NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics

MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 2

Topics: Continuity (Anton et al. 1.5.1, 1.5.2, 1.5.3, 1.5.4, 1.5.5, 1.5.6) Intermediate-Value Theorem, Squeezing Theorem (1.5.8, 1.5.9, 1.6.2, 1.6.3). Differentiability, Product and Quotient Rules, Chain Rule (Anton et al. 2.2.2, 2.2.3, 2.4.1, 2.4.2, 2.6.1) Implicit differentiation (Anton et al. Sec 2.7), Relative extremum and absolute extremum, First Derivative Test, Second Derivative Test (Anton 3.2.1, 3.2.2, 3.2.3, 3.2.4, 3.4.1, 3.4.2), Rolle's theorem, Mean value theorem (Anton et al. 3.8.1, 3.8.2, 3.1.2), L'Hôpital's Rule (Anton 6.5.1, 6.5.2).

1. Suppose that a function f is continuous everywhere and that $f(-2) = 3$, $f(-1) = -1$, $f(0) = -4$, $f(1) = 1$, and $f(2) = 5$. Does the Intermediate-Value Theorem guarantee that f has a root on the following intervals?

   (a) $[-2, -1]$
   (b) $[-1, 0]$
   (c) $[-1, 1]$
   (d) $[0, 2]$

2. Determine whether the statement is true or false. Explain your answer.

   (a) If $f(x)$ is continuous at $x = c$, then so is $|f(x)|$.
   (b) If $|f(x)|$ is continuous at $x = c$, then so is $f(x)$.
   (c) If $f(x)$ is continuous at $x = c$, then so is $\sqrt{f(x)}$.
   (d) If f and g are discontinuous at $x = c$, then so is $f + g$.
   (e) If f and g are discontinuous at $x = c$, then so is fg.
   (f) If $\sqrt{f(x)}$ is continuous at $x = c$, then so is $f(x)$.

3. Find a value of the constant k, if possible, that will make the function continuous everywhere.

   (a) $f(x) = \begin{cases} 7x - 2, & x \leq 1 \\ kx^2, & x > 1. \end{cases}$
   (b) $f(x) = \begin{cases} kx^2, & x \leq 2 \\ 2x + k, & x > 2. \end{cases}$
   (c) $f(x) = \begin{cases} 9 - x^2, & x \geq -3 \\ k/x^2, & x < -3. \end{cases}$
   (d) $f(x) = \begin{cases} 9 - x^2, & x \geq 0 \\ k/x^2, & x < 0. \end{cases}$

4. Find values of the constants k and m, if possible, that will make the function f continuous everywhere.

$$f(x) = \begin{cases} x^2 + 5, & x > 2 \\ m(x+1) + k, & -1 < x \leq 2 \\ 2x^3 + x + 7, & x \leq -1 \end{cases}$$

5. Find the values of x (if any) at which f is not continuous, and determine whether the function can be made continuous by changing the value of the function appropriately.

   (a) $f(x) = \dfrac{|x|}{x}, x \neq 0; f(0) = 0.$
   (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}, x \neq -3; f(-3) = 0$
   (c) $f(x) = \dfrac{x - 2}{|x| - 2}, x \neq \pm 2; f(2) = f(-2) = 0.$
   (d) $f(x) = \frac{\sin x}{x}, x \neq 0; f(0) = 0.$

6. Prove that a rational function is continuous at every point where the denominator is nonzero, and has discontinuities at the points where the denominator is zero.

7. Determine whether the statement is true or false. Explain your answer

   (a) Suppose that M is a positive number and that for all real numbers x, a function f satisfies $-M \leq f(x) \leq M$. Then $\lim_{x \to 0} x f(x) = 0$ and $\lim_{x \to \infty} \frac{f(x)}{x} = 0$.

(b) Suppose that for all real numbers x, a function f satisfies $|f(x)+5| \le |x+1|$. Then $\lim\limits_{x\to -1} f(x) = -5$.

(c) For $0 < x < \pi/2$, the graph of $y = \sin x$ lies below the graph of $y = x$ and above the graph of $y = x\cos x$.

8. Answer the following questions by referring to the function

$$f(x) = \begin{cases} x^2-1, & -1 \le x < 0 \\ 2x, & 0 < x < 1 \\ 1, & x = 1 \\ -2x+4, & 1 < x < 2 \\ 0, & 2 < x < 3 \end{cases}$$

(a) Does $f(-1)$ exist?

(b) Does $\lim\limits_{x\to -1^+} f(x)$ exists?

(c) Does $\lim\limits_{x\to -1^+} f(x) = f(-1)$?

(d) Is f continuous at $x = -1$?

(e) Does $f(1)$ exist?

(f) Does $\lim\limits_{x\to 1} f(x)$ exists?

