Harmonic Morphisms with One-dimensional Fibres and Milnor Fibrations

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Review

Definition

A function f is harmonic if it satisfies the Laplace equation:

$$\Delta f = 0$$

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$$\Delta f = 0$$

Remark

For a flat (Euclidean) manifold the Laplacian takes the familiar form:

$$\Delta f = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x^2}$$

Jacobi's Problem

In 1848, Jacobi asked the following question:

Jacobi's Problem

In 1848, Jacobi asked the following question:

Let $\varphi : \mathbb{R}^3 \to \mathbb{C}$ be a harmonic function and let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function.

The Question

Under what conditions on φ is $f \circ \varphi$ harmonic?

$$\Delta(f \circ \varphi) = 0$$

Jacobi's Answer

Defining $z := (\varphi_1, \varphi_2)$, the expression for the Laplacian can be shown to be:

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Laplacian of Composition

$$\Delta(f \circ \varphi) = \frac{\mathrm{d}f}{\mathrm{d}z}\Delta\varphi + \frac{\mathrm{d}^2 f}{\mathrm{d}z^2}|\vec{\nabla}\varphi|^2 = 0$$

Jacobi's Answer

Defining $z := (\varphi_1, \varphi_2)$, the expression for the Laplacian can be shown to be:

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$$\Delta(f \circ \varphi) = \frac{\mathrm{d}f}{\mathrm{d}z}\Delta\varphi + \frac{\mathrm{d}^2 f}{\mathrm{d}z^2}|\vec{\nabla}\varphi|^2 = 0$$

Thus if we want the above to be identically zero we need:

Conditions

$$\Delta\varphi = 0 \quad and \quad |\vec{\nabla}\varphi|^2 = 0$$

1st Condition

Observing $\varphi = \varphi_1 + i\varphi_2$

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Laplacian of map φ

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Laplacian of map φ

$$\Delta\varphi = \Delta\varphi_1 + i\Delta\varphi_2$$

Harmonic Components

$$\Delta\varphi_1 = 0 \quad and \quad \Delta\varphi_2 = 0$$

2nd Condition

Observing that $\varphi = \varphi_1 + i\varphi_2$:

$|\vec{\nabla}\varphi|^2 = 0$

$$|\vec{\nabla}\varphi_1|^2 + 2i\langle\vec{\nabla}\varphi_1, \vec{\nabla}\varphi_2\rangle - |\vec{\nabla}\varphi_2|^2 = 0$$

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$\mathfrak{Re}|\vec{\nabla}\varphi|^2 = 0$

$$|\vec{\nabla}\varphi_1|^2 = |\vec{\nabla}\varphi_2|^2$$

2nd Condition

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$\mathfrak{Re}|\vec{\nabla}\varphi|^2 = 0$

$$|\vec{\nabla}\varphi_1|^2 = |\vec{\nabla}\varphi_2|^2$$

$\mathfrak{Im}|\vec{\nabla}\varphi|^2 = 0$

$$\langle\vec{\nabla}\varphi_1, \vec{\nabla}\varphi_2\rangle = 0$$

A Complete Characterization

Since f is an arbitrary harmonic function, shows that the two conditions completely characterize harmonic morphisms from $\mathbb{R}^3$ to $\mathbb{C}$:

Characterization

$\varphi : \mathbb{R}^3 \to \mathbb{C}$ is a harmonic morphism if and only if

1. $\Delta\varphi_1 = 0$ and $\Delta\varphi_2 = 0$
2. $\langle \vec{\nabla}\varphi_1, \vec{\nabla}\varphi_2 \rangle = 0$ and $|\vec{\nabla}\varphi_1|^2 = |\vec{\nabla}\varphi_1|^2 = \lambda(x)$ for $i = 1, 2$ and some scalar function λ (called the dilation function)

A Complete Solution

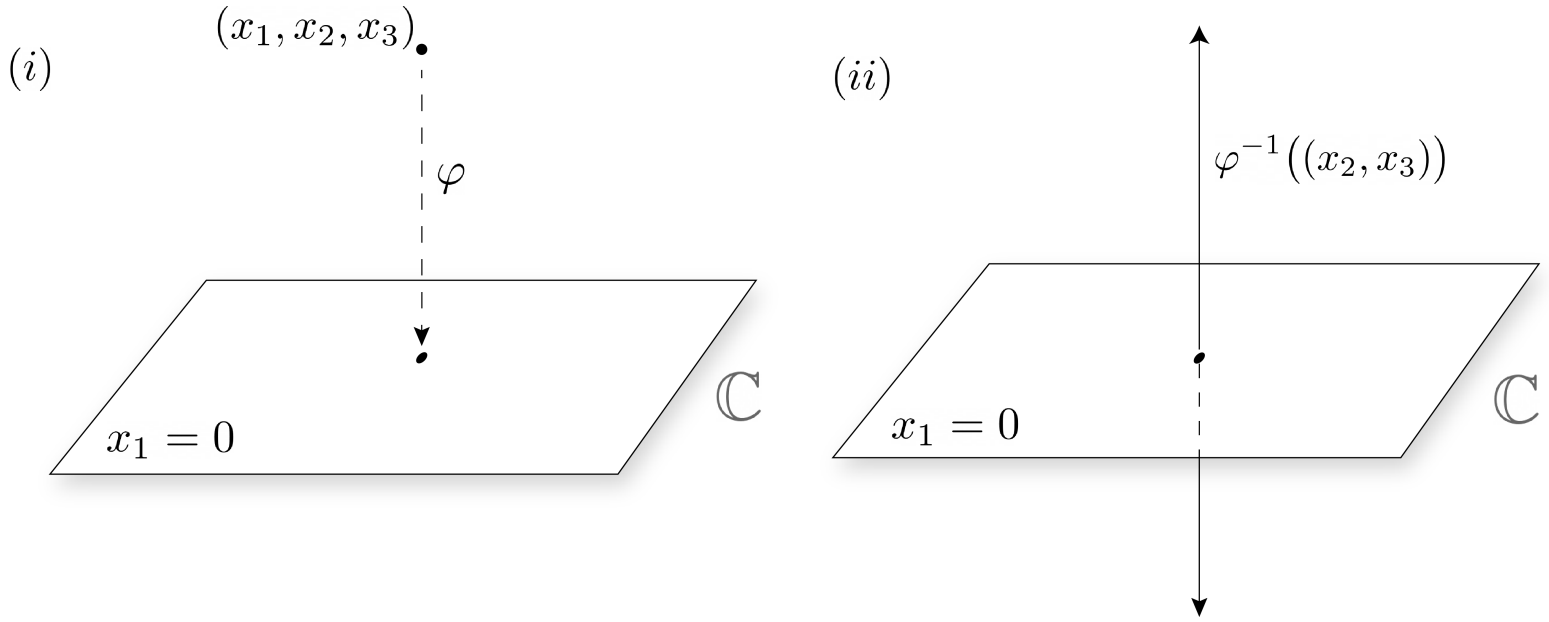
Let g and h be meromorphic functions satisfying: $\lim_{z \to z_0} h(z)/g(z)^2 = L$, where z_0 is a pole of h and $L \in \mathbb{R}$. Then any smooth local solution $\varphi : U \to \mathbb{C}, z = \varphi(x)$ to the equation

Theorem (Jacobi's Solution)

$$-2g(z)x_1 + (1 - g(z)^2)x_2 + i(1 + g(z)^2)x_3 = 2h(z)$$

on a convex open set of $\mathbb{R}^3$ is a submersive harmonic morphism with connected fibres not all in the direction of the negative x_1 axis;

