Nyquist & Shannon's Sampling Theorem

ME461 - Group: Little Daisies

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Nyquist Shannon Sampling Theorem

Sampling: The process of converting a CT signal to a discrete-time (DT) signal (i.e., a discrete sequence of numbers)

$$x(t) \longrightarrow \ldots, x(-2T), x(-T), x(0), x(T), x(2T), \ldots$$

Why is sampling theorem very important/useful?

- It provides a bridge between continuous-time signals and discrete-time signals.
- Sampling theorem allows us to represent continuous-time signals with discrete samples.
- With the development of digital technology, it became much easier to process discrete-time signals.

Nyquist Shannon Sampling Theorem

Theorem (Nyquist Shannon Sampling Theorem)

The theorem states that a continuous-time signal $x(t)$ *that is bandlimited to a maximum angular frequency* ω_M *can be completely reconstructed from its samples if the sampling frequency* ω_s *satisfies*

$$\omega_s > 2\omega_M \quad \text{or equivalently} \quad f_s > 2f_M$$

where f_s *is the sampling frequency and* f_M *is the highest frequency component of* $x(t)$*.*

Nyquist Shannon Sampling Theorem

Definition: A signal $x(t)$ is <u>bandlimited</u> if

$$X(j\omega) = 0 \quad \text{for } |\omega| > \omega_m \text{ for some } \omega_m$$

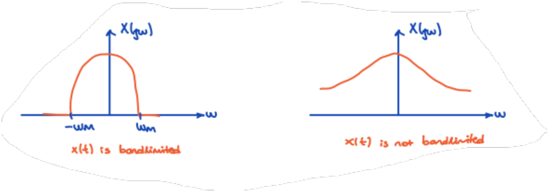


Figure: Example for Bandlimited Signal

Nyquist Shannon Sampling Theorem

Ideal impulse-train sampling:

Periodic impulse train

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

Sampling of a CT signal $x(t)$:

$$x_p(t) = x(t)\,p(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t - nT)$$

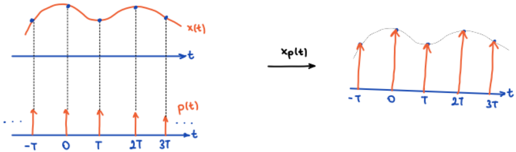


Figure: Example for an Impulse Train

Nyquist Shannon Sampling Theorem

Find the spectrum of $x_p(t)$:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \Longleftrightarrow \quad P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\begin{aligned} X_p(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega) &= \frac{1}{2\pi} X(j\omega) * \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s) \\ &= \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \end{aligned}$$

where $X(j(\omega - k\omega_s))$ represents shifted copies of $X(j\omega)$

Nyquist Shannon Sampling Theorem

Assume $x(t)$ is bandlimited with maximal frequency ω_M. When $\omega_s > 2\omega_M$,

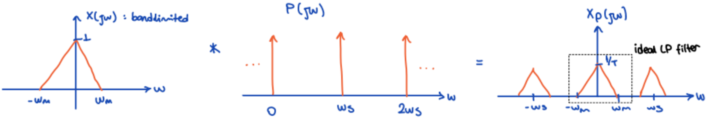


Figure: Example of the Method

By using a low-pass (LP) filter, we can get $X(j\omega)$ from $X_p(j\omega)$.

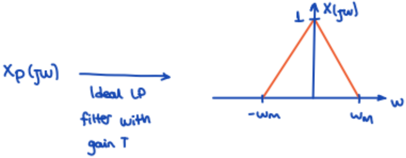


Figure: Recovery

Aliasing

The problem of resulting overlap in the frequency domain is referred to as **aliasing**. If $\omega_s < 2\omega_M$ (undersampling): We lose information $\Rightarrow$ **Aliasing** occurs if $\omega_s < 2\omega_M$. $X(j\omega)$ cannot be recovered from its samples when $\omega_s < 2\omega_M$.

