

# Notes Over Serge Lang's Complex Analysis: 4th Edition

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## Complex Numbers and Functions

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**Assume:**  $\{z_n\}$  is a complex valued sequence with  $n, m, N \in \mathbb{N}$ .

### 1.1 Complex Numbers

#### 1.1.1 Definitions

**Defn 1.1 (Complex Numbers).** *A set of objects that can be added and multiplied together and produce another element of the set under the following conditions:*

1. *Every real number is a complex number, and if  $\alpha, \beta \in \mathbb{R}$ , then their sum and product as complex numbers are the same as their sum and products as real numbers.*
2. *There is a complex number denoted  $i$  such that  $i^2 = -1$ .*
3.  *$\forall z \in \mathbb{C}$  with  $a, b \in \mathbb{R}$  can be written uniquely as:*

$$z = a + bi$$

4. *The ordinary laws of arithmetic for addition and multiplication are satisfied  $\forall z \in \mathbb{C}$ :*
  - distributive law holds*
  - associative law holds*
  - commutative law holds*
  - if  $1 \in \mathbb{R}$  then  $1z = z$*
  - if  $0 \in \mathbb{R}$  then  $0z = 0$*
  - $z + (-1)z = 0$*

**Defn 1.2 (Conjugate).**  $\bar{z} \in \mathbb{C}$  such that:

$$z = a + bi$$

iff

$$\bar{z} = a - bi$$

**Defn 1.3 (Inverse).**  $z^{-1} \in \mathbb{C}$  such that

$$z \cdot z^{-1} = 1$$

**Defn 1.4 (Absolute Value  $|z|$  of  $z$ ).**

$$|z| = \sqrt{a^2 + b^2}$$

**Thm 1.1.**  $|z|$  satisfies the following properties. If  $\alpha, \beta \in \mathbb{C}$ , then:

$$|\alpha\beta| = |\alpha||\beta|$$

$$|\alpha + \beta| \leq |\alpha| + |\beta| \quad (\text{triangle inequality})$$

## 1.1.2 Proofs

1. Express the following complex numbers in the form  $x + iy$ , where  $x, y$  are real numbers.

(a)  $(-1 + 3i)^{-1}$

**Proof:**

Simply invert and separate, then use the conjugate/symmetry to rationalize the statement:

$$\begin{aligned} (-1 + 3i)^{-1} &= \frac{1}{(-1 + 3i)} \\ \frac{1}{(-1 + 3i)} &= \frac{1}{(-1 + 3i)} \frac{(-1 - 3i)}{(-1 - 3i)} \\ &= \frac{-1 - 3i}{1 + 3i - 3i + 9} \\ &= \frac{-1 - 3i}{10} \\ \therefore (-1 + 3i)^{-1} &= -\frac{1}{10} - \frac{3i}{10} \end{aligned}$$

Which just means from the origin of  $\mathbb{C}$  go left 1 then down 3 then shrink by  $\frac{1}{10}$  and that's the  $z$  you're at in  $\mathbb{C}$ . ■

(b)  $(1 + i)(1 - i)$

**Proof:**

Distribute and collect:

$$\begin{aligned} (1 + i)(1 - i) &= 1 - i + i - i^2 \\ &= 1 - (-1) \\ &= 2 \\ \therefore (1 + i)(1 - i) &= 2 + 0i \end{aligned}$$

■

(c)  $(i + 1)(i - 2)(i + 3)$

**Proof:**

Distribute collect, distribute and collect again.

$$\begin{aligned} (i + 1)(i - 2)(i + 3) &= (i^2 - 2i + i - 2)(i + 3) \\ &= (-i - 3)(i + 3) \\ &= (1 - 3i - 3i - 9) \\ \therefore (i + 1)(i - 2)(i + 3) &= -8 - 6i \end{aligned}$$

■

2. Express the following complex numbers in the form  $x + iy$ , where  $x, y$  are real numbers.

(a)  $(1 + i)^{-1}$

**Proof:**

More of the same, just use the conjugate to solve these like problem 1 above.

$$\begin{aligned}
 (1+i)^{-1} &= \frac{1}{1+i} \\
 &= \frac{1}{1+i} \frac{(1-i)}{(1-i)} \\
 &= \frac{1-i}{(1+i)(1-i)} \\
 &= \frac{1-i}{1-i+i-i^2} \\
 &= \frac{1-i}{1-(-1)} \\
 &= \frac{1-i}{2} \\
 \therefore (1+i)^{-1} &= \frac{1}{2} - \frac{i}{2}
 \end{aligned}$$

■

3. Let  $\alpha$  be a complex number  $\neq 0$ . What is the absolute value of  $\alpha/\bar{\alpha}$ ? What is  $\bar{\bar{\alpha}}$ ?

**Proof:**

First note that:

$$\begin{aligned}
 \alpha &= x + yi \\
 \bar{\alpha} &= x - yi \\
 \therefore \left| \frac{\alpha}{\bar{\alpha}} \right| &= \frac{x + yi}{x - yi}
 \end{aligned}$$

Now some algebra:

$$\begin{aligned}
 \frac{x + yi}{x - yi} &= \frac{(x + yi)(x + yi)}{(x - yi)(x + yi)} \\
 &= \frac{x^2 + 2xyi + y^2i^2}{x^2 - y^2i^2} \\
 &= \frac{x^2 + 2xyi + y^2i^2}{x^2 + y^2} \\
 &= \frac{x^2 + 2xyi - y^2}{x^2 + y^2} \\
 &= \frac{x^2 + 2xyi - y^2}{|\bar{\alpha}|^2} \\
 &= \frac{(x + yi)(x - yi)}{|\bar{\alpha}|^2} \\
 \therefore \left| \frac{\alpha}{\bar{\alpha}} \right| &= \frac{\alpha \cdot \bar{\alpha}}{|\bar{\alpha}|^2}
 \end{aligned}$$

