SOME FIXED POINT THEOREMS AND THEIR APPLICATIONS

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CERTIFICATION

This is to certify that this project was carried out by OGUNLEYE OLAOLUWA ADEFEMI under my supervision.

DEDICATION

I dedicate this project to the Almighty God who has been gracious to me through out my undergraduate studies in the University of Lagos, Nigeria.

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All thanks be to God Almighty for his ever existing presence in my life and for the wisdom he has bestowed upon me throughout my stay on this campus.

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ABSTRACT

The theory of fixed point is an important tool in the arsenal of a pure and applied mathematician which in the last five decades has received unusual attention and have attracted a vast number of researchers across the globe. Speaking of application, it has enormous applications.

This project work presents a research and review of fixed point theory beginning from the popular Banach Contraction Principle and Browler's fixed point theorem to the Kannan's Fixed Point Theorem and ends as we discuss the Zamfirescu's Fixed Point Theorem. Furthermore, a new fixed point theory is proved.

List of Figures

Contents

Chapter 1

INTRODUCTION

1.1 Background of Study

Fixed point theory is one of the important thrust areas of research in Non-linear Analysis. Since last one century, fixed point theory has grown tremendously because of its applicability in different areas of mathematical sciences. Fixed point theory has played a central role in the problem solving techniques of Non-Linear Functional Analysis. Let X be a set and $f : X \to X$, a point $x \in X$ is said to be a fixed point of f if it satisfies the equation $f(x) = x$ or we can say that a point whose position is not changed by transformation is called a fixed point.

In a wide range of mathematical problems the existence of a solution is equivalent to the existence of a fixed point for suitable maps. The existence of a fixed point is therefore of paramount importance in several areas of mathematics and other sciences. Fixed point results provide conditions under which maps have solutions. The theory itself is a beautiful mixture of analysis (pure and applied), topology, and geometry. Over the last century the theory of fixed points has been revealed as a very powerful and important tool in the study of non-linear phenomena. In particular, fixed point techniques have been applied in such diverse fields as biology, chemistry, economics, engineering, game theory, and physics.

Examples of fixed points: A continuous function that maps $[0, 1]$ into itself has a fixed point.

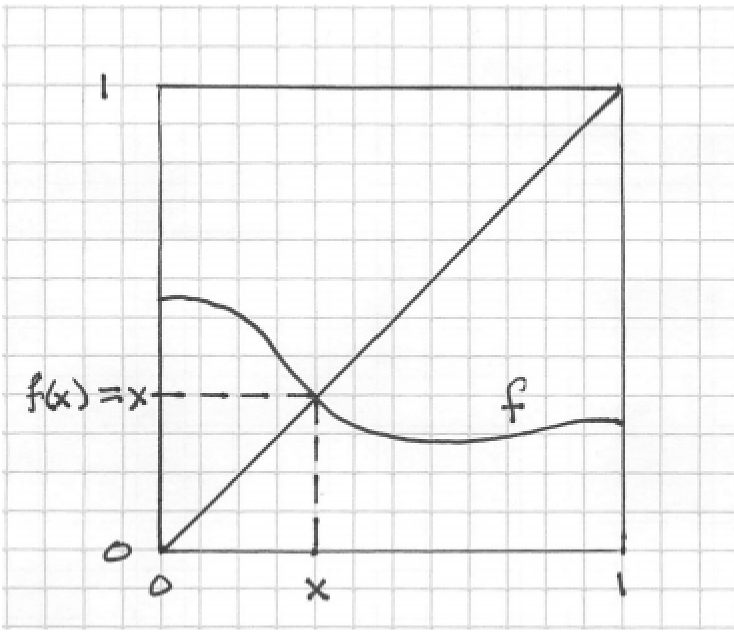


Figure 1.1: Fixed point of f on [0,1]

A topological space that is such that any continuous mapping of it onto itself have a fixed point is said to have the fixed point property. Not all topological spaces have the fixed point property.

A physical example of fixed point of a mapping is the center of a whirlpool in a cup of tea when stirred. (*The fact that the center of the whirlpool moves over time is just due to the fact that the mapping is changing over time*). The aim of this project is to study existing fixed point theorems and investigate their respective applications.

1.2 Statement of the Problem

Meaning of Fixed Points

A fixed point of a function is a point that is mapped to itself by the function. Given any equation $g(x)$, a fixed point of $g(x)$ is not a root of the equation $g(x) = 0$, but it is a solution of the equation $g(x) = x$.

Geometrically, the fixed points of a function $g(x)$ are the point of intersection of the curve $y = g(x)$ and the line $y = x$. This is illustrated in Figure(1.2) below.

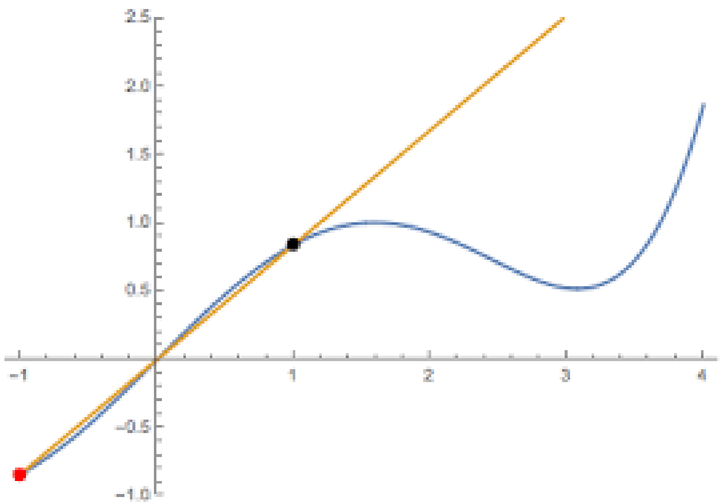


Figure 1.2: Geometric Representation of Fixed Point

One of the most natural ways to prove that an equation has a solution is to pose it as a fixed point problem, that is, to find a function f such that x is a solution of the equation $f(x) = x$. Then, the fixed point theorems can be used to show that f has a fixed point.

To illustrate this, let us consider the simple quadratic equation $x^2 - 11x + 10 = 0$. Here, $x = 1$ and $x = 10$ are the roots of this equation.

Now, we can also write this equation in the following form: $\quad x = \dfrac{x^2 + 10}{11}$

Taking $f(x) = \dfrac{x^2+10}{11}$, we obtained that $x = 1$ and $x = 10$ are two fixed points of f. So it can be concluded that the problem of finding the solution of an equation $F(x) = 0$ is the same as deducing the fixed point of a function $f(x)$, where $f(x) = F(x) + x$.

However, the results of fixed point theory are often non-constructive, that is, they guarantee that a fixed point exists but do not help in finding the fixed point.

In this paper, we study some already established fixed point theorems and investigate their applications in differential equations, computational methods, iterated systems.

1.3 Objective of the Study

The presence or absence of fixed point is an intrinsic property of a function. However many necessary and/or sufficient conditions for the existence of such points involve a mixture of algebraic order theoretic or topological properties of mapping or its domain. Fixed point theory concerns itself with a very simple and basic mathematical setting.

This research work gives a progressive discussion of the Picard iterative scheme for Contraction and Kannan maps alongside the theorem which guarantees fixed point of both maps and provides explicit proof of the theorems.

Finally, in the later chapters of the papers we find the applications of the various fixed point theorem in mathematics and other field of studies.

1.4 Research Questions

A point is often called fixed point when it remains invariant, irrespective of the type of transformation it undergoes. For a function f that has a set X as both domain and range, a fixed point is a point $x\ in\ X$ for which $f(x) = x$. The following are relevant question that arises in Metric fixed point theory

1. *What guarantees the existence of fixed point of Contraction and Kannan maps in a metric space?*

2. *Is completeness a necessary and sufficient condition for any map to have a fixed point?*

3. *Does all contraction maps in a metric space have a fixed point?*

4. *Does all Kannan maps in a metric space have a fixed point?*

5. *What is the rate of convergence of the Picard iteration of a contraction map in a complete metric space?*

6. *Are there Picard iteration for Kanaan maps?*

1.5 Scope and Delimitation of Study

As there are many fixed point theorems, we consider only the Contraction Fixed Point Theory by Stefan Banach (1922) , Kanaan fixed point theory by R. Kannan(1968) and the Zamfirescu Fixed Point Theory by Tudor Zamfirescu(1971).

1.6 Significance of Study

The classical fixed point theory is one of the important branches of mathematics. The theory of fixed points has been revealed as one of the very powerful and important tool in non-linear analysis and has tremendous applications in various branches of pure and applied mathematics. Due to its applicability, the theory has gain a remarkable scope of research in non-linear analysis for more than one century. Fixed point theory has played central role in the problems of Non-linear Functional Analysis and have provided powerful tools in demonstrating the existence of solution to a large variety of problems in applied mathematics. Fixed point theorems are mainly useful in existence theory for the solutions of Differential equations, Integral equations, Partial differential equations and Random differential equations. Also the theory has numerous applications in other related areas like Control theory, Game theory, Economics etc. Besides this, fixed point theory has very fruitful applications in Eigen value problems, Boundary value problems and Best approximation problems.

