Gielis' Superformula and Point Cloud Generation

So, our parametric surface is given by Gielis' three dimensional surperformula.

$$\vec{r}(\theta,\phi) = r_1(\theta)cos(\theta)r_2(\phi)cos(\phi)\hat{\imath}+r_1(\theta)sin(\theta)r_2(\phi)cos(\phi)\hat{\jmath} + r_2(\phi)sin(\phi)\hat{k}$$

$$= \langle\, r_1(\theta)cos(\theta)r_2(\phi)cos(\phi), r_1(\theta)sin(\theta)r_2(\phi)cos(\phi), r_2(\phi)sin(\phi)\rangle$$

Where

$$r_1(\theta) = \left[\left|\frac{cos\left(\frac{m_1\theta}{4}\right)}{a}\right|^{n_2} + \left|\frac{sin\left(\frac{m_1\theta}{4}\right)}{b}\right|^{n_3}\right]^{-\frac{1}{n_1}}$$

And

$$r_2(\phi) = \left[\left|\frac{cos\left(\frac{m_2\phi}{4}\right)}{c}\right|^{n_5} + \left|\frac{sin\left(\frac{m_2\phi}{4}\right)}{d}\right|^{n_6}\right]^{-\frac{1}{n_4}}$$

And the range of our parameters is $\theta \in [-\pi,\pi]$ (longitude measured from the x-axis in the xy-plane) and $\phi \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ (latitude measured from xy-plane)

and $m_1, m_2, n_1, n_2, n_3, n_4, n_5, n_6, a, b, c, d \in \mathbb{R}^+ \cup \{0\}$, although of course we don't want $n_1 = n_4 = a = b = c = d = 0$ or

Now, we are going to need to give the Poisson surface reconstructor the normal vector to a given point on the parametric surface which is given by the following equation

$$\vec{n}(\theta,\phi) = \frac{\vec{r}_\theta(\theta,\phi) \times \vec{r}_\phi(\theta,\phi)}{\left|\vec{r}_\theta(\theta,\phi) \times \vec{r}_\phi(\theta,\phi)\right|}$$

Where $\times$ denotes the vector cross product, $|\cdot|$ denotes the modulus of the vector, and the subscripts denote the partial derivatives with respect to the given variables.

So computing the normal

$$\vec{r}_\theta(\theta,\phi) = \langle r_2(\phi)cos(\phi)\frac{\partial\big(r_1(\theta)cos(\theta)\big)}{\partial\theta}, r_2(\phi)cos(\phi)\frac{\partial\big(r_1(\theta)sin(\theta)\big)}{\partial\theta}, 0\rangle$$

$$= \langle r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}cos(\theta)+\frac{\partial(cos(\theta))}{\partial\theta}r_1(\theta)\right), r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}sin(\theta)+\frac{\partial(sin(\theta))}{\partial\theta}r_1(\theta)\right), 0\rangle$$

$$= \langle r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}cos(\theta)-sin(\theta)r_1(\theta)\right), r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}sin(\theta)+cos(\theta)r_1(\theta)\right), 0\rangle$$

And

$$\vec{r}_\phi(\theta,\phi) = \langle r_1(\theta)cos(\theta)\frac{\partial(r_2(\phi)cos(\phi))}{\partial\phi}, r_1(\theta)sin(\theta)\frac{\partial(r_2(\phi)cos(\phi))}{\partial\phi}, \frac{\partial(r_2(\phi)sin(\phi))}{\partial\phi}\rangle$$

$$= \begin{array}{c} \langle r_1(\theta)cos(\theta)\left(\frac{\partial(r_2(\phi))}{\partial\phi}cos(\phi)+\frac{\partial(cos(\phi))}{\partial\phi}r_2(\phi)\right), \\ r_1(\theta)sin(\theta)\left(\frac{\partial(r_2(\phi))}{\partial\phi}cos(\phi)+\frac{\partial(cos(\phi))}{\partial\theta}r_2(\phi)\right), \left(\frac{\partial(r_2(\phi))}{\partial\phi}sin(\phi)+\frac{\partial(sin(\phi))}{\partial\theta}r_2(\phi)\right)\rangle \end{array}$$

$$= \begin{array}{c} \langle r_1(\theta)cos(\theta)\left(\frac{\partial(r_2(\phi))}{\partial\phi}cos(\phi)-sin(\phi)r_2(\phi)\right), \\ r_1(\theta)sin(\theta)\left(\frac{\partial(r_2(\phi))}{\partial\phi}cos(\phi)-sin(\phi)r_2(\phi)\right), \left(\frac{\partial(r_2(\phi))}{\partial\phi}sin(\phi)+cos(\phi)r_2(\phi)\right)\rangle \end{array}$$

