# Polytope

URL, Mathtician,

February 2021

## 1 Introduction

Welcome to the Polytope Discord!

## 2 What is a polytope?

Roughly speaking, a **polytope** is an $n$-dimensional shape. As with many terms used on the Polytope Discord, the word "polytope" can have a few definitions which are not completely equivalent. This section will list four properties of a polytope from least to most controversial. The most common definition is the strictest, requiring all four.

1. A polytope in $n$ dimensions (known hereafter as an $n$-polytope) is made of **facets** which are $(n-1)$-polytopes.

   - **Points** (0-polytopes) are the facets of **line segments** (1-polytopes),
   - which are the facets of **polygons** (2-polytopes),
   - which are the facets of **polyhedra** (3-polytopes),
   - which are the facets of **polychora** (4-polytopes), etc.

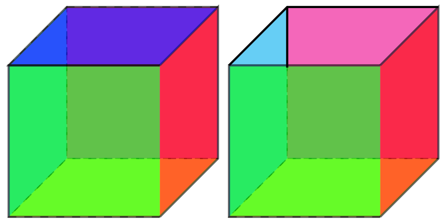
   Note that an $n$-polytope must lie in $n$-dimensional space. A hexagon with vertices jutting out of the plane is usually not a bona fide polygon but is instead given the name "**skew** polygon." If the skew hexagon is a face of a larger 3D shape, said 3D shape is not considered a polyhedron either, since its faces are not all polygons. Readers coming from jan Misali's video on regular polyhedra may find fault with this, but the fact of the matter is that skew polytopes are rarely mentioned or included in our lists.

   For the next criterion, consider the following figures ABC and DEF.

```
  B       E
 / \     /
A — C   D — F
```

ABC is a triangle, a polygon with three line segments, or **edges**, as they are known when mentioned as part of a larger polytope. It also contains three points, or **vertices**, or even **verts** for short. (In the polytope world, abbreviations are everywhere!) DEF, on the other hand, is not a triangle; it is missing an edge, leaving "open ends" at E and F which are each connected to only one edge. To exclude DEF and figures like it, we require that within a polygon, every vertex be connected to exactly two edges. This condition also excludes "branches" where more than two edges meet at a vertex.

Let's generalize this rule to $n$-polytopes. Two 3D figures are shown below.

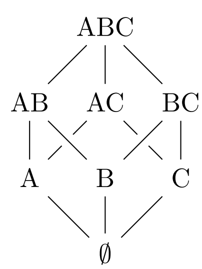


The left figure is a cube, a polyhedron with six square **faces**, twelve edges, and eight vertices. The right is the same, but with the top face removed. The right figure is not a polyhedron because, like DEF, it leaves "open ends." However, in this case, the open ends are not vertices but the top four edges, which are each connected to only one face. We require that within a polyhedron, every edge be connected to exactly two faces. [1]

Are you beginning to see a pattern? In an $n$-polytope, removing an $(n-1)$-dimensional facet creates "open ends" in the $(n-2)$-dimensional "facets of facets," or **ridges**. Ridges are the vertices of polygons, the edges of polyhedra, the faces of polychora, and so on. Thus the next trait of a polytope is:

2. Every ridge must be connected to exactly two facets.

Collectively, a polytope's vertices, edges, faces, and so on are known as its **elements**. We may keep track of which elements have which others as facets using a **Hasse diagram**, shown below for the triangle ABC.



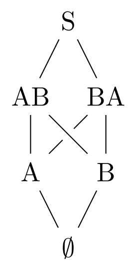

Each node of the Hasse diagram represents an element of ABC, including both the whole triangle at the top and the **null element** $\emptyset$ at the bottom. [2] Whenever two nodes of the diagram are connected, the higher node's element contains the lower node's element as a facet. For example, the edge AC contains the vertex C, so their nodes are connected in the diagram with AC above

[1] The previous rule for polygons does not apply to the cube nor to other polyhedra; every vertex of the cube is connected to three edges, not two. However, the rule does apply to each of the cube's square faces.

[2] $\emptyset$, also known as the **nullitope**, is a convenient edge case. It has no vertices or other elements and is considered to be $-1$-dimensional!

C. Notice that the structure of the diagram does not depend on where the vertices of the original figure are; a scalene triangle would have the same Hasse diagram as the equilateral ABC.