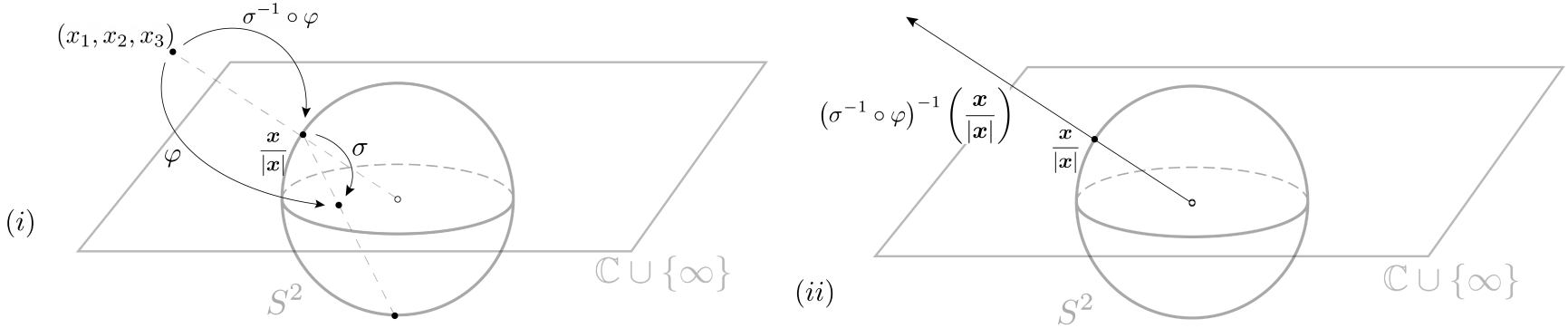
$g \equiv 0$, $h = \frac{1}{2}z$



Orthogonal Projection

$$\varphi\big((x_1, x_2, x_3)\big) = (x_2, x_3)$$

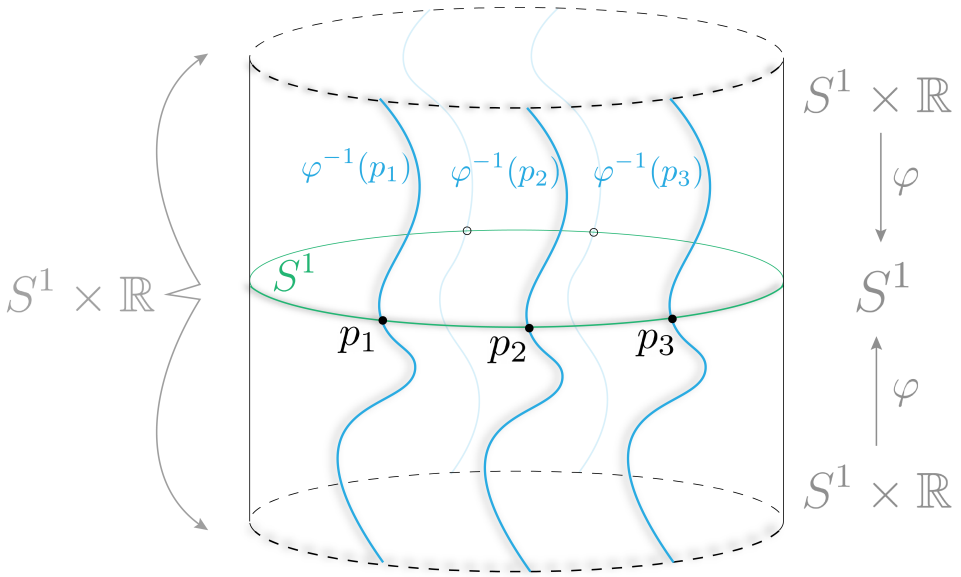
$g = z \ , \ h \equiv 0$



Radial Projection

$$(\sigma^{-1} \circ \varphi)\big((x_1, x_2, x_3)\big) = \sigma^{-1}\left(\frac{-x_1 + |\mathbf{x}|}{x_2 - ix_3}\right) = \frac{\mathbf{x}}{|\mathbf{x}|}$$

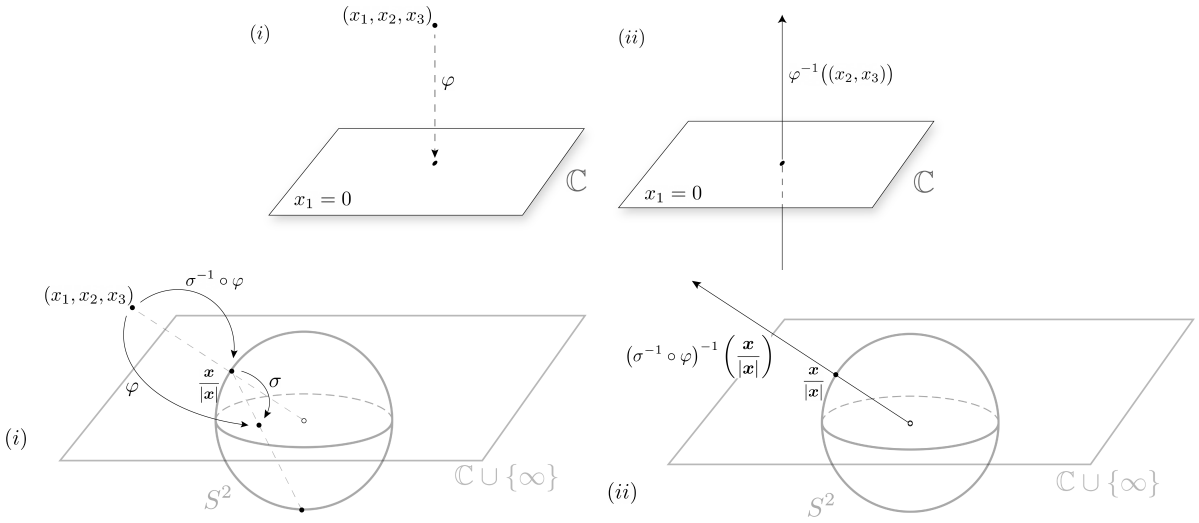
Solutions Generate Regular Foliations



Regular Foliation

A regular foliation is a decomposition of a manifold into submanifolds all of equal dimension. The leaves of the foliation can be thought of as fibers $\varphi^{-1}(p)$ of a submersion φ

Fibres Are Straight Lines: $\mathbb{R}^3 \to \mathbb{R}^2$



Theorem (Jacobi's Solution)

$$-2g(z)x_1 + (1 - g(z)^2)x_2 + i(1 + g(z)^2)x_3 = 2h(z)$$

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- Harmonic Maps from $\mathbb{R}^m$ to $\mathbb{R}^n$
- HWC maps from $\mathbb{R}^m$ to $\mathbb{R}^n$
- Characterization of Harmonic Morphisms (Euclidean)
- Examples: $\mathbb{R}^m \to \mathbb{R}^n$

Definition

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A map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is called a harmonic morphism if for any harmonic function $f : U \subseteq \mathbb{R}^n \to \mathbb{R}$, defined on an open subset U of $\mathbb{R}^n$ with $\varphi^{-1}(U) \subseteq \mathbb{R}^m$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

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In Other Words

Harmonic morphisms pullback harmonic functions on $\mathbb{R}^n$ to harmonic functions on $\mathbb{R}^m$

Harmonic Maps (Euclidean)

Definition

A map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is called harmonic if for each of its component functions $(\varphi_1, \ldots, \varphi_n)$:

$$\Delta\varphi_i = \sum_{j=1}^{m} \frac{\partial^2 \varphi_i}{\partial x_j^2} = 0$$

HWC Maps (Euclidean)

Horizontally Weakly Conformal Maps (Euclidean)

In the Euclidean case, a map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ whose component functions $(\varphi_1, \ldots, \varphi_n)$ satisfy the two following conditions is called horizontally weakly conformal (HWC).

HWC Maps (Euclidean)

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(i) $|\vec{\nabla}\varphi_i|^2 = |\vec{\nabla}\varphi_j|^2 \leftrightsquigarrow$ the magnitude of the components are equal at every point in the domain.

HWC Maps (Euclidean)

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(i) $|\vec{\nabla}\varphi_i|^2 = |\vec{\nabla}\varphi_j|^2 \leftrightsquigarrow$ the magnitude of the components are equal at every point in the domain.

(ii) $\langle \vec{\nabla}\varphi_i, \vec{\nabla}\varphi_j \rangle = 0 \leftrightsquigarrow$ the component functions are mutually orthogonal at each point of the domain.

A General Characterization

Theorem

Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that $\varphi = (\varphi_1, \ldots, \varphi_n)$. Then φ is a harmonic morphism if and only if the following conditions are satisfied:

(i) $\Delta\varphi_i = 0$ for each component function of φ

(ii) $\vec{\nabla}\varphi_i \cdot \vec{\nabla}\varphi_j = \lambda(\mathbf{x})\delta_{ij}$ for each $\mathbf{x} \in \mathbb{R}^m$

where $\lambda(\mathbf{x})$ is called the dilation function of φ.

Examples: $\mathbb{R}^m \to \mathbb{R}^n$

A List of Examples

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- Any constant function is a harmonic morphism.

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- Any constant function is a harmonic morphism.
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- Orthogonal projection $\varphi((x_1, \ldots x_{n-k}, x_{n-k+1}, \ldots, x_n)) = (x_{n-k+1}, \ldots, x_n)$ is a harmonic morphism.

