NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics

MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 8

Topics: Vector field, Scalar potential (15.1.1, 15.1.3), Divergence and Curl (15.1.4, 15.1.5). Line integral of scalar valued functions (15.2.1, (9), (10), (12), (14)), Line integral of vector valued functions (15.2.2, (29)) Work (15.2.3).Line integral independent of path (15.3.1, 15.3.2), Conservative Field (15.3.3, (19)), Physical interpretation of Divergence and Curl (Section 16.4 of Thomas), Green's Theorem for plane (15.4.1, Theorem 4 of Sec 16.4 of Thomas), finding areas using Green's theorem ((6)).

1. Determine whether the statement about the vector field $\mathbf{F}(x,y)$ is true or false. If false, explain why.
   (a) $\mathbf{F}(x,y) = x^2\mathbf{i} - y\mathbf{j}$.
      i. $||\mathbf{F}(x,y)|| \to 0$ as $(x,y) \to (0,0)$.
      ii. If (x,y) is on the positive y-axis, then the vector points in the negative y-direction.
      iii. If (x,y) is in the first quadrant, then the vector points down and to the right.
   (b) $\mathbf{F}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\mathbf{i} - \dfrac{y}{\sqrt{x^2+y^2}}\mathbf{j}$
      i. As (x,y) moves away from the origin, the lengths of the vectors decrease.
      ii. If (x,y) is a point on the positive x-axis, then the vector points up.
      iii. If (x,y) is a point on the positive y-axis, the vector points to the right.

2. Confirm that ϕ is a potential function for $\mathbf{F}(\mathbf{r})$ on some region, and state the region.
   (a) $\phi(x,y) = \tan^{-1} xy$; $\mathbf{F}(x,y) = \dfrac{y}{1+x^2y^2}\mathbf{i} + \dfrac{x}{1+x^2y^2}\mathbf{j}$
   (b) $\phi(x,y,z) = x^2 - 3y^2 + 4z^2$; $\mathbf{F}(x,y,z) = 2x\mathbf{i} - 6y\mathbf{j} + 8z\mathbf{k}$
   (c) $\phi(x,y) = 2y^2 + 3x^2y - xy^3$; $\mathbf{F}(x,y) = (6xy - y^3)\mathbf{i} + (4y + 3x^2 - 3xy^2)\mathbf{j}$
   (d) $\phi(x,y,z) = x\sin z + y\sin x + z\sin y$; $\mathbf{F}(x,y,z) = (\sin z + y\cos x)\mathbf{i} + (\sin x + z\cos y)\mathbf{j} + (\sin y + x\cos z)\mathbf{k}$

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3. Find $\operatorname{div}\mathbf{F}$ and $\operatorname{curl}\mathbf{F}$.
   (a) $\mathbf{F}(x,y,z) = 7y^3z^2\mathbf{i} - 8x^2z^5\mathbf{j} - 3xy^4\mathbf{k}$
   (b) $\mathbf{F}(x,y,z) = e^{xy}\mathbf{i} - \cos y\mathbf{j} + \sin^2 z\mathbf{k}$
   (c) $\mathbf{F}(x,y,z) = \dfrac{1}{\sqrt{x^2+y^2+z^2}}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$
   (d) $\mathbf{F}(x,y,z) = \ln x\mathbf{i} + e^{xyz}\mathbf{j} + \tan^{-1}(z/x)\mathbf{k}$

4. Find $\nabla \cdot (\mathbf{F} \times \mathbf{G})$.
   (a) $\mathbf{F}(x,y,z) = 2x\mathbf{i} + \mathbf{j} + 4y\mathbf{k}$; $\mathbf{G}(x,y,z) = x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$
   (b) $\mathbf{F}(x,y,z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$; $\mathbf{G}(x,y,z) = xy\mathbf{j} + xyz\mathbf{k}$

5. Find $\nabla \cdot (\nabla \times \mathbf{F})$.
   (a) $\mathbf{F}(x,y,z) = \sin x\mathbf{i} + \cos(x-y)\mathbf{j} + z\mathbf{k}$
   (b) $\mathbf{F}(x,y,z) = e^{xz}\mathbf{i} + 3xe^{y}\mathbf{j} - e^{yz}\mathbf{k}$

