PD controllers

An ideal PD (proportional-derivative) controller has the transfer function

$$C(s) = K_p + K_d s ,$$

where K_p is the proportional term, and $K_d s$ is the derivative term. The coefficients K_p, K_d must be tuned to achieve the desired performance of the feedback system. In the time domain, if e is the error signal and v is the controller output, then

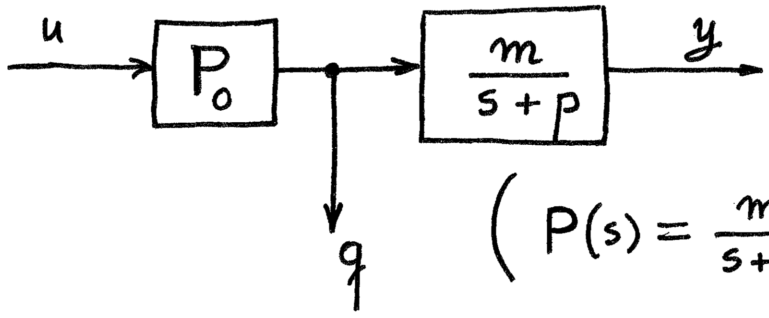
$$v = K_p e + K_d \dot{e} .$$

Note that the gain $|C(i\omega)| = \sqrt{K_p^2 + \omega^2 K_d^2}$ tends to infinity as $\omega \to \infty$ (this controller is not proper). Such a behavior is difficult (or impossible) to achieve in practice, so that we often use an approximate realisation of the PD controller:

$$C(s) = K_p + \frac{K_d s}{1 + Ts} ,$$

where $T > 0$ is small. Then for low frequencies (i.e., for $|s| << \frac{1}{T}$) we have $C(s)$ approximately equal to $K_p + K_d s$.

Sometimes we have access to a measurement in the plant that makes it unnecessary to evaluate $\dot{e}$. If the plant P can be factored as in the figure

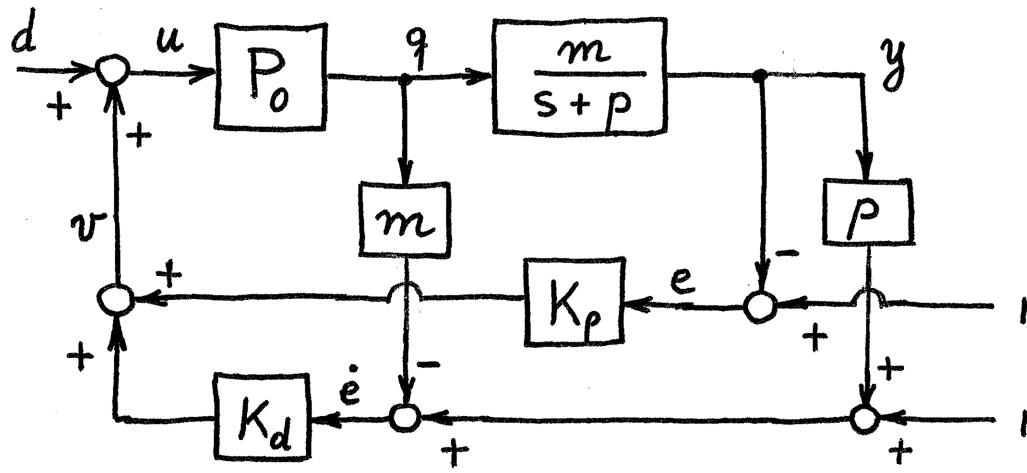


$$\left(P(s) = \frac{m}{s+p} \cdot P_0(s) \right)$$

and we can measure both y and q, then we can compute $\dot{y}$ without having to differentiate, from $\dot{y} + py = mq$.

