NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics

MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 1

Topics: Definition of a function (Anton et al. Def 0.1.1, 0.1.2), natural domain and range of a real valued function of real variable (Anton et al. Def 0.1.5, 0.2.1, 0.2.2), equation of family of functions (Anton et al. Sec 0.3), inverse functions (Anton et al. Sec 0.4), intuitive notion and the precise definition of the limit of a function (Anton et al. Def 1.1.1, 1.1.2, 1.1.3, 1.4.1), limit laws (Anton et al. 1.2.2), infinite limit and limit at infinity (Thomas Section 2.6 definitions).

1. Find the natural domain and determine the range of each of the following functions.

   (a) $f(x) = \dfrac{1}{x-3}$

   (b) $F(x) = \dfrac{x}{|x|}$

   (c) $h(x) = \dfrac{1}{1-\sin x}$

   (d) $F(x) = \sqrt{4-x^2}$

   (e) $g(x) = 3 + \sqrt{x}$

   (f) $H(x) = (\sin\sqrt{x})^{-2}$

2. Let$f(x) = 3\sqrt{x}-2$ and $g(x) = |x|$. In each part, give the formula for the function and state the corresponding domain.

   (a) $f+g : ------$ Domain: $------$

   (b) $f-g : ------$ Domain:$------$

   (c) $fg : ------$ Domain: $------$

   (d) $f/g : ------$ Domain: $------$

3. Find formulas for $f\circ g$ and $g\circ f$, and state the domains of the compositions.

   (a) $f(x) = x^2,\ g(x) = \sqrt{1-x}$

   (b) $f(x) = \sqrt{x-3},\ g(x) = \sqrt{x^2+3}$

   (c) $f(x) = \dfrac{1+x}{1-x},\ g(x) = \dfrac{x}{1-x}$

   (d) $f(x) = \dfrac{x}{1+x^2},\ g(x) = \dfrac{1}{x}$

   (e) $f(x) = 2-x^2, g(x) = \sqrt{x}$

4. Complete the following table

	$g(x)$	$f(x)$	$f\circ g(x)$
$a.$	$x-7$	$\sqrt{x}$	?
$b.$	$x+2$	$3x$	?
$c.$	?	$\sqrt{x-5}$	$\sqrt{x^2-5}$
$d.$	$\frac{x}{x-1}$	$\frac{x}{x-1}$	?
$e.$	?	$1+\frac{1}{x}$	x
$f.$	$\frac{1}{x}$	?	x

5. Find a formula for $f\circ g\circ h$

   (a) $f(x) = x^2+1,\ g(x) = \frac{1}{x},\ h(x) = x^3$

   (b) $f(x) = \dfrac{1}{1+x},\ g(x) = \sqrt[3]{x},\ h(x) = \dfrac{1}{x^3}$

6. Let $f(x) = x^2-2x,\ x\le 1$. Find $f^{-1}(x)$ and identify the domain and range of f^{-1}. As a check, show that $f^{-1}(f(x)) = f(f^{-1}(x)) = x$.

7. Show that the graph of the inverse of $f(x) = mx + b$, where m and b are constants and $m \neq 0$, is a line with slope $\frac{1}{m}$ and y-intercept $\frac{-b}{m}$.

8. (a) Find the inverse of $f(x) = -x + 1$. Graph the line $y = -x + 1$ together with the line $y = x$. At what angle do the lines intersect?
   (b) Find the inverse of $f(x) = -x + b$ (b constant). What angle does the line $y = -x + b$ make with the line $y = x$?
   (c) What can you conclude about the inverses of functions whose graphs are lines perpendicular to the line $y = x$?

9. Let $f(x) = \frac{ax+b}{cx+d}$, $c \neq 0$, $ad - bc \neq 0$.
   (a) Give a convincing argument that f is one-to-one.
   (b) Find a formula for the inverse of f.

10. If $(3, -2)$ is a point on the graph of an odd invertible function f , then$------$ and $------$ are points on the graph of f^{-1}.