6. Find $\nabla \times (\nabla \times \mathbf{F})$.
   (a) $\mathbf{F}(x,y,z) = xy\mathbf{j} + xyz\mathbf{k}$
   (b) $\mathbf{F}(x,y,z) = y^2x\mathbf{i} - 3yz\mathbf{j} + xy\mathbf{k}$

7. Let k be a constant, $\mathbf{F} = \mathbf{F}(x,y,z)$, $\mathbf{G} = \mathbf{G}(x,y,z)$, and $\phi = \phi(x,y,z)$. Prove the following identities, assuming that all derivatives involved exist and are continuous.
   (a) $\operatorname{div}(k\mathbf{F}) = k\operatorname{div}\mathbf{F}$
   (b) $\operatorname{curl}(k\mathbf{F}) = k\operatorname{curl}\mathbf{F}$

(c) $\text{div}(\mathbf{F}+\mathbf{G}) = \text{div}\,\mathbf{F} + \text{div}\,\mathbf{G}$

(d) $\text{curl}(\mathbf{F}+\mathbf{G}) = \text{curl}\,\mathbf{F} + \text{curl}\,\mathbf{G}$

(e) $\text{div}(\phi\mathbf{F}) = \phi\,\text{div}\,\mathbf{F} + \nabla\phi\cdot\mathbf{F}$

(f) $\text{curl}(\phi\mathbf{F}) = \phi\,\text{curl}\,\mathbf{F} + \nabla\phi\times\mathbf{F}$

(g) $\text{div}(\text{curl}\mathbf{F}) = 0$

(h) $\text{curl}(\nabla\phi) = 0$

8. Let $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $r = ||\mathbf{r}||$, f be a differentiable function of one variable and let $\mathbf{F(r)} = f(r)\mathbf{r}$.

   (a) Verify that the radius vector $\mathbf{r}$ has the stated property.

   (1) $\text{curl}\ \mathbf{r} = 0$ (2) $\nabla||\mathbf{r}|| = \dfrac{\mathbf{r}}{||\mathbf{r}||}$ (3) $\text{div}\ \mathbf{r} = 3$ (4) $\nabla\dfrac{1}{||\mathbf{r}||} = -\dfrac{\mathbf{r}}{||\mathbf{r}||^3}$

   (b) Use the chain rule and part (a) to show that $\nabla f(r) = \dfrac{f^{'}(r)}{r}\mathbf{r}$

   (c) Use the results in part (a), part(b) and the result $\text{div}(\phi\mathbf{F}) = \phi\ \text{div}\mathbf{F} + \nabla\phi\ .\ \mathbf{F}$ to show that $\text{div}\mathbf{F} = 3f(r) + rf^{'}(r)$.

   (d) Use part (a), (b), and the result $\text{curl}(\phi\ \mathbf{F}) = \phi\ \text{curl}\ \mathbf{F} + \nabla\phi\ \times\ \mathbf{F}$ to show that curl $\mathbf{F} = \mathbf{0}$.

   (e) Use part (a), (b) and the result $\text{div}(\phi\mathbf{F}) = \phi\ \text{div}\mathbf{F} + \nabla\phi\ .\ \mathbf{F}$ to show that $\nabla^2 f(r) = 2\dfrac{f^{'}(r)}{r} + f^{''}(r)$.

   (f) Use part (c) to show that the divergence of the inverse-square field $\mathbf{F} = \dfrac{\mathbf{r}}{||\mathbf{r}||^3}$ is zero.

   (g) Use part (c) to show that if $\mathbf{F}$ is a vector field of the form $\mathbf{F} = f(||\mathbf{r}||)\mathbf{r}$ and if $\text{div}\mathbf{F} = 0$, then $\mathbf{F}$ is an inverse-square field. [Suggestion: Let $r = ||\mathbf{r}||$ and multiply $3f(r) + rf^{'}(r) = 0$ through by r^2. Then write the result as a derivative of a product.]

