NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics
MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 5

Topics: Functions of several variables: Level curves and level surfaces (Definition in Sec 14.1 of Thomas), Limit along curves (Equation (1) and (2) in Section 13.2), Limit (13.2.1, 13.2.2), Limit laws (Thomas: THEOREM 1— Properties of Limits of Functions of Two Variables). Continuity (13.2.3, 13.2.4, 13.2.5 Continuity at the boundary point not needed), Partial derivative (13.3.1), Relation between partial derivatives and continuity, Higher order partial derivatives, Theorem on equality of mixed partials (13.3.2), Chain rule (13.5.1, 13.5.2), Implicit differentiation (13.5.3, 13.5.4).

1. (a) Let $f(x,y) = e^{x+y}$.
   i. For what values of k will the graph of the level curve $f(x,y) = k$ be nonempty?
   ii. Describe the level curves $f(x,y) = k$ for the values of k obtained in part (a).

   (b) Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2+1}$
   i. For what values of k will the graph of the level surface $f(x,y,z) = k$ be nonempty?
   ii. Describe the level surfaces $f(x,y,z) = k$ for the values of k obtained in part (a).

   (c) Sketch the level curve $z(x,y) = k$ for the specified values of k.
   i. $z = x^2 + y^2;\quad k = 0,1,2,3,4$
   ii. $z = x^2 + y;\quad k = -2,-1,0,1,2$
   iii. $z = y\csc x;\quad k = -2,-1,0,1,2$
   iv. $z = x^2 - y^2;\quad k = -2,-1,0,1,2$
   v. $z = 1 - |y|;\quad k = -2,-1,0,1,2$
   vi. $z = 1 - |x| - |y|;\quad k = -2,-1,0,1,2$

2. Find an equation for the level surface of the following functions through the given points:
   (a) $f(x,y,z) = \sqrt{x-y} - \ln z;\quad (3,-1,1)$
   (b) $f(x,y,z) = \ln(x^2+y+z^2);\quad (-1,2,1)$
   (c) $g(x,y,z) = \sqrt{x^2+y^2+z^2};\quad (1,-1,\sqrt{2})$
   (d) $g(x,y,z) = \dfrac{x-y+z}{2x+y-z};\quad (1,0,-2)$

3. Let $f(x,y) = \dfrac{x^2-y^2}{x^2+y^2}$. Determine the limit of $f(x,y)$ as (x,y) approaches $(0,0)$ along the curve C.
   (a) $C:\ x = 0$ (b) $C:\ y = 0$ (c) $C:\ y = x$ (d) $C:\ y = x^2$

4. (a) Each of the following exercises gives a function $f(x,y)$ and a positive number ϵ. In each exercise, show that there exists a $\delta > 0$ such that for all (x,y), $\sqrt{x^2+y^2} < \delta \implies |f(x,y) - f(0,0)| < \epsilon$.
   i. $f(x,y) = x^2 + y^2,\quad \epsilon = 0.01$
   ii. $f(x,y) = \dfrac{y}{x^2+1},\quad \epsilon = 0.05$
   iii. $f(x,y) = \dfrac{x+y}{x^2+1},\quad \epsilon = 0.01$
   iv. $f(x,y) = \dfrac{x+y}{2+\cos x},\quad \epsilon = 0.02$

   (b) Each of the following exercises gives a function $f(x,y,z)$ and a positive number ϵ. In each exercise, show that there exists a $\delta > 0$ such that for all (x,y,z), $\sqrt{x^2+y^2+z^2} < \delta \implies |f(x,y,z) - f(0,0,0)| < \epsilon$.
   i. $f(x,y,z) = x^2+y^2+z^2,\quad \epsilon = 0.015$
   ii. $f(x,y,z) = xyz,\quad \epsilon = 0.008$

5. (a) Show that the limit does not exist by considering the limits as $(x,y) \to (0,0)$ along the coordinate axes.
   i. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3}{x^2+2y^2}$ ii. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x+y}{2x^2+y^2}$ iii. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x-y}{x^2+y^2}$ iv. $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\cos xy}{x^2+y^2}$

(b) By considering different paths of approach, show that the following functions have no limit as $(x, y) \to (0, 0)$.

