Notes on Serge Lang's Complex Analysis (4^{ed})

By: Travis Ravert

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Contents

Chapter 1

Complex Numbers and Functions

Assume: $\{z_n\}$ is a complex valued sequence with $n, m, N \in \mathbb{N}$.

1.1 Complex Numbers

D 1.1 (Complex Numbers). *A set of objects, denoted $\mathbb{C}$, closed under addittion and multipliplication that satisfy the following conditions:*

1. $\alpha, \beta \in \mathbb{R} \subseteq \mathbb{C}$.
2. *There exists an element of the set $\mathbb{C}$ denoted i such that $i^2 = -1$.*
3. $\forall z \in \mathbb{C}$ *can be written uniquely as:*
$$z = \alpha + \beta i$$
4. *Arithmetic laws of addition and multiplication are satisfied $\forall z \in \mathbb{C}$:*

   distributive law holds.

   associative law holds.

   commutative law holds.

   (multiplicative identity exists) $1z = z$.

   (additive identity exists) $0 + z = z$.

   (additive inverses exist) $z + (-1)z = 0 \;\; \forall z$

D 1.2 (Conjugate). *The symmetric partner of z denoted $\bar{z} \in \mathbb{C}$ such that:*

$$z = a + bi$$

iff

$$\bar{z} = a - bi$$

D 1.3 (Multiplicative Inverses). $\forall z \in \mathbb{C}$ *there exists z^{-1} such that*

$$z \cdot z^{-1} = 1$$

D 1.4 (Modulus of z).

$$|z| = \sqrt{a^2 + b^2}$$

T 1.1. *$|z|$ satisfies the following properties. If $\alpha, \beta \in \mathbb{C}$, then:*

$$|\alpha\beta| = |\alpha||\beta|$$

$$|\alpha + \beta| \;\leq\; |\alpha| + |\beta| \quad \textit{(triangle inequality)}$$

1.1.1 Proofs

1. Express the following complex numbers in the form $x + iy$, where x, y are real numbers.

   (a) $(-1+3i)^{-1}$

   Proof:

   Simply invert and separate, then use the conjugate/symmetry to rationalize the statement:

$$
\begin{aligned}
(-1+3i)^{-1} &= \frac{1}{(-1+3i)} \\
\frac{1}{(-1+3i)} &= \frac{1}{(-1+3i)}\frac{(-1-3i)}{(-1-3i)} \\
&= \frac{-1-3i}{1+3i-3i+9} \\
&= \frac{-1-3i}{10}
\end{aligned}
$$

$$
\therefore\ (-1+3i)^{-1} = -\frac{1}{10} - \frac{3i}{10}
$$

   Which just means from the origin of $\mathbb{C}$ go left 1 then down 3 then shrink by $\frac{1}{10}$ and that's the z you're at in $\mathbb{C}$. ■

   (b) $(1+i)(1-i)$

   Proof: Distribute and collect:

$$
\begin{aligned}
(1+i)(1-i) &= 1 - i + i - i^2 \\
&= 1 - (-1) \\
&= 2 \\
\therefore\ (1+i)(1-i) &= 2 + 0i
\end{aligned}
$$

■

   (c) $(i+1)(i-2)(i+3)$

   Proof: Distribute collect, distribute and collect again.

$$
\begin{aligned}
(i+1)(i-2)(i+3) &= (i^2 - 2i + i - 2)(i+3) \\
&= (-i-3)(i+3) \\
&= (1 - 3i - 3i - 9)
\end{aligned}
$$

$$
\therefore\ (i+1)(i-2)(i+3) = -8 - 6i
$$

■

2. Express the following complex numbers in the form $x + iy$, where x, y are real numbers.

   (a) $(1+i)^{-1}$

Proof: More of the same, just use the conjugate to solve these like problem 1 above.

$$\begin{aligned}
(1+i)^{-1} &= \frac{1}{1+i} \\
&= \frac{1}{1+i}\frac{(1-i)}{(1-i)} \\
&= \frac{1-i}{(1+i)(1-i)} \\
&= \frac{1-i}{1-i+i-i^2} \\
&= \frac{1-i}{1-(-1)} \\
&= \frac{1-i}{2} \\
\therefore\ (1+i)^{-1} &= \frac{1}{2}-\frac{i}{2}
\end{aligned}$$

■

3. Let α be a complex number $\neq 0$. What is the absolute value of $\alpha/\bar{\alpha}$? What is $\bar{\bar{\alpha}}$?

Proof: First note that:

$$\begin{aligned}
\alpha &= x+yi \\
\bar{\alpha} &= x-yi \\
\therefore\ \left|\frac{\alpha}{\bar{\alpha}}\right| &= \frac{x+yi}{x-yi}
\end{aligned}$$

Now some algebra:

$$\begin{aligned}
\frac{x+yi}{x-yi} &= \frac{(x+yi)}{(x-yi)}\frac{(x+yi)}{(x+yi)} \\
&= \frac{x^2+2xyi+y^2i^2}{x^2-y^2i^2} \\
&= \frac{x^2+2xyi+y^2i^2}{x^2+y^2} \\
&= \frac{x^2+2xyi-y^2}{x^2+y^2} \\
&= \frac{x^2+2xyi-y^2}{|\bar{\alpha}|^2} \\
&= \frac{(x+yi)(x-yi)}{|\bar{\alpha}|^2} \\
\therefore\ \left|\frac{\alpha}{\bar{\alpha}}\right| &= \frac{\alpha\cdot\bar{\alpha}}{|\bar{\alpha}|^2}
\end{aligned}$$

■

Part 3b: What is $\bar{\bar{\alpha}}$?

Proof: Note

$$\alpha = x + yi \quad \Leftrightarrow \quad \bar{\alpha} = x - yi.$$
$$\therefore \ \bar{\bar{\alpha}} = \overline{x - yi}$$

So now because the conjugate operation just changes the sign on the *imaginary* part of α we have the straightforward result of:

$$\begin{aligned} \bar{\bar{\alpha}} &= \overline{x - yi} \\ &= x + yi \\ \therefore \ \bar{\bar{\alpha}} &= \alpha \end{aligned}$$

■

4. Let α, β be two complex numbers. Show that:

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$$

and that:

$$\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$$

Proof: First is easy since we just distribute out $\alpha \cdot \beta$ and gather reals and imaginary parts together and see it is the same result as if we had simply taken the conjugate of each component.

Algebraically, with $\alpha_n, \beta_n, \rho \in \mathbb{R}$:

$$\begin{aligned} \overline{\alpha\beta} &= \overline{(\alpha_1 + \alpha_2 i)(\beta_1 + \beta_2 i)} \\ &= \overline{(\alpha_1\beta_1 + \alpha_1\beta_2 i + \beta_1\alpha_2 i + \alpha_2\beta_2 i^2)} \\ &= \overline{(\alpha_1\beta_1 + i(\alpha_1\beta_2 + \beta_1\alpha_2) + \alpha_2\beta_2 i^2)} \\ &= \overline{(\alpha_1\beta_1 + \alpha_2\beta_2 i^2 + i(\alpha_1\beta_2 + \beta_1\alpha_2))} \\ &= \overline{(\alpha_1\beta_1 - \alpha_2\beta_2 + i(\alpha_1\beta_2 + \beta_1\alpha_2))} \\ &= \overline{\rho_1 + i\rho_2} \\ \overline{\alpha\beta} &= \rho_1 - i\rho_2 \end{aligned}$$

Now we go the other way:

$$\begin{aligned} \bar{\alpha}\bar{\beta} &= \overline{(\alpha_1 + \alpha_2 i)} \cdot \overline{(\beta_1 + \beta_2 i)} \\ &= (\alpha_1 - \alpha_2 i) \cdot (\beta_1 - \beta_2 i) \\ &= (\alpha_1\beta_1 - \alpha_1\beta_2 i - \alpha_2\beta_1 i + \alpha_2\beta_2 i^2) \\ &= (\alpha_1\beta_1 - \alpha_1\beta_2 i - \alpha_2\beta_1 i - \alpha_2\beta_2) \\ &= (\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_1\beta_2 i - \alpha_2\beta_1 i) \\ &= (\alpha_1\beta_1 - \alpha_2\beta_2 - i(\alpha_1\beta_2 + \alpha_2\beta_1)) \\ \bar{\alpha}\bar{\beta} &= \rho_1 - i\rho_2 \end{aligned}$$

$$\therefore \ \overline{\alpha\beta} = \bar{\alpha}\bar{\beta} \quad ■$$

Second is easier as we only convert the sign inside the complex numbers, and do nothing with the operation between the two complex numbers, only on the reals in the number.

Again, basically just some algebra of converting to the real and imaginary parts and gathering terms. ■

5. Justify the assertion that the real part of a complex number is $\leq$ its absolute value.

   Proof: The value can be equal to the absolute value if it happens to be positive, in which case it coincides with the absolute value.