1.7 Definition of Terms

We start with the basic concepts related to metric spaces, fixed points and different contraction conditions.

Definition 1.7.1 (Metric Space)**.** *Let X be a non-empty set together with a distance function $d : X \times X \to \mathbb{R}_+$. The function d is said to be a metric if and only if $\forall x, y, z \in X$ the following are satisfied*

i $d(x, y) \geq 0$ *and* $d(x, y) = 0$ *if and only if* $x = y$

ii $d(x, y) = d(y, x)$

iii $d(x, y) \leq d(x, z) + d(z, y)$

The pair (X, d) is called a ***metric space****.*

Definition 1.7.2. *Let (X, d) be a metric space and d the metric defined on X. A sequence $\{x_n\}_{n=1}^{\infty}$ of element in X is said to be a* ***Cauchy sequence*** *if for every $\epsilon \geq 0$, there exists $n_0 \in \mathbb{N}$ such that $d(x_n, x_m) < \epsilon$, for all $n, m \geq n_0$.*

Definition 1.7.3. *A metric space (X, d) is said to be complete if every Cauchy sequence converges in it.*

That is, if $\{x_n\}_{n=1}^{\infty}$ is Cauchy sequence in X, then it converges, and not only that , it converges to and element $x_0 \in X$.

Definition 1.7.4. *The diameter of a set A denoted by $\delta(A)$ is defined as* $\delta(A) = \sup\{d(a, b) : a, b \in A\}$.

It means that the diameter of a set is the least upper bound of the distances between the points of the set A. If the diameter is finite, i.e, $\delta(A) < \infty$, then A is bounded.

Definition 1.7.5. *Let (X, d) be a metric space and $T : X \to X$. Then T has a fixed point if $\exists\ x \in X$ such that $T(x) = x$. The point x is called a fixed point of T.*

Definition 1.7.6. *Let (X, d) be a metric space and d the metric defined on X. A mapping $T : X \to X$ is called*

a. Lipschitzian (c-lipschitzian) if $\exists\ c > 0$ such that $d(Tx, Ty) \leq cd(x, y) \qquad \forall x, y \in X$

b. Contraction if $\exists\ a \in$ (0,1] such that T is $a -$ Lipshcitzian

c. Non-expansive if T is 1-Lipschitzian

d. Isometry if $d(Tx, Ty) = cd(x, y) \qquad \forall x, y \in X$

Definition 1.7.7. *Let (X, d) be a metric space. A function $f : X \to X$ is continuous at $a \in X$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that $d(x, a) < \delta$ implies that $d(f(x), f(a)) < \epsilon$.*

Definition 1.7.8. *Let X be a metric space and d a metric defined on it. A function $f : X \to X$ is sequentially continuous at $a \in X$ if $x_n \to a$ in X implies that $f(x_n) \to f(a)$ in Y.*

Proposition 1.7.1. *Let X be a metric space and d a metric defined on it. A function $f : X \to X$ is continuous at a if and only if it is sequentially continuous at a.*

Proof. Suppose that f is continuous at a. Let $\epsilon > 0$ be given and suppose that $x_n \to a$. Then there exists $\delta > 0$ such that $d(f(x), f(a)) < \epsilon$ for $d(x, a) < \delta$, and there exists $N \in \mathbb{N}$ such that $d(x_n, a) < \delta \; for \; n > N$. It follows that $d(f(x_n), f(a)) < \epsilon \; for \; n > N$, so $f(x_n) \to f(a)$ and f is sequentially continuous at a.

Conversely, suppose that f is not continuous at a. Then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$ there exists $x_n \in X$ with $d(x_n, a) < \frac{1}{n}$ and $d(f(x_n), f(a)) \geq \epsilon$.

Then $x_n \to a$ but $f(x_n) \to f(a)$, so f is not sequentially continuous at a. $\square$

Chapter 2

LITERATURE REVIEW

"Fixed points and fixed point theorems have always been a major theoretical tool in fields as widely apart as differential equations, topology, economics, game theory, dynamics, optimal control, and functional analysis. Moreover, more or less recently, the usefulness of the concept for applications increased enormously by the development of accurate and efficient techniques for computing fixed points, making fixed point methods a major weapon in the arsenal of the applied mathematician."

M. Hazewinkel

The literature of fixed point theory abounds with research papers which established fixed points for maps satisfying a variety of contraction/contractive conditions. This chapter, which is a purely descriptive, presents a brief introduction of the subject and development in the theory during last many decades.

2.1 Historical Development

Let us begin with the definitions of “Fixed Point” and “Fixed Point Theorem”. The concept of fixed point comprises of a triplet (x, X, T) where x is a member of a set X and T is a self-mapping of X such that $x = Tx$, i.e., a point x in a set X is said to be a fixed point of a self-mapping T of X if $x = Tx$; in other words, x remains invariant under the mapping T. Theorems concerning the existence and properties of fixed point are known as fixed point theorems. Also, by a fixed point theorem, one will understand a statement which asserts that under what conditions on the mapping T and or on the set X, the self-mapping T of X admits one or more fixed points.

The origin of fixed point theory lies in the method of successive approximations used for proving existence of solutions of differential equations introduced independently by Joseph Liouville [19] in 1837 and Charles Emile Picard [26] in 1890. The idea of fixed point of an operator was flashed in the mind of Cauchy[9] while dealing with the existence and uniqueness of solution of certain differential equation and by this notion, a new light in the research arena appeared as Fixed Point Theory. This has two-fold-valuation one from the classical analysis point of view and the other is its application to many branches of sciences and economics. Here we are ardently interested in the investigation on the classical part and therefore, we shall focus broadly on this part of fixed point theory.

After Cauchy [9], R. Lipschitz [20] simplified Cauchy’s proof in 1877 using “Lipschitz condition” and G. Peano [25] proved a deeper result in 1890 which relates mostly to the modern fixed point theory; in the same year Picard [26] applied this method to Ordinary and Partial differential equations. In 1886, Poincarẽ [27] proved a fixed point theorem for a continuous self-mapping f on $\mathbb{R}^n$ satisfying condition $f(x) + \alpha x = r,\ \|x\| = r$, for all x in $\mathbb{R}^n$, for some $r > 0$ and for every $\alpha > 0$. This theorem was rediscovered by P. Bohl [3] in 1904. For a long period of time, this branch remain suppressed, which was redeemed and re-cultivated by the Dutch mathematician Luitzen Egbertus Jan Brouwer(1881-1966) and put this branch of mathematics in the front-line of the research arena. In 1912, [7], he proved the well-known "Brouwer fixed point theorem" for a continuous self-map on a closed unit ball in $\mathbb{R}^n$ which exactly states that "a continuous map on a closed unit ball in $\mathbb{R}^n$ has a fixed point". Moreover, this theorem appears with varieties of proofs in literature till recently, some of which are very short but based on certain results in Algebraic Topology. This theorem of Brouwer, concerned with the subsets of $\mathbb{R}^n$, is not of much use in Functional Analysis where one is generally concerned with the infinite dimensional subsets of some function spaces.

An important extension of the Brouwer's fixed point theorem is the Schauders fixed point theorem [30] by a Polish mathematician Juliusz Schauder in 1930 stating that "a continuous map on a convex compact subspace of a Banach space has a fixed point". This theorem is an extension of Brouwers fixed point theorem to topological vector space, which may be of infinite dimension. Similarly, in 1935, a Russian-Soviet mathematician Andrey Tychnoff extended Brouwer's result to a compact, convex subset of a locally convex linear topological space.

In 1922, a Polish mathematician Stefan Banach(1892-1945) launched a new concept of mapping called "contraction mapping" and he showed in [1] that a contraction self-mapping on a complete metric space has a unique fixed point. Banach used the idea of shrinking map to obtain this fundamental result. Appearance of this theorem of Banach made many theorems on differential equations regarding the existence and uniqueness of solutions short, lucid and elegant; since then it has been applied to solve many such problems in differential and integral equations. Furthermore, this theorem of Banach created a big jolt to the heart of the mathematicians of that age, which, thereafter, led to start developing this branch by introducing various contraction type conditions, sometimes attributing new constraints to the mappings; by putting numerous restrictions to the structure of the spaces.

The Banach contraction principle was the only main tool to establish the existence and uniqueness of fixed points until 1968. This principle has been considered as the key of metric fixed point theory, but it suffers from one drawback, i.e., it requires the mapping to be continuous at all points of its domain. In 1968, Kannan [17] introduced a contractive condition which possessed a unique fixed point like that of Banach. However, unlike the Banach condition, Kannan [17] proved that there are mappings that have a discontinuity in their domain but still have fixed point, although such mappings are continuous at their fixed point. Following the appearance of Kannan [17] many researchers started working along this line and presented numerous contractive conditions not requiring continuity of the mapping. Various authors have defined a number of contractive type mappings on a complete metric space which are generalizations of the well-known Banach contraction and have the property that each mappings have unique fixed point.

