\bar{\mathbf{x}}(t')}{d(t')} = \mathbf{f}(\bar{\mathbf{x}}(t'), \bar{\mathbf{u}}(t')) , \quad \bar{\mathbf{x}}(\tau - \Delta) = \mathbf{x}_0.$$

This shows that the function $\bar{\mathbf{x}}(\cdot)$ is the solution of the differential equation corresponding to the initial condition $\bar{\mathbf{x}}(\tau - \Delta) = \mathbf{x}_0$ and the input $(\sigma^\Delta \mathbf{u})(\cdot)$, and at time $t - \Delta$ one has $\bar{\mathbf{x}}(t - \Delta) = (\sigma^\Delta \mathbf{x})(t - \Delta) = \mathbf{x}(t)$.

As far as the output is concerned, the time-invariance is an immediate consequence of the structure of the output map (3.3).

3.2 Equilibrium points and their properties

We first consider autonomous (i.e., without inputs) time-invariant discrete-time state-space models

$$\mathbf{x}(t + 1) = \mathbf{f}(\mathbf{x}(t)) \tag{3.9}$$

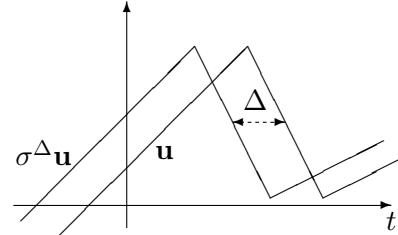


Figure 3.1.1

or continuous-time ones

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)), \quad (3.10)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ for every $t \geq 0$.

Definition 3.2.1 [EQUILIBRIUM POINT] A state vector \mathbf{x}_e is an equilibrium point (or fixed point) of the system (3.9) or (3.10) if, when the initial state is $\mathbf{x}(0) = \mathbf{x}_e$, it holds $\mathbf{x}(t) = \mathbf{x}_e$ for all $t \geq 0$.

Clearly, in the discrete-time case \mathbf{x}_e is an equilibrium point if and only if

$$\mathbf{f}(\mathbf{x}_e) = \mathbf{x}_e; \quad (3.11)$$

in the continuous-time case, condition

$$\mathbf{f}(\mathbf{x}_e) = \mathbf{0} \quad (3.12)$$

is necessary to guarantee that \mathbf{x}_e is an equilibrium point, and it is also sufficient to guarantee that the system remains in \mathbf{x}_e if (as we will assume henceforth) the differential equation (3.10) admits a unique solution for any given initial condition.

Definition 3.2.1 extends to the non-autonomous systems (3.7) and (3.8) controlled by a constant input $\mathbf{u}(t) = \bar{\mathbf{u}}$, $\forall t$.

Definition 3.2.2 [EQUILIBRIUM POINT CORRESPONDING TO A CONSTANT INPUT] We say that the state vector \mathbf{x}_e is an equilibrium point of the system (3.7) or (3.8) corresponding to the constant input $\mathbf{u} = \bar{\mathbf{u}}$ if, when the initial state is $\mathbf{x}(0) = \mathbf{x}_e$ and the input is $\mathbf{u}(t) = \bar{\mathbf{u}}$ for every $t \geq 0$, it holds $\mathbf{x}(t) = \mathbf{x}_e$ for every $t \geq 0$.

In the discrete-time case \mathbf{x}_e is an equilibrium point of (3.7) corresponding to the constant input $\mathbf{u} = \bar{\mathbf{u}}$ if and only if

$$\mathbf{f}(\mathbf{x}_e, \bar{\mathbf{u}}) = \mathbf{x}_e; \quad (3.13)$$

while in the continuous-time case (assuming, again, that the differential equation (3.10) admits a unique solution for any given initial condition), \mathbf{x}_e is an equilibrium point of (3.7) corresponding to the constant input $\mathbf{u} = \bar{\mathbf{u}}$ if and only if

$$\mathbf{f}(\mathbf{x}_e, \bar{\mathbf{u}}) = \mathbf{0}. \quad (3.14)$$

We now introduce the concepts of stability, attractiveness and asymptotic stability of an equilibrium point. Since all these concepts refer to the case when the system is autonomous, in the sequel we will concentrate only on the state-space models (3.9) and (3.10).

Intuitively, the notion of *stability* of an equilibrium point \mathbf{x}_e requires that the evolution of the system satisfies the following conditions:

- a “small” displacement of the initial state $\mathbf{x}(0)$ of the system from the equilibrium point \mathbf{x}_e determines a “perturbed” trajectory whose points are all at close distance from \mathbf{x}_e ,

- the neighborhood of \mathbf{x}_e in which the perturbed trajectory is contained can be made arbitrarily small by reducing the displacement of $\mathbf{x}(0)$ from the equilibrium state \mathbf{x}_e .

Such conditions are formalized in the following definition.

Definition 3.2.3 [STABLE EQUILIBRIUM] Let \mathbf{x}_e be an equilibrium for the discrete-time system (3.9) or for the continuous-time system (3.10). We say that \mathbf{x}_e is a stable equilibrium point if, for any real number $\epsilon > 0$, there exist a real number $\delta > 0$ such that if

$$\|\mathbf{x}(0) - \mathbf{x}_e\| < \delta, \quad (3.15)$$

then the evolution $\mathbf{x}(t)$ starting from $\mathbf{x}(0)$ satisfies for any $t \geq 0$ the inequality

$$\|\mathbf{x}(t) - \mathbf{x}_e\| < \epsilon. \quad (3.16)$$

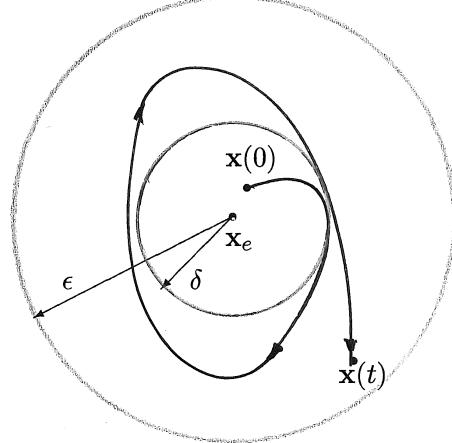


Figura 3.1.1

Observe that verifying the stability of the equilibrium point \mathbf{x}_e requires to:

1. set a ball (for scalar systems an interval) of radius $\epsilon > 0$, arbitrarily small, centered in the equilibrium point;
2. show that there exists a corresponding ball, centered in the equilibrium point and of radius δ , such that any trajectory of the system starting in this second ball remains confined inside the first ball.

Example 3.2.1 [AFFINE SCALAR SYSTEM] The system

$$x(t+1) = ax(t) + b \quad (3.17)$$

has as unique equilibrium point

$$x_e = \frac{b}{1-a}$$

if $a \neq 1$. If, on the other hand, $a = 1$ and $b \neq 0$, it does not have any equilibrium point, while every state is an equilibrium if $a = 1, b = 0$.

