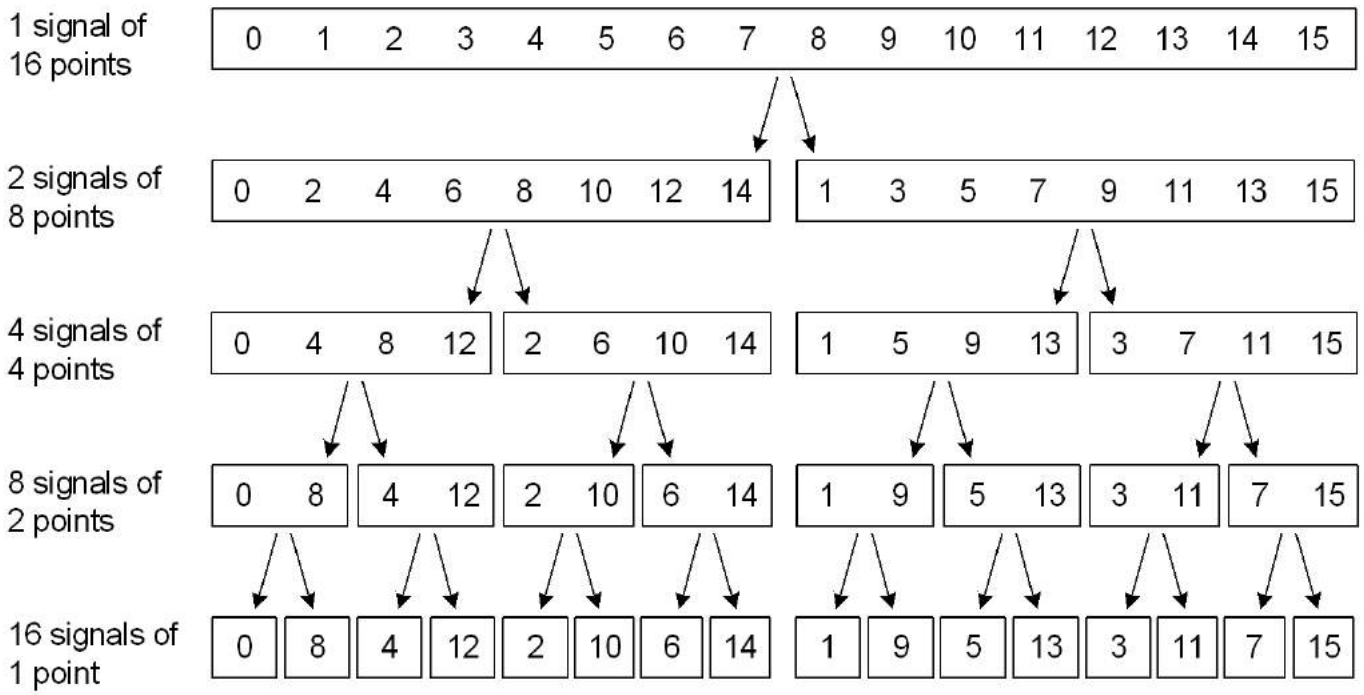
Chapter 55: Fast Fourier Transform

The Real and Complex form of DFT (**D**iscrete **F**ourier **T**ransforms) can be used to perform frequency analysis or synthesis for any discrete and periodic signals. The FFT (**F**ast **F**ourier **T**ransform) is an implementation of the DFT which may be performed quickly on modern CPUs.

Section 55.1: Radix 2 FFT

The simplest and perhaps best-known method for computing the FFT is the Radix-2 Decimation in Time algorithm. The Radix-2 FFT works by decomposing an N point time domain signal into N time domain signals each composed of a single point