11. Determine whether f and g are inverse functions
    (a) $f(x) = 4x,\ g(x) = \frac{1}{4}x$
    (b) $f(x) = 3x + 1,\ g(x) = 3x - 1$
    (c) $f(x) = \sqrt[3]{x-2},\ g(x) = x^3 + 2$
    (d) $f(x) = x^4,\ g(x) = \sqrt[4]{x}$

12. Find a formula for $f^{-1}(x)$
    (a) $f(x) = 7x - 6$
    (b) $f(x) = \frac{x+1}{x-1}$
    (c) $f(x) = 3/x^2,\ x < 0$
    (d) $f(x) = \begin{cases} \frac{5}{2} - x, & x < 2 \\ \frac{1}{x}, & x \geq 2 \end{cases}$
    (e) $f(x) = \begin{cases} 2x, & x \leq 0 \\ x^2, & x > 0 \end{cases}$

13. (a) Find an equation for the family of lines whose members have slope $m = 3$.
    (b) Find an equation for the member of the family that passes through $(-1, 3)$.
    (c) Find an equation for the family of lines with y-intercept $b = 2$.
    (d) Find an equation for the family of lines that pass through the point $(1, -2)$
    (e) Find an equation for the family of lines parallel to $2x + 4y = 1$.
    (f) Find all lines through $(6, -1)$ for which the product of the x- and y-intercepts is 3.

14. Sketch the graph of the following functions
    (a) $f(x) = \frac{1}{x-3}, x \neq 3$
    (b) $f(x) = \frac{x}{|x|}, x \neq 0$
    (c) $f(x) = x \sin x$
    (d) $f(x) = x^2 \sin x$
    (e) $f(x) = \frac{x+1}{x-1}, x \neq 1$

15. (a) Suppose that ϵ is any positive number. Find the largest value of δ such that $|5x - 10| < \epsilon$, if $0 < |x - 2| < \delta$.
    (b) Find the smallest positive number N such that for each $x > N$, the value of $f(x) = 1/\sqrt{x}$ is within 0.01 of 0.
    (c) Find the largest open interval, centered at the origin on the x-axis, such that for each x in the interval the value of the function $f(x) = x + 2$ is within 0.1 unit of the number $f(0) = 2$.
    (d) Find the largest open interval, centered at $x = 3$, such that for each x in the interval the value of the function $f(x) = 4x - 5$ is within 0.01 unit of the number $f(3) = 7$.

16. In each part, find the largest open interval, centered at $x = 0$, such that for each x in the interval the value of $f(x) = 2x + 3$ is within ϵ units of the number $f(0) = 3$.
    (a) $\epsilon = 0.1$
    (b) $\epsilon = 0.01$

17. A positive number ϵ and the limit L of a function f at a are given. Find a number δ such that $|f(x) - L| < \epsilon$ if $0 < |x - a| < \delta$.

(a) $\lim_{x\to 4} 2x = 8;\ \epsilon = 0.1$

(b) $\lim_{x\to -1/2} \frac{4x^2-1}{2x+1} = -2;\ \epsilon = 0.05$

(c) $\lim_{x\to 4} \sqrt{x} = 2;\ \epsilon = 0.001$

(d) $\lim_{x\to 0} |x| = 0;\ \epsilon = 0.05$

18. Find the limits.

(a) $\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

(b) $\lim_{y\to 4} \frac{4-y}{2-\sqrt{y}}$

(c) $\lim_{x\to +\infty} (-2x)$

(d) $\lim_{x\to -\infty} \frac{x}{|x|}$

(e) $\lim_{x\to -\infty} (3-x)$

(f) $\lim_{x\to +\infty} \left(5 - \frac{1}{x}\right)$

(g) $\lim_{x\to 3} f(x)$ for $f(x) = \begin{cases} x-1, & x \le 3 \\ 3x-7, & x > 3. \end{cases}$

(h) $\lim_{t\to +\infty} \frac{6-t^3}{7t^3+3}$

(i) $\lim_{y\to +\infty} \frac{2-y}{\sqrt{7+6y^2}}$

19. Determine whether the statement is true or false. Explain your answer.

(a) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then so does $\lim_{x\to a} [f(x) + g(x)]$

(b) If $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x)$ exists, $\lim_{x\to a} [f(x)/g(x)]$ does not exist.

(c) If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are equal, then $\lim_{x\to a} [f(x)/g(x)] = 1$.

(d) If $f(x)$ is a rational function and $x = a$ is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

20. Let $f(x) = \begin{cases} \frac{x^2-9}{x+3}, & x \ne -3 \\ k, & x = -3. \end{cases}$

(a) Find k so that $f(-3) = \lim_{x\to -3} f(x)$.

(b) With k assigned the value $\lim_{x\to -3} f(x)$, show that $f(x)$ can be expressed as a polynomial.

21. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ x, & \text{if } x \text{ is irrational} \end{cases}$. Prove that $\lim_{x\to 0} f(x) = 0$.

22. Let $f(x) = \begin{cases} 0, & \text{if } x \text{ is rational} \\ 1, & \text{if } x \text{ is irrational} \end{cases}$. Prove that $\lim_{x\to 0} f(x)$ does not exist.

23. If $\lim_{x\to 1} f(x) = 5$, must f be defined at $x = 1$? If it is, must $f(1) = 5$? Can we conclude anything about the values of f at $x = 1$? Explain.

24. If $f(1) = 5$, must $\lim_{x\to 1} f(x)$ exist? If it does, then must $\lim_{x\to 1} f(x) = 5$? Can we conclude anything about $\lim_{x\to 1} f(x)$? Explain.

25. If $\sqrt{5-2x^2} \le f(x) \le \sqrt{5-x^2}$ for $-1 \le x \le 1$, find $\lim_{x\to 0} f(x)$.

26. If $2 - x^2 \le g(x) \le 2\cos x$ for all x, find $\lim_{x\to 0} g(x)$.

27. Find the limits that exist.

(a) $\displaystyle\lim_{x\to -\infty} \frac{2x^2+x}{4x^2-3}$.

(b) $\displaystyle\lim_{x\to +\infty} \frac{1}{2+\sin x}$.

28. Given that $\lim_{x\to +\infty} f(x) = 2$ and $\lim_{x\to +\infty} g(x) = -3$. Find the limits that exist.

(a) $\lim_{x\to +\infty} [3f(x) - g(x)]$.

(b) $\displaystyle\lim_{x\to +\infty} \frac{f(x)}{g(x)}$.

(c) $\lim_{x\to+\infty} \dfrac{2f(x)+3g(x)}{3f(x)+2g(x)}$.

29. Given that $\lim_{x\to-\infty} f(x) = 0$ and $\lim_{x\to+\infty} f(x) = +\infty$. Evaluate the limit using an appropriate substitution.

(a) $\lim_{x\to 0^+} f(1/x)$

(b) $\lim_{x\to 0^-} f(1/x)$

30. A positive number ϵ and the limit L of a function f at $+\infty$ are given. Find a positive number N such that $|f(x)-L| < \epsilon$ if $x > N$.

(a) $\lim_{x\to+\infty} \frac{1}{x^2} = 0; \epsilon = 0.01$

(b) $\lim_{x\to+\infty} \frac{x}{x+1} = 1; \epsilon = 0.001$

(c) $\lim_{x\to+\infty} \frac{4x-1}{2x+5} = 2; \epsilon = 0.1$

31. A positive number ϵ and the limit L of a function f at $-\infty$ are given. Find a negative number N such that $|f(x)-L| < \epsilon$ if $x < N$.

(a) $\lim_{x\to-\infty} \frac{1}{x+2} = 0; \epsilon = 0.005$

(b) $\lim_{x\to-\infty} \frac{x}{x+1} = 1; \epsilon = 0.001$

32. Find the following limits

(a) $\lim_{x\to\infty} (\sqrt{x+9} - \sqrt{x+4})$

(b) $\lim_{x\to-\infty} (\sqrt{x^2+3} + x)$

(c) $\lim_{x\to\infty} (\sqrt{x^2+x} - \sqrt{x^2-x})$

(d)* $\lim_{x\to\infty} \dfrac{x^{\frac{2}{3}} + x^{-1}}{x^{\frac{2}{3}} + \cos^2 x}$

(e) $\lim_{t\to\infty} \ln(1+\frac{1}{t})$

(f) $\lim_{x\to\infty} e^{\frac{1}{x}} \cos \dfrac{1}{x}$

(g) $\lim_{t\to-\infty} e^{3t} \sin^{-1}\left(\frac{1}{t}\right)$

(h) $\lim_{x\to a} \dfrac{x^2-a^2}{x^4-a^4}$

(i) $\lim_{x\to 0} \dfrac{8x}{3\sin x - x}$

References

[1] Anton, H., Bivens, I. C., Davis, S., *Calculus*, Lecture Notes, United Kingdom: Wiley, 10-th Edition, 2012.

[2] Weir, Maurice D. Haefliger, *THOMAS'CALCULUS EARLY TRANSCENDENTALS*, Thomas, George B.(George Brinton), 12-th Edition, 2004.

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