For $a \neq 1$, if we denote $\Delta x := x - x_e$ then

$$\Delta x(t+1) = x(t+1) - x_e = ax(t) + b - x_e = a\Delta x(t) + [(a-1)x_e + b] = a\Delta x(t).$$

Hence, for $|a| \leq 1$ the equilibrium point x_e is stable: given $\epsilon > 0$, it is sufficient to choose $\delta = \epsilon$ for any initial state at a distance less than δ from the equilibrium point to yield an evolution which remains at a distance less than ϵ from the equilibrium point.

Example 3.2.2 Consider the scalar function $f : \mathbb{R} \rightarrow \mathbb{R}$ defined as

$$f(x) = \begin{cases} 0, & \text{if } x < 0 \\ 2x, & \text{if } 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \text{if } \frac{1}{2} \leq x \leq 1 \\ 0, & \text{if } x > 1. \end{cases} \quad (3.18)$$

The system

$$x(t+1) = f(x(t))$$

has the equilibrium points $x_{e1} = 0$ and $x_{e2} = \frac{2}{3}$, that is, the abscissae of the intersections between the graph of f and the bisectrix of the first and third quadrants.

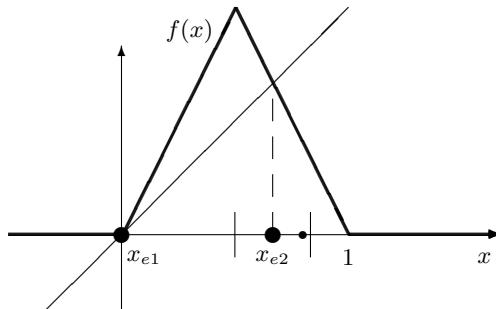


Figure 3.2.2

The equilibrium point x_{e2} is not stable, since if we choose $\epsilon < \frac{1}{6}$, then every trajectory starting in an arbitrary point $x(0)$ different from the equilibrium point and belonging to the interval

$$I_\epsilon = \{x : |x - x_{e2}| < \epsilon\}$$

leaves the interval.

To show this, set $\Delta x = x - x_{e2}$. Since in I_ϵ the displacement Δx updates according to the law

$$\Delta x(t+1) = -2\Delta x(t),$$

if $0 < |\Delta x(0)| < \frac{1}{6}$ then there exists a time instant in which $|\Delta x(t)| \geq \frac{1}{6}$ and $x(t+1)$ leaves the interval I_ϵ .

- EXERCISE 3.2.1 [STABILITY DEFINITION AND EQUILIBRIUM CONDITION] Let \mathbf{x}_0 be an arbitrary state of (3.9) or (3.10) and assume that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\|\mathbf{x}(0) - \mathbf{x}_0\| < \delta$ and $\mathbf{x}(t)$ is the trajectory which starts in $\mathbf{x}(0)$, then $\|\mathbf{x}(t) - \mathbf{x}_0\| < \epsilon$, $\forall t \geq 0$. Show that \mathbf{x}_0 is an equilibrium point of the system.

Suggestion: Assume that \mathbf{x}_0 is not an equilibrium point. In the discrete-time case, we get $\mathbf{f}(\mathbf{x}_0) \neq \mathbf{x}_0$ and it is sufficient to choose $\epsilon = \frac{1}{2}\|\mathbf{f}(\mathbf{x}_0) - \mathbf{x}_0\|$ to reach a contradiction. In the continuous-time case, instead, the trajectory starting in \mathbf{x}_0 passes through a state $\mathbf{x}(t) \neq \mathbf{x}_0$. Choosing $\epsilon = \frac{1}{2}\|\mathbf{x}(t) - \mathbf{x}_0\|$ we then get a contradiction.

The concept of *convergence to an equilibrium point* is independent of that of stability. It deals with the case in which, as t diverges, an equilibrium point \mathbf{x}_e is an asymptotic “attractor” for any trajectory of the system beginning in a state $\mathbf{x}(0)$ sufficiently close to \mathbf{x}_e .

Definition 3.2.4 [ATTRACTIVE EQUILIBRIUM POINT] Let \mathbf{x}_e be an equilibrium point for the system (3.9) or for the system (3.10). The equilibrium point in \mathbf{x}_e is attractive if there exists a real $\delta > 0$ such that, if

$$\|\mathbf{x}(0) - \mathbf{x}_e\| < \delta,$$

then for $t \rightarrow +\infty$ the state $\mathbf{x}(t)$ tends to \mathbf{x}_e

$$\lim_{t \rightarrow +\infty} \mathbf{x}(t) = \mathbf{x}_e.$$

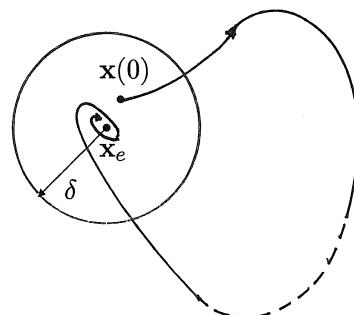


Figura 3.1.3

The definition of stability assures that the radius ϵ of the ball around the equilibrium point \mathbf{x}_e in which the trajectory is confined can be reduced arbitrarily, provided that the

radius δ of the neighborhood of \mathbf{x}_e in which the initial state lies is also reduced.

In general, convergence to the equilibrium does not imply that by reducing the radius $\delta > 0$ of the neighborhood of \mathbf{x}_e in which the trajectory begins it is possible to reduce arbitrarily the radius ϵ of the neighborhood of \mathbf{x}_e in which the entire trajectory is confined. It only means that the displacement of the points $\mathbf{x}(t)$ of the trajectory from the equilibrium state \mathbf{x}_e can be made arbitrarily small from some sufficiently large time t .

When stability and convergence coexist, we have *asymptotic stability* of the equilibrium point.

Definition 3.2.5. [ASYMPTOTICALLY STABLE EQUILIBRIUM POINT] Let \mathbf{x}_e be an equilibrium state for the system (3.9) or for the system (3.10). The equilibrium point \mathbf{x}_e is asymptotically stable if it is stable and attractive.

Example 3.2.3 For the discrete-time system of order one

$$x(t+1) = \begin{cases} 2x(t), & \text{if } |x(t)| < 1 \\ 0, & \text{if } |x(t)| \geq 1 \end{cases}$$

the equilibrium in the origin is attractive but not stable.

Example 3.2.4 For the discrete-time system $\mathbf{x}(t+1) = \mathbf{x}(t)$ the equilibrium in any state is stable but not attractive.

We conclude this part by observing that the convergence to an equilibrium is defined as a “local” property: it holds corresponding to sufficiently small perturbations of the equilibrium state, confined inside a neighborhood of \mathbf{x}_e of which it is useful to establish the existence but not to determine the size.

When it is important to know the *domain of attraction* of \mathbf{x}_e , that is, the set of all points in the state space X from which the state evolution converges to \mathbf{x}_e , then we have a stability problem “at large”. In particular, an equilibrium point is said to be *globally asymptotically stable* if it is stable and if every evolution of the system converges to \mathbf{x}_e when $t \rightarrow +\infty$. Stability problems at large are normally much harder than those pertaining local stability. Linear systems are an exception: as we will see, for them local and global stability are equivalent properties.