   Or, it can be the symmetric partner if it is negative and therefore equal in magnitude but opposite in direction, therefore ordered as $\leq$ the absolute value by definition of well-ordering in $\mathbb{R}$. Because the reals are symmetric like shoes, the have a left-handedness and a right-handedness. $\blacksquare$

6. If $\alpha = a + ib$ with $a, b \in \mathbb{R}$ then b is called the **imaginary part** of α and we write:

   $$\mathfrak{Im}(\alpha) = b.$$

   (a) Show that:

   $$\alpha - \bar{\alpha} = 2i\ \mathfrak{Im}(\alpha)$$

   Proof: Just do the algebra:

   $$\begin{aligned} \alpha - \bar{\alpha} &= (a + ib) - (a - ib) \\ &= 2ib \\ \therefore\ \alpha - \bar{\alpha} &= 2i\ \mathfrak{Im}(\alpha) \end{aligned}$$

   $\blacksquare$

   (b) Show that:

   $$\mathfrak{Im}(\alpha) \leq |\mathfrak{Im}(\alpha)| \leq |\alpha|$$

   Proof: Again, with some algebra we see the answer by considering the case of the imaginary part being either positive or negative while the absolute value will always be positive and therefore will be equal to this value or greater than it if it is negative.

   Next, think of whether part of α along just the real part or that part plus another would always make it larger than or equal to? If they always have the same imaginary part, then adding a real only increases the size of α while leaving the imaginary part at its maximum value. I'm too lazy to type this out right now, maybe later. $\blacksquare$

7. Find the real and imaginary parts of $(1 + i)^{100}$.

   Proof: Working with a base of $(1 + i)$ we just find useful factors to work with:

   $$\begin{aligned} (1 + i)^2 &= 2i \\ (1 + i)^4 &= 2i^2 \\ &= -4 \\ (1 + i)^{10} &= (1 + i)^4(1 + i)^4(1 + i)^2 \\ &= (-4)(-4)(2i) \\ &= 32i \end{aligned}$$

   Now just plug and play:

   $$\begin{aligned} (1 + i)^{100} &= ((1 + i)^{10})^{10} \\ &= (32i)^{10} \\ &= i^{10}32^{10} \\ &= -(32)^{10} \end{aligned}$$

   $$\therefore\ (1 + i)^{100} = -(32)^{10} + 0i$$

   $\blacksquare$

8. Prove that for any two complex numbers z, w we have:

   (a) $|z| \leq |z - w| + |w|$

   Proof: Consider the three cases we could have have:

$$
\begin{aligned}
w &< 0 \\
w &= 0 \\
w &> 0
\end{aligned}
$$

   If $w < 0$:

$$
\begin{aligned}
|z - (-w)| + |-w| &= |z + w| + |w| \\
\therefore\ z &< |z - w| + |w|
\end{aligned}
$$

   If $w = 0$:

$$
\begin{aligned}
|z - w| + |w| &= |z - 0| + |0| = z \\
\therefore\ z &= |z - w| + |w|
\end{aligned}
$$

   If $w > 0$, with $z_w + w = z$:

$$
\begin{aligned}
|z - w| + |w| &= z_w + |w| = z \\
\therefore\ z &= |z - w| + |w|
\end{aligned}
$$

   By these three cases combined we have:

$$|z| \leq |z - w| + |w|$$

   ■

   (b) $|z| - |w| \leq |z - w|$

   Proof: By (a) above we just subtract $|w|$ off the right and left, and have a logically equivalent statement. ■

   (c) $|z| - |w| \leq |z + w|$

   Proof: If the above were not true, then (b) would be false, but (b) is true, so then:

$$|z| - |w| \leq |z + w|$$

   ■

9. Let $\alpha = a + ib$ and $z = x + iy$. Let $c \in \mathbb{R} > 0$. Transform the condition:

$$|z - \alpha| = c$$

   into an equation involving only x, y, a, b and c, and describe in a simple way what geometric figure is represented by this equation.

   Proof: The equation comes out as:

$$
\begin{aligned}
|z - \alpha| &= |x + iy - (a + ib)| \\
&= |(x - a) + i(y - b)| \\
&= c
\end{aligned}
$$

Which is a vector always moved from the origin of $\mathbb{C}$ by α and this vector forms a circular perimeter as we consider the values of x, y and see that no matter what you select you must be on that circular perimeter a distance of c from whatever origin we are at. ∎

10. Describe geometrically the sets of points z satisfying the following conditions:

    (a) $|z - i + 3| = 5$

    The perimeter of the circle that has a radius of 5 and with an origin $|z - i + 3|$ from the origin of $\mathbb{C}$.

    (b) $|z - i + 3| > 5$

    The complex plane outside a set that sits $|z - i + 3|$ from the origin of $\mathbb{C}$ with a radius of 5 with no points *on* the perimeter of the radius.

    (c) $|z - i + 3| \leq 5$

    The disc of points in $\mathbb{C}$ centered at $|z - i + 3|$ with a radius of 5.

    (d) $|z + 2i| \leq 1$

    The disc of radius 1 that is centered at z and moved vertically by $2i$.

    (e) $\mathfrak{Im}(z) > 0$

    The set of points in $\mathbb{C}$ not including 0 that have real parts $= 0$ and imaginary parts > 0, so the y-axis.

    (f) $\mathfrak{Im}(z) \geq 0$

    The set of points along the positive axis of $\mathbb{C}$ including 0.

    (g) $\mathfrak{Re}(z) > 0$

    The set of points along the positive axis of $\mathbb{R} \subset \mathbb{C}$ not including 0.

    (h) $\mathfrak{Re}(z) \geq 0$

    The set of points along the positive axis of $\mathbb{R} \subset \mathbb{C}$ including 0.

1.2 Polar Form

Let $z = x + iy$.

D 1.5 (Polar Coordinates). *An ordered pair* (r, θ) *with* $r = radius$ *and* θ *rotating from the* x*-axis such that:*

1. $r \in \mathbb{R}$ *and* $r = |z| = \sqrt{x^2 + y^2}$.
2. $\theta \in [0, 2\pi]$.

D 1.6 (Polar Form).

$$\begin{aligned} re^{i\theta} &= r\cos\theta + ir\sin\theta \\ &= x + iy \\ \therefore\ re^{i\theta} &\in \mathbb{C} \end{aligned}$$

Note that:

$$x = r\cos\theta, \quad y = r\sin\theta$$
$$\theta = \cos^{-1}\left(\frac{x}{r}\right), \quad \theta = \sin^{-1}\left(\frac{y}{r}\right)$$

1.2.1 Factors of Pi

D 1.7 (Factors of Pi). *"If you don't know your factors of pi you don't know squat" - Big Rick Feynman*

$$n \in \mathbb{N} \qquad z, w \in \mathbb{C}$$

$$\begin{aligned} e^0 &= 1 + i0 & e^0 &= (1, 0) \\ e^{i\pi/6} &= \frac{\sqrt{3}}{2} + i\frac{1}{2} & e^{i\pi/6} &= \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right) \\ e^{i\pi/4} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} & e^{i\pi/4} &= \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \\ e^{i\pi/3} &= \frac{1}{2} + i\frac{\sqrt{3}}{2} & e^{i\pi/3} &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \\ e^{i\pi/2} &= 0 + i & e^{i\pi/2} &= (0, 1) \\ e^{i\pi} &= -1 + i0 & e^{i\pi} &= (-1, 0) \\ e^{2i\pi} &= 1 + i0 & e^{2i\pi} &= (1, 0) \\ \\ e^{2n\pi i} &= 1 & \textit{If } \ e^z = e^w &\ \textit{ then } \ z = w + 2k\pi i \end{aligned}$$

Now take sums and multiples to build more factors from these.

T 1.2. *Let* $\theta, \varphi \in \mathbb{R}$ *then:*

$$e^{i\theta + i\varphi} = e^{i\theta}e^{i\varphi}$$

T 1.3. *Let* $\alpha, \beta \in \mathbb{C}$ *then:*

$$e^{\alpha+\beta} = e^{\alpha}e^{\beta}$$

T 1.4 (1.2 and 1.3 together)**.** *Let* $z_1 = r_1e^{i\theta}$ *and* $z_2 = r_2e^{i\varphi}$ *then:*

$$z_1 \cdot z_2 = r_1r_2e^{i(\theta+\varphi)}$$

i.e. multiply the absolute values and add the angles.

1.2.2 Proofs

1. Put the following complex numbers in polar form.

   (a) $z = 1 + i$

   Change base. Note that:

$$\begin{aligned} e^0 &= e^{2i\pi} = 1 \\ e^{i\pi/2} &= i \end{aligned}$$

   Then note:

$$\begin{aligned} r = |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{1+1} \\ \therefore\ r &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \therefore\ 1 + i &= \sqrt{2}e^{2i\pi}e^{(i\pi/2)} \\ &= \sqrt{2}e^{i\pi/2} \end{aligned}$$

■

   (b) $1 + i\sqrt{2}$

   Note:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + \sqrt{2}^2} \\ &= \sqrt{1+2} \\ \therefore\ r &= \sqrt{3} \end{aligned}$$

   Previously we selected the factor of π which gave us equal x and y pieces, but here something else is going on.

   We need to go right along the x-axis by 1 then up the y-axis by $\sqrt{2}$.