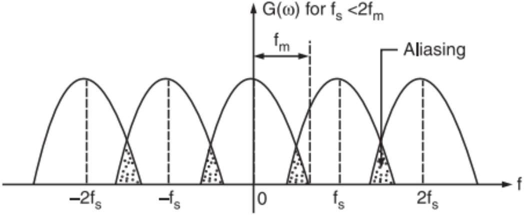
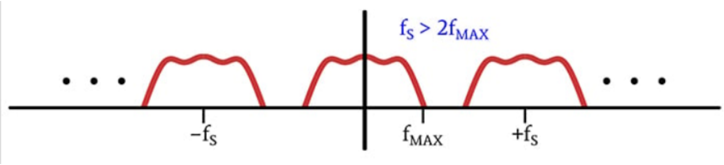
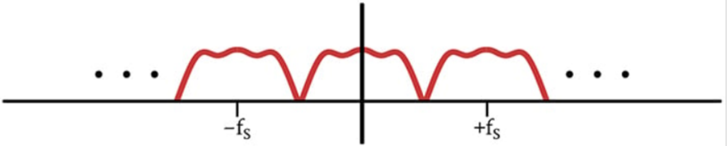


Figure: Aliasing

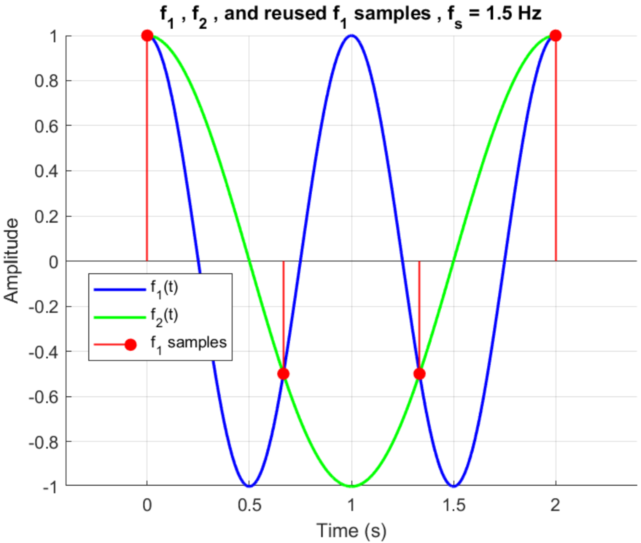
Aliasing Examples

As the sampling frequency decreased the below results are obtained.

