MA1002E MATHEMATICS -I

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- Differentiability
- Product Rule
- Quotient Rule

The Chain Rule

If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x.
Moreover, if $y = f(g(x))$ and $u = g(x)$ then $y = f(u)$ and $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

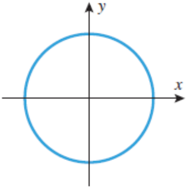
1. Find $\frac{dy}{dx}$ if $y = \cos(x^3)$
2. Find $\frac{dw}{dt}$ if $w = \tan x$ and $x = 4t^3 + t$.

- An equation of the form $y = f(x)$ is said to define y **explicitly** as a function of x because the variable y appears alone on one side of the equation and does not appear at all on the other side.
- Sometimes functions are defined by equations of the form $f(x, y) = 0$ in which y is not alone on one side.

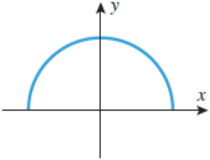
Definition

We will say that a given equation in x and y defines the function f **implicitly** if the graph of $y = f(x)$ coincides with a portion of the graph of the equation

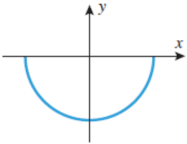
Example:



$x^2 + y^2 = 1$



$y = \sqrt{1 - x^2}$



$y = -\sqrt{1 - x^2}$

Implicit Differentiation

In general, it is not necessary to solve an equation for y in terms of x in order to differentiate the functions defined implicitly by the equation.

Consider $xy = 1$

$$\frac{d}{dx}[xy] = \frac{d}{dx}[1]$$
$$x\frac{d}{dx}[y] + y\frac{d}{dx}[x] = 0$$
$$x\frac{dy}{dx} + y = 0$$
$$\frac{dy}{dx} = -y$$

Now substitute $y = \frac{1}{x}$ into the last expression,
we obtain $\frac{dy}{dx} = \frac{-1}{x^2}$

1. Use implicit differentiation to find $\frac{dy}{dx}$ if $5y^2 + \sin y = x^2$.
2. Use implicit differentiation to find $\frac{d^2y}{dx^2}$ if $4x^2 - 2y^2 = 9$
3. Find the slopes of the tangent lines to the curve $y^2 - x + 1 = 0$ at the points $(2, -1)$ and $(2, 1)$.

Theorem

Let f be a function that is continuous on a closed interval $[a, b]$ and differentiable on the open interval (a, b).

(a) If $f'(x) > 0$ for every value of x in (a, b), then f is increasing on $[a, b]$.

(b) If $f'(x) < 0$ for every value of x in (a, b), then f is decreasing on $[a, b]$.

(c) If $f'(x) = 0$ for every value of x in (a, b), then f is constant on $[a, b]$.

1. Find the intervals on which $f(x) = x^2 - 4x + 3$ is increasing and the intervals on which it is decreasing.
2. Find the intervals on which $f(x) = x^3$ is increasing and the intervals on which it is decreasing.

Relative extremum and absolute extremum

Definition

A function f is said to have a **relative maximum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the largest value, that is, $f(x_0) \geq f(x)$ for all x in the interval.

Similarly, f is said to have a **relative minimum** at x_0 if there is an open interval containing x_0 on which $f(x_0)$ is the smallest value, that is, $f(x_0) \leq f(x)$ for all x in the interval.

If f has either a relative maximum or a relative minimum at x_0, then f is said to have a **relative extremum** at x_0.

Example: $f(x) = x^2$ has a relative minimum at $x = 0$ but no relative maxima.
$f(x) = x^3$ has no relative extrema.

Concavity

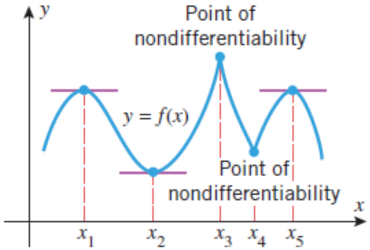
If f is differentiable on an open interval, then f is said to be **concave up** on the open interval if f' is increasing on that interval, and f is said to be **concave down** on the open interval if f' is decreasing on that interval.