(g) Does $\lim\limits_{x\to 1} f(x) = f(1)$?

(h) Is f continuous at $x = 1$?

(i) Is f defined at $x = 2$?

(j) Is f continuous at $x = 2$?

(k) At what values of x is f continuous?

(l) What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

(m) To what new value should $f(1)$ be changed to remove the discontinuity?

9. At what points are the functions continuous?

(a) $y = \dfrac{1}{x-2} - 3x$

(b) $y = \dfrac{x+1}{x^2-4x+3}$

(c) $y = \dfrac{\cos x}{x}$

(d) $y = \dfrac{\sqrt{x^4+1}}{1+\sin^2 x}$

10. Show that f is continuous at $x = 0$, where $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$

11. Find a formula (or formulas) for computing the composite function $h(x) = f[g(x)]$ where f and g are

(a) $f(x) = \dfrac{x+|x|}{2}$ for all x, $\quad g(x) = \begin{cases} x & \text{for } x < 0, \\ x^2 & \text{for } x \ge 0. \end{cases}$

(b) $f(x) = \begin{cases} 1 & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1, \end{cases} \quad g(x) = \begin{cases} 2-x^2 & \text{if } |x| \le 2, \\ 2 & \text{if } |x| > 2. \end{cases}$

In (a) and (b), for what values of x is h continuous?

12. Prove: If f and g are continuous on $[a,b]$, and $f(a) > g(a)$, $f(b) < g(b)$, then there is at least one solution of the equation $f(x) = g(x)$ in (a,b). [Hint: Consider $f(x) - g(x)$]

13. Prove: If $p(x)$ is a polynomial of odd degree, then the equation $p(x) = 0$ has at least one real solution.

14. Use the Intermediate Value Theorem to show that the equation $x = \cos x$ has a solution in the interval $\left[0, \dfrac{\pi}{2}\right]$.

15. (a) Show that $f(x) = \begin{cases} x^2+x+1, & x \le 1 \\ 3x, & x > 1 \end{cases}$ is continuous at $x = 1$. Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there. Sketch the graph of f .

(b) Let $f(x) = \begin{cases} x^2-16x, & x < 9 \\ \sqrt{x}, & x \ge 9. \end{cases}$

Is f continuous at $x = 9$? Determine whether f is differentiable at $x = 9$. If so, find the value of the derivative there.

16. Let $f(x) = x^8 - 2x + 3$; find $\lim\limits_{w\to 2} \dfrac{f'(w) - f'(2)}{w-2}$.

17. Find all values of x at which the tangent line to the given curve satisfies the stated property.

(a) $y = \dfrac{x^2+1}{x+1}$; parallel to the line $y = x$

(b) $y = \dfrac{1}{x+4}$; passes through the origin

(c) $y = \dfrac{2x+5}{x+2}$; y-intercept 2.

(d) $y = \dfrac{\sqrt{12-3x^2}}{2}$; perpendicular to $y = 0$.

18. Use a derivative to evaluate each limit.

(a) $\displaystyle\lim_{h\to 0} \frac{\sin(\frac{\pi}{2}+h)-1}{h}$

(b) $\displaystyle\lim_{h\to 0} \frac{\csc(x+h)-\csc(x)}{h}$

19.* Suppose that f is a function that is differentiable everywhere. Explain the relationship, if any, between the periodicity of f and that of f' . That is, if f is periodic, must f' also be periodic? If f' is periodic, must f also be periodic?

20. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	5	-2	5	7
5	3	-1	12	4

(a) $F'(3)$, where $F(x) = f(g(x))$

(b) $G'(3)$, where $G(x) = g(f(x))$

21. (a) Find $f'(x^2)$ if $\dfrac{d}{dx}\left[f(x^2)\right] = x^2$.

(b) Find $\dfrac{d}{dx}[f(x)]$, if $\dfrac{d}{dx}[f(3x)] = 6x$.

22.* Recall that a function f is even if $f(-x) = f(x)$ and odd if $f(-x) = -f(x)$, for all x in the domain of f . Assuming that f is differentiable, prove:

(a) f' is odd if f is even

(b) f' is even if f is odd.

23. (a) Suppose that $z = x^3y^2$, where both x and y are changing with time. At a certain instant when $x = 1$ and $y = 2$, x is decreasing at the rate of 2 units/s, and y is increasing at the rate of 3 units/s. How fast is z changing at this instant? Is z increasing or decreasing?

(b) Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

24. Suppose that $f(x) = \begin{cases} x^2-1, & x \le 1 \\ k(x-1), & x > 1. \end{cases}$ For what values of k is f continuous? For what values of k is f differentiable?

25. Suppose that a function f is differentiable at $x = 1$ and $\lim_{h\to 0} \frac{f(1+h)}{h} = 5$. Find $f(1)$ and $f'(1)$.

