AnalyticalCases

July 6, 2022

We will compare different prediction schemes, using exact analytical values of $\tilde{T}(r)$ and $\langle T_r \rangle = \frac{1-\tilde{T}(r)}{r\tilde{T}(r)}$.

The first chosen distribution is the inverse Gaussian distribution, $\phi(t) = \frac{L}{\sqrt{4\pi D t^3}} exp\left(-\frac{(L-Vt)^2}{4Dt}\right)$, with $L = 1000000, D = 12500000, V = 1$. The mean and CV are $\mu = 1ns, \frac{\sigma}{\mu} = 5$. The best improvement factor is ~ 8.5 for $r^* \approx 31ns^{-1}$.

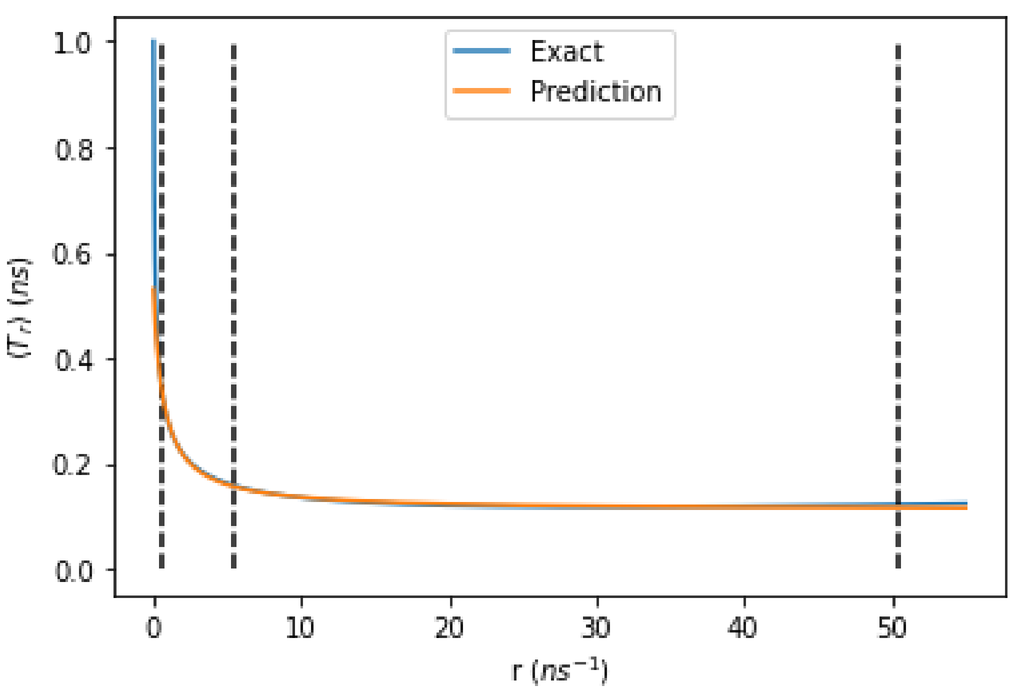
We will compare different schemes for different starting points.

0.1 Pade approximation for $\langle T_r \rangle$:

We will approximate $\langle T_r \rangle$ using $f(r) = \frac{U_3 r^3 + U_2 r^2 + U_1 r + U_0}{D_3 r^3 + D_2 r^2 + D_1 r + D_0}$. We will use 10000 points, from $r = r^*$ to $r_{max} = (C+1)r^*, C = 1, 10, 100, 1000$.

For example, here is the case for points between $r^* = 0.5ns^{-1}$ and $C = 101$ (The dashed lines sign $r = r^*, 11r^*, 101r^*$):

```
Text(0, 0.5, '$\\langle T_r \\rangle$ ($ns$)')
```