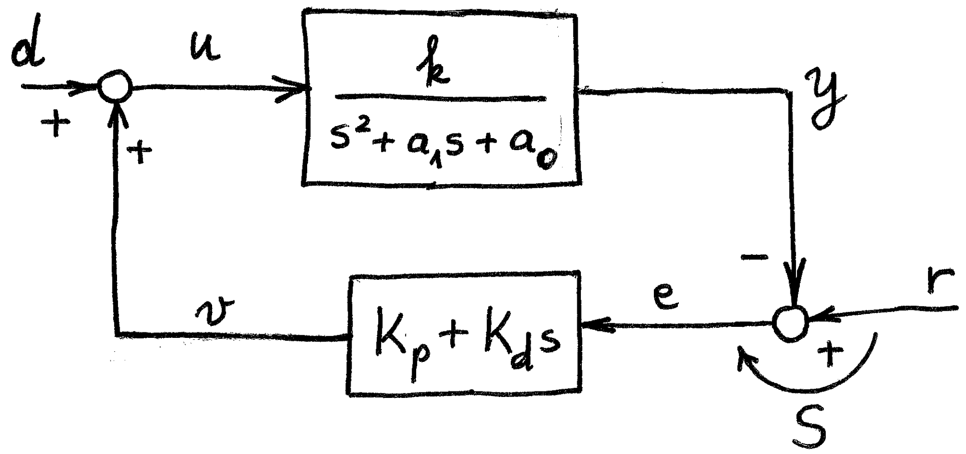
If $\dot{r}$ is given (this is reasonable, since we generate r) then $\dot{e} = \dot{r} - \dot{y} = \dot{r} + py - mq$.

Thus, we can build the equivalent of a PD controller as follows:



For example, if q represents a speed that can be measured directly, and y is the corresponding position, then we can use this arrangement with $p=0$ and $m=1$. We mention that if, in the block diagram on p. 2, we replace $\dot{r}$ with 0 (i.e., we do not supply $\dot{r}$), this will have no effect on the location of the poles of the feedback system (this should be clear).

We show how a PD controller can be used to control a second order plant:



It is easy to check that the closed-loop transfer function from r to y is

$$G(s) = 1 - S(s) = \frac{kK_d s + kK_p}{s^2 + (a_1 + kK_d)s + (a_0 + kK_p)}.$$

We see that by changing K_p and K_d, we can move the two poles of G anywhere we want (the only restriction being that if the poles are non-real, they must be conjugate). As explained before, to have a small overshoot and a small settling time, a good choice is to take the poles equal and of large absolute value (large ω_n and $\zeta = 1$, see WEEK 4). Thus, we would choose K_p large and then K_d such that

$$a_1 + k K_d = 2\sqrt{a_0 + k K_p}\,.$$

PD controllers are a good choice of controller when the plant can be approximated well by a second order plant. This means that

$$P(s) = \frac{k}{s^2 + a_1 s + a_0} + \underbrace{Q(s)}_{\text{usually this contains the poles with large negative real part}}$$

where $|Q(i\omega)|$ is small in the frequency range of interest (usually, this is an interval, $\omega \in [-\omega_b, \omega_b]$). "Small" is a relative concept, of course. If P can be decomposed as above, then the poles of the first term (i.e., the zeros of $s^2 + a_1 s + a_0$) are called the <u>dominant poles of **P**</u>.

We mention that in industry we often encounter <u>PID controllers</u> (an obvious combination of PI and PD described earlier).

Root locus plots

Consider the standard feedback configuration with plant P and controller C, with sensitivity $S = (1+PC)^{-1}$. We often want to know how the poles of S would change if we changed the gain of the controller, i.e., if we replaced C with αC, where $\alpha > 0$. To investigate this problem, we introduce the following notation:

$$P(s)\,C(s) = k\,\frac{n(s)}{d(s)} \quad \begin{matrix} \longleftarrow (\text{"numerator"}) \\ \longleftarrow (\text{"denominator"}) \end{matrix}$$