9. Evaluate $\displaystyle\int_C \mathbf{F}.d\mathbf{r}$ along the line segment C from P to Q.

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   (a) $\mathbf{F}(x,y) = 8\mathbf{i} + 8\mathbf{j}$; $P(-4,4)$, $Q(-4,5)$

   (b) $\mathbf{F}(x,y) = 2\mathbf{i} + 5\mathbf{j}$; $P(1,-3)$, $Q(4,-3)$

   (c) $\mathbf{F}(x,y) = 2x\mathbf{j}$; $P(-2,4)$, $Q(-2,11)$

   (d) $\mathbf{F}(x,y) = -8x\mathbf{i} + 3y\mathbf{j}$; $P(-1,0)$, $Q(6,0)$

10. (a) Let C be the curve represented by the equations $x = 2t$, $y = t^2$ $(0 \le t \le 1)$.
   In each part, evaluate the line integral along C.

   i. $\displaystyle\int_C (x-\sqrt{y})ds$ ii. $\displaystyle\int_C (x-\sqrt{y})dx$ iii. $\displaystyle\int_C (x-\sqrt{y})dy$

   (b) Let C be the curve represented by the equations $x = t$, $y = 3t^2$, $z = 6t^3$ $(0 \le t \le 1)$.
   In each part, evaluate the line integral along C.

   i. $\displaystyle\int_C xyz^2 ds$ ii. $\displaystyle\int_C xyz^2 dx$ iii. $\displaystyle\int_C xyz^2 dy$ iv. $\displaystyle\int_C xyz^2 dz$

11. (a) In each part evaluate the integral $\displaystyle\int_C (3x+2y)dx + (2x-y)dy$ along the stated curve.

   i. The line segment from $(0,0)$ to $(1,1)$.

   ii. The parabolic arc $y = x^2$ from $(0,0)$ to $(1,1)$.

   iii. The curve $y = \sin\left(\dfrac{\pi x}{2}\right)$ from $(0,0)$ to $(1,1)$.

   iv. The curve $x = y^3$ from $(0,0)$ to $(1,1)$.

   (b) In each part, evaluate the integral $\displaystyle\int_C ydx + zdy - xdz$ along the stated curve.

   i. The line segment from $(0,0,0)$ to $(1,1,1)$.

   ii. The twisted cubic $x = t$, $y = t^2$, $z = t^3$ from $(0,0,0)$ to $(1,1,1)$.

   iii. The helix $x = \cos\pi t$, $y = \sin\pi t$, $z = t$ from $(1,0,0)$ to $(-1,0,1)$.

12. Evaluate the line integral with respect to s along the curve C.

(a) $\displaystyle\int_C \frac{1}{1+x} ds,$ $\quad C$: $\mathbf{r}(t) = t\mathbf{i} + \frac{2}{3}t^{\frac{3}{2}}\mathbf{j} \quad (0 \le t \le 3)$

(b) $\displaystyle\int_C \frac{x}{1+y^2} ds$ $\quad C$: $x = 1 + 2t,\ y = t \quad (0 \le t \le 1)$

(c) $\displaystyle\int_C 3x^2yz ds$ $\quad C$: $x = t,\ y = t^2,\ z = \frac{2}{3}t^3 \quad (0 \le t \le 1)$

(d) $\displaystyle\int_C \frac{e^{-z}}{x^2+y^2} ds$ $\quad C$: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k} \quad (0 \le t \le 2\pi)$

13. Evaluate the line integral along the curve C.

(a) $\displaystyle\int_C (x+2y)dx + (x-y)dy$ $\quad C: x = 2\cos t,\ y = 4\sin t \quad (0 \le t \le \frac{\pi}{4})$

(b) $\displaystyle\int_C -ydx + xdy$ $\quad C: y^2 = 3x$ from $(3,3)$ to $(0,0)$

(c) $\displaystyle\int_C (x^2+y^2)dx - xdy$ $\quad C: x^2 + y^2 = 1$, counterclockwise from $(1,0)$ to $(0,1)$

