Notes Over Serge Lang's Complex Analysis: 4th Edition

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Contents

Chapter 1

Complex Numbers and Functions

<u>**Assume**</u>: $\{z_n\}$ is a complex valued sequence with $n, m, N \in \mathbb{N}$.

1.1 Complex Numbers

1.1.1 Definitions

Defn 1.1 (Complex Numbers). *A set of objects that can be added and multiplied together and produce another element of the set under the following conditions:*

1. *Every real number is a complex number, and if $\alpha, \beta \in \mathbb{R}$, then their sum and product as complex numbers are the same as their sum and products as real numbers.*
2. *There is a complex number denoted i such that $i^2 = -1$.*
3. *$\forall z \in \mathbb{C}$ with $a, b \in \mathbb{R}$ can be written uniquely as:*

$$z = a + bi$$

4. *The ordinary laws of arithmetic for addition and multiplication are satisfied $\forall z \in \mathbb{C}$:*

   distributive law holds

   associative law holds

   commutative law holds

   if $1 \in \mathbb{R}$ then $1z = z$

   if $0 \in \mathbb{R}$ then $0z = 0$

   $z + (-1)z = 0$

Defn 1.2 (Conjugate). *$\bar{z} \in \mathbb{C}$ such that:*

$$z = a + bi$$

iff

$$\bar{z} = a - bi$$

Defn 1.3 (Inverse). *$z^{-1} \in \mathbb{C}$ such that*

$$z \cdot z^{-1} = 1$$

Defn 1.4 (Absolute Value $|z|$ of z).

$$|z| = \sqrt{a^2 + b^2}$$

Thm 1.1. *$|z|$ satisfies the following properties. If $\alpha, \beta \in \mathbb{C}$, then:*

$$|\alpha\beta| = |\alpha||\beta|$$

$$|\alpha + \beta| \le |\alpha| + |\beta| \quad \textit{(triangle inequality)}$$

1.1.2 Proofs

1. Express the following complex numbers in the form $x + iy$, where x, y are real numbers.

   (a) $(-1+3i)^{-1}$

   Proof: Simply invert and separate, then use the conjugate/symmetry to rationalize the statement:

$$\begin{aligned}(-1+3i)^{-1} &= \frac{1}{(-1+3i)} \\ \frac{1}{(-1+3i)} &= \frac{1}{(-1+3i)}\frac{(-1-3i)}{(-1-3i)} \\ &= \frac{-1-3i}{1+3i-3i+9} \\ &= \frac{-1-3i}{10}\end{aligned}$$

$$\therefore\ (-1+3i)^{-1} = -\frac{1}{10} - \frac{3i}{10} \quad \blacksquare$$

   Which just means from the origin of $\mathbb{C}$ go left 1 then down 3 then shrink by $\frac{1}{10}$ and that's the z you're at in $\mathbb{C}$.

   (b) $(1+i)(1-i)$

   Proof: Distribute and collect:

$$\begin{aligned}(1+i)(1-i) &= 1 - i + i - i^2 \\ &= 1-(-1) \\ &= 2\end{aligned}$$

$$\therefore\ (1+i)(1-i) = 2 + 0i \quad \blacksquare$$

   Which is a bit strange because that's the same result of $(1+i)+(1-i)$.

   (c) $(i+1)(i-2)(i+3)$

   Proof: Distribute collect, distribute and collect again.

$$\begin{aligned}(i+1)(i-2)(i+3) &= (i^2-2i+i-2)(i+3) \\ &= (-i-3)(i+3) \\ &= (1-3i-3i-9)\end{aligned}$$

$$\therefore\ (i+1)(i-2)(i+3) = -8-6i \quad \blacksquare$$

2. Express the following complex numbers in the form $x + iy$, where x, y are real numbers.

   (a) $(1+i)^{-1}$

Proof: More of the same, just use the conjugate to solve these like problem 1 above.

$$\begin{aligned}
(1+i)^{-1} &= \frac{1}{1+i} \\
&= \frac{1}{1+i}\frac{(1-i)}{(1-i)} \\
&= \frac{1-i}{(1+i)(1-i)} \\
&= \frac{1-i}{1-i+i-i^2} \\
&= \frac{1-i}{1-(-1)} \\
&= \frac{1-i}{2} \\
\therefore\ (1+i)^{-1} &= \frac{1}{2} - \frac{i}{2} \quad \blacksquare
\end{aligned}$$

3. Let α be a complex number $\neq 0$. What is the absolute value of $\alpha/\bar{\alpha}$? What is $\bar{\bar{\alpha}}$?

