Why does Rejection Sampling Make Sense?

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Rejection sampling is a basic sampling method when the CDF is unknown. Given random variable X with PDF $p(x)$, pick the sampling density q. Let $M = \sup \frac{p}{q}$. The accept rate is

$$P(u < \frac{p}{Mq}) = \mathbb{E}[\frac{p}{Mq}] = \int q \frac{p}{Mq} = \frac{1}{M}, \tag{1}$$

where $u \sim U(0,1)$. Consider the CDF of X,

$$\begin{aligned}
P(X \leq x) &= P(Y \leq x | u < \frac{p(Y)}{Mq(Y)}) && (2)\\
&= MP(Y \leq x, u < \frac{p(Y)}{Mq(Y)}) && (3)\\
&= M \int q(y) P(Y \leq x, u < \frac{p(Y)}{Mq(Y)} | Y = y) dy && (4)\\
&= M \int q(y) P(y \leq x, u < \frac{p(y)}{Mq(y)}) dy && (5)\\
&= M \int q(y) I(y \leq x) P(u < \frac{p(y)}{Mq(y)}) dy && (6)\\
&= M \int_{-\infty}^{x} q(y) \frac{p(y)}{Mq(y)} dy && (7)\\
&= \int_{-\infty}^{x} p(y) dy. && (8)
\end{aligned}$$