Examples: $\mathbb{R}^m \to \mathbb{R}^n$

A List of Examples

- Any constant function is a harmonic morphism.
- Let f and g be harmonic morphisms. Then $f \circ g$ is a harmonic morphism.
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- Orthogonal projection $\varphi((x_1, \ldots x_{n-k}, x_{n-k+1}, \ldots, x_n)) = (x_{n-k+1}, \ldots, x_n)$ is a harmonic morphism.
- Radial projection $\varphi(\mathbf{x}) = \frac{\mathbf{x}}{|\mathbf{x}|}$ is a harmonic morphism.

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3. Harmonic Morphisms $(M, g) \to (N, h)$
   - Definitions
   - HWC maps (Riemannian)
   - Harmonic Maps (Riemannian)
   - Characterization of Harmonic Morphism (Riemannian)
   - Fundamental Equation of Harmonic Morphisms

Basic Definition

Let (M, g) and (N, h) be manifolds with their associated Riemannian metrics (symmetric, positive-definite).

Harmonic Morphism between Riemannian Manifolds

A map $\varphi : M \to N$ is called a harmonic morphism if for any harmonic function $f : U \subseteq N \to \mathbb{R}$, defined on an open subset U of N with $\varphi^{-1}(U) \subseteq M$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

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Remark

This is precisely the same definition as in the Euclidean case. But it's important to note that The Laplacian is metric-dependent:

The Laplacian of a Function for Riemannian Manifolds

The Laplacian

$$\Delta f = \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{1}{\sqrt{\det g}} \frac{\partial}{\partial x_i} \left(\sqrt{\det g} \, (g^{-1})_{ij} \, \frac{\partial f}{\partial x_j} \right)$$

The components of the metric tensor are given by

$$(g_p)_{ij} = g \left(\left. \frac{\partial}{\partial x_i} \right|_p, \left. \frac{\partial}{\partial x_j} \right|_p \right)$$

Horizontal and Vertical Spaces

Let $\varphi : M \to N$ be a smooth map. Then the differential of φ at a point $x \in M$ is a map $\mathrm{d}\varphi_x : T_x M \to T_{\varphi(x)} N$ such that for a vector field $X \in \Gamma(TM)$ and smooth function f on N

Differential Map

$$\mathrm{d}\varphi_x(X)(f) := X(f \circ \varphi)(x)$$

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Differential Map

$$\mathrm{d}\varphi_x(X)(f) := X(f \circ \varphi)(x)$$

Vertical Space $\mathcal{V}_x$ and Horizontal space $\mathcal{H}_x$

$$\mathcal{V}_x := \ker \mathrm{d}\varphi_x = \{X \in T_xM \mid \mathrm{d}\varphi_x X = 0\}$$
$$\mathcal{H}_x := (\ker \mathrm{d}\varphi_x)^\perp$$

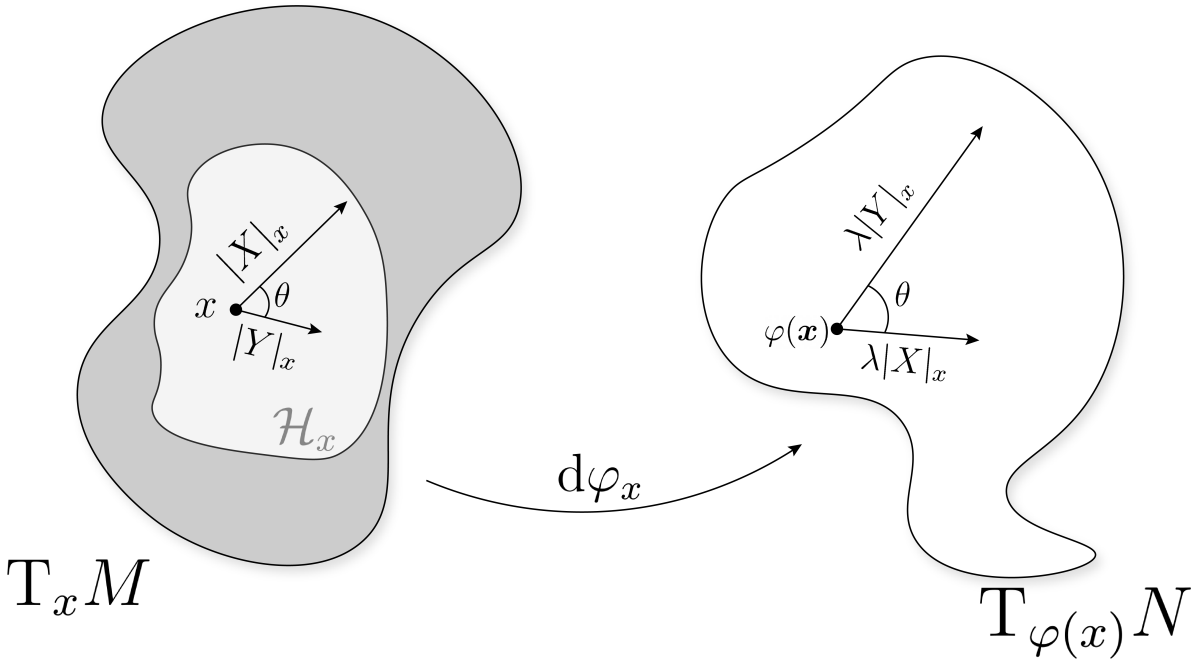
HWC maps

$\varphi : (M, g) \to (N, h)$ is said to be horizontally (weakly) conformal if there exists a function $\lambda : M \setminus C_\varphi \to \mathbb{R}^+$ such that for all $X, Y \in \mathcal{H}_x$ and $x \in M \setminus C_\varphi$.

Horizontally Weakly Conformal

$$\lambda^2(x) g(X, Y) = h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))$$

HWC maps



Horizontally Weakly Conformal

$$\lambda^2(x) g(X, Y) = h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))$$

Generalizing the Laplace Equation

Remark

For a general map $\varphi : M \to N$ between Riemannian manifolds, the Laplacian is a tensor, not a scalar. Need to develop a proper generalization.

Generalizing the Laplace Equation

Second Fundamental Form of φ

$$\nabla \mathrm{d}\varphi(X, Y) := \nabla^N_{\mathrm{d}\varphi X}(\mathrm{d}\varphi Y) - \mathrm{d}\varphi(\nabla^M_X Y)$$

where $X, Y \in \Gamma(TM)$

Generalizing the Laplace Equation

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where $X, Y \in \Gamma(TM)$

The Tension Field

$$\tau(\varphi) := \mathrm{Tr}\ \nabla \mathrm{d}\varphi = \sum_{i=1}^{m} \nabla \mathrm{d}\varphi(E_i, E_i)$$

where $\{E_i\}$ is an orthonormal frame of M.

Harmonic Maps

Energy Density of φ

$$e(\varphi) = \frac{1}{2}\|\mathrm{d}\varphi\|^2$$

where $||\mathrm{d}\varphi_x|| := \sum_{i=1}^{m} h\big(\mathrm{d}\varphi_x(E_i), \mathrm{d}\varphi_x(E_i)\big)$ for an orthonormal frame $\{E_i\}$ of M at x

Harmonic Maps

Energy Density of φ

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where $||\mathrm{d}\varphi_x|| := \sum_{i=1}^m h\big(\mathrm{d}\varphi_x(E_i), \mathrm{d}\varphi_x(E_i)\big)$ for an orthonormal frame $\{E_i\}$ of M at x

Total Energy

The total energy of φ is given by integration over M of the energy density of φ

$$E(\varphi) = \frac{1}{2}\int_M \|\mathrm{d}\varphi\|^2 dV$$

Harmonic Maps

Let $E(\varphi_t; D)$ be the energy functional.