   Note that we can normalize these with $\frac{1}{r}$ or use the Euler formula relating cosine to x and r to start.

$$\frac{1}{\sqrt{3}}(1 + i\sqrt{2}) = \frac{1}{\sqrt{3}} + \frac{i\sqrt{2}}{\sqrt{3}}$$

   Try the Euler method here instead:

$$1 + i\sqrt{2} = \sqrt{3}\cos\theta + i\sqrt{3}\sin\theta$$

   Then

$$\begin{aligned} \frac{x}{r} &= \frac{1}{\sqrt{3}} = \cos\theta \\ \frac{y}{r} &= \frac{\sqrt{2}}{\sqrt{3}} = \sin\theta \end{aligned}$$

(c) -3

Go left on the real line in the complex plane:

$$-3 = 3e^{i\pi}$$

■

(d) $4i$

Go up by $4i$ in the complex plane:

$$4i = 4e^{i\pi/2}$$

■

(e) $1 - i\sqrt{2}$

Go right by 1 and down by $\sqrt{2}$ in the complex plane:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + \sqrt{2}^2} \\ &= \sqrt{1+2} \\ \therefore\ r &= \sqrt{3} \end{aligned}$$

And then:

$$\theta = \cos^{-1}\frac{1}{\sqrt{3}}$$

So finally:

$$1 - i\sqrt{2} = \sqrt{3}e^{i\pi\cdot\cos^{-1}\frac{1}{\sqrt{3}}}$$

■

(f) $5i$

Go up by $5i$ in the complex plane:

$$5i = 5e^{i\pi/2}$$

■

(g) -7

Go left by -7 in the complex plane:

$$-7 = 7e^{i\pi}$$

■

(h) $-1 - i$

Go left by -1 and down by -1 in the complex plane:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + 1^2} \\ &= \sqrt{1+1} \\ \therefore\ r &= \sqrt{2} \end{aligned}$$

So then:

$$-1 - i = \sqrt{2}e^{5i\pi/4}$$

■

2. Put the following complex numbers in the ordinary form $x + iy$.

(a) $e^{3i\pi}$
Simple, use Theorems 1.2 and 1.3 and change base!

$$\begin{aligned} e^{3i\pi} &= e^{i(2\pi+\pi)} \\ &= e^{i2\pi}e^{i\pi} \\ \therefore\ e^{3i\pi} &= (1)(-1) = -1 \end{aligned}$$

■

(b) $e^{2\pi i/3}$
Use Theorem 1.3 and notice:

$$e^{2\pi i/3} = e^{\pi i/3}e^{\pi i/3}$$

Now check the Factors of Pi for $e^{\pi i/3}$ to see that we have:

$$e^{\pi i/3} = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

Then we have:

$$\begin{aligned} e^{2\pi i/3} &= e^{(\pi i/3\,+\,\pi i/3)} \\ &= e^{\pi i/3}e^{\pi i/3} \\ e^{2\pi i/3} &= \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right) \\ &= \frac{1}{2}\frac{1}{2} + \frac{1}{2}\frac{i\sqrt{3}}{2} + \frac{1}{2}\frac{i\sqrt{3}}{2} + \frac{i\sqrt{3}}{2}\frac{i\sqrt{3}}{2} \\ &= \frac{1}{4} + \frac{i\sqrt{3}}{4} + \frac{i\sqrt{3}}{4} - \frac{3}{4} \\ &= \frac{1}{4} + \frac{2i\sqrt{3}}{4} - \frac{3}{4} \\ \therefore\ e^{2\pi i/3} &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{aligned}$$

■

(c) $3e^{-i\pi/4}$

Here we just see that we are using $r = |z| = 3$ and moving *downward* from the x-axis to start with rotation $\frac{\pi}{4}$.

$$\therefore\ 3e^{-i\pi/4} = 3\left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

■

(d) $\pi e^{-i\pi/3}$

Here we just have $r = |z| = \pi$ and rotate *down* again by factor of $\frac{\pi}{3}$.

Looking at the Factors of Pi and moving down we get:

$$e^{i\pi/3} = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$
$$\therefore\ \pi e^{-\,i\pi/3} = \pi\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$$

■

(e) $e^{2i\pi/6}$

Lets factor, get some basis and then rotate to what we want:

$$e^{2i\pi/6} = e^{i\pi/3}$$
$$\therefore\ e^{2i\pi/6} = \left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

(f) $e^{-\,i\pi/2}$

Simple, factor we know but just going in different y direction.

$$e^{-\,i\pi/2} = \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)$$

■

(g) $e^{-i\pi}$

We know this already, just going in different y direction.

$$e^{-i\pi} = (-1 + 0i)$$
$$= -1$$

■

(h) $e^{-\,5i\pi/4}$

Just break up the factors and use theorem 1.2 and 1.3.

$$e^{-\,5i\pi/4} = e^{-\,i\pi/4}e^{-\,4i\pi/4}$$
$$= \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)e^{-i\pi}$$
$$= \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)(-1)$$
$$\therefore\ e^{-\,5i\pi/4} = \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right)$$

■

3. Let α be a complex number $\neq 0$. Show there are two distinct complex numbers whose square is α.

Proof: For any $z \in \mathbb{C}$ with $z = a + bi$ we have a symmetric partner such that:

$$z = (a + bi)$$
$$\bar{z} = a - bi$$
$$\therefore\ z \neq \bar{z}$$

So z and $\bar{z}$ are distinct.

Now take their square:

$$\begin{aligned} z\bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - i^2b^2 \\ &= a^2 + b^2 \in \mathbb{C} \\ &= \zeta \in \mathbb{C} \\ &\neq 0 \end{aligned}$$

Which demonstrates $\forall z \neq 0$ and $\forall z \in \mathbb{C}$ we get:

$$\begin{aligned} z &\neq \bar{z} \\ z\bar{z} &= \zeta \in \mathbb{C} \\ &\neq 0 \end{aligned}$$

■

4. Let $a + bi$ be a complex number. Find real numbers x, y such that:

$$(x+iy)^2 = a + bi$$

expressing x, y in terms of a and b.

Proof:
Let $x, y \in \mathbb{R}$ then:

$$\begin{aligned} (x+iy)(x+iy) &= x^2 + 2xyi + i^2y^2 \\ &= (x^2 - y^2) + 2xyi \\ &= a + bi \\ \therefore\ (x+iy)^2 &= a + bi\ \ \forall x, y \in \mathbb{R} \end{aligned}$$

■

5. Plot all the complex numbers z such that $z^n = 1$ for $n = 2, 3, 4$ and 5.

Just use $e^{2\pi i/n}$ to cut the circle up so that you then just tile that slice n times to get back to 1.

So then:

$$z_n = e^{2\pi i/n}$$

Which would then give:

$$\begin{aligned} (z_n)^n &= (e^{2\pi i/n})^n \\ &= e^{2\pi i} \\ &= 1 \\ &= (1, 0) \end{aligned}$$

Which satisfies the fist condition. Now to give the plots for the remaining z's.

Suppose we start with $n = 2$:

$$\begin{aligned} z_2 &= e^{\pi i} \\ &= (-1, 0) \end{aligned}$$

Now with $n = 3$:

$$\begin{aligned} z_3 &= e^{2\pi i/3} \\ &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right) \end{aligned}$$

For $n = 4$:

$$\begin{aligned} z_4 &= e^{\pi i/2} \\ &= (0, 1) \end{aligned}$$

For $n = 5$:

$$\begin{aligned} z_5 &= e^{2\pi i/5} \\ &= (\cos(2\pi i/5),\ \sin(2\pi i/5)) \end{aligned}$$

■

6. Let α be a complex number $\neq 0$. Let n be a positive integer. Show that there are n distinct complex numbers z such that $z^n = \alpha$. Write these complex numbers in polar form.

   Proof: Suppose:

$$\begin{aligned} &\alpha \in \mathbb{C} \\ &\alpha = \alpha_1 + \alpha_2 i \\ &\quad = |\alpha| e^{i\theta} \\ &\quad = re^{i\theta} \\ &\quad \neq 0 \end{aligned}$$

   Now let:

$$\begin{aligned} &z_n \in \mathbb{C} \\ &z_n = r^{1/n} e^{i\theta/n} \\ &z_n \neq 0 \end{aligned}$$

   And note for some other $m \in \mathbb{N}$ with $m \neq n$:

$$\begin{aligned} z_m &\neq z_n \\ (z_m)^m &= (r^{1/m} e^{i\pi/m})^m \\ &= re^{i\theta} \end{aligned}$$

   Then we have $\forall n, m \in \mathbb{N}$ with $m \neq n$:

$$\begin{aligned} z_m &\neq z_n \\ (z_n)^n &= r^{(1/n)^n} e^{(i\theta/n)^n} \\ (z_m)^m &= r^{(1/n)^m} e^{(i\theta/m)^m} \\ &= re^{i\theta} \\ &= \alpha \end{aligned}$$

   And so we have $n \in \mathbb{N}$ distinct complexe numbers such that:

$$z^n = \alpha$$

■

7. Find the real and imaginary parts of $i^{1/4}$, taking the fourth root such that its angle lies in $[0, \pi/2]$.

   Proof: Change of base and Euler's identity.

$$i = e^{i\pi/2}$$

   So then:

$$\begin{aligned} i^{1/4} &= e^{(i\pi/2)^{1/4}} \\ &= e^{(i\pi/8)} \end{aligned}$$

   Which then just means $\theta = \pi/8$ and we can get the real and imaginary parts with:

$$x = \cos(\pi/8) \text{ and } y = \sin(\pi/8)$$

$$\therefore\ i^{1/4} = \cos(\pi/8) + i\sin(\pi/8)$$

■

8. (a) Describe all complex numbers z such that $e^z = 1$.

   Proof: Because we have $e^{2\pi ik} = 1 \quad \forall k \in \mathbb{N}$ we then see this will happen in the cases when $z = 2i\pi k$ ■

   (b) Let w be a complex number. Let α be a complex number such that $e^\alpha = w$. Describe all complex numbers α such that $e^\alpha = w$.