i. $f(x,y) = -\dfrac{x}{\sqrt{x^2+y^2}}$ ii. $f(x,y) = \dfrac{xy}{|xy|}$ iii. $h(x,y) = \dfrac{x^2+y}{y}$ iv. $h(x,y) = \dfrac{x^2 y}{x^4+y^2}$

6. Find the following limits:

(a) $\lim\limits_{(x,y)\to(3,2)} x\cos \pi y =$ ____________

(b) $\lim\limits_{(x,y)\to(0,1)} e^{xy^2} =$ ____________

(c) $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\dfrac{1}{x^2+y^2}\right) =$ ____________

7. Evaluate the limit using the substitution $z = x^2 + y^2$ and observing that $z \to 0^+$ if and only if $(x, y) \to (0, 0)$.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$

(b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-\cos(x^2+y^2)}{x^2+y^2}$

(c) $\lim\limits_{(x,y)\to(0,0)} e^{-1/(x^2+y^2)}$

(d) * $\lim\limits_{(x,y)\to(0,0)} \dfrac{e^{-1/(\sqrt{x^2+y^2})}}{\sqrt{x^2+y^2}}$

8. Determine whether the limit exists for the following. If so, find their value.

(a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^4-y^4}{x^2+y^2}$

(b) * $\lim\limits_{(x,y)\to(0,0)} \dfrac{1-x^2-y^2}{x^2+y^2}$

(c) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{\sin(x^2+y^2+z^2)}{(x^2+y^2+z^2)}$

(d) $\lim\limits_{(x,y,z)\to(0,0,0)} \tan^{-1}\left[\dfrac{1}{(x^2+y^2+z^2)}\right]$

9. Evaluate the following limits by converting to polar coordinates:

(a) $\lim\limits_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\ln(x^2+y^2)$

(b) * $\lim\limits_{(x,y)\to(0,0)} y\ln(x^2+y^2)$

(c) $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^2}{\sqrt{x^2+y^2}}$

(d) $\lim\limits_{(x,y)\to(0,0)} \dfrac{xy}{\sqrt{x^2+2y^2}}$

10. (a) Show that the value of $\dfrac{x^3y}{2x^6+y^2}$ approaches 0 as $(x, y) \to (0, 0)$ along any straight line $y = mx$, or along any parabola $y = kx^2$.

(b) Show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3y}{2x^6+y^2}$ does not exist by letting $(x, y) \to (0, 0)$ along the curve $y = x^3$.

(c) Show that the value of $\dfrac{xyz}{x^2+y^4+z^4}$ approaches 0 as $(x, y, z) \to (0, 0, 0)$ along any line $x = at,\ y = bt,\ z = ct$.

(d) Show that $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^4+z^4}$ does not exist by letting $(x, y, z) \to (0, 0, 0)$ along the curve $x = t^2,\ y = t,\ z = t$.

11. Find the limits for the following:

(a) $\lim\limits_{(x,y)\to(0,4)} \dfrac{x}{\sqrt{y}}$ (b) $\lim\limits_{(x,y)\to(2,-3)} \left(\dfrac{1}{x}+\dfrac{1}{y}\right)^2$ (c) $\lim\limits_{(x,y)\to(1,1)} \ln|1+x^2y^2|$ (d) $\lim\limits_{(x,y)\to(1,\pi/6)} \dfrac{x\sin y}{x^2+1}$

12. (a) Let $f(x,y) = \begin{cases} 1, & y \ge x^4 \\ 1, & y \le 0 \\ 0, & \text{otherwise.} \end{cases}$ Find each of the following limits, or explain that the limit does not exist.

i. $\lim\limits_{(x,y)\to(0,1)} f(x,y)$ ii. $\lim\limits_{(x,y)\to(2,3)} f(x,y)$ iii. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

(b) Let $f(x,y) = \begin{cases} x^2, & x \geq 0 \\ x^3, & x < 0 \end{cases}$. Find the following limits.

i. $\lim\limits_{(x,y)\to(3,-2)} f(x,y)$ ii. $\lim\limits_{(x,y)\to(-2,1)} f(x,y)$ iii. $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

13. (a) At what points (x, y) in the plane are the following functions continuous?