A diagram considered on its own, without mention of the vertices' locations, is also known as an **abstract polytope**. For example, consider the following diagram:




It represents an abstract "polytope" with one null element, two vertices, two edges, and one face (reading from the bottom up). It satisfies the two conditions given thus far and is often called the digon (two-sided polygon). However, its two sides AB and BA contain all of the same elements (A, B, and $\emptyset$) and will lie on top of each other when drawn on a sheet of paper. For this reason, the digon is not considered a polytope. Likewise, a quadrilateral with two vertices in the same spot, which would look like a triangle when drawn, is also not a polytope. This leads us to the third criterion:

3. No two elements of a polytope may coincide:
   (a) no two elements (other than vertices and $\emptyset$) may have the same facets and
   (b) no two vertices may have the same location.

Figures which pass the first and second tests but not this third, such as the digon, are called **fissaries**. Note than condition 3(b) is the first which cannot be tested just from the Hasse diagram. Condition 1 is equivalent to requiring that the Hasse diagram be organized into "layers," with one element on the top (the whole polytope) and another on the bottom ($\emptyset$).

Finally, perhaps the most contentious part of the definition:

4. A polytope is *connected*, i.e. it is possible to reach any facet from any other facet by repeatedly jumping to adjacent facets.

Two triangles next to each other a polygon do not make! Figures which fail to satisfy this criterion are known as **compounds**. The question "Are compounds polytopes?" sparked lively discussusions the first few times it was brought up on the server and weary sighs thereafter. With that, the definition of a polytope is complete!

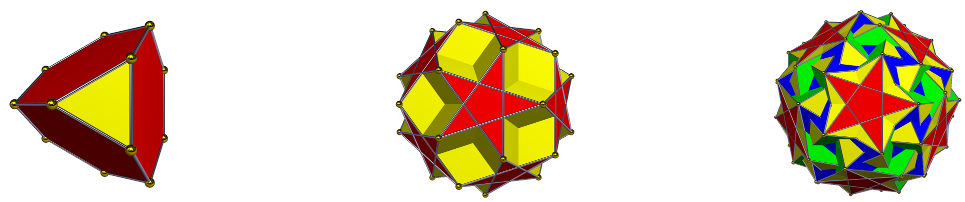
# 3 Regular polytopes

There are multiple definitions for when a polytope is **regular**, but they all require every element (vertices, edges, faces, etc.) to "look the same."

# 4 Uniform polytopes

General polytopes can be very complicated. Therefore, we only tend to study specific categories of polytopes. The type of polytopes we study most in this server are **uniform polytopes**.

Uniformity is defined recursively. In 2D, uniform polytopes are simply the regular polygons. In higher dimensions, uniform polytopes are those whose facets are all uniform. To see what we mean, let's look at a few examples.



(a) Truncated tetrahedron (b) Dodecadodecahedron (c) Snub icosidodecahedron

Figure 1: Three examples of uniform polyhedra.

In 3D, the uniform polytopes have already been enumerated. It turns out that, aside from the infinite families of **prisms** and **antiprisms**, there's exactly 75 uniform polyhedra.

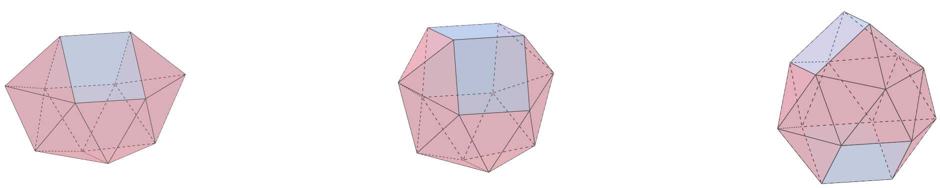
(a) Heptagonal prism (b) Heptagonal antiprism

Figure 2: An example of a prism and an antiprism. These can be built from any regular polygon, and made uniform in all cases.

In 4D and higher up, the problem remains unsolved. In 4D, we know of two infinte families and 2194 uniform polychora. In 5D and up, we haven't yet done a thorough examination, though we know of various families of uniforms such as **Wythoffians** (those generated by a Coxeter diagram), and **multiprisms**.

# 5 CRF polytopes

A polytope is called **convex regular-faced**, or **CRF** for short, when it is convex (without dents, holes or self-intersections) and all of its faces are regular. Let's look at a few examples.



(a) Sphenomegacorona (b) Hebesphenomegacorona (c) Disphenocingulum

Figure 3: Test images!