   Proof: This is true for any $z \in \mathbb{C}$ and $k \in \mathbb{N}$ such that $z = \alpha + 2i\pi k$ or else $e^{2\pi i} \neq 1$, which is absurd. ■

9. If $e^z = e^w$ show that there is an integer k such that $z = w + 2\pi ki$.

   Proof: Let $z, w \in \mathbb{C}$ with $z \neq w$ and $e^z = e^w$.

   Using the results of problem 8.b above, we see this can only be true when $z = w + 2\pi ik$ which means $\exists k \in \mathbb{N}$.

■

10. (a) If θ is real, show that:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

   Proof: Just expand the exponentials to the polar coordinates of $\sin\theta$ and $\cos\theta$ and then do the algebra.

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}$$

   Then we have:

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta \\ &= \cos\theta + \cos\theta \\ \therefore\ \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} \end{aligned}$$

   We see a similar argument doing it for $\sin\theta$ and noting the sign change between the e terms. ■

(b) For arbitrary complex z, suppose we define $\cos z$ and $\sin z$ by replacing θ with z in the above formula. Show that the only values of z for which $\cos z = 0$ and $\sin z = 0$ are the usual real values from trigonometry.

Proof: We need to use the one-to-one correspondence between the ordered pairs $z = (x, y)$ in the complex plane and (r, θ) in the real plane.

Notice first some maps:

$$\begin{aligned}
|z| &= \sqrt{x^2 + y^2} \\
&= r \\
z &= (x, y) = x + iy \\
&= (r, \theta) = re^{i\theta} \\
\cos(\theta) &= \frac{x}{r} \\
\cos(z) &= \frac{x}{|z|} \\
\cos(\theta) &= \frac{e^{i\theta} + e^{-i\theta}}{2} \\
\cos(z) &= \frac{e^{iz} + e^{-iz}}{2} \\
\sin(\theta) &= \frac{y}{r} \\
\sin(z) &= \frac{y}{|z|} \\
\sin(\theta) &= \frac{e^{i\theta} - e^{-i\theta}}{2i} \\
\sin(z) &= \frac{e^{iz} - e^{-iz}}{2i} \\
|r|e^{i\theta} &= r\cos(\theta) + ir\sin(\theta)
\end{aligned}$$

This gives the maps we need between using $\forall z \in \mathbb{C}$ and $0 \leq \theta \leq 2\pi$.

Notice that asking when $cos(z) = 0$ is precisely when $\cos(z) = \frac{e^{iz}+e^{-iz}}{2} = 0$ as well.

11. Prove that for any complex number $z \neq 1$ we have

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}$$

Proof: Strategy: Just add the z^{n+1} term to both sides and notice that you end up back to the original statement on the right.

Base case: let $n = 1$:

$$\begin{aligned}
\frac{z^{1+1} - 1}{z - 1} &= \frac{z^2 - 1}{z - 1} \\
&= \frac{(z + 1)(z - 1)}{z - 1}
\end{aligned}$$

$$\therefore \quad \frac{z^{1+1} - 1}{z - 1} = z + 1 \tag{1.1}$$

And the result is true for the sum as well:

$$\sum_{n=1}^{1}(1+z^n) = 1+z \tag{1.2}$$

By (1.1) and (1.2) we see the *base case* is true.
Now, suppose it is true for $n \in \mathbb{N}$ and let $k = n+1$, then:

$$\begin{aligned} 1+z+\cdots+z^n+z^k &= \frac{z^{n+1}-1}{z-1} + z^k \\ &= \frac{z^{n+1}-1}{z-1} + \frac{z^k(z-1)}{(z-1)} \\ &= \frac{z^{n+1}-1}{z-1} + \frac{z^{k+1}-z^k}{(z-1)} \\ &= \frac{z^{n+1}-1+z^{k+1}-z^k}{z-1} \end{aligned}$$

Note that $z^{n+1} = z^k$ and we have:

$$\begin{aligned} 1+z+\cdots+z^n+z^k &= \frac{-1+z^{k+1}}{z-1} \\ &= \frac{z^{k+1}-1}{z-1} \end{aligned}$$

Noting again that k is the $n+1$ case, we prove the statement holds $\forall n \in \mathbb{N}$. ■

12. Using the preceding exercise, and taking real parts, prove:

$$1+\cos\theta+\cos 2\theta+\cdots+\cos n\theta = \frac{1}{2} + \frac{\sin[(n+\frac{1}{2})\theta]}{2\sin\frac{\theta}{2}}$$

For $0 < \theta < 2\pi$.

Proof: Need help here. There's some kind of identity I'm missing to get each $\cos^n$ term to map respectively to each $\cos n$.

The trick is *de Moivre's Formula*!

$$(\cos x + i\sin x)^n = \cos nx + i\sin nx$$

And since we only use the real part we have $\sin nx = 0$ and so:

$$(\cos x)^n = \cos nx$$

This completes the mapping of terms on the left side.

Now, we need to show:

$$\frac{1}{2} + \frac{\sin(n+\frac{1}{2}\theta)}{2\sin\frac{\theta}{2}} = \frac{z^{n+1}-1}{z-1}$$

13. Let z, w be two complex numbers such that $\bar{z}w \neq 1$. Prove that

$$\left|\frac{z-w}{1-\bar{z}w}\right| < 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1,$$
$$\left|\frac{z-w}{1-\bar{z}w}\right| = 1 \quad \text{if } |z| = 1 \text{ and } |w| = 1,$$

Proof: Not sure.

1.2.3 Incomplete Proofs

- 10.b, 12, 13

1.3 Complex Valued Functions

Denote the association of the **value** $f(z)$ to z as:

$$z \mapsto f(z)$$

D 1.8 (Complex Valued Function).

$$f(z) = u(z) + iv(z)$$
$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$
$$z \mapsto u(z) \ \ \textit{and} \ \ z \mapsto v(z)$$

So a complex valued function can be represented as functions of 2 *real variables.*

Example:

$$\begin{aligned} f(z) &= (x - iy)^2 \\ &= (x - iy)(x - iy) \end{aligned}$$

$$\therefore \ f(z) = x^2 - y^2 - i2xy$$

With u and v given by:

$$\begin{aligned} u(z) &= u(x, y) = x^2 - y^2 \\ v(z) &= v(x, y) = -2xy \end{aligned}$$

$$\begin{aligned} \therefore \ f(z) &= u(x, y) + iv(x, y) \\ &= u(z) + iv(z) \end{aligned}$$

1.3.1 Power Function

The most important example of complex valued functions.

D 1.9 (Power Function). *Any function of the form:*

$$f(z) = z^n$$

1.3.2 Polar coordinates

Write z in polar coordinates using $|z| = r \in \mathbb{R}$ and $\theta \in [0, 2\pi]$.

D 1.10 (Complex Numbers and Functions in Polar Form).

$$\begin{aligned} z = re^{i\theta} \qquad &= \cos\theta + i\sin\theta \\ f(z) = r^n e^{in\theta} \quad &= r^n(\cos n\theta + i \sin n\theta) \end{aligned}$$

1.3.3 Closed/Open Disc

D 1.11 (Set of the Closed Disc $\overline{D}$).

$$\overline{D} = \{z \in \mathbb{C} \ \mid \ \forall z \leq 1\}$$

$$\forall z \in \overline{D}, \ n \in \mathbb{N} : \quad z \mapsto z^n \in \overline{D}$$

The **Open Disc** changes $z < 1$.

1.3.4 Unit Disc / Roots of Unity

Intent: Break the unit circle up into n sectors/tiles/slices.

Motivation: Allows mapping solutions for polynomials, which themselves represent complex numbers or analytic functions, to roots of unity over $[0, 2\pi]$.

D 1.12 (Sector of Unit Circle).

$$S = \left\{ z \in \mathbb{C} \;\mid\; z = e^{i\theta}, \quad 0 \leq \theta \leq \frac{2\pi}{n} \right\}$$

Function of a Real Variable

Suppose $r \in \mathbb{R}$ with a function defined by:

$$r \mapsto r^n$$

This maps the unit interval $[0, 1]$ onto itself:

$$[0, 1] \to [0, 1]$$

Function of Theta

Suppose $\theta \in [0, 2\pi]$ with a function defined by:

$$\theta \mapsto n\theta$$

This maps the interval $[0, \frac{2\pi}{n}]$ onto the circumference of a circle $[0, 2\pi]$:

$$\left[0, \frac{2\pi}{n}\right] \to [0, 2\pi]$$

Function of a Complex Value

Combining above the function $f(z) = z^n$ maps the sector S onto the full disc of numbers w where:

$$w = te^{i\varphi}$$

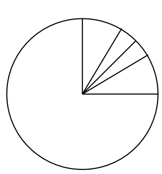
$$0 \leq t \leq 1$$

$$0 \leq \varphi \leq 2\pi$$

Things to note:

- The power function tiles the unit disc D_1 by sector S a total of n times.
- So $z \mapsto z^n$ wraps the disc n times around.

Diagram of this tiling for a few factors of π with $\theta = 0, \pi/6, \pi/4, \pi/3, \pi/2$:



This gives 4 seperate choices of tile to then multiply by n to cover the unit disc.