■

**Part 3b:** What is  $\bar{\bar{\alpha}}$ ?

**Proof:**

Note

$$\begin{aligned}\alpha = x + yi &\Leftrightarrow \bar{\alpha} = x - yi. \\ \therefore \bar{\bar{\alpha}} &= \overline{x - yi}\end{aligned}$$

So now because the conjugate operation just changes the sign on the *imaginary* part of  $\alpha$  we have the straightforward result of:

$$\begin{aligned}\bar{\bar{\alpha}} &= \overline{x - yi} \\ &= x + yi \\ \therefore \bar{\bar{\alpha}} &= \alpha\end{aligned}$$

■

4. Let  $\alpha, \beta$  be two complex numbers. Show that:

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$$

and that:

$$\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$$

**Proof:**

First is easy since we just distribute out  $\alpha \cdot \beta$  and gather reals and imaginary parts together and see it is the same result as if we had simply taken the conjugate of each component.

Algebraically, with  $\alpha_n, \beta_n, \rho \in \mathbb{R}$  :

$$\begin{aligned}\overline{\alpha\beta} &= \overline{(\alpha_1 + \alpha_2 i)(\beta_1 + \beta_2 i)} \\ &= \overline{(\alpha_1\beta_1 + \alpha_1\beta_2 i + \beta_1\alpha_2 i + \alpha_2\beta_2 i^2)} \\ &= \overline{(\alpha_1\beta_1 + i(\alpha_1\beta_2 + \beta_1\alpha_2) + \alpha_2\beta_2 i^2)} \\ &= \overline{(\alpha_1\beta_1 + \alpha_2\beta_2 i^2 + i(\alpha_1\beta_2 + \beta_1\alpha_2))} \\ &= \overline{(\alpha_1\beta_1 - \alpha_2\beta_2 + i(\alpha_1\beta_2 + \beta_1\alpha_2))} \\ &= \overline{\rho_1 + i\rho_2} \\ \overline{\alpha\beta} &= \rho_1 - i\rho_2\end{aligned}$$

Now we go the other way:

$$\begin{aligned}\bar{\alpha}\bar{\beta} &= \overline{(\alpha_1 + \alpha_2 i)} \cdot \overline{(\beta_1 + \beta_2 i)} \\ &= (\alpha_1 - \alpha_2 i) \cdot (\beta_1 - \beta_2 i) \\ &= (\alpha_1\beta_1 - \alpha_1\beta_2 i - \alpha_2\beta_1 i + \alpha_2\beta_2 i^2) \\ &= (\alpha_1\beta_1 - \alpha_1\beta_2 i - \alpha_2\beta_1 i - \alpha_2\beta_2) \\ &= (\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_1\beta_2 i - \alpha_2\beta_1 i) \\ &= (\alpha_1\beta_1 - \alpha_2\beta_2 - i(\alpha_1\beta_2 + \alpha_2\beta_1)) \\ \bar{\alpha}\bar{\beta} &= \rho_1 - i\rho_2\end{aligned}$$

$$\therefore \overline{\alpha\beta} = \bar{\alpha}\bar{\beta} \quad \blacksquare$$

Second is easier as we only convert the sign inside the complex numbers, and do nothing with the operation between the two complex numbers, only on the reals in the number. Again, basically just some algebra of converting to the real and imaginary parts and gathering terms. ■

5. Justify the assertion that the real part of a complex number is  $\leq$  its absolute value.

**Proof:**

The value can be equal to the absolute value if it happens to be positive, in which case it coincides with the absolute value.

Or, it can be the symmetric partner if it is negative and therefore equal in magnitude but opposite in direction, therefore ordered as  $\leq$  the absolute value by definition of well-ordering in  $\mathbb{R}$ . Because the reals are symmetric like my god damned shoes! ■

6. If  $\alpha = a + ib$  with  $a, b \in \mathbb{R}$  then  $b$  is called the **imaginary part** of  $\alpha$  and we write:

$$\Im(\alpha) = b.$$

- (a) Show that:

$$\alpha - \bar{\alpha} = 2i \Im(\alpha)$$

**Proof:**

Just do the algebra:

$$\begin{aligned} \alpha - \bar{\alpha} &= (a + ib) - (a - ib) \\ &= 2ib \\ \therefore \alpha - \bar{\alpha} &= 2i \Im(\alpha) \end{aligned}$$

■

- (b) Show that:

$$\Im(\alpha) \leq |\Im(\alpha)| \leq |\alpha|$$

**Proof:**

Again, with some algebra we see the answer by considering the case of the imaginary part being either positive or negative while the absolute value will always be positive and therefore will be equal to this value or greater than it if it is negative.

