Geometric and Frequency-Domain Resonance of the Riemann Zeta Function

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Abstract

We introduce a geometric and frequency-domain framework in which the Riemann zeta function $\zeta(s)$ exhibits a reproducible resonance uniquely along the critical line $\mathrm{Re}(s) = \frac{1}{2}$. By representing $\zeta(s)$ in a complex base $b = \frac{1}{2}i$, we define a "digit-expansion residual"—the remainder after expressing $\zeta(s)$ as a finite series in powers of b. Numerical evaluation across known nontrivial zeros reveals that this residual collapses by several orders of magnitude exactly at $\mathrm{Re}(s) = \frac{1}{2}$, and increases rapidly off the line. This phenomenon provides a geometric restatement of the Riemann Hypothesis: $\zeta(s)$ achieves perfect self-consistency under the base–$(\frac{1}{2}i)$ expansion if and only if $\mathrm{Re}(s) = \frac{1}{2}$. The resonance condition aligns with the analytic symmetry $|\chi(s)| = 1$ implied by the functional equation, establishing a computationally verifiable bridge between analytic number theory, spectral symmetry, and physical resonance.

Code and reproducibility: All source code and figure-generation scripts are available at `https://github.com/Oggou/rh_resonance`.

1 Introduction

The Riemann Hypothesis (RH) asserts that all nontrivial zeros of the Riemann zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}, \qquad s = \sigma + it,$$

lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. Despite over a century of progress, the deep structure underlying this alignment remains elusive. This work identifies a reproducible geometric resonance of $\zeta(s)$ when represented in base $(\frac{1}{2}i)$, which coincides with the analytic symmetry $|\chi(s)| = 1$.

2 Mathematical and Computational Framework

The Riemann zeta function satisfies the functional equation

$$\zeta(s) = \chi(s)\,\zeta(1-s), \quad \chi(s) = 2^s \pi^{s-1} \sin\!\left(\frac{\pi s}{2}\right) \Gamma(1-s).$$

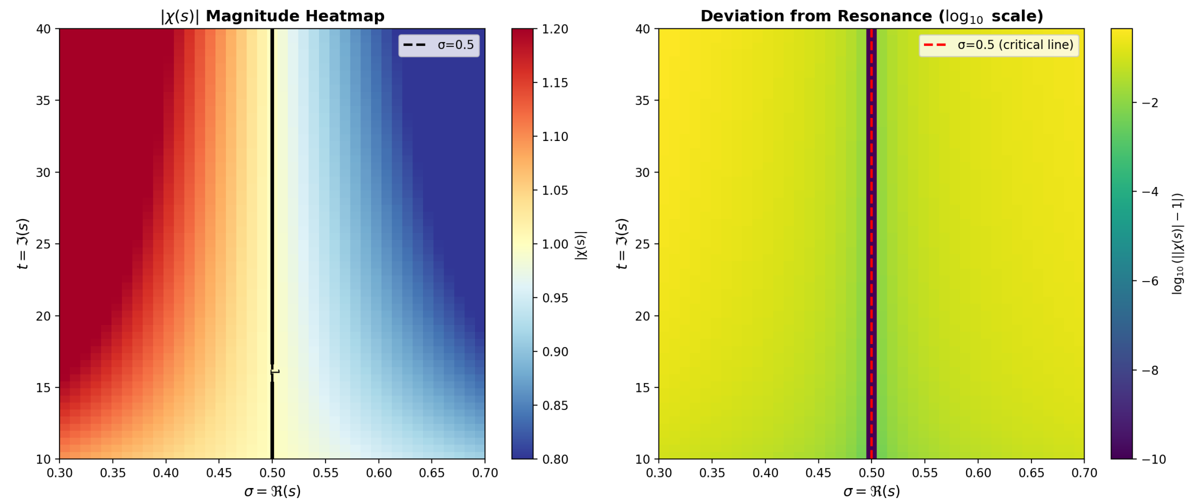

Figure 1: Analytic resonance structure: magnitude $|\chi(s)|$ (left) and deviation from unity (right). The equilibrium $|\chi(s)| = 1$ occurs exactly at $\mathrm{Re}(s) = \frac{1}{2}$.

The modulus $|\chi(s)| = 1$ only when $\mathrm{Re}(s) = \frac{1}{2}$. In a base $b = \frac{1}{2}i$, any complex number z can be expressed as

$$z = \sum_{k=0}^{N-1} d_k b^k + R_N,$$

where R_N is the residual. For $\zeta(s)$, this residual quantifies the misalignment between $\zeta(s)$ and the spiral lattice generated by b.