3.3 Stability of linear systems

For linear systems, the detection of equilibrium points and the analysis of stability and convergence can all be formulated as problems of linear algebra and modal analysis.

The set X_e of equilibrium points is a subspace of the state space. For a discrete-time system $\mathbf{x}(t+1) = F\mathbf{x}(t)$, in fact, one has

$$X_e = \{\mathbf{x} : \mathbf{x} = F\mathbf{x}\} = \{\mathbf{x} : (F - I_n)\mathbf{x} = \mathbf{0}\} = \ker(F - I_n). \quad (3.19)$$

Apart from the origin, which is always an equilibrium point, there can be other equilibrium points if and only if the matrix $F - I_n$ is singular. In this case, the subspace of equilibrium points coincides with the eigenspace of the matrix F corresponding to the eigenvalue $\lambda = 1$, and hence has dimension equal to the geometric multiplicity of the eigenvalue $\lambda = 1$.

For a continuous-time system $\dot{\mathbf{x}}(t) = F\mathbf{x}(t)$ one finds

$$X_e = \{\mathbf{x} : F\mathbf{x} = \mathbf{0}\} = \ker F. \quad (3.20)$$

When the matrix F is singular, there are equilibrium states different from zero, and the subspace X_e has dimension equal to the algebraic multiplicity of the eigenvalue $\lambda = 0$.

- EXERCISE 3.3.1 If $\tilde{\mathbf{x}}(\cdot)$ and $\tilde{\mathbf{x}}(\cdot) + \mathbf{c}$, $\mathbf{c} \neq \mathbf{0}$, are both evolutions of an autonomous linear system, then $\mathbf{c} \in X_e$, and hence the space of equilibrium points of the system does not contain only the origin.

Viceversa, if $\mathbf{c} \in X_e$ and $\tilde{\mathbf{x}}(\cdot)$ is an evolution of the system, also $\tilde{\mathbf{x}}(\cdot) + \mathbf{c}$ is an evolution.

Definition 3.3.1 [HURWITZ AND SCHUR MATRICES] A matrix $F \in \mathbb{R}^{n \times n}$ is said to be *Hurwitz* if all its eigenvalues have negative real part. It is called *Schur* if all its eigenvalues have modulus smaller than 1.

The unforced evolution of a linear system, in both continuous and discrete time, is expressed as $\mathbf{x}(t) = \Phi(t)\mathbf{x}(0)$, with $\Phi(t) = e^{Ft}$ in the continuous-time case and $\Phi(t) = F^t$ in the discrete-time case. The following proposition summarizes the fundamental facts concerning the stability and convergence of the equilibrium points of a linear system.

Proposition 3.3.2 [CONVERGENCE AND STABILITY OF EQUILIBRIUM POINTS OF LINEAR SYSTEMS] In a linear autonomous system

1. if the system has an attractive equilibrium point then it must necessarily be the origin. In this case the system cannot have any other equilibrium point, that is $X_e = \{\mathbf{0}\}$;
2. the equilibrium in the origin is attractive if and only if
 - in the continuous-time case the matrix F is Hurwitz,
 - in the discrete-time case the matrix F is Schur;
3. if the origin is an attractive equilibrium then convergence is global;
4. if the system has a stable equilibrium point, then any other equilibrium point has the same stability property;
5. the equilibrium in the origin is stable if and only if
 - in the continuous-time case the matrix F has all eigenvalues with real part which is non-positive and the eigenvalues with zero real part are simple roots of the minimal polynomial,
 - in the discrete-time case the matrix F has all eigenvalues with modulus less or equal than 1, and the eigenvalues with modulus equal to 1 are simple roots of the minimal polynomial;
6. if the origin is an attractive equilibrium point, then it is stable and hence also asymptotically stable.

PROOF 1. If $\mathbf{x}_e \neq \mathbf{0}$ is an equilibrium point, so are all the points of the line $\{\alpha\mathbf{x}_e, \alpha \in \mathbb{R}\}$. Then the equilibrium point in \mathbf{x}_e cannot be attractive because every neighborhood of \mathbf{x}_e , of arbitrarily small radius δ , contains other equilibrium points which obviously do not converge to \mathbf{x}_e . For the same reason, if $\mathbf{x}_e \neq \mathbf{0}$ is an equilibrium point, then $\mathbf{0}$ cannot be an attractive equilibrium. Hence the existence of an attractive equilibrium state implies $X_e = \{\mathbf{0}\}$.

2. When F has at least one eigenvalue with nonnegative real part (modulus smaller than 1 in the discrete-time case), at least one element of the matrix e^{Ft} (of F^t) does not converge to zero¹ as t grows. If this element is the (i, j) -entry, for any $\delta \neq 0$ the initial state $\delta\mathbf{e}_j$ induces a state evolution that does not converge to $\mathbf{0}$. Hence the origin cannot be an attractive equilibrium point.

If instead all the eigenvalues of F have negative real part (all have modulus smaller than 1 in the discrete-time case), every element of e^{Ft} (of F^t) converges to zero as t grows, and any initial state $\mathbf{x}(0)$ gives a state evolution $e^{Ft}\mathbf{x}(0)$ ($F^t\mathbf{x}(0)$) which converges to the origin as $t \rightarrow +\infty$.

3. This is obvious, given that all elements of $\Phi(t)$ are infinitesimal for $t \rightarrow +\infty$.

4. Let us observe first that if $\mathbf{x}_e \neq \mathbf{0}$ is an equilibrium point and if $\mathbf{x}(\cdot)$ is the evolution from the initial state $\mathbf{x}(0)$, then $\mathbf{x}(\cdot) + \mathbf{x}_e$ is the evolution from the initial state $\mathbf{x}(0) + \mathbf{x}_e$. Assume then that the origin is a stable equilibrium point. For every choice of $\epsilon > 0$, by definition of stability there exists a positive δ such that

$$\|\mathbf{x}(0)\| < \delta \Rightarrow \|\mathbf{x}(t)\| < \epsilon, \quad \forall t \geq 0. \quad (3.21)$$

But then, given a ball $\mathcal{S}(\mathbf{x}_e, \epsilon)$ centered in the equilibrium point \mathbf{x}_e and given the same δ for which (3.21) holds, it follows from the reasoning above that every evolution beginning in $\mathcal{S}(\mathbf{x}_e, \delta)$ remains confined inside $\mathcal{S}(\mathbf{x}_e, \epsilon)$, and this shows that also \mathbf{x}_e is stable.