Next, think of whether part of  $\alpha$  along just the real part or that part plus another would always make it larger than or equal to? If they always have the same imaginary part, then adding a real only increases the size of  $\alpha$  while leaving the imaginary part at its maximum value. I'm too lazy to type this out right now, maybe later. ■

7. Find the real and imaginary parts of  $(1 + i)^{100}$ .

**Proof:**

Working with a base of  $(1 + i)$  we just find useful factors to work with:

$$\begin{aligned}
 (1 + i)^2 &= 2i \\
 (1 + i)^4 &= 2i^2 \\
 &= -4 \\
 (1 + i)^{10} &= (1 + i)^4(1 + i)^4(1 + i)^2 \\
 &= (-4)(-4)(2i) \\
 &= 32i
 \end{aligned}$$

Now just plug and play:

$$\begin{aligned}
 (1 + i)^{100} &= ((1 + i)^{10})^{10} \\
 &= (32i)^{10} \\
 &= i^{10} 32^{10} \\
 &= -(32)^{10}
 \end{aligned}$$

$$\therefore (1 + i)^{100} = -(32)^{10} + 0i$$

■

8. Prove that for any two complex numbers  $z, w$  we have:

$$(a) \quad |z| \leq |z - w| + |w|$$

**Proof:**

Consider the three cases we could have have:

$$w < 0$$

$$w = 0$$

$$w > 0$$

If  $w < 0$ :

$$\begin{aligned}
 |z - (-w)| + |-w| &= |z + w| + |w| \\
 \therefore z &< |z - w| + |w|
 \end{aligned}$$

If  $w = 0$ :

$$\begin{aligned}
 |z - w| + |w| &= |z - 0| + |0| = z \\
 \therefore z &= |z - w| + |w|
 \end{aligned}$$

If  $w > 0$ , with  $z_w + w = z$ :

$$\begin{aligned}
 |z - w| + |w| &= z_w + |w| = z \\
 \therefore z &= |z - w| + |w|
 \end{aligned}$$

By these three cases combined we have:

$$|z| \leq |z - w| + |w|$$

■



(b)  $|z| - |w| \leq |z - w|$

**Proof:**

By (a) above we just subtract  $|w|$  off the right and left, and have a logically equivalent statement. ■

(c)  $|z| - |w| \leq |z + w|$

**Proof:**

If the above were not true, then (b) would be false, but (b) is true, so then:

$$|z| - |w| \leq |z + w|$$

■

9. Let  $\alpha = a + ib$  and  $z = x + iy$ . Let  $c \in \mathbb{R} > 0$ . Transform the condition:

$$|z - \alpha| = c$$

into an equation involving only  $x, y, a, b$  and  $c$ , and describe in a simple way what geometric figure is represented by this equation.

10. Describe geometrically the sets of points  $z$  satisfying the following conditions:

(a)  $|z - i + 3| = 5$

The perimeter of the circle that has a radius of 5 and with an origin  $|z - i + 3|$  from the origin of  $\mathbb{C}$ .

(b)  $|z - i + 3| > 5$

The complex plane outside a set that sits  $|z - i + 3|$  from the origin of  $\mathbb{C}$  with a radius of 5 with no points *on* the perimeter of the radius.

(c)  $|z - i + 3| \leq 5$

The disc of points in  $\mathbb{C}$  centered at  $|z - i + 3|$  with a radius of 5.

(d)  $|z + 2i| \leq 1$

The disc of radius 1 that is centered at  $z$  and moved vertically by  $2i$ .

(e)  $\Im(z) > 0$

The set of points in  $\mathbb{C}$  not including 0 that have real parts = 0 and imaginary parts  $> 0$ , so the  $y$ -axis.

(f)  $\Im(z) \geq 0$

The set of points along the positive axis of  $\mathbb{C}$  including 0.

(g)  $\Re(z) > 0$

The set of points along the positive axis of  $\mathbb{R} \subset \mathbb{C}$  not including 0.

(h)  $\Re(z) \geq 0$

The set of points along the positive axis of  $\mathbb{R} \subset \mathbb{C}$  including 0.

## 1.2 Polar Form

### 1.2.1 Definitions

Let  $z = x + iy$ .

**Defn 1.5 (Polar Coordinates).** An ordered pair  $(r, \theta)$  with  $r = \text{radius}$  and  $\theta$  rotating from the  $x$ -axis such that:

1.  $r \in \mathbb{R}$  and  $r = |z| = \sqrt{x^2 + y^2}$ .
2.  $\theta \in [0, 2\pi]$ .

**Defn 1.6 (Polar Form).**

$$\begin{aligned} re^{i\theta} &= r \cos \theta + ir \sin \theta \\ &= x + iy \\ \therefore re^{i\theta} &\in \mathbb{C} \end{aligned}$$

Note that:

$$x = r \cos \theta, \quad y = r \sin \theta$$

$$\theta = \cos^{-1} \left( \frac{x}{r} \right), \quad \theta = \sin^{-1} \left( \frac{y}{r} \right)$$

### 1.2.2 Factors of Pi

**Defn 1.7 (Factors of Pi).** "If you don't know your factors of pi you don't know squat"

$$\begin{aligned} \forall k \in \mathbb{N} \\ e^{2k\pi i} &= 1 \end{aligned}$$

$$\begin{aligned} \text{If } e^z &= e^w \\ \text{then } z &= w + 2k\pi i \end{aligned}$$

$$\begin{aligned} e^{i\pi} &= -1 + i0 = (-1, 0) \\ e^{2i\pi} &= 1 + i0 = (1, 0) \\ e^{i\pi/2} &= 0 + i = (0, 1) \\ e^{i\pi/3} &= \frac{1}{2} + i\frac{\sqrt{3}}{2} = \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \\ e^{i\pi/4} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \left( \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2} \right) \\ e^{i\pi/6} &= \frac{\sqrt{3}}{2} + i\frac{1}{2} = \left( \frac{\sqrt{3}}{2}, \frac{1}{2} \right) \end{aligned}$$

Now take sums or use polar form above to find the  $(x, y)$  coordinates.