In 1930, an Italian mathematician Renato Caccioppoli [8] remarked on Banach contraction principle that the contraction condition may be replaced by the assumption of the convergence of the sequence of iterates, which led to open another direction of studying fixed point

theory, known as “approximation of fixed point of an operator”. So, to speak on iterative sequence, Picard iterative scheme has a wide range of applications in different branches of sciences; nevertheless, it has been found to have some crucial drawback that the iterative sequence obtained by this method may not always converge, which was pointed out and rectified by W. R. Mann [21] in 1953 by introducing a new type of iteration scheme, called the Mann iterative process, formed by certain regular type of infinite matrices, with which he proved some theorems on approximation of fixed point for continuous mapping.
The simplicity and usefulness of Banach contraction principle inspired many mathematicians to analyse it further, these studies had led to a number of generalisations and modifications of this principle.

From the class of contraction mappings of Banach, Michael Edelstein in 1962 [11] singled out a new class of mappings, called “contractive mappings” and he found that such a mapping may fail to have a fixed point in complete metric space, but admits a unique fixed point in compact metric space. This theorem appears to be one sort of development of Banach contraction principle. The result of Edelstein is restrictive in the sense that it requires the space to be compact. In 1962, E. Rakotch [29] generalised the Banach theorem, which was further generalised by D. W. Boyd and J. S. W. Wong [6] in 1969. It may happen that neither the iterative sequence nor the sequence of fixed points for non-expansive mappings typically converges. There are examples of fixed point free non-expansive mappings for which none of the iterative sequences contain convergent subsequences. However, Edelstein showed in 1964 that, even if such convergent subsequences exist, there is a non-expansive mapping which remain fixed point free. Moreover, the fact that the relationship between the behaviour of a sequence of mappings and that of the sequence of their fixed points was first investigated by F. F. Bonsai [5] in 1962; and was subsequently studied by S. B. Nadler, Jr. [23] in 1968.

As many real life phenomenon as to do with more than one variables, i.e multivalued functions, results concerning the existence of fixed points for contraction and non-expansive single valued mappings has be extended partially to the case of set-valued (or multi-valued) mappings. The definition of fixed point for multi-valued mapping has been modified thus: *a point x in a set X is said to be a fixed point for a multi-valued mapping T on X into itself, if x is in Tx i.e. $x \in Tx$.*

The first extension of topological fixed point theory for continuous mapping to the case of set-valued mapping was made by J. V. Neumann [24] in 1935. In 1941 [16] Brouwer fixed point theorem was extended by S. Kakutani to multivalued mapping on a compact convex set in the Euclidean space. The extension of Schauder fixed point theorem to the set-valued case was given independently by H. Bohnenblust and S. Karlin [4] in 1950 and by I. Glicksberg [13] in 1952. Tychonoff theorem was extended for multi-valued mapping by Ky Fan [14] in 1952. In 1953, Strother [31] showed that every continuous multivalued mapping of the unit interval $I = [0, 1]$ into the non-empty compact subsets of I has a fixed point, but that the analogous result may not hold in $\mathbb{R}^2$. Further results in this direction were obtained by F. E. Browder and others. Some theorems on contraction type set-valued mappings were studied by S. B. Nadler, Jr. and R. B. Fraser [12] in 1969 and also by S. B. Nadler, Jr. [32] in the same year.

In recent years the study on multi-valued fixed point theory has been developed extensively by many authors, with application to mathematical economics, optimization and optimal control, game theory etc. More informations on multi-valued fixed point theorems can be readily available from the survey article written by Downing and Ray [10] in 1981.

Chapter 3

SOME GENERALIZATION OF BANACH CONTRACTION PRINCIPLE

3.1 The Contraction Mapping Theorem

We will discuss here the most basic fixed-point theorem in analysis. It is due to Banach and appeared in his Ph.D. thesis (1920, published in 1922).

Definition 3.1.1 (Contraction Map). *Let X be a metric space equipped with a distance d. A map $T : X \to X$ is said to be Lipschitz continuous if there exist $\lambda \in (0,1)$ such that*

$$d(T(x_1), T(x_2)) \leq \lambda d(x_1, x_2), \qquad \forall x_1, x_2 \in X.$$

The smallest λ for which the above inequality holds is the Lipschitz constant of T.

If $\lambda = 1$ T is said to be non-expansive, if $\lambda < 1$ T is said to be a contraction.

Proposition 3.1.1 (Contraction Map is Continuous). *Let X be a complete metric space* *and let $T : X \to X$ be a contraction mapping with contractivity coefficient λ. Then T is continuous.*

Proof. Let $\epsilon > 0$. Suppose that $d(x, y) < \dfrac{\delta}{\lambda} \quad \forall x, y \in X$.

Then we have that,

$$
\begin{aligned}
d(T(x), T(y)) &\leq \lambda d(x, y) && \text{(by contraction property)} \\
&< \lambda \, \frac{\delta}{\lambda} \\
&= \delta
\end{aligned}
$$

Choose δ such that $0 < \delta < \epsilon$

Therefore $\forall \epsilon > 0 \quad \exists \delta > 0 : \quad d(x, y) < \delta \implies d(T(x), T(y)) < \epsilon \quad \forall x, y \in X$ $\square$

Proposition 3.1.2 (Uniqueness of Fixed Point)**.** *Let X be a complete metric space and let $T : X \to X$ be a contraction mapping with contractivity coefficient λ. If T have a fixed point then the fixed point is unique.*

Proof. Suppose that $x_1, x_2 \in X$ are fixed points of T, then

$T(x_1) = x_1$ and $T(x_2) = x_2$

$$
\begin{aligned}
d(x_1, x_2) &= d(T(x_1), T(x_2)) \\
&\leq \lambda d(x_1, x_2) && \text{(by contraction axiom)}
\end{aligned}
$$

This implies that $\lambda = 0$ and that $d(x_1, x_2) = 0$, which in turn implies that $x_1 = x_2$ by metric property of d. $\square$

Proposition 3.1.3 (Fundamental Contraction Inequality)**.** *Let X be a complete metric space and let $T : X \to X$ be a contraction mapping with contractivity coefficient λ, then for all $x_1, x_2 \in X$,*

$$d(x_1, x_2) \leq \frac{1}{(1 - \lambda)}[d(x_1, T(x_1)) + d(x_2, T(x_2))]$$

Proof. By the triangle inequality,

$$\begin{aligned}
d(x_1,x_2) &\leq d(x_1,T(x_1)) + d(T(x_1),T(x_2)) + d(T(x_2),x_2) \\
&\leq d(x_1,T(x_1)) + \lambda d(x_1,x_2) + d(T(x_2),x_2) \\
(1-\lambda)d(x_1,x_2) &\leq d(x_1,T(x_1)) + d(T(x_2),x_2) \\
d(x_1,x_2) &\leq \frac{1}{(1-\lambda)}[d(x_1,T(x_1)) + d(T(x_2),x_2)] \qquad \text{(as required)}
\end{aligned}$$

□

This is a very important inequality: It says that we can estimate how far apart any two points x_1 and x_2 are just from knowing how far x_1 is from its image $T(x_1)$ and how far x_2 is from its image $T(x_2)$.

We now move to the most important and basic fixed point theorem- The Contraction Mapping Theorem also known as the Banach Contraction Principle.

Theorem 3.1.1 (Banach Contraction Theorem)**.** *Let X be a complete metric space and let $T : X \to X$ be a contraction mapping with contractivity coefficient λ. Let $x_0 \in X$ and inductively define $x_{n+1} = T(x_n)$ for $n \geq 0$. Then T has a unique fixed point a and the sequence x_n converges to a*

Proof. Choose now any $x_0 \in X$, and define the iterative sequence $x_{n+1} = T(x_n)$.

So,

$$\begin{aligned}
d(x_2,x_1) &= d(T(x_1),T(x_0)) \\
&\leq \lambda d(x_1,x_0) \\
d(x_3,x_2) &= d(T(x_2),T(x_1)) \\
&\leq \lambda d(x_2,x_1) \leq \lambda^2 d(x_1,x_0)
\end{aligned}$$

By induction on n,

$$d(x_{n+1}, x_n) \leq \lambda^n d(x_1, x_0).$$

Thus, for $n < m$, we have

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \\ &\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(x_1, x_0) \\ &\leq (\lambda^n + \lambda^{n+1} + \dots) d(x_1, x_0) \\ &\leq \frac{\lambda^n}{(1-\lambda)} d(x_1, x_0) && \text{(Sum to Infinity)} \end{aligned}$$

Where we have made use of the triangle inequality and the properties of sums. Since $|\lambda| < 1$, $\frac{\lambda^n}{1-\lambda} \to 0$ as $n \to \infty$. Hence $\{x_n\}$ is Cauchy and has a limit $a \in X$ by completeness property of X. Contraction maps are continuous, so it follows that

$$T(a) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = a;$$

as desired.

By proposition (3.1.2) the fixed point a of T is unique. $\square$

Remark 3.1.1. *When applicable, the Banach fixed point theorem is especially useful as it both guarantees the existence and uniqueness of a fixed point. Unfortunately, the requirement that our map be a contraction limits its utility.*

The proof of the contraction mapping theorem yields useful information about the rate of convergence towards the fixed point, as follows.