   Proof: First note that:

$$\begin{aligned}
\alpha &= x + yi \\
\bar{\alpha} &= x - yi \\
\therefore\ \left|\frac{\alpha}{\bar{\alpha}}\right| &= \frac{x+yi}{x-yi}
\end{aligned}$$

   Now some algebra:

$$\begin{aligned}
\frac{x+yi}{x-yi} &= \frac{(x+yi)}{(x-yi)}\frac{(x+yi)}{(x+yi)} \\
&= \frac{x^2+2xyi+y^2i^2}{x^2-y^2i^2} \\
&= \frac{x^2+2xyi+y^2i^2}{x^2+y^2} \\
&= \frac{x^2+2xyi-y^2}{x^2+y^2} \\
&= \frac{x^2+2xyi-y^2}{|\bar{\alpha}|^2} \\
&= \frac{(x+yi)(x-yi)}{|\bar{\alpha}|^2} \\
\therefore\ \left|\frac{\alpha}{\bar{\alpha}}\right| &= \frac{\alpha\cdot\bar{\alpha}}{|\bar{\alpha}|^2} \quad \blacksquare
\end{aligned}$$

Part 3b: What is $\bar{\bar{\alpha}}$?

Proof: Note

$$\alpha = x + yi \quad \Leftrightarrow \quad \bar{\alpha} = x - yi.$$
$$\therefore \ \bar{\bar{\alpha}} = \overline{x - yi}$$

So now because the conjugate operation just changes the sign on the *imaginary* part of α we have the straightforward result of:

$$\begin{aligned} \bar{\bar{\alpha}} &= \overline{x - yi} \\ &= x + yi \\ \therefore \ \bar{\bar{\alpha}} &= \alpha \quad \blacksquare \end{aligned}$$

4. Let α, β be two complex numbers. Show that:

$$\overline{\alpha\beta} = \bar{\alpha}\bar{\beta}$$

and that:

$$\overline{\alpha + \beta} = \bar{\alpha} + \bar{\beta}$$

Proof: First is easy since we just distribute out $\alpha \cdot \beta$ and gather reals and imaginary parts together and see it is the same result as if we had simply taken the conjugate of each component.

Algebraically, with $\alpha_n, \beta_n, \rho \in \mathbb{R}$:

$$\begin{aligned} \overline{\alpha\beta} &= \overline{(\alpha_1 + \alpha_2 i)(\beta_1 + \beta_2 i)} \\ &= \overline{(\alpha_1\beta_1 + \alpha_1\beta_2 i + \beta_1\alpha_2 i + \alpha_2\beta_2 i^2)} \\ &= \overline{(\alpha_1\beta_1 + i(\alpha_1\beta_2 + \beta_1\alpha_2) + \alpha_2\beta_2 i^2)} \\ &= \overline{(\alpha_1\beta_1 + \alpha_2\beta_2 i^2 + i(\alpha_1\beta_2 + \beta_1\alpha_2))} \\ &= \overline{(\alpha_1\beta_1 - \alpha_2\beta_2 + i(\alpha_1\beta_2 + \beta_1\alpha_2))} \\ &= \overline{\rho_1 + i\rho_2} \\ \overline{\alpha\beta} &= \rho_1 - i\rho_2 \end{aligned}$$