(d) $\displaystyle\int_C x^2dx + xydy + z^2dz$ $\quad C: x = \sin t,\ y = \cos t,\ z = t^2 \quad (0 \le t \le \frac{\pi}{2})$

14. Evaluate $\displaystyle\int_C \mathbf{F}.d\mathbf{r}$ along the curve C.

(a) $\mathbf{F}(x,y) = x^2\mathbf{i} + xy\mathbf{j}$ $\quad C$: $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} \quad (0 \le t \le \pi)$

(b) $\mathbf{F}(x,y) = x^2y\mathbf{i} + 4\mathbf{j}$ $\quad C$: $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} \quad (0 \le t \le 1)$

(c) $\mathbf{F}(x,y) = (x^2+y^2)^{\frac{3}{2}}(x\mathbf{i} + y\mathbf{j})$ $\quad C$: $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{j} \quad (0 \le t \le 1)$

(d) $\mathbf{F}(x,y,z) = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ $\quad C$: $\mathbf{r}(t) = \sin t\mathbf{i} + 3\sin t\mathbf{j} + \sin^2 t\mathbf{k} \quad (0 \le t \le \frac{\pi}{2})$

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15. (a) Find the mass of a thin wire shaped in the form of the curve $x = e^t\cos t, y = e^t\sin t$ $(0 \le t \le 1)$ if the density function δ is proportional to the distance from the origin.

(b) Find the mass of a thin wire shaped in the form of the helix $x = 3\cos t, y = 3\sin t, z = 4t$ $(0 \le t \le \pi/2)$ if the density function is $\delta = \dfrac{kx}{(1+y^2)}$; $(k > 0)$.

16. Find the work done by the force field $\mathbf{F}$ on a particle that moves along the curve C.

(a) $\mathbf{F}(x,y) = xy\mathbf{i} + x^2\mathbf{j}$
$C\ :\ x = y^2$ from $(0,0)$ to $(1,1)$

(b) $\mathbf{F}(x,y,z) = (x+y)\mathbf{i} + xy\mathbf{j} - z^2\mathbf{k}$
$C\ :$ along line segments from $(0,0,0)$ to $(1,3,1)$ to $(2,-1,4)$

17. Use a line integral to find the area of the surface.

(a) The surface that extends upward from the parabola $y = x^2$ $(0 \le x \le 2)$ in the xy-plane to the plane $z = 3x$.

(b) The surface that extends upward from the semicircle $y = \sqrt{4-x^2}$ in the xy-plane to the surface $z = x^2y$.

18. Determine whether $\mathbf{F}$ is a conservative vector field. If so, find a potential function for it.

(a) $\mathbf{F}(x,y) = 3y^2\ \mathbf{i} + 6xy\ \mathbf{j}$

(b) $\mathbf{F}(x,y) = e^x\cos y\ \mathbf{i} - e^x\sin y\ \mathbf{j}$

(c) $\mathbf{F}(x,y) = (\cos y + y\cos x)\ \mathbf{i} + (\sin x - x\sin y)\ \mathbf{j}$

(d) $\mathbf{F}(x,y) = x\ln y\ \mathbf{i} + y\ln x\ \mathbf{j}$

19. (a) Show that the line integral $\displaystyle\int_C (y\sin x\ dx - \cos x\ dy)$ is independent of the path.

(b) Evaluate the integral in part (a) along the line segment from $(0,1)$ to $(\pi,-1)$.

(c) Evaluate the integral $\displaystyle\int_{(0,1)}^{(\pi,-1)} (y\sin x\,dx - \cos x\,dy)$ using the fundamental theorem of line integrals, and confirm that the value is the same as that obtained in part (b).

20. Show that the line integral is independent of the path, and use the fundamental theorem of line integrals to find its value.