5. If the eigenvalues satisfy the given assumption, then every elementary mode of the system must be bounded, and hence all elements $\Phi_{ij}(\cdot)$ of the transition matrix must be bounded, regardless of the chosen basis. Letting

$$M := \sup_{\substack{t \in [0, +\infty) \\ i, j = 1, 2, \dots, n}} |\Phi_{ij}(t)|$$

and denoting by $\|\cdot\|_2$ the Euclidean norm of a vector, we have²

$$\|\mathbf{x}(t)\|_2 = \|\Phi(t)\mathbf{x}(0)\|_2 = \left\| \begin{bmatrix} \sum_{j=1}^n \Phi_{1j}(t)x_j(0) \\ \vdots \\ \sum_{j=1}^n \Phi_{nj}(t)x_j(0) \end{bmatrix} \right\|_2 \leq \left\| \begin{bmatrix} M \sum_{j=1}^n |x_j(0)| \\ \vdots \\ M \sum_{j=1}^n |x_j(0)| \end{bmatrix} \right\|_2$$

¹If all elements of e^{Ft} or of F^t were converging to 0, changing basis and moving to a Jordan form it would turn out that all modes of the system would be converging.

²See Exercise A.11.2.

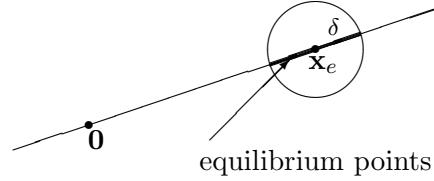


Figure 3.2.1

$$\leq M\|\mathbf{x}(0)\|_1 \left\| \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right\|_2 = M\sqrt{n}\|\mathbf{x}(0)\|_1 \leq Mn\|\mathbf{x}(0)\|_2.$$

In order for the evolution to be contained in a ball centered at the origin and of radius ϵ , it is sufficient to choose the initial state inside a sphere $\mathcal{S}(\mathbf{0}, \delta)$ of radius $\delta \leq \frac{\epsilon}{nM}$.

Viceversa, if not every eigenvalue satisfies the assumption, the system must have at least a mode which is not bounded. Then in the transition matrix at least one element is not bounded. Repeating the argument at item 2, in an arbitrarily small neighborhood of the origin, it is possible to find an initial state that induces an unbounded evolution. Hence the equilibrium point in the origin cannot be stable.

6. Follows from items 2 and 5. ■

3.4 Lyapunov equations

Lyapunov equations are linear matrix equations widely used in control theory and in filtering.

3.4.1 Equations for linear continuous-time systems

We have just seen that the origin is an asymptotically stable equilibrium point for a continuous-time linear system $\dot{\mathbf{x}}(t) = F\mathbf{x}(t)$ if and only if the eigenvalues of F have all negative real part. Based on this result we have the following

Definition 3.4.1 [POSITIVE OR NEGATIVE (SEMI)DEFINITE MATRIX] Let $P \in \mathbb{R}^{n \times n}$ be a real square matrix and suppose it is symmetric, namely $P = P^T$. The matrix P

- is positive semidefinite ($P \geq 0$) if $\mathbf{x}^T P \mathbf{x} \geq 0$ for every $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$;
- is positive definite ($P > 0$) if $\mathbf{x}^T P \mathbf{x} > 0$ for every $\mathbf{x} \in \mathbb{R}^n, \mathbf{x} \neq 0$;
- is negative (semi)definite ($P \leq 0$ or $P < 0$) if $-P$ is positive (semi)definite.

Theorem 3.4.2 [LYAPUNOV EQUATIONS FOR CONTINUOUS-TIME SYSTEMS] A necessary and sufficient condition for the asymptotic stability of the system

$$\dot{\mathbf{x}}(t) = F\mathbf{x}(t) \tag{3.22}$$

is that for any positive definite matrix Q the linear matrix equation

$$F^T X + X F = -Q \tag{3.23}$$

in the matrix unknown X of dimension $n \times n$ admits a positive definite solution P .

The following proposition provides an important result when one want to use Lyapunov equation as stability criterion: according to it, the existence of multiple solutions of the equation rules out the asymptotic stability of the system, with no need to check the nature

of these solutions (notice that if asymptotic stability were compatible with the existence of multiple solutions one should have had to verify whether the set of such solutions contains at least one which is positive definite).

Proposition 3.4.3 *Given a positive definite matrix Q , if the system is asymptotically stable the solution of (3.23) is unique.*

PROOF Assume that P_1 and P_2 are two solutions of (3.23). Premultiplying by $e^{F^T t}$ and postmultiplying by e^{Ft} both members of

$$F^T(P_1 - P_2) + (P_1 - P_2)F = 0, \quad (3.24)$$

one gets

$$0 = e^{F^T t} F^T (P_1 - P_2) e^{Ft} + e^{F^T t} (P_1 - P_2) F e^{Ft} = \frac{d}{dt} \left(e^{F^T t} (P_1 - P_2) e^{Ft} \right), \forall t.$$

Hence the matrix $e^{F^T t} (P_1 - P_2) e^{Ft}$ is independent of t , and its value can be obtained setting $t = 0$

$$e^{F^T t} (P_1 - P_2) e^{Ft} = P_1 - P_2, \quad \forall t. \quad (3.25)$$

When $t \rightarrow +\infty$, the modes of the system, and with them e^{Ft} , tend to zero and (3.25) gives

$$0 = P_1 - P_2.$$

■

- EXERCISE 3.4.1 [LYAPUNOV EQUATION AND CHANGE OF BASIS] Let $X = M$ be a solution of the equation $F^T X + X F = L$. If $\bar{F} = T^{-1}FT$ and $\bar{L} = T^T LT$, the equation $\bar{F}^T X + X \bar{F} = \bar{L}$ has $T^T MT$ as a solution.
- EXERCISE 3.4.2 [SYMMETRIC SOLUTIONS OF THE LYAPUNOV EQUATION] Let $F \in \mathbb{R}^{n \times n}$ and $S = S^T \in \mathbb{R}^{n \times n}$. Assume that the equation $F^T X + X F = S$ admits a solution R (not necessarily symmetric). Then
 - $(R + R^T)/2$ is a symmetric solution of the same equation;
 - if the solution is unique, then it is necessarily symmetric;
 - if $F = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $S = 0$, the symmetric solution of the equation is not unique.

3.4.2 Equation for discrete-time linear systems

Also for discrete-time linear systems asymptotic stability can be studied with an appropriate Lyapunov equation, as stated in the following theorem.

Theorem 3.4.4 [LYAPUNOV EQUATION FOR DISCRETE-TIME SYSTEMS] *A necessary and sufficient condition for the asymptotic stability of the discrete-time linear system*

$$\mathbf{x}(t+1) = F\mathbf{x}(t) \quad (3.26)$$

is that for every positive definite matrix Q the linear matrix equation

$$F^T X F - X = -Q \quad (3.27)$$

in the matrix unknown $X \in \mathbb{R}^{n \times n}$ admits a positive definite solution P .

Remark It is worth remarking that if the system is asymptotically stable, namely F is a Schur matrix, for every $Q = Q^T > 0$ the solution of (3.27) is expressed as

$$P := \sum_{t=0}^{+\infty} (F^T)^t Q F^t. \quad (3.28)$$

It is also worth noticing that if F is Schur, then for every positive semi-definite Q , the matrix (3.28) provides a positive semidefinite solution of (3.27).