So now we're just left with the less than ideal computation of $\frac{\partial(r_1(\theta))}{\partial\theta}$ and $\frac{\partial(r_2(\phi))}{\partial\phi}$. With a little bit of work one can arrive at the following expressions

$$\frac{\partial(r_1(\theta))}{\partial\theta} = \frac{1}{n_1}\left[\left|\frac{cos\left(\frac{m_1\theta}{4}\right)}{a}\right|^{n_2}+\left|\frac{sin\left(\frac{m_1\theta}{4}\right)}{b}\right|^{n_3}\right]^{-\left(\frac{1}{n_1}+1\right)}\left(\frac{m_1n_2\left|\frac{cos\left(\frac{m_1\theta}{4}\right)}{a}\right|^{n_2}sin\left(\frac{m_1\theta}{4}\right)}{4cos\left(\frac{m_1\theta}{4}\right)}-\frac{m_1n_3\left|\frac{sin\left(\frac{m_1\theta}{4}\right)}{b}\right|^{n_3}cos\left(\frac{m_1\theta}{4}\right)}{4sin\left(\frac{m_1\theta}{4}\right)}\right)$$

$$\frac{\partial(r_2(\phi))}{\partial\phi} = \frac{1}{n_3}\left[\left|\frac{cos\left(\frac{m_2\phi}{4}\right)}{c}\right|^{n_5}+\left|\frac{sin\left(\frac{m_2\phi}{4}\right)}{d}\right|^{n_6}\right]^{-\left(\frac{1}{n_3}+1\right)}\left(\frac{m_2n_5\left|\frac{cos\left(\frac{m_2\phi}{4}\right)}{c}\right|^{n_5}sin\left(\frac{m_2\phi}{4}\right)}{4cos\left(\frac{m_2\phi}{4}\right)}-\frac{m_2n_6\left|\frac{sin\left(\frac{m_2\phi}{4}\right)}{d}\right|^{n_6}cos\left(\frac{m_2\phi}{4}\right)}{4sin\left(\frac{m_2\phi}{4}\right)}\right)$$

What is important to note here is that the derivative of abs(x) is undefined at x=0 which in this context will translate to the zero's of the trig functions $cos\left(\frac{m_1\theta}{4}\right)$, $sin\left(\frac{m_1\theta}{4}\right)$, $cos\left(\frac{m_2\phi}{4}\right)$, and $sin\left(\frac{m_2\phi}{4}\right)$ that is to say that the above expressions are only valid (not to mention non-singular) provided that $\theta \neq$

$0, \pm\frac{2\pi}{m_i}, \pm\frac{4\pi}{m_i}$ and $\phi \neq 0, \pm\frac{2\pi}{m_i}$ for $i = 1, 2$. So we will have to careful of this when sampling the parametric surface to generate the point cloud.

So, now having the above expressions we can toil on

$$\vec{r}_\theta(\theta,\phi) \times \vec{r}_\phi(\theta,\phi) =$$

$$det\begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}cos(\theta) - sin(\theta)r_1(\theta)\right) & r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}sin(\theta) + cos(\theta)r_1(\theta)\right) & 0 \\ r_1(\theta)cos(\theta)\left(\frac{\partial(r_2(\phi))}{\partial\phi}cos(\phi) - sin(\phi)r_2(\phi)\right) & r_1(\theta)sin(\theta)\left(\frac{\partial(r_2(\phi))}{\partial\phi}cos(\phi) - sin(\phi)r_2(\phi)\right) & \left(\frac{\partial(r_2(\phi))}{\partial\phi}sin(\phi) + cos(\phi)r_2(\phi)\right) \end{bmatrix}$$

$$= \langle\, r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}sin(\theta) + cos(\theta)r_1(\theta)\right)\left(\frac{\partial(r_2(\phi))}{\partial\phi}sin(\phi) + cos(\phi)r_2(\phi)\right),$$

$$-r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}cos(\theta) - sin(\theta)r_1(\theta)\right)\left(\frac{\partial(r_2(\phi))}{\partial\phi}sin(\phi) + cos(\phi)r_2(\phi)\right),$$

$$r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}cos(\theta) - sin(\theta)r_1(\theta)\right)r_1(\theta)sin(\theta)\left(\frac{\partial(r_2(\phi))}{\partial\phi}cos(\phi) - sin(\phi)r_2(\phi)\right)$$
$$- r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}sin(\theta) + cos(\theta)r_1(\theta)\right)r_1(\theta)cos(\theta)\left(\frac{\partial(r_2(\phi))}{\partial\phi}cos(\phi) - sin(\phi)r_2(\phi)\right)\rangle$$

