A Puzzle about Random Walk over the Circle

Yanhua Huang

Feb 2019

Consider a particle that moves along a set of $m + 1$ nodes, labeled $0, 1, ..., m$, that are arranged around a circle. At each step the particle is equally likely to move one position in either the clockwise or counterclockwise direction. Suppose now that the particle starts at 0 and continues to move around according to the preceding rules until all the nodes have been visited. What is the probability that node i is the last one visited?

Let us first define two event. Event A_i: node i is visted at the last. Event $A_{i-1,i+1}$: node $i - 1$ is visited before node $i + 1$. It is obvious that

$$P(A_{i-1,i+1}) + P(A_{i+1,i-1}) = 1 \qquad (1)$$

and

$$P(A_i | A_{i-1,i+1}) = P(A_i | A_{i+1,i-1}) := c. \qquad (2)$$

Furthermore, with the Bayes' formula,

$$P(A_i) = P(A_{i-1,i+1}) * P(A_i | A_{i-1,i+1}) + P(A_{i+1,i-1}) * P(A_i | A_{i+1,i-1}), \qquad (3)$$

so $P(A_i) = c$ for any $i = 1, ..., m$ or $P(A_i) = \frac{1}{m}$.