NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics

MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 6

Topics: Directional derivatives (13.6.1, 13.6.2, 13.6.3), Gradient (13.6.4, 13.6.5, 13.6.6, Th 9 in Sec 14.5 of Thomas), normal vector to surfaces and tangent planes (13.7.1, 13.7.2). Local maxima and local minima of functions of two variables, critical point, (13.8.1, 13.8.2, 13.8.4, 13.8.5), Saddle point (Definition in Sec 14.7 of Thomas). Hessian/second Partial derivative test (13.8.6). Method of Lagrange multipliers (13.9.3, 13.9.4).

1. Find $D_u f$ at P.

   (a) $f(x,y) = \ln(1+x^2+y)$; $P(0,0)$; $u = \dfrac{-1}{\sqrt{10}}i - \dfrac{3}{\sqrt{10}}j$

   (b) $f(x,y,z) = 4x^5y^2z^3$; $P(2,-1,1)$; $u = \dfrac{1}{3}i + \dfrac{2}{3}j - \dfrac{2}{3}k$

2. Find the directional derivative of f at P in the direction of a.

   (a) $f(x,y,z) = xy + z^2$; $P(-3,0,4)$; $a = i+j+k$

   (b) $f(x,y,z) = e^{x+y+3z}$; $P(-2,2,-1)$; $a = 20i - 4j + 5k$

   $P(0, \frac{\pi}{4})$ in the direction of the origin.

3. Find the directional derivative of $f(x,y) = \sqrt{xy}e^y$ at $P(1,1)$ in the direction of the negative y-axis.

4. Let $f(x,y) = \dfrac{y}{(x+y)}$. Find a unit vector u for which $D_u f(2,3) = 0$.

5. Find the directional derivative of $f(x,y,z) = \dfrac{y}{(x+z)}$ at $P(2,1,-1)$ in the direction from P to $Q(-1,2,0)$.

6. Find the gradient of f at the indicated point.

   (a) $f(x,y) = (x^2+y^2)^{\frac{-1}{2}}$; $(3,4)$

   (b) $f(x,y,z) = y^2 z \tan^3 x$; $(\frac{\pi}{4}, -3, 1)$.

7. Sketch the level curve of $f(x,y)$ that passes through P and draw the gradient vector at P.

   (a) $f(x,y) = 4x - 2y + 3; P(1,2)$

   (b) $f(x,y) = \frac{y}{x^2}; P(-2,2)$

   (c) $f(x,y) = x^2 + 4y^2; P(-2,0)$

   (d) $f(x,y) = x^2 - y^2; P(2,-1)$.

8. Find a unit vector in the direction in which f increases most rapidly at P, and find the rate of change of f at P in that direction.

   (a) $f(x,y,z) = \dfrac{x}{z} + \dfrac{z}{y^2}; P(1,2,-2)$

   (b) $f(x,y,z) = \tan^{-1}\left(\dfrac{x}{(y+z)}\right); P(4,2,2)$

9. Check whether this statement is true or false: If u is a fixed unit vector and $D_u f(x,y) = 0$ for all points (x,y), then f is a constant function.

10. Check whether this statement is true or false: If the displacement vector from (x_0,y_0) to (x_1,y_1) is a positive multiple of $\nabla f(x_0,y_0)$, then $f(x_0,y_0) \leq f(x_1,y_1)$.

11. Given that $\nabla f(x_0,y_0) = \mathbf{i} - 2\mathbf{j}$ and $D_{\mathbf{u}} f(x_0,y_0) = -2$, find $\mathbf{u}$ (two answers).

12. A particle moves along a path C given by the equations $x = t$ and $y = -t^2$. If $z = x^2 + y^2$, find $\dfrac{dz}{ds}$ along C at the instant when the particle is at the point $(2,-4)$.

13. Given that the directional derivative of $f(x,y,z)$ at the point $(3,-2,1)$ in the direction of $\mathbf{a} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$ is -5 and that $||\nabla f(3,-2,1)|| = 5$, find $\nabla f(3,-2,1)$.

14. Let $r = \sqrt{x^2 + y^2}$.

(a) Show that $\nabla r = \dfrac{\mathbf{r}}{r}$, where $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$.

(b) Show that $\nabla f(r) = f'(r)\nabla r = \dfrac{f'(r)}{r}\mathbf{r}$.