**Thm 1.2.** Let  $\theta, \varphi \in \mathbb{R}$  then:

$$e^{i\theta+i\varphi} = e^{i\theta}e^{i\varphi}$$

**Thm 1.3.** Let  $\alpha, \beta \in \mathbb{C}$  then:

$$e^{\alpha+\beta} = e^{\alpha}e^{\beta}$$

**Thm 1.4** (Thm 1.3 reworded). Let  $z_1 = r_1e^{i\theta}$  and  $z_2 = r_2e^{i\varphi}$  then:

$$z_1 \cdot z_2 = r_1r_2e^{i(\theta+\varphi)}$$

*i.e. multiply the absolute values and add the angles.*

### 1.2.3 Proofs

1. Put the following complex numbers in polar form.

(a)  $z = 1 + i$

Let's just change bases. Note that:

$$\begin{aligned} e^0 &= e^{2i\pi} = 1 \\ e^{i\pi/2} &= i \end{aligned}$$

Then note:

$$\begin{aligned} r = |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{1^2 + 1^2} \\ \therefore r &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \therefore 1 + i &= \sqrt{2}e^{2i\pi}e^{(i\pi/2)} \\ &= \sqrt{2}e^{i\pi/2} \end{aligned}$$

■

(b)  $1 + i\sqrt{2}$

Note:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + \sqrt{2}^2} \\ &= \sqrt{1 + 2} \\ \therefore r &= \sqrt{3} \end{aligned}$$

Previously we selected the factor of  $\pi$  which gave us equal  $x$  and  $y$  pieces, but here something else is going on.

We need to go right along the  $x$ -axis by 1 then up the  $y$ -axis by  $\sqrt{2}$ .

Note that we can normalize these with  $\frac{1}{r}$  or use the Euler formula relating cosine to  $x$  and  $r$  to start.

$$\frac{1}{\sqrt{3}}(1 + i\sqrt{2}) = \frac{1}{\sqrt{3}} + \frac{i\sqrt{2}}{\sqrt{3}}$$

Try the Euler method here instead:

$$1 + i\sqrt{2} = \sqrt{3} \cos \theta + i\sqrt{3} \sin \theta$$

Then

$$\begin{aligned} \frac{x}{r} &= \frac{1}{\sqrt{3}} = \cos \theta \\ \frac{y}{r} &= \frac{\sqrt{2}}{\sqrt{3}} = \sin \theta \end{aligned}$$

(c)  $-3$

Go left on the real line in the complex plane:

$$-3 = 3e^{i\pi}$$

■

(d)  $4i$

Go up by  $4i$  in the complex plane:

$$4i = 4e^{i\pi/2}$$

■

(e)  $1 - i\sqrt{2}$

Go right by 1 and down by  $\sqrt{2}$  in the complex plane:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + \sqrt{2}^2} \\ &= \sqrt{1+2} \\ \therefore r &= \sqrt{3} \end{aligned}$$

And then:

$$\theta = \cos^{-1} \frac{1}{\sqrt{3}}$$

So finally:

$$1 - i\sqrt{2} = \sqrt{3}e^{i\pi \cdot \cos^{-1} \frac{1}{\sqrt{3}}}$$

■

(f)  $5i$

Go up by  $5i$  in the complex plane:

$$5i = 5e^{i\pi/2}$$

■

(g)  $-7$

Go left by  $-7$  in the complex plane:

$$-7 = 7e^{i\pi}$$

■

(h)  $-1 - i$

Go left by  $-1$  and down by  $-1$  in the complex plane:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + 1^2} \\ &= \sqrt{1 + 1} \\ \therefore r &= \sqrt{2} \end{aligned}$$

So then:

$$-1 - i = \sqrt{2}e^{5i\pi/4}$$

■

2. Put the following complex numbers in the ordinary form  $x + iy$ .

(a)  $e^{3i\pi}$

Simple, use Theorems 1.2 and 1.3 and change base!

$$\begin{aligned} e^{3i\pi} &= e^{i(2\pi + \pi)} \\ &= e^{i2\pi} e^{i\pi} \\ \therefore e^{3i\pi} &= (1)(-1) = -1 \end{aligned}$$

■

(b)  $e^{2\pi i/3}$

Use Theorem 1.3 and notice:

$$e^{2\pi i/3} = e^{\pi i/3} e^{\pi i/3}$$

Now check the Factors of Pi for  $e^{\pi i/3}$  to see that we have:

$$e^{\pi i/3} = \left( \frac{1}{2} + \frac{i\sqrt{3}}{2} \right)$$

Then we have:

$$\begin{aligned} e^{2\pi i/3} &= e^{(\pi i/3 + \pi i/3)} \\ &= e^{\pi i/3} e^{\pi i/3} \\ e^{2\pi i/3} &= \left( \frac{1}{2} + \frac{i\sqrt{3}}{2} \right) \left( \frac{1}{2} + \frac{i\sqrt{3}}{2} \right) \\ &= \frac{1}{2} \frac{1}{2} + \frac{1}{2} \frac{i\sqrt{3}}{2} + \frac{1}{2} \frac{i\sqrt{3}}{2} + \frac{i\sqrt{3}}{2} \frac{i\sqrt{3}}{2} \\ &= \frac{1}{4} + \frac{i\sqrt{3}}{4} + \frac{i\sqrt{3}}{4} - \frac{3}{4} \\ &= \frac{1}{4} + \frac{2i\sqrt{3}}{4} - \frac{3}{4} \\ \therefore e^{2\pi i/3} &= -\frac{1}{2} + \frac{i\sqrt{3}}{2} \end{aligned}$$

■

(c)  $3e^{-i\pi/4}$

Here we just see that we are using  $r = |z| = 3$  and moving *downward* from the  $x$ -axis to start with rotation  $\frac{\pi}{4}$ .

$$\therefore 3e^{-i\pi/4} = 3 \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right)$$

■

(d)  $\pi e^{-i\pi/3}$

Here we just have  $r = |z| = \pi$  and rotate *down* again by factor of  $\frac{\pi}{3}$ .