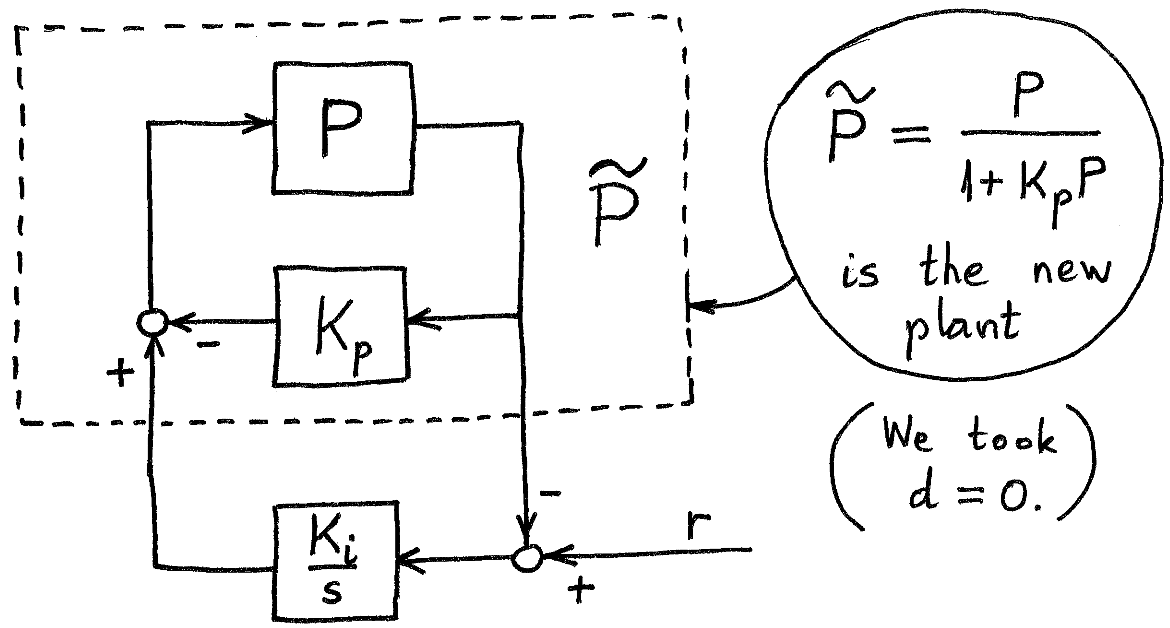
where
$$n(s) = s^w + b_{w-1}\,s^{w-1} + \dots + b_1 s + b_0\,,$$
$$d(s) = s^m + a_{m-1}\,s^{m-1} + \dots + a_1 s + a_0\,,$$

b_j and a_j are real coefficients and $m \geq w$.

Then the problem described earlier is equivalent to plotting the poles of S as a function of k, either for all $k>0$ or for all $k<0$. These plots are called <u>the root locus of n/d</u>. If the controller has several parameters and we only want to investigate the change of the poles of S as one of these parameters changes, we can usually reduce this to a standard root locus problem by rearranging the block diagram. For example, if C is a PI controller,

$$C(s) = K_p + \frac{K_i}{s}$$

and we want to see the effect of K_i on the poles of S, then we can rearrange our feedback system like this:



This system is not equivalent to the previous one in the sense that its sensitivity $\tilde{S} = \left(1 + \tilde{P}\frac{K_i}{s}\right)^{-1}$ is not equal to S. However, the poles of $\tilde{S}$ and S are the same (verify this fact as an exercise – you may use either a transfer function reasoning, or a state space reasoning).

In 1942, W. R. Evans worked out a list of "rules" that enable an engineer to construct the root locus plots, based on the polynomials n and d, without using a computer (which has not yet been invented). Learning and applying all these rules requires a lot of effort and training, but it is much less relevant today, because MATLAB (and other software) can plot the root locus of n/d fast and accurately. Thus, we shall present here only 6 rules and some examples, which would enable the student to guess the shape of the root locus plots for simple systems. More detailed info on the root locus plotting rules and on their application in feedback control design can be found in all the books listed at the end of WEEK 1.

RULE 1. The root locus (RL) of $\frac{n}{d}$ has m branches (remember that m is the degree of the polynomial d). In other words, for each value of k, S has m poles. These poles are distributed symmetrically with respect to the real axis $\mathbb{R}$.

Indeed, this is clear from

$$S = \frac{d}{d + kn}$$

which shows that the poles of S are the zeros of the polynomial $d + kn$.