i. $f(x,y) = \ln(x^2+y^2)$ iii. $g(x,y) = \dfrac{x+y}{2+\cos x}$ v. $f(x,y) = -\dfrac{x}{\sqrt{x^2+y^2}}$ vii. $h(x,y) = \dfrac{x^2+y}{y}$

ii. $g(x,y) = \sin\dfrac{1}{xy}$ iv. $g(x,y) = \dfrac{1}{x^2-y}$ vi. $f(x,y) = \dfrac{xy}{|xy|}$ viii. $h(x,y) = \dfrac{x^2y}{x^4+y^2}$

(b) At what points (x,y,z) in space are the following functions continuous?

i. $h(x,y,z) = xy\sin\dfrac{1}{z}$ ii. $h(x,y,z) = \dfrac{1}{x^2+z^2-1}$ iii. $h(x,y,z) = \dfrac{1}{|y|+|z|}$ iv. $h(x,y,z) = \dfrac{1}{|xy|+|z|}$

14. Determine whether the following statements are true or false. Explain your answers.

(a) If $f(x,y)\to L$ as (x,y) approaches $(0,0)$ along the x-axis, and if $f(x,y)\to L$ as (x,y) approaches $(0,0)$ along the y-axis, then $\lim\limits_{(x,y)\to(0,0)} f(x,y) = L$.

(b) If f and g are functions of two variables such that $f+g$ and fg are both continuous, then f and g are themselves continuous.

(c) If $\lim\limits_{x\to 0^+} f(x) = L \neq 0$, then $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2+y^2}{f(x^2+y^2)} = 0$.

15. Let $z = e^{2x}\sin y$. Find

(a) $\dfrac{\partial z}{\partial x}$ (b) $\dfrac{\partial z}{\partial y}$ (c) $\left.\dfrac{\partial z}{\partial x}\right|_{(0,y)}$ (d) $\left.\dfrac{\partial z}{\partial x}\right|_{(x,0)}$ (e) $\left.\dfrac{\partial z}{\partial y}\right|_{(0,y)}$ (f) $\left.\dfrac{\partial z}{\partial y}\right|_{(x,0)}$ (g) $\left.\dfrac{\partial z}{\partial x}\right|_{(\ln 2,0)}$ (h) $\left.\dfrac{\partial z}{\partial y}\right|_{(\ln 2,0)}$

16. If $f(x,y) = \begin{cases} \dfrac{\sin(x^3+y^4)}{x^2+y^2}, & (x,y)\neq(0,0) \\ 0, & (x,y) = (0,0) \end{cases}$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ at (0,0)

17. Let $f(x,y) = xe^{-y} + 5y$.

(a) Find the slope of the surface $z = f(x,y)$ in the x-direction at the point $(3,0)$.

(b) Find the slope of the surface $z = f(x,y)$ in the y-direction at the point $(3,0)$.

18. Let $z = (x+y)^{-1}$.

(a) Find the rate of change of z with respect to x at the point $(-2,4)$ with y held fixed.

(b) Find the rate of change of z with respect to y at the point $(-2,4)$ with x held fixed.

19. Determine whether the following statements are true or false. Explain your answer.

(a) If the line $y = 2$ is a contour of $f(x,y)$ through $(4,2)$, then $f_x(4,2) = 0$.

(b) If the plane $x = 3$ intersects the surface $z = f(x,y)$ in a curve that passes through $(3,4,16)$ and satisfies $z = y^2$, then $f_y(3,4) = 8$.

(c) If the graph of $z = f(x,y)$ is a plane in 3-space, then both f_x and f_y are constant functions.

(d) There exists a polynomial $f(x,y)$ that satisfies the equations $f_x(x,y) = 3x^2+y^2+2y$ and $f_y(x,y) = 2xy+2y$.

20. Let $f(x,y,z) = x^2y^4z^3 + xy + z^2 + 1$. Find the following:

(a) $f_x(x,y,z)$ (b) $f_y(x,y,z)$ (c) $f_z(x,y,z)$ (d) $f_x(1,y,z)$ (e) $f_y(1,2,z)$ (f) $f_z(1,2,3)$

21. (a) A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $y = 1$. At what rate is z changing with respect to x when the point is at $(2, 1, 7)$?

(b) A point moves along the intersection of the elliptic paraboloid $z = x^2 + 3y^2$ and the plane $x = 2$. At what rate is z changing with respect to y when the point is at $(2, 1, 7)$?

