NATIONAL INSTITUTE OF TECHNOLOGY CALICUT
Department of Mathematics
MA 1002E: Mathematics 1

Monsoon Semester 2023-24: Tutorial 7

Topics: Evaluation of double integral(14.1.2), Fubini's theorem (14.1.3, 14.2.2), Polar coordinates (14.3.3), Evaluation of triple integral (14.5.1, 14.5.2), cylindrical and spherical coordinates (14.6.1, Equation (10), Definition and Equations Relating Spherical Coordinates to Cartesian and Cylindrical Coordinates in Thomas Sec 15.7), Change of variables, Jacobian (14.7.1, 14.7.2, 14.7.3, 14.7.4), Improper integrals. Applications: mass of a lamina (14.8.1), center of gravity and centroid (Eqn (9–17)), Mass and first moment (Thomas TABLE 15.1), Moments of inertia (Thomas TABLE 15.2).

1. Evaluate the following double integral in two ways using iterated integrals:
   (a) $\iint\limits_R x^2\, dA$; R is the region bounded by $y = 16/x$, $y = x$, and $x = 8$.
   (b) $\iint\limits_R y\, dA$; R is the region in the first quadrant enclosed between the circle $x^2 + y^2 = 25$ and the line $x + y = 5$.
2. Evaluate the following double integrals:
   (a) $\iint\limits_R x(1 + y^2)^{-1/2}\, dA$; R is the region in the first quadrant enclosed by $y = x^2, y = 4$, and $x = 0$.
   (b) $\iint\limits_R \sin(y^3)\, dA$; where R is the region bounded by $y = \sqrt{x}, y = 2$, and $x = 0$. [Hint: Choose the order of integration carefully.]
3. Use double integration to find the area of the plane region enclosed by the given curves.
   (a) $y = \sin x$ and $y = \cos x$, for $0 \leq x \leq /4$.
   (b) $y = \cosh x, y = \sinh x, x = 0$, and $x = 1$.
4. Use a double integral in polar coordinates to find the area of the regions described below:
   (a) The region enclosed by the rose $r = \sin 2\theta$
   $r = \sin 2\theta$, with $\pi/4 \leq \theta \leq \pi/2$.
   (b) The region common to the interior of the cardioids $r = 1 + \cos\theta$ and $r = 1 - \cos\theta$.
5. Use polar coordinates to evaluate the double integral.
   (a) $\iint_R \sin(x^2 + y^2) dA$, where R is the region enclosed by the circle $x^2 + y^2 = 9$. bounded by $y = 0, y = x$, and $x^2 + y^2 = 4$.
   (b) $\iint_R 2\ y\ dA$, where R is the region in the first quadrant bounded above by the circle $(x - 1)^2 + y^2 = 1$ and below by the line $y = x$.
6. Use **double integration** to find the volume of each of the following solids.
   (a) The solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3 - x$.
   (b) The solid in the first octant bounded above by $z = 9 - x^2$, below by $z = 0$, and laterally by $y^2 = 3x$.
7. What region R in the xy-plane maximizes the value of $\iint_R (4 - x^2 - 2y^2)\ dA$? What region R in the xy-plane minimizes the value of $\iint_R (x^2 + y^2 - 9)\ dA$? Give the reason for your answer.
8. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \leq x \leq 2$, $0 \leq y \leq 2$.
9. Show, by changing to polar coordinates, that $\displaystyle\int_0^{a\sin\beta} \int_{y\cot\beta}^{\sqrt{a^2-y^2}} \ln\ (x^2 + y^2)\ dx\ dy = a^2\beta(\ln\ a - 1/2)$, where $a > 0$ and $0 < \beta < \pi/2$. Rewrite the Cartesian integral with the order of integration reversed.

10. By changing the order of integration, show that : $\int_0^x \int_0^u e^{m(x-t)} f(t)\, dt\, du = \int_0^x (x-t)\, e^{m(x-t)} f(t)\, dt.$

11. (a) Find the volume of the solid in the first octant bounded above by the surface $z = r\sin\theta$, below by the xy-plane, and laterally by the plane $x = 0$ and the surface $r = 3\sin\theta$.