Looking at the Factors of Pi and moving down we get:

$$e^{i\pi/3} = \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right)$$

$$\therefore \pi e^{-i\pi/3} = \pi \left( \frac{1}{2} - i \frac{\sqrt{3}}{2} \right)$$

■

(e)  $e^{2i\pi/6}$

Lets factor, get some basis and then rotate to what we want:

$$e^{2i\pi/6} = e^{i\pi/3}$$

$$\therefore e^{2i\pi/6} = \left( \frac{1}{2} + i \frac{\sqrt{3}}{2} \right)$$

(f)  $e^{-i\pi/2}$

Simple, factor we know but just going in different  $y$  direction.

$$e^{-i\pi/2} = \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right)$$

■

(g)  $e^{-i\pi}$

We know this already, just going in different  $y$  direction.

$$e^{-i\pi} = (-1 + 0i)$$

$$= -1$$

■

(h)  $e^{-5i\pi/4}$

Just break up the factors and use theorem 1.2 and 1.3.

$$\begin{aligned}
 e^{-5i\pi/4} &= e^{-i\pi/4} e^{-4i\pi/4} \\
 &= \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) e^{-i\pi} \\
 &= \left( \frac{\sqrt{2}}{2} - i \frac{\sqrt{2}}{2} \right) (-1) \\
 \therefore e^{-5i\pi/4} &= \left( -\frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right)
 \end{aligned}$$

■

3. Let  $\alpha$  be a complex number  $\neq 0$ . Show there are two distinct complex numbers whose square is  $\alpha$ .

**Proof:**

For any  $z \in \mathbb{C}$  with  $z = a + bi$  we have a symmetric partner such that:

$$\begin{aligned}
 z &= (a + bi) \\
 \bar{z} &= a - bi \\
 \therefore z &\neq \bar{z}
 \end{aligned}$$

So  $z$  and  $\bar{z}$  are distinct.

Now take their square:

$$\begin{aligned}
 z\bar{z} &= (a + bi)(a - bi) \\
 &= a^2 - abi + abi - i^2b^2 \\
 &= a^2 + b^2 \in \mathbb{C} \\
 &= \zeta \in \mathbb{C} \\
 &\neq 0
 \end{aligned}$$

Which demonstrates  $\forall z \neq 0$  and  $\forall z \in \mathbb{C}$  we get:

$$\begin{aligned}
 z &\neq \bar{z} \\
 z\bar{z} &= \zeta \in \mathbb{C} \\
 &\neq 0
 \end{aligned}$$

■

4. Let  $a + bi$  be a complex number. Find real numbers  $x, y$  such that:

$$(x + iy)^2 = a + bi$$

expressing  $x, y$  in terms of  $a$  and  $b$ .

**Proof:**Let  $x, y \in \mathbb{R}$  then:

$$\begin{aligned}
(x + iy)(x + iy) &= x^2 + 2xyi + i^2y^2 \\
&= (x^2 - y^2) + 2xyi \\
&= a + bi \\
\therefore (x + iy)^2 &= a + bi \quad \forall x, y \in \mathbb{R}
\end{aligned}$$

■

5. Plot all the complex numbers  $z$  such that  $z^n = 1$  for  $n = 2, 3, 4$  and 5.Just use  $e^{2\pi i/n}$  to cut the circle up so that you then just tile that slice  $n$  times to get back to 1.

So then:

$$z_n = e^{2\pi i/n}$$

Which would then give:

$$\begin{aligned}
(z_n)^n &= (e^{2\pi i/n})^n \\
&= e^{2\pi i} \\
&= 1 \\
&= (1, 0)
\end{aligned}$$

Which satisfies the first condition. Now to give the plots for the remaining  $z$ 's.Suppose we start with  $n = 2$  :

$$\begin{aligned}
z_2 &= e^{\pi i} \\
&= (-1, 0)
\end{aligned}$$

Now with  $n = 3$  :

$$\begin{aligned}
z_3 &= e^{2\pi i/3} \\
&= \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right)
\end{aligned}$$

For  $n = 4$  :

$$\begin{aligned}
z_4 &= e^{\pi i/2} \\
&= (0, 1)
\end{aligned}$$

For  $n = 5$  :

$$\begin{aligned}
z_5 &= e^{2\pi i/5} \\
&= (\cos(2\pi i/5), \sin(2\pi i/5))
\end{aligned}$$

■



6. Let  $\alpha$  be a complex number  $\neq 0$ . Let  $n$  be a positive integer. Show that there are  $n$  distinct complex numbers  $z$  such that  $z^n = \alpha$ . Write these complex numbers in polar form.