To express every complex number $w^n = z$ one ends up with the following generalization:

D 1.13 (Root of Unity).

$$\zeta^k = e^{2\pi ik/n}$$

Which has the nth power given as:

$$(\zeta^k)^n = (e^{2\pi ik/n})^n = e^{2\pi ik} = 1$$

The points w_k are just the product of $e^{\frac{i\theta}{n}}$ with all the n-th roots of unity:

$$w_k = e^{i\theta/n}\zeta^k$$

One of the **major results of the theory of complex variables** is to reduce the study of certain functions, including most of the common functions like exponentials, logarithms, sine, cosine; to power series, which can be approximated by polynomials.

Thus the power function is in some sense the unique basic function out of which the others are constructed.

1.3.5 Proofs

1. Let $f(z) = {}^1\!/\!_z$. Describe what f does to the inside and ouside of the unit circle, and also, what it does to points on the uit circle. This map is called **inversion** through the unit circle.

   Proof: Take the points $|z| < 1$ and move them to outside the unit disc.

   $$\begin{aligned} z &< 1 \\ z &\mapsto \frac{1}{z} > 1 \end{aligned}$$

   Then, Take the points outside the unit circle $|z| > 1$ and put them in the unit circle:

   $$\begin{aligned} z &> 1 \\ z &\mapsto \frac{1}{z} < 1 \end{aligned}$$

   ■

2. Let $f(z) = {}^1\!/\!_{\bar{z}}$. Describe f in the same manner as Exercise 1.

   This map is called **reflection** through the unit circele.

   Proof: Take a point outside the unit circle, $|z| > 1$, reflect it across the line of symmetry to $\bar{z}$ and then map it to the same quadrant inside the unit circle:

   $$z \mapsto {}^1\!/\!_{\bar{z}}$$

   Take a point inside the unit circle, $0 < |z| < 1$, reflect it across the line of symmetry to $\bar{z}$ and then map it to the same quadrant outside the unit circle. ■

3. Let $f(z) = e^{2\pi i z}$. Describe the image under f of the set consisting of the points $x + iy$ with:

   $$\frac{-1}{2} \leq x \leq \frac{1}{2}$$

   and

   $$y \geq B := \{z = x + iy \mid y > 0\}$$

   Proof:

4. Let $f(z) = e^z$. Describe th image under f of the following sets:

   (a) The set of $z = x + iy$ such that $x \leq 1$ and $0 \leq y \leq \pi$.

   Proof:

   (b) The set of $z = x + iy$ such that $0 \leq y \leq \pi$ and no condition on x.

   Proof:

1.3.6 Incomplete Proofs

- 3, 4

1.4 Limits and Compact Sets

1.4.1 Limits In $\mathbb{C}$

D 1.14 (Open Disc of radius $r > 0$ centered at α). *The set $S \subset \mathbb{C}$ with elements z such that:*

$$|z - \alpha| < r$$

Denoted $D(\alpha, r)$.

D 1.15 (Open Set $U \subset \mathbb{C}$). *U is an open set if for every point $\alpha \in U$:*

1. *there is a disc $D(\alpha, r)$ centered at α*
2. *there is a disc of some radius $r > 0$ such that this disc $D(\alpha, r)$ is contained in U.*

D 1.16 (Boundary Point of S). *A point α such that every disc $D(\alpha, r)$ centered at α and of radius $r > 0$ contains both points of S and points not in S.*

D 1.17 (Adherent Point to S). *α is adherent to S if:*

- *every disc $D(\alpha, r)$ with $r > 0$ contains some element of S.*

D 1.18 (Interior point to S). *α is an interior point of S if:*

- *there exists a disc $D(\alpha, r)$ which is contained in S.*

Note that:

- An adherent point can be a boundary point
- An adherent point can be an interior point

D 1.19 (Closed Set). *A set is closed if it contains all its boundary points.*

D 1.20 (Bounded Set). *S is bounded if there exists a number $C > 0$ such that:*

$$|z| \leq C \qquad \forall z \in S$$

D 1.21 (Closure of a Set S). *The unioin of S and all its boundary points.*

- *Denoted $\bar{S}$*

D 1.22 (Limit of f).

- Let f be a function on $S \subseteq \mathbb{C}$.
- Let α be an adherent point of S.
- Let w be a complex number.

Say that:

$$z \in S$$

$$\lim_{z \to \alpha} f(z) = w$$

If given $\epsilon > 0$ there exists $\delta > 0$ such that if $z \in S$ and $|z - \alpha| < \delta$, then:

$$|f(z) - w| < \epsilon$$

D 1.23 (Continuous). *f is continuous at α if:*

$$\lim_{z \to \alpha} f(z) = f(\alpha)$$

D 1.24 (**Cauchy Sequence**). *If* $\epsilon > 0$ *then* $\exists\ N$ *such that if:*

$$m, n \geq N$$

then:

$$\mid z_n - z_m \mid < \epsilon$$

1.4.2 Compact Sets

This section is a bit peculiar and full of theorems I've not had a use for yet, but may be useful later. The main point is this:

T 1.5. *A set* S *is compact* $\iff$ *it is closed and bounded.*

1.4.3 Sequence of Complex Numbers (My own section)

D 1.25 (**Sequence of Complex Numbers**). *A function defined on the set of Natural numbers whose range is contained in the set of Complex numbers.*

$$n \to z$$

If a sequence has a limit, it converges. Else, it diverges.

1.4.4 Proofs

1. a. Let α be a complex number of absolute value < 1. What is $\lim_{n\to\infty} \alpha^n$? Proof?

Proof: Just use a Cauchy Sequence.

$|\alpha| < 1$ iff $|\alpha| = \frac{1}{|z|}$.

So then check $\lim_{n\to\infty} |\alpha|^n$:

$$\lim_{n\to\infty} |\alpha|^n = \lim_{n\to\infty} \frac{1}{|z|^n}$$

This sequence has different behavior for $z < 1$ vs $z > 1$, but we must have $|\alpha| < 1$ therefore $z > 1$ only here.

$$\lim_{n\to\infty} |\alpha|^n = \lim_{n\to\infty} \frac{1}{|z|^n}$$

Since $z > 1$ and by the Archimedean principal we can always find $m > n$ then clearly:

$$\alpha^m < \alpha^n \iff \frac{1}{z^m} < \frac{1}{z^n}$$

Therefore this sequence is monotonic, it only decreases each new term, and the difference between m, n terms is small but always > 0 which gives us a cauchy sequence.

So then:

$$|\alpha^n - \alpha^m| < \epsilon$$

■

b. Let α be a complex number of absolute value > 1. What is $\lim_{n\to\infty} \alpha^n$? Proof?

Proof: $\alpha = |z| > 1$ and $\alpha_n = |z|^n$. Looking at the *limit of the sequence* we see it increasing arbitrarily, and so each term is larger than the previous which means we can't ever satisfy our definition of a limit since we *can* choose some m, n that would have some arbitrarily large difference between them $> \epsilon$. ■

2. Show that for any complex number $z \neq 1$, we have

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}$$

If $|z| < 1$, show that

$$\lim_{n\to\infty} (1 + z + \cdots + z^n) = \frac{1}{1 - z}$$

Proof: Now we need to look at the *limit of a sequence of partial sums* and test convergence.

$$\begin{aligned}
S_1 &= 1 \\
S_2 &= 1 + z \\
S_3 &= 1 + z + z^2 \\
\vdots &= \vdots \\
S_n &= 1 + z + z^2 + \cdots + z^n
\end{aligned}$$

Now play with the algebra and multiply both sides by z for shits and giggles

$$\begin{aligned}
zS_n &= z + z^2 + \cdots + z^{n+1} \\
zS_n - S_n &= z + z^2 + \cdots + z^{n+1} - S_n \\
S_n(z-1) &= z + z^2 + \cdots + z^{n+1} - S_n
\end{aligned}$$

The telescoping series starts to look more obvious

$$S_n(z-1) = z + z^2 + \cdots + z^{n+1} - (1 + z + z^2 + \cdots + z^n)$$

Matching up the pairs we see the result we need

$$\begin{aligned}
S_n(z-1) &= z + z^2 + \cdots + z^{n+1} - 1 - z - z^2 - \cdots - z^n) \\
S_n(z-1) &= z^{n+1} - 1 + z - z + z^2 - z^2 + z^3 - z^3 + \cdots + z^n - z^n) \\
\therefore S_n &= \frac{z^{n+1} - 1}{z - 1}
\end{aligned}$$

And now we do as we said, *take the limit of a sequence of partial sums*

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \frac{z^{n+1} - 1}{z - 1}$$

And clearly this sequence diverges if $|z| > 1$ since the numerator grows unbounded over a fixed denominator.

Now consider if $|z| < 1$ and

$$\begin{aligned}
\lim_{n\to\infty} S_n &= \lim_{n\to\infty} \frac{z^{n+1} - 1}{z - 1} \\
&= \frac{-1}{z-1} \\
&= \frac{1}{1-z} \\
\therefore \lim_{n\to\infty}(1 + z + \cdots + z^n) &= \frac{1}{1-z}
\end{aligned}$$

■

3. Let f be the function defined by

$$f(z) = \lim_{n\to\infty} \frac{1}{1 + n^2 z}$$

Show that f is the characteristic function of the set $\{0\}$, that is, $f(0) = 1$, and $f(z) = 0$ if $z \neq 0$.

