Polytope

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1 Introduction

Welcome to the Polytope Discord!

2 What is a polytope?

A **polytope** is a general name for **polygons** (2D), **polyhedra** (3D), **polychora** (4D), and so on for any dimension. As with many terms used on the Polytope Discord, the word "polytope" can have a few definitions which are not completely equivalent. All of the commonly-used ones, however, agree on the following:

- A polytope in n dimensions (known hereafter as an n-polytope) is made of **facets** which are $(n-1)$-polytopes.
  - Polychora (4-polytopes) have facets which are polyhedra (3-polytopes),
  - whose facets are polygons (2-polytopes),
  - whose facets are line segments (1-polytopes),
  - whose facets are points (0-polytopes)!
- More stuff here, although the next section might break up the list, so it might be good not to format the whole def in terms of a list so that we can fit in more explanations.

For example, consider the following triangle ΔABC.

$$\begin{array}{ccc} & \text{B} & \\ / & & \backslash \\ \text{A} & — & \text{C} \end{array}$$

It is a polygon which contains three line segments, or **edges**, as they are known when mentioned as part of a larger polytope. It also contains three points, or **vertices**, or even **verts** for short. (In the polytope world, abbreviations are everywhere!) More specifically, its facets are the edges $\overline{\text{AB}}$, $\overline{\text{AC}}$, and $\overline{\text{BC}}$, and *their* facets are the vertices A, B, and C. Collectively, the vertices and edges of ΔABC are its **elements**. We keep track of which elements have which others as facets using a **Hasse diagram**, shown below.

$$
\begin{tikzcd}
& \Delta\mathrm{ABC} \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\overline{\mathrm{AB}} \arrow[d, dash] \arrow[dr, dash] & \overline{\mathrm{AC}} \arrow[dl, dash] \arrow[dr, dash] & \overline{\mathrm{BC}} \arrow[dl, dash] \arrow[d, dash] \\
\mathrm{A} \arrow[dr, dash] & \mathrm{B} \arrow[d, dash] & \mathrm{C} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}
$$

Each node of the Hasse diagram represents an element of ΔABC, which includes both the whole triangle at the top and the **null face** $\emptyset$ at the bottom. [1] Whenever two nodes of the diagram are connected, the higher node's element contains the lower element as a facet. For example, the edge $\overline{\mathrm{AC}}$ contains the vertex C, so their nodes are connected in the diagram with $\overline{\mathrm{AC}}$ above. Notice that the structure of the diagram does not depend on where the vertices are; a scalene triangle would have the same Hasse diagram as the equilateral ΔABC. A diagram considered on its own, without mention of the vertices' locations, is also known as an **abstract polytope**.

Although every polytope can be represented by a Hasse diagram, not every diagram represents a polytope. For example, consider the following arrangement in 2D and its Hasse diagram:

$$
\begin{tikzcd}
& \mathrm{B} \arrow[dl, dash] \\
\mathrm{A} \arrow[r, dash] & \mathrm{C}
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathrm{ABC} \arrow[dl, dash] \arrow[d, dash] & \\
\mathrm{AB} \arrow[d, dash] \arrow[dr, dash] & \mathrm{AC} \arrow[dl, dash] \arrow[dr, dash] & \\
\mathrm{A} \arrow[dr, dash] & \mathrm{B} \arrow[d, dash] & \mathrm{C} \arrow[dl, dash] \\
& \emptyset &
\end{tikzcd}
$$

The left figure is identical to the polygon ΔABC, but with the edge $\overline{\mathrm{BC}}$ missing. Likewise, the Hasse diagram on the right is identical to that of ΔABC, but without the node corresponding to $\overline{\mathrm{BC}}$ and all of its connections. Intuitively speaking, this is not a polygon because it is not "closed," i.e. its edges do not form a closed loop. More formally, we require that within a polygon, every vertex be contained in exactly two edges.

3 Regular polytopes

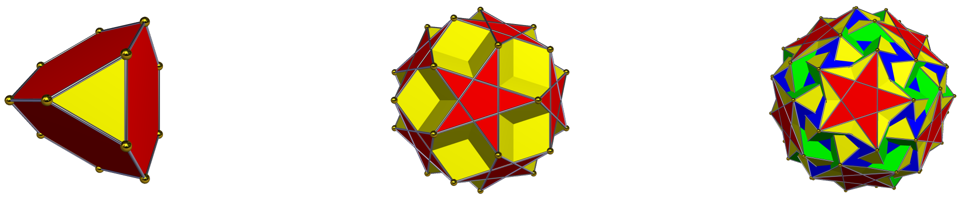
There are multiple definitions for when a polytope is **regular**, but they all require every element (vertices, edges, faces, etc.) to "look the same."

4 Uniform polytopes

General polytopes can be very complicated. Therefore, we only tend to study specific categories of polytopes. The type of polytopes we study most in this server are **uniform polytopes**.

[1] $\emptyset$, also known as the **nullitope**, is a convenient edge case. It has no vertices or other elements, and it is considered to be -1-dimensional(!)

Uniformity is defined recursively. In 2D, uniform polytopes are simply the regular polygons. In higher dimensions, uniform polytopes are those whose facets are all uniform. To see what we mean, let's look at a few examples.



(a) Truncated tetrahedron (b) Dodecadodecahedron (c) Snub icosidodecahedron

Figure 1: Three examples of uniform polyhedra.

In 3D, the uniform polytopes have already been enumerated. It turns out that, aside from the infinite families of **prisms** and **antiprisms**, there's exactly 75 uniform polyhedra.

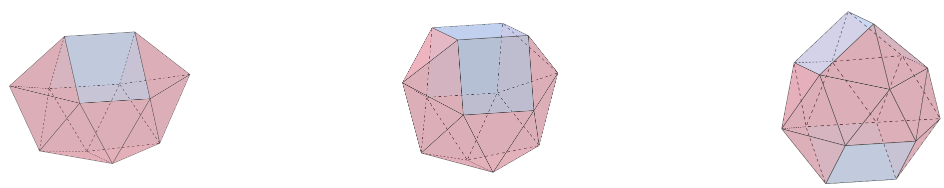
(a) Heptagonal prism (b) Heptagonal antiprism

Figure 2: An example of a prism and an antiprism. These can be built from any regular polygon, and made uniform in all cases.

In 4D and higher up, the problem remains unsolved. In 4D, we know of two infinte families and 2194 uniform polychora. In 5D and up, we haven't yet done a thorough examination, though we know of various families of uniforms such as **Wythoffians** (those generated by a Coxeter diagram), and **multiprisms**.

5 CRF polytopes

A polytope is called **convex regular-faced**, or **CRF** for short, when it is convex (without dents, holes or self-intersections) and all of its faces are regular. Let's look at a few examples.



(a) Sphenomegacorona (b) Hebesphenomegacorona (c) Disphenocingulum

Figure 3: Test images!