RULE 2. A point $\alpha \in \mathbb{R}$ belongs to the RL if and only if

- there is an odd number of poles or zeros of n/d to the right of α, if $k > 0$,
- there is an even number of poles or zeros of n/d to the right of α, if $k < 0$.

Note that every real point is on the RL of n/d, either on the plot for $k > 0$ or on the plot for $k < 0$. A double pole (or a double zero) is counted as two poles (or zeros).

RULE 3. The RL starts at the poles of n/d and it ends at the zeros of n/d or at infinity. In other words, as $k \to 0$ the poles of S tend to the poles of n/d, while as $k \to \pm\infty$, the poles of S tend to either the zeros of n/d or to ∞. As $k \to \pm\infty$, w poles of S tend to zeros of n/d, and $m - w$ poles tend to ∞ in $\mathbb{C}$.

You may skip these explanations of why RULES 3 & 4 hold.

We give an explanation about why this rule holds. From what we said on top of p. 8, the RL contains the poles of S, i.e., the zeros of the polynomial $d+kn$. If $k \to 0$, then clearly these converge to the zeros of d. The case $k \to \pm\infty$ is a bit more difficult. The zeros of $d+kn$ are the same as the zeros of $\frac{1}{k}d+n$. If a specific zero of $\frac{1}{k}d+n$ stays bounded as $k \to +\infty$, (or $k \to -\infty$), then $\frac{1}{k}d \to 0$, so that this zero tends to a zero of n (i.e., a zero of n/d). If a specific zero of $\frac{1}{k}d+n$ (which is a function of k) tends to ∞ in the complex plane, then $\frac{1}{k}d$ does not tend to zero. We look at all the terms of $\frac{1}{k}d$ and n and notice that the first term (both in $\frac{1}{k}d$ and in n) becomes much larger then the remaining terms. Thus, we can rewrite the equation approximately, by keeping only the first terms: $\frac{1}{k}s^m + s^w = 0$. This gives $s^{m-w} = -k$, which shows that we have $m-w$ zeros of $d+kn$ that behave like the solutions of the equation $s^{m-w} = -k$. This can happen only if $m > w$. This explains also the fourth rule, given below:

RULE 4. The $m-w$ poles of S that tend to ∞ in $\mathbb{C}$ do so along $m-w$ straight asymptotes. The angles of these asymptotes are $\frac{\ell \cdot 180°}{m-w}$, where ℓ are odd numbers if $k>0$, and ℓ are even numbers if $k<0$.

For example, if $m-w=1$ and $k>0$, then there will be only one asymptote, the negative real axis. If $m-w=3$ and $k>0$, then the angles of the asymptotes will be $60°$, $180°$, $300°$. Note that the angle between two adjecent asymptotes is always $360°/m-w$.

RULE 5. All the asymptotes of the RL meet in one point γ on the real axis, given by

$$\gamma = \frac{\sum_{j=1}^{m} \operatorname{Re} p_j - \sum_{j=1}^{w} \operatorname{Re} z_j}{m-w},$$

You may omit to write "Re" the formula remains valid.

where $p_1, p_2, \dots p_m$ are the poles of n/d, and $z_1, z_2, \dots z_w$ are the zeros of n/d.

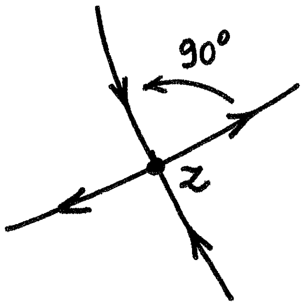
RULE 6. If $z \in \mathbb{C}$ is a point on the RL where two or more poles come together, then

$$d'(z)\,n(z) - n'(z)\,d(z) = 0.$$

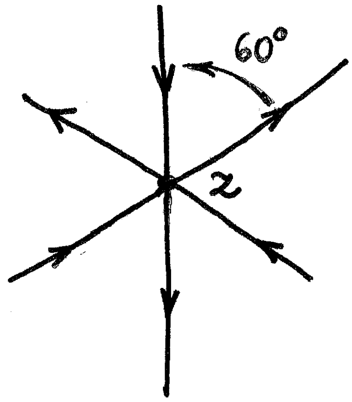
We give some comments and explanations about RULE 6. If a point s lies on the RL and it corresponds to some gain k, then clearly

$$d(s) + k\,n(s) = 0.$$

Thus, we can express $k = -\frac{d(s)}{n(s)}$. When poles get together in a point z, they continue in a different direction, see the figures.



