NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics
MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 4

Topics: Parameterised curves in space (Sec 12.1, 12.2), Arc length (12.3.1, 12.3.3, 12.3.4). Unit Tangent, Normal, and Bi-normal Vectors (Sec 12.4), curvature (12.5.1, 12.5.2), radius of curvature, velocity and acceleration (12.6.1, 12.6.2, 12.6.3).

1. Sketch the curve of intersection of the surfaces, and find a vector equation for the curve in terms of the parameter $x = t$

   (a) $9x^2 + y^2 + 9z^2 = 81, \;\; y = x^2 \;\; (z > 0)$ (b) $y = x, \;\; x + y + z = 1$

2. (a) Show that the graph of $\mathbf{r} = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j} + t^2\,\mathbf{k}$ lies on the paraboloid $z = x^2 + y^2$.

   (b) Show that the graph of $\mathbf{r} = t\mathbf{i} + \frac{1+t}{t}\mathbf{j} + \frac{1-t^2}{t}\mathbf{k}, \;\; t > 0$ lies in the plane $x - y + z + 1 = 0$.

3. Find parametric equations of the line tangent to the graph of $\mathbf{r}(t)$ at the point where $t = t_0$.

   (a) $\mathbf{r}(t) = e^{2t}\,\mathbf{i} - 2\cos 3t\,\mathbf{j}; \;\; t_0 = 0$ (b) $\mathbf{r}(t) = 2\cos \pi t\,\mathbf{i} + 2\sin \pi t\,\mathbf{j} + 3t\,\mathbf{k}; \;\; t_0 = \frac{1}{3}$

4. Show that the graphs of $\mathbf{r_1}(t)$ and $\mathbf{r_2}(t)$ intersect at the point P . Find, to the nearest degree, the acute angle between the tangent lines to the graphs of $\mathbf{r_1}(t)$ and $\mathbf{r_2}(t)$ at the point P.

   (a) $\mathbf{r_1}(t) = t^2\mathbf{i} + t\mathbf{j} + 3t^3\mathbf{k}$
   $\mathbf{r_2}(t) = (t-1)\mathbf{i} + \frac{1}{4}t^2\mathbf{j} + (5-t)\mathbf{k}; \;\; P(1,1,3)$

   (b) $\mathbf{r_1}(t) = 2e^{-t}\mathbf{i} + \cos t\mathbf{j} + (t^2+3)\mathbf{k}$
   $\mathbf{r_2}(t) = (1-t)\mathbf{i} + t^2\mathbf{j} + (t^3+4)\mathbf{k}; \;\; P(2,1,3)$

5. Find the arc length of the parametric curve.

   (a) $x = \cos^3 t, \; y = \sin^3 t, \; z = 2; \;\; 0 \le t \le \pi/2$ (b) $x = e^t, \; y = e^{-t}, \; z = \sqrt{2}\,t; \;\; 0 \le t \le 1$

6. Find the exact arc length of the curve over the interval.

   (a) $x = \frac{1}{3}(y^2+2)^{\frac{3}{2}}$ from $y = 0$ to $y = 1$ (b) $y = \dfrac{x^6+8}{16x^2}$ from $x = 2$ to $x = 3$

7. Calculate $\dfrac{d\mathbf{r}}{d\tau}$ by the chain rule, and then check your result by expressing $\mathbf{r}$ in terms of τ and differentiating.

   (a) $\mathbf{r} = t\mathbf{i} + t^2\mathbf{j}; \;\; t = 4\tau + 1$ (b) $\mathbf{r} = \mathbf{i} + 3t^{3/2}\mathbf{j} + t\mathbf{k}; \;\; t = 1/\tau$

8. (a) Find the arc length parametrization of the line $x = t, \; y = t$ that has the same orientation as the given line and has reference point (0, 0).

   (b) Find the arc length parametrization of the line $x = t, \; y = t, \; z = t$ that has the same orientation as the given line and has reference point $(0, 0, 0)$.

9. (a) Find the arc length parametrization of the line $x = 1 + t, \; y = 3 - 2t, \; z = 4 + 2t$ that has the same direction as the given line and has reference point $(1, 3, 4)$.

   (b) Use the parametric equations obtained in part (a) to find the point on the line that is 25 units from the reference point in the direction of increasing parameter.

10. Find an arc length parametrization of the curve that has the same orientation as the given curve and for which the reference point corresponds to $t = 0$.

    (a) $\mathbf{r}(t) = (3 + \cos t)\mathbf{i} + (2 + \sin t)\mathbf{j}\;; \; 0 \le t \le 2\pi$ (b) $\mathbf{r}(t) = \frac{1}{3}t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}; \; t \ge 0$

11. Find $\mathbf{T}(t)$ and $\mathbf{N}(t)$ at the given point.

(a) $\mathbf{r}(t) = \ln t\,\mathbf{i} + t\,\mathbf{j};\ t = e$

(b) $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k}; t = 0$

12. Find parametric equations for the tangent line to the graph of $\mathbf{r}(t)$ at $t = 0$ in terms of an arc length parameter s.

