Polytope

URL, Mathtician,

February 2021

1 Introduction

Welcome to the Polytope Discord!

2 What is a polytope?

Roughly speaking, a **polytope** is an n-dimensional shape. As with many terms used on the Polytope Discord, the word "polytope" can have a few definitions which are not completely equivalent. This section will list four properties of a polytope from least to most controversial. The most common definition is the strictest, requiring all four.

1. A polytope in n dimensions (known hereafter as an n-polytope) is made of **facets** which are $(n-1)$-polytopes.

    - **Points** (0-polytopes) are the facets of **line segments** (1-polytopes),
    - which are the facets of **polygons** (2-polytopes),
    - which are the facets of **polyhedra** (3-polytopes),
    - which are the facets of **polychora** (4-polytopes), etc.

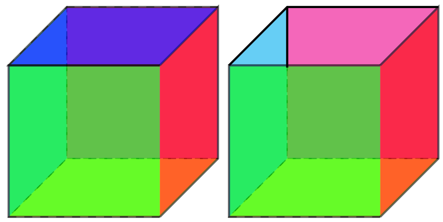
    Note that an n-polytope must lie in n-dimensional space. A square with a vertex jutting out of the plane is usually not a bona fide polygon but is instead given the name "**skew** polygon." If the skew square is a face of a larger 3D shape, said 3D shape is not considered a polyhedron either, since its faces are not all polygons. Readers coming from jan Misali's video on regular polyhedra may find fault with this, but the fact of the matter is that skew polytopes are rarely mentioned or included in our lists.

    For the next criterion, consider the following figures ABC and DEF.

```
  B       E
 / \     /
A — C   D — F
```

ABC is a triangle, a polygon with three line segments, or **edges**, as they are known when mentioned as part of a larger polytope. It also contains three points, or **vertices**, or even **verts** for short. (In the polytope world, abbreviations are everywhere!) DEF, on the other hand, is not a triangle; it is missing an edge, leaving "open ends" at E and F which are each connected to only one edge. To exclude DEF and figures like it, we require that within a polygon, every vertex be connected to exactly two edges. This condition also excludes "branches" where more than two edges meet at a vertex.

Let's generalize this rule to n-polytopes. Two 3D figures are shown below.

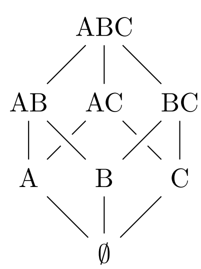


The left figure is a cube, a polyhedron with six square **faces**, twelve edges, and eight vertices. The right is the same, but with the top face removed. The right figure is not a polyhedron because, like DEF, it leaves "open ends." However, in this case, the open ends are not vertices but the top four edges, which are each connected to only one face. We require that within a polyhedron, every edge be connected to exactly two faces. [1]

Are you beginning to see a pattern? In an n-polytope, removing an $(n-1)$-dimensional facet creates "open ends" in the $(n-2)$-dimensional "facets of facets," or **ridges**. Ridges are the vertices of polygons, the edges of polyhedra, the faces of polychora, and so on. Thus the next trait of a polytope is:

2. Every ridge must be connected to exactly two facets.

Collectively, a polytope's vertices, edges, faces, and so on are known as its **elements**. We may keep track of which elements have which others as facets using a **Hasse diagram**, shown below for the triangle ABC.



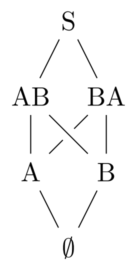
Each node of the Hasse diagram represents an element of ABC, including both the whole triangle at the top and the **null element** $\emptyset$ at the bottom. [2] Whenever two nodes of the diagram are connected, the higher node's element contains the lower node's element as a facet. For example, the edge AC contains the vertex C, so their nodes are connected in the diagram with AC above

[1] The previous rule for polygons does not apply to the cube nor to other polyhedra; every vertex of the cube is connected to three edges, not two. However, the rule does apply to each of the cube's square faces.

[2] $\emptyset$, also known as the **nullitope**, is a convenient edge case. It has no vertices or other elements and is considered to be -1-dimensional!

C. Notice that the structure of the diagram does not depend on where the vertices of the original figure are; a scalene triangle would have the same Hasse diagram as the equilateral ABC.

A diagram on its own, without the locations of the vertices, is also known as an **abstract polytope**. For example, consider the following diagram:



It represents an abstract "polytope" with one null element, two vertices, two edges, and one face (reading from the bottom up). It satisfies the two conditions given thus far and is often called the digon (two-sided polygon). However, its two edges AB and BA contain the same vertices and will lie on top of each other when drawn on a sheet of paper. For this reason, the digon is not considered a polytope. Likewise, a quadrilateral with two vertices in the same spot, which would look like a triangle when drawn, is also not a polytope. This leads us to the third criterion:

3. No two elements of a polytope may coincide:
   (a) no two elements (other than vertices and $\emptyset$) may have the same facets and
   (b) no two vertices may have the same location.