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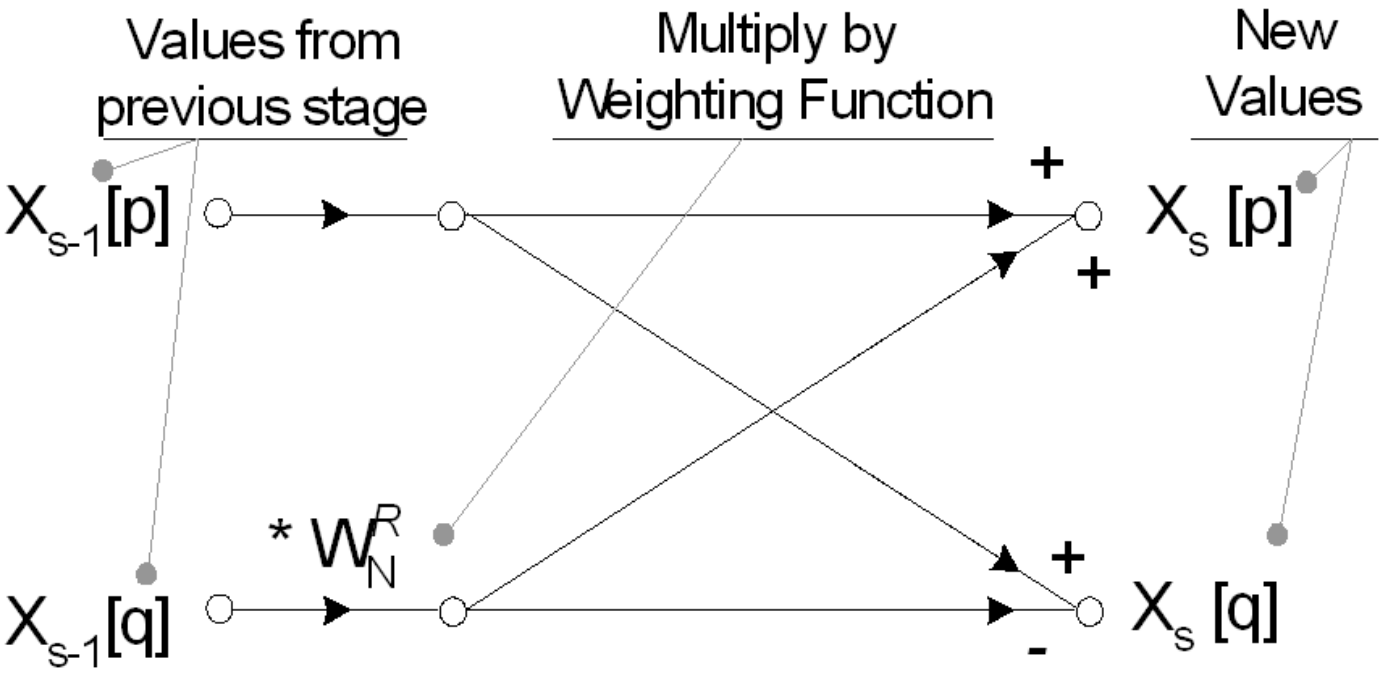
Signal decomposition, or 'decimation in time' is achieved by bit reversing the indices for the array of time domain data. Thus, for a sixteen-point signal, sample 1 (Binary 0001) is swapped with sample 8 (1000), sample 2 (0010) is swapped with 4 (0100) and so on. Sample swapping using the bit reverse technique can be achieved simply in software, but limits the use of the Radix 2 FFT to signals of length N = 2^M.

The value of a 1-point signal in the time domain is equal to its value in the frequency domain, thus this array of decomposed single time-domain points requires no transformation to become an array of frequency domain points. The N single points; however, need to be reconstructed into one N-point frequency spectra. Optimal reconstruction of the complete frequency spectrum is performed using butterfly calculations. Each reconstruction stage in the Radix-2 FFT performs a number of two point butterflies, using a similar set of exponential weighting functions, Wn^R.

