NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics

# MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 9

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**Topics: Parametric representation of surfaces, Smooth surface, area, surface area differential, (Sec 15.5 of Anton), tangent planes to parametric surfaces (14.4.1), Orientation, flux (15.6.1, 15.6.2), Moments and Masses (Table 16.3 in Sec 16.6 of Thomas), Gauss' divergence theorem (15.7.1), Relative orientation of curves and surfaces, Stokes' theorem (15.8.1).**

1. Express the area of the given surface as an iterated double integral, and then find the surface area.
   (a) The portion of the plane $2x + 2y + z = 8$ in the first octant.
   (b) The portion of the cone $z^2 = 4x^2 + 4y^2$ that is above the region in the first quadrant bounded by the line $y = x$ and the parabola $y = x^2$.

2. Express the area of the given surface as an iterated double integral in polar coordinates, and then find the surface area.
   (a) The portion of the cone $z = \sqrt{x^2 + y^2}$ that lies inside the cylinder $x^2 + y^2 = 2x$.
   (b) The portion of the surface $z = xy$ that is above the sector in the first quadrant bounded by the lines $y = x/\sqrt{3}$, $y = 0$, and the circle $x^2 + y^2 = 9$.

3. (a) Find parametric equations for the portion of the cylinder $x^2 + y^2 = 5$ that extends between the planes $z = 0$ and $z = 1$.
   (b) Find parametric equations for the portion of the plane $y - 2z = 5$ that extends between the planes $x = 0$ and $x = 3$.

4. (a) Find parametric equations for the surface generated by revolving the curve $y = \sin x$ about the $x$-axis.
   (b) Find parametric equations for the surface generated by revolving the curve $y - e^x = 0$ about the $x$-axis.

5. Find a parametric representation of the surface in terms of the parameters $r$ and $\theta$ , where $(r, \theta, z)$ are the cylindrical coordinates of a point on the surface.
   (a) $z = \frac{1}{1+x^2+y^2}$
   (b) $z = x^2 - y^2$

6. Find the equation of the tangent plane to the parametric surface at the stated point.
   (a) $x = u$, $y = v$, $z = u^2 + v^2$; $(1, 2, 5)$
   (b) $r = uvi + ue^v j + ve^u k$; $u = \ln 2$, $v = 0$

7. Find the area of the given surface.
   (a) The portion of the paraboloid $r(u, v) = u \cos vi + u \sin vj + u^2 k$ for which $1 \le u \le 2$, $0 \le v \le 2\pi$.
   (b) The portion of the cone $r(u, v) = u \cos vi + u \sin vj + uk$ for which $0 \le u \le 2v$, $0 \le v \le \pi/2$.

8. For the function $f(x, y) = ax + by$, prove that the area of the surface $z = f(x, y)$ over a rectangle $\boldsymbol{R}$ in the $xy$-plane is the product of $\|\langle 1, 0, a\rangle \times \langle 0, 1, b\rangle\|$ and the area of $\boldsymbol{R}$.

9. (a) Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.
   (b) Find the area of the portion of the surface $x^2 - 2z = 0$ that lies above the triangle bounded by the lines $x = \sqrt{3}, y = 0$ and y=x in the xy-plane.

10. Evaluate the surface integral $\iint_\sigma f(x, y, z)\, dS$
    (a) $f(x, y, z) = (x^2 + y^2)z; \sigma$ is the portion of sphere $x^2 + y^2 + z^2 = 4$ above the plane $z = 1$.
    (b) $f(x, y, z) = x + y + z; \sigma$ is the surface of the cube defined by the inequalities $0 \le x \le 1, 0 \le y \le 1, 0 \le z \le 1$. [Hint: Integrate over each face separately.]

11. Find the mass of the lamina with constant density $\delta_0$.

(a) The lamina that is the portion of the circular cylinder $x^2 + z^2 = 4$ that lies directly above the rectangle $R = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 4\}$ in the xy-plane.

(b) The lamina that is the portion of the paraboloid $2z = x^2 + y^2$ inside the cylinder $x^2 + y^2 = 8$.

12. (a) Find the mass of the lamina that is the portion of the surface $y^2 = 4 - z$ between the planes $x = 0, x = 3, y = 0$, and y = 3 if the density is $\delta(x, y, z) = y$.

(b) Show that if the density of the lamina $x^2 + y^2 + z^2 = a^2$ at each point is equal to the distance between that point and the xy-plane, then the mass of the lamina is $2\pi a^3$.