3.5 Linearization of a nonlinear state-space model around an equilibrium condition

Given a discrete-time, time-invariant system, described by the equations (3.7), assume that \mathbf{x}_e is an *equilibrium state corresponding to the constant input $\mathbf{u} = \bar{\mathbf{u}}$* , by this meaning that if the initial state of the system is $\mathbf{x}(0) = \mathbf{x}_e$ and we apply the input $\mathbf{u}(t) = \bar{\mathbf{u}}, \forall t \geq 0$, to the system, then the state of the system indefinitely remains in the state \mathbf{x}_e . As we have seen, a necessary and sufficient condition for this to happen is that \mathbf{x}_e satisfies the equation

$$\mathbf{x}_e = \mathbf{f}(\mathbf{x}_e, \bar{\mathbf{u}}). \quad (3.29)$$

Note that, in this situation, the system output is constant and takes value $\mathbf{y}_e := \mathbf{h}(\mathbf{x}_e, \bar{\mathbf{u}})$. Suppose, in addition, that the functions $f_i, i = 1, \dots, n$, and $h_j, j = 1, \dots, p$, are differentiable, and their first order derivatives are continuous in a neighbourhood of $(\mathbf{x}_e, \bar{\mathbf{u}})$. We want to describe the dynamics of the system when the initial condition $\mathbf{x}(0)$ and the input sequence $\mathbf{u}(t), t \geq 0$, are obtained by slightly perturbing the equilibrium values \mathbf{x}_e and $\bar{\mathbf{u}}$. To this end, denote by

- $\Delta \mathbf{x}(t)$ the difference between the state at time t and the equilibrium value \mathbf{x}_e

$$\Delta \mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_e,$$

- $\Delta \mathbf{u}(t)$ the difference between the input at time t and the constant value $\bar{\mathbf{u}}$

$$\Delta \mathbf{u}(t) = \mathbf{u}(t) - \bar{\mathbf{u}},$$

- $\Delta \mathbf{y}(t)$ the difference between the output at time t and the output value corresponding to the equilibrium conditions \mathbf{y}_e

$$\Delta \mathbf{y}(t) = \mathbf{y}(t) - \mathbf{y}_e.$$

The one-step update equation of the system, corresponding to an input \mathbf{u} and a state \mathbf{x} belonging to a neighbourhood of the equilibrium condition $(\mathbf{x}_e, \bar{\mathbf{u}})$, can be represented by expanding the function \mathbf{f} in Taylor series and approximating it with its first order term:

$$\mathbf{x}(t+1) = \mathbf{x}_e + \Delta \mathbf{x}(t+1) = \mathbf{f}(\mathbf{x}_e + \Delta \mathbf{x}(t), \bar{\mathbf{u}} + \Delta \mathbf{u}(t))$$

$$= \mathbf{f}(\mathbf{x}_e, \bar{\mathbf{u}}) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta \mathbf{x}(t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta \mathbf{u}(t) + \varepsilon. \quad (3.30)$$

Analogously, for the output equation one has:

$$\begin{aligned} \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}_e + \Delta \mathbf{x}(t), \bar{\mathbf{u}} + \Delta \mathbf{u}(t)) \\ &= \mathbf{h}(\mathbf{x}_e, \bar{\mathbf{u}}) + \left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta \mathbf{x}(t) + \left[\frac{\partial \mathbf{h}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta \mathbf{u}(t) + \eta. \end{aligned} \quad (3.31)$$

The constant matrices

$$F := \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \quad G := \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_m} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_m} \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_m} \end{bmatrix}_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \quad (3.32)$$

are the Jacobian matrices of the function \mathbf{f} with respect to \mathbf{x} and \mathbf{u} , evaluated at $\mathbf{x} = \mathbf{x}_e$ and $\mathbf{u} = \bar{\mathbf{u}}$. The matrices

$$H := \left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}}, \quad D := \left[\frac{\partial \mathbf{h}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}}$$

have an analogous meaning. The terms ε and η are infinitesimal of higher order with respect to $(\|\Delta \mathbf{x}(t)\|^2 + \|\Delta \mathbf{u}(t)\|^2)^{\frac{1}{2}}$, and hence, keeping in mind (3.29) and neglecting ε and η in a neighbourhood of the equilibrium condition, from (3.30) and (3.31), one obtains the approximate model

$$\begin{cases} \Delta \mathbf{x}(t+1) &= F \Delta \mathbf{x}(t) + G \Delta \mathbf{u}(t), \\ \Delta \mathbf{y}(t) &= H \Delta \mathbf{x}(t) + D \Delta \mathbf{u}(t). \end{cases} \quad (3.33)$$

Equations (3.33) show that the dynamics of a non-linear system in a neighbourhood of \mathbf{x}_e and for input values that are close to $\bar{\mathbf{u}}$ ($\Delta \mathbf{u}$ is small) satisfies the equations of a linear, time-invariant system. In general, equations (3.33) do not provide a reasonable approximation unless $\Delta \mathbf{u}$ and $\Delta \mathbf{x}$ are sufficiently small.

Example 3.5.1 [MODEL OF AN EPIDEMIC]. Consider two animal populations \mathcal{P}_1 and \mathcal{P}_2 , susceptible to the same infection. Suppose that

- The infection can be contracted only in a well-defined time of the year, so that it is reasonable to resort to a discrete-time model;
- The infection is transmitted through contact between an infected individual belonging to one population and a susceptible one (i.e., not yet or no longer infective) belonging to the other population, but not between individuals of the same population;
- Healing does not confer immunity;
- Populations remain constant in time.

The previously described situation can model, for instance, some epidemics of non-fatal sexual diseases: \mathcal{P}_1 and \mathcal{P}_2 represent the male and female populations of the same species having a short and precise breeding period during the year.

Let us denote by $x_i(t)$, $0 \leq x_i \leq 1$, $i = 1, 2$, the fraction of the population \mathcal{P}_i that, at the beginning of the t -th period has been affected by the infection and can transmit it to individuals of the other population. The complementary quantities $1 - x_i(t)$ are the fractions of the susceptible individuals. Consider first the situation in which the two populations \mathcal{P}_1 and \mathcal{P}_2 are isolated from each other, so that the disease can not be transmitted. In these conditions, one can assume that the infected individuals decrease with the time, due to therapies and/or spontaneous healing, and the fractions of infected individuals in the time unit vary according to a law of the type:

$$\begin{aligned} x_1(t+1) &= (1 - \beta_1)x_1(t), \quad 0 < \beta_1 < 1, \\ x_2(t+1) &= (1 - \beta_2)x_2(t), \quad 0 < \beta_2 < 1, \end{aligned} \quad (3.34)$$

where $\beta_1 x_1(t)$ and $\beta_2 x_2(t)$ are the fractions of the populations \mathcal{P}_1 and \mathcal{P}_2 that are healed (and are again susceptible) in between t and $t + 1$. Clearly, as t grows to $+\infty$,