Corollary 3.1.1 (Approximate Inequalities)**.** *Let T be a contraction mapping on a complete metric space X, with contraction constant λ and fixed point a.*

For any $x_0 \in X$, with T-iterates $\{x_n\}$, we have the estimates

$$d(x_n, a) \leq \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)) \tag{3.1}$$

$$d(x_n, a) \leq \lambda d(x_{n-1}, a) \tag{3.2}$$

$$d(x_n, a) \leq \frac{\lambda}{1-\lambda} d(x_{n-1}, x_n) \tag{3.3}$$

Proof. For $m > n$ we have

$$\begin{aligned}
d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, d_m) \\
&\leq (\lambda^n + \lambda^{n+1} + \cdots + \lambda^{m-1}) d(x_1, x_0) \\
&\leq (\lambda^n + \lambda^{n+1} + \ldots) d(x_1, x_0) \\
&\leq \frac{\lambda^n}{1-\lambda} d(x_1, x_0) && \text{(Sum to Infinity)} \\
d(x_n, x_m) &\leq \frac{\lambda^n}{1-\lambda} d(x_0, x_1) = \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)) \\
d(x_n, x_m) &\leq \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)) && (3.4)
\end{aligned}$$

Since the right side of (3.4) is independent of m, let $m \to \infty$

$$\begin{aligned}
\lim_{m\to\infty} d(x_n, x_m) &\leq \lim_{m\to\infty} \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)) \\
d(x_n, a) &\leq \frac{\lambda^n}{1-\lambda} d(x_0, T(x_0)) && \text{(we obtain (3.1) as required)}
\end{aligned}$$

To derive (3.2), from a being a fixed point we get

$$d(x_n, a) = d(T(x_{n-1}), T(a)) \leq \lambda d(x_{n-1}, a) \tag{3.5}$$

$$d(x_n, a) \leq \lambda d(x_{n-1}, a) \qquad \text{(we obtain (3.2) as required)}$$

Applying the triangle inequality on the right side of (3.5) using the three points x_{n-1}, x_n and

a,

$$\begin{aligned} d(x_n, a) &\leq \lambda(d(x_{n-1}, x_n) + d(x_n, a)) \\ (1-\lambda)d(x_n, a) &\leq \lambda d(x_{n-1}, x_n) \\ d(x_n, a) &\leq \frac{\lambda}{(1-\lambda)} d(x_{n-1}, x_n) \qquad \text{(we obtain (3.3) as required)} \end{aligned}$$

□

The three inequalities in Corollary (3.1.1) serve different purposes. The inequality (3.1) tells us, in terms of the distance between x_0 and $T(x_0) = x_1$, how many times we need to iterate T starting from x_0 to be certain that we are within a specified distance from the fixed point. This is an upper bound on how long we need to compute. It is called an a **priori estimate**. Inequality (3.2) shows that once we find a term by iteration within some desired distance of the fixed point, all further iterates will be within that distance. However, (3.2) is not so useful as an error estimate since both sides of (3.2) involve the unknown fixed point. The inequality (3.3) tells us, after each computation, how much closer we are to the fixed point in terms of the previous two iterations. This kind of estimate, called an a **posteriori estimate**, is very important because if two successive iterations are nearly equal, (3.3) guarantees that we are very close to the fixed point.

Corollary 3.1.2. *When $T : X \to X$ is a contraction with constant λ, any iterate T^n is a contraction with constant λ^n; the unique fixed point of T will also be the unique fixed point of any T^n.*

Example 3.1.1. *Let us consider the space*

$$\mathbb{X} = \{x \in \mathbb{R} : x \geq 1\}$$

with metric

$$d(x,y) = |x - y|, \qquad \forall\ x,y \in \mathbb{X}$$

and let $T : \mathbb{X} \to \mathbb{X}$ *be given by*

$$T(x) = x + \frac{1}{x}$$

Then, an easy computation shows that

$$d(T(x), T(y)) = \frac{xy - 1}{xy}|x - y| < |x - y| = d(x,y)$$

On the other hand, there do not exist $0 \leq \lambda < 1$ *such that*

$$d(T(x), T(y)) \leq \lambda d(x,y), \qquad \forall x,y \in \mathbb{X}$$

and T *has no fixed point in* $\mathbb{X}$. *If it does, then*

$$T(x) = x \implies x = x + \frac{1}{x} \implies \frac{1}{x} = 0.$$

There is no such $x \in \mathbb{R}$ *such that* $\frac{1}{x} = 0$, *hence* T *has no fixed points.*

The above example shows that if we replace the assumption of the Contraction Mapping Theorem that T be a contraction mapping by the less restrictive hypothesis that

$$d(T(x), T(y)) < d(x,y), \qquad \forall x,y \in \mathbb{X}$$

then T need not have a fixed point.

Example 3.1.2. *Let us consider the space*

$$\mathbb{X} = \{x \in \mathbb{R} : x \geq 1\}$$

with metric

$$d(x,y) = |x - y|, \qquad \forall\ x,y \in \mathbb{X}$$

and let $T : \mathbb{X} \to \mathbb{X}$ *be given by*

$$T(x) = \frac{x}{2} + \frac{1}{x}$$

Then, we wish to show that T *is contraction map. Consider the derivative* $T' : \mathbb{X} \to \mathbb{X}$ *given by*

$$T'(x) = \frac{1}{2} - \frac{1}{x^2}$$

This implies that $|T'(x)| \leq \frac{1}{2}$ *for all* $x \in \mathbb{X}$. *Now, from the mean value theorem, we have that given* $x, y \in \mathbb{X}$, *with* $x < y$ *we can find* $c \in \mathbb{X}$ *with* $x < c < y$ *such that*

$$d(T(x), T(y)) = |T(x) - T(y)| = |T'(c)(x - y)| = |T'(c)||(x - y)| \leq \frac{1}{2}|(x - y)| = \frac{1}{2}d(x, y)$$

Hence T *is a contraction mapping on* $\mathbb{X}$. *Since* $\mathbb{X} \subset \mathbb{R}$ *equipped with the usual metric induced by the absolute value* $(d(x, y) = |xy|)$ *is a complete metric space, then the contraction mapping principle implies that* T *has a unique fixed point on* $\mathbb{X}$.

The fixed point satisfies

$$T(x) = x \implies \frac{x}{2} + \frac{1}{x} = x \implies \frac{x^2}{2} + 1 = x^2$$
$$x^2 + 2 = 2x^2 \implies x^2 = 2 \implies x = \sqrt{2}$$

Hence the unique fixed point is $x = \sqrt{2}$

Example 3.1.3. *Graphs of* $y = cosx$ *and* $y = x$ *intersect once (see the figure below), which means the cosine function has a unique fixed point in* $\mathbb{R}$.

From the graph, this fixed point lies in $[0, 1]$.

Example 3.1.4. *The graphs of* $y = e^{-x}$ *and* $y = x$ *intersect once, so* $e^{-a} = a$ *for unique real number* $a \in \mathbb{R}$. *See the graph below.*

From the graph, this fixed point lies in $[0, 1]$.

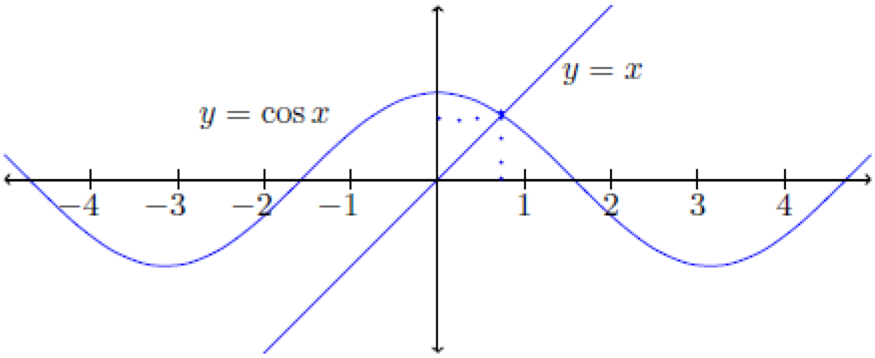


Figure 3.1: Fixed Point of Cosine Function

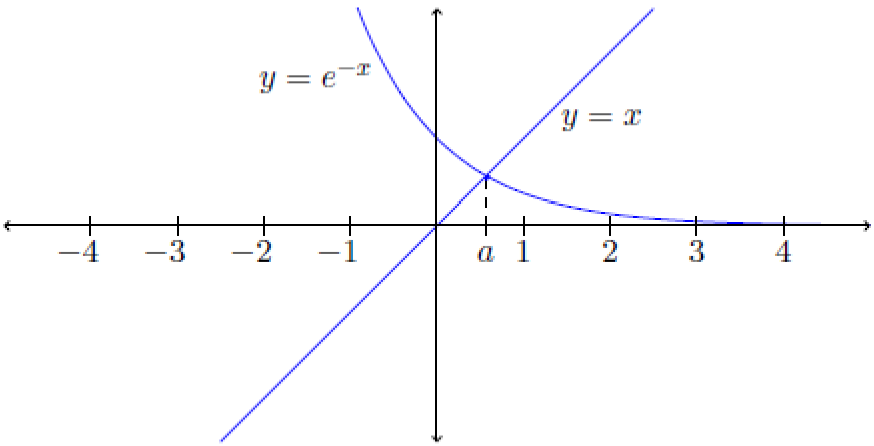


Figure 3.2: Fixed Point of Exponential Function

The continuous functions $f(x) = cosx$ of Example 3.1.3 and $f(x) = e^{-x}$ of Example 3.1 is not a contraction on $\mathbb{R}$, and yet have unique fixed point. The next theorem presents a statement that guarantees the existence of fixed point(s) of a continuous functions in a complete metric space.