Proof: Start by plugging in:

$$\begin{aligned} f(z) &= \lim_{n\to\infty} \frac{1}{1+n^2 z} \\ f(0) &= \lim_{n\to\infty} \frac{1}{1+n^2 0} \\ f(0) &= 1 \end{aligned}$$

Now consider all other cases:

$$\begin{aligned} f(+\infty) &= \lim_{n\to\infty} \frac{1}{1+n^2\infty} \\ f(+\infty) &= 0 \\ f(-\infty) &= \lim_{n\to-\infty} \frac{1}{1+n^2(-\infty)} \\ f(-\infty) &= 0 \end{aligned}$$

Then consider $b < 1 < c$:

$$\begin{aligned} f(c) &= \lim_{n\to\infty} \frac{1}{1+n^2 c} \\ f(c) &= 0 \\ f(b) &= \lim_{n\to\infty} \frac{1}{1+n^2 b} \\ f(b) &= 0 \end{aligned}$$

And so the function f satisfies the definition and so can be called the characteristic function of the set $\{0\}$. ■

4. For $|z| \neq 1$ show that the following limit exists:

$$f(z) = \lim_{n\to\infty} \left(\frac{z^n - 1}{z^n + 1}\right)$$

Is it possible to define $f(z)$ when $|z| = 1$ in such a way to make f continuous?

Proof: Take the limits for the cases of z.

First for $z > 1$:

$$\begin{aligned} f(z) &= \lim_{n\to\infty}\left(\frac{z^n-1}{z^n+1}\right) \\ &= \lim_{n\to\infty}\frac{z^n}{z^n}\left(\frac{1-\frac{1}{z^n}}{1+\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{1-\frac{1}{z^n}}{1+\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{-\frac{1}{z^n}}{\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{-z^n}{z^n}\right) \\ &= -1 \end{aligned}$$

For $0 < z < 1$:

$$\begin{aligned} f(z) &= \lim_{n\to\infty}\left(\frac{z^n-1}{z^n+1}\right) \\ &= \lim_{n\to\infty}\frac{z^n}{z^n}\left(\frac{1-\frac{1}{z^n}}{1+\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{1-\frac{1}{z^n}}{1+\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{-\frac{1}{z^n}}{\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{-z^n}{z^n}\right) \\ &= -1 \end{aligned}$$

For $z = 0$

$$\begin{aligned} f(z) &= \lim_{n\to\infty}\left(\frac{0-1}{0+1}\right) \\ &= -1 \end{aligned}$$

For $-1 < z < 0$:

$$\begin{aligned} f(z) &= \lim_{n\to\infty}\left(\frac{z^n-1}{z^n+1}\right) \\ &= \lim_{n\to\infty}\frac{z^n}{z^n}\left(\frac{1-\frac{1}{z^n}}{1+\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{1-\frac{1}{z^n}}{1+\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{-\frac{1}{z^n}}{\frac{1}{z^n}}\right) \\ &= -1 \end{aligned}$$

And lastly $z < -1$:

$$\begin{aligned} f(z) &= \lim_{n\to\infty}\left(\frac{z^n-1}{z^n+1}\right) \\ &= \lim_{n\to\infty}\frac{z^n}{z^n}\left(\frac{1-\frac{1}{z^n}}{1+\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{1-\frac{1}{z^n}}{1+\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{-\frac{1}{z^n}}{\frac{1}{z^n}}\right) \\ &= \lim_{n\to\infty}\left(\frac{-z^n}{z^n}\right) \\ &= -1 \end{aligned}$$

And so the limit exists for $|z| \neq 1$.

Can we make the function continuous? Yes, by defining $f(z)$ as:

$$f(z) = \begin{cases} \lim_{n\to\infty}\left(\frac{z^n-1}{z^n+1}\right) & \text{if } |z| \neq 1 \\ -1 & \text{if } |z| = 1 \end{cases}$$

■

5. Let

$$f(z) = \lim_{n\to\infty}\frac{z^n}{1+z^n}$$

(a) What is the domain of definition of f, that is, for which complex numbers z does the limit exist?

Proof: The domain clearly exists for the cases of $|z| < 1$:

$$\begin{aligned} f(z) &= \lim_{n\to\infty}\frac{z^n}{1+z^n} \\ &= \frac{0}{1+0} \\ &= 0 \end{aligned}$$

And then for $z > -1$:

$$\begin{aligned} f(z) &= \lim_{n\to\infty}\frac{z^n}{1+z^n} \\ &= \frac{-\infty}{1+-\infty} \\ &= 1 \end{aligned}$$

Now to consider $z = -1$. Here we have an issue of never settling on a sign which is a divergent sequence. That is enough to show that a singularity exists at this point, which is to say the limit does not exist at $z = -1$.

$\therefore$ The domain of definition is $(-\infty, -1) \cup (-1, \infty)$. ■

(b) Give explicitly the values of $f(z)$ for the various z in the domain of f.

Proof: See above problem. ■

6. Show that the series
$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})}$$
Converges to
$$\frac{1}{(1-z)^2} \quad |z| < 1$$
and
$$\frac{1}{z(1-z)^2} \quad |z| > 1$$
Prove that the convergence is uniform for $|z| \leq c < 1$ in the first case, and $|z| \geq b > 1$ in the second case.

[*Hint:* multiply and divide each term by $1 - z$, and do a partial fraction decomposition, getting a telescoping effect.]

Proof:

For this problem, just follow along with Lang's suggestion and watch the telescope emerge. Let's look at the nth term for z and see if there's something there.

$$\begin{aligned}
z_n &= \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} \\
z_n \cdot \frac{(1-z)}{(1-z)} &= \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} \\
z_n(1-z) &= (1-z) \cdot \frac{z^{n-1}}{(1-z^n)(1-z^{n+1})} \\
z_n(1-z) &= \frac{(1-z)z^{n-1}}{(1-z^n)(1-z^{n+1})} \\
z_n(1-z) &= \frac{z^{n-1} - z^n}{(1-z^n)(1-z^{n+1})}
\end{aligned}$$

Now use a **partial fraction decomposotion** here:

$$\frac{z^{n-1} - z^n}{(1-z^n)(1-z^{n+1})} = \frac{z^{n-1}}{(1-z^n)} - \frac{z^n}{(1-z^{n+1})}$$

So then

$$\begin{aligned}
z_n(1-z) &= \frac{z^{n-1}}{(1-z^n)} - \frac{z^n}{(1-z^{n+1})} \\
\therefore \quad z_n &= \frac{1}{(1-z)} \left(\frac{z^{n-1}}{1-z^n} - \frac{z^n}{1-z^{n+1}} \right)
\end{aligned}$$

And then we see the terms create telescoping terms between them using this new expression

$$
\begin{aligned}
z_1 &= \frac{1}{(1-z)}\left(\frac{z^{1-1}}{1-z^1} - \frac{z^1}{1-z^{1+1}}\right) \\
&= \frac{1}{(1-z)}\left(\frac{1}{1-z} - \frac{z}{1-z^2}\right) \\
&= \frac{1}{(1-z)^2} - \frac{z}{(1-z)(1-z^2)}
\end{aligned}
$$

And then note

$$
\begin{aligned}
z_2 &= \frac{1}{(1-z)}\left(\frac{z^{2-1}}{1-z^2} - \frac{z^2}{1-z^{2+1}}\right) \\
&= \frac{1}{(1-z)}\left(\frac{z}{1-z^2} - \frac{z^2}{1-z^3}\right) \\
&= \frac{z}{(1-z)(1-z)^2} - \frac{z}{(1-z)(1-z^2)}
\end{aligned}
$$

And so

$$
\begin{aligned}
z_1 + z_2 &= \frac{1}{(1-z)^2} - \frac{z}{(1-z)(1-z^2)} + \frac{z}{(1-z)(1-z)^2} - \frac{z}{(1-z)(1-z^2)} \\
z_1 + z_2 &= \frac{1}{(1-z)^2} - \frac{z}{(1-z)(1-z^2)}
\end{aligned}
$$

And so now we see how the terms will match up and we are left with

$$
\begin{aligned}
z_1 + z_2 + \cdots + z_n &= \frac{1}{(1-z)^2} \\
&- \frac{z}{(1-z)(1-z^2)} + \frac{z}{(1-z)(1-z)^2} \\
&- \frac{z^2}{(1-z)(1-z^2)} + \frac{z^2}{(1-z)(1-z^2)} + \cdots \\
&- \frac{z^n - 1}{(1-z)(1-z^n)} + \frac{z^n - 1}{(1-z)(1-z^n)} - \frac{z^n}{(1-z)(1-z^{n+1})} \\
&= \frac{1}{(1-z)^2} - \frac{z^n}{(1-z)(1-z^{n+1})}
\end{aligned}
$$

And so clearly for the case $|z| < 1$ and taking the $\lim n \to \infty$ the last term goes to 0 and we are left with the desired result.

This finishes the first case of $z < 1$.

