MASS OF A LAMINA

14.8.1 MASS OF A LAMINA If a lamina with a continuous density function $\delta(x, y)$ occupies a region R in the xy-plane, then its total mass M is given by

$$M = \iint_R \delta(x, y)\, dA \tag{3}$$

Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$\bar{x} = \frac{\displaystyle\iint_R x\delta(x, y)\, dA}{\displaystyle\iint_R \delta(x, y)\, dA}, \qquad \bar{y} = \frac{\displaystyle\iint_R y\delta(x, y)\, dA}{\displaystyle\iint_R \delta(x, y)\, dA} \tag{9–10}$$

Alternative Formulas for Center of Gravity $(\bar{x}, \bar{y})$ of a Lamina

$$\bar{x} = \frac{M_y}{M} = \frac{1}{\text{mass of } R} \iint_R x\delta(x, y)\, dA \tag{11}$$

$$\bar{y} = \frac{M_x}{M} = \frac{1}{\text{mass of } R} \iint_R y\delta(x, y)\, dA \tag{12}$$

Centroid of a Region R

$$\bar{x} = \frac{\displaystyle\iint_R x\, dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R} \iint_R x\, dA \tag{13}$$

$$\bar{y} = \frac{\displaystyle\iint_R y\, dA}{\displaystyle\iint_R dA} = \frac{1}{\text{area of } R} \iint_R y\, dA \tag{14}$$

$$M = \text{mass of } G = \iiint_G \delta(x, y, z)\, dV \tag{15}$$

Center of Gravity $(\bar{x}, \bar{y}, \bar{z})$ of a Solid G

$$\bar{x} = \frac{1}{M} \iiint_G x\delta(x, y, z)\, dV$$

$$\bar{y} = \frac{1}{M} \iiint_G y\delta(x, y, z)\, dV$$

$$\bar{z} = \frac{1}{M} \iiint_G z\delta(x, y, z)\, dV$$

Centroid $(\bar{x}, \bar{y}, \bar{z})$ of a Solid G

$$\bar{x} = \frac{1}{V} \iiint_G x\, dV$$

$$\bar{y} = \frac{1}{V} \iiint_G y\, dV$$

$$\bar{z} = \frac{1}{V} \iiint_G z\, dV$$

(16–17)

TABLE 15.1 Mass and first moment formulas

(Thomas)

THREE-DIMENSIONAL SOLID

Mass: $M = \iiint_D \delta \, dV$ $\quad \delta = \delta(x, y, z)$ is the density at (x, y, z).

First moments about the coordinate planes:

$$M_{yz} = \iiint_D x \, \delta \, dV, \qquad M_{xz} = \iiint_D y \, \delta \, dV, \qquad M_{xy} = \iiint_D z \, \delta \, dV$$

Center of mass:

$$\bar{x} = \frac{M_{yz}}{M}, \qquad \bar{y} = \frac{M_{xz}}{M}, \qquad \bar{z} = \frac{M_{xy}}{M}$$

TWO-DIMENSIONAL PLATE

Mass: $M = \iint_R \delta \, dA$ $\quad \delta = \delta(x, y)$ is the density at (x, y).

First moments: $M_y = \iint_R x \, \delta \, dA, \qquad M_x = \iint_R y \, \delta \, dA$

Center of mass: $\bar{x} = \dfrac{M_y}{M}, \qquad \bar{y} = \dfrac{M_x}{M}$

TABLE 15.2 Moments of inertia (second moments) formulas

(Thomas)

THREE-DIMENSIONAL SOLID

About the x-axis: $I_x = \iiint (y^2 + z^2) \, \delta \, dV$ $\quad \delta = \delta(x, y, z)$

About the y-axis: $I_y = \iiint (x^2 + z^2) \, \delta \, dV$

About the z-axis: $I_z = \iiint (x^2 + y^2) \, \delta \, dV$

About a line L: $I_L = \iiint r^2(x, y, z) \, \delta \, dV$ $\quad r(x, y, z) =$ distance from the point (x, y, z) to line L

TWO-DIMENSIONAL PLATE

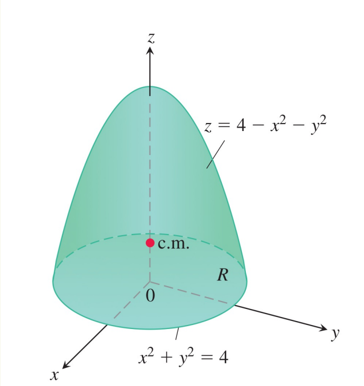
About the x-axis: $I_x = \iint y^2 \, \delta \, dA$ $\quad \delta = \delta(x, y)$

About the y-axis: $I_y = \iint x^2 \, \delta \, dA$

About a line L: $I_L = \iint r^2(x, y) \, \delta \, dA$ $\quad r(x, y) =$ distance from (x, y) to L

About the origin (polar moment): $I_0 = \iint (x^2 + y^2) \, \delta \, dA = I_x + I_y$

EXAMPLE 1 Find the center of mass of a solid of constant density δ bounded below by the disk $R: x^2 + y^2 \leq 4$ in the plane $z = 0$ and above by the paraboloid $z = 4 - x^2 - y^2$ (Figure 15.36).