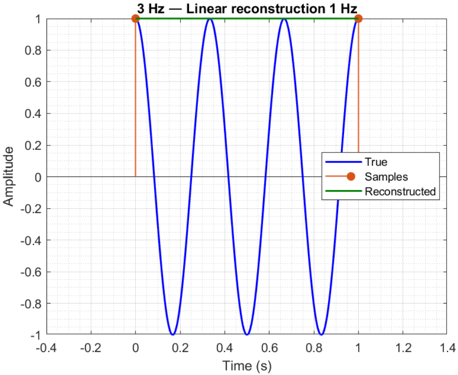


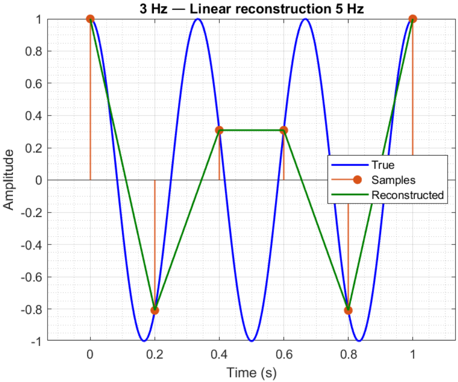


Sampling with a lower frequency then required

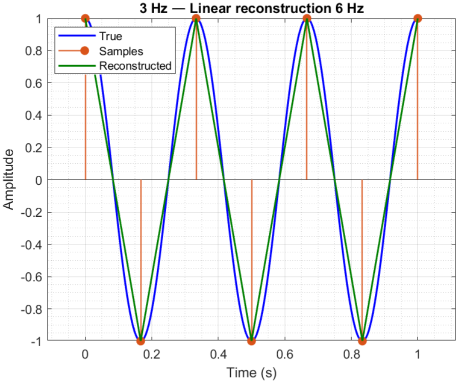


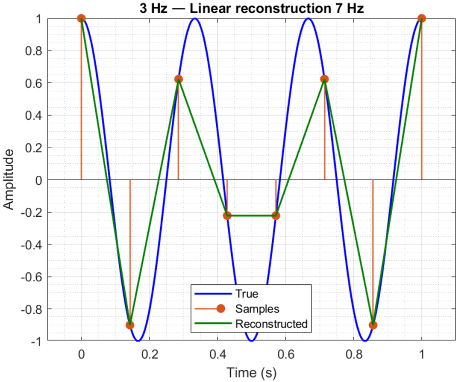
Sampling with different frequencies



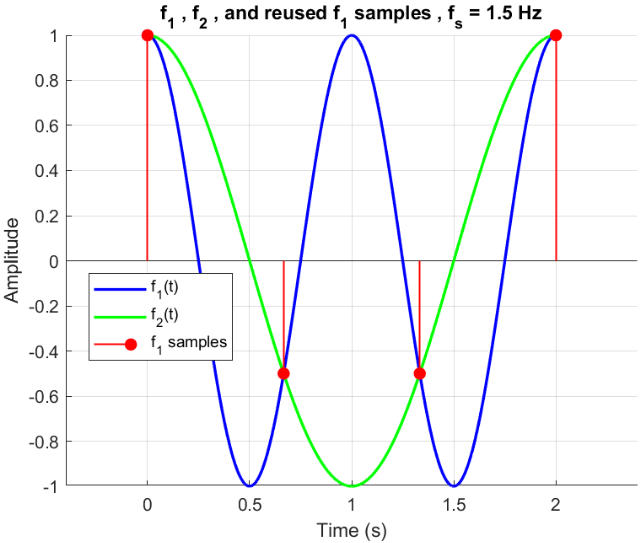


Sampling with different frequencies





Sampling with different frequencies



Quiz

- **Q1:** If our sampling rate is at least twice the rate of the original signal, is it guaranteed to obtain the original signal? If not, why?

Quiz

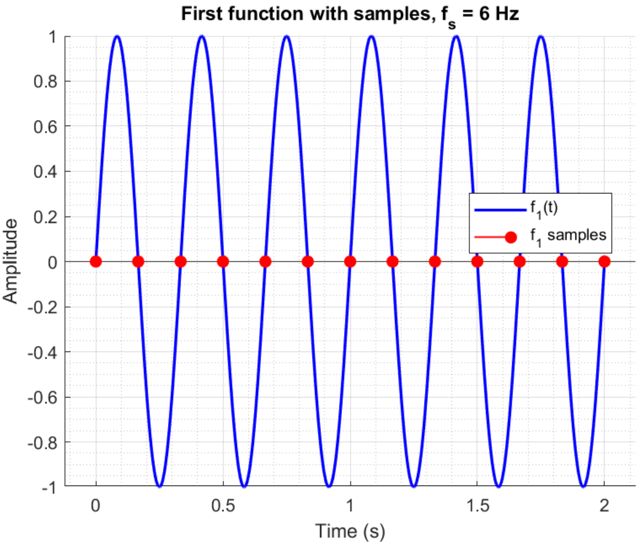


Figure: Hint

Quiz

- **Q1:** If our sampling rate is at least twice the rate of the original signal, is it guaranteed to obtain the original signal? If not, why?
- **Answer:** No. We shouldn't sample with the integer multiples of the original sample's frequency. If the sampling rate is an integer multiple of the tone every sample hits the same phase.

$$x[n] = x(nT_s) = \cos\left(2\pi \frac{f_0}{f_s} n + \phi\right).$$

$$\text{If } \frac{f_0}{f_s} = k \in \mathbb{Z}, \quad x[n] = \cos(2\pi kn + \phi) = \cos(\phi).$$

Quiz

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- **Answer:** Human's range of hearing is between 20 - 20000 Hz.

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- **Answer:** Yes, if when we choose a lower frequency then the required sampling frequency, we first lose the higher frequencies (refer to aliasing figure). In fact, when you communicate with your phone, your voice is sampled with approximately 8kHz, because human speech is mostly around 3-4 kHz range.

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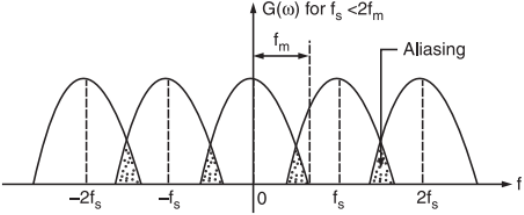

Figure: Aliasing