(a) $\displaystyle\int_{(0,0)}^{(1,\pi/2)} (e^x \sin y\,dx + e^x \cos y\,dy)$

(b) $\displaystyle\int_{(-1,2)}^{(0,1)} (3x - y + 1)\,dx - (x + 4y + 2)\,dy$

(c) $\displaystyle\int_{(2,-2)}^{(-1,0)} 2xy^3\,dx + 3x^2y^2\,dy$

(d) $\displaystyle\int_{(1,1)}^{(3,3)} (e^x \ln y - \frac{e^y}{x})\,dx + (\frac{e^x}{y} - e^y \ln x)\,dy$, where x and y are positive.

21. Confirm that the force field $\mathbf{F}$ is conservative in some open connected region containing the points P and Q, and then find the work done by the force field on a particle moving along an arbitrary smooth curve in the region from P to Q.

(a) $\mathbf{F}(x,y) = xy^2\mathbf{i} + x^2y\mathbf{j};\;\; P(1,1), Q(0,0)$

(b) $\mathbf{F}(x,y) = e^{-y}\cos x\mathbf{i} - e^{-y}\sin x\mathbf{j};\;\; P\left(\frac{\pi}{2}, 1\right), Q\left(\frac{-\pi}{2}, 0\right)$

22. Let $\mathbf{F}(x,y) = f(x,y)\mathbf{i} + g(x,y)\mathbf{j} = \dfrac{y}{x^2+y^2}\mathbf{i} - \dfrac{x}{x^2+y^2}\mathbf{j}$.

(a) Show that $\displaystyle\int_{C_1} \mathbf{F}\cdot d\mathbf{r} \neq \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$ if C_1 and C_2 are the semicircular paths from (1.0) to $(-1,0)$ given by
$C_1 \;:\; x = \cos t,\;\; y = \sin t\;\; (0 \le t \le \pi)$
$C_2 \;:\; x = \cos t,\;\; y = -\sin t\;\; (0 \le t \le \pi)$

(b) Show that the components of $\mathbf{F}$ satisfy $\dfrac{\partial f}{\partial y} = \dfrac{\partial g}{\partial x}$

(c) Do the results in parts (a) and (b) contradict the theorem of conservative field test? Explain.

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23. For what values of b and c will $\mathbf{F} = (y^2 + 2czx)\,\mathbf{i} + y(bx + cz)\,\mathbf{j} + (y^2 + cx^2)\,\mathbf{k}$ be a gradient field?

24. Show that the work done by a constant force field $\mathbf{F} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$ in moving a particle along any path from A to B is $W = \mathbf{F}.\,\overrightarrow{AB}$

25. (a) Find a potential function for the gravitational field $\mathbf{F} = -GmM\dfrac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$ where G, m, M are constants.

(b) Let P_1 and P_2 be points at distance s_1 and s_1 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is $GmM\left(\dfrac{1}{s_2} - \dfrac{1}{s_1}\right)$.

26. Verify Green's Theorem for $\oint_C y^2dx + x^2dy$, where C is the square with vertices $(0,0), (1,0), (1,1)$, and $(0,1)$ oriented counterclockwise.

27. Verify Green's Theorem for $\oint_C ydx + xdy$, where C is the unit circle oriented counterclockwise.

28. (a) Use Green's Theorem to evaluate the integral $\oint_C 3xydx + 2xydy$, where C is the rectangle bounded by $x = -2, x = 4, y = 1$, and $y = 2$ oriented counterclockwise.

(b) Use Green's Theorem to evaluate the integral $\oint_C x\cos ydx - y\sin xdy$, where C is the square with vertices $(0,0), (\frac{\pi}{2}, 0), (\frac{\pi}{2}, \frac{\pi}{2})$, and $(0, \frac{\pi}{2})$ oriented counterclockwise.

(c) Use Green's Theorem to evaluate the integral $\oint_C \ln(1+y)dx - \frac{xy}{1+y}dy$, where C is the triangle with vertices $(0,0), (2,0)$, and $(0,4)$ oriented counterclockwise.

(d) Use Green's Theorem to evaluate the integral $\displaystyle\oint_C \tan^{-1} y\,dx - \frac{y^2x}{1+y^2}dy$, where C is the square with vertices $(0,0), (1,0), (1,1)$, and $(0,1)$ oriented counterclockwise.