13. Evaluate the integral $\iint_\sigma f(x, y, z)\, dS$ over the surface $\sigma$ represented by $\mathbf{r}(u, v)$.

(a) $f(x, y, z) = \frac{x^2+z^2}{y}; \mathbf{r}(u, v) = 2\cos v\mathbf{i} + u\mathbf{j} + 2\sin v\mathbf{k} (1 \leq u \leq 3, 0 \leq v \leq 2\pi)$

(b) $f(x, y, z) = \frac{1}{\sqrt{1+4x^2+4y^2}}; \mathbf{r}(u, v) = u\cos v\mathbf{i} + u\sin v\mathbf{j} + u^2\mathbf{k} (0 \leq u \leq \sin v, 0 \leq v \leq \pi)$

14. (a) Find the flux of $\mathbf{F}(x, y, z) = 2\mathbf{i} + 3\mathbf{j}$ through a disk of radius 5 in the plane $y = 3$ oriented in the direction of increasing $y$.

(b) Find the flux of $\mathbf{F}$ across the surface $\sigma$ by expressing $\sigma$ parametrically where $\mathbf{F}(x, y, z) = \mathbf{i}+\mathbf{j}+\mathbf{k}$ and the surface $\sigma$ is the portion of the cone $z = \sqrt{x^2 + y^2}$ between the planes $z = 1$ and $z = 2$, oriented by downward unit normals

15. Let $x, y$ and $z$ be measured in meters, and suppose that $\mathbf{F}(x, y, z) = 2x\mathbf{i} - 3y\mathbf{j} + z\mathbf{k}$ is the velocity vector (in m/s) of a fluid particle at the point $(x, y, z)$ in a steady-state incompressible fluid flow.

(a) Find the net volume of fluid that passes in the upward direction through the portion of the plane $x + y + z = 1$ in the first octant in 1 s.

(b) Assuming that the fluid has a mass density of 806 kg/m$^3$, find the net mass of fluid that passes in the upward direction through the surface in part (a) in 1 s.

16. (a) Integrate $G(x, y, z) = z - x$ over the portion of the graph of $z = x + y^2$ above the triangle in the $xy$-plane having vertices (0, 0, 0), (1, 1, 0) and (0, 1, 0).

(b) Integrate $G(x, y, z) = x$ over the surface given by $z = x^2 + y$ for $0 \leq x \leq 1,\ -1 \leq y \leq 1$.

17. Find the center of mass and the moment of inertia about the $z$-axis of a thin shell of constant density $\delta$ cut from the cone $x^2 + y^2 - z^2 = 0$ by the planes $z = 1$ and $z = 2$.

18. Find the moment of inertia about the $z$-axis of a thin shell of constant density $\delta$ cut from the cone $4x^2 + 4y^2 - z^2 = 0, z \geq 0$, by the circular cylinder $x^2 + y^2 = 2x$

19. (a) Verify the Divergence Theorem for $F(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $\sigma$ is the surface of the cube bounded by the planes $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1$.

(b) Verify the Divergence Theorem for $F(x, y, z) = 5\mathbf{j} + 7\mathbf{k}$ where $\sigma$ is the spherical surface $x^2 + y^2 + z^2 = 1$.

20. (a) Use the Divergence Theorem to find the flux of $F(x, y, z) = (x^2+y)\mathbf{i}+z^2\mathbf{j}+(e^y - z)\mathbf{k}$ across $\sigma$ which is the surface of the rectangular solid bounded by the coordinate planes and the planes $x = 3, y = 1$, and $z = 2$, with outward orientation.

(b) Use the Divergence Theorem to find the flux of $F(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across $\sigma$ which is the surface of the solid bounded by the paraboloid $z = 1 - x^2 - y^2$ and the $xy$-plane, with outward orientation.

21. Prove that if $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\sigma$ is the surface of a solid $G$ oriented by outward unit normals, then $vol(G) = \frac{1}{3}\iint_\sigma \mathbf{r} \cdot \mathbf{n}\, dS$ where $vol(G)$ is the volume of $G$.

22. Use the result in question (21) to find the outward flux of the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ across the surface $\sigma$ of the cylindrical solid bounded by $x^2 + 4x + y^2 = 5$, $z = -1$, and $z = 4$.

23. Prove the following identities assuming that $\mathbf{F}, \sigma$, and $G$ satisfy the hypotheses of the Divergence Theorem and that all necessary differentiability requirements for the functions $f(x, y, z)$ and $g(x, y, z)$ are met.