$$= \langle n_1(\theta,\phi), n_2(\theta,\phi), n_3(\theta,\phi)\rangle$$

$$\vec{n}(\theta,\phi) = \frac{\vec{r}_\theta(\theta,\phi) \times \vec{r}_\phi(\theta,\phi)}{\left|\vec{r}_\theta(\theta,\phi) \times \vec{r}_\phi(\theta,\phi)\right|} = \frac{\langle n_1(\theta,\phi), n_2(\theta,\phi), n_3(\theta,\phi)\rangle}{\sqrt{\left(n_1(\theta,\phi)\right)^2 + \left(n_2(\theta,\phi)\right)^2 + \left(n_3(\theta,\phi)\right)^2}}$$

Where

$$\boldsymbol{n_1(\theta,\phi)} = r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}sin(\theta) + cos(\theta)r_1(\theta)\right)\left(\frac{\partial(r_2(\phi))}{\partial\phi}sin(\phi) + cos(\phi)r_2(\phi)\right)$$

$$\boldsymbol{n_2(\theta,\phi)} = -r_2(\phi)cos(\phi)\left(\frac{\partial(r_1(\theta))}{\partial\theta}cos(\theta) - sin(\theta)r_1(\theta)\right)\left(\frac{\partial(r_2(\phi))}{\partial\phi}sin(\phi) + cos(\phi)r_2(\phi)\right)$$

$$\boldsymbol{n_3(\theta,\phi)} = r_1(\theta)r_2(\phi)cos(\phi)sin(\theta)\left(\frac{\partial\big(r_1(\theta)\big)}{\partial\theta}cos(\theta) - sin(\theta)r_1(\theta)\right)\left(\frac{\partial\big(r_2(\phi)\big)}{\partial\phi}cos(\phi) - sin(\phi)r_2(\phi)\right) -$$
$$r_1(\theta)r_2(\phi)cos^2(\phi)\left(\frac{\partial\big(r_1(\theta)\big)}{\partial\theta}sin(\theta) + cos(\theta)r_1(\theta)\right)\left(\frac{\partial\big(r_2(\phi)\big)}{\partial\phi}cos(\phi) - sin(\phi)r_2(\phi)\right)$$

In order to sample the surface without losing any detail in regions of high curvature, we'll have to increment the sampling by equal amounts of arc length. First, remember our variables θ and ϕ are the longitude and the latitude, respectively. So, what I think will work is we'll first fix our latitude to lie in the xy-plane, that is to say, at the pseudo-equator of our surface, so $\phi = 0$, but remembering that we cannot set $\phi = 0$ exactly due to fear of singularity in the expression for our normal vector, we'll have to let $\phi = \varepsilon$, where ε is some infinitesimal amount.

So, $\vec{r}(\theta,\varepsilon) = \langle\, r_1(\theta)cos(\theta)r_2(\varepsilon)cos(\varepsilon), r_1(\theta)sin(\theta)r_2(\varepsilon)cos(\varepsilon), r_2(\varepsilon)sin(\varepsilon)\rangle$

$$= \langle\, Ar_1(\theta)cos(\theta), Ar_1(\theta)sin(\theta), B\rangle$$

So in this case, since one of our components is constant, we can apply the familiar expression for arc length.

$$S(\theta) = \int_{\theta_1}^{\theta_2} \sqrt{\big(x'(\theta)\big)^2 + \big(y'(\theta)\big)^2}\, d\theta$$

$$= \int_{\theta_1}^{\theta_2} \sqrt{\left(A\left(\frac{\partial\big(r_1(\theta)\big)}{\partial\theta}cos(\theta) - sin(\theta)r_1(\theta)\right)\right)^2 + \left(A\left(\frac{\partial\big(r_1(\theta)\big)}{\partial\theta}sin(\theta) + cos(\theta)r_1(\theta)\right)\right)^2}\, d\theta$$

$$= A\int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\partial\big(r_1(\theta)\big)}{\partial\theta}\right)^2 cos^2(\theta) + sin^2(\theta)\big(r_1(\theta)\big)^2 + \left(\frac{\partial\big(r_1(\theta)\big)}{\partial\theta}\right)^2 sin^2(\theta) + cos^2(\theta)\big(r_1(\theta)\big)^2}\, d\theta$$

$$= A\int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{\partial\big(r_1(\theta)\big)}{\partial\theta}\right)^2 + \big(r_1(\theta)\big)^2}\, d\theta$$