Theorem 3.1.2 (Fixed Point of Continuous Maps)**.** *Let X be a complete metric space and let $T : X \to X$ be a continuous map. Let $x_0 \in X$ and inductively define*

$$x_{n+1} = T(x_n) :$$

If

$$\sum_{n=0}^{\infty} d(x_n, x_{n+1}) < \infty$$

then the sequence $\{x_n\}_{n\geq 0}$ *converges to a fixed point* a *of* T*. Moreover*

$$d(a, x_n) \leq \sum_{k=n}^{\infty} d(x_k, x_{k+1})$$

Proof. By the triangle inequality if $0 \leq n < m$ then

$$\begin{aligned} d(x_n, x_m) &\leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) \qquad (1) \\ &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \cdots = \sum_{k=n}^{\infty} d(x_k, x_{k+1}) \\ d(x_n, x_m) &\leq \sum_{k=n}^{\infty} d(x_k, x_{k+1}) \ < \ \infty \end{aligned}$$

Thus $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence. And since X is a complete metric space then $\{x_n\}_{n=0}^{\infty}$ converges to a point $a \in X$.

By continuity of T we have

$$T(a) = T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = a;$$

So $a = T(a)$.

Recall from 1,

$$d(x_n, x_m) \leq d(x_n, x_{n+1}) + \cdots + d(x_{m-1}, x_m) = \sum_{k=n}^{m-1} d(x_k, x_{k+1})$$

Now taking the limit as $m \to \infty$

$$\begin{aligned} \lim_{m\to\infty} d(x_n, x_m) &\leq \lim_{m\to\infty} \sum_{k=n}^{m-1} d(x_k, x_{k+1}) \\ d(x_n, a) &\leq \sum_{k=n}^{\infty} d(x_k, x_{k+1}) \end{aligned}$$

We obtain the required estimate. □

Remark 3.1.2 (Theorem 3.1.2 only requires the continuity of the map T and the summation restriction in 3.1.2 to ascertain a fixed of T. However we cannot tell if the fixed point is unique or not)**.**

Corollary 3.1.3 (Converse of the Contraction Mapping Theorem)**.** *The contraction mapping theorem admits a converse which states that if* X *is any set (not yet a metric space),* $\lambda \in (0,1)$*, and* $T : X \to X$ *is a function such that each iterate* $T^n : X \to X$ *has a unique fixed point then there is a metric* d *on* X *making it a complete metric space such that, for this metric,* T *is a contraction with contraction constant* λ*.*

Chapter 4

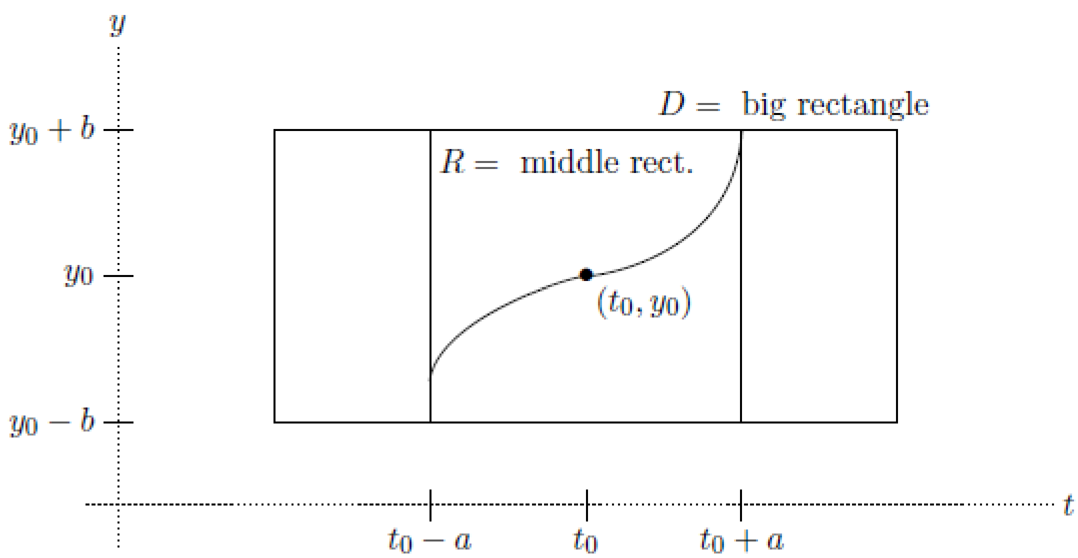
ANALYSIS

4.1 Picard's Existence and Uniqueness Theorem

Theorem 4.1.1 (Existence and Uniqueness). *Suppose* $f(t, y)$ *and* $\dfrac{\partial f}{\partial y}(t, y)$ *are continuous on a rectangle* D *as shown. Then we can choose a smaller rectangle* R *(as shown) so that the IVP*

$$\frac{dy}{dt} = f(t, y(t)), \quad y(t_0) = y_0 \tag{4.1}$$

has a unique solution defined on $[t_0 - a, t_0 + a]$ *whose graph is entirely inside* R.



Proof. The proof is partitioned in stages.

Stage (1) : Let $M = max_D \left|\frac{\partial f}{\partial y}(t, y)\right|$

Using the mean value theorem

$$f(t, y_2) - f(t, y_1) = \frac{\partial f}{\partial y}(t, c)(y_2 - y_1) \qquad \text{(for some } c \text{ between } y_1 \text{ and } y_2\text{)}$$

$$\implies \quad |f(t, y_2) - f(t, y_1)| < L\,|y_2 - y_1| \qquad \text{(Lipshcitz condition).}$$

Stage (2) : **Choosing the rectangle** R

Choose $a < min\left(\frac{b}{M}, \frac{1}{2L}\right)$. We will use this in stages (3) and (5).

Stage (3): **The operator** T

Let Y be the space of all functions y which are continuous on $[t_0 - a, t_0 + a]$ and whose graph is entirely inside R. For any $y \in Y$ define

$$Ty = z(t) = y_0 + \int_{t_0}^{t} f(s, y(s))ds \tag{4.2}$$

We Observe a number of facts about T

(a) Ty = z(t) is well defined on $[t_0 - a, t_0 + a]$.

(b) z(t) is continuous. (since both y and f are continuous and composition of continuous function is continuous).

(c) The graph of z(t) is entirely in R.

Proof:

$$|z(t) - y_0| = \left|\int_{t_0}^{t} f(s, y(s))ds\right| \leq M|t - t_0| \leq Ma < b$$

(This follows from the choice of a in Stage (2))

Facts a-c show that T maps the space **Y** into itself.

(d) z'(t) = f(t,y(t)) (This follows by fundamental theorem of calculus).

(e) **Claim:** y is a solution to the IVP (4.1) $\iff$ y is a fixed point of T.

Proof: Suppose y is a solution, i.e., $y(t_0) = y_0$ and $y'(t) = f(t, y(t))$.

Then

$$Ty = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$
$$= y_0 + \int_{t_0}^{t} y'(s)ds = y_0 + y(s)|_{t_0}^{t} = y(t)$$

So y is a fixed point of T.

Conversely, suppose y is a fixed point, then

$$y = Ty = y_0 + \int_{t_0}^{t} f(s, y(s))ds$$

$$\implies y(t_0) = y_0 \text{ and } y' = f(t, y(t))$$

That is, y satisfies the IVP (4.1).

The above claims shows that proving existence and uniqueness is equivalent to proving that T has a unique fixed point. (This is proved in (8) and (9) below).

Stage 4 : **The metric on Y**

For y_1 and y_2 in Y define

$$\delta(y_1, y_2) = max\{|y_1(t) - y_2(t)| : \quad t \in [t_0 - a, t_0 + a]\}$$

Clearly,

(a) $\delta(y_1, y_2) = 0 \implies y_1 = y_2$.

(b) δ satisfies the triangle inequality: $\delta(y_1, y_2) + \delta(y_2, y_3) \geq \delta(y_1, y_3)$.

(c) δ tells how to measure 'closeness' between 'points' of Y.

(d) Y has no 'holes' .

Stage 5. **Claim:** $\delta(Ty_1, Ty_2) \leq \frac{1}{2}\delta(y_1, y_2)$.

