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Abstract Linear Algebra

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Orthogonal Complements and Minimization Problems

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1 Introduction

For the orthogonal complement we will present a definition, some basic properties, and then move on to three theorems. For the orthogonal projection we will present a definition, some basic properties, and then move on to an application to minimization problems.

2 Orthogonal Complements

Definition 2.1. If U is a subset of V, then the **orthogonal complement** of U, denoted $U^{\perp}$, is the set of all vectors in V that are orthogonal to every vector in U:

$$U^{\perp} = \{v \in V : \langle v, u \rangle = 0 \ \forall u \in U\}$$

Example 2.1. If U is a plane in $\mathbb{R}^3$ then $U^{\perp}$ is a line perpendicular to the plane that passes through the origin.

We now present some basic properties of the orthogonal complement.

Theorem 2.1.

1. If $U \subset V$, then $U^{\perp} < V$
2. $\{0\}^{\perp} = V$
3. $V^{\perp} = \{0\}$
4. If $U \subset V$, then $U \cap U^{\perp} \subset \{0\}$.
5. If $U \subset W \subset V$, then $W^{\perp} \subset U^{\perp}$

Proof.

1. Since $\langle 0, u \rangle = 0 \ \forall u \in U, 0 \in U^{\perp}$. Thus $U^{\perp} \neq \emptyset$. Let $w_1, w_2 \in U^{\perp}$. Then for any $u \in U$, $\langle w_1 + w_2, u \rangle = \langle w_1, u \rangle + \langle w_2, u \rangle = 0$ by the definition of the inner product and $w_1, w_2 \in U^{\perp}$. Thus $w_1 + w_2 \in U^{\perp}$. Take $c \in \mathbb{F}, w \in U^{\perp}$. Then for any $u \in U$, $\langle cw, u \rangle = c \cdot \langle w, u \rangle = c \cdot 0 = 0$ by the definition of the inner product and $w \in U^{\perp}$. Thus $cw \in U^{\perp}$ and $U^{\perp} < V$.

2. The orthogonal complement of any subset of V is necessarily a subset of V. If you prefer, we could also argue that since $\{0\} \subset V$, by (1) we have that $\{0\}^{\perp} < V$. Thus $\{0\}^{\perp} \subset V$. Now we show that $V \subset \{0\}^{\perp}$. Take any $v \in V$. Then $\langle v, 0 \rangle = 0$ by the properties of inner products. Thus $v \in \{0\}^{\perp}$. Thus $V \subset \{0\}^{\perp}$ and it follows that $V = \{0\}^{\perp}$.

3. Since $\langle 0, v \rangle = 0$ for all $v \in V$, $\{0\} \subset V^{\perp}$. Now take any $v \in V^{\perp}$. Note that $V^{\perp} \subset V$, so v is also in V. Since $v \in V^{\perp}$, v is orthogonal to every vector in V. Symbolically we can see this as $\langle v, v' \rangle = 0$ for all $v' \in V$. But v itself is in V. Therefore $\langle v, v \rangle = 0$. By the definition of inner product, this implies that $v = 0$. Therefore $V^{\perp} \subset \{0\}$ and it follows that $V^{\perp} = \{0\}$.

4. Take any $u \in U \cap U^{\perp}$. Then $u \in U$ and $\langle u, u' \rangle = 0$ for all $u' \in U$. Thus $\langle u, u \rangle = 0$ which implies that $u = 0$. Thus $U \cap U^{\perp} \subset \{0\}$.

   Note that if $U < V$ then since $U^{\perp} < V$ we have that $U \cap U^{\perp} < V$. Therefore $\{0\} \subset U \cap U^{\perp}$ and we have $U \cap U^{\perp} = \{0\}$

5. Take $v \in W^{\perp}$. Then $\langle v, w \rangle = 0$ for all $w \in W$. But since $U \subset W$, all $u \in U$ are in W. Thus $\langle v, u \rangle = 0$ for all $u \in U$. Thus $v \in U^{\perp}$. Thus $W^{\perp} \subset U^{\perp}$.

□

We now present a useful fact about how a subspace and its orthogonal complement sit inside the whole space.