$$\lim_{t \rightarrow +\infty} x_i(t) = 0,$$

namely infected individuals asymptotically disappear in the two isolated populations. When the two populations interact, the susceptible part of the population \mathcal{P}_1 may contract the infection from the infected part of the population \mathcal{P}_2 and the increase of the infected individuals in the population \mathcal{P}_1 , in the unit of time, will be proportional to the number of encounters, namely to the product $(1 - x_1(t))x_2(t)$. A similar reasoning applies to the increase of infected individuals in the population \mathcal{P}_2 . The equations (3.34) are then modified as follows

$$\begin{aligned} x_1(t+1) &= (1 - \beta_1)x_1(t) + \alpha_1 x_2(t)(1 - x_1(t)) = f_1(x_1(t), x_2(t)) \quad 0 < \beta_1, \alpha_1 \leq 1, \\ x_2(t+1) &= (1 - \beta_2)x_2(t) + \alpha_2 x_1(t)(1 - x_2(t)) = f_2(x_1(t), x_2(t)) \quad 0 < \beta_2, \alpha_2 \leq 1, \end{aligned} \quad (3.35)$$

where the conditions on α_i , $i = 1, 2$, ensure that x_1 and x_2 necessarily belong at every time instant to the interval $[0, 1]$. System (3.35) is “autonomous”, i.e., with no inputs, and it obviously has an equilibrium point $\mathbf{x}_e^{(1)} = \mathbf{0}$, corresponding to the lack of infected individuals in both populations. By solving the algebraic equation (3.29), namely

$$\begin{aligned} x_{1e} &= (1 - \beta_1)x_{1e} + \alpha_1 x_{2e}(1 - x_{1e}), \\ x_{2e} &= (1 - \beta_2)x_{2e} + \alpha_2 x_{1e}(1 - x_{2e}), \end{aligned} \quad (3.36)$$

one can determine another equilibrium point $\mathbf{x}_e^{(2)} \neq \mathbf{0}$. As a nonzero solution of (3.36) has both coordinates different from zero, we can assume $\xi_1 = 1/x_{1e}$, $\xi_2 = 1/x_{2e}$, $\gamma_1 = \beta_1/\alpha_1$, $\gamma_2 = \beta_2/\alpha_2$. From (3.36) we obtain a linear system of equations in the unknown variables ξ_1 and ξ_2 :

$$\begin{aligned} \gamma_1 \xi_2 - \xi_1 + 1 &= 0 \\ \gamma_2 \xi_1 - \xi_2 + 1 &= 0 \end{aligned} \quad (3.37)$$

from which we derive

$$\mathbf{x}_e^{(2)} = \begin{bmatrix} \frac{1}{\xi_1} \\ \xi_1 \end{bmatrix} = \begin{bmatrix} \frac{1 - \gamma_1 \gamma_2}{1 + \gamma_1} \\ \frac{1 - \gamma_1 \gamma_2}{1 + \gamma_2} \end{bmatrix}.$$

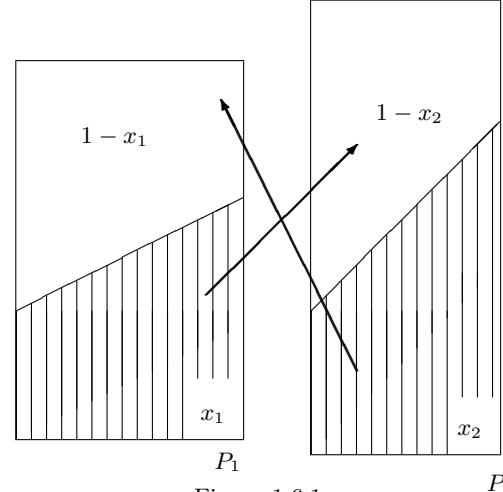


Figure 1.6.1

Clearly $\mathbf{x}_e^{(2)}$ has a physical meaning and does not coincide with $\mathbf{x}_e^{(1)}$ only if both its coordinates are positive, namely if $\alpha_1\alpha_2 - \beta_1\beta_2 > 0$.

In a neighbourhood of the origin, namely when the populations have very few infected individuals, the dynamics of x_1 and x_2 is described by the linearised system

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_e^{(1)}=\mathbf{0}} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 1 - \beta_1 & \alpha_1 \\ \alpha_2 & 1 - \beta_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}. \quad (3.38)$$

In a neighbourhood of the second equilibrium point, namely when the norm of $\Delta\mathbf{x} = \mathbf{x} - \mathbf{x}_e^{(2)}$ is small, the dynamics is described by the linearised system

$$\begin{bmatrix} \Delta x_1(t+1) \\ \Delta x_2(t+1) \end{bmatrix} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{\mathbf{x}=\mathbf{x}_e^{(2)}} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix} = \begin{bmatrix} 1 - \beta_1 - \alpha_1 x_{2e}^{(2)} & \alpha_1(1 - x_{1e}^{(2)}) \\ \alpha_2(1 - x_{2e}^{(2)}) & 1 - \beta_2 - \alpha_2 x_{1e}^{(2)} \end{bmatrix} \begin{bmatrix} \Delta x_1(t) \\ \Delta x_2(t) \end{bmatrix}.$$

Also for a continuous-time system, described by the equations (3.8), it is possible to determine a linear system which approximates the dynamics in a neighbourhood of an equilibrium state obtained corresponding to a constant input $\bar{\mathbf{u}}$. In this case, a state \mathbf{x}_e is an equilibrium point corresponding to the input $\mathbf{u}(t) = \bar{\mathbf{u}}, \forall t \geq 0$, if it satisfies the algebraic equation

$$\mathbf{0} = \mathbf{f}(\mathbf{x}_e, \bar{\mathbf{u}}). \quad (3.39)$$

With the same notations introduced for discrete-time systems, one obtains that the quantities

$$\Delta\mathbf{x}(t) = \mathbf{x}(t) - \mathbf{x}_e, \quad \Delta\mathbf{u}(t) = \mathbf{u}(t) - \bar{\mathbf{u}}, \quad \Delta\mathbf{y}(t) = \mathbf{y}(t) - \mathbf{y}_e = \mathbf{y}(t) - \mathbf{h}(\mathbf{x}_e, \bar{\mathbf{u}}),$$

satisfy the equations

$$\begin{aligned} \frac{d(\Delta\mathbf{x})}{dt} &= \frac{d(\mathbf{x}_e + \Delta\mathbf{x})}{dt} = \mathbf{f}(\mathbf{x}_e + \Delta\mathbf{x}, \bar{\mathbf{u}} + \Delta\mathbf{u}) \\ &= \mathbf{f}(\mathbf{x}_e, \bar{\mathbf{u}}) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta\mathbf{x}(t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta\mathbf{u}(t) + \varepsilon \\ &= \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta\mathbf{x}(t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta\mathbf{u}(t) + \varepsilon \\ \Delta\mathbf{y}(t) &= \mathbf{h}(\mathbf{x}_e + \Delta\mathbf{x}, \Delta\mathbf{u}) - \mathbf{h}(\mathbf{x}_e, \bar{\mathbf{u}}) \\ &= \left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta\mathbf{x}(t) + \left[\frac{\partial \mathbf{h}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_e \\ \mathbf{u}=\bar{\mathbf{u}}}} \Delta\mathbf{u}(t) + \eta \end{aligned} \quad (3.40)$$

and hence, in a neighbourhood of the equilibrium point, they satisfy the approximate equations

$$\frac{d\Delta\mathbf{x}(t)}{dt} = F \Delta\mathbf{x}(t) + G \Delta\mathbf{u}(t),$$

$$\Delta \mathbf{y}(t) = H \Delta \mathbf{x}(t) + D \Delta \mathbf{u}(t). \quad (3.41)$$

Remark The linearisation procedure can be extended to a more general case, to study the dynamics of a non-linear system corresponding to small displacements with respect to a predetermined nominal dynamics.