Now we need to consider the case $|z| > 1$ and continue on:

$$
\begin{aligned}
z_1 + z_2 + \cdots + z_n &= \frac{1}{(1-z)^2} - \frac{z^n}{(1-z)(1-z^{n+1})} \\
&= \frac{1}{(1-z)^2} - \frac{z^n}{(1-z)(1-z)(z^n)} \\
&= \frac{1}{(1-z)^2} - \frac{1}{(1-z)(1-z)} \\
&= \frac{1}{(1-z)^2} - \frac{1}{(z^2-z)} \\
&= \frac{1}{(1-z)^2} - \frac{1}{z(z-1)} \\
&= \frac{z(z-1)}{z(z-1)(1-z)^2} - \frac{(1-z)^2}{z(z-1)(1-z)^2} \\
&= \frac{z(z-1)-(1-z)^2}{z(z-1)(1-z)^2} \\
&= \frac{z^2-z-[(1-z)(1-2)]}{z(z-1)(1-z)^2} \\
&= \frac{z^2-z-(1-2z+z^2)}{z(z-1)(1-z)^2} \\
&= \frac{z^2-z-1+2z-z^2}{z(z-1)(1-z)^2} \\
&= \frac{z-1}{z(z-1)(1-z)^2} \\
&= \frac{1}{z(1-z)^2}
\end{aligned}
$$

Which finishes the proof. ■

1.5 Complex Differentiability

(There are no exercises in this section.)

(There are no exercises in this section.)

- Let U be an open set.
- Let f be a function on U.

D 1.26 (**f Complex Differentiable at z**). *If the limit exists:*

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

denoted by $f'(z)$ or df/dz.

- *Note:* If f is differentiablre at z then f is continuous at z.

All that really matters in this section is that all the usual rules for sums, products, quotients, and functions of functions are the same wrt complex Differentiability as they were with real Differentiability.

1.5.1 Holomorphic Function

A function f defined on an open set U is said to be **differentiable** if it is differentiable at every point.

- Also say that f is **Holomorphic** on U.
- Holomorphic is usually used to specify *complex* differentiability as distinguised from *real* differentiability.

D 1.27 (**Holomorphic Isomorphism**). *A holomorphic function*

$$f : U \to V$$

*From an open set into another open set is a **holomorphic isomorphism** if there exists a holomorphic function*

$$g : V \to U$$

such that g is the inverse of f. That is;

$$g \circ f = id_u \quad and \quad f \circ g = id_v$$

D 1.28 (**Holomorphic Automorphism**). *A holomorphic isomorphism of an open set U with itself.*

1.5.2 Proofs

Nothing here?

1.6 The Cauchy-Reimann Equations

In this section:

- Let f be a function on an open set U.
- Write f in terms of its *real* and *imaginary* parts.

$$f(x+iy) = u(x,y) + iv(x,y)$$

- Derive the equivalent conditions on u and v for f to be holomorphic.

At a fixed z:

- let $f'(z) = a + bi$.
- let $w = h + ik \quad h, k \in \mathbb{R}$
- Suppose:

$$\lim_{w \to 0} \sigma(w) = 0$$
$$f(z+w) - f(z) = f'(z)w + \sigma(w)w$$

- Let:

$$\vec{F} : U \to \mathbb{R}^2$$

- such that:

$$\vec{F}(x,y) = (u(x,y), v(x,y))$$

- Call $\vec{F}$ the (real) **Field Associated with** f.
- If one assumes that f is holomorphic then $\vec{F}$ is differentiable, and its derivative is represented by the **Jacobian Matrix**:

$$J_{\vec{F}}(x,y) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

- This shows:

$$f'(z) = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y}$$

This all culminates to the incredibly important result of:

D 1.29 (Cauchy-Riemann Equations).

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \textit{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Last bit over Jacobian Determinant $\triangle_{\vec{F}}$ needed.

1.6.1 Proofs

1. Prove in detail that if u, v satisfy the Cauch-Riemann equations, then the function

$$f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

is holomorphic.

Proof. Let u, v satisfy the Cauchy-Riemann equations. The aim is to show that since the equations satisfy cauchy-riemann, then they *must* be everywhere differentiable and therefore holomorphic, or else they could not satisfy the equation. ■

1.6.2 Incomplete Proofs

Only proof is incomplete, but you just need to work backwards from the proof in the section to arrive at the result.

1.7 Angles Under Holomorphic Maps

(There are no exercises in this section.)

What is important here is a simple geometric property of holomorphic maps. Roughly speaking, they preserve angles.

Chapter 2

Power Series

We've already been dancing around with these in some previous proofs, now let's really dig in.

2.1 Formal Power Series

D 2.1 (Formal Power Series). *Using a neutral letter T:*

$$\sum_{n=0}^{\infty} a_n T^n = a_0 + a_1 T + a_2 T^2 + \cdots$$

The important part of this definition are the *coefficients* $a_0, a_1, a_2, ...$ which we take as complex numbers.

- You could think of this as *a map from the integers ≥ 0 to the complex numbers.*

$$n \mapsto a_n$$

Main points:

- if you wanna compose functions and maps, do it term by term with their series expansions.
- Often when doing all this you can even get a telescoping series and then arrive at a closed form of the expression.

For the below definitions refer to the **formal expression of a power series:**

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} a_n T^n \\ &= a_0 + a_1 T + a_2 T^2 + \cdots \end{aligned}$$

D 2.2 (Constant Term). *The leading term of f denoted a_0.*

D 2.3 (Order r of f). *r is the smallest integer n such that $a_n \neq 0$*

$$r = \text{ord } f$$

T 2.1. *Any power series has order 0 if and only if it starts with constant term $\neq 0$.*

T 2.2. *ord fg = ord f + ord g*

D 2.4 (Inverse). *Let* $g = \sum b_n T^n$

$$gf = 1$$

This leads to the following Theorem:

T 2.3. *If f has a non-zero constant term, then f has an inverse as a power series.*

2.1.1 Proofs

1. Give the terms of order ≤ 3 in the power series: The trick for all of these is to just use their power series expansions or use ones we've found and stuff more fs into them to get out more that one needs.

   (a) $e^z \sin(z)$

   (b) $\sin z \cos z$

   (c) $\frac{e^z - 1}{z}$

   (d) $\frac{e^z - \cos z}{z}$

   (e) $\frac{1}{\cos z}$

   (f) $\frac{\cos z}{\sin z}$

   (g) $\frac{\sin z}{\cos z}$

   (h) $\frac{e^z}{\sin z}$

2. Let $f(z) = \sum a_n z^n$.

   Define $f(-z) = \sum a_n (-z)^n = \sum a_n (-1)^n z^n$.

   Define f as **even** if $a_n = 0$ for n odd.

   Define f as **odd** if $a_n = 0$ for n even.

   Verify that f is even if and only if $f(-z) = f(z)$ and f is odd if and only if $f(-z) = -f(z)$.

   Proof: Use power series expansions. Proving the **even** case first, but the same type of argument will work for the odd.

   If f is **even** then n is odd, so:

$$
\begin{aligned}
f(-z) &= \sum a_n z^n (-1)^n \\
&= a_1 z(-1) + a_2 z^2 (-1)^2 + a_3 z^3 (-1)^3 \\
&+ a_4 z^4 (-1)^4 + a_5 z^5 (-1)^5 + a_6 z^6 (-1)^6 + \cdots + a_n z^n (-1)^n
\end{aligned}
$$

   Recall that since this is **even** then all the $a_n = 0$ for odd n which leaves us with (note $k \in \mathbb{N}$ here):

$$f(-z) = a_2 z^2 (-1)^2 + a_4 z^4 (-1)^4 + a_6 z^6 (-1)^6 + \cdots + a_n z^{2k} (-1)^{2k}$$

   Noticing that all of the factors of (-1) have the form $(-1)^{2k}$ one sees:

$$f(-z) = f(z)$$

   Conversely, if you assume $f(-z) = f(z)$ then f is **even** simply by definition, which completes the proof for the **even** case.

   A similar argument can be applied for the **odd** case. ■

3. Define the **Bernoulli numbers** B_n by the power series:

$$\frac{z}{e^z - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} z^n$$

Prove the recursion formula:

$$\frac{B_0}{n!\,0!} + \frac{B_1}{(n-1)!\,1!} + \cdots + \frac{B_{n-1}}{1!\,(n-1)!} + = \begin{cases} 1 & \text{if } n = 1. \\ 0 & \text{if } n > 1. \end{cases}$$

Then $B_0 = 1$.

Compute B_1, B_2, B_3, B_4.

Show that $B_n = 0$ if n is odd and $n \neq 1$.

Proof: First, transform the given function into a geometric series, then expand the series to find the coefficients a_n for the terms, which will correspond to $a_n n! = B_n$ in the recursion formula.

$$\begin{aligned}
\frac{z}{e^z - 1} &= \frac{z}{-1 + e^z} \\
&= \frac{z}{-1 + \left(1 + z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}\right)} \\
&= \frac{z}{\left(z + \frac{z^2}{2!} + \cdots + \frac{z^n}{n!}\right)} \\
&= \frac{1}{\left(1 + \frac{z}{2!} + \cdots + \frac{z^{n-1}}{n!}\right)} \\
&= \frac{1}{1 - \left(-\frac{z}{2!} + \cdots + \frac{z^{n-1}}{n!}\right)} \\
&= 1 + \left(-\frac{z}{2!} + \cdots + \frac{z^{n-1}}{n!}\right) + \left(-\frac{z}{2!} + \cdots + \frac{z^{n-1}}{n!}\right)^2 + \cdots + \text{higher terms}
\end{aligned}$$

Now use the terms above to solve for the needed coefficients.

Then:

$$a_0 = 1$$

There is only one term with base z which is $\frac{-1}{2!}$. So:

$$a_1 = -1/2$$

For a_2 check the z^2 base's coefficients.