Theorem

Let f be twice differentiable on an open interval.
(a) If $f''(x) > 0$ for every value of x in the open interval, then f is concave up on that interval.
(b) If $f''(x) < 0$ for every value of x in the open interval, then f is concave down on that interval

Theorem

Suppose that f is a function defined on an open interval containing the point x_0. If f has a relative extremum at $x = x_0$, then $x = x_0$ is a **critical point** of f ; that is, either $f'(x_0) = 0$ or f is not differentiable at x_0.



1. Find all critical points of $f(x) = x^3 - 3x + 1$
2. Find all critical points of $f(x) = 3x^{5/3} - 15x^{2/3}$.

First Derivative Test

Theorem

Suppose that f is continuous at a critical point x_0.

(a) If $f'(x) > 0$ on an open interval extending left from x_0 and $f'(x) < 0$ on an open interval extending right from x_0, then f has a relative maximum at x_0.

(b) If $f'(x) < 0$ on an open interval extending left from x_0 and $f'(x) > 0$ on an open interval extending right from x_0, then f has a relative minimum at x_0.

(c) If $f'(x)$ has the same sign on an open interval extending left from x_0 as it does on an open interval extending right from x_0, then f does not have a relative extremum at x_0.

Proof:

Example:

$f(x) = 3x^{5/3} - 15x^{2/3}$ has a relative maximum at $x = 0$ and a relative minimum at $x = 2$.

$f'(x) = \frac{5(x-2)}{x^{\frac{1}{3}}}$

Interval	$\frac{5(x-2)}{x^{\frac{1}{3}}}$	$f'(x)$
$x < 0$	$(-)/(-)$	$+$
$0 < x < 2$	$(-)/(+)$	-
$x > 2$	$(+)/(+)$	$+$

Second Derivative Test

Theorem

Suppose that f is twice differentiable at the point x_0.

(a) If $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a relative minimum at x_0.

(b) If $f'(x_0) = 0$ and $f''(x_0) < 0$, then f has a relative maximum at x_0.

(c) If $f'(x_0) = 0$ and $f''(x_0) = 0$, then the test is inconclusive; that is, f may have a relative maximum, a relative minimum, or neither at x_0.

Proof:

Example: Find the relative extrema of $f(x) = 3x^{5/3} - 15x^{2/3}$
We have

$$f'(x) = 15x^4 - 15x^2 = 15x^2\left(x^2 - 1\right) = 15x^2(x+1)(x-1)$$
$$f''(x) = 60x^3 - 30x = 30x\left(2x^2 - 1\right)$$

STATIONARY POINT	$30x\left(2x^2-1\right)$	$f''(x)$	SECOND DERIVATIVE TEST
$x = -1$	-30	-	f has a relative maximum
$x = 0$	0	0	Inconclusive
$x = 1$	30	$+$	f has a relative minimum

The test is inconclusive at $x = 0$, so we will try the first derivative test at that point. A sign analysis of f' is given in the following table:

INTERVAL	$15x^2(x+1)(x-1)$	$f'(x)$
$-1 < x < 0$	$(+)(+)(-)$	-
$0 < x < 1$	$(+)(+)(-)$	-

Absolute Extrema

Consider an interval in the domain of a function f and a point x_0 in that interval.
We say that f has an **absolute maximum** at x_0 if $f(x) \leq f(x_0)$ for all x in the interval, and we say that f has an **absolute minimum** at x_0 if $f(x_0) \leq f(x)$ for all x in the interval.
We say that f has an absolute extremum at x_0 if it has either an absolute maximum or an absolute minimum at that point.

Extreme Value Theorem

If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

Rolle 's Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = 0$ and $f(b) = 0$ then there is at least one point c in the interval (a, b) such that $f'(c) = 0$.

Proof

1. Find the two x-intercepts of the function $f(x) = x^2 - 5x + 4$ and confirm that $f(c) = 0$ at some point c between those intercepts.