Suppose to know the state and output evolutions, $\mathbf{x}_N(\cdot)$ and $\mathbf{y}_N(\cdot)$, respectively, corresponding to a “nominal” initial state $\mathbf{x}_N(0)$ and some “nominal” input $\mathbf{u}_N(\cdot)$. We want to evaluate the state and the output responses, $\mathbf{x}_P(\cdot)$ and $\mathbf{y}_P(\cdot)$, corresponding to the “perturbed” versions $\mathbf{x}_P(0)$ of the initial condition and $\mathbf{u}_P(\cdot)$ of the input trajectory. Set

$$\begin{aligned} \Delta \mathbf{x}(t) &= \mathbf{x}_P(t) - \mathbf{x}_N(t), \\ \Delta \mathbf{u}(t) &= \mathbf{u}_P(t) - \mathbf{u}_N(t), \\ \Delta \mathbf{y}(t) &= \mathbf{y}_P(t) - \mathbf{y}_N(t). \end{aligned} \quad (3.42)$$

If we refer to a discrete-time system, for instance, we get:

$$\begin{aligned} \Delta \mathbf{x}(t+1) &= \mathbf{f}(\mathbf{x}_P(t), \mathbf{u}_P(t)) - \mathbf{f}(\mathbf{x}_N(t), \mathbf{u}_N(t)) \\ &= \mathbf{f}(\mathbf{x}_N(t) + \Delta \mathbf{x}(t), \mathbf{u}_N(t) + \Delta \mathbf{u}(t)) - \mathbf{f}(\mathbf{x}_N(t), \mathbf{u}_N(t)) \\ &= \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t)}} \Delta \mathbf{x}(t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t)}} \Delta \mathbf{u}(t) + \epsilon, \\ \Delta \mathbf{y}(t) &= \mathbf{h}(\mathbf{x}_P(t), \mathbf{u}_P(t)) - \mathbf{h}(\mathbf{x}_N(t), \mathbf{u}_N(t)) \\ &= \mathbf{h}(\mathbf{x}_N(t) + \Delta \mathbf{x}(t), \mathbf{u}_N(t) + \Delta \mathbf{u}(t)) - \mathbf{h}(\mathbf{x}_N(t), \mathbf{u}_N(t)) \\ &= \left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t)}} \Delta \mathbf{x}(t) + \left[\frac{\partial \mathbf{h}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t)}} \Delta \mathbf{u}(t) + \eta. \end{aligned} \quad (3.43)$$

By neglecting the terms ϵ and η and by assuming

$$F(t) := \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t)}}, \quad G(t) := \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t)}}, \quad H(t) = \left[\frac{\partial \mathbf{h}}{\partial \mathbf{x}} \right]_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t)}}, \quad D(t) = \left[\frac{\partial \mathbf{h}}{\partial \mathbf{u}} \right]_{\substack{\mathbf{x}=\mathbf{x}_N(t) \\ \mathbf{u}=\mathbf{u}_N(t)}}$$

one can obtain the linearised system

$$\begin{aligned} \Delta \mathbf{x}(t+1) &= F(t) \Delta \mathbf{x}(t) + G(t) \Delta \mathbf{u}(t), \\ \Delta \mathbf{y}(t) &= H(t) \Delta \mathbf{x}(t) + D(t) \Delta \mathbf{u}(t), \end{aligned} \quad (3.44)$$

where F, G, H and D are time-varying matrices. This is due to the fact that the Jacobian matrices are not evaluated corresponding to constant values of the input and the state, but along the nominal trajectories $\mathbf{x}_N(t)$ and $\mathbf{u}_N(t)$, that vary with t .

3.6 Stability analysis through the linearization method

Proposition 3.6.1 [REDUCED LYAPUNOV CRITERION: CONTINUOUS-TIME CASE]. Let us assume that in the nonlinear autonomous system (3.10) \mathbf{f} is continuous with its first derivatives, and that $\mathbf{x}_e \in \mathbb{R}^n$ is an equilibrium point.

(i) If in the system

$$\dot{\mathbf{x}}(t) = F\mathbf{x}(t), \quad (3.45)$$

obtained linearizing (3.10) around \mathbf{x}_e , the Jacobian matrix F is Hurwitz (that is, all its eigenvalues have negative real parts), then \mathbf{x}_e is an asymptotically stable equilibrium point for the nonlinear system (3.10).

(ii) If at least one of the eigenvalues of F has positive real part, then \mathbf{x}_e is not a stable equilibrium point of the nonlinear system (3.10).

The reduced criterion does not cover situations in which all the eigenvalues of the Jacobian matrix belong to the closed left half of the complex plane with one or more of them on the imaginary axis. In this case, while the stability character of the linearized system is determined by the multiplicity of the imaginary roots of the minimal polynomial $\psi_F(s)$, the stability of the equilibrium of the nonlinear system cannot be deduced by the behavior of the linearized system.

Lyapunov reduced criterion can be extended to discrete-time systems in a completely analogous way.

Proposition 3.6.2 [LYAPUNOV REDUCED CRITERION: DISCRETE-TIME CASE]. Assume that in the nonlinear autonomous system (3.9) $\mathbf{f}(\cdot)$ is continuous with its first derivatives and that $\mathbf{x}_e \in \mathbb{R}^n$ is an equilibrium point.

i) If in the system

$$\mathbf{x}(t+1) = F\mathbf{x}(t),$$

obtained linearizing (3.9) in a neighborhood of the origin, the Jacobian matrix F is Schur (that is, all its eigenvalues have moduli smaller than 1), then \mathbf{x}_e is an asymptotically stable equilibrium point for the nonlinear system (3.9).

ii) If at least one of the eigenvalues of F has modulus greater than 1, then \mathbf{x}_e is not a stable equilibrium point for the nonlinear system (3.9). ■

Example 3.6.1 [COBWEB MODEL]. The cobweb model represents the dynamics of the iteration between the demand and the offer of a perishable product (for instance an agricultural product). Under suitable assumptions, it can be obtained from static models representing how the price of the product influences the market demand and the producers' offer.

Let $x(t)$ denote the price of the product in the t -th period and assume that

1. the producers predict that the price will not change in the time period that follows, and hence decide their production level for the $(t+1)$ -th period based on the current price $x(t)$. The production level p is then governed by a law of the type

$$p(t+1) = \alpha(x(t)), \quad (3.46)$$

where $\alpha(\cdot)$ is a monotonically increasing function of its argument.