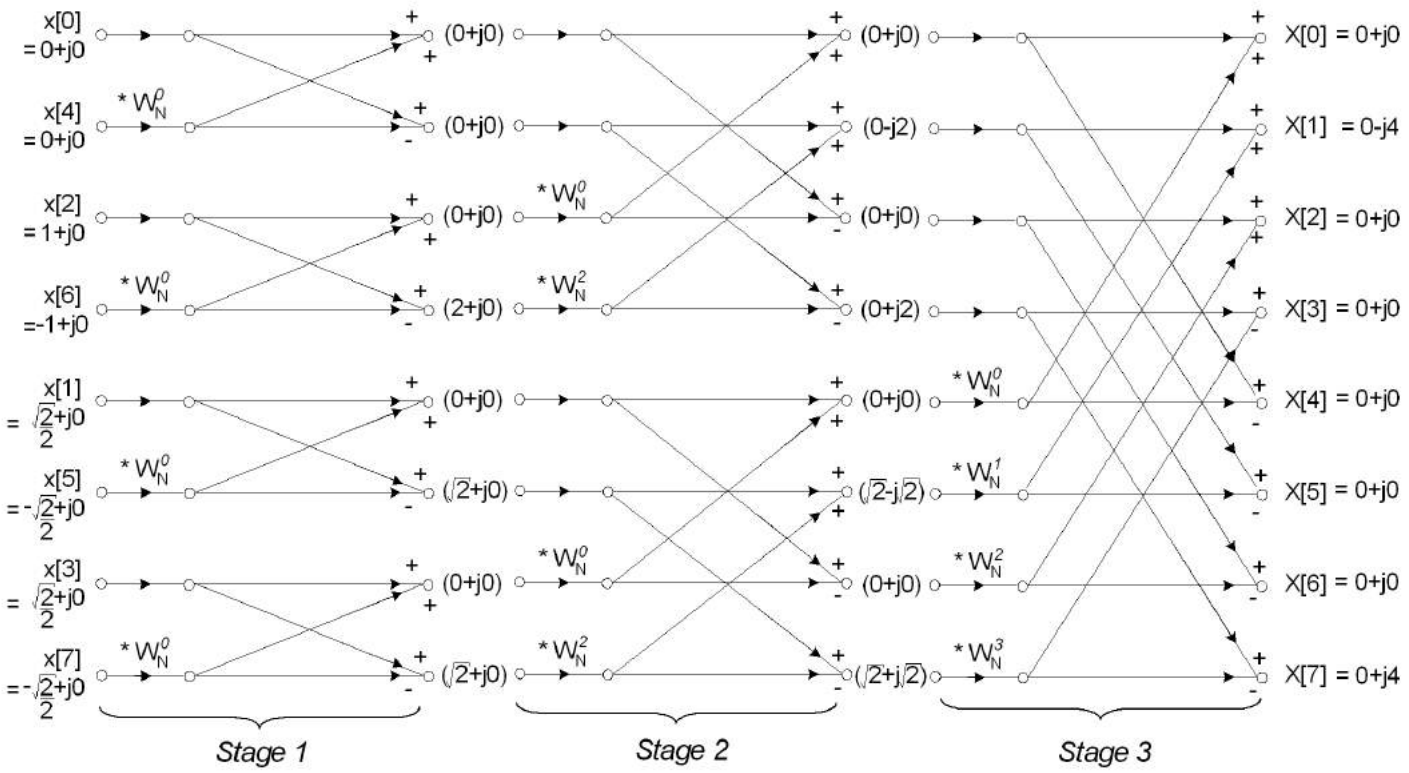


The FFT removes redundant calculations in the Discrete Fourier Transform by exploiting the periodicity of Wn^R. Spectral reconstruction is completed in log2(N) stages of butterfly calculations giving X[K]; the real and imaginary frequency domain data in rectangular form. To convert to magnitude and phase (polar coordinates) requires finding the absolute value, √(Re2 + Im2), and argument, tan-1(Im/Re).

Exponential Weighting Factor: $W_N^R = e^{-j(2\pi R/N)} = \cos(2\pi R/N) - j\sin(2\pi R/N)$

N:	*Number of points in the FFT*
R:	*Current WN Factor: depends on N, current FFT stage and separation of butterflies in that stage*

The complete butterfly flow diagram for an eight point Radix 2 FFT is shown below. Note the input signals have previously been reordered according to the decimation in time procedure outlined previously.



The FFT typically operates on complex inputs and produces a complex output. For real signals, the imaginary part may be set to zero and real part set to the input signal, x[n], however many optimisations are possible involving the transformation of real-only data. Values of Wn^R used throughout the reconstruction can be determined using the exponential weighting equation.

The value of R (the exponential weighting power) is determined the current stage in the spectral reconstruction and the current calculation within a particular butterfly.