Now we go the other way:

$$\begin{aligned} \bar{\alpha}\bar{\beta} &= \overline{(\alpha_1 + \alpha_2 i)} \cdot \overline{(\beta_1 + \beta_2 i)} \\ &= (\alpha_1 - \alpha_2 i) \cdot (\beta_1 - \beta_2 i) \\ &= (\alpha_1\beta_1 - \alpha_1\beta_2 i - \alpha_2\beta_1 i + \alpha_2\beta_2 i^2) \\ &= (\alpha_1\beta_1 - \alpha_1\beta_2 i - \alpha_2\beta_1 i - \alpha_2\beta_2) \\ &= (\alpha_1\beta_1 - \alpha_2\beta_2 - \alpha_1\beta_2 i - \alpha_2\beta_1 i) \\ &= (\alpha_1\beta_1 - \alpha_2\beta_2 - i(\alpha_1\beta_2 + \alpha_2\beta_1)) \\ \bar{\alpha}\bar{\beta} &= \rho_1 - i\rho_2 \end{aligned}$$

$$\therefore \ \overline{\alpha\beta} = \bar{\alpha}\bar{\beta} \quad \blacksquare$$

Second is easier as we only convert the sign inside the complex numbers, and do nothing with the operation between the two complex numbers, only on the reals in the number. Again, basically just some algebra of converting to the real and imaginary parts and gathering terms. $\blacksquare$

5. Justify the assertion that the real part of a complex number is $\leq$ its absolute value.

   Proof: The value can be equal to the absolute value if it happens to be positive, in which case it coincides with the absolute value.

   Or, it can be the symmetric partner if it is negative and therefore equal in magnitude but opposite in direction, therefore ordered as $\leq$ the absolute value by definition of well-ordering in $\mathbb{R}$. Because the reals are symmetric like my god damned shoes! ■

6. If $\alpha = a + ib$ with $a, b \in \mathbb{R}$ then b is called the **imaginary part** of α and we write:

$$\mathfrak{Im}(\alpha) = b.$$

   (a) Show that:

$$\alpha - \bar{\alpha} = 2i\ \mathfrak{Im}(\alpha)$$

   Proof: Just do the algebra:

$$\begin{aligned} \alpha - \bar{\alpha} &= (a + ib) - (a - ib) \\ &= 2ib \\ \therefore\ \alpha - \bar{\alpha} &= 2i\ \mathfrak{Im}(\alpha) \quad \blacksquare \end{aligned}$$

   (b) Show that:

$$\mathfrak{Im}(\alpha) \leq |\mathfrak{Im}(\alpha)| \leq |\alpha|$$

   Proof: Again, with some algebra we see the answer by considering the case of the imaginary part being either positive or negative while the absolute value will always be positive and therefore will be equal to this value or greater than it if it is negative.

   Next, think of whether part of α along just the real part or that part plus another would always make it larger than or equal to? If they always have the same imaginary part, then adding a real only increases the size of α while leaving the imaginary part at its maximum value. I'm too lazy to type this out right now, maybe later. ■

7. Find the real and imaginary parts of $(1 + i)^{100}$.

   Proof: Working with a base of $(1 + i)$ we just find useful factors to work with:

$$\begin{aligned} (1+i)^2 &= 2i \\ (1+i)^4 &= 2i^2 \\ &= -4 \\ (1+i)^{10} &= (1+i)^4(1+i)^4(1+i)^2 \\ &= (-4)(-4)(2i) \\ &= 32i \end{aligned}$$

   Now just plug and play:

$$\begin{aligned} (1+i)^{100} &= ((1+i)^{10})^{10} \\ &= (32i)^{10} \\ &= i^{10}32^{10} \\ &= -(32)^{10} \end{aligned}$$

$$\therefore\ (1+i)^{100} = -(32)^{10} + 0i \quad \blacksquare$$

8. Prove that for any two complex numbers z, w we have:

   (a) $|z| \leq |z - w| + |w|$

   Proof: Consider the three cases we could have have:

$$w < 0$$
$$w = 0$$
$$w > 0$$

If $w < 0$:

$$|z - (-w)| + |-w| = |z + w| + |w|$$
$$\therefore\ z < |z - w| + |w|$$

If $w = 0$:

$$|z - w| + |w| = |z - 0| + |0| = z$$
$$\therefore\ z = |z - w| + |w|$$

If $w > 0$, with $z_w + w = z$:

$$|z - w| + |w| = z_w + |w| = z$$
$$\therefore\ z = |z - w| + |w|$$

By these three cases combined we have:

$$|z| \leq |z - w| + |w| \quad \blacksquare$$

   (b) $|z| - |w| \leq |z - w|$

   Proof: By (a) above we just subtract $|w|$ off the right and left, and have a logically equivalent statement. $\blacksquare$