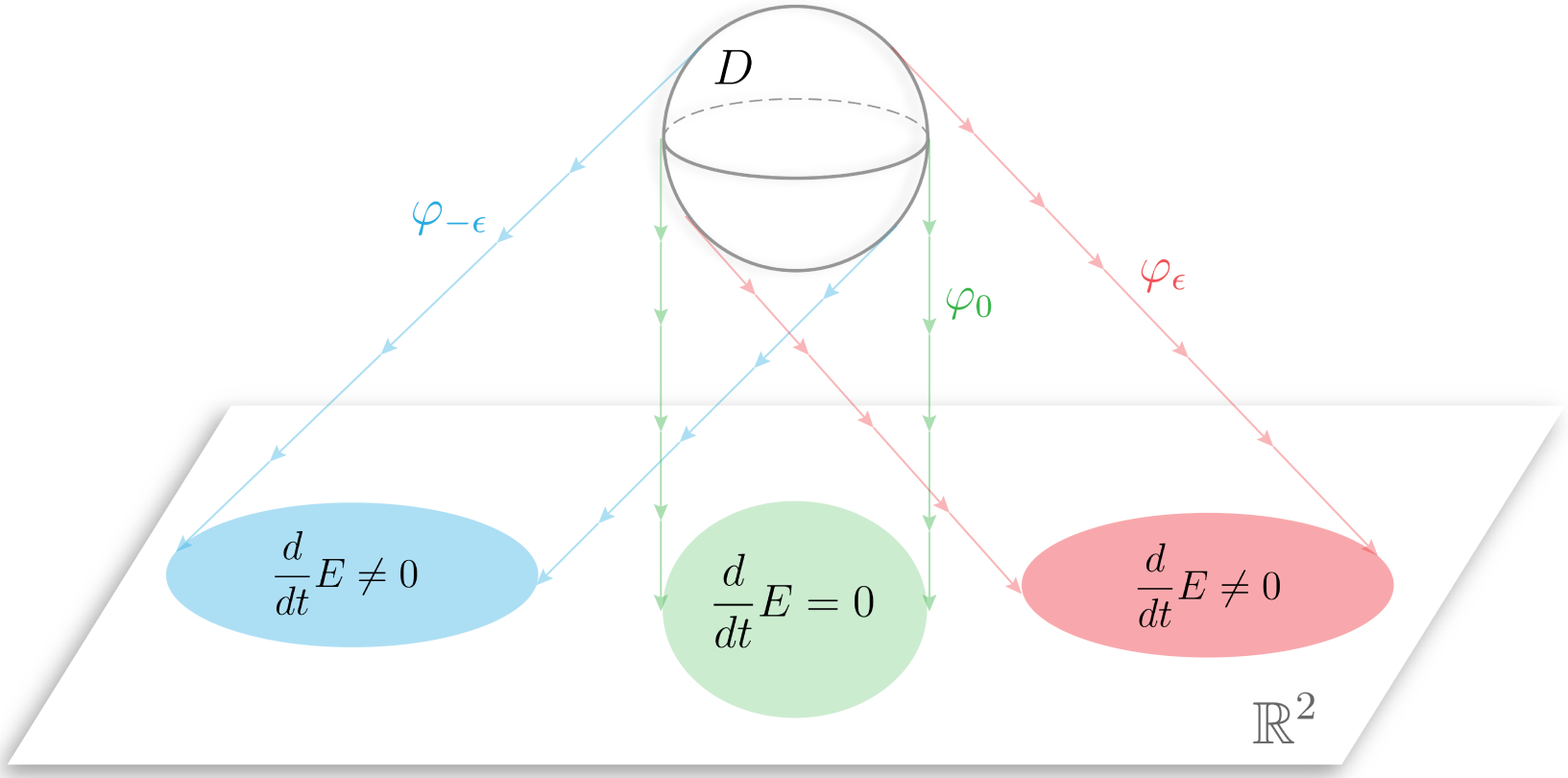
Definition

A map φ is consider harmonic if

$$\frac{d}{dt} E(\varphi_t; D)\Big|_{t=0} = 0$$

over all compact domains D of M and all smooth variations $\{\varphi_t\}$ of φ where $\varphi_0 = \varphi$

Minimizing the Energy Functional



Harmonic Maps

It can be shown that

Derivative of Energy Functional

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\varphi_t; D)\Big|_{t=0} = -\int_M \langle v, \tau(\varphi)\rangle \, dV$$

where $v(x) := \frac{\partial \varphi_t}{\partial t}(x)\Big|_{t=0} \in \Gamma(\varphi^{-1}TN)$ and $\langle \cdot, \cdot \rangle$ is the metric on the pullback bundle.

Harmonic Maps

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Derivative of Energy Functional

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where $v(x) := \frac{\partial \varphi_t}{\partial t}(x)\Big|_{t=0} \in \Gamma(\varphi^{-1}TN)$ and $\langle \cdot, \cdot \rangle$ is the metric on the pullback bundle.

Theorem

A map φ is harmonic if and only if

$$\tau(\varphi) = 0$$

Characterization of Harmonic Morphisms (Riemannian)

Theorem

A map $\varphi : (M^m, g) \to (N^n, h)$ is a harmonic morphism if and only if both of the following conditions hold:

(i) φ is an harmonic map

(ii) φ is horizontally weakly conformal

Fundamental Equation

Let φ be a HWC submersion (meaning no critical points).

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Tension Field for HWC Submersion

$$\tau(\varphi) = -\mathrm{d}\varphi\Big[(n-2)\,\pi^{\mathcal{H}}\big(\vec{\nabla}\ln\lambda\big) + (m-n)\,\mu^{\mathcal{V}}\Big]$$

where $\pi^{\mathcal{H}}$ is the projection on the horizontal space, and $\mu^{\mathcal{V}}$ is the mean curvature of the fibres of φ

$$\mu^{\mathcal{V}} = \sum_{i=1}^{m-n} \pi^{\mathcal{H}}(\nabla_{E_i} E_i)$$

Fundamental Equation

Theorem (Fundamental Equation)

Let $\varphi : (M^m, g) \to (N^n, h)$ be a non-constant HWC submersion. Then φ is a harmonic morphism if and only if

$$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla} \ln \lambda) + (m-n)\,\mu^{\mathcal{V}} = 0$$

[see [8]]

$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) + (m-n)\,\mu^{\mathcal{V}} = 0$

Solutions

(i) $m = n = 2 \Rightarrow$ All HWC submersions from M^2 to N^2 are harmonic morphisms.

$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) + (m-n)\,\mu^{\mathcal{V}} = 0$

Solutions

(i) $m = n = 2 \Rightarrow$ All HWC submersions from M^2 to N^2 are harmonic morphisms.

(ii) $n = 2, m \neq n \Rightarrow \mu^{\mathcal{V}} \overset{!}{=} 0.$

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(iii) $n \neq 2, m = n \Rightarrow \pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) \stackrel{!}{=} 0.$

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(ii) $n = 2, m \neq n \Rightarrow \mu^{\mathcal{V}} \overset{!}{=} 0.$

(iii) $n \neq 2, m = n \Rightarrow \pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) \overset{!}{=} 0.$

(iv) $m, n \geq 3, m \neq n \Rightarrow$

(a) $\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) = \mu^{\mathcal{V}} = 0$

$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) + (m-n)\,\mu^{\mathcal{V}} = 0$

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(iv) $m, n \geq 3, m \neq n \Rightarrow$

(a) $\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) = \mu^{\mathcal{V}} = 0$

(b) $(n-2)\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) = -(m-n)\mu^{\mathcal{V}}$

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- Killing Type
- Warped Product Type
- T Type
- Interrelation of Three Types

Harmonic Morphisms with One-dimensional Fibres

One-dimensional Fibres

Let $\varphi : M^{n+1} \to N^n$ be harmonic morphism. Then $\dim \varphi^{-1}(p) = (n+1) - n = 1$. Hence, we say φ is a harmonic morphism with one-dimensional fibres

Classification

The Three Types

Harmonic morphisms with one-dimensional fibres have been completely characterized. They are divided into three groups:

(i) Killing type
(ii) warped product type
(iii) T type

Classification

The Three Types

Harmonic morphisms with one-dimensional fibres have been completely characterized. They are divided into three groups:

(i) Killing type
(ii) warped product type
(iii) T type

Differences

All three types can be characterized by the geometric relationship of the level-sets of the dilation function λ to the fibres and horizontal submanifolds generated by the map φ

Killing Type

Killing Vector Field

Refered to as Killing type because the fundamental vertical vector field V is a generator for a Killing vector field i. e.