26. Suppose that a function f is differentiable at $x = 2$ and $\lim_{x\to 2} \frac{x^3 f(x)-24}{x-2} = 28$. Find $f(2)$ and $f'(2)$.

27. (a) Does the graph of $f(x) = \begin{cases} x^2\sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$ have a tangent at the origin? Give reasons for your answer.

(b) Does the graph of $g(x) = \begin{cases} x\sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$ have a tangent at the origin? Give reasons for your answer.

28. Determine if the piecewise defined function $g(x) = \begin{cases} x^{2/3}, & x \ge 0 \\ x^{1/3}, & x < 0 \end{cases}$ is differentiable at the origin.

29.* Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that f is differentiable at $x = 0$ and find $f^{'}(0)$.

30. Find the values of a and b that make the following function differentiable for all x-values. $f(x) = \begin{cases} ax+b, & x > -1 \\ bx^2-3, & x \le -1 \end{cases}$

31. (a) Graph the function $f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$

(b) Is f continuous at $x = 0$?

(c) Is f differentiable at $x = 0$?

32. (a) Graph the function $f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$

(b) Is f continuous at $x = 0$?

(c) Is f differentiable at $x = 0$?

33. For what value or values of the constant m, if any, is $f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0. \end{cases}$

(a) continuous at $x = 0$?

(b) differentiable at $x = 0$?

34. (a) Show that if the position x of a moving point is given by a quadratic function of t, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1,\ t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

(b) What is the geometric significance of the result in part (a).

35. Show that the following functions are differentiable at $x = 0$.

(a) $|x| \sin x$

(b) $x^{2/3} \sin x$

(c) $\sqrt[3]{x}(1 - \cos x)$

(d) $h(x) = \begin{cases} x^2 \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$

36. Is the derivative of $h(x) = \begin{cases} x^2 \sin(1/x), & x \ne 0 \\ 0, & x = 0 \end{cases}$ continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

37. Suppose that a function f satisfies the following conditions for all real values of x and y:

(a) $f(x + y) = f(x)f(y)$

(b) $f(x) = 1 + xg(x)$, where $\lim_{x\to 0} g(x) = 1$

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

38. A function f is defined as follows:

$$f(x) = \begin{cases} x^2 & \text{if } x \le c, \\ ax + b & \text{if } x > c, \end{cases} \quad (a,\ b,\ c \text{ constants}).$$

Find values of a and b (in terms of c) such that $f'(c)$ exists.

39. Solve the above question when $f(x) = \begin{cases} \dfrac{1}{|x|} & \text{if } |x| > c, \\ a + bx^2 & \text{if } |x| \le c. \end{cases}$

40. Given that the derivative $f'(a)$ exists. State which of the following statements are true.

(a) $f'(a) = \lim_{h\to a} \dfrac{f(h) - f(a)}{h - a}$.

(b) $f'(a) = \lim_{h\to 0} \dfrac{f(a) - f(a - h)}{h}$.

(c) $f'(a) = \lim_{t\to 0} \dfrac{f(a + 2t) - f(a)}{t}$.

(d) $f'(a) = \lim_{t\to 0} \dfrac{f(a + 2t) - f(a + t)}{2t}$.

41.**Find the relative extrema of the function f.

(a) $f(x) = x^4 - 4x^3 + 4x^2$

(b) $f(x) = \dfrac{x + 3}{x - 2}$

(c) $f(x) = |3x - x^2|$

(d) $f(x) = |1 + \sqrt[3]{x}|$

42. Find the absolute maximum and minimum values of f , if any, on the given interval, and state where those values occur.

(a) $f(x) = x^2 - x - 2;\quad (-\infty, +\infty)$

(b) $f(x) = x^4 + 4x;\quad (-\infty, +\infty)$

(c) $f(x) = x^3 - 9x + 1;\quad (-\infty, +\infty)$

(d) $f(x) = \dfrac{x^2+1}{x+1};\quad (-5, -1)$

(e) $f(x) = \dfrac{x-2}{x+1};\quad (-1, 5]$

43. Find the absolute maximum and minimum values of attained on $\left[\frac{1}{2}, \frac{7}{2}\right]$ by the function

$$f(x) = \begin{cases} 4x - 2, & x < 1 \\ (x-2)(x-3), & x \geq 1. \end{cases}$$

44. (a) Let $f(x) = x^2 + px + q$. Find the values of p and q such that $f(1) = 3$ is an extreme value of f on $[0, 2]$. Is this value a maximum or minimum?

(b) Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point (1,2).

(c) Determine the values of constants a, b, c and d so that $f(x) = ax^3 + bx^2 + cx + d$ has a local maximum at the point (0,0) and a local minimum at the point (1,-1).