   (c) $|z| - |w| \leq |z + w|$

   Proof: If the above were not true, then (b) would be false, but (b) is true, so then:

$$|z| - |w| \leq |z + w| \quad \blacksquare$$

9. Let $\alpha = a + ib$ and $z = x + iy$. Let $c \in \mathbb{R} > 0$. Transform the condition:

$$|z - \alpha| = c$$

into an equation involving only x, y, a, b and c, and describe in a simple way what geometric figure is represented by this equation.

10. Describe geometrically the sets of points z satisfying the following conditions:

   (a) $|z - i + 3| = 5$

   The perimeter of the circle that has a radius of 5 and with an origin $|z - i + 3|$ from the origin of $\mathbb{C}$.

(b) $|z - i + 3| > 5$

The complex plane outside a set that sits $|z - i + 3|$ from the origin of $\mathbb{C}$ with a radius of 5 with no points *on* the perimeter of the radius.

(c) $|z - i + 3| \leq 5$

The disc of points in $\mathbb{C}$ centered at $|z - i + 3|$ with a radius of 5.

(d) $|z + 2i| \leq 1$

The disc of radius 1 that is centered at z and moved vertically by $2i$.

(e) $\mathfrak{Im}(z) > 0$

The set of points in $\mathbb{C}$ not including 0 that have real parts $= 0$ and imaginary parts > 0, so the y-axis.

(f) $\mathfrak{Im}(z) \geq 0$

The set of points along the positive axis of $\mathbb{C}$ including 0.

(g) $\mathfrak{Re}(z) > 0$

The set of points along the positive axis of $\mathbb{R} \subset \mathbb{C}$ not including 0.

(h) $\mathfrak{Re}(z) \geq 0$

The set of points along the positive axis of $\mathbb{R} \subset \mathbb{C}$ including 0.

1.2 Polar Form

1.2.1 Definitions

Let $z = x + iy$.

Defn 1.5 (Polar Coordinates). *An ordered pair (r, θ) with $r = radius$ and θ rotating from the x-axis such that:*

1. $r \in \mathbb{R}$ *and* $r = |z| = \sqrt{x^2 + y^2}$.
2. $\theta \in [0, 2\pi]$.

Defn 1.6 (Polar Form).

$$\begin{aligned} re^{i\theta} &= r\cos\theta + ir\sin\theta \\ &= x + iy \\ \therefore\ re^{i\theta} &\in \mathbb{C} \end{aligned}$$

Note that:

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$\theta = \cos^{-1}\left(\frac{x}{r}\right), \quad \theta = \sin^{-1}\left(\frac{y}{r}\right)$$

1.2.2 Factors of Pi

Defn 1.7 (**Factors of Pi**)**.** *If you don't know your factors of pi you don't know squat:*

$$\begin{aligned} \textit{If} \quad e^z &= e^w \\ \textit{then} \quad z &= w + 2ik\pi \\ e^{k2i\pi} &= 1 \ \ \forall k \in \mathbb{N} \\ \\ e^{i\pi} &= -1 + i0 \\ e^{2i\pi} &= 1 + i0 \\ e^{i\pi/2} &= 0 + i \\ e^{i\pi/3} &= \frac{1}{2} + i\frac{\sqrt{3}}{2} \\ e^{i\pi/4} &= \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} \\ e^{i\pi/6} &= \frac{\sqrt{3}}{2} + i\frac{1}{2} \end{aligned}$$