$$g(\nabla_V X, Y) + g(X, \nabla_V Y) = 0$$

where $X, Y \in \Gamma(TM)$

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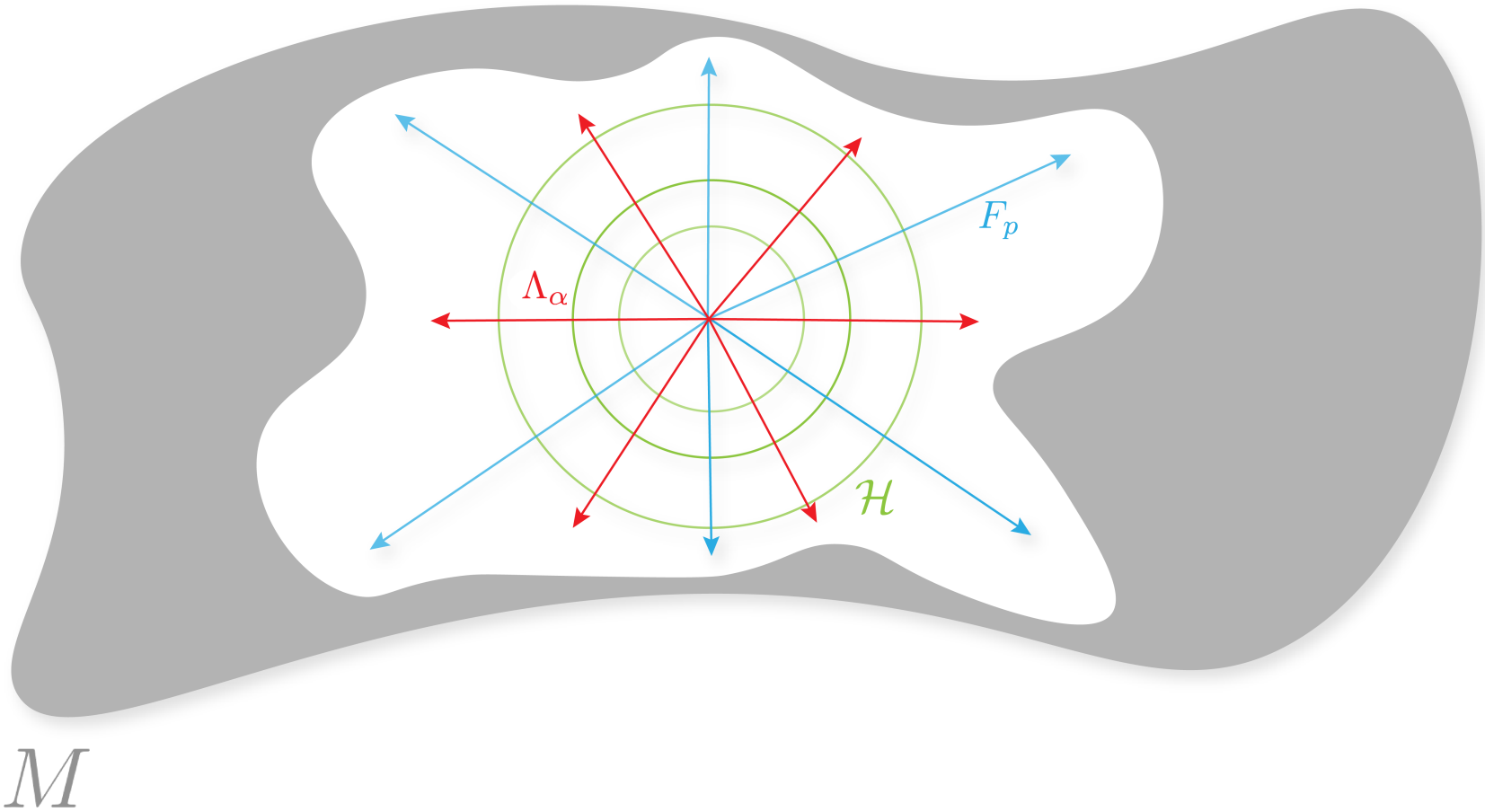
$$g(\nabla_V X, Y) + g(X, \nabla_V Y) = 0$$

where $X, Y \in \Gamma(TM)$

Remark

In the Killing type, the fibres of φ are parallel to the level sets of λ. In other words, $\pi^{\mathcal{V}}(\vec{\nabla}\lambda) = 0$

Killing Type



Examples

Killing Type Harmonic Morphisms

1. Orthogonal projection.
2. Projection onto the first or second factor of a Riemannian product $\mathbb{R}^m \times \mathbb{R}^n$.
3. Riemannian submersions.

Warped Product Type

Warped Product Manifold

A manifold $W = M \times_f N$ with associated Riemannian metric $w = g + f^2\, h$ is called a warped product manifold with warping function $f : M \to \mathbb{R}^+$.

Warped Product Type

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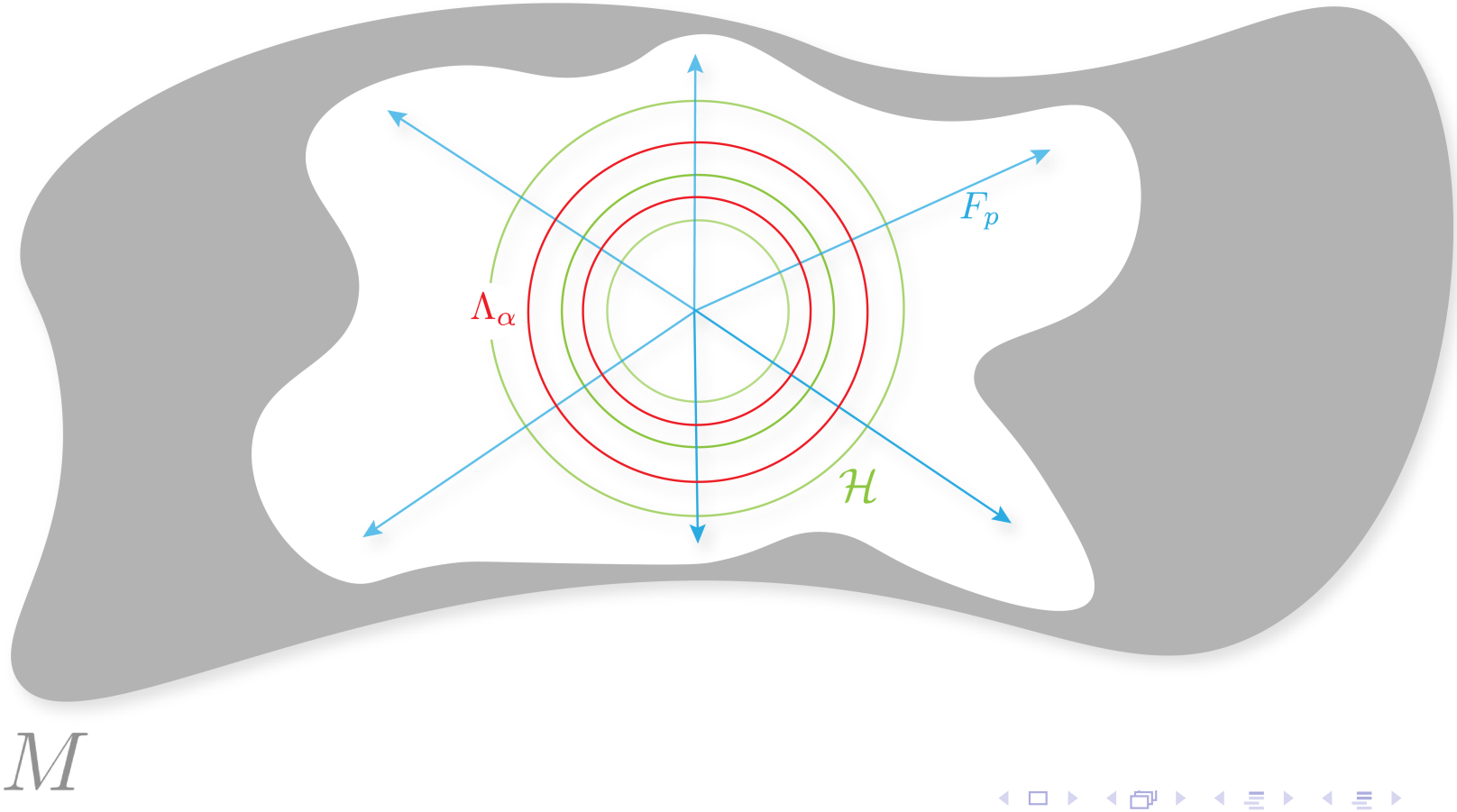
A manifold $W = M \times_f N$ with associated Riemannian metric $w = g + f^2\, h$ is called a warped product manifold with warping function $f : M \to \mathbb{R}^+$.

Warped Product Type

The projection $\varphi : M \times_f N \to N$ onto the second factor of a warped product manifold is a horizontally homothetic submersion with minimal fibres. In other words, $\pi^{\mathcal{H}}(\vec{\nabla}\lambda) = 0$ and $\mu^{\mathcal{V}} = 0 \Rightarrow$ (by the fundamental equation) φ is a harmonic morphism.