45. (a) A piece of wire 40 cm long is cut into two pieces. One piece is bent into the shape of a square and the other is bent into the shape of a circle. How should the wire be cut so that the total area enclosed is a (a) maximum and (b) minimum?

(b) Express 18 as a sum of two positive numbers such that the product of the first and square of the second is as large as possible.

46. Find the absolute maximum value of $f(x) = x^2 \ln\left(\dfrac{1}{x}\right)$ and say where it is assumed.

47. Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

(a) $f(x) = x^2 - 8x + 15;\quad [3, 5]$

(b) $f(x) = \cos x;\quad [\pi/2, 3\pi/2]$

(c) $f(x) = \dfrac{(x^2-1)}{(x-2)};\quad [-1, 1]$

(d) $f(x) = 2^x - 30x + 82;\quad [3, 7]$

48. Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

(a) $f(x) = x^2 - x;\quad [-3, 5]$

(b) $f(x) = \sqrt{25 - x^2};\quad [-5, 3]$

(c) $f(x) = x - \dfrac{1}{x};\quad [3, 4]$

(d) $f(x) = -2x^3 + 6x - 2;\quad [-3, 5]$

49. Determine whether the following statements are true or false. Explain.

(a) Rolle's Theorem says that if f is a continuous function on $[a, b]$ and $f(a) = f(b)$, then there is a point between a and b at which the curve $y = f(x)$ has a horizontal tangent line.

(b) If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is a point between a and b at which the instantaneous rate of change of f matches the average rate of change of f over $[a, b]$.

50. Let $f(x) = x^{\frac{2}{3}}$, $\quad a = -1$, and $b = 8$.

(a) Show that there is no point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

(b) Explain why the result in part (a) does not contradict the Mean-Value Theorem.

51. An automobile travels 4 mi along a straight road in 5 min. Show that the speedometer reads exactly 48 mi/h at least once during the trip.

52. Use the Mean-Value Theorem to show that $\sqrt{y} - \sqrt{x} < \dfrac{y - x}{2\sqrt{x}}$ if $0 < x < y$.

53. (a) Show that if f and g are functions for which $f'(x) = g(x)$ and $g'(x) = -f(x)$ for all x, then $f^2(x) + g^2(x)$ is a constant.

(b) Show that if f and g are functions for which $f'(x) = g(x)$ and $g'(x) = f(x)$ for all x, then $f^2(x) - g^2(x)$ is a constant.

(c) Give an example of functions f and g with this property

54.* Prove the following result using the Mean-Value Theorem: Let f be continuous at x_0 and suppose that $\lim_{x\to x_0} f'(x)$ exists. Then f is differentiable at x_0 and $f'(x_0) = \lim_{x\to x_0} f'(x)$ [Hint: The derivative $f'(x_0)$ is given by $f'(x_0) = \lim_{x\to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$ provided this limit exists.]

55. (a) If

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$$

Show that

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0^+} f'(x)$$

but that $f'(0)$ does not exist.

(b) If

$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$$

Show that $f'(0)$ exists but $f''(0)$ does not exist.

56. (a) Show that if f is differentiable on an open interval and $f'(x) \neq 0$ on the interval, the equation $f(x) = 0$ can have at most one real root in the interval.

(b) Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.

(c) Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?

57. (a) Suppose that $0 < f'(x) < \dfrac{1}{2}$ for all x-values. Show that $f(-1) < f(1) < f(-1) + 1$.

(b) Show that $|\cos x - 1| \le |x|$ for all x-values.(Hint: Consider $f(t) = \cos t$ on $[0, x]$.)

(c) Show that for any numbers a and b, the sine inequality $|\sin b - \sin a| \le |b - a|$ is true.

58. Use L'Hôpital's rule find the limits:

(a) $\displaystyle\lim_{x\to 2} \frac{x-2}{x^2-4}$

(b) $\displaystyle\lim_{t\to 0} \frac{\sin t^2}{t}$

(c) $\displaystyle\lim_{\theta\to\frac{\pi}{2}} \frac{1-\sin\theta}{1+\cos 2\theta}$

(d) $\displaystyle\lim_{x\to 0^+} (\ln x - \ln\sin x)$

(e) $\displaystyle\lim_{h\to 0} \frac{e^h - (1+h)}{h^2}$

(f) $\displaystyle\lim_{x\to\infty} x^2 e^{-x}$

59. Find the following limits: (L'Hôpital's rule does not help with these limits)

(a) $\displaystyle\lim_{x\to 0^+} \frac{\sqrt{x}}{\sqrt{\sin x}}$

(b) $\displaystyle\lim_{x\to\infty} \frac{e^{x^2}}{xe^x}$

(c) $\displaystyle\lim_{x\to 0^+} \frac{x}{e^{(-1/x)}}$