Thm 1.2. *Let $\theta, \varphi \in \mathbb{R}$ then:*

$$e^{i\theta + i\varphi} = e^{i\theta}e^{i\varphi}$$

Thm 1.3. *Let $\alpha, \beta \in \mathbb{C}$ then:*

$$e^{\alpha+\beta} = e^{\alpha}e^{\beta}$$

Thm 1.4 (Thm 1.3 reworded)**.** *Let $z_1 = r_1e^{i\theta}$ and $z_2 = r_2e^{i\varphi}$ then:*

$$z_1 \cdot z_2 = r_1r_2e^{i(\theta+\varphi)}$$

i.e. multiply the absolute values and add the angles.

1.2.3 Proofs

1. Put the following complex numbers in polar form.

   (a) $z = 1 + i$

   Let's just change bases. Note that:

$$\begin{aligned} e^0 = e^{2i\pi} &= 1 \\ e^{i\pi/2} &= i \end{aligned}$$

   Then note:

$$\begin{aligned} r = |z| &= \sqrt{x^2 + y^2} \\ &= \sqrt{1+1} \\ \therefore \ r &= \sqrt{2} \end{aligned}$$

$$\begin{aligned} \therefore \ 1 + i &= \sqrt{2}e^{2i\pi}e^{(i\pi/2)} \\ &= \sqrt{2}e^{i\pi/2} \quad \blacksquare \end{aligned}$$

(b) $1 + i\sqrt{2}$

Note:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + \sqrt{2}^2} \\ &= \sqrt{1+2} \\ \therefore\ r &= \sqrt{3} \end{aligned}$$

Previously we selected the factor of π which gave us equal x and y pieces, but here something else is going on.

We need to go right along the x-axis by 1 then up the y-axis by $\sqrt{2}$.

Note that we can normalize these with $\frac{1}{r}$ or use the Euler formula relating cosine to x and r to start.

$$\frac{1}{\sqrt{3}}(1 + i\sqrt{2}) = \frac{1}{\sqrt{3}} + \frac{i\sqrt{2}}{\sqrt{3}}$$

Try the Euler method here instead:

$$1 + i\sqrt{2} = \sqrt{3}\cos\theta + i\sqrt{3}\sin\theta$$

Then

$$\begin{aligned} \frac{x}{r} &= \frac{1}{\sqrt{3}} = \cos\theta \\ \frac{y}{r} &= \frac{\sqrt{2}}{\sqrt{3}} = \sin\theta \end{aligned}$$

(c) -3

Go left on the real line in the complex plane:

$$-3 = 3e^{i\pi} \quad \blacksquare$$

(d) $4i$

Go up by $4i$ in the complex plane:

$$4i = 4e^{i\pi/2} \quad \blacksquare$$

(e) $1 - i\sqrt{2}$

Go right by 1 and down by $\sqrt{2}$ in the complex plane:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + \sqrt{2}^2} \\ &= \sqrt{1+2} \\ \therefore\ r &= \sqrt{3} \end{aligned}$$

And then:

$$\theta = \cos^{-1}\frac{1}{\sqrt{3}}$$

So finally:

$$1 - i\sqrt{2} = \sqrt{3}e^{i\pi\cdot\cos^{-1}\frac{1}{\sqrt{3}}} \quad \blacksquare$$

(f) $5i$

Go up by $5i$ in the complex plane:

$$5i = 5e^{i\pi/2} \quad \blacksquare$$

(g) -7

Go left by -7 in the complex plane:

$$-7 = 7e^{i\pi} \quad \blacksquare$$

(h) $-1 - i$

Go left by -1 and down by -1 in the complex plane:

$$\begin{aligned} r = |z| &= \sqrt{1^2 + 1^2} \\ &= \sqrt{1+1} \\ \therefore\ r &= \sqrt{2} \end{aligned}$$

So then:

$$-1 - i = \sqrt{2}e^{5i\pi/4} \quad \blacksquare$$

2. Put the following complex numbers in the ordinary form $x + iy$.

(a) $e^{3i\pi}$
Simple, use Theorems 1.2 and 1.3 and change base!