15. Use the formula in part (b) of Exercise 15 to find

(a) $\nabla f(r)$ if $f(r) = re^{-3r}$

(b) $f(r)$ if $\nabla f(r) = 3r^2\mathbf{r}$ and $f(2) = 1$.

16. Prove: If f and g are differentiable, then

(a) $\nabla(f + g) = \nabla f + \nabla g$

(b) $\nabla(cf) = c\nabla f$ (c constant)

(c) $\nabla(fg) = f\nabla g + g\nabla f$

(d) $\nabla(\frac{f}{g}) = \dfrac{g\nabla f - f\nabla g}{g^2}$

(e) $\nabla(f^n) = nf^{n-1}\nabla f$.

17. A heat-seeking particle is located at the point P on a flat metal plate whose temperature at a point (x, y) is $T(x, y)$. Find parametric equations for the trajectory of the particle if it moves continuously in the direction of maximum temperature increase.

(a) $T(x, y) = 5 - 4x^2 - y^2$; $P(1, 4)$

(b) $T(x, y) = 100 - x^2 - 2y^2$; $P(5, 3)$

18. Consider the ellipsoid $x^2 + y^2 + 4z^2 = 12$.

(a) Find an equation of the tangent plane to the ellipsoid at the point $(2, 2, 1)$.

(b) Find the parametric equations of the line that is normal to the ellipsoid at the point $(2, 2, 1)$.

(c) Find the acute angle that the tangent plane at the point $(2, 2, 1)$ makes with the xy-plane.

19. Find an equation for the tangent plane and parametric equations for the normal line to the surface at the point P.

(a) $x^2 - xyz = 56$; $P(-4, 5, 2)$

(b) $z = x^{1/2} + y^{1/2}$; $P(4, 9, 5)$

20. Find all points on the surface at which the tangent plane is horizontal.

(a) $z = x^3y^2$

(b) $z = x^2 - xy + y^2 - 2x + 4y$

21. Find a point on the surface $z = 3x^2 - y^2$ at which the tangent plane is parallel to the plane $6x + 4y - z = 5$.

22. Show that the surfaces

$$z = \sqrt{x^2 + y^2} \text{ and } z = \frac{1}{10}(x^2 + y^2) + \frac{5}{2}$$

intersect at (3, 4, 5) and have a common tangent plane at that point.

23. Show that every line that is normal to the sphere $x^2 + y^2 + z^2 = 1$ passes through the origin.

24. Locate all relative maxima, relative minima, and saddle points, if any.

(a) $f(x, y) = x^2 + xy - 2y - 2x + 1$

(b) $f(x, y) = x^2 + xy + y^2 - 3x$

(c) $f(x, y) = xy - x^3 - y^2$

(d) $f(x, y) = x^2 + y - e^y$

(e) $f(x, y) = xy + \frac{2}{x} + \frac{4}{y}$

(f) $f(x, y) = e^x \sin y$

(g) $f(x, y) = e^{-(x^2+y^2+2x)}$

(h) $f(x, y) = xy + \frac{a^3}{x} + \frac{b^3}{y}$ $(a \neq 0, b \neq 0)$

25. (a) Show that the second partials test provides no information about the critical points of the function $f(x, y) = x^4 + y^4$.

(b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

26. (a) Show that the second partials test provides no information about the critical points of the function $f(x, y) = x^4 - y^4$.

    (b) Classify all critical points of f as relative maxima, relative minima, or saddle points.

27. Recall that if a continuous function of one variable has exactly one relative extremum on an interval, then that relative extremum is an absolute extremum on the interval. The following exercises shows that this result does not extend to functions of two variables.

    (a) Show that $f(x, y) = 3xe^y - x^3 - e^{3y}$ has only one critical point and that a relative maximum occurs there.

    (b) Show that f does not have an absolute maximum.

28. If f is a continuous function of one variable with two relative maxima on an interval, then there must be a relative minimum between the relative maxima. (Convince yourself of this by drawing some pictures.) The purpose of this exercise is to show that this result does not extend to functions of two variables.
    Show that $f(x, y) = 4x^2e^y - 2x^4 - e^{4y}$ has two relative maxima but no other critical points.

29. Find the absolute extrema of the function $f(x, y) = xy^2$ on a region R that satisfies the inequalities $x \geq 0$, $y \geq 0$, and $x^2 + y^2 \leq 1$.