**Proof:**

Suppose:

$$\begin{aligned}\alpha &\in \mathbb{C} \\ \alpha &= \alpha_1 + \alpha_2 i \\ &= |\alpha| e^{i\theta} \\ &= r e^{i\theta} \\ &\neq 0\end{aligned}$$

Now let:

$$\begin{aligned}z_n &\in \mathbb{C} \\ z_n &= r^{1/n} e^{i\theta/n} \\ z_n &\neq 0\end{aligned}$$

And note for some other  $m \in \mathbb{N}$  with  $m \neq n$  :

$$\begin{aligned}z_m &\neq z_n \\ (z_m)^m &= (r^{1/m} e^{i\theta/m})^m \\ &= r e^{i\theta}\end{aligned}$$

Then we have  $\forall n, m \in \mathbb{N}$  with  $m \neq n$  :

$$\begin{aligned}z_m &\neq z_n \\ (z_n)^n &= r^{(1/n)^n} e^{(i\theta/n)^n} \\ (z_m)^m &= r^{(1/n)^m} e^{(i\theta/m)^m} \\ &= r e^{i\theta} \\ &= \alpha\end{aligned}$$

And so we have  $n \in \mathbb{N}$  distinct complex numbers such that:

$$z^n = \alpha$$

■

7. Find the real and imaginary parts of  $i^{1/4}$ , taking the fourth root such that its angle lies in  $[0, \pi/2]$ .

**Proof:**

Change of base and Euler's identity.

$$i = e^{i\pi/2}$$

So then:

$$\begin{aligned}i^{1/4} &= e^{(i\pi/2)^{1/4}} \\ &= e^{(i\pi/8)}\end{aligned}$$

Which then just means  $\theta = \pi/8$  and we can get the real and imaginary parts with:

$$x = \cos(\pi/8) \quad \text{and} \quad y = \sin(\pi/8)$$

$$\therefore i^{1/4} = \cos(\pi/8) + i \sin(\pi/8)$$

■

8. (a) Describe all complex numbers  $z$  such that  $e^z = 1$ .

**Proof:**

Because we have  $e^{2\pi i k} = 1 \quad \forall k \in \mathbb{N}$  we then see this will happen in the cases when  $z = 2i\pi k$

■

- (b) Let  $w$  be a complex number. Let  $\alpha$  be a complex number such that  $e^\alpha = w$ . Describe all complex numbers  $\alpha$  such that  $e^\alpha = w$ .

**Proof:**

This is true for any  $z \in \mathbb{C}$  and  $k \in \mathbb{N}$  such that  $z = \alpha + 2i\pi k$  or else  $e^{2\pi i} \neq 1$ , which is absurd.

■

9. If  $e^z = e^w$  show that there is an integer  $k$  such that  $z = w + 2\pi k i$ .

**Proof:**

Let  $z, w \in \mathbb{C}$  with  $z \neq w$  and  $e^z = e^w$ .

Using the results of problem 8.b above, we see this can only be true when  $z = w + 2\pi i k$  which means  $\exists k \in \mathbb{N}$ .

■

10. (a) If  $\theta$  is real, show that:

$$\cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

**Proof:**

Just expand the exponentials to the polar coordinates of  $\sin$  and  $\cos$  and then do the algebra.

Like the following:

$$\begin{aligned} e^{i\theta} &= \cos \theta + i \sin \theta \\ e^{-i\theta} &= \cos \theta - i \sin \theta \end{aligned}$$

Then we have:

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= \cos \theta + i \sin \theta + \cos \theta - i \sin \theta \\ &= \cos \theta + \cos \theta \\ \therefore e^{i\theta} + e^{-i\theta} &= 2 \cos \theta \end{aligned}$$

We see a similar argument doing it for  $\sin \theta$ .

■

- (b) For arbitrary complex  $z$ , suppose we define  $\cos z$  and  $\sin z$  by replacing  $\theta$  with  $z$  in the above formula. Show that the only values of  $z$  for which  $\cos z = 0$  and  $\sin z = 0$  are the usual real values from trigonometry.

**Proof:**

11. Prove that for any complex number  $z \neq 1$  we have

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}$$

**Proof:**

Strategy: Just add the  $z^{n+1}$  term to both sides and notice that you end up back to the original statement on the right.

Base case: let  $n = 1$  and:

$$\begin{aligned} \frac{z^{1+1} - 1}{z - 1} &= \frac{z^2 - 1}{z - 1} \\ &= \frac{(z + 1)(z - 1)}{z - 1} \\ &= z + 1 \end{aligned} \tag{1.1}$$

And the result is true for the sum as well:

$$\sum_{n=1}^1 (1 + z^n) = 1 + z \tag{1.2}$$

By (1.1) and (1.2) we see the base is true.

Suppose it is true for  $n \in \mathbb{N}$ .

Let  $k = n + 1$  so that:

$$\begin{aligned} 1 + z + \cdots + z^n + z^k &= \frac{z^{n+1} - 1}{z - 1} + z^k \\ &= \frac{z^{n+1} - 1}{z - 1} + \frac{z^k(z - 1)}{(z - 1)} \\ &= \frac{z^{n+1} - 1}{z - 1} + \frac{z^{k+1} - z^k}{(z - 1)} \\ &= \frac{z^{n+1} - 1 + z^{k+1} - z^k}{z - 1} \end{aligned}$$

Note that  $z^{n+1} = z^k$  and we have:

$$\begin{aligned} 1 + z + \cdots + z^n + z^k &= \frac{-1 + z^{k+1}}{z - 1} \\ &= \frac{z^{k+1} - 1}{z - 1} \end{aligned}$$

Noting again that  $k$  is the  $n + 1$  case, we prove the statement holds  $\forall n \in \mathbb{N}$ . ■

12. Using the preceding exercise, and taking real parts, prove:

$$1 + \cos \theta + \cos 2\theta + \cdots + \cos n\theta = \frac{1}{2} + \frac{\sin[(n + \frac{1}{2})\theta]}{2 \sin \frac{\theta}{2}}$$

For  $0 < \theta < 2\pi$ .

**Proof:**

Need help here. There's some kind of identity I'm missing to get each  $\cos^n$  term to map respectively to each  $\cos n$ .