(a) $\iint_\sigma \text{curl } \mathbf{F} \cdot \mathbf{n}\ dS = 0$

(b) $\iint_\sigma \nabla f \cdot \mathbf{n}\ dS = \iiint_G \nabla^2 f dV$, where $\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

(c) $\iint_\sigma (f\nabla g) \cdot \mathbf{n}\ dS = \iiint_G (f\nabla^2 g + \nabla f \cdot \nabla g)\ dV$

(d) $\iint_\sigma (f\nabla g - g\nabla f)\cdot \mathbf{n}\; dS = \iiint_G (f\nabla^2 g - g\nabla^2 f)\; dV$ (Hint: Interchange $f$ and $g$ in (c))

(e) $\iint_\sigma (f\mathbf{n})\cdot \mathbf{v}\; dS = \iiint_G \nabla f \cdot \mathbf{v}\; dV$ ($\mathbf{v}$ a fixed vector)

24. (a) Show that the outward flux of the position vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through a smooth closed surface $S$ is three times the volume of the region enclosed by the surface.

    (b) Let $\mathbf{n}$ be the outward unit normal vector field on $S$. Show that it is not possible for $\mathbf{F}$ to be orthogonal to $\mathbf{n}$ at every point of $S$.

25. **Volume of a solid region** Let $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and suppose that the surface $S$ and region $D$ satisfy the hypotheses of the Divergence Theorem. Show that the volume of $D$ is given by the formula

$$\text{volume of } (D) = \tfrac{1}{3}\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma$$

26. Verify Formula in Stokes' Theorem by evaluating the line integral and the surface integral. Assume that the surface has an upward orientation.

    (a) $\mathbf{F}(x,y,z) = (x-y)\mathbf{i} + (y-z)\mathbf{j} + (z-x)\mathbf{k}$; $\sigma$ is the portion of the plane $x+y+z=1$ in the first octant.

    (b) $\mathbf{F}(x,y,z) = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$; $\sigma$ is the portion of the cone $z = \sqrt{x^2+y^2}$ below the plane $z=1$.
    $z = \sqrt{a^2 - x^2 - y^2}$ .

27. Use Stokes' Theorem to evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$

    (a) $\mathbf{F}(x,y,z) = z^2\mathbf{i} + 2x\mathbf{j} - y^3\mathbf{k}$; $C$ is the circle $x^2+y^2=1$ in the $xy$-plane with counterclockwise orientation looking down the positive $z$-axis.

    (b) $\mathbf{F}(x,y,z) = -3y^2\mathbf{i} + 4z\mathbf{j} + 6x\mathbf{k}$; $C$ is the triangle in the plane $z = \frac{1}{2}y$ with vertices (2, 0, 0), (0, 2, 1) and (0, 0, 0) with a counterclockwise orientation looking down the positive $z$-axis.

28. Let $\mathbf{F}(x,y) = y\mathbf{i} - 2x\mathbf{j}$

    (a) Find a nonzero function $h(x)$ such that $h(x)\mathbf{F}(x,y)$ is a conservative vector field.

    (b) Find a nonzero function $g(y)$ such that $g(y)\mathbf{F}(x,y)$ is a conservative vector field.

29. Find a vector field with twice-differentiable components whose curl is $xi + yj + zk$ or prove that no such field exists.

30. **Center of mass of an arch** A slender metal arch lies along the semicircle $y = \sqrt{a^2 - x^2}$ in the $xy$-plane. The density at the point $(x,y)$ on the arch is $\delta(x,y) = 2a - y$. Find the center of mass.

31. **Inertia and center of mass of a shell** Find $I_z$ and the center of mass of a thin shell of density $\delta(x,y,z) = z$ cut from the upper portion of the sphere $x^2+y^2+z^2 = 25$ by the plane $z=3$.

32. **Moment of inertia of a cube** Find the moment of inertia about the $z$-axis of the surface of the cube cut from the first octant by the planes $x=1$, $y=1$, and $z=1$ if the density is $\delta = 1$.

33. Find all points $(a,b,c)$ on the sphere $x^2+y^2+z^2 = R^2$ where the vector field $\mathbf{F} = yz^2 i + xz^2 j + 2xyzk$ is normal to the surface and $\mathbf{F}(a,b,c) \neq 0$.

34. Among all rectangular regions $0 \le x \le a$, $0 \le y \le b$, find the one for which the total outward flux of $\mathbf{F} = (x^2+4xy)i - 6yj$ across the four sides is least. What is the least flux?

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