Saddlepoint Approximation Explanation

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This document is prepared as a supplement to the source code available in `https://github.com/OguzKislal/ErrorDetection_InfoTheory` for the manuscript "Undetected Error Probability in the Short Blocklength Regime: Approaching Finite-Blocklength Bounds with Polar Codes" to explain the simulation procedure in detail.

In our simulation, we have simulated Theorem 2 and Theorem 3 as follows:

- For a given $\mathbf{x}$ and $\mathbf{y}$ search for ζ^* that $\left(\gamma^{(n)}(\zeta^*)\right)' - \omega <= 10^{-3}$. Note that ω here depends on $\mathbf{x}$ and $\mathbf{y}$, and slightly differs for Theorem 2 and Theorem 3.
- Evaluate CGF $\left(\gamma^{(n)}(\zeta)\right)$ and its first two derivatives , $\left(\gamma^{(n)}(\zeta)\right)'$ and $\left(\gamma^{(n)}(\zeta)\right)''$ for $\zeta = \zeta^*$.
- Evaluate the pairwise error probability using the saddlepoint approx. which is given in Theorem 4.
- Evaluate the RCU bound using the approximated pairwise error probability.
- Monte-Carlo simulations required to evaluate the outer expectation that appears in (12) for Theorem 2 and (17) for Theorem 3. To do so, repeat the previous steps a suitable number of times and evaluate the empirical average of the RCU bound obtained for different $\mathbf{x}$ and $\mathbf{y}$.

In fact, one of the most time-consuming steps of this process is to search for ζ^*, as it needs to be repeated for every Monte Carlo simulation run. However, for the BiAWGN channel, it is shown in [R1] that a uniquely selected ζ^* can be used for all $\mathbf{x}$, $\mathbf{y}$ without a loss in approximation accuracy. For the implementation of Theorem 3 for the BiAWGN channel (in UER_RCU_Saddlepoint_fixedTau.m), we used this method to have a shorter simulation time. Authors also showed that in [R1] a second saddlepoint approximation may be used to evaluate the RCU bound which exactly coincides with what we have in Theorem 2. Thus, for the implementation of Theorem 2 for the BiAWGN channel (in rcu_saddle_biawgn.m), we used double saddlepoint approximation to have an even shorter simulation time. Note that none of these improvements are applicable for the case of block-phase channel.

References

[R1] J. Font-Segura, G. Vazquez-Vilar, A. Martinez, A. Guillén i Fàbregas and A. Lancho, "Saddlepoint approximations of lower and upper bounds to the error probability in channel coding," *52nd Annual Conference on Information Sciences and Systems (CISS)*, Princeton, NJ, USA, 2018.