Distribute out the squared term to get the z^2 based coefficients:

$$\left(-\frac{z}{2!} + \cdots + \frac{z^{n-1}}{n!}\right)^2 = \left(-\frac{z}{2!} + \cdots + \frac{z^{n-1}}{n!}\right)\left(-\frac{z}{2!} + \cdots + \frac{z^{n-1}}{n!}\right)$$

Distribute out and notice how we need nothing above z^2 and can stop once we have all its terms:

$$\begin{aligned}\left(-\frac{z}{2!}+\cdots+\frac{z^{n-1}}{n!}\right)^2 &= \left(-\frac{z}{2!}+\cdots+\frac{z^{n-1}}{n!}\right)\left(-\frac{z}{2!}+\cdots+\frac{z^{n-1}}{n!}\right)\\ &= \left(-\frac{z^2}{2!2!}+(\text{terms} > z^2)\right)\end{aligned}$$

Combining the matching based terms results in:

$$\begin{aligned}a_2 &= \frac{-1}{6}+\frac{1}{4}\\ &= \frac{1}{12}\end{aligned}$$

Keeping this up results in the first four coefficients coming out to:

$$\begin{aligned}a_0 &= 1\\ a_1 &= \frac{-1}{2}\\ a_2 &= \frac{1}{12}\\ a_3 &= \frac{-1}{24}+\frac{1}{12}+\frac{1}{12}-\frac{1}{8}\\ &= 0\end{aligned}$$

Now using the relationship given the first four Bernoulli Numbers B_n come out to:

$$\begin{aligned}B_n &= a_n \cdot n!\\ B_0 &= 1 \cdot 0!\\ B_1 &= \frac{-1}{2}\cdot 1!\\ B_2 &= \frac{1}{12}\cdot 2!\\ &= \frac{1}{6}\\ B_3 &= 0 \cdot 3!\\ &= 0\end{aligned}$$

Plugging these into the recursion results in a true statements. So, the base case is proved, assume true for n, and now show it's true for $n+1$.

4. Show that:

$$\frac{z}{2}\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}}=\sum_{n=0}^{\infty}\frac{B_n}{(2n)!}z^{2n}\ .$$

Replace z by $2\pi iz$ to show that:

$$\pi z\cot(\pi z)=\sum_{n=0}^{\infty}(-1)^n\frac{(2\pi)^{2n}}{(2n)!}z^{2n}B_{2n}\ .$$

5. Express the power series for $\tan z$, $\frac{z}{\sin z}$, $z\cot z$, in terms of **Bernoulli numbers**.
6. (**Difference Equation**) given complex numbers a_0, a_1, u_1, u_2 define a_n for $2 \leq n$ by:

$$a_n = u_1 a_{n-1} + u_2 a_{n-2}\ .$$

If you have a factorization: **STOPPED**

2.2 Convergent Power Series

2.2.1 Sequences and Series of Complex Numbers

D 2.5 (Sequence).

$$\{z_n\} \;\; or \;\; z_n$$

D 2.6 (Series).

$$\sum_{n=1}^{\infty} z_n$$

D 2.7 (Partial Sum).

$$s_n = \sum_{k=1}^{n} z_k = z_1 + z_2 + \cdots z_n$$

2.2.2 Convergence of Complex Numbers

D 2.8 (Convergent Series). *The series s_n converges if there is some $w \in \mathbb{C}$ such that the following limit exists:*

$$\lim_{n\to\infty} s_n = w$$

In which case we say that w is equal to the *sum of the series.*

D 2.9 (Sum of the Series). *Formally:*

$$w = \sum_{n=1}^{\infty} z_n$$

Now consider and deal with questions of *uniformity.*

2.2.3 Sequences of Functions

Suppose:

- $S \subseteq \mathbb{C}$
- f is a *bounded function* on S
- $n \in \mathbb{N}$

D 2.10 (Sequence).

$$\{f_n\} \; or \; f_n$$

D 2.11 (Sup norm). *Let sup be the least upper bound, then:*

$$\|f\|_S = \|f\| = \sup_{z\in S} |f(z)|$$

2.2.4 Uniform Convergence of Complex Functions

D 2.12 (Uniform Convergence). *The sequence f_n converges uniformly on S if there exists a function f on S satisfying the following properties.*

Given ϵ, there exists N such that if $n \geq N$ then:

$$\|f_n - f\| < \epsilon$$

D 2.13 (Cauchy Sequence). *We call f_n Cauchy if given ϵ there exists N such that if $m, n \geq N$ then:*

$$\|f_n - f_m\| < \epsilon$$

T 2.4. *If a sequence f_n of functions on S is Cauchy, then f_n converges uniformly.*

T 2.5. *If the functions f_n in the theorem are bounded, then the limiting function f is bounded.*

2.2.5 Series of Functions

D 2.14 (Partial Sum).

$$s_n = \sum_{k=1}^{n} f_k = f_1 + f_2 + \cdots f_n$$

D 2.15 (Uniform Convergence of Series). *The series $\sum_{k=1}^{n} f_k$ converges uniformly if the sequence of partial sums s_n converges uniformly.*

D 2.16 (Absolute Convergence of Series). *A series $\sum f_n$ converges absolutely if $\forall z \in S$ the following series converges:*

$$\sum |f_n(z)|$$

T 2.6 (Comparison Test). *Let c_n be a sequence of real numbers ≥ 0 and assume that*

$$\sum c_n$$

converges. Let f_n be a sequence of functions on S such that $||f_n|| \leq c_n$ for all n. Then $\sum f_n$ converges uniformly and absolutely.

Power Series

Now consider the functions f_n in the form

$$a_n \in \mathbb{C} \quad n \in \mathbb{N}$$

$$f_n(z) = a_n z^n$$

T 2.7 (Absolute Convergence). *Let a_n be a sequence of complex numbers, and let r be a number > 0 such that the series*

$$\sum |a_n| r^n$$

converges. Then for $|z| \leq r$ the series $\sum a_n z^n$ converges absolutely and uniformly.

T 2.8 (Existence of the Radius of Convergence). *Let $\sum a_n z^n$ be a power series. If it does not converge absolutely for all z, then there exists a number r such that the series converges absolutely for $|z| < r$ and does not converge for $|z| > r$.*

D 2.17 (Radius of Convergence r). *The number $r \neq |z|$ such that $\sum a_n z^n$ converges for $|z| < r$.*

- *If the power series converges for all z then the radius of convergence is infinity.*
- *If the radius of convergence is $r = 0$ then the series converges absolutely only for $z = 0$.*
- *The radius of convergence can be determined by the coefficients.*

D 2.18 (Convergent Power Series). *Any power series that has a non-zero radius of convergence.*

D 2.19 (Power Series Converging on a Disk D). *If D is a disk centered at the origin contained in the disk $\mathbb{D}(0, r)$, where r is the radius of convergence for some power series g, then g converges on D.*

D 2.20 (Point of Accumulation). *Given a sequence t_n of real numbers, the number t is the point of accumulation of t_n if given ϵ, there exist infinitely many indices n such that*

$$|t_n - t| < \epsilon$$

So, infinitely many points of the sequence lie in a given interval centered at t. The Weierstrass-Balzano theorem proves that, *in the reals*, every bounded sequence has a point of accumulation. There can be many points of accumulation as well.

Assume

- t_n is *bounded*
- Let S be the set of *points of accumulation*

D 2.21 (Limit Superior: lim sup t_n**).** *The least upper bound of S.*

Let $\lambda \in S$ be this least upper bound, then one writes: $\lambda =$ lim sup $t_n =$ least upper bound of S.

Notice that λ has the following properties:

T 2.9 (Epsilon Around Lamda). *Given ϵ, there exist only finitely many n such that*

$$t_n \geq \lambda + \epsilon$$

There exist infinitely many n such that

$$t_n \geq \lambda - \epsilon$$

2.2.6 Proofs

2.3 Relations Between Formal and Convergent Series

Sums and Products

Let $f = f(T)$ and $g = g(t)$ be formal power series.

If f converges absolutely for some complex number z, then we have the value $f(z)$ and similarly for $g(z)$.

And so the big theorem here is that if f and g converge on the disc, then so do their sum and products.

There are no definitions in this section.

2.3.1 Proofs

2.4 Analytic Functions

2.4.1 Proofs

2.5 Differentiation of Power Series

2.5.1 Proofs

2.6 The Inverse and Open Mapping Theorems

2.6.1 Proofs

2.7 The Local Maximum Modulus Principle

2.7.1 Proofs

Chapter 3

Cauchy's Theorem, First Part

3.1 Holomorphic Functions on Connected Sets

Let $[a, b]$ be a closed interval of real numbers.

D 3.1 (Curve). *A function of class C^1 defined on $[a, b]$ with range $\mathbb{C}$*

$$\gamma : [a, b] \to \mathbb{C}$$

3.1.1 proofs

3.2 Integrals Over Paths

3.2.1 proofs

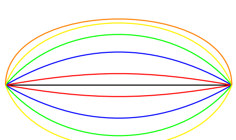
3.3 Local Primitive for a Holomorphic Function

3.3.1 proofs

3.4 Anothe Description of the Integral Along a Path

3.4.1 proofs

3.5 The Homotopy Form of Cauchy's Theorem



3.5.1 proofs

3.6 Existence of Global Primitives. Definition of the Logarithm

3.6.1 proofs

3.7 The Local Cauchy Formula

3.7.1 proofs