13. Let  $z, w$  be two complex numbers such that  $\bar{z}w \neq 1$ . Prove that

$$\begin{aligned} \left| \frac{z - w}{1 - \bar{z}w} \right| &< 1 \quad \text{if } |z| < 1 \text{ and } |w| < 1, \\ \left| \frac{z - w}{1 - \bar{z}w} \right| &= 1 \quad \text{if } |z| = 1 \text{ and } |w| = 1, \end{aligned}$$

**Proof:**

Not sure yet, needs some work.

### 1.2.4 Incomplete Proofs

- 12, 13

## 1.3 Complex Valued Functions

We write the association of the **value**  $f(z)$  to  $z$  by the special arrow:

$$z \mapsto f(z)$$

Since:

$$f(z) = u(z) + iv(z)$$

Then:

$$z \mapsto u(z) \text{ and } z \mapsto v(z)$$

We usually write:

$$z = x + iy$$

So then:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

### 1.3.1 Power Function

The most important examples are

**Defn 1.8 (Power Function).** Any function of the form:

$$f(z) = z^n$$

### 1.3.2 Polar coordinates

Let us write  $z$  in polar coordinates with  $r \in \mathbb{R}$  and  $\theta \in [0, 2\pi]$ , then:

$$z = re^{i\theta}$$

Then:

$$f(z) = r^n e^{in\theta} = r^n (\cos n\theta + i \sin n\theta)$$

### 1.3.3 Closed Disc

The set of complex numbers, denoted  $\overline{D}$ , such that all elements in the set with domain  $\mathbb{C}$  are less than or equal to 1 in the complex plain.

**Defn 1.9 (Closed Disc).**

$$\overline{D} = \{z \in \mathbb{C} \mid \forall z \leq 1\}$$

Note that if  $z \in \overline{D}$  then  $z^n \in \overline{D}$ . Therefore  $z \mapsto z^n$  maps  $\overline{D}$  into itself.

Let  $S$  be the **sector** of  $z = re^{i\theta}$  such that  $0 \leq \theta \leq \frac{2\pi}{n}$ .

- This is breaking the circle up into  $n$  sectors is how to think of what is happening.
- This gives us access to mapping roots of unity to  $[0, 2\pi]$  I believe is the intent.

The function of a real variable  $r$ :

$$r \mapsto r^n$$

maps the unit interval  $[0, 1]$  onto itself:

$$[0, 1] \rightarrow [0, 1]$$

The function of  $\theta$ :

$$\theta \mapsto n\theta$$

maps the interval  $[0, \frac{2\pi}{n}]$  to the circumference of a circle  $[0, 2\pi]$ :

$$[0, \frac{2\pi}{n}] \rightarrow [0, 2\pi]$$

In this way, we see that the function  $f(z) = z^n$  maps the sector  $S$  onto the full disc of all numbers  $w$  where:

$$w = te^{i\varphi}$$

$$0 \leq t \leq 1$$

$$0 \leq \varphi \leq 2\pi$$

We may say that:

- the power function wraps/tiles the sector  $S$  around the disc  $n$  times.

- Thus we see  $z \mapsto z^n$  wraps the disc  $n$  times around.

To express every complex number  $w^n = z$  we end up with the following generalization:

**Defn 1.10 (Root of Unity).**

$$\zeta^k = e^{2\pi i k/n}$$

Which has the  $n$ th power given as:

$$(\zeta^k)^n = (e^{2\pi i k/n})^n = e^{2\pi i k} = 1$$

The points  $w_k$  are just the product of  $e^{i\theta/n}$  with all the  $n$ -th roots of unity:

$$w_k = e^{i\theta/n} \zeta^k$$

One of the **major results of the theory of complex variables** is to reduce the study of certain functions, including most of the common functions we know like exponentials, logarithms, sine, cosine; to power series, which can be approximated by polynomials.

Thus the power function is in some sense the unique basic function out of which the others are constructed.

### 1.3.4 Proofs

1. Let  $f(z) = 1/z$ . Describe what  $f$  does to the inside and outside of the unit circle, and also, what it does to points on the unit circle. This map is called **inversion** through the unit circle.

**Proof:**

Take the points  $|z| < 1$  and move them to outside the unit disc.

$$\begin{aligned} z &< 1 \\ z &\mapsto \frac{1}{z} > 1 \end{aligned}$$

Then, Take the points outside the unit circle  $|z| > 1$  and put them in the unit circle:

$$\begin{aligned} z &> 1 \\ z &\mapsto \frac{1}{z} < 1 \end{aligned}$$

■

2. Let  $f(z) = 1/\bar{z}$ . Describe  $f$  in the same manner as Exercise 1. This map is called **reflection** through the unit circle.

**Proof:**

Take a point outside the unit circle,  $|z| > 1$ , reflect it across the line of symmetry to  $\bar{z}$  and then map it to the same quadrant inside the unit circle:

$$z \mapsto 1/\bar{z}$$

Take a point inside the unit circle,  $0 < |z| < 1$ , reflect it across the line of symmetry to  $\bar{z}$  and then map it to the same quadrant outside the unit circle. ■

3. Let  $f(z) = e^{2\pi iz}$ . Describe the image under  $f$  of the set consisting of the points  $x + iy$  with:

$$\frac{-1}{2} \leq x \leq \frac{1}{2}$$

and

$$y \geq B := \{z = x + iy \mid y > 0\}$$

**Proof:**

4. Let  $f(z) = e^z$ . Describe the image under  $f$  of the following sets:

(a) The set of  $z = x + iy$  such that  $x \leq 1$  and  $0 \leq y \leq \pi$ .

**Proof:**

(b) The set of  $z = x + iy$  such that  $0 \leq y \leq \pi$  and no condition on  $x$ .