22. Show that $f_x(0, 0)$ and $f_y(0, 0)$ exist at (0,0) but f is not continuous at (0,0).

(a) $f(x,y) = \begin{cases} \dfrac{xy^2}{x^2+y^4}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ (b) $f(x,y) = \begin{cases} 0, & x^2 < y < 2x^2 \\ 1, & \text{otherwise.} \end{cases}$

23. Let $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0). \end{cases}$. Find $\dfrac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\dfrac{\partial^2 f}{\partial y \partial x}(0,0)$.

24. (a) Calculate $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ using implicit differentiation. Leave your answers in terms of x, y, and z.

i. $(x^2 + y^2 + z^2)^{3/2} = 1$ ii. $x^2 + z \sin xyz = 0$

(b) Find $\dfrac{\partial w}{\partial x}$, $\dfrac{\partial w}{\partial y}$ and $\dfrac{\partial w}{\partial z}$ using implicit differentiation. Leave your answers in terms of x, y, z, and w.

i. $(x^2 + y^2 + z^2 + w^2)^{3/2} = 4$ ii. $w^2 + w \sin xyz = 1$

25. Let a and b denote two sides of a triangle and let θ denote the included angle. Suppose that a, b, and θ vary with time in such a way that the area of the triangle remains constant. At a certain instant $a = 5$ cm, $b = 4$ cm, and $\theta = \pi/6$ radians, and at that instant both a and b are increasing at a rate of 3 cm/s. Estimate the rate at which θ is changing at that instant.

26. (a) Use an appropriate form of the chain rule to find dw/dt for the following:

i. $w = 3x^2y^3$; $x = t^4$, $y = t^2$
ii. $w = e^{1-xy}$; $x = t^{1/3}$, $y = t^3$
iii. $w = \ln(3x^2 - 2y + 4z^3)$; $x = t^{1/2}$, $y = t^{2/3}$, $z = t^{-2}$
iv. $w = 5\cos xy - \sin xz$; $x = 1/t$, $y = t$, $z = t^3$

(b) Suppose that $z = f(x, y)$ is a function with $f_x(4, 8) = 3$ and $f_y(4, 8) = -1$. If $x = t^2$ and $y = t^3$, find dz/dt when $t = 2$.

(c) Suppose that $w = f(x, y, z)$ is a function with $f_x(1, 0, 2) = 1$, $f_y(1, 0, 2) = 2$, and $f_z(1, 0, 2) = 3$. If $x = t$, $y = \sin(\pi t)$, and $z = t^2 + 1$, find dw/dt when $t = 1$.

27. Use appropriate forms of the chain rule to find $\dfrac{\partial z}{\partial u}$ and $\dfrac{\partial z}{\partial v}$ for the following:

(a) $z = 8x^2y - 2x + 3y$; $x = uv$, $y = u - v$
(b) $z = x/y$; $x = 2\cos u$, $y = 3\sin v$
(c) $z = e^{x^2 y}$; $x = \sqrt{uv}$, $y = 1/v$
(d) $z = \cos x \sin y$; $x = u - v$, $y = u^2 + v^2$

28. Let f be a differentiable function of one variable and

(a) let $z = f(x + 2y)$. Show that $2\dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y} = 0$.

(b) let $z = f(x^2 + y^2)$. Show that $y\dfrac{\partial z}{\partial x} - x\dfrac{\partial z}{\partial y} = 0$.

(c) let $w = f(\rho)$, where $\rho = (x^2 + y^2 + z^2)^{1/2}$. Show that $\left(\dfrac{\partial w}{\partial x}\right)^2 + \left(\dfrac{\partial w}{\partial y}\right)^2 + \left(\dfrac{\partial w}{\partial z}\right)^2 = \left(\dfrac{dw}{d\rho}\right)^2$.

29. Let f be a differentiable function of three variables and suppose that $w = f(x - y, y - z, z - x)$. Show that $\dfrac{\partial w}{\partial x} + \dfrac{\partial w}{\partial y} + \dfrac{\partial w}{\partial z} = 0$.

30. Find $\dfrac{\partial w}{\partial t}$ and $\dfrac{\partial w}{\partial s}$ for the following:

(a) $w = f(s^3 + t^2)$ and $f'(x) = e^x$
(b) $w = f\left(ts^2, \dfrac{s}{t}\right)$, $\dfrac{\partial f}{\partial x}(x,y) = xy$ and $\dfrac{\partial f}{\partial y}(x,y) = \dfrac{x^2}{2}$