Code Example (C/C++)

A C/C++ code sample for computing the Radix 2 FFT can be found below. This is a simple implementation which works for any size N where N is a power of 2. It is approx 3x slower than the fastest FFTw implementation, but still a very good basis for future optimisation or for learning about how this algorithm works.

```
#include <math.h>

#define PI       3.14159265358979323846264338327950    // PI for sine/cos calculations
#define TWOPI    6.28318530717958647692528676655900    // 2*PI for sine/cos calculations
#define Deg2Rad  0.01745329251994329576923690768488612  // Degrees to Radians factor
#define Rad2Deg  57.29577951308232087679815481410517    // Radians to Degrees factor
#define log10_2  0.30102999566398119521373889472449    // Log10 of 2
#define log10_2_INV 3.32192809488736234787031942949483 // 1/Log10(2)

// complex variable structure (double precision)
struct complex
{
public:
    double  Re, Im;         // Not so complicated after all
};

// Returns true if N is a power of 2
bool isPwrTwo(int N, int *M)
{
    *M = (int)ceil(log10((double)N) * log10_2_INV);// M is number of stages to perform. 2^M = N
    int NN = (int)pow(2.0, *M);
```

```
    if ((NN != N) || (NN == 0)) // Check N is a power of 2.
        return false;

    return true;
}

void rad2FFT(int N, complex *x, complex *DFT)
{
    int M = 0;

    // Check if power of two. If not, exit
    if (!isPwrTwo(N, &M))
        throw "Rad2FFT(): N must be a power of 2 for Radix FFT";

    // Integer Variables

    int BSep;                   // BSep is memory spacing between butterflies
    int BWidth;                 // BWidth is memory spacing of opposite ends of the butterfly
    int P;                      // P is number of similar Wn's to be used in that stage
    int j;                      // j is used in a loop to perform all calculations in each stage
    int stage = 1;              // stage is the stage number of the FFT. There are M stages in total
(1 to M).
    int HiIndex;                // HiIndex is the index of the DFT array for the top value of each
butterfly calc
    unsigned int iaddr;         // bitmask for bit reversal
    int ii;                     // Integer bitfield for bit reversal (Decimation in Time)
    int MM1 = M - 1;

    unsigned int i;
    int l;
    unsigned int nMax = (unsigned int)N;

    // Double Precision Variables
    double TwoPi_N = TWOPI / (double)N;    // constant to save computational time.  = 2*PI / N
    double TwoPi_NP;

    // complex Variables (See 'struct complex')
    complex WN;                 // Wn is the exponential weighting function in the form a + jb
    complex TEMP;               // TEMP is used to save computation in the butterfly calc
    complex *pDFT = DFT;        // Pointer to first elements in DFT array
    complex *pLo;               // Pointer for lo / hi value of butterfly calcs
    complex *pHi;
    complex *pX;                // Pointer to x[n]


    // Decimation In Time - x[n] sample sorting
    for (i = 0; i < nMax; i++, DFT++)
    {
        pX = x + i;             // Calculate current x[n] from base address *x and index i.
        ii = 0;                 // Reset new address for DFT[n]
        iaddr = i;              // Copy i for manipulations
        for (l = 0; l < M; l++) // Bit reverse i and store in ii...
        {
            if (iaddr & 0x01)     // Detemine least significant bit
                ii += (1 << (MM1 - l));    // Increment ii by 2^(M-1-l) if lsb was 1
            iaddr >>= 1;                // right shift iaddr to test next bit. Use logical
operations for speed increase
            if (!iaddr)
                break;
        }
        DFT = pDFT + ii;        // Calculate current DFT[n] from base address *pDFT and bit
reversed index ii
```

```
        DFT->Re = pX->Re;       // Update the complex array with address sorted time domain signal
x[n]
        DFT->Im = pX->Im;       // NB: Imaginary is always zero
    }

    // FFT Computation by butterfly calculation
    for (stage = 1; stage <= M; stage++) // Loop for M stages, where 2^M = N
    {
        BSep = (int)(pow(2, stage)); // Separation between butterflies = 2^stage
        P = N / BSep;              // Similar Wn's in this stage = N/Bsep
        BWidth = BSep / 2;     // Butterfly width (spacing between opposite points) = Separation /
2.

        TwoPi_NP = TwoPi_N*P;

        for (j = 0; j < BWidth; j++) // Loop for j calculations per butterfly
        {
            if (j != 0)             // Save on calculation if R = 0, as WN^0 = (1 + j0)
            {
                //WN.Re = cos(TwoPi_NP*j)
                WN.Re = cos(TwoPi_N*P*j);     // Calculate Wn (Real and Imaginary)
                WN.Im = -sin(TwoPi_N*P*j);
            }

            for (HiIndex = j; HiIndex < N; HiIndex += BSep) // Loop for HiIndex Step BSep
butterflies per stage
            {
                pHi = pDFT + HiIndex;                   // Point to higher value
                pLo = pHi + BWidth;                     // Point to lower value (Note VC++ adjusts
for spacing between elements)

                if (j != 0)                             // If exponential power is not zero...
                {
                    //CMult(pLo, &WN, &TEMP);           // Perform complex multiplication of Lovalue
with Wn
                    TEMP.Re = (pLo->Re * WN.Re) - (pLo->Im * WN.Im);
                    TEMP.Im = (pLo->Re * WN.Im) + (pLo->Im * WN.Re);

                    //CSub (pHi, &TEMP, pLo);
                    pLo->Re = pHi->Re - TEMP.Re;        // Find new Lovalue (complex subtraction)
                    pLo->Im = pHi->Im - TEMP.Im;

                    //CAdd (pHi, &TEMP, pHi);           // Find new Hivalue (complex addition)
                    pHi->Re = (pHi->Re + TEMP.Re);
                    pHi->Im = (pHi->Im + TEMP.Im);
                }
                else
                {
                    TEMP.Re = pLo->Re;
                    TEMP.Im = pLo->Im;

                    //CSub (pHi, &TEMP, pLo);
                    pLo->Re = pHi->Re - TEMP.Re;        // Find new Lovalue (complex subtraction)
                    pLo->Im = pHi->Im - TEMP.Im;

                    //CAdd (pHi, &TEMP, pHi);           // Find new Hivalue (complex addition)
                    pHi->Re = (pHi->Re + TEMP.Re);
                    pHi->Im = (pHi->Im + TEMP.Im);
                }
            }
        }
    }
```

```
    pLo = 0;    // Null all pointers
    pHi = 0;
    pDFT = 0;
    DFT = 0;
    pX = 0;
}
```