(b) Find the volume of the solid inside the surface $r^2 + z^2 = 4$ and outside the surface $r = 2\cos\theta$.

12. Evaluate the following improper integrals as iterated integrals.

(a) $\int_1^{\infty} \int_{e^{-x}}^{1} \frac{1}{x^3 y}\, dy\, dx$

(b) $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{(x^2+1)(y^2+1)}\, dx\, dy$

(c) $\int_{-1}^{1} \int_{\frac{-1}{\sqrt{1-x^2}}}^{\frac{1}{\sqrt{1-x^2}}} (2y+1)\, dy\, dx$

(d) $\int_0^{\infty} \int_0^{\infty} x e^{-(x+2y)}\, dx\, dy$

13. Evaluate the following triple integrals:

(a) $\iiint\limits_G xy\sin(yz)\, dV$, where G is the rectangular box defined by the inequalities $0 \le x \le \pi$, $0 \le y \le 1, 0 \le z \le \pi/6$.

(b) $\iiint\limits_G xyz\, dV$, where G is the solid in the first octant that is bounded by the parabolic cylinder $z = 2 - x^2$ and the planes $z = 0, y = x$, and $y = 0$.

(c) $\iiint\limits_G \cos(z/y)\, dV$, where G is the solid defined by the inequalities $\pi/6 \le y \le \pi/2, y \le x \le \pi/2, 0 \le z \le xy$.

14. Use a triple integral to find the volume of the solid.

(a) The solid in the first octant bounded by the coordinate planes and the plane $3x + 6y + 4z = 12$.

(b) The wedge in the first octant that is cut from the solid cylinder $y^2 + z^2 \le 1$ by the planes $y = x$ and $x = 0$.

15. Use cylindrical coordinates to find the volume of the following solids.

(a) The solid enclosed by the paraboloid $z = x^2 + y^2$ and the plane $z = 9$.

(b) The solid enclosed between the cone $z = (hr)/a$ and the plane $z = h$.

16. Use spherical coordinates to find the volume of the following solids.

(a) The solid bounded above by the sphere $\rho = 4$ and below by the cone $\phi = \pi/3$.

(b) The solid within the sphere $x^2 + y^2 + z^2 = 9$, outside the cone $z = \sqrt{x^2 + y^2}$, and above the xy-plane.

17. Let G be the solid in the first octant bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the coordinate planes. Evaluate $\iiint\limits_G xyz\, dV$

(a) using rectangular coordinates (b) using cylindrical coordinates (c) using spherical coordinates.

18. (a) Solve for a: $\int_0^1 \int_0^{4-a-x^2} \int_a^{4-x^2-y} dzdydx = \frac{4}{15}$

(b) For what value of c is the volume of the ellipsoid $x^2 + (y/2)^2 + (z/c)^2 = 1$ equal to 8π?

(c) What domain D in the space minimizes the value of the integral $\iiint\limits_D (4x^2 + 4y^2 + z^2 - 4)\, dV$? What domain D in the space maximizes the value of the integral $\iiint_D (1 - x^2 - y^2 - z^2)\, dV$? Give reason for your answer.

19. For the following find a transformation $u = f(x, y),\ v = g(x, y)$ that when applied to the region R in the xy-plane has as its image the region S in the uv-plane.