Mean-Value Theorem

Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Then there is at least one point c in (a, b) such that $f(c) = \frac{f(b)-f(a)}{b-a}$

Proof:

Show that the function $f(x) = 1/4x^3 + 1$ satisfies the hypotheses of the Mean-Value Theorem over the interval $[0, 2]$, and find all values of c in the interval $(0, 2)$ at which the tangent line to the graph of f is parallel to the secant line joining the points $(0, f(0))$ and $(2, f(2))$.

L'Hopital's Rule

L'Hopital's Rule for Form ∞/∞

Suppose that f and g are differentiable functions on an open interval containing $x = a$, except possibly at $x = a$, and that $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$ If $\lim_{x\to a} f(x)/g(x)$ exists or if this limit is $+\infty$ or $-\infty$ then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Moreover, this statement is also true in the case of a limit as $x \to a-$, $x \to a+$, $x \to -\infty$, or as $x \to +\infty$

1. $\lim\limits_{x\to+\infty} \frac{x}{e^x}$
2. $\lim\limits_{x\to 0+} \frac{\log x}{x}$
3. $\lim\limits_{x\to 0+} x\log x$
4. $\lim\limits_{x\to \frac{\pi}{4}} (1-\tan x)\sec 2x$
5. $\lim\limits_{x\to 0} (1+\sin x)^{\frac{1}{x}}$

Definition

A function F is called an antiderivative of a function f on a given open interval if $F'(x) = f(x)$ for all x in the interval.

For example, the function $F(x) = \frac{1}{3}x^3$ is an antiderivative of $f(x) = x^2$ on the interval $(-\infty, +\infty)$ because for each x in this interval

$$F'(x) = \frac{d}{dx}\left[\frac{1}{3}x^3\right] = x^2 = f(x)$$

However, $F(x) = \frac{1}{3}x^3$ is not the only antiderivative of f on this interval. If we add any constant C to $\frac{1}{3}x^3$, then the function $G(x) = \frac{1}{3}x^3 + C$ is also an antiderivative of f on $(-\infty, +\infty)$, since

$$G'(x) = \frac{d}{dx}\left[\frac{1}{3}x^3 + C\right] = x^2 + 0 = f(x)$$

In general, once any single antiderivative is known, other antiderivatives can be obtained by adding constants to the known antiderivative. Thus,

$$\frac{1}{3}x^3, \quad \frac{1}{3}x^3 + 2, \quad \frac{1}{3}x^3 - 5, \quad \frac{1}{3}x^3 + \sqrt{2}$$

are all antiderivatives of $f(x) = x^2$.

Theorem

If $F(x)$ is any antiderivative of $f(x)$ on an open interval, then for any constant C the function $F(x) + C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval can be expressed in the form $F(x) + C$ by choosing the constant C appropriately.

Definition(Area Under a Curve)

If the function f is continuous on $[a,b]$ and if $f(x) \geq 0$ for all x in $[a,b]$, then the area A under the curve $y = f(x)$ over the interval $[a,b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\,\Delta x$$

where $\Delta x = \frac{b-a}{n}$ and $x_0 = a, x_1, x_2 \cdots, x_n = b$ are partition points of $[a,b]$ with length Δx and $x_1^*, x_2^*, \cdots x_n^*$ denote the points selected in each subintervals

Use Definition of area under a curve with x_k^* as the right endpoint of each subinterval to find the area between the graph of $f(x) = x^2$ and the interval [0, 1].

Definition

A function f is said to be integrable on a finite closed interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

exists and does not depend on the choice of partitions or on the choice of the points x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f\left(x_k^*\right) \Delta x_k$$

which is called the definite integral of f from a to b. The numbers a and b are called the lower limit of integration and the upper limit of integration, respectively, and $f(x)$ is called the integrand.

Sketch the region whose area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry.