$$\begin{aligned} e^{3i\pi} &= e^{i(2\pi+\pi)} \\ &= e^{i2\pi}e^{i\pi} \\ \therefore\ e^{3i\pi} &= (1)(-1) = -1 \quad \blacksquare \end{aligned}$$

(b) $e^{2\pi i/3}$
Use Theorem 1.3 and notice:

$$e^{2\pi i/3} = e^{\pi i/3}e^{\pi i/3}$$

Now check the Factors of Pi for $e^{\pi i/3}$ to see that we have:

$$e^{\pi i/3} = \left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

Then we have:

$$\begin{aligned}
e^{2\pi i/3} &= e^{(\pi i/3\,+\,\pi i/3)} \\
&= e^{\pi i/3}e^{\pi i/3} \\
e^{2\pi i/3} &= \left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)\left(\frac{1}{2}+\frac{i\sqrt{3}}{2}\right) \\
&= \frac{1}{2}\frac{1}{2}+\frac{1}{2}\frac{i\sqrt{3}}{2}+\frac{1}{2}\frac{i\sqrt{3}}{2}+\frac{i\sqrt{3}}{2}\frac{i\sqrt{3}}{2} \\
&= \frac{1}{4}+\frac{i\sqrt{3}}{4}+\frac{i\sqrt{3}}{4}-\frac{3}{4} \\
&= \frac{1}{4}+\frac{2i\sqrt{3}}{4}-\frac{3}{4} \\
\therefore\ e^{2\pi i/3} &= -\frac{1}{2}+\frac{i\sqrt{3}}{2} \quad \blacksquare
\end{aligned}$$

(c) $3e^{-\,i\pi/4}$

Here we just see that we are using $r = |z| = 3$ and moving *downward* from the x-axis to start with rotation $\frac{\pi}{4}$.

$$\therefore\ 3e^{-\,i\pi/4} = 3\left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right) \quad \blacksquare$$

(d) $\pi e^{-\,i\pi/3}$

Here we just have $r = |z| = \pi$ and rotate *down* again by factor of $\frac{\pi}{3}$.

Looking at the Factors of Pi and moving down we get:

$$\begin{aligned}
e^{i\pi/3} &= \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\
\therefore\ \pi e^{-\,i\pi/3} &= \pi\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right) \quad \blacksquare
\end{aligned}$$

(e) $e^{2i\pi/6}$

Lets factor, get some basis and then rotate to what we want:

$$\begin{aligned}
e^{2i\pi/6} &= e^{i\pi/3} \\
\therefore\ e^{2i\pi/6} &= \left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \quad \blacksquare
\end{aligned}$$

(f) $e^{-\,i\pi/2}$

Simple, factor we know but just going in different y direction.

$$e^{-\,i\pi/2} = \left(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}\right) \quad \blacksquare$$

(g) $e^{-i\pi}$

We know this already, just going in different y direction.

$$\begin{aligned} e^{-i\pi} &= (-1+0i) \\ &= -1 \quad \blacksquare \end{aligned}$$

(h) $e^{-5i\pi/4}$

Just break up the factors and use theorem 1.2 and 1.3.

$$\begin{aligned} e^{-5i\pi/4} &= e^{-i\pi/4}e^{-4i\pi/4} \\ &= \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)e^{-i\pi} \\ &= \left(\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right)(-1) \\ \therefore\ e^{-5i\pi/4} &= \left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) \quad \blacksquare \end{aligned}$$

3. Let α be a complex number $\neq 0$. Show there are two distinct complex numbers whose square is α.

Proof: For any $z \in \mathbb{C}$ with $z = a + bi$ we have a symmetric partner such that $z \neq \bar{z} = a - bi$.

So then

$$\begin{aligned} z\bar{z} &= (a+bi)(a-bi) \\ &= a^2 - abi + abi - i^2b^2 \\ &= a^2 + b^2 \\ &= \zeta \end{aligned}$$

where $\zeta \in \mathbb{C}$

4. Let $a + bi$ be a complex number. Find real numbers x, y such that:

$$(x+iy)^2 = a + bi$$

expressing x, y in terms of a and b.