(a) $\mathbf{r}(t) = \sin t\mathbf{i} + \cos t\mathbf{j} + \frac{1}{2}t^2\mathbf{k};\ t_0 = 0$

(b) $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\mathbf{k};\ t_0 = 1$

13. Use the formula $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ to find $\mathbf{B}(t)$ and then verify your answer using the formula $\mathbf{B}(t) = \dfrac{\mathbf{r}'(t) \times \mathbf{r}''(t)}{||\mathbf{r}'(t) \times \mathbf{r}''(t)||}$ by calculating $\mathbf{B}(t)$ directly from $\mathbf{r}(t)$.

(a) $\mathbf{r}(t) = e^t \sin t\mathbf{i} + e^t \cos t\mathbf{j} + 3\mathbf{k}$

(b) $\mathbf{r}(t) = (\sin t - t\cos t)\,\mathbf{i} + (\cos t + t\sin t)\,\mathbf{j} + \mathbf{k}$

14. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, and $\mathbf{B}(t)$ for the given value of t. Then find equations for the osculating, normal, and rectifying planes at the point that corresponds to that value of t.

(a) $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} +\ \mathbf{k};\ t = \frac{\pi}{4}$

(b) $\mathbf{r}(t) = e^t\,\mathbf{i} + e^t\cos t\,\mathbf{j} + e^t\sin t\,\mathbf{k};\ t = 0$

15. Find the curvature and the radius of curvature at the stated point.

(a) $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j} + t\,\mathbf{k};\ t = 0$

(b) $x = e^t\cos t, y = e^t\sin t, z = e^t;\ t = 0$

16. Find the curvature at the stated point.

(a) $x = t,\ y = \dfrac{1}{t};\ t = 1$

(b) $x = e^{3t},\ y = e^{-t};\ t = 0$

17. At what point(s) does $y = e^x$ have maximum curvature?

18. At what point(s) does $4x^2 + 9y^2 = 36$ have minimum radius of curvature?

19. Find the maximum and minimum values of the radius of curvature for the curve $x = \cos t, y = \sin t, z = \cos t$.

20. Find the velocity, speed, and acceleration at the given time t of a particle moving along the given curve.

(a) $x = 1 + 3t, y = 2 - 4t, z = 7 + t;\ t = 2$

(b) $x = 2\cos t,\ y = 2\sin t,\ z = t;\ t = \frac{\pi}{4}$

21. (a) Suppose that the position vector of a particle moving in the plane is $\mathbf{r} = 12\sqrt{t}\,\mathbf{i} + t^{\frac{3}{2}}\,\mathbf{j},\ t > 0$. Find the minimum speed of the particle and its location when it has this speed.

(b) Suppose that the motion of a particle is described by the position vector $\mathbf{r} = (t - t^2)\,\mathbf{i} - t^2\,\mathbf{j}$. Find the minimum speed of the particle and its location when it has this speed.

22. (a) Find, to the nearest degree, the angle between $\mathbf{v}$ and $\mathbf{a}$ for $\mathbf{r} = t^3\,\mathbf{i} + t^2\,\mathbf{j}$ when $t = 1$.

(b) Show that the angle between $\mathbf{v}$ and $\mathbf{a}$ is constant for the position vector $\mathbf{r} = e^t\cos t\,\mathbf{i} + e^t\sin t\,\mathbf{j}$. Find the angle.

23. The position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ of two particles are given. Show that the particles move along the same path but the speed of the first is constant and the speed of the second is not.

(a) $\mathbf{r}_1 = 2\cos 3t\,\mathbf{i} + 2\sin 3t\,\mathbf{j}$
$\mathbf{r}_2 = 2\cos(t^2)\,\mathbf{i} + 2sin(t^2)\,\mathbf{j} \quad (t \geq 0)$

(b) $\mathbf{r}_1 = (3 + 2t)\,\mathbf{i} + t\,\mathbf{j} + (1 - t)\,\mathbf{k}$
$\mathbf{r}_2 = (5 - 2t^3)\,\mathbf{i} + (1 - t^3)\,\mathbf{j} + t^3\,\mathbf{k}$

24. The position function of a particle is given. Find (a) the scalar tangential and normal components of acceleration at the stated time t; (b) the vector tangential and normal components of acceleration at the stated time t; (c) the curvature of the path at the point where the particle is located at the stated time t.

(a) $\mathbf{r} = (t^3 - 2t)\,\mathbf{i} + (t^2 - 4)\,\mathbf{j}; t = 1$

(b) $\mathbf{r} = e^t i + e^{-2t} j + tk; t = 0$

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