Proof:

$$\begin{aligned}
|Ty_1 - Ty_2(t)| &= \left| \int_{t_0}^{t} f(s, y_1(s)) - f(s, y_2(s)) ds \right| \\
&\leq \int_{t_0}^{t} |f(s, y_1(s)) - f(s, y_2(s)) ds| \\
&\leq L \int_{t_0}^{t} |y_1(s) - y_2(s)| ds && \text{(Lipschitz condition)} \\
&\leq L\delta(y_1, y_2) \int_{t_0}^{t} ds && \text{(pull out } max(y_1(s) - y_2(s)) \text{)} \\
&= L\delta(y_1, y_2)(t - t_0) \leq \delta(y_1, y_2) L \cdot a < \frac{1}{2}\delta(y_1, y_2)
\end{aligned}$$

The last inequality uses the choice of a in step (2).

Note: since T shrinks distances it is called a contraction mapping.

Stage 6. **Claim**: T has at most one fixed point.

Proof: Suppose there were two different fixed points y_1 and y_2. Then since $Ty_j = y_j$ for fixed point y_j, we get

$$\delta(y_1, y_2) = \delta(Ty_1, Ty_2) \leq \frac{1}{2}\delta(y_1, y_2)$$

Hence a contradiction i.e. T can have at most one fixed point.

Stage (7). **Claim**: If the sequence $y_0, y_1, y_2, \ldots$ converges to y then $Ty_0, Ty_1, Ty_2, \ldots$ converges to Ty. This follows since T is a continuous map of Y to itself.

Stage (8). **Picard iteration**

Start with $y_0(t) = y_0$. Let $y_1 = Ty_0, y_2 = Ty_1, \ldots, y_{n+1} = Ty_n = T^n y_0$.

Claim: The sequence $y_0, y_1, y_2, \ldots$ converges.

Proof:

$$\begin{aligned}
\delta(y_2, y_1) &= \delta(Ty_1, Ty_0) \leq \frac{1}{2}\delta(y_1, y_0) \\
\delta(y_3, y_2) &= \delta(Ty_2, Ty_1) \leq \frac{1}{2}\delta(y_2, y_1) \leq \frac{1}{4}\delta(y_1, y_0)
\end{aligned}$$

Inductively, $$\delta(y_{n+1}, y_n) \leq \left(\frac{1}{2}\right)^n \delta(y_1, y_0)$$

$$\implies \sum_{n=0}^{\infty} \delta(y_{n+1}, y_n) \leq \delta(y_1, y_0) \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$$

Clearly, $\{y_n\}$ is a Cauchy sequence, so y_n converges to a function $y \in Y$ by completeness axiom.

Stage (9). Take the sequence from stage (8) and let $\lim_{n\to\infty} y_n = y$

Claim: y is a fixed point of T.

Proof: Since $y = \lim_{n\to\infty} T^n y_0$ we have that

$$Ty = \lim_{n\to\infty} T^{n+1} y_0 = y$$

so, $Ty = y \implies y$ is a solution of the integral equation.

Therefore the IVP (4.1) has unique solution. $\square$

Example 4.1.1. *Consider the IVP*

$$y' = 3y^{\frac{2}{3}}, \quad y(2) = 0$$

Then $f(x, y) = 3y^{\frac{2}{3}}$ *and* $\frac{\partial f}{\partial y} = 2y^{\frac{-1}{3}}$, *so* $f(x, y)$ *is continuous when* $y = 0$ *but* $\frac{\partial f}{\partial y}$ *is not. Hence the hypothesis of Picard's Theorem does not hold, neither does the conclusion; the IVP has two analytic solutions,* $y^{\frac{1}{3}} = x - 2$ *and* $y \equiv 0$

Example 4.1.2. *Consider the IVP*

$$y' = 2y, \quad y(0) = 1$$

This IVP is equivalent to $y = 1 + \int_0^x 2y dt$, *so the Picard iterates are*

$$\begin{aligned}
y_0(x) &\equiv 1 \\
y_1(x) &= 1 + \int_0^x 2y_0(t)dt = 1 + 2x \\
y_2(x) &= 1 + \int_0^x 2(1+2t)dt = 1 + 2x + \frac{(2x)^2}{2!}
\end{aligned}$$

and so on. It can be shown by induction that the nth iterate is

$$y_n(x) = 1 + 2x + \frac{(2x)^2}{2!} + \cdots + \frac{(2x)^n}{n!}$$

which is th nth partial sum of the Maclaurin series for e^{2x}. *Thus, as* $n \to \infty$, $y_n(x) \to e^{2x}$

4.2 Iterative Solutions of Non-linear Equation

There are a variety of techniques used in obtaining solutions of non-linear equations, these include the Newton Raphson Method, Secant method, etc. We discuss briefly the Newton Raphson iterative scheme for obtaining solutions of non-linear equations.

4.2.1 Newton-Raphson Iteration

The Newton-Raphson method is a powerful technique for solving equations numerically.

Derivation:

Let x_0 be a good estimate of r and let $r = x_0 + h$. Since the true root is r, and $h = r - x_0$, the number h measures how far the estimate x_0 is from the true root.

Since h is 'small' we can use the Taylor's series approximation to conclude that

$$0 = f(r) = f(x_0 + h) \approxeq f(x_0) + hf'(x_0),$$

and therefore, unless $f'(x_0)$ is close to 0,

$$h \approxeq -\frac{f(x_0)}{f'(x_0)}.$$

It follows that

$$r = x_0 + h \approxeq x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The new improved estimate x_1 of r is therefore given by

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

The next estimate x_2 is obtained from x_1 in exactly the same way as x_1 was obtained from x_0:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}.$$

Continue in this way. If x_n is the current estimate, then the next estimate x_{n+1} is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The importance of this section is to answer the question " Under what condition are we guaranteed that a non linear equation $g(x)$ defined on a complete metric space have a solution(s)".

The question is answered in the theorem below

Theorem 4.2.1. *Let $\mathbb{R}$ be the set of real numbers endowed with the usual metric $(d(x,y) = |x - y|, \ \forall x, y \in \mathbb{R})$, and if $g : \mathbb{R} \to \mathbb{R}$ is a real-valued function such that $-2 < \frac{g(x)-g(y)}{x-y} < 0 \ \ \forall x, y \in \mathbb{R}$. Then the equation $g(x) = 0$ has a unique solution.*

Proof. Define $f(x) = g(x) + x$. Clearly, the proof of the existence of solution of $g(x)$ is equivalent to showing that $\exists x \in \mathbb{R} \ : f(x) = x$.

For all $x, y \in \mathbb{R}$

$$\begin{aligned} |f(x) - f(y)| &= |g(x) + x - g(y) - y| \\ &= |g(x) - g(y) + x - y| \\ &= \left| \left(\frac{g(x) - g(y)}{x - y} + 1 \right) (x - y) \right| \\ &= \left| \frac{g(x) - g(y)}{x - y} + 1 \right| |x - y| \qquad (4.3) \end{aligned}$$

But

$$\frac{g(x) - g(y)}{x - y} \in (-2, 0) \Longrightarrow \left| \frac{g(x) - g(y)}{x - y} + 1 \right| < 1$$

Setting $\delta = \left| \frac{g(x) - g(y)}{x - y} + 1 \right|$, we have $\delta < 1$ and from 4.3

$$|f(x) - f(y)| \leq \delta |x - y|, \qquad \forall x, y \in \mathbb{R}$$

Hence f is a contraction map defined on a complete metric space $\mathbb{R}$. By the Banach Contraction Principle f has a unique fixed point which implies that the equation $g(x) = 0$ have a unique solution. $\square$

Example 4.2.1. *Let $\mathbb{R}$ be the set of real numbers and d the usual metric defined by*

$$d(x, y) = |x - y|, \qquad \forall\ x, y \in \mathbb{R}$$

and let $g : \mathbb{R} \to \mathbb{R}$ be given by

$$g(x) = -\frac{3}{2}x + 5$$

Then, an easy computation shows that

$$\begin{aligned} g(x) - g(y) &= -\frac{3}{2}(x - y) \\ \frac{g(x) - g(y)}{(x - y)} &= -\frac{3}{2} \qquad \forall\ x, y \in \mathbb{R} \end{aligned}$$

That is, $\dfrac{g(x) - g(y)}{(x - y)} \in (-2, 0) \quad \forall\ x, y \in \mathbb{R}$. *Now by Theorem (4.2.1), we have that the linear equation $g(x) = 0$ has a unique solution.*

The solution is obtained thus

$$\begin{aligned} g(x) = 0 &\implies -\frac{3x}{2} + 5 = 0 \\ &\implies -\frac{3x}{2} = -5 \\ &\implies x = \frac{10}{3}. \end{aligned}$$

Remark 4.2.1. *It should be observed that the map is not a contraction map and yet has a unique fixed point.*

4.3 Kannan Fixed Point Theorem

Definition 4.3.1 (Kannan Map). *Let (X, d) be a complete metric space. $T : X \to X$ is a Kannan map if there exists $a \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \quad \textit{for all } x, y \in X.$$

Theorem 4.3.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping for which there exists $a \in \left[0, \frac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq a[d(x, Tx) + d(y, Ty)], \quad \textit{for all } x, y \in X. \tag{4.4}$$

Then T has a unique fixed point.