Warped Product Type



Examples

Warped Product Type Harmonic Morphisms

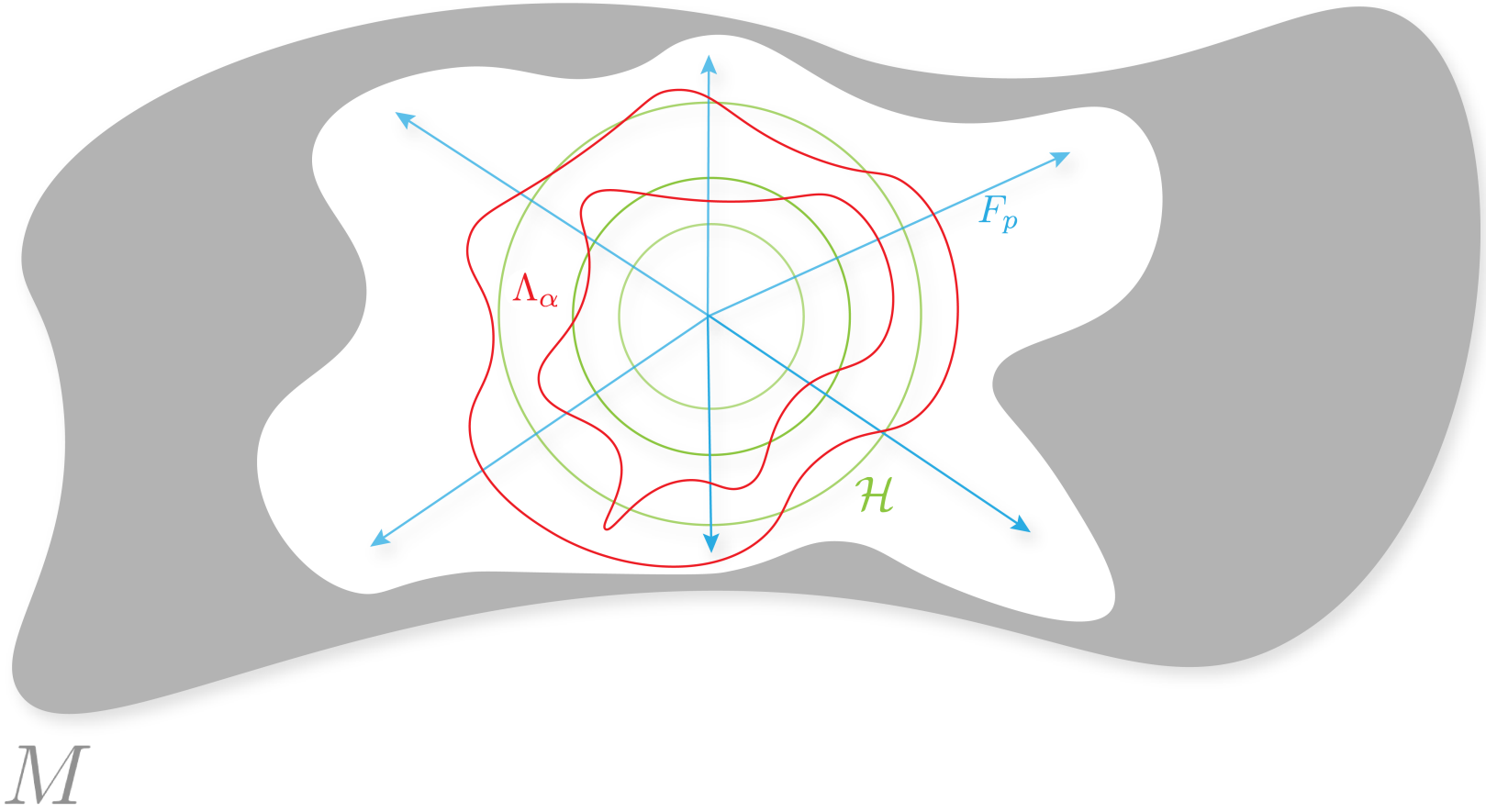
1. Radial Projection from $\mathbb{R}^n$ to S^{n-1}
2. Radial Projection from $S^n \setminus \{(\pm 1, 0, \ldots, 0)\}$ to S^{n-1}
3. Projection onto the second factor of a warped product manifold

T Type

T Type

A T Type harmonic morphisms φ is characterized by $\pi^{\mathcal{V}}(\vec{\nabla}\lambda) \neq 0$ and by the fact that the level-sets of λ are transversal to the fibres of φ.