20. Find the Jacobian $\partial(x, y)/\partial(u, v)$ if

(a) $x = u + 4v, y = 3u - 5v$

(b) $x = u + 2v^2, y = 2u^2 - v$

(c) $x = \sin u + cosv, y = -\cos u + \sin v$

(d) $x = \frac{2u}{u^2 + v^2}, y = -\frac{2v}{u^2 + v^2}$

21. Solve for x and y in terms of u and v, and then find the Jacobian $\partial(x,y)/\partial(u,v)$.

(a) $u = 2x - 5y, v = x + 2y$

(b) $u = e^x, v = ye^{-x}$

(c) $u = x^2 - y^2, v = x^2 + y^2 \quad (x > 0, y > 0)$

(d) $u = xy, v = xy^3 \quad (x > 0, y > 0)$

22. Find the Jacobian $\partial(x,y,z)/\partial(u,v,w)$.

(a) $x = 3u + v, y = u - 2w, z = v + w$

(b) $x = u - uv, y = uv - uvw, z = uvw$

(c) $u = xy, v = y, w = x + z$

(d) $u = x + y + z, v = x + y - z, w = x - y + z$

23. (a) Use the transformation $u = x - 2y, v = 2x + y$ to find $\displaystyle\iint_R \frac{x-2y}{x+2y}\, dA$, where R is the rectangular region enclosed by the lines $x - 2y = 1,\ x - 2y = 4,\ 2x + y = 1,\ 2x + y = 3$.

(b) Use the transformation $u = y/x, v = xy$ to find $\displaystyle\iint_R xy^3 dA$, over the region R in the first quadrant enclosed by $y = x, y = 3x, xy = 1, xy = 4$.

(c) Use the transformation $x = u/v, y = uv$ to evaluate the integral sum $\displaystyle\int_1^2 \int_{1/y}^{y} (x^2 + y^2)\, dxdy + \int_2^4 \int_{y/4}^{4/y} (x^2 + y^2)\, dxdy$

24. Perform the integration by transforming the ellipsoidal region of integration into a spherical region of integration and then evaluating the transformed integral in spherical coordinates.

(a) $\displaystyle\iiint_G x^2\ dV$, where G is the region enclosed by the ellipsoid $9x^2 + 4y^2 + z^2 = 36$.

(b) $\displaystyle\iiint_G (y^2 + z^2)\ dV$, where G is the region enclosed by the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$

25. (a) Use the transformation $u = x,\ v = z - y,\ w = xy$ to find $\int\int\int_G (z-y)^2 xy\ dV$, where G is the region enclosed by the surfaces $x = 1, x = 3, z = y,\ z = y + 1,\ xy = 2,\ xy = 4$.

(b) Use the transformation $u = xy,\ v = yz,\ w = xz$ to find the volume of the region in the first octant that is enclosed by the hyperbolic cylinders $xy = 1,\ xy = 2,\ yz = 1,\ yz = 3, xz = 1,\ xz = 4$.

26. For the following find the mass and center of gravity of the lamina.
x-axis, the line $x = 1$, and the curve $y = \sqrt{x}$. y = 0, x = 0,and x = π.A lamina with density $\delta(x,y) = xy$ is in the first quadrant and is bounded by the circle $x^2 + y^2 = a^2$ and the coordinate axes. A lamina with density $\delta(x,y) = x^2 + y^2$ is bounded by the x-axis and the upper half of the circle $x^2 + y^2 = 1$

27. Find the centroid of the following regions.

(a) The region enclosed by the cardioid $r = a(1 + \sin\theta)$. $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2 (a < b)$.

(b) The region enclosed between the y-axis and the right half of the circle $x^2 + y^2 = a^2$.

28. Find the mass and center of gravity of the solid.
by the inequalities, $0 \le x \le a,\ 0 \le y \le a$, and $0 \le z \le a$. is enclosed by $x^2 + y^2 = a^2, z = 0$, and $z = h$.The solid that has density $\delta(x,y,z) = yz$ and is enclosed by $z = 1 - y^2$ (for $y \ge 0$), $z = 0, y = 0, x = -1$, and $x = 1$. The solid that has density $\delta(x,y,z) = xz$ and is enclosed by $y = 9 - x^2$ (for $x \ge 0$), $x = 0, y = 0, z = 0$, and $z = 1$.

29. (a) Find the moments of inertia about the coordinate axes of a thin rectangular plate of constant density δ bounded by the lines $x = 3$ and $y = 3$ in the first quadrant. and the interval $\pi \le x \le 2\pi$ of the x-axis.

(b) Find the moment of inertia about the x axes of a thin plate bounded by the parabola $x = y - y^2$ and the line $x + y = 0$ if $\delta(x,y) = x + y + 1$.

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