Proof. Let $x_0 \in X$, and $x_n = T^n x_0$, $n = 0, 1, 2, \ldots$ be the Picard iteration.

Then by (4.4) we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq a \cdot [d(x_{n-1}, x_n) + d(x_n, x_{n+1})], \tag{4.5}$$

which implies

$$d(x_n, x_{n+1}) \leq \frac{a}{1-a} \cdot d(x_{n-1}, x_n) \leq \cdots \leq \left(\frac{a}{1-a}\right)^n \cdot d(x_0, x_1)$$

Since $0 \leq \dfrac{a}{1-a} < 1$, for $a \in \left[0, \frac{1}{2}\right)$, we deduce that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence, and hence a convergent sequence.

Let $x^* \in X$ be its limit. Then we have that

$$d(x^*, Tx^*) \leq d(x^*, x_n) + d(x_n, Tx^*) \leq d(x^*, x_n) + a[d(x_{n-1}, x_n) + d(x^*, Tx^*)],$$

and hence

$$d(x^*, Tx^*) \leq \frac{1}{a} \cdot d(x^*, x_n) + \frac{a}{1-a} \cdot d(x_{n-1}, x_n)$$
$$d(x^*, Tx^*) \leq \frac{1}{a} \cdot d(x^*, x_n) + \left(\frac{a}{1-a}\right)^n \cdot d(x_0, x_1) \tag{4.6}$$

Now, letting $n \to \infty$ in (4.6), we obtain

$$d(x^*, Tx^*) = 0 \Longleftrightarrow x^* = Tx^*$$

and therefore, $\quad x_n \to x^*(n \to \infty)$, for each $x_0 \in X$. $\square$

4.3.1 Extended Kannan Theorem

Definition 4.3.2. *Let (X, d) be a metric space. A mapping $T : X \to X$ is said to be sequentially convergent if we have, for every sequence $\{y_n\} \to y_0$, $\{Ty_n\} \to Ty_0$.*

Definition 4.3.3. *Let X be a nonempty set. Suppose that the mapping $d : X \to X$, satisfies:*

(i) $d(x, y) \geq 0$, *for all* $x, y \in X$ *and* $d(x, y) = 0$ *if and only if* $x = y$;

(ii) $d(x, y) = d(y, x)$ *for all* $x, y \in X$;

(iii) $d(x, y) \leq d(x, w) + d(w, z) + d(z, y)$, *for all* $x, y \in X$ *and for all distinct points* $w, z \in X \backslash \{x, y\}$.

Then d is called a generalized metric and (X, d) is a generalized metric space.

Theorem 4.3.2 (Extended Kannnan's Theorem)**.** *Let (X, d) be a complete metric space and $T, S : X \to X$ be mappings such that T is continuous, one-to-one and subsequentially convergent. If $\lambda \in [0, \frac{1}{2})$ and*

$$(TSx, TSy) \leq \lambda[(Tx, TSx) + d(Ty, TSy)] \tag{4.7}$$

for all $x, y \in X$, then S has a unique fixed point. Also if T is sequentially convergent then for every $x_0 \in X$ the sequence of iterates $\{S^n x_0\}$ converges to this fixed point.

Proof. Let x_0 be an arbitrary point in X. Define the iterative sequence $\{x_n\}$ by $x_{n+1} = Sx_n$

That is,

$$x_1 = Sx_0$$

$$x_2 = Sx_1 = SSx_0 = S^2x_0$$

$$x_3 = Sx_2 = SS^2x_0 = s^3x_0$$

Inductively $x_n = S^n x_0 \quad \forall\, n$

Using inequality 4.7, we have

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &= d(TSx_{n1}, TSx_n) \\ &\leq \lambda[d(Txn1, TSxn1) + d(Txn, TSxn)], \end{aligned} \tag{2}$$

$$d(Tx_n, Tx_{n+1}) \leq \lambda d(Tx_{n1}, TSx_{n1}) + \lambda d(Tx_n, TSx_n)$$

$$(1-\lambda)d(Tx_n, Tx_{n+1}) = \lambda d(Tx_{n1}, Tx_n)$$

so,

$$d(Tx_n, Tx_{n+1}) \leq \frac{\lambda}{1-\lambda} d(Tx_{n1}, Tx_n) \tag{3}$$

By the same argument,

$$\begin{aligned} d(Tx_n, Tx_{n+1}) &\leq d(Tx_{n-1}, Tx_n) \leq d(Tx_{n-2}, Tx_{n-1}) \\ &\leq ... \leq (\frac{\lambda}{1-\lambda})^n d(Tx_0, Tx_1) \end{aligned} \tag{4}$$

By 4, for every $m, n \in \mathbb{N}$ such that $m > n$ we have,

$$\begin{aligned} d(Tx_m, Tx_n) &\leq d(Tx_m, Tx_{m-1}) + d(Tx_{m-1}, Tx_{m-2}) + ... + d(Tx_{n+1}, Tx_n) \\ &\leq [(\frac{\lambda}{1-\lambda})^{m-1} + (\frac{\lambda}{1-\lambda})^{m-2} + ... + (\frac{\lambda}{1-\lambda})^n] d(Tx_0, Tx_1) \\ &\leq [(\frac{\lambda}{1-\lambda})^n + (\frac{\lambda}{1-\lambda})^{n+1} + ...] d(Tx_0, Tx_1) \\ &= (\frac{\lambda}{1-\lambda})^n \frac{1}{1-(\frac{\lambda}{1-\lambda})} d(Tx_0, Tx_1) \end{aligned} \tag{5}$$

Letting $m, n \to \infty$ in 5, we have $\{Tx_n\}$ is Cauchy sequence, and since X is a complete metric space, there exists $v \in X$ such that

$$\lim_{n\to\infty} Tx_n = v \tag{6}$$

Since T is a subsequentially convergent, $\{x_n\}$ has a convergent subsequence. So there exists $u \in X$ and $\{x_{n(k)}\}_{k=1}^{\infty}$such that $\lim_{n\to\infty} x_{n(k)} = u$.

Since T is continuous and $\lim_{n\to\infty} x_{n(k)} = u$, $\lim_{n\to\infty} Tx_{n(k)} = Tu$ By 6 we conclude that $Tu = v$. So,

$$\begin{aligned}
d(TSu, Tu) &\leq d(TSu, TS^{n(k)}x_0) + d(TS^{n(k)}x_0, TS^{n(k)+1}x_0) + d(TS^{n(k)+1}x_0, Tu) \\
&\leq \lambda[d(Tu, TSu) + d(TS^{n(k)-1}x_0, TS^{n(k)}x_0)] \\
&+ (\frac{\lambda}{1-\lambda})^{n(k)} d(TSx_0, Tx_0) + d(Tx_{n(k)+1}, Tu) \\
&= \lambda(Tu, TSu) + \lambda(Tx_{n(k)-1}, Tx_{n(k)}) \\
&+ (\frac{\lambda}{1-\lambda})^{n(k)} d(Tx_1, Tx_0) + d(Tx_{n(k)+1}, Tu),
\end{aligned} \tag{7}$$

$$(1-\lambda)d(Tu, TSu) \leq \lambda(Tx_{n(k)-1}, Tx_{n(k)}) + (\frac{\lambda}{1-\lambda})^{n(k)} d(Tx_1, Tx_0) + d(Tx_{n(k)+1}, Tu)$$

hence,

$$\begin{aligned}
d(TSu, Tu) &\leq \frac{\lambda}{1-\lambda}(Tx_{n(k)-1}, Tx_{n(k)}) + \frac{1}{1-\lambda}(\frac{\lambda}{1-\lambda})^{n(k)} d(Tx_1, Tx_0) \\
&+ \frac{1}{1-\lambda} d(Tx_{n(k)+1}, Tu) \longrightarrow 0 \qquad \text{as} \quad k \longrightarrow \infty
\end{aligned} \tag{8}$$

Therefore $d(TSu, Tu) = 0$.

Since T is one-to-one $Su = u$. So S has a fixed point. Since 4.7 holds and T is one-to-one, S has a unique fixed point.

Now if T is sequentially convergent, by replacing $\{n\}$ with $\{n(k)\}$ we conclude that $\lim_{n\to\infty} x_n = u$ and this shows that $\{x_n\}$ converges to the fixed point of S. $\square$

4.4 Zamfirescu Fixed Point Theorem

Definition 4.4.1 (Zamfirescu Map)**.** *Let (X, d) be a complete metric space. $T : X \to X$ is a Zamfirescu map if there exists $\gamma \in \left[0, \dfrac{1}{2}\right)$ such that*

$$d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)], \quad \textit{for all } x, y \in X.$$

Theorem 4.4.1. *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping for which there exist the real numbers α, β and γ satisfying $0 \leq \alpha < 1$, $0 \leq \beta < \frac{1}{2}$ and $0 \leq \gamma < \frac{1}{2}$, such that, for each $x, y \in X$, at least one of the following is true:*

(a) $d(Tx, Ty) \leq \alpha d(x, y)$;

(b) $d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]$;

(c) $d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]$.