1. $\int_1^4 2dx$
2. $\int_{-1}^2 (x+2)dx$
3. $\int_0^1 \sqrt{1-x^2}dx$

The Fundamental Theorem of Calculus, Part 1

If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proof

1. Evaluate $\int_1^2 x dx$.
The function $F(x) = \frac{1}{2}x^2$ is an antiderivative of $f(x) = x$; thus,

$$\int_1^2 x dx = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(2)^2 - \frac{1}{2}(1)^2 = 2 - \frac{1}{2} = \frac{3}{2}$$

2. Evaluate $A = \int_0^3 \left(9 - x^2\right) dx$

$$A = \int_0^3 \left(9 - x^2\right) dx = \left[9x - \frac{x^3}{3}\right]_0^3 = \left(27 - \frac{27}{3}\right) - 0 = 18$$

3. (a) Find the area under the curve $y = \cos x$ over the interval $[0, \pi/2]$
(b) Make a conjecture about the value of the integral $\int_0^\pi \cos x dx$ and confirm your conjecture using the Fundamental Theorem of Calculus.

The Mean-Value Theorem for Integrals

If f is continuous on a closed interval $[a, b]$, then there is at least one point x in $[a, b]$ such that

$$\int_a^b f(x)dx = f(x^*)(b-a)$$

Proof

The Fundamental Theorem of Calculus, Part 2

If f is continuous on an interval, then f has an antiderivative on that interval. In particular, if a is any point in the interval, then the function F defined by

$$F(x) = \int_a^x f(t)dt$$

is an antiderivative of f; that is, $F'(x) = f(x)$ for each x in the interval, or in an alternative notation

$$\frac{d}{dx}\left[\int_a^x f(t)dt\right] = f(x)$$

Proof

Example 1. Find $\frac{d}{dx}\left[\int_1^x t^3 dt\right]$.
The integrand is a continuous function, so from

$$\frac{d}{dx}\left[\int_1^x t^3 dt\right] = x^3$$

Alternatively, evaluating the integral and then differentiating yields

$$\int_1^x t^3 dt = \frac{t^4}{4}\bigg]_{t=1}^{x} = \frac{x^4}{4} - \frac{1}{4}, \quad \frac{d}{dx}\left[\frac{x^4}{4} - \frac{1}{4}\right] = x^3$$

so the two methods for differentiating the integral agree.

Example 2. Since $f(x) = \frac{\sin x}{x}$ is continuous on any interval that does not contain the origin, it follows from (11) that on the interval $(0, +\infty)$ we have

$$\frac{d}{dx}\left[\int_1^x \frac{\sin t}{t} dt\right] = \frac{\sin x}{x}$$

Area between the curves

Area formula

If f and g are continuous functions on the interval $[a, b]$, and if $f(x) \geq g(x)$ for all x in $[a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is

$$A = \int_a^b [f(x) - g(x)]dx$$

1. Find the area of the region bounded above by $y = x + 6$, bounded below by $y = x^2$, and bounded on the sides by the lines $x = 0$ and $x = 2$
2. Find the area of the region that is enclosed between the curves$y = x^2$ and $y = x + 6$

Area formula

If w and v are continuous functions and if $w(y) \geq v(y)$ for all y in $[c, d]$, then the area of the region bounded on the left by $x = v(y)$, on the right by $x = w(y)$, below by $y = c$, and above by $y = d$ is

$$A = \int_c^d [w(y) - v(y)]dy$$

1. Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

Volume Formula

Let S be a solid bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$. If, for each x in $[a, b]$, the cross-sectional area of S perpendicular to the x-axis is $A(x)$, then the volume of the solid is

$$V = \int_a^b A(x)dx$$

provided $A(x)$ is integrable. There is a similar result for cross sections perpendicular to the y-axis.

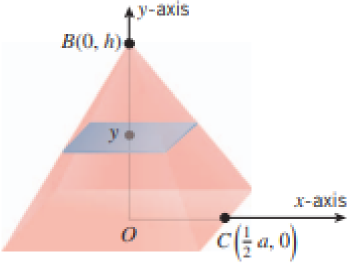
Volume Formula

Let S be a solid bounded by two parallel planes perpendicular to the y-axis at $y = c$ and $y = d$. If, for each y in $[c, d]$, the cross-sectional area of S perpendicular to the y-axis is $A(y)$, then the volume of the solid is

$$V = \int_c^d A(y)dy$$

provided $A(y)$ is integrable.