Proof: Let $x, y \in \mathbb{R}$ then:

$$\begin{aligned} (x+iy)(x+iy) &= x^2 + 2xyi + i^2y^2 \\ &= (x^2 - y^2) + 2xyi \\ &= a + bi \\ \therefore\ (x+iy)^2 &= a + bi \ \ \forall x, y \in \mathbb{R} \quad \blacksquare \end{aligned}$$

5. Plot all the complex numbers z such that $z^n = 1$ for $n = 2, 3, 4$ and 5.

Just use $e^{2\pi i/n}$ to cut the circle up so that you then just tile that slice n times to get back to 1.

So then:

$$z_n = e^{2\pi i/n}$$

Which would then give:

$$\begin{aligned}(z_n)^n &= (e^{2\pi i/n})^n \\ &= e^{2\pi i} \\ &= 1 \\ &= (1,0)\end{aligned}$$

Which satisfies the fist condition. Now to give the plots for the remaining z's.

Suppose we start with $n = 2$:

$$\begin{aligned}z_2 &= e^{\pi i} \\ &= (-1,0)\end{aligned}$$

Now with $n = 3$:

$$\begin{aligned}z_3 &= e^{2\pi i/3} \\ &= \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\end{aligned}$$

For $n = 4$:

$$\begin{aligned}z_4 &= e^{\pi i/2} \\ &= (0,i)\end{aligned}$$

For $n = 5$:

$$\begin{aligned}z_5 &= e^{2\pi i/5} \\ &= e^{(\pi i/5\ +\ \pi i/5)}\end{aligned}$$

6. Let α be a complex number $\neq 0$. Let n be a positive integer. Show that there are n distinct complex numbers z such that $z^n = \alpha$. Write these complex numbers in polar form.

Proof:

Let:

$$\begin{aligned}\alpha &= \alpha_1 + \alpha_2 i \\ &= |\alpha| e^{i\theta} \\ &= re^{i\theta}\end{aligned}$$

Note:

$$\begin{aligned}\theta &= {}^{2\pi}/_{n} \\ z &= x + iy \\ &= re^{i\theta}\end{aligned}$$

Then:

$$\begin{aligned} &\forall n \in \mathbb{N} \\ z^n &= (re^{i\theta})^n \\ &= r^n (e^{2\pi i/n})^n \\ &= r^n e^{2\pi i} \end{aligned}$$

And so we have n distinct roots..

7. Find the real and imaginary parts of $i^{i/4}$, taking the fourth root such that its angle lies in $[0, \pi/2]$.

   Proof: This requires some algebra and breaking identities.

8. (a) Describe all complex numbers z such that $e^z = 1$.

   Proof:

   This will happen in the cases when $z = 2i\pi$

   (b) Let w be a complex number. Let α be a complex number such that $e^\alpha = w$. Describe all complex numbers α such that $e^\alpha = w$.

   Proof:

9. If $e^z = e^w$ show that there is an integer k such that $z = w + 2\pi k i$.

   Proof: Use the fact that every 2π we come back around.

10. (a) If θ is real, show that:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \quad \text{and} \quad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

   Proof: Just expand the exponentials to the polar coordinates of $\sin$ and $\cos$ and then do the algebra.

   Like the following:

$$\begin{aligned} e^{i\theta} &= \cos\theta + i\sin\theta \\ e^{-i\theta} &= \cos\theta - i\sin\theta \end{aligned}$$

   Then we have:

$$\begin{aligned} e^{i\theta} + e^{-i\theta} &= \cos\theta + i\sin\theta + \cos\theta - i\sin\theta \\ &= \cos\theta + \cos\theta \\ \therefore\ e^{i\theta} + e^{-i\theta} &= 2\cos\theta \quad \blacksquare \end{aligned}$$

   We see a similar argument doing it for $\sin\theta$.