Theorem 2.2. Suppose U is a finite-dimensional subspace of V. Then

$$V = U \oplus U^{\perp}$$

Proof. We will first show that $V = U + U^{\perp}$. Take any $v \in V$. Let $e_1, e_2, ..., e_m$ be an orthonormal basis of U. Let $u = \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \cdots + \langle v, e_m \rangle e_m$. Then we can write v as

$$\begin{aligned} v &= u + v - u \\ &= \langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \cdots + \langle v, e_m \rangle e_m + v - (\langle v, e_1 \rangle e_1 + \langle v, e_2 \rangle e_2 + \cdots + \langle v, e_m \rangle e_m). \end{aligned}$$

Let $w = v - u$. We note that $u \in U$ because u is a linear combination of basis elements of U. We claim that $w \in U^{\perp}$. We note that for each $j = 1, ..., m$ we have

$$\begin{aligned}
\langle w, e_j \rangle &= \langle v - u, e_j \rangle && \text{(definition of w)} \\
&= \langle v, e_j \rangle - \langle u, e_j \rangle && \text{(inner product property)} \\
&= \langle v, e_j \rangle - \langle \langle v, e_1 \rangle e_1 + \cdots + \langle v, e_m \rangle e_m, e_j \rangle && \text{(definition of } u\text{)} \\
&= \langle v, e_j \rangle - \langle \langle v, e_1 \rangle e_1, e_j \rangle - \cdots - \langle \langle v, e_m \rangle e_m, e_j \rangle && \text{(inner product property)} \\
&= \langle v, e_j \rangle - \langle v, e_1 \rangle \langle e_1, e_j \rangle - \cdots - \langle v, e_m \rangle \langle e_m, e_j \rangle && \text{(inner product property)} \\
&= \langle v, e_j \rangle - \langle v, e_1 \rangle \cdot 0 - \cdots - \langle v, e_j \rangle \langle e_j, e_j \rangle - \langle v, e_m \rangle \cdot 0 && \text{(properties of orthonormal basis)} \\
&= \langle v, e_j \rangle - \langle v, e_j \rangle \cdot 1 \\
&= 0.
\end{aligned}$$

Thus w is orthogonal to every basis element of U. Take any $x \in U$. Then $x = c_1 e_1 + \cdots + c_m e_m$ for some $c_1, ..., c_m \in \mathbb{F}$. Then

$$\begin{aligned}
\langle w, x \rangle &= \langle w, c_1 e_1 + \cdots c_m e_m \rangle && \text{(definition of x)} \\
&= \langle w, c_1 e_1 \rangle + \cdots + \langle w, c_m e_m \rangle && \text{(property of inner product)} \\
&= \bar{c_1} \langle w, e_1 \rangle + \cdots + \bar{c_m} \langle w, e_m \rangle && \text{(property of inner product)} \\
&= \bar{c_1} \cdot 0 + \cdots + \bar{c_m} \cdot 0 && \text{(by claim above)} \\
&= 0.
\end{aligned}$$

Thus $w \in U^{\perp}$. Therefore we have that $V = U + U^{\perp}$. We have previously shown that $U \cap U^{\perp} \subset \{0\}$. Then if $U{<}V$, then $U, U^{\perp}{<}V$ implies that $\{0\} \subset U \cap U^{\perp}$. Thus $U \cap U^{\perp} = \{0\}$ and we have the direct sum as desired. $\square$

The next result follows directly from the previous.

Theorem 2.3. Suppose V is finite-dimensional and U is a subspace of V. Then

$$\dim(U^{\perp}) = \dim(V) - \dim(U)$$

Proof. From the result above we have that $V = U \oplus U^{\perp}$ and using a previous theorem we then have that $\dim(V) = \dim(U) + \dim(U^{\perp})$. Therefore $\dim(U^{\perp}) = \dim(V) - \dim(U)$. $\square$

Our last result of this section shows that the orthogonal complement of the orthogonal complement is just the space itself.

Theorem 2.4. Suppose U is a finite-dimensional subspace of V. Then

$$U = (U^{\perp})^{\perp}$$

Proof. First want to show that $U \subset (U^{\perp})^{\perp}$
To do this, suppose $u \in U$. Then $\langle u, v\rangle = 0$ for every $v \in U^{\perp}$ by the definition of orthogonal complement. Because u is orthogonal to every vector in $U^{\perp}$, we have $u \in (U^{\perp})^{\perp}$, So, $U \subset (U^{\perp})^{\perp}$.

To prove the inclusion in the other direction, suppose $v \in (U^{\perp})^{\perp}$. Since $V = U \oplus U^{\perp}$, we can write $v = u + w$, where $u \in U$ and $w \in U^{\perp}$. We have $v - u = w \in U^{\perp}$. Because $v \in (U^{\perp})^{\perp}$ and $u \in (U^{\perp})^{\perp}$, we have $v - u \in (U^{\perp})^{\perp}$. Thus $v - u \in U^{\perp} \cap (U^{\perp})^{\perp}$, which implies that $v - u$ is orthogonal to itself, which implies that $v - u = 0$, which implies that $v = u$, which implies that $v \in U$. Thus $(U^{\perp})^{\perp} \subset U$, which completes the proof. $\square$

3 Orthogonal Projection

Definition 3.1. Suppose U is a finite-dimensional subspace of V. The **orthogonal projection** of V onto U is the operator $P_U \in L(V)$ defined as follows: For $v \in V$, write $v = u + w$, where $u \in U$ and $w \in U^{\perp}$. Then $P_U v = u$.