T Type



Interrelation of Three Types

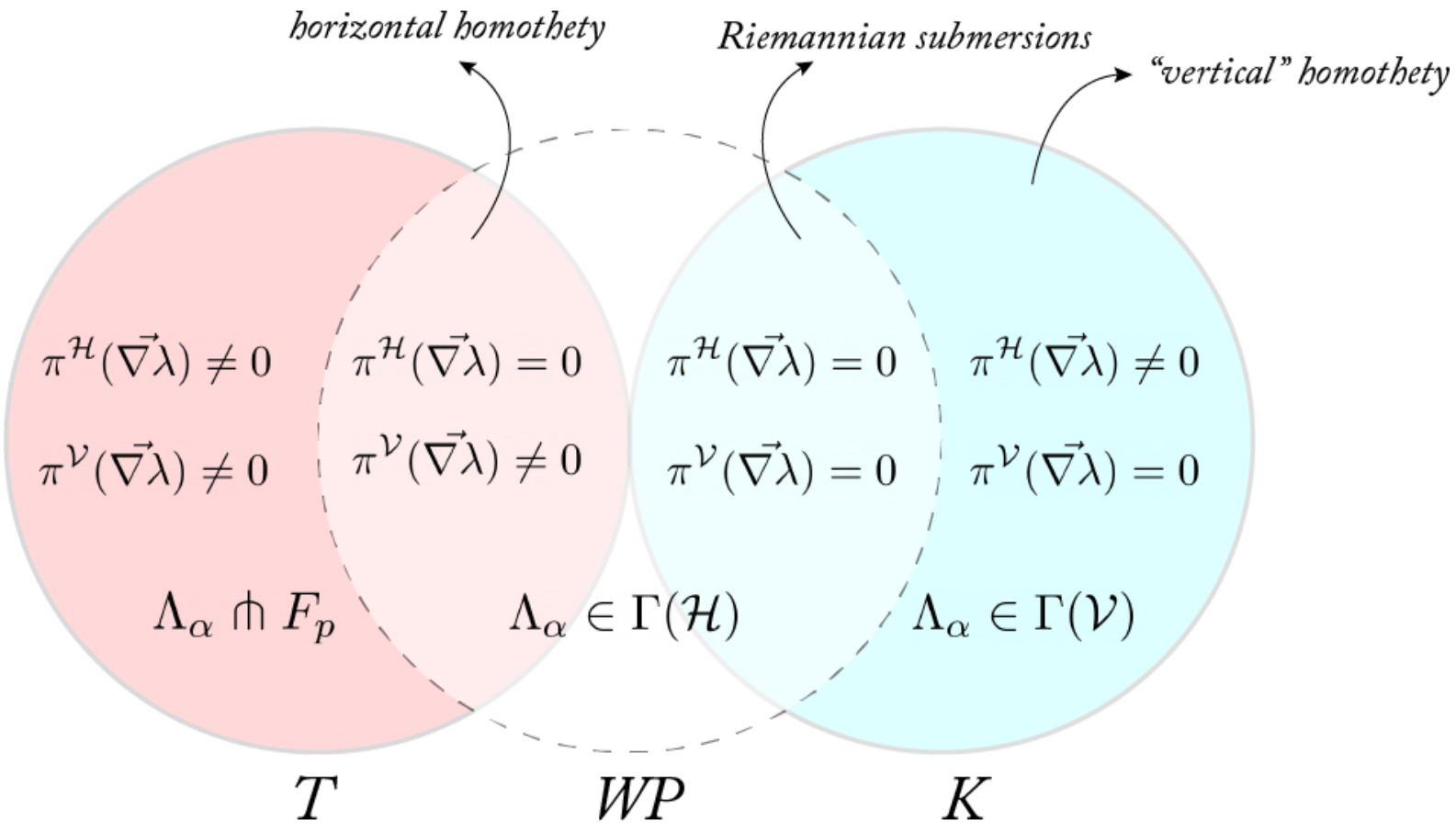


Table of Contents

5 Milnor Fibration Problem
- The Set Up
- The Problem
- Proposition
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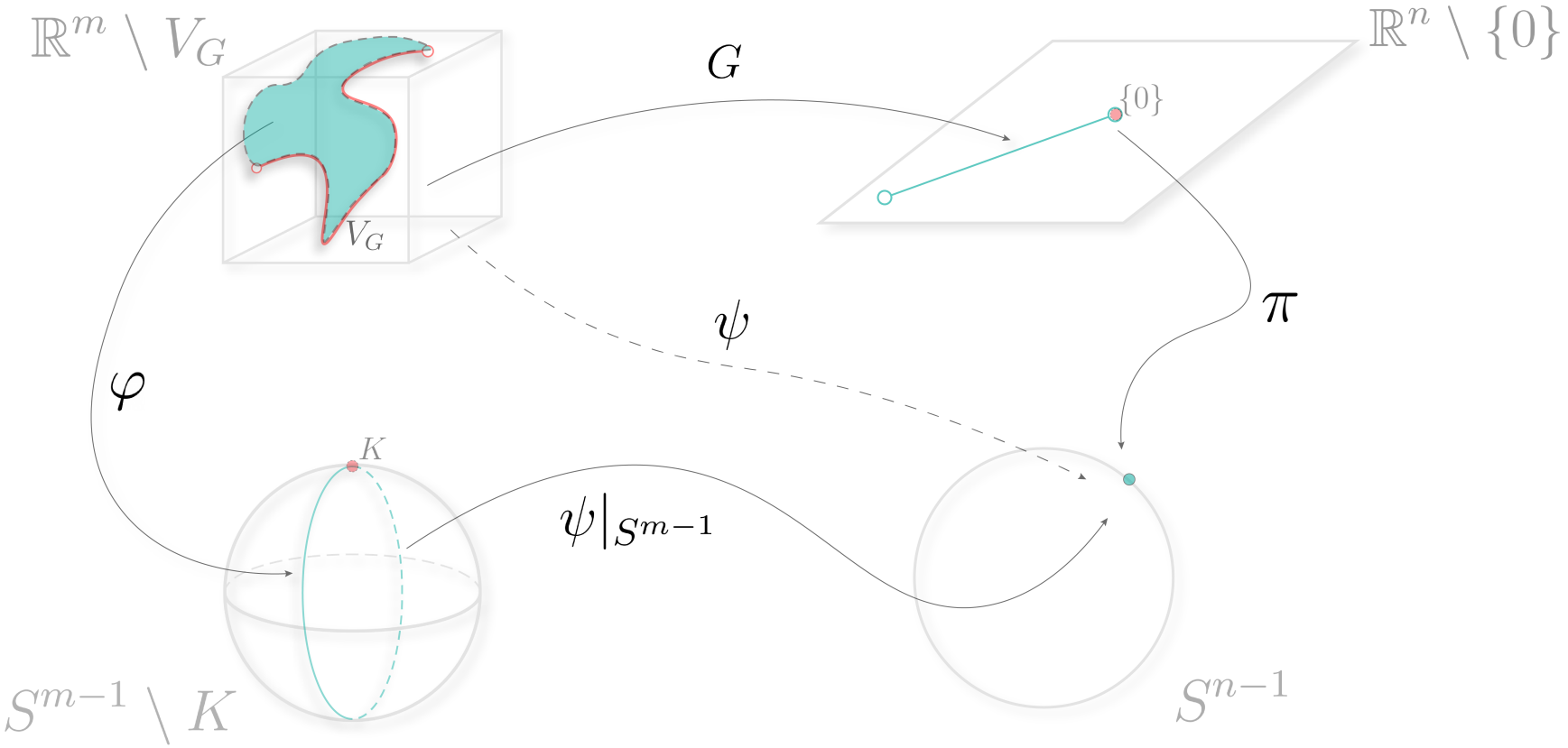
Milnor Fibration

$$\begin{tikzcd}
\mathbb{R}^m \setminus V_G \arrow[r, "G"] \arrow[d, "\varphi"'] \arrow[dr, dashed, "\Psi"] & \mathbb{R}^n \setminus \{0\} \arrow[d, "\pi"] \\
S_\epsilon^{m-1} \setminus K_\epsilon \arrow[r, "\Psi|"'] & S^{n-1}
\end{tikzcd}$$

1. $G : \mathbb{R}^m \to \mathbb{R}^n$ is a map with an isolated singularity ($\mathrm{d}G = 0$) at the origin
2. $V_G := G^{-1}(0) \leftrightsquigarrow$ The Variety of G
3. $K_\epsilon := S_\epsilon^{m-1} \cap V_G \leftrightsquigarrow$ The Link
4. π is radial projection
5. φ is a submersion
6. $\psi| = \frac{G}{|G|}$ is the Milnor fibration map

A Diagram


$\mathbb{R}^m \setminus V_G$
V_G
G
$\mathbb{R}^n \setminus \{0\}$
$\{0\}$
ψ
π
φ
K
$\psi|_{S^{m-1}}$
$S^{m-1} \setminus K$
S^{n-1}

The Problem

$$\begin{tikzcd}
\mathbb{R}^m \setminus V_G \arrow[r, "G"] \arrow[d, "\varphi"'] \arrow[dr, dashed, "\Psi"] & \mathbb{R}^n \setminus \{0\} \arrow[d, "\pi"] \\
S^{m-1}_\epsilon \setminus K_\epsilon \arrow[r, "\Psi|"'] & S^{n-1}
\end{tikzcd}$$

Theorem

It can be shown that if G is a homogeneous polynomial harmonic morphism, then it retracts to a map $\psi| : S^{m-1} \setminus K \to S^{n-1}$ that is also a harmonic morphism which in turn defines a Milnor fibration [See [6]].

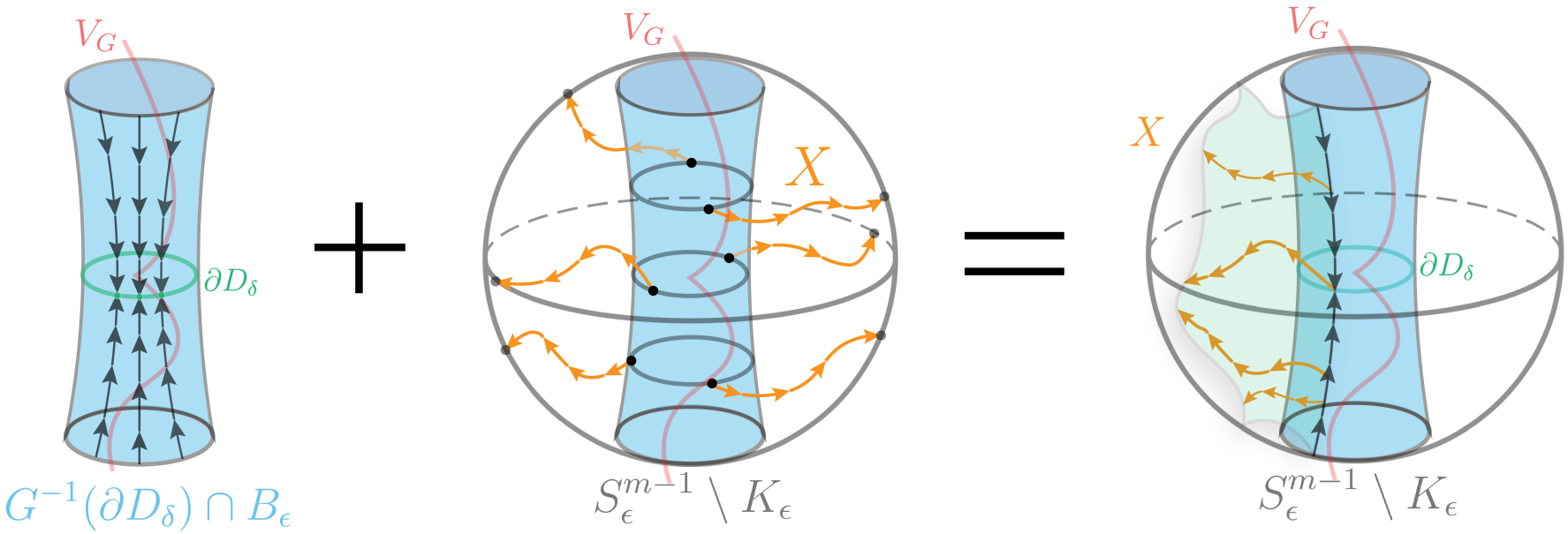
The Problem

$$\begin{tikzcd}
\mathbb{R}^m \setminus V_G \arrow[r, "G"] \arrow[d, "\varphi"'] \arrow[dr, dashed, "\Psi"] & \mathbb{R}^n \setminus \{0\} \arrow[d, "\pi"] \\
S_\epsilon^{m-1} \setminus K_\epsilon \arrow[r, "\Psi|"'] & S^{n-1}
\end{tikzcd}$$