   (b) For arbitrary complex z, suppose we define $\cos z$ and $\sin z$ by replacing θ with z in the above formula. Show that the only values of z for which $\cos z = 0$ and $\sin z = 0$ are the usual real values from trigonometry.

   Proof:

1.3 Complex Valued Functions

We write the association of the **value** $f(z)$ to z by the special arrow:

$$z \mapsto f(z)$$

Since:

$$f(z) = u(z) + iv(z)$$

Then:

$$z \mapsto u(z) \text{ and } z \mapsto v(z)$$

We usually write:

$$z = x + iy$$

So then:

$$f(z) = f(x + iy) = u(x, y) + iv(x, y)$$

1.3.1 Power Function

The most important examples are

Defn 1.8 (Power Function). *Any function of the form:*

$$f(z) = z^n$$

1.3.2 Polar coordinates

Let us write z in polar coordinates with $r \in \mathbb{R}$ and $\theta \in [0, 2\pi]$, then:

$$z = re^{i\theta}$$

Then:

$$f(z) = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$$

1.3.3 Closed Disc

The set of complex numbers, denoted $\overline{D}$, such that all elements in the set with domain $\mathbb{C}$ are less than or equal to 0 in the complex plain.

Defn 1.9 (Closed Disc).

$$\overline{D} = \{z \in \mathbb{C} \mid \forall z \leq 1\}$$

Note that if $z \in \overline{D}$ then $z^n \in \overline{D}$. Therefore $z \mapsto z^n$ maps $\overline{D}$ into itself.

Let S be the **sector** of $z = re^{i\theta}$ such that $0 \leq \theta \leq \frac{2\pi}{n}$.

- This is breaking the circle up into n sectors is how to think of what is happening.
- This gives us access to mapping roots of unity to $[0, 2\pi]$ I believe is the intent.

The function of a real variable r:

$$r \mapsto r^n$$

maps the unit interval $[0, 1]$ onto itself:

$$[0, 1] \to [0, 1]$$

The function of θ:

$$\theta \mapsto n\theta$$

maps the interval $[0, \frac{2\pi}{n}]$ to the circumference of a circle $[0, 2\pi]$:

$$[0, \frac{2\pi}{n}] \to [0, 2\pi]$$

In this way, we see that the function $f(z) = z^n$ maps the sector S onto the full disc of all numbers w where:

$$w = te^{i\varphi}$$
$$0 \leq t \leq 1$$
$$0 \leq \varphi \leq 2\pi$$

We may say that:

- *the power function wraps/tiles the sector S around the disc n times.*
- Thus we see $z \mapsto z^n$ wraps the disc n times around.

To express every complex number $w^n = z$ we end up with the following generalization:

Defn 1.10 (Root of Unity).

$$\zeta^k = e^{2\pi ik/n}$$

Which has the nth power given as:

$$(\zeta^k)^n = (e^{2\pi ik/n})^n = e^{2\pi ik} = 1$$

The points w_k are just the product of $e^{\frac{i\theta}{n}}$ with all the n-th roots of unity:

$$w_k = e^{i\theta/n}\zeta^k$$

One of the **major results of the theory of complex variables** is to reduce the study of certain functions, including most of the common functions we know like exponentials, logarithms, sine, cosine; to power series, which can be approximated by polynomials.

Thus the power function is in some sense the unique basic function out of which the others are constructed.

1.4 Limits and Compact Sets

1.5 Complex Differentiability

1.6 The Cauchy-Reimann Equations

1.7 title

1.8 Cauchy Criterion

Defn 1.11 (Cauchy Sequence). *If $\epsilon > 0$ then $\exists$ N such that if $m, n \geq N$ then:*

$$\mid z_n - z_m \mid < \epsilon$$

Chapter 2

Power Series

2.1 Formal Power Series

2.2 Convergent Power Series

2.3 Relations Between Formal and Convergent Series

2.4 Analytic Functions

2.5 Differentiation of Power Series

2.6 The Inverse and Open Mapping Theorems

2.7 The Local Maximum Modulus Principle