Section 55.2: Radix 2 Inverse FFT

Due to the strong duality of the Fourier Transform, adjusting the output of a forward transform can produce the inverse FFT. Data in the frequency domain can be converted to the time domain by the following method:

1. Find the complex conjugate of the frequency domain data by inverting the imaginary component for all instances of K.
2. Perform the forward FFT on the conjugated frequency domain data.
3. Divide each output of the result of this FFT by N to give the true time domain value.
4. Find the complex conjugate of the output by inverting the imaginary component of the time domain data for all instances of n.

***Note**: both frequency and time domain data are complex variables. Typically the imaginary component of the time domain signal following an inverse FFT is either zero, or ignored as rounding error. Increasing the precision of variables from 32-bit float to 64-bit double, or 128-bit long double significantly reduces rounding errors produced by several consecutive FFT operations.*

Code Example (C/C++)

```
#include <math.h>

#define PI       3.14159265358979323846264338327795    // PI for sine/cos calculations
#define TWOPI    6.28318530717958647692528676655900     // 2*PI for sine/cos calculations
#define Deg2Rad  0.01745329251994329576923690768488612  // Degrees to Radians factor
#define Rad2Deg  57.29577951308232087679815481410517    // Radians to Degrees factor
#define log10_2  0.30102999566398119521373889472449    // Log10 of 2
#define log10_2_INV 3.32192809488736234787031942949483 // 1/Log10(2)

// complex variable structure (double precision)
struct complex
{
public:
    double  Re, Im;         // Not so complicated after all
};

void rad2InverseFFT(int N, complex *x, complex *DFT)
{
    // M is number of stages to perform. 2^M = N
    double Mx = (log10((double)N) / log10((double)2));
    int a = (int)(ceil(pow(2.0, Mx)));
    int status = 0;
    if (a != N) // Check N is a power of 2
    {
        x = 0;
        DFT = 0;
        throw "rad2InverseFFT(): N must be a power of 2 for Radix 2 Inverse FFT";
    }

    complex *pDFT = DFT;         // Reset vector for DFT pointers
    complex *pX = x;             // Reset vector for x[n] pointer
    double NN = 1 / (double)N;   // Scaling factor for the inverse FFT
```

```
    for (int i = 0; i < N; i++, DFT++)
        DFT->Im *= -1;          // Find the complex conjugate of the Frequency Spectrum

    DFT = pDFT;                 // Reset Freq Domain Pointer
    rad2FFT(N, DFT, x); // Calculate the forward FFT with variables switched (time & freq)

    int i;
    complex* x;
    for ( i = 0, x = pX; i < N; i++, x++){
        x->Re *= NN;    // Divide time domain by N for correct amplitude scaling
        x->Im *= -1;    // Change the sign of ImX
    }
}
```