$$A\int_{\theta_1}^{\theta_2}\sqrt{\left(\frac{1}{n_1}\left[\left|\frac{cos\left(\frac{m_1\theta}{4}\right)}{a}\right|^{n_2}+\left|\frac{sin\left(\frac{m_1\theta}{4}\right)}{b}\right|^{n_3}\right]^{-\left(\frac{1}{n_1}+1\right)}\left(\frac{m_1 n_2\left|\frac{cos\left(\frac{m_1\theta}{4}\right)}{a}\right|^{n_2} sin\left(\frac{m_1\theta}{4}\right)}{4cos\left(\frac{m_1\theta}{4}\right)}-\frac{m_1 n_3\left|\frac{sin\left(\frac{m_1\theta}{4}\right)}{b}\right|^{n_3} cos\left(\frac{m_1\theta}{4}\right)}{4sin\left(\frac{m_1\theta}{4}\right)}\right)\right)^2+\left(\left[\left|\frac{cos\left(\frac{m_1\theta}{4}\right)}{a}\right|^{n_2}+\left|\frac{sin\left(\frac{m_1\theta}{4}\right)}{b}\right|^{n_3}\right]^{-\frac{1}{n_1}}\right)^2}\,d\theta$$

So just one glance at the above integrand can motivate the following approximation.

We can assume that for a sufficiently small change in our angle $\Delta\theta$ that the arclength can be approximated by a straight line as is stated mathematically below. Mind you, that $\vec{r}(\theta)=\vec{r}(\theta,\varepsilon)$ below.

$$S(\theta)^2=\left|\vec{r}\big(\theta+\Delta_\theta(\theta)\big)-\vec{r}(\theta)\right|^2$$

The right hand side can be approximated to the first order by

$$\left|\vec{r}\big(\theta+\Delta_\theta(\theta)\big)-\vec{r}(\theta)\right|^2\approx\left(A\frac{\partial}{\partial\theta}\big(r_1(\theta)cos(\theta)\big)\Delta_\theta(\theta)\right)^2+\left(A\frac{\partial}{\partial\theta}\big(r_1(\theta)sin(\theta)\big)\Delta_\theta(\theta)\right)^2$$

Where A is as above.

Then doing all the fun calculus and algebra, we can solve for $\Delta_\theta(\theta)$

$$\boxed{\Delta_\theta(\theta)=\frac{S(\theta)}{A\sqrt{\left(\frac{\partial\big(r_1(\theta)\big)}{\partial\theta}\right)^2+\big(r_1(\theta)\big)^2}}}$$

So then we choose a sufficiently small constant value for $S(\theta)$ so that we sample plenty of points and then we can generate the necessary values of theta recursively as follows

$$\theta_i=\theta_{i-1}+\Delta_\theta(\theta_{i-1})\qquad \theta_0=\varepsilon,\ i\in\{1,\dots,N\}\mid\theta_N<\pi$$

$$\theta_i=\theta_{i-1}-\Delta_\theta(\theta_{i-1})\qquad \theta_{N+1}=-\varepsilon,\ i\in\{N+2,\dots,2N\}\ (assuming\ symmetry)\mid\theta_{2N}>-\pi$$

And then we'll be interested in the set $\boxed{\{\,(Ar_1(\theta_i)cos(\theta_i),Ar_1(\theta_i)sin(\theta_i),B)\mid\ i\in\{1,\dots,2N\}\}}$

So, we've still got more work to do. Having found a set of roughly equal distant points on the pseudo-equator of our surface, we need to use the values in the set $\{\theta_i, \pm\varepsilon \mid i \in \{1, \ldots, 2N\}\}$ to generate all the longitudinal points.

What we will do is fix our longitude at one of the points in $\{\theta_i, \pm\varepsilon \mid i \in \{1, \ldots, 2N\}\}$ and then perform the following change of coordinate basis to reduce the subsequent problems to 2D problems.

If we think of our parametric surface as being in the normal coordinate axis system x, y, z then we can rotate the x and y coordinates by some arbitrary angle while leaving the z-axis fixed using the following equations (Note:x' and y' do not denote derivatives, but the axes of the new coordinate system)

$$\begin{cases} x'(\theta_2, \phi) = x(\theta_1, \phi)cos(\theta_2) + y(\theta_1, \phi)sin(\theta_2) \\ y'(\theta_2, \phi) = -x(\theta_1, \phi)sin(\theta_2) + y(\theta_1, \phi)cos(\theta_2) \end{cases}$$

Where $x(\theta_1, \phi) = r_1(\theta_1)cos(\theta_1)r_2(\phi)cos(\phi)$ and $y(\theta_1, \phi) = r_1(\theta_1)sin(\theta_1)r_2(\phi)cos(\phi)$.