Example 3.1. Suppose $x \in V$ with $x \neq 0$ and $U = span(x)$ Show that

$$P_U v = \frac{\langle v, x\rangle}{\|x\|^2} x$$

for every $v \in V$.

Proof.

$$\begin{aligned} v &= \frac{\langle v, x\rangle}{\|x\|^2}x + (v - \frac{\langle v, x\rangle}{\|x\|^2}x) \\ &\frac{\langle v, x\rangle}{\|x\|^2}x \in \text{ span}(x) \\ &(v - \frac{\langle v, x\rangle}{\|x\|^2}x) \in U^{\perp} \qquad \text{since } \langle v - \frac{\langle v, x\rangle}{\|x\|^2}x, x\rangle = 0 \end{aligned}$$

$$\text{Therefore,} \qquad P_U v = \frac{\langle v, x\rangle}{\|x\|^2}x$$

□

3.1 Properties of the orthogonal projection

Theorem 3.1. Suppose U is a finite-dimensional subspace of V and $v \in V$. Then

1. $P_U \in L(U)$
2. $P_U u = u$ for every $u \in U$
3. $P_U w = 0$ for every $w \in U^{\perp}$
4. range $P_U = U$
5. null$P_U = U^{\perp}$
6. $v - P_U v \in U^{\perp}$
7. $P_U^2 = P_U$
8. $\|P_U v\| \leq \|v\|$

Proof. 1. Suppose $v_1 = u_1 + w_1, v_2 = u_2 + w_2 \in V$ with $u_1, u_2 \in U$ and $w_1, w_2 \in U^{\perp}$.
Thus $P_U v_1 = u_1$ and $P_U v_2 = u_2$
But $v_1 + v_2 = (u_1 + u_2) + (w_1 + w_2)$
where $u_1 + u_2 \in U$ and $w_1 + w_2 \in U^{\perp}$

$\implies P_U(v_1 + v_2) = u_1 + u_2 = P_U v_1 + P_U v_2$

Similarly, suppose $\lambda \in \mathbb{F}$

$\lambda v_1 = \lambda u_1 + \lambda w_1$ where $\lambda u_1 \in U$ and $\lambda w_1 \in U^\perp$

$\implies P_U(\lambda v_1) = \lambda u_1 = \lambda P_U v_1$

2. $P_U u = u$ for every $u \in U$

   Suppose $u \in U$. We can write $u = u + 0$, where $u \in U, 0 \in U^\perp$ Therefore, $P_U u = u$.

3. $P_U w = 0$ for every $w \in U^\perp$

   Suppose $w \in U$. We can write $w = 0 + w$, where $0 \in U, w \in U$ Therefore, $P_U w = 0$

4. range $P_U = U$

   The definition of P_U implies that range $P_U \subset U$.

   From (2) above, $P_U u = u$ for every $u \in U \implies U \subset$ range P_U

   Therefore, range $P_U = U$.

5. null $P_U = U^\perp$

   From (3) above, $P_U w = 0$ for every $w \in U^\perp \implies U^\perp \subset$ null P_U

   Next, suppose $v \in$ null $P_U \implies P_U v = 0 \implies v = 0 + v$

   where $0 \in U, v \in U^\perp \implies$ null $P_U \subset U^\perp$.

   Therefore, null $P_U = U^\perp$

6. $v - P_U v \in U^\perp$

   Let $v = u + w,$ with $u \in U$ and $w \in U^\perp$ then

   $$v - P_U v = v - u = w \in U^\perp$$

7. $P_U^2 = P_U$

   If $v = u + w$ with $u \in U, w \in U^\perp$, then

   $$(P_U^2)v = P_U(P_U v) = P_U u = u = P_U v$$

8. $\|P_U v\| \leq \|v\|$

   let $v = u + w$ with $u \in U, w \in U \perp$,then

$$\|P_U v\|^2 = \|u\|^2 \leq \|u\|^2 + \|w\|^2 = \|v\|^2;$$

   where the last equality comes from the Pythagorean Theorem.

$\square$

4 Minimization Problems

Let us now apply the inner product to minimization problems. Given a subspace $U \subset V$ and a vector $v \in V$, find the $u \in U$ that is closest to the vector V. In other words, we want to make $\|v - u\|$ as smallest as possible.

The next theorem shows that $P_U v$ is the closest point in U to the vector v.