two poles getting together and then departing in different directions



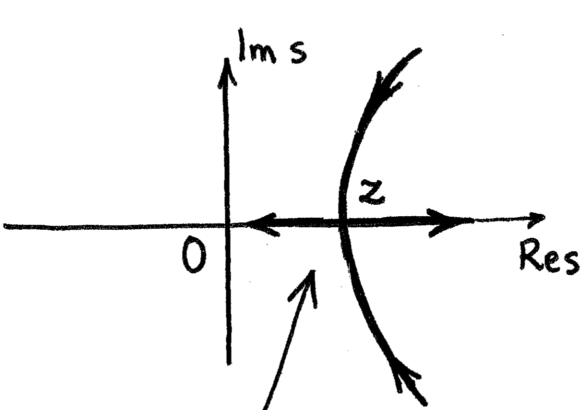
the same for three poles

Note that both pictures are (rotationally) symmetric around the meeting point z, if we only look at a small area around the meeting point z.

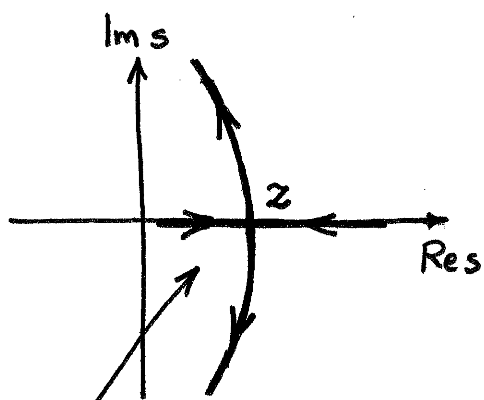
A better argument: $dk = -\left[\frac{d}{n}\right]' ds \in \mathbb{R}$. At a meeting point of branches we can take different directions for ds, hence $\left[\frac{d}{n}\right]' = 0$.

Since the direction of the RL curves changes at the meeting point z, their derivative is ∞ (as functions of k). Since $\frac{ds}{dk} = 1 / \frac{dk}{ds}$, we conclude that $\frac{dk}{ds} = 0$. From $k = -\frac{d}{n}$ we have $k' = \frac{n'd - d'n}{n^2}$, where n', d' are the derivatives of the polynomials n, d. From here, RULE 6 follows.

Usually we apply RULE 6 for two poles meeting on the real axis. We have two possible situations:



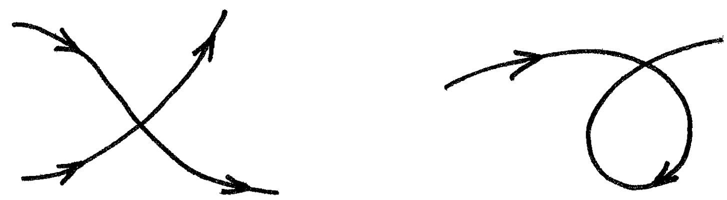
"break-in point"



"break-away point"

Remark about RL curves:

The RL curves never intersect each other. In other words, if a point z is on the RL curves for some $k \in \mathbb{R}$, then this point cannot belong to the RL curves for any other value $k \in \mathbb{R}$. This is clear from the formula $k = -d(z)/n(z)$, which determines a unique k for z. Thus, we cannot have loops or intersections, like these:



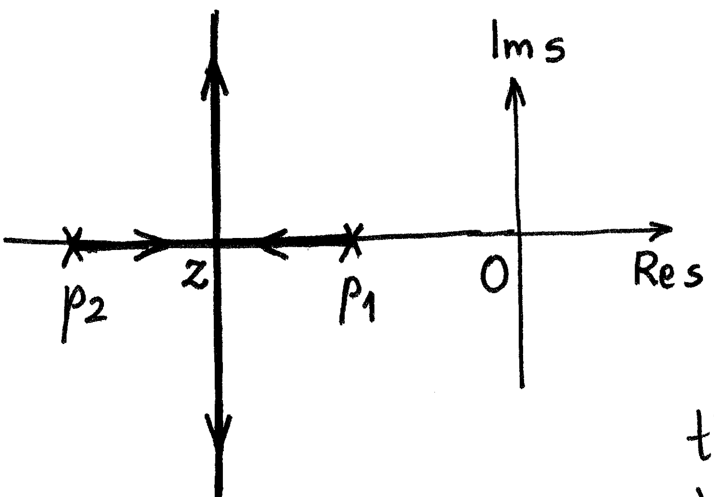
If $w = m$ and $k < 0$, then for $k = -1$ a pole of S (a branch of the RL) goes to ∞ and then comes back. This does not count as an asymptote.

Examples of RL curves

Example 1

$$\frac{n(s)}{d(s)} = \frac{1}{(s-p_1)(s-p_2)} .$$

In the figure, we assume $p_2 < p_1 < 0$, $k > 0$.



Note that now the meeting point of the asymptotes

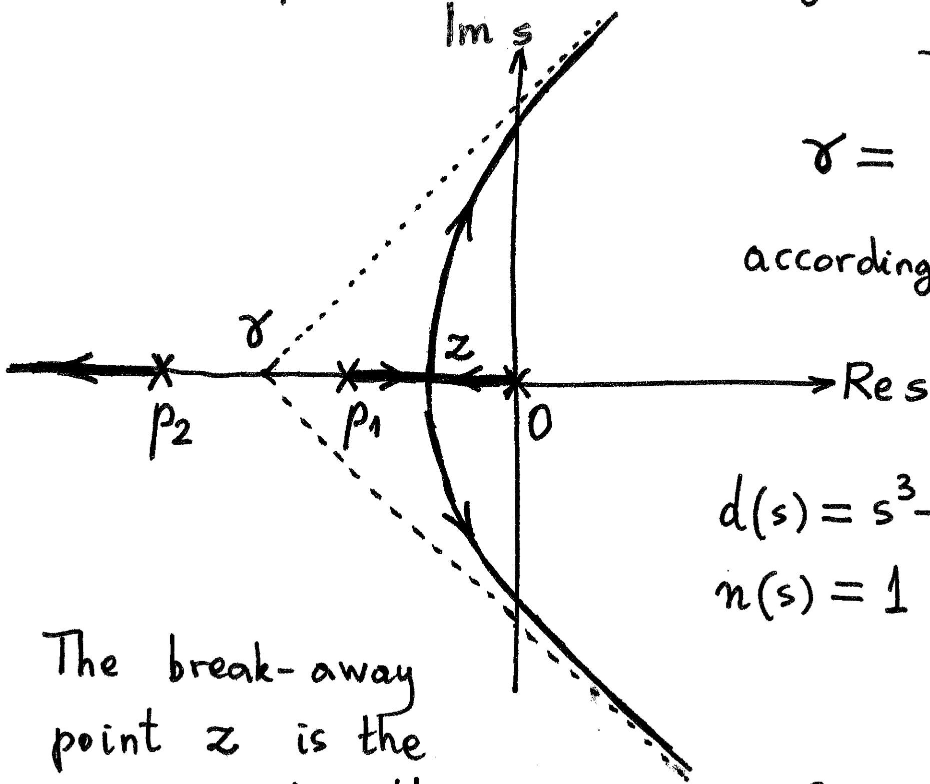
$$z = \frac{1}{2}(p_1 + p_2)$$

coincides with the break-away point. Verify the 6 rules!

Example 2

$$\frac{n(s)}{d(s)} = \frac{1}{s(s-p_1)(s-p_2)}.$$

In the plots, we assume again $p_2 < p_1 < 0$, $k > 0$.



$$\gamma = \frac{p_1 + p_2}{3}$$

according to RULE 5.