2. The demand d in the t -th period is a strictly monotonically decreasing function of the price in the same period

$$d(t) = \beta(x(t)). \quad (3.47)$$

3. The market determines the price so that the demand absorbs exactly the production, that is, neither it allows for some product to remain unsold (when the price is too high) nor it allows the demand to remain unsatisfied (when the price is too low):

$$d(t) = p(t) \quad (3.48)$$

From (3.46)-(3.48) one obtains the one-step price update equation

$$x(t+1) = \beta^{-1}(d(t+1)) = \beta^{-1}(p(t+1)) = \beta^{-1}(\alpha(x(t))). \quad (3.49)$$

The smooth curves that represent the functions $\alpha(x)$ and $\beta(x)$ belong to the first quadrant and they meet in at most one point.

If such a point x_e exists and if $x(t) = x_e$, then

$$x(t+1) = \beta^{-1}(\alpha(x_e)) = x_e = x(t)$$

and x_e is an equilibrium point for the system (3.49).

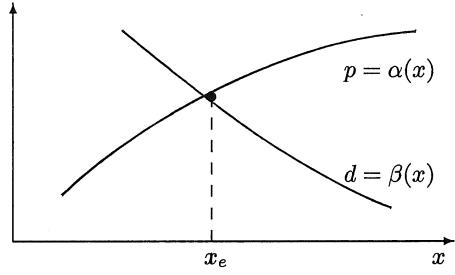


Figura 3.6.2

The behavior of the prices for small displacements Δx of the price from its equilibrium point can be analyzed using the linearization of (3.49) in a neighborhood of x_e :

$$\Delta x(t+1) = \frac{d}{dx}[\beta^{-1}(\alpha(x))]_{x=x_e} \Delta x(t).$$

Upon denoting $a = \left(\frac{d\alpha}{dx}\right)_{x=x_e}$, $-b = \left(\frac{d\beta}{dx}\right)_{x=x_e}$, with $a \geq 0$ and $b > 0$, in x_e one has

$$\frac{d}{dx}\beta^{-1}(\alpha(x)) = \frac{1}{d\beta} \frac{d\alpha}{dx} = \frac{a}{-b}.$$

In a neighborhood of the equilibrium point, the curves of Figure 3.6.2 are approximated by the corresponding tangent lines, and the dynamics of the system beginning from the initial price x_0 , is represented as the "cobweb" shown in Figure 3.6.3.

Since the linearized system has the structure

$$\Delta x(t+1) = -\frac{a}{b} \Delta x(t), \quad (3.50)$$

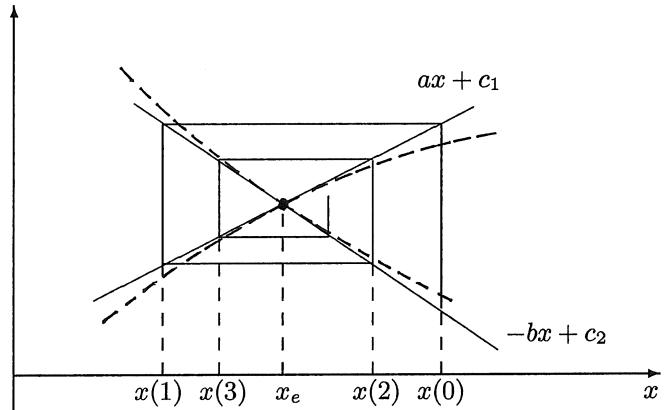


Figura 3.6.3

by the Lyapunov reduced criterion, the equilibrium x_e of the nonlinear system is asymptotically stable if $|a/b| < 1$, while it is unstable if $|a/b| > 1$. When the ratio is 1, the reduced criterion does not provide any information on the stability of the equilibrium of the nonlinear system.

Notice that the asymptotic stability condition, $a < b$, corresponds to assuming that the producers are less "sensitive" than the consumers to price changes.

Figure 3.6.4 shows the sequences $x(t)$ e $x'(t)$ constructed from two initial prices x_0 and x'_0 . As it can be seen, the sequence beginning in x_0 converges to the equilibrium value x_e , while the one beginning from the price x'_0 does not converge. The asymptotic stability of x_e , which can be deduced from the analysis of the linearized system, has in fact only a local character. When investigating the prices induced by an initial condition such as x'_0 , which is quite distant from the equilibrium point, we would have reached an erroneous conclusion if we had adopted the linearized model.

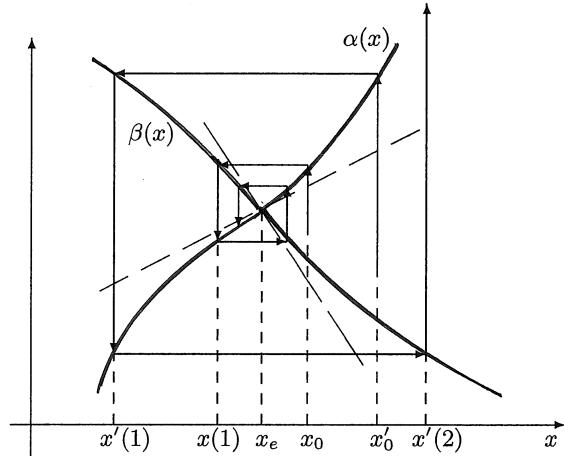


Figura 3.6.4

Example 3.6.2 [NEWTON-RAPHSON METHOD] The Newton-Raphson method is one of the simplest numerical algorithms for the determination of the roots of a scalar equation

$$g(x) = 0. \quad (3.51)$$

Assume that in an interval (a, b) the function g has a first derivative, which is everywhere nonzero, and also has a second derivative, and let us assume

$$f(x) = x - \frac{g(x)}{g'(x)}.$$

The discrete-time system

$$x(t+1) = f(x(t)) \quad (3.52)$$

has in $x_e \in (a, b)$ an equilibrium point if and only if x_e is a root of (3.51).

Furthermore, the derivative of the state update function

$$f'(x) = 1 - \frac{g'(x)^2 - g(x)g''(x)}{g'(x)^2} = \frac{g(x)g''(x)}{g'(x)^2}$$

vanishes in x_e and hence the equilibrium in x_e is asymptotically stable.

This implies that the sequence $x(0), x(1), x(2), \dots$ converges to x_e if the initial state $x(0)$ of (3.52) is sufficiently close to x_e , that is, if a sufficiently accurate initial estimate $x(0)$ of the sought root is available.

3.7 Bibliographical notes

An introduction to several topics in stability theory - some of which are discussed in the next chapter - is provided by the book

- 1 J.L.Willems "*Stability theory of dynamical systems*" Nelson, 1970

Stability is extensively discussed also in several chapters of

- 2 S.Rinaldi "*Teoria dei sistemi*", Hoepli, 1977
- 3 D.G.Luenberger "*Introduction to dynamic systems*", Wiley, 1979

For the stability criteria of the discrete-time unidimensional systems see

- 4 S.Elaydi "*An introduction to difference equations*" Springer Undergraduate Texts in Mathematics, 2005