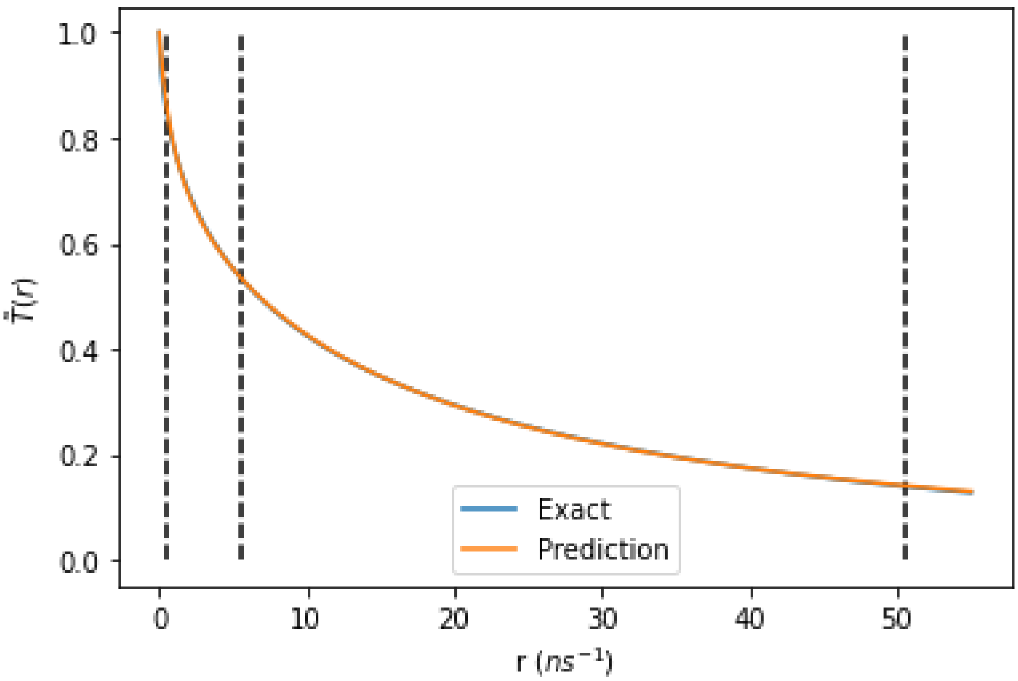
0.2 Pade approximation for $\tilde{T}(r)$:

We will approximate $\tilde{T}(r)$ using $f(r) = \frac{U_3 r^3 + U_2 r^2 + U_1 r + A}{D_4 r^4 + D_3 r^3 + D_2 r^2 + D_1 r + A}$. We will use 10000 points, from $r = r^*$ to $r_{max} = (C+1)r^*, C = 1, 10, 100, 1000$, and an additional point, $\tilde{T}(0) = 1$.

Then, we can approximate $\langle T_0 \rangle$ by $\langle T_0 \rangle = -\frac{\partial \tilde{T}(0)}{\partial r} \approx \frac{D_1 - U_1}{A}$

For example, here is the case for points between $r^* = 0.5 ns^{-1}$ and $C = 101$ (The dashed lines sign $r = r^*, 11r^*, 101r^*$):

```
Text(0, 0.5, '$\\tilde{T}(r)$')
```