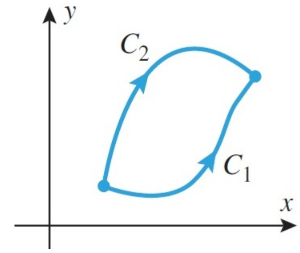
29. (a) Use the formula $A = \frac{1}{2} \oint_C -ydx + xdy$, to find the area of the region swept out by the line from the origin to the ellipse $x = a\cos t, y = b\sin t$ if t varies from $t = 0$ to $t = t_0$ $(0 \le t_0 \le 2\pi)$.

(b) Use a line integral to find the area of the region enclosed by the astroid $x = a\cos^3\phi$ and $y = a\sin^3\phi, (0 \le \phi \le 2\pi)$.

(c) Use a line integral to find the area of the triangle with vertices $(0,0), (a,0)$, and $(0,b)$, where $a > 0$ and $b > 0$.

(d) Use a line integral to find the area of the region swept out by the line from the origin to the hyperbola $x = a\cosh t, y = b\sinh t$ if t varies from $t = 0$ to $t = t_0$ $(t_0 \ge 0)$.

30. Suppose that $F(x,y) = f(x,y)\widehat{\mathbf{i}} + g(x,y)\widehat{\mathbf{j}}$ is a vector field on the xy-plane and that f and g have continuous first partial derivatives with $f_y = g_x$ everywhere. Use Green's Theorem to explain why $\int_{C_1} F.dr = \int_{C_2} F.dr$ where C_1 and C_2 are the oriented curves as shown in the figure.



31. (a) Use Green's Theorem to find the work done by the force field $F(x,y) = xy\widehat{\mathbf{i}} + (\frac{1}{2}x^2 + xy)\widehat{\mathbf{j}}$ on a particle that moves along the described path: the particle starts at $(5,0)$, traverses the upper semicircle $x^2 + y^2 = 25$, and returns to its starting point along the x-axis.

(b) Use Green's Theorem to find the work done by the force field $F(x,y) = \sqrt{y}\widehat{\mathbf{i}} + \sqrt{x}\widehat{\mathbf{j}}$ on a particle that moves along the described path: the particle moves counterclockwise one time around the closed curve given by the equations $y = 0, x = 2$, and $y = \frac{x^3}{4}$.

32. Find a simple closed curve C with counterclockwise orientation that maximizes the value of $\oint_C \frac{1}{3}y^3dx + (x - \frac{1}{3}x^3)dy$ and explain your reasoning.

33. Let C be the line segment from a point (a,b) to a point (c,d). Show that $\int_C -ydx + xdy = ad - bc$.

34. Among all smooth, simple closed curves in the plane, oriented counterclockwise, find the one along which the work done by $F = (\frac{1}{4}x^2y + \frac{1}{3}y^3)\widehat{\mathbf{i}} + x\widehat{\mathbf{j}}$ is greatest. (Hint: Where is $(curlF).k$ positive?)

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35. (a) Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $F = (x-y)\widehat{\mathbf{i}} + (y-x)\widehat{\mathbf{j}}$ and curve C: The square bounded by $x = 0, x = 1, y = 0, y = 1$.

(b) Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $F = (x^2+4y)\widehat{\mathbf{i}} + (x+y^2)\widehat{\mathbf{j}}$ and curve C: The square bounded by $x = 0, x = 1, y = 0, y = 1$.

(c) Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $F = (y^2 - x^2)\widehat{\mathbf{i}} + (x^2 + y^2)\widehat{\mathbf{j}}$ and curve C: The triangle bounded by $y = 0, x = 3$ and $y = x$.

(d) Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $F = (x+y)\widehat{\mathbf{i}} - (x^2+y^2)\widehat{\mathbf{j}}$ and curve C: The triangle bounded by $y = 0, x = 1$ and $y = x$.

36. Show that the value of $\oint_C xy^2dx + (x^2y + 2x)dy$ around any square depends only on the area of the square and not on its location in the plane.

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