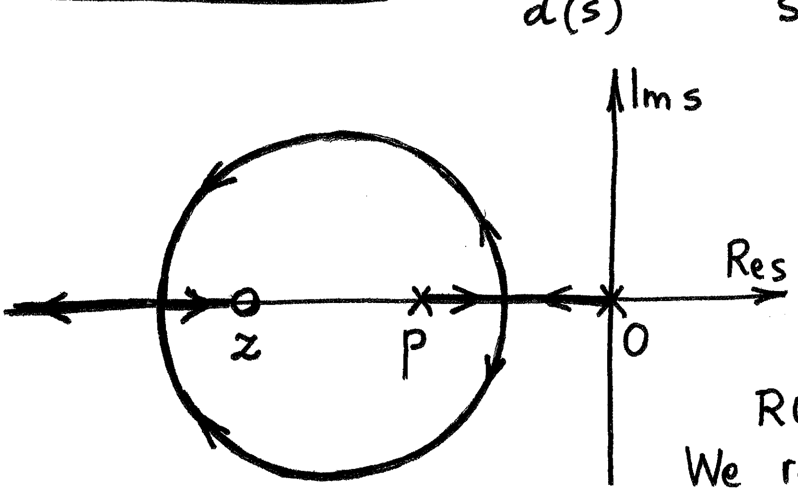
$$d(s) = s^3 - (p_1+p_2)s^2 + p_1 p_2 s$$
$$n(s) = 1$$

Exercise: determine the intersection of the RL with the imaginary axis, and the corresponding gain k.

The break-away point z is the solution of $d' = 0 \Longrightarrow 3z^2 - 2(p_1+p_2)z + p_1 p_2 = 0$ which lies in the interval $[p_1, 0]$. (The other solution of this equation is in the interval $[p_2, p_1]$ and it represents the break-away point of the RL curves for $k < 0$.)

Example 3

$$\frac{n(s)}{d(s)} = \frac{s-z}{s(s-p)}, \quad z < p < 0, \quad k > 0.$$

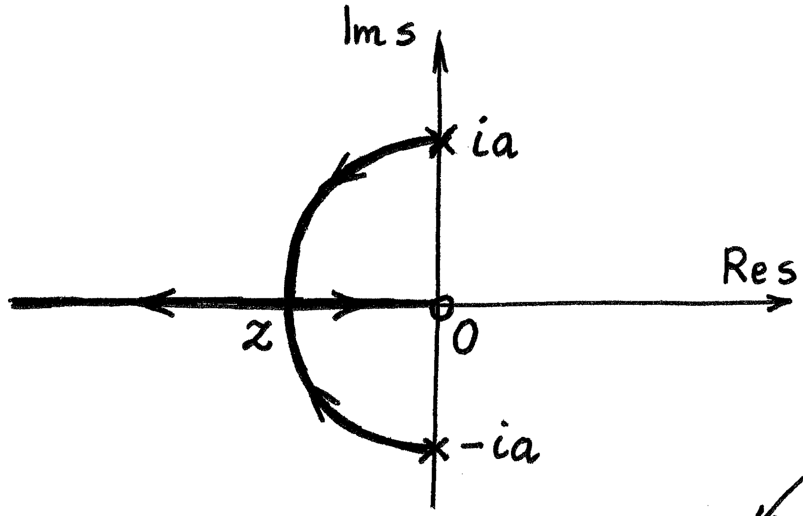


Note that there is only one asymptote, one break-away point and one break-in point. RULE 5 is pointless.

We remark that the non-real part of the RL is a circle.

Example 4

$$\frac{n(s)}{d(s)} = \frac{s}{s^2 + a^2}, \quad a > 0.$$



Intended for a study group or homework

Example 5

Consider a plant with transfer function

$$P(s) = \frac{m}{s^2(s-p)}, \quad p < 0, \; m > 0.$$

Show, using RL techniques, that we cannot stabilize this plant with a proportional or with a PI controller, no matter how we choose the coefficients. However, we can stabilize the plant with a PD controller

$$C(s) = K_p\,(s - z), \quad p < z < 0, \; K_p > 0.$$

Draw the RL plot for the feedback connection of P and C, using $k = K_p m$ as the parameter of the curves. Apply the six rules to approximate the RL plot. Estimate the 5% settling time for a step r, for large K_p.