Derive the formula for the volume of a right pyramid whose altitude is h and whose base is a square with sides of length a



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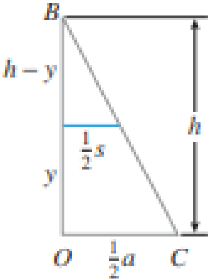
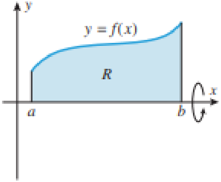
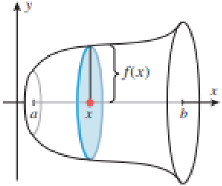


Volumes by disk perpendicular to the x-axis

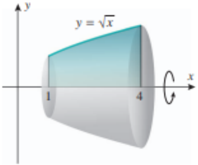
Problem: Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the x-axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.



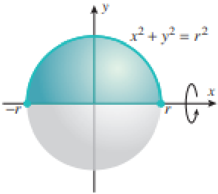


$$V = \int_a^b \pi[f(x)]^2 dx$$

1. Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the x-axis

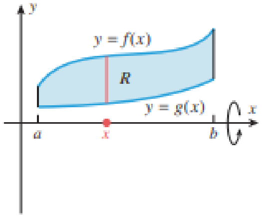


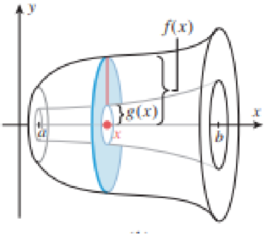
2. Derive the formula for the volume of a sphere of radius r



Volumes by washers perpendicular to the x-axis

Problem: Let f and g be continuous and nonnegative on $[a,b]$, and suppose that $f(x) \geq g(x)$ for all x in the interval $[a,b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the x-axis.



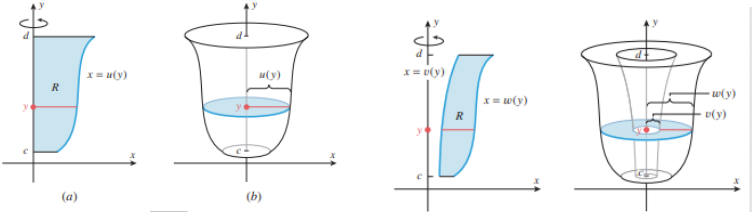


$$V = \int_a^b \pi([f(x)]^2 - [g(x)]^2)dx$$

Example: Find the volume of the solid generated when the region between the graphs of the equations $f(x) = 1/2 + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the x-axis

The methods of disks and washers have analogs for regions that are revolved about the y-axis

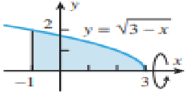
$$V = \int_c^d \pi[u(y)]^2 dy \quad V = \int_c^d \pi\left([w(y)]^2 - [v(y)]^2\right) dy$$



Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.

Find the volume of the solid that results when the shaded region is revolved about the indicated axis

1.



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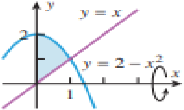

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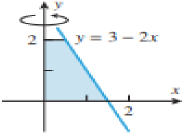

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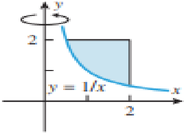

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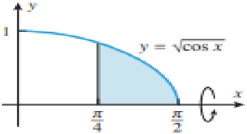

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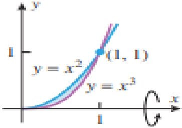

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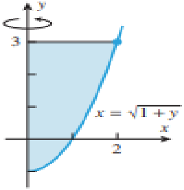

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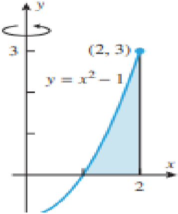

Arc Length

If $y = f(x)$ is a smooth curve on the interval $[a, b]$, then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$

This result provides both a definition and a formula for computing arc lengths. Also it can be expressed as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Moreover, for a curve expressed in the form $x = g(y)$, where g' is continuous on $[c, d]$, the arc length L from $y = c$ to $y = d$ can be expressed as

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

1. Find the arc length of the curve $y = x^{3/2}$ from $(1,1)$ to $(2, 2\sqrt{2})$ in two ways:
2. Find the arc length of $\frac{1}{3}(y^2+2)^{3/2}$ from $y = 0$ to $y = 1$
3. Find the arc length of $y = x^{2/3}$ from $x = 1$ to $x = 8$