Figures which pass the first and second tests but not this third, such as the digon, are called **fissaries**. Note than condition 3(b) is the first which cannot be tested just from the Hasse diagram. Condition 1 is equivalent to requiring that the Hasse diagram be organized into "layers," with one element on the top (the whole polytope) and another on the bottom ($\emptyset$).

Finally, perhaps the most contentious part of the definition:

4. A polytope is *connected*, i.e. it is possible to reach any facet from any other facet by repeatedly jumping to adjacent facets. [3]

Two triangles next to each other a polygon do not make! Figures which fail to satisfy this criterion are known as **compounds**. The question "Are compounds polytopes?" sparked lively discussusions the first few times it was brought up on the server and weary sighs thereafter. With that, the definition of a polytope is complete! [4]

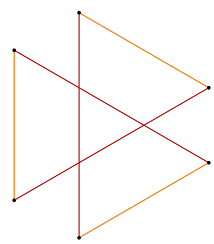
[3] Here, "adjacent" means "sharing a ridge," e.g. two edges of a polygon which share a point, or two faces of a polyhedron which share an edge.

[4] Perhaps the most common requirement for a polytope seen elsewhere but not here is convexity: not having dents, holes, or self-intersections. All of those are perfectly fine and, in fact, overwhelmingly common.

3 Regular polytopes

If you ask a server member what a regular polytope is, they might reply "a regular polytope is **flag-transitive**." This raises two questions: what is a flag, and what is "-transitive?" To answer the first question, reconsider the Hasse diagram of triangle ABC from earlier. A flag of the triangle is a sequence of elements that starts with $\emptyset$ and ends with ABC, where each entry in the sequence contains the previous one. In other words, a flag is a path drawn through the diagram from bottom to top. For example, $\emptyset \to$ B $\to$ AB $\to$ ABC is a flag. ABC happens to have six of them: starting from $\emptyset$, there are three choices of vertex, then two choices of which other vertex to include in the edge, then only one choice of ABC as the final element.

Next, what is "-transitive?" Note that I keep the hyphen because "transitivity" on its own is not a property; something needs to come before the word. A polytope is x-transitive if every x of the polytope can be moved to any other x by some combination of rotation, reflection, and occasionally translation, so that the polytope as a whole looks the same afterward. For example, the following hexagon is vertex-transitive (or **isogonal**), but not edge-transitive.

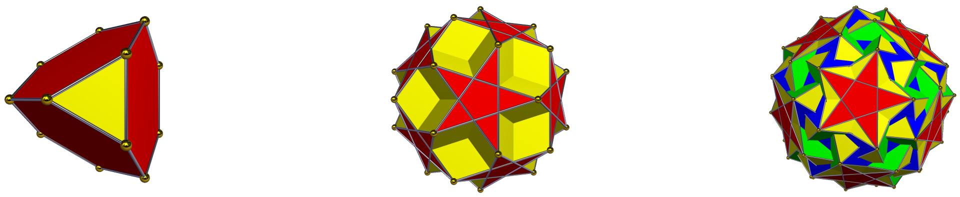


Any of the six black vertices (intersection points don't count!) can be moved to any other vertex with rotation or reflection. However, an orange edge cannot be moved to a red one without changing the shape, size, or orientation of the hexagon.

4 Uniform polytopes

General polytopes can be very complicated. Therefore, we only tend to study specific categories of polytopes. The type of polytopes we study most in this server are **uniform polytopes**.

Uniformity is defined recursively. In 2D, uniform polytopes are simply the regular polygons. In higher dimensions, uniform polytopes are the vertex-transitive polytopes whose facets are all uniform. To see what we mean, let's look at a few examples.



(a) Truncated tetrahedron (tut)

(b) Dodecadodecahedron (did)

(c) Snub icosidodecahedron (snid)

Figure 1: Three examples of uniform polyhedra. They all have regular polygonal faces (corresponding to the 2D uniforms) and are vertex-transitive.

In 3D, the uniform polytopes have already been enumerated. It turns out that, aside from the infinite families of **prisms** and **antiprisms**, there's exactly 75 uniform polyhedra.

(a) Heptagonal prism (hep)

(b) Heptagonal antiprism (heap)

Figure 2: An example of a prism and an antiprism. These can be built from any regular polygon, and made uniform in all cases.

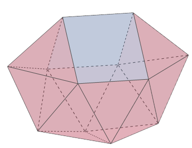
In 4D and higher up, the problem of enumerating the uniforms remains unsolved. As of February 2021, we know of two infinte families plus 2194 uniform polychora. In 5D and up, we haven't yet done a thorough examination, though we know of various constructions that generate uniforms in any dimension.

4.1 Coxeter Diagrams

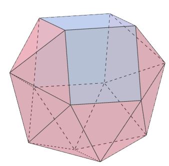
4.2 Multiprisms

5 CRF polytopes

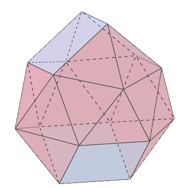
A polytope is called **convex regular-faced**, or **CRF** for short, when it is convex (without dents, holes or self-intersections) and all of its faces are regular. Let's look at a few examples.



(a) Sphenomegacorona



(b) Hebesphenomegacorona



(c) Disphenocingulum

Figure 3: Test images!