Solution By symmetry $\bar{x} = \bar{y} = 0$. To find $\bar{z}$, we first calculate

$$M_{xy} = \iint_R \int_{z=0}^{z=4-x^2-y^2} z\,\delta\,dz\,dy\,dx = \iint_R \left[\frac{z^2}{2}\right]_{z=0}^{z=4-x^2-y^2} \delta\,dy\,dx$$

$$= \frac{\delta}{2} \iint_R (4 - x^2 - y^2)^2\,dy\,dx$$

$$= \frac{\delta}{2} \int_0^{2\pi} \int_0^2 (4 - r^2)^2 r\,dr\,d\theta$$ Polar coordinates simplify the integration.

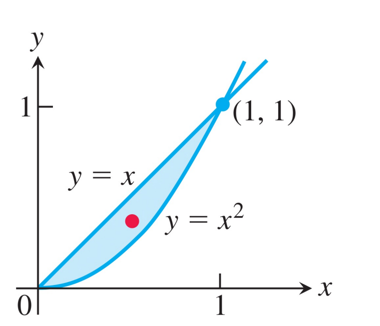
$$= \frac{\delta}{2} \int_0^{2\pi} \left[-\frac{1}{6}(4 - r^2)^3\right]_{r=0}^{r=2} d\theta = \frac{16\delta}{3} \int_0^{2\pi} d\theta = \frac{32\pi\delta}{3}.$$

A similar calculation gives the mass

$$M = \iint_R \int_0^{4-x^2-y^2} \delta\,dz\,dy\,dx = 8\pi\delta.$$

Therefore $\bar{z} = (M_{xy}/M) = 4/3$ and the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, 4/3)$.

EXAMPLE 2 Find the centroid of the region in the first quadrant that is bounded above by the line $y = x$ and below by the parabola $y = x^2$.



Solution We sketch the region and include enough detail to determine the limits of integration (Figure 15.37). We then set δ equal to 1 and evaluate the appropriate formulas from Table 15.1:

$$M = \int_0^1 \int_{x^2}^{x} 1\, dy\, dx = \int_0^1 \Big[y \Big]_{y=x^2}^{y=x} dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_0^1 = \frac{1}{6}$$

$$M_x = \int_0^1 \int_{x^2}^{x} y\, dy\, dx = \int_0^1 \left[\frac{y^2}{2} \right]_{y=x^2}^{y=x} dx$$

$$= \int_0^1 \left(\frac{x^2}{2} - \frac{x^4}{2} \right) dx = \left[\frac{x^3}{6} - \frac{x^5}{10} \right]_0^1 = \frac{1}{15}$$

$$M_y = \int_0^1 \int_{x^2}^{x} x\, dy\, dx = \int_0^1 \Big[xy \Big]_{y=x^2}^{y=x} dx = \int_0^1 (x^2 - x^3)\, dx = \left[\frac{x^3}{3} - \frac{x^4}{4} \right]_0^1 = \frac{1}{12}.$$

From these values of M, M_x, and M_y, we find

$$\bar{x} = \frac{M_y}{M} = \frac{1/12}{1/6} = \frac{1}{2} \quad \text{and} \quad \bar{y} = \frac{M_x}{M} = \frac{1/15}{1/6} = \frac{2}{5}.$$

The centroid is the point $(1/2, 2/5)$. ■

EXAMPLE 3 Find I_x, I_y, I_z for the rectangular solid of constant density δ shown in Figure 15.40.

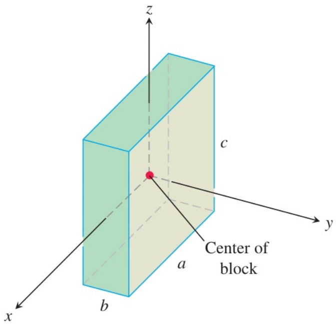


FIGURE 15.40 Finding I_x, I_y, and I_z for the block shown here. The origin lies at the center of the block (Example 3).

Solution The formula for I_x gives

$$I_x = \int_{-c/2}^{c/2}\int_{-b/2}^{b/2}\int_{-a/2}^{a/2}(y^2 + z^2)\,\delta\,dx\,dy\,dz.$$

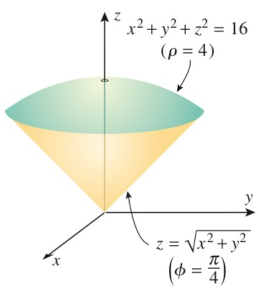
We can avoid some of the work of integration by observing that $(y^2 + z^2)\delta$ is an even function of x, y, and z since δ is constant. The rectangular solid consists of eight symmetric pieces, one in each octant. We can evaluate the integral on one of these pieces and then multiply by 8 to get the total value.