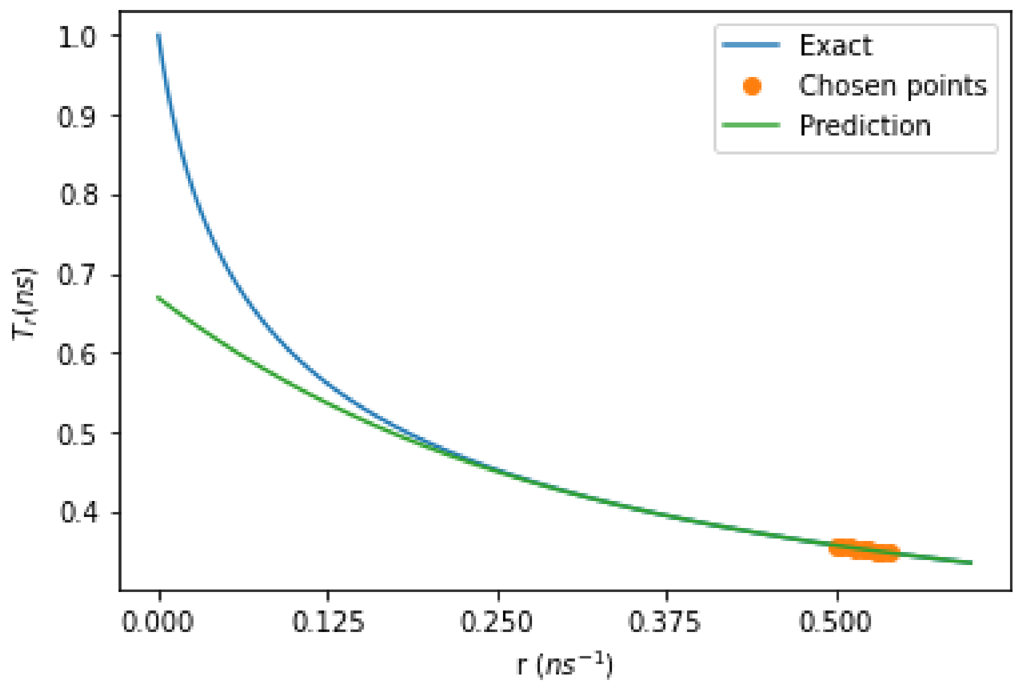
0.3 Taylor expansion

We will use forward finite difference approximations for the first four derivatives of $\langle T_r \rangle$ at r^*, and evaluate $\langle T_r \rangle$ using a fourth order Taylor expansion. We will use $\delta = 0.001r^*, 0.01r^*, 0.1r^*, r^*$ for the distance between grid points for the derivatives.

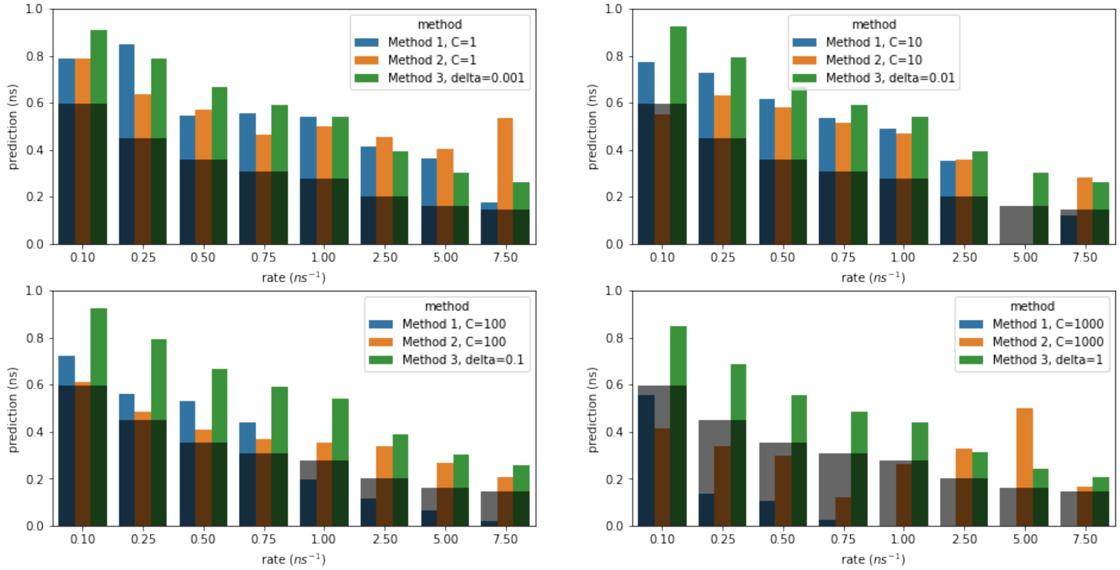
For example, here is the case for $r^* = 0.5ns^{-1}$ and $\delta = 0.01r^*$:

```
([<matplotlib.axis.XTick at 0x7f6f98ab8ca0>,
  <matplotlib.axis.XTick at 0x7f6f98ab8c70>,
  <matplotlib.axis.XTick at 0x7f6f98ab83a0>,
```

```
 <matplotlib.axis.XTick at 0x7f6f98a12eb0>,
 <matplotlib.axis.XTick at 0x7f6f98a16640>],
[Text(0, 0, ''),
 Text(0, 0, ''),
 Text(0, 0, ''),
 Text(0, 0, ''),
 Text(0, 0, '')])
```

