NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics
MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 3

Topics: Integration, anti-derivative, Fundamental Theorem of Calculus, Mean-Value Theorem for Integrals (Anton 4.2.1, 4.2.2, 4.6.1, 4.6.2, 4.6.3), Area between curves, volume of solid with a given cross-sectional area, volume of solid of revolution, arc-length, area of surface of revolution (5.1.2, 5.1.4, 5.2.2, 5.2.3, Q 5.2.5 -A (6), (7-8), 5.4.2, 5.5.2), improper integrals (7.8.1, 7.8.2, 7.8.3, 7.8.4, 7.8.5), Gamma function (Kreyzig, Apostol)

1. Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed.

   (a) $\displaystyle\int_0^3 x\ dx$ (b) $\displaystyle\int_2^3 (1-\frac{1}{2}x)\ dx$ (c) $\displaystyle\int_{-1}^2 |2x-3|\ dx$

2. Evaluate the integral $\displaystyle\int_{-2}^2 (1-3|x|)dx$

3. If $f(x) = \begin{cases} |x-2|, & x \geq 0 \\ x+2, & x < 0 \end{cases}$ evaluate the following integrals:

   (a) $\displaystyle\int_{-2}^0 f(x)\ dx$ (b) $\displaystyle\int_{-2}^2 f(x)\ dx$

4. If $f(x) = \begin{cases} 2x, & x \leq 1 \\ 2, & x > 1 \end{cases}$ evaluate the following integrals:

   (a) $\displaystyle\int_0^1 f(x)\ dx$ (b) $\displaystyle\int_{-1}^1 f(x)\ dx$

5. Find all values of x^* in the stated interval that satisfy Equation in the Mean-Value Theorem for Integrals and explain what these numbers represent.

   (a) $f(x) = \sqrt{x}; [0,3]$ (b) (a) $f(x) = \sin x; [-\pi, \pi]$ (c) $f(x) = 1/x^2; [1,3]$

6. Evaluate the following integrals

   (a) $\int_{-1}^1 |2x-1|dx$ (b) $\int_0^{3\pi/4} |\cos x|dx$ (c) $\int_{-1}^2 \sqrt{2+|x|}dx$

7. A function $f(x)$ is defined piecewise on an interval. In these exercises find an antiderivative of the given function $f(x)$ on the interval and verify Part 1 of the Fundamental Theorem of Calculus.

   (a) $f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ x^2, & 1 < x \leq 2 \end{cases}$ (b) $f(x) = \begin{cases} \sqrt{x}, & 0 \leq x < 1 \\ 1/x^2, & 1 \leq x \leq 4 \end{cases}$

8. Determine whether the statement is true or false. Explain your answer.

   (a) There does not exist a differentiable function $F(x)$ such that $F'(x) = |x|$.

   (b) If $f(x)$ is continuous on the interval $[a,b]$, and if the definite integral of $f(x)$ over this interval has value 0 , then the equation $f(x) = 0$ has at least one solution in the interval $[a,b]$.

   (c) If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then

   $$\int_a^b f(x)dx = \int_a^b g(x)dx$$

   if and only if $G(a) + F(b) = F(a) + G(b)$.

9. Let $F(x) = \int_4^x \sqrt{t^2+9}dt$. Find

(a) $F(4)$ (b) $F'(4)$ (c) $F''(4)$.

10. Let $F(x) = \int_0^x \frac{\cos t}{t^2+3t+5} dt$. Find

    (a) $F(0)$ (b) $F'(0)$ (c) $F''(0)$.

11. Let $F(x) = \int_0^x \frac{t-3}{t^2+7} dt$ for $-\infty < x < +\infty$.

    (a) Find the value of x where F attains its minimum value.
    (b) Find intervals over which F is only increasing or only decreasing.
    (c) Find open intervals over which F is only concave up or only concave down.

12. Sketch the region enclosed by the curves and find its area.

    (a) $y = x^2$, $y = \sqrt{x}$, $x = \dfrac{1}{4}$, $x = 1$
    (b) $y = \cos 2x$, $y = 0$, $x = \dfrac{\pi}{4}$, $x = \dfrac{\pi}{2}$
    (c) $x = \sin y$, $x = 0$, $y = \dfrac{\pi}{4}$, $y = \dfrac{3\pi}{4}$
    (d) $y = 2 + |x-1|$, $y = -\dfrac{1}{5}x + 7$

13. (a) * Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.
    (b) * Find the volume of the solid whose base is the region enclosed between the curve $x = 1 - y^2$ and the y-axis and whose cross sections taken perpendicular to the y-axis are squares.

14. Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

    (a) $y = \sqrt{25 - x^2}$, $y = 3$
    (b) $y = 9 - x^2$, $y = 0$
    (c) $x = \sqrt{y}$, $x = \dfrac{y}{4}$
    (d) $y = \sin x$, $y = \cos x$, $x = 0$, $x = \dfrac{\pi}{4}$

15. A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of a radius 2 ft. Find the volume of material removed from the sphere.

16. Find the exact arc length of the curve over the interval.

    (a) $y = 3x^{\frac{3}{2}} - 1$ from $x = 0$ to $x = 1$
    (b) $y = \dfrac{x^6+8}{16x^2}$ from $x = 2$ to $x = 3$
    (c) $24xy = y^4 + 48$ from $y = 2$ to $y = 4$
    (d) $x = \frac{1}{3}\left(y^2+2\right)^{3/2}$ from $y = 0$ to $y = 1$
    (e) $x = \frac{1}{8}y^4 + \frac{1}{4}y^{-2}$ from $y = 1$ to $y = 4$

17. Find the area of the surface generated by revolving the given curve about the stated axis.

    (a) $y = 7x$, $0 \le x \le 1$ about the x-axis
    (b) $y = \sqrt{4-x^2}$, $-1 \le x \le 1$ about the x-axis
    (c) $x = y^3$, $0 \le y \le 1$ about the y-axis
    (d) $x = 2\sqrt{1-y}$, $-1 \le y \le 0$ about the y-axis

18.** A solid is generated by revolving about the x-axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x-axis, and the fixed line $x = a$ and the variable line $x = b$, $b > a$. Its volume, for all b, is $b^2 - ab$. Find $f(x)$.

19. In each part, determine all values of p for which the integral is improper.

    (a) $\displaystyle\int_0^1 \frac{dx}{x^p}$ (b) $\displaystyle\int_1^2 \frac{dx}{x-p}$ (c) $\displaystyle\int_0^1 e^{-px}\,dx$

20. In each part, determine all values of s for which the integral converges and evaluate the integral.

    (a) $\displaystyle\int_0^\infty e^{-st}\sin t\,dt$ (b) $\displaystyle\int_0^\infty e^{-st}e^{2t}\,dt$ (c) $\displaystyle\int_0^\infty e^{-t}t^{s-1}\,dt$

21. Evaluate the integrals that converge.

(a) $\displaystyle\int_0^{\infty} e^{-2x}\,dx$

(b) $\displaystyle\int_e^{\infty} \frac{dx}{x\ln^3 x}$

(c) $\displaystyle\int_2^{\infty} \frac{dx}{x\sqrt{\ln x}}$

(d) $\displaystyle\int_0^{\infty} xe^{-x^2}\,dx$

(e) $\displaystyle\int_{-\infty}^{3} \frac{dx}{x^2+9}$

(f) $\displaystyle\int_{-\infty}^{0} \frac{e^x\,dx}{3-2e^x}$

(g) $\displaystyle\int_{-1}^{1} \frac{dx}{x^{2/3}}$

(h) $\displaystyle\int_{-1}^{1} \ln|x|\,dx$

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{1+e^{-2x}}$

22. Use L'Hospital's rule to evaluate the following improper integrals.

(a) $\int_0^1 \ln x\,dx$

(b) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^2}\,dx$

(c) $\displaystyle\int_1^{\infty} \frac{\ln x}{x^3}\,dx$

23. Show that $\displaystyle\int_0^{\infty} \frac{2x\,dx}{x^2+1}$ diverges even though $\displaystyle\lim_{b\to\infty}\int_b^{b} \frac{2x\,dx}{x^2+1} = 0$.

24. Find the area of the region between the x-axis and the curve

(a) $y = e^{-3x}$ for $x \geq 0$.

(b) $y = \frac{8}{x^2-4}$ for $x \geq 4$.

25. Suppose that the region between the x-axis and the curve $y = e^{-x}$ for $x \geq 0$ is revolved about the x-axis. Find the volume and the surface area of the solid that is generated.

26. Show that

(a) $\int_0^1 (\ln x)^n dx = (-1)^n\Gamma(n+1), \quad n > 0$ [Hint: Let $t = -\ln x$.]

(b) $\int_0^{+\infty} e^{-x^n} dx = \Gamma\left(\frac{n+1}{n}\right), n > 0$. [Hint: Let $t = x^n$.]