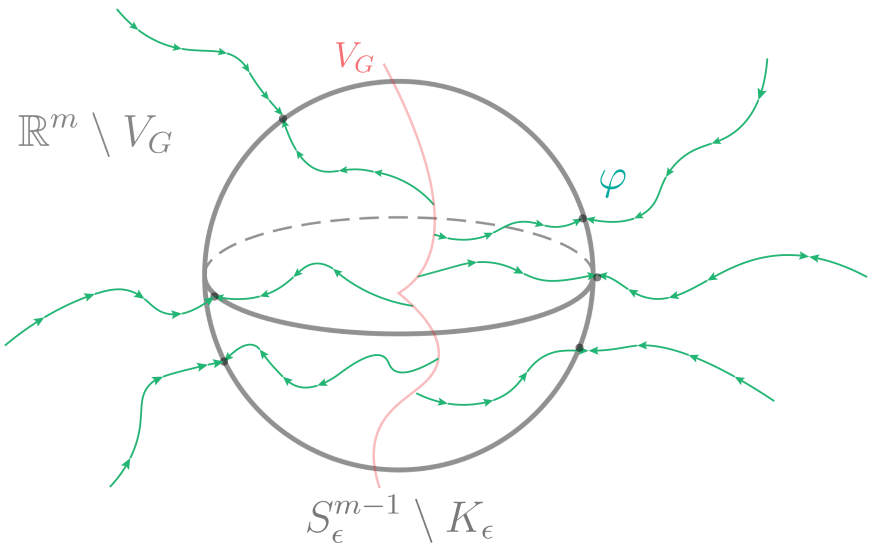
The Question

Can we relax the requirement that G needs to be a homogeneous map?

Milnor Fibration Theorems [5] , [7]



Harmonic Morphism



A map φ

The integral curves of the vector field X can also function as the fibres of a map $\varphi : \mathbb{R}^m \setminus V_G \to S_\epsilon^{m-1}$. Since $\arg G$ is preserved means $\forall p \in S^{m-1} \subset \mathbb{R}^m$ we have $\varphi(p) = p$

Proposition

φ is radial projection

Let $\varphi : \mathbb{R}^m \setminus V_G \to S_\epsilon^{m-1} \setminus K_\epsilon$ be a harmonic morphism such that for all $p \in S_\epsilon^{m-1} \subset \mathbb{R}^m$ we have $\varphi(p) = p$. Then φ is radial projection.

Proposition

φ is radial projection

Let $\varphi : \mathbb{R}^m \setminus V_G \to S_\epsilon^{m-1} \setminus K_\epsilon$ be a harmonic morphism such that for all $p \in S_\epsilon^{m-1} \subset \mathbb{R}^m$ we have $\varphi(p) = p$. Then φ is radial projection.

Proposition

Proof

- Since φ preserves the sphere $S_\epsilon^{m-1} \Rightarrow$ the dilation $\lambda(x)$ is constant on the sphere.

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- Assuming the dilation in general is non-constant $\Rightarrow \vec{\nabla}\lambda$ is non-zero on S_ϵ^{m-1} which means φ is locally of warped product type and S_ϵ^{m-1} is a horizontal submanifold.

Proposition

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- Assuming the dilation in general is non-constant $\Rightarrow \vec{\nabla}\lambda$ is non-zero on S_ϵ^{m-1} which means φ is locally of warped product type and S_ϵ^{m-1} is a horizontal submanifold.
- Since φ is of warped product type $\Rightarrow$ The horizontal submanifolds of φ must constitute a parallel family of umbilic submanifolds $\Rightarrow$ all horizontal submanifolds of φ are spheres

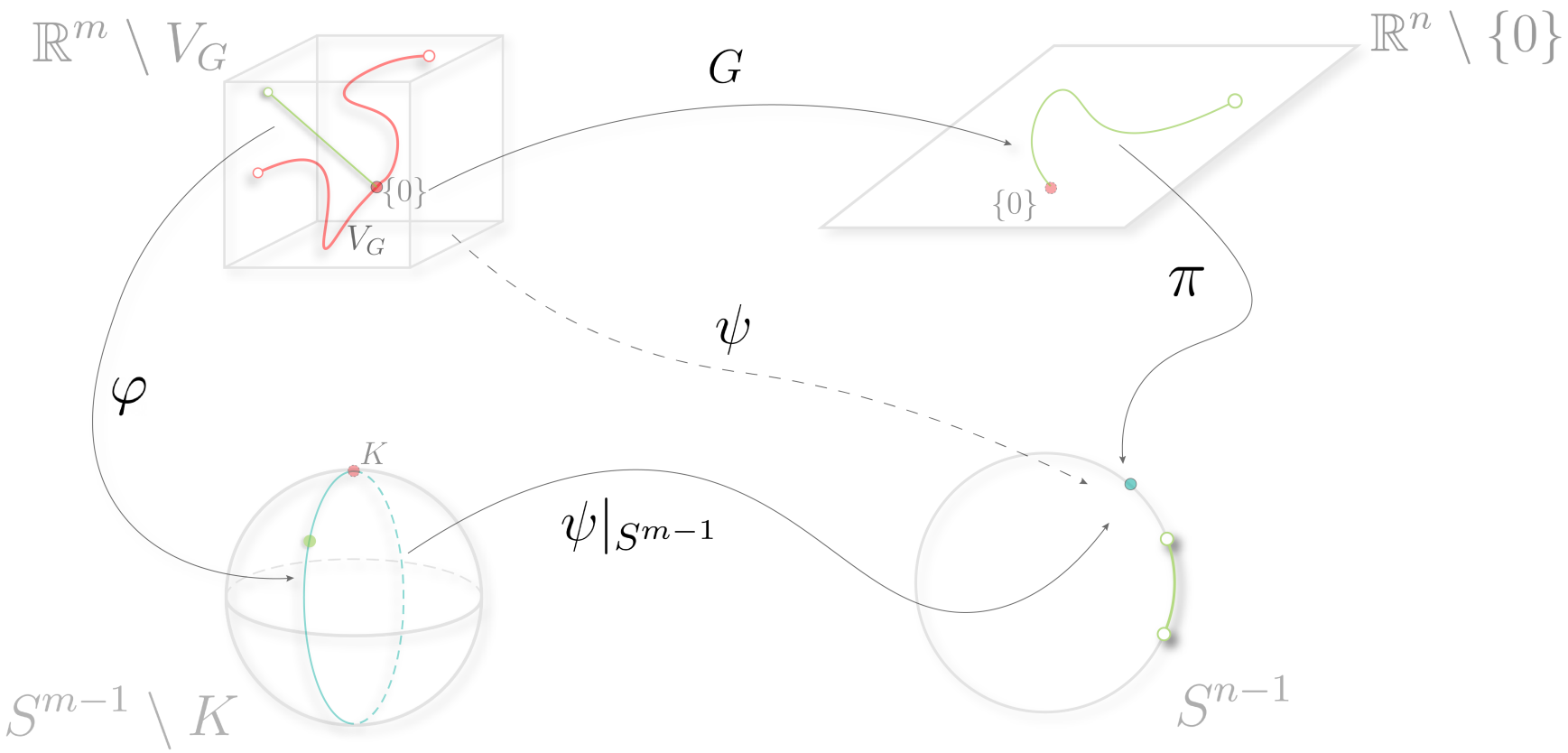
Proposition

Proof

- The fibres (vertical submanifolds) of φ must be orthogonal to all the horizontal submanifolds $\Rightarrow \varphi$ is radial projection

[See [3] , [4], [2], [1] , [8]]

If G is Not Homogeneous



Final Result

$$\begin{tikzcd}
\mathbb{R}^m \setminus V_G \arrow[r, "G"] \arrow[d, "\varphi"'] \arrow[dr, dashed, "\Psi"] & \mathbb{R}^n \setminus \{0\} \arrow[d, "\pi"] \\
S_\epsilon^{m-1} \setminus K_\epsilon \arrow[r, "\Psi|"'] & S^{n-1}
\end{tikzcd}$$

Conclusion

Ultimately, this means that the class of harmonic morphisms that retract down harmonic morphisms is limited to homogeneous polynomial maps

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