Now, something very convenient happens if we set $\theta_1 = \theta_2$ in the above equations, $y'(\theta_1 = \theta_2, \phi) = 0$! This is great because this leads to the expression of our parametric surface in the new coordinate system to be two dimensional, allowing us to apply the same analysis as above

$$\vec{r}_{x',y'}(\theta_i, \phi) = \langle\, x(\theta_i, \phi)cos(\theta_i) + y(\theta_i, \phi)sin(\theta_i), 0, r_2(\phi)sin(\phi)\rangle$$

So now,

$$S(\phi)^2 \approx \left(\frac{\partial}{\partial\phi}\big(\, x(\theta_i, \phi)cos(\theta_i) + y(\theta_i, \phi)sin(\theta_i)\big)\Delta_\phi(\phi)\right)^2 + \left(\frac{\partial}{\partial\phi}\big(r_2(\phi)sin(\phi)\big)\Delta_\phi(\phi)\right)^2$$

$$= \Bigg(r_1(\theta_i)cos^2(\theta_i)\left(\frac{\partial}{\partial\phi}\big(r_2(\phi)\big)cos(\phi) - r_2(\phi)sin(\phi)\right)$$

$$+ r_1(\theta_i)sin^2(\theta_i)\left(\frac{\partial}{\partial\phi}\big(r_2(\phi)\big)cos(\phi) - r_2(\phi)sin(\phi)\right)\Bigg)^2 \Delta_\phi(\phi)^2$$

$$+ \left(\frac{\partial}{\partial\phi}\big(r_2(\phi)\big)sin(\phi) + r_2(\phi)cos(\phi)\right)^2 \Delta_\phi(\phi)^2$$

Then, following some manipulations and simplications, we get

$$\boxed{\Delta_\phi(\phi) = \frac{S(\phi)}{\sqrt{\left(r_1(\theta_i)\left(\frac{\partial}{\partial\phi}\big(r_2(\phi)\big)cos(\phi) - r_2(\phi)sin(\phi)\right)\right)^2 + \left(\frac{\partial}{\partial\phi}\big(r_2(\phi)\big)sin(\phi) + r_2(\phi)cos(\phi)\right)^2}}}$$

So then we choose a sufficiently small constant value for $S(\phi)$so that we sample plenty of points and then we can generate the necessary values of phi recursively as follows

$$\phi_i = \phi_{i-1} + \Delta_\phi(\phi_{i-1}) \qquad \phi_0 = \varepsilon, i \in \{1, \dots, N\} \,\Big|\, \phi_N < \frac{\pi}{2}$$

$$\phi_i = \phi_{i-1} - \Delta_\phi(\phi_{i-1}) \qquad \phi_{N+1} = -\varepsilon, i \in \{N+2, \dots, 2N\} \,\Big|\, \phi_{2N} > -\frac{\pi}{2}$$

Note: N above is not necessarily the same as the N for the pseudo-equatorial point set. Also, as before, there does not necessarily have to be the same number of points on either side of 0, but assuming radial symmetry then I suppose there would be.

We're interested in the set

$$\left\{ \left(x(\theta_i, \phi_j) cos(\theta_i) + y(\theta_i, \phi_j) sin(\theta_i), 0, r_2(\phi_j) sin(\phi_j) \right) \,\middle|\, i \in \{1, \dots, 2N_1\}, j \in \{1, \dots, 2N_{2,i}\} \right\}$$

Note (1): The range of j is dependent on i. It may happen that for a given value of θ_i, the longitudinal cross section will possess more or less curvature leading to a larger or smaller range of j, respectively.

Note (2): We are really interested in the original points prior to the change of basis, but since the coordinate change should not have affected the way the angles were measured. So the point set were truly interested in is

$$\boxed{\{ (r_1(\theta_i) cos(\theta_i) r_2(\phi_j) cos(\phi_j), r_1(\theta_i) sin(\theta_i) r_2(\phi_j) cos(\phi_j), r_2(\phi_j) sin(\phi_j)) \,|\, i \in \{1, \dots, 2N_1\}, j \in \{1, \dots, 2N_{2,i}\} \}}$$

Assuming all the above analysis is correct, the union of the two highlighted point sets should constitute the entirety of our point cloud, in which case we can feed the point cloud to the Poisson surface reconstructor which will derive an approximate implicit formula for our parametric surface which then, in turn, the implicit formula is fed to the mesh generator and – hopefully – out pops a nice curvature sensitive mesh ☺