$$\begin{aligned}
I_x &= 8\int_0^{c/2}\int_0^{b/2}\int_0^{a/2}(y^2 + z^2)\,\delta\,dx\,dy\,dz = 4a\delta\int_0^{c/2}\int_0^{b/2}(y^2 + z^2)\,dy\,dz \\
&= 4a\delta\int_0^{c/2}\left[\frac{y^3}{3} + z^2 y\right]_{y=0}^{y=b/2} dz \\
&= 4a\delta\int_0^{c/2}\left(\frac{b^3}{24} + \frac{z^2 b}{2}\right) dz \\
&= 4a\delta\left(\frac{b^3 c}{48} + \frac{c^3 b}{48}\right) = \frac{abc\delta}{12}(b^2 + c^2) = \frac{M}{12}(b^2 + c^2). \qquad M = abc\delta
\end{aligned}$$

Similarly,

$$I_y = \frac{M}{12}(a^2 + c^2) \qquad \text{and} \qquad I_z = \frac{M}{12}(a^2 + b^2).$$

► Example 5 Find the centroid of the solid G bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the sphere $x^2 + y^2 + z^2 = 16$.



▲ Figure 14.8.9

Solution. The solid G is sketched in Figure 14.8.9. In Example 3 of Section 14.6, spherical coordinates were used to find that the volume of G is

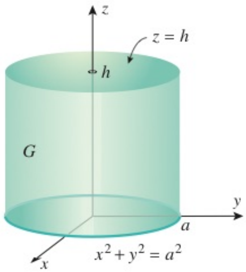
$$V = \frac{64\pi}{3}(2 - \sqrt{2})$$

By symmetry, the centroid $(\bar{x}, \bar{y}, \bar{z})$ is on the z-axis, so $\bar{x} = \bar{y} = 0$. In spherical coordinates, the equation of the sphere $x^2 + y^2 + z^2 = 16$ is $\rho = 4$ and the equation of the cone $z = \sqrt{x^2 + y^2}$ is $\phi = \pi/4$, so from (17) we have

$$\begin{aligned}
\bar{z} &= \frac{1}{V}\iiint_G z\,dV = \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\int_0^4 (\rho\cos\phi)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\
&= \frac{1}{V}\int_0^{2\pi}\int_0^{\pi/4}\left[\frac{\rho^4}{4}\cos\phi\sin\phi\right]_{\rho=0}^4 d\phi\,d\theta \\
&= \frac{64}{V}\int_0^{2\pi}\int_0^{\pi/4}\sin\phi\cos\phi\,d\phi\,d\theta = \frac{64}{V}\int_0^{2\pi}\left[\frac{1}{2}\sin^2\phi\right]_{\phi=0}^{\pi/4} d\theta \\
&= \frac{16}{V}\int_0^{2\pi} d\theta = \frac{32\pi}{V} = \frac{3}{2(2 - \sqrt{2})}
\end{aligned}$$

The centroid of G is

$$(\bar{x}, \bar{y}, \bar{z}) = \left(0, 0, \frac{3}{2(2 - \sqrt{2})}\right) \approx (0, 0, 2.561)$$



▲ **Figure 14.8.8**

▶ **Example 4** Find the mass and the center of gravity of a cylindrical solid of height h and radius a (Figure 14.8.8), assuming that the density at each point is proportional to the distance between the point and the base of the solid.

Solution. Since the density is proportional to the distance z from the base, the density function has the form $\delta(x, y, z) = kz$, where k is some (unknown) positive constant of proportionality. From (15) the mass of the solid is

$$\begin{aligned} M = \iiint\limits_G \delta(x, y, z)\, dV &= \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h kz\, dz\, dy\, dx \\ &= k \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{2} h^2\, dy\, dx \\ &= kh^2 \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx \\ &= \tfrac{1}{2} kh^2 \pi a^2 \end{aligned}$$

Interpret the integral as the area of a semicircle.

Without additional information, the constant k cannot be determined. However, as we will now see, the value of k does not affect the center of gravity.

From (16),

$$\begin{aligned} \bar{z} = \frac{1}{M} \iiint\limits_G z\delta(x, y, z)\, dV &= \frac{1}{\frac{1}{2}kh^2\pi a^2} \iiint\limits_G z\delta(x, y, z)\, dV \\ &= \frac{1}{\frac{1}{2}kh^2\pi a^2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_0^h z(kz)\, dz\, dy\, dx \\ &= \frac{k}{\frac{1}{2}kh^2\pi a^2} \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \frac{1}{3} h^3\, dy\, dx \\ &= \frac{\frac{1}{3}kh^3}{\frac{1}{2}kh^2\pi a^2} \int_{-a}^{a} 2\sqrt{a^2 - x^2}\, dx \\ &= \frac{\frac{1}{3}kh^3\pi a^2}{\frac{1}{2}kh^2\pi a^2} = \frac{2}{3}h \end{aligned}$$

Similar calculations using (16) will yield $\bar{x} = \bar{y} = 0$. However, this is evident by inspection, since it follows from the symmetry of the solid and the form of its density function that the center of gravity is on the z-axis. Thus, the center of gravity is $\left(0, 0, \frac{2}{3}h\right)$. ◀