Area of Surface of Revolution

If f is a smooth, nonnegative function on $[a, b]$, then the surface area S of the surface of revolution that is generated by revolving the portion of the curve $y = f(x)$ between $x = a$ and $x = b$ about the x-axis is defined as,

$$S = \int_x^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx$$

1. Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the x-axis.
2. Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the y-axis.
3. Find the area of the surface generated by revolving the given curve about the x-axis $y = \sqrt{4 - x^2}$, $-1 \leq x \leq 1$

Improper Integral

The improper integral of f over the interval $[a, +\infty)$ is defined to be

$$\int_a^{+\infty} f(x)dx = \lim_{b \to +\infty} \int_a^b f(x)dx$$

In the case where the limit exists, the improper integral is said to **converge**, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to **diverge**, and it is not assigned a value.

1. Evaluate $\int_1^\infty \frac{1}{x^3}dx$
2. Evaluate $\int_1^\infty \frac{1}{x}dx$
3. For what values of p does the integral $\frac{1}{x^p}dx$ converge?

Theorem

$$\int_1^{+\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if} \quad p > 1 \\ \text{diverges} & \text{if} \quad p \leq 1 \end{cases}$$

1. Evaluate $\int_0^{\infty} (1-x)e^{-x} dx$.

The improper integral off over the interval $(-\infty, b]$ is defined to be

$$\int_{-\infty}^{b} f(x)dx = \lim_{a \to -\infty} \int_{a}^{b} f(x)dx$$

The integral is said to converge if the limit exists and diverge if it does not. The improper integral of fover the interval $(-\infty, +\infty)$ is defined as

$$\int_{-\infty}^{+\infty} f(x)dx = \int_{-\infty}^{c} f(x)dx + \int_{c}^{+\infty} f(x)dx$$

where c is any real number. The improper integral is said to converge if both terms converge and diverge if either term diverges.

1. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{1+x^2}$

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at b, then the improper integral of f over the interval $[a, b]$ is defined as

$$\int_a^b f(x)dx = \lim_{k \to b^-} \int_a^k f(x)dx$$

In the case where the indicated limit exists, the improper integral is said to converge, and the limit is defined to be the value of the integral. In the case where the limit does not exist, the improper integral is said to diverge, and it is not assigned a value.

1. Evaluate $\int_0^1 \frac{dx}{\sqrt{1-x}}$

If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a, then the improper integral of fover the interval $[a, b]$ is defined as

$$\int_a^b f(x)dx = \lim_{k \to a^+} \int_k^b f(x)dx$$

The integral is said to converge if the indicated limit exists and diverge if it does not.
If f is continuous on the interval $[a, b]$, except for an infinite discontinuity at a point c in (a, b), then the improper integral of fover the interval $[a, b]$ is defined as

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

where the two integrals on the right side are themselves improper. The improper integral on the left side is said to converge if both terms on the right side converge and diverge if either term on the right side diverges.

1. Evaluate $\int_1^2 \frac{dx}{1-x}$
2. Evaluate $\int_1^4 \frac{dx}{x-2}^{2/3}$
3. Derive the formula for the circumference of a circle of radius r

An example of improper integral

Gamma Function

The gamma function $\Gamma(\alpha)$ is defined by the integral

$$\Gamma(\alpha) = \int_0^\infty e^{-\infty} t^{\alpha-1} \qquad \alpha > 0$$

Properties of gamma function

- $\Gamma(1) = 1$
- $\Gamma(\alpha + 1) = \alpha\Gamma(\alpha)$
- The gamma function can be regarded as a generalization of the elementary factorial function. i.e., $\Gamma(k+1) = k!$ where $k = 0, 1, 2, ...$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

Show that
(a) $\int_0^1 (\ln x)^n dx = (-1)^n \Gamma(n+1), \quad n > 0$ [Hint: Let $t = -\ln x$.]
(b) $\int_0^{+\infty} e^{-x^n} dx = \Gamma\left(\frac{n+1}{n}\right), n > 0$. [Hint: Let $t = x^n$.]