Theorem 4.1. Suppose U is a finite dimensional subspace of V, $v \in V$ and $u \in U$. Then,

$$\|v - P_U v\| \leq \|v - u\|$$

Furthermore, the inequality above is an equality if and only if $u = P_U v$

Proof. Consider,

$$\|v - P_U v\|^2 \leq \|v - P_U v\|^2 + \|P_U v - u\|^2$$

Because $\|P_U v - u\| \geq 0$

$v - P_U v \in U^\perp$ and $P_U v - u \in U \implies v - P_U v$ and $P_U v - u$ are orthogonal to each other.

Therefore, by Pythagorean theorem,

$$\|v - P_U v\|^2 \leq \|(v - P_U v) + (P_U v - u)\|^2 = \|v - u\|^2$$

$\implies \|v - P_U v\| \leq \|v - u\|$

If $P_U v = u$ then $P_U v - u = 0$. Then,

$$\|v - P_U v\|^2 = \|v - P_U v\|^2 + \|P_U v - u\|^2 = \|v - u\|^2$$

$\square$

Example 4.1. Find a polynomial u with real coefficients and degree at most 5 that approximate $sin(x)$ as well as possible on the interval $[-\pi, \pi]$ in the sense that,

$$\int_{-\pi}^{\pi} \mid sin(x) - u(x) \mid^2 dx$$

is as well as possible. Compare this result to the Taylor series approximation.

***Solution*:**

$C_{\mathbf{R}}[-\pi, \pi]$ denotes continuous real valued functions on $[-\pi, \pi]$. Let $f, g \in C_{\mathbf{R}}[-\pi, \pi]$. Then

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx - - - - - (1)$$

Let $v \in C_{\mathbf{R}}[-\pi, \pi]$ be the function defined by $v(x) = sin(x)$. Let U denote the subspace of $C_{\mathbf{R}}[-\pi, \pi]$ consisting polynomials with real coefficients and degree at most 5. Now we can reformulate our problem as follows,
Find $U \in U$ such that $\|v - u\| = (\int_{-\pi}^{\pi} \mid sin(x) - u(x) \mid^2 dx)^2$ is smallest as possible. $B = \{1, x^1, x^2, x^3, x^4 . x^5, x^6\}$ basis for U. By applying Gram-Schmidt procedure we can find Orthonormal basis $B_0 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ of U as follows,

$$e_1 = \frac{1}{\sqrt{2\pi}}, e_2 = \frac{\sqrt{6}x}{2\pi\sqrt{\pi}}, e_3 = \frac{1}{8}\frac{\sqrt{8}\sqrt{45}(x^2 - \frac{1}{3}\pi^2)}{\sqrt{\pi^5}}$$

$$e_4 = \frac{1}{8}\frac{\sqrt{175}\sqrt{8}(x^3 - \frac{3}{5}\pi^2 x)}{\sqrt{\pi^7}}$$

$$e_5 = \frac{1}{128}\frac{\sqrt{11025}\sqrt{128}(x^4 - \frac{1}{5}x^4 - \frac{6}{7}\pi^2(x^2 - \frac{1}{3}\pi^2))}{\sqrt{\pi^9}}$$

$$e_6 = \frac{1}{128}\frac{\sqrt{128}\sqrt{43659}(x^5 - \frac{3}{7}\pi^4 x - \frac{10}{9}\pi^2(x^3 - \frac{3}{5}\pi^2 x))}{\sqrt{\pi^{11}}}$$

Since U is finite dimensional, we can use $\|v - P_U v\| \leq \|v - u\|$

Now want to find $P_U v = P_U sin(x)$,

$$U(x) = P_U sin(x) = \langle sin(x), e_1 \rangle e_1 + \langle sin(x), e_2 \rangle e_2 + ... + \langle sin(x), e_6 \rangle e_6$$

by using the inner product given by (1),

$$u(x) = (\int_{-\pi}^{\pi} sin(x) e_1 dx) e_1 + (\int_{-\pi}^{\pi} sin(x) e_2 dx) e_2 + ...(\int_{-\pi}^{\pi} sin(x) e_6 dx) e_6$$

$$u(x) = 0.987862x - 0.155271x^3 + 0.00564312x^5$$

For comparison,

There is a better known 5^{th} degree polynomial approximation for $sin(x)$.

$$T_5(x) = f(0) + \frac{f'(0)x}{1!} + \frac{f''(0)x^2}{2!} + ... + \frac{f^5(0)x^5}{5!}$$

For $f(x) = sin(x)$, $sin(x) = T_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$

$$T_5(x) = x - 0.1667x^3 + 0.00833x^5$$

As we moved from 0 towards $\pm\pi$, the approximation from $T_5(x)$ is not good as the $u(x)$ approximation.