Then has a unique fixed point.

Proof. Let $x, y \in X$. At least one of (a), (b) or (c) is true.

If (a) holds, we have the Banach Contraction Theorem (3.1.1), which yields that T have a unique fixed point.

If (b) holds, the we have

$$\begin{aligned} d(Tx, Ty) &\leq \beta[d(x, Tx) + d(y, Ty)] \\ &\leq \beta\{d(x, Tx) + [d(y, x) + d(x, Tx) + d(Tx, Ty)]\} \\ (1 - \beta)d(Tx, Ty) &\leq 2\beta d(x, Tx) + \beta d(x, y), \end{aligned}$$

Which yields

$$d(Tx, Ty) \leq \frac{2\beta}{1 - \beta} d(x, Tx) + \frac{\beta}{1 - \beta} d(x, y). \tag{4.8}$$

If (c) holds, then similarly we get

$$d(Tx,Ty) \leq \frac{2\gamma}{1-\gamma} d(x,Ty) + \frac{\gamma}{1-\gamma} d(x,y) \tag{4.9}$$

Therefore, denoting

$$\delta = max\left\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\right\}$$

we have $0 \leq \delta < 1$ and then, for all $x, y \in X$, the following inequality

$$d(Tx,Ty) \leq 2\delta \cdot d(x,Tx) + \delta \cdot d(x,y) \tag{4.10}$$

holds. In a similar manner we obtain

$$d(Tx,Ty) \leq 2\delta \cdot d(x,Ty) + \delta \cdot d(x,y) \tag{4.11}$$

valid for all $x, y \in X$.

We will now show that T has a (unique) fixed point. Let $x_0 \in X$ and $\{x_n\}_{n=0}^{\infty}$,

$$x_n = T^n x_0 \ , \quad n = 0, 1, 2, \dots$$

be a Picard iteration associated to T

If $x := x_n$, $y := x_{n-1}$ are two successive approximations, then by (4.11) we have

$$d(x_{n+1}, x_n) \leq \delta \cdot d(x_n, x_{n-1}) \leq \delta^n d(x_0, x_1). \tag{4.12}$$

From this we deduce that $\{x_n\}_{n=0}^{\infty}$ is a Cauchy sequence, and hence a convergent sequence, too. Let $x^* \in X$ be its limit. In particular we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{4.13}$$

By triangle rule and (4.10) we get

$$\begin{aligned} d(x^*, Tx^*) &\leq d(x^*, x_{n+1}) + d(Tx_n, Tx^*) \\ &\leq d(x^*, x_{n+1}) + \delta d(x^*, x_n) + 2\delta d(x_n, Tx_n), \end{aligned}$$

which, by letting $n \to \infty$, yields

$$d(x^*, Tx^*) = 0 \Longleftrightarrow x^* = Tx^*,$$

Therefore x^* is a fixed point of T

To check for uniqueness, let $a, b \in X$ be two fixed point of T.

If (a) holds,

$$d(a,b) = d(Ta,Tb) \leq \alpha d(a,b) \qquad \text{for } 0 \leq \alpha < 1$$

$$\implies \alpha = 0 \implies d(a,b) = 0 \implies a = b$$

If (b) holds,

$$\begin{aligned} d(a,b) = d(Ta,Tb) &\leq \beta[d(a,Ta) + d(b,Tb)] \qquad \text{for } 0 \leq \beta < \frac{1}{2} \\ &= \beta[d(a,a) + d(b,b)] = 0 \\ \implies d(a,b) = 0 \implies a = b \end{aligned}$$

If (c) holds,

$$\begin{aligned} d(a,b) = d(Ta,Tb) &\leq \gamma[d(a,Tb) + d(b,Ta)] \qquad \text{for } 0 \leq \gamma < \frac{1}{2} \\ &= \gamma[d(a,b) + d(b,a)] \\ &= \gamma[d(a,a)] = 0 \\ \implies d(a,b) = 0 \implies a = b \end{aligned}$$

Therefore T have a unique fixed point. $\square$

Reference

[1] Banach, S. (1922). Sur les operations dans ensembles abstraits et leur application aux equations untegrales, *Fund. Math.*, 3: 133 - 181.

[2] Berinde, V. (2002). Iterative Approximation of Fixed Points, *Editura Efemeride, Baia Mare.*

[3] Bohl, P. (1904). Uber die Beweging eines mechanischen systems in der Nahe einer Gleichgewicht stage, J. Reine. Angew. Math., 127: 279 - 286.

[4] Bohnenblust, H. F. & Karlin, S. (1950): On a theorem of Ville, Contribution to the theory of games, *Princeton Univ. Press, Princeton, N. J.*, 1:155 - 160.

[5] Bonsai, F. F. (1962). Lectures on fixed point theorems of functional analysis, *TIFR, Bombay, India.*

[6] Boyd, D. W. & Wong, J. S. W. (1969). On non-linear contractions, *Proc. Amer. Math. Soc.*, 20: 456 - 464.

[7] Brouwer, L. E. J. (1911/12). Uber Abbildungen von Mannigfaltigkeithen, *Math. Ann.*, 17: 97-115.

[8] Caccioppoli, R. (1930). Unateorem generale sullexistenzadi elementi uniti in una transformazione fimzionale, *Rend. Akad. Nazionale deli*, dincei, 11: 794-799.

[9] Cauchy, A. L. (1884). Lecons sur les calculas differentiel et integral, vols. 1 and 2, Paris .

[10] Downing, D. & Ray, W. (1981). Some remarks on set-valued mappings, non-linear Analysis, 5: 245 -249.

[11] Edelstein, M. (1962). On fixed points and periodic points under contractive mappings, *J. Lond. Math. Soc.*, 37 : 74 - 79.

[12] Fraser, R. B. & Nadler, S. B. (1969). Sequences of contraction maps and fixed points, *Pacific J. Math.*, 31: 659-667.

[13] Glicksberg, I. L. (1952). A further generalisation of the Kakutani fixed point theorem with application to Nash Equilibrium points, *Proc. Amer. Math. Soc.*, 3: 170 - 174.

[14] Goebel, K & Kirk, W. A. (1972). A fixed point theorem for asymptotically non-expansive mappings, *Proc. Amer. Math. Soc.*, 35: 171 - 174.

[15] Jeremy O. (2018). Existence and Uniqueness Theorem, *http://web.mit.edu/jorloff/www/18.03-esg/notes/existAndUniq.pdf.*

[16] Kakutani, S. (1941): A generalisation of Brouwer fixed point theorem, *Duke Math. J.*, 8: 457-459.

[17] Kannan, R. (1968). Some results on fixed point, *Bulletin of Calcutta Mathematical Soceity*, 60: 71 76.

[18] Keith C. The Contraction Mapping Theorem, *http://www.math.uconn.edu/kconrad/blurbs/analysis/contraction.pdf.*

[19] Liouville, J. (1837). Second mmoire sur le dveloppement des fonctions ou parties de fonctions en sries dont divers termes sont assujettis satisfaire a une mme quation diffrentielle du second ordre contenant un paramtre variable, *J. Math. Pures Appl.*, 2: 16-35.

[20] Lipschitz, R. (1877) . Lehrbuch der Analyse, Bonn,

[21] Mann, W. R. (1953). Mean value methods in iteration, *Proc. Amer. Math. Soc.*, 4: 506-510.

[22] Moradi S. (2011). New Extensions of Kannan Fixed-Point Theorem on Complete Metric and Generalized Metric Spaces, *Int. Journal of Math. Analysis*, 2(47): 2313-2320.

[23] Nadler, S. B., Jr. (1968). Sequences of contraction mappings and fixed points, *Pacific J. Math.*,27: 579-585.

[24] Neumann, J. V. (1935). Uber ein okonomisches Gleichungssystem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, *Ergebnisse eines Mathematischen Kolloquiums*, 8: 73-83.

[25] Peano, G.(1890). Demonstration de Lintegrabilite des equations differentilles ordinarres, *Math. Ann.*, 37: 182-228

[26] Picard, E. (1890). Memoire sur la theorie des equations aux derivees partielles et la method des approximations successives, *J. de. Math.*, 6: 145-210.

[27] Poincareare,H. (1886). Suries courbes defines par les equations differentielles, J. de. Math., 2.

[28] Olaleru, J. O. (2006). A Generalization of Gregus Fixed Point Theorem, *Journal of Applied Sciences*, 6: 3160-3163.

[29] Rakotch, E. (1962): A note on contraction mappings, *Proc. Amer. Math. Soc.*, 13: 459.

[30] Schauder, J. (1930). Der fixpunktsatz in funktionalrumen, *Studia Math.*, 2: 171 180.

[31] Strother, W. L. (1953). On an open question concerning fixed points, *Proc. Amer. Math. Soc.*, 4: 988-993.

[32] Nadler Sam B.(1969). Multi-valued contraction mappings, *Pacific. J. Math.*,30: 475-488.