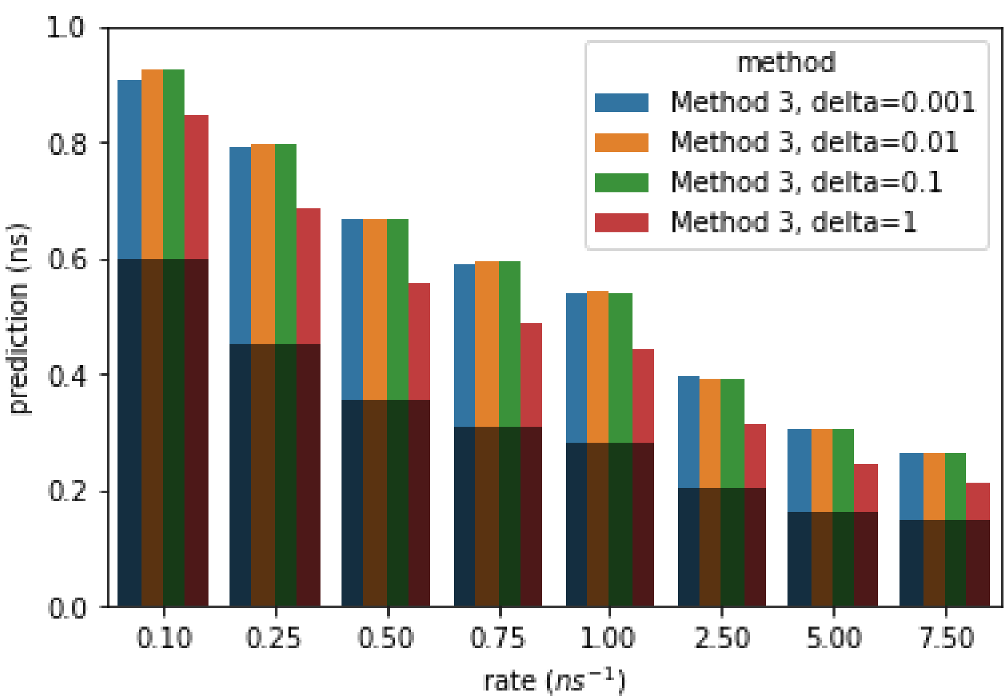


We will now plot the predictions of the different methods for several reset rates. The shaded bars represent $\langle T_r \rangle$.



It seems that the third method is the most consistent and precise. Other methods yield very different results for different C values, while this method yields similar results for different δ values:

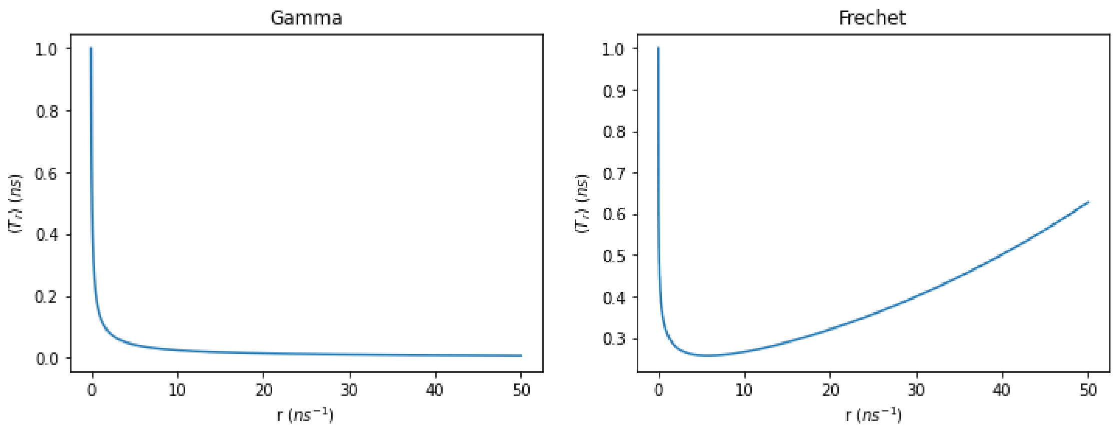
```
(0.0, 1.0)
```



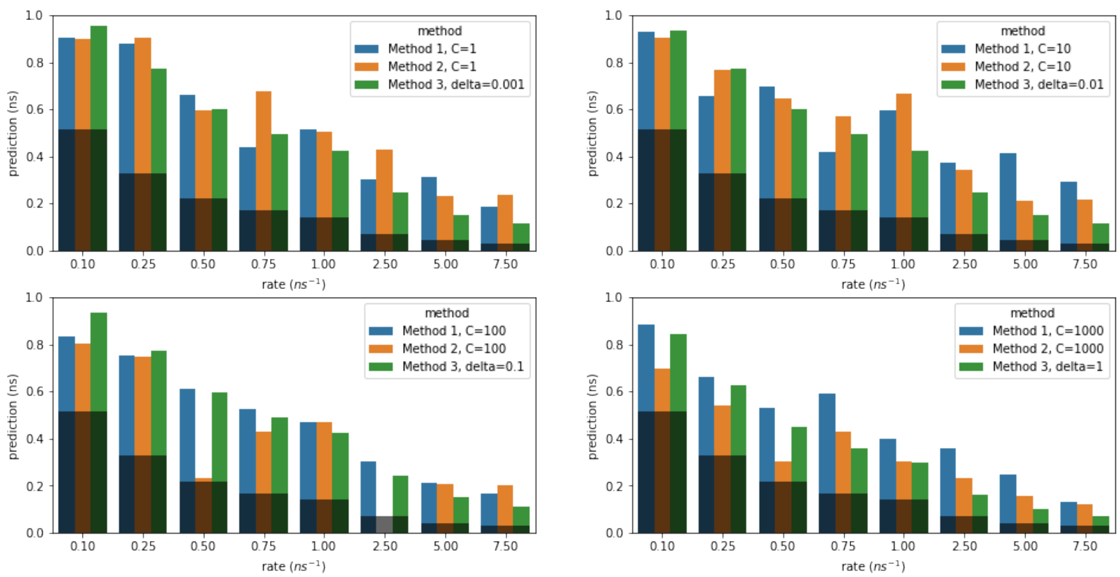
We will now show similar plots for additional distributions:

1. Gamma distribution, $\phi(t) = \frac{1}{\Gamma(k)\theta^k} t^{k-1} exp\left(-\frac{t}{\theta}\right)$, using the same mean and CV.
2. Frechet distribution, $\phi(t) = t^{-2} exp\left(-t^{-1}\right)$, with $\langle T_{0.0005} \rangle = 1ns$. We will predict $\langle T_{0.0005} \rangle$ instead of $\langle T_0 \rangle$ because $\langle T_0 \rangle$ diverges.

Below are plots of $\langle T_r \rangle$ against r for these distributions.

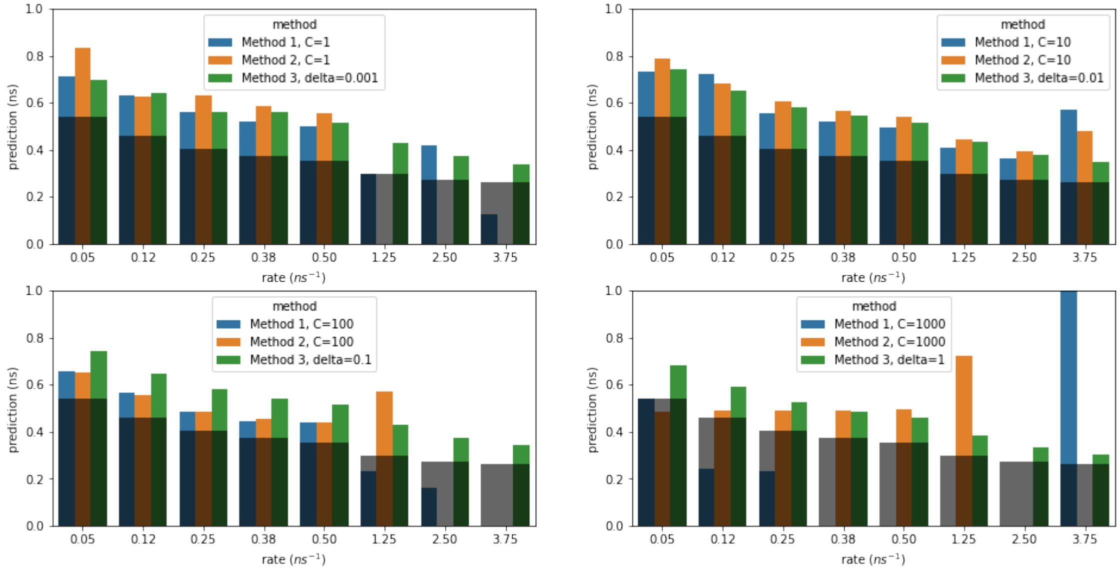


Below are the results for the Gamma distribution.



For this distribution, the results are not as conclusive. For many cases, methods 1 and 2 performed better than method 3.

Below are the results for the Frechet distribution.



These results don't show significantly better performence for a single method. Method 3 is more stable regarding to changes in the parameters.