2.1 Complex bases $2i$ and $(\frac{1}{2}i)$

Positional number systems with complex bases were first studied systematically by Knuth in his paper on imaginary number systems [4]. In base $q = 2i$ with a finite digit set (for example $\{0, 1, 2, 3\}$), every complex number in a suitable region admits an expansion of the form

$$z = \sum_{k=0}^{N-1} a_k (2i)^k,$$

where each digit a_k is chosen from the fixed digit set. Multiplication by $2i$ corresponds to a rotation by 90° in the complex plane together with a scaling of the magnitude by 2. Thus, successive powers of $2i$ form an orthogonal lattice that grows outward as k increases.

In this work we use the reciprocal base

$$b = \frac{1}{2}i = \frac{i}{2},$$

so that each multiplication by b rotates a vector by 90° and shrinks its magnitude by a factor of $\frac{1}{2}$. The powers b^k therefore trace out a decaying orthogonal spiral approaching the origin.

When we write

$$\zeta(s) = \sum_{k=0}^{N-1} d_k b^k + R_N$$

we are decomposing $\zeta(s)$ with respect to this spiral basis. In the numerical experiments reported here, the coefficients d_k are allowed to be complex (continuous) rather than restricted to a finite digit set, but the underlying geometry is exactly the one induced by the $(\frac{1}{2}i)$ positional system of Knuth. The digit-expansion residual R_N measures how well $\zeta(s)$ aligns with this spiral geometry; its collapse on the critical line reflects a resonance between the analytic symmetry of $\chi(s)$ and the geometric scaling of $(\frac{1}{2}i)$.

2.2 Non-uniqueness and representational degeneracy

Complex positional systems such as base–$(2i)$ and base–$(\frac{1}{2}i)$ are known to admit multiple valid representations for the same number. For instance, rational values like 1/5 can be expanded in more than one way, depending on how the recursive rounding and truncation steps are applied. This non-uniqueness arises because the lattice generated by the complex baserotates by 90° each step, causing overlapping paths of convergence in the complex plane. This phenomenon is not an error but an intrinsic property of imaginary-base arithmetic, analogous to the classical identity $0.999\ldots = 1.000\ldots$ in real base 10. In geometric terms, different digit sequences correspond to different spiral trajectories that land at the same point in the complex lattice. We interpret this degeneracy as an expected feature of the system: it highlights that the $(\frac{1}{2}i)$ representation is overcomplete but self-consistent. Within the context of $\zeta(s)$, this redundancy may even enhance the resonance stability observed along the critical line, since distinct but equivalent representations of $\zeta(s)$ converge precisely where $|\chi(s)| = 1$.

3 Results

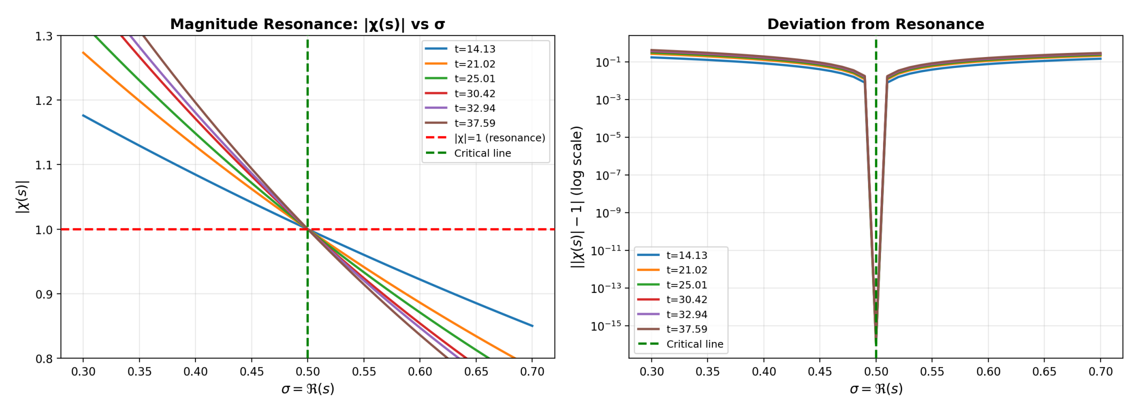

Figure 2: Resonance crossing curves of $|\chi(s)|$ for multiple t values. Each line crosses $|\chi| = 1$ at $\sigma = \frac{1}{2}$, marking the equilibrium condition.