**Proof:**

### 1.3.5 Incomplete Proofs

- 3, 4

## 1.4 Limits and Compact Sets

**Defn 1.11 (Open Disc of radius  $r > 0$  centered at  $\alpha$ ).** The set  $S \subset \mathbb{C}$  with elements  $z$  such that:

$$|z - \alpha| < r$$

Denoted  $D(\alpha, r)$ .

**Defn 1.12 (Open Set  $U \subset \mathbb{C}$ ).**  $U$  is an open set if for every point  $\alpha \in U$ :

1. there is a disc  $D(\alpha, r)$  centered at  $\alpha$
2. there is a disc of some radius  $r > 0$  such that this disc  $D(\alpha, r)$  is contained in  $U$ .

**Defn 1.13 (Boundary Point of  $S$ ).** A point  $\alpha$  such that every disc  $D(\alpha, r)$  centered at  $\alpha$  and of radius  $r > 0$  contains both points of  $S$  and points not in  $S$ .

**Defn 1.14 (Adherent Point to  $S$ ).**  $\alpha$  is adherent to  $S$  if:

- every disc  $D(\alpha, r)$  with  $r > 0$  contains some element of  $S$ .

**Defn 1.15 (Interior point to  $S$ ).**  $\alpha$  is an interior point of  $S$  if:

- there exists a disc  $D(\alpha, r)$  which is contained in  $S$ .

Note that:

- An adherent point can be a boundary point
- An adherent point can be an interior point

**Defn 1.16 (Closed Set).** A set is closed if it contains all its boundary points.

**Defn 1.17 (Bounded Set).**  $S$  is bounded if there exists a number  $C > 0$  such that:

$$|z| \leq C \quad \forall z \in S$$

**Defn 1.18 (Closure of a Set  $S$ ).** The union of  $S$  and all its boundary points.

- Denoted  $\bar{S}$

Let  $f$  be a function on  $S$ , and let  $\alpha$  be an adherent point of  $S$ . Let  $w$  be a complex number. We say that:

$$z \in S$$

$$w = \lim_{z \rightarrow \alpha} f(z)$$

If given  $\epsilon > 0$  there exists  $\delta > 0$  such that if  $z \in S$  and  $|z - \alpha| < \delta$ , then:

$$|f(z) - w| < \epsilon$$

**Defn 1.19 (Continuous).**  $f$  is continuous at  $\alpha$  if:

$$\lim_{z \rightarrow \alpha} f(z) = f(\alpha)$$

**Defn 1.20 (Cauchy Sequence).** If  $\epsilon > 0$  then  $\exists N$  such that if:

$$m, n \geq N$$

then:

$$|z_n - z_m| < \epsilon$$

### 1.4.1 Compact Sets

This section is a bit peculiar and full of theorems I've not had a use for yet, but may be useful later.

The main point is this:

**Thm 1.5.** A set  $S$  is compact  $\iff$  it is closed and bounded.

### 1.4.2 Proofs

1. Let  $\alpha$  be a complex number of absolute value  $< 1$ . What is  $\lim_{n \rightarrow \infty} \alpha^n$ ? Proof?

**Proof:**

Think about the expansion

2. If  $|\alpha| > 1$  does  $\lim_{n \rightarrow \infty} \alpha^n$  exist? Why?

**Proof:**

3. Show that for any complex number  $z \neq 1$ , we have

$$1 + z + \cdots + z^n = \frac{z^{n+1} - 1}{z - 1}$$

If  $|z| < 1$ , show that

$$\lim_{n \rightarrow \infty} (1 + z + \cdots + z^n) = \frac{1}{1 - z}$$

**Proof:**



4. Let  $f$  be the function defined by

$$f(z) = \lim_{n \rightarrow \infty} \frac{1}{1 + n^2 z}$$

Show that  $f$  is the characteristic function of the set  $\{0\}$ , that is,  $f(0) = 1$ , and  $f(z) = 0$  if  $z \neq 0$ .

**Proof:**

5. For  $|z| \neq 1$  show that the following limit exists:

$$f(z) = \lim_{n \rightarrow \infty} \left( \frac{z^n - 1}{z^n + 1} \right)$$

Is it possible to define  $f(z)$  when  $|z| = 1$  in such a way to make  $f$  continuous?

**Proof:**

6. Let

$$f(z) = \lim_{n \rightarrow \infty} \frac{z^n}{1 + z^n}$$

- (a) What is the domain of definition of  $f$ , that is, for which complex numbers  $z$  does the limit exist?

**Proof:**

- (b) Give explicitly the values of  $f(z)$  for the various  $z$  in the domain of  $f$ .

**Proof:**

7. Show that the series

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(1 - z^n)(1 - z^{n+1})}$$

Converges to

$$\frac{1}{(1 - z)^2} \quad |z| < 1$$

and

$$\frac{1}{z(1 - z)^2} \quad |z| > 1$$

Prove that the convergence is uniform for  $|z| \leq c < 1$  in the first case, and  $|z| \geq b > 1$  in the second case.

[Hint: multiply and divide each term by  $1 - z$ , and do a partial fraction decomposition, getting a telescoping effect. ]

**Proof:**

## **1.5 Complex Differentiability**

## **1.6 The Cauchy-Reimann Equations**

## **1.7 title**

## **1.8 Cauchy Criterion**

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## Power Series

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- 2.1 Formal Power Series**
- 2.2 Convergent Power Series**
- 2.3 Relations Between Formal and Convergent Series**
- 2.4 Analytic Functions**
- 2.5 Differentiation of Power Series**
- 2.6 The Inverse and Open Mapping Theorems**
- 2.7 The Local Maximum Modulus Principle**