30. Find three positive numbers whose sum is 27 and such that the sum of their squares is as small as possible.

31. Find all points on the portion of the plane $x + y + z = 5$ in the first octant at which $f(x, y, z) = xy^2z^2$ has a maximum value.

32. Consider the function
$$f(x, y) = 4x^2 - 3y^2 + 2xy$$
    over the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$.

    (a) Find the maximum and minimum values of f on each edge of the square.

    (b) Find the maximum and minimum values of f on each diagonal of the square.

    (c) Find the maximum and minimum values of f on the entire square.

33. Determine the dimensions of a rectangular box, open at the top, having volume V, and requiring the least amount of material for its construction.

34. Find two numbers a and b with $a \leq b$ such that
$$\int_a^b (6 - x - x^2)dx$$
    has its largest value.

35. Find two numbers a and b with $a \leq b$ such that
$$\int_a^b (24 - 2x - x^2)^{\frac{1}{3}} dx$$
    has its largest value.

36. Show that $(0, 0)$ is a critical point of $f(x, y) = x^2 + kxy + y^2$ no matter what value the constant k has. (Hint: Consider two cases: $k = 0$ and $k \neq 0$.)

37. For what values of the constant k does the Second Derivative Test guarantee that $f(x, y) = x^2 + kxy + y^2$ will have a saddle point at $(0, 0)$? A local minimum at $(0, 0)$? For what values of k is the Second Derivative Test inconclusive? Give reasons for your answers.

38. Use Lagrange multipliers to find the maximum and minimum values of f subject to the given constraint. Also, find the points at which these extreme values occur.

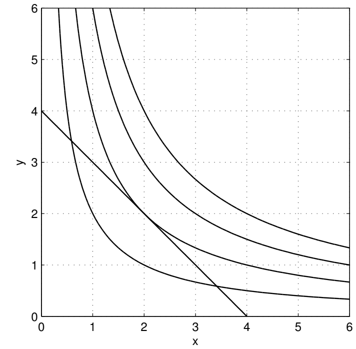
    (a) $f(x, y) = xy$; $4x^2 + 8y^2 = 16$

    (b) $f(x, y) = x - 3y - 1$; $x^2 + 3y^2 = 16$

    (c) $f(x, y, z) = 3x + 6y + 2z$; $2x^2 + 4y^2 + z^2 = 70$

    (d) $f(x, y, z) = x^4 + y^4 + z^4$; $x^2 + y^2 + z^2 = 1$

39. The accompanying figure shows graphs of the line $x + y = 4$ and the level curves of height $c = 2$, 4, 6, and 8 for the function $f(x, y) = xy$.

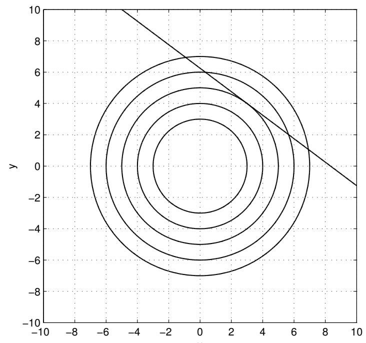


(a) Use the figure to find the maximum value of the function $f(x, y) = xy$ subject to $x + y = 4$, and explain

(b) How can you tell from the figure that your answer to part (a) is not the minimum value of f subject to the constraint? your reasoning.

(c) Use Lagrange multipliers to check your work.

40. The accompanying figure shows the graphs of the line $3x + 4y = 25$ and the level curves of height $c = 9$, 16, 25, 36, and 49 for the function $f(x, y) = x^2 + y^2$.



(a) Use the accompanying figure to find the minimum value of the function $f(x, y) = x^2 + y^2$ subject to $3x + 4y = 25$, and explain your reasoning.

(b) How can you tell from the accompanying figure that your answer to part (a) is not the maximum value of f subject to the constraint?

(c) Use Lagrange multipliers to check your work.

41. Solve using Lagrange multipliers.

(a) Find the point on the line $2x - 4y = 3$ that is closest to the origin.

(b) Suppose that the temperature at a point (x, y) on a metal plate is $T(x, y) = 4x^2 - 4xy + y^2$. An ant, walking on the plate, traverses a circle of radius 5 centered at the origin. What are the highest and lowest temperatures encountered by the ant?

42. Solve the problems 30,31,32,33 using the method of Lagrange multipliers.