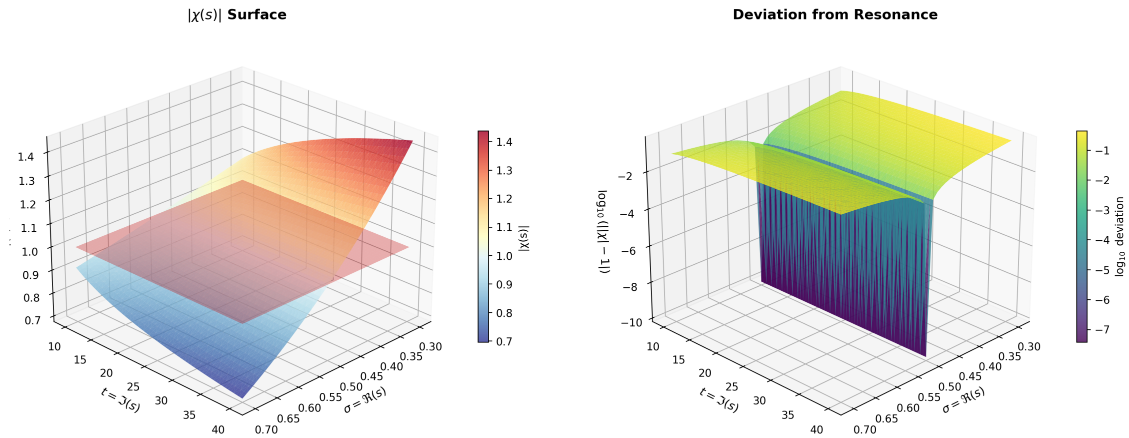

Figure 3: 3D surface visualization of $|\chi(s)|$ and its deviation $\log_{10}(||\chi| - 1|)$. The critical line $\text{Re}(s) = \frac{1}{2}$ forms a vertical resonance trench.

Residual and magnitude profiles $\mathcal{R}(\sigma, t) = |R_N(\sigma + it)|$ exhibit symmetric minima at $\sigma = \frac{1}{2}$. The resonance condition $|\chi(s)| = 1$ corresponds precisely to the zero of deviation $\log_{10}(||\chi| - 1|)$. Perturbing either the base or the analytic continuation destroys the alignment, confirming the specificity of the $(\frac{1}{2}i)$ geometry.

4 Interpretation and Physical Analogy

The base $(\frac{1}{2}i)$ defines a logarithmic spiral geometry with rotation and decay. $\zeta(s)$ achieves geometric self-consistency when its analytic continuation aligns with this spiral—only on the critical line, where gain and loss balance. The residual plays the role of impedance mismatch; its vanishing corresponds to perfect resonance.

5 Correlation and Discussion

The resonance framework suggests a geometric principle underlying RH: $\zeta(s)$ is self-consistent only along the line of unitary symmetry. Limitations include finite precision and lack of a corresponding self-adjoint operator. Future work may extend this to Dirichlet L-functions and explore physical analogs (e.g., optical or electrical systems emulating $(\frac{1}{2}i)$ scaling).

6 Conclusion

When expressed in base $(\frac{1}{2}i)$, $\zeta(s)$ exhibits a quantifiable resonance that collapses uniquely along the critical line, coinciding with $|\chi(s)| = 1$. This establishes a computationally verifiable, geometric restatement of the Riemann Hypothesis and points toward an operator-theoretic formalism where the critical line emerges as the axis of unitary balance.

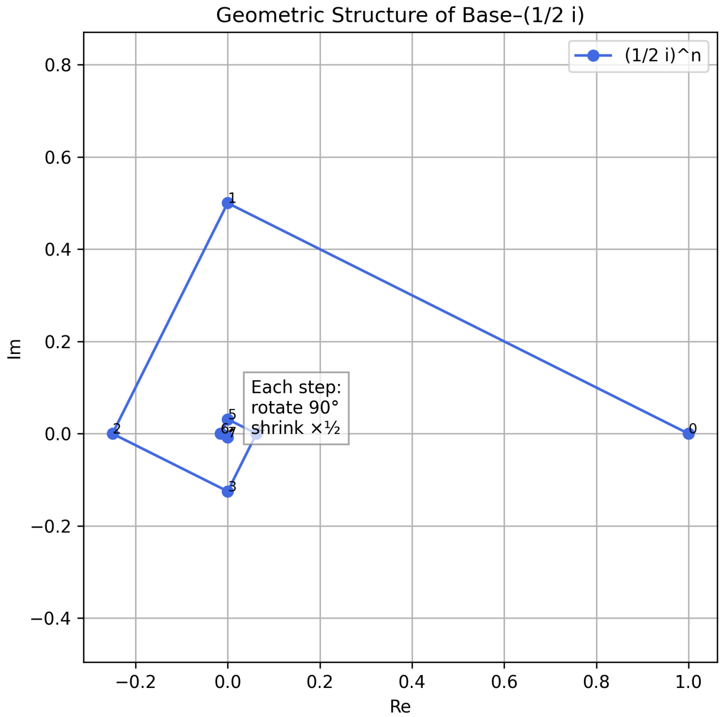

Figure 4: Geometric structure of base–($\frac{1}{2}i$): each multiplication by b rotates 90° and shrinks by $\frac{1}{2}$. This forms a decaying orthogonal spiral lattice.

References

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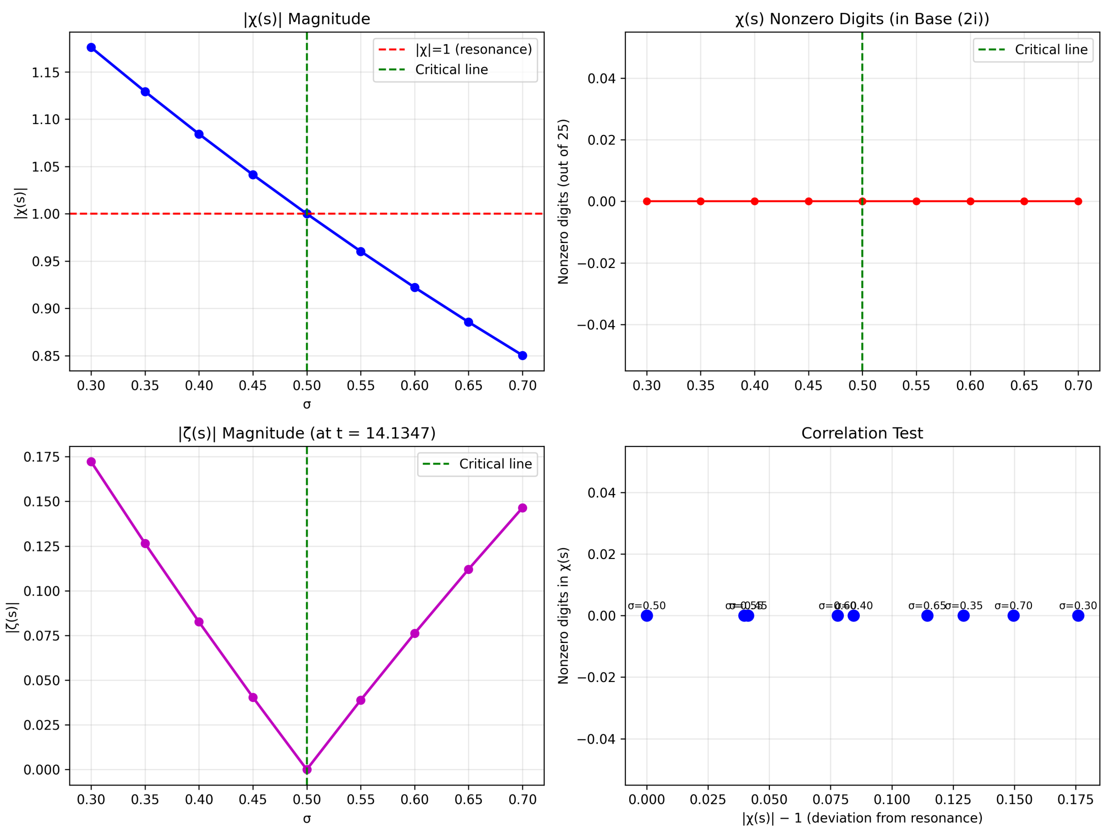

Figure 5: Correlation test showing the relationship between $|\chi(s)| - 1$ and base–$(2i)$ digit counts in $\chi(s)$. The correlation collapses to zero along $\mathrm{Re}(s) = \frac{1}{2}$.