Introduction to control theory

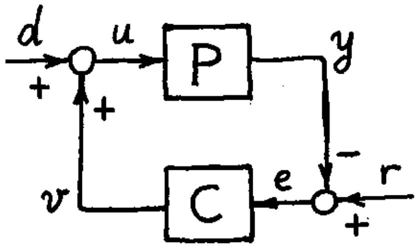
WEEK 4

Eliminating the steady-state error for constant reference and disturbance

We have seen in WEEK 2 that if we apply proportional control to a first order system and the reference r and/or the disturbance d converge to non-zero constants, then also the error e given by $\hat{e} = S\hat{r} - PS\hat{d}$ converges (as $t \to \infty$) to a non-zero value e_{ss} called the <u>steady-state error</u>. We would like to achieve $e_{ss} = 0$ by using a different controller C.

P = plant transfer function

C = controller transfer function



d = disturbance

r = reference

e = tracking error

y = plant output

u = plant input

If r converges to a constant (as $t \to \infty$) (for example, if r is a step) and similarly for d, then denoting $r_{ss} = \lim_{t\to\infty} r(t)$, $d_{ss} = \lim_{t\to\infty} d(t)$, by the final value theorem

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} s\hat{e}(s) = \lim_{s\to 0} S(s)s\hat{r}(s)$$

$$-\lim_{s\to 0} P(s)S(s)s\hat{d}(s) = S(0)r_{ss} - P(0)S(0)d_{ss},$$

where S is the sensitivity of the feedback system, $S = (1+PC)^{-1}$ (the transfer function from r to e).

In order to obtain $e_{ss} = 0$ (for all r_{ss}, d_{ss}), we need $S(0) = 0$. It is easy to see (and we discussed this in an earlier lecture) that the zeros of S are the poles of the loop gain PC. Thus, we obtain:

$$\boxed{\text{For } e_{ss}=0, \; C \text{ should have a pole at } s=0.}$$

In the above discussion, we had in mind the first order system from WEEK 2: $P(s) = \dfrac{k}{1+Ts}$.

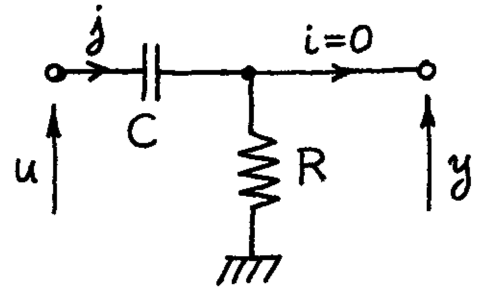
However, the arguments (and the conclusion) remain valid for any rational transfer function **P**. There is a little problem that has to be clarified if P has a pole at $s=0$. In this case **PC** will have a higher order pole at $s=0$, and we still have $\lim_{s\to 0} P(s)S(s) = 0$, so that the scheme will work also in this case.

Naturally, the other requirements for a controller must be satisfied as well: stability and w-stability. This implies the following:

$$\boxed{\text{If } \mathbf{P}(0)=0, \text{ then we cannot achieve } e_{ss}=0.}$$

Indeed, in this case, there would be a pole-zero cancellation at $s=0$ in the product **PC**. According to the Proposition on p.7 of WEEK 2, the feedback connection of P and C would not be stable in this case.

<u>Example</u>. Consider the circuit below, with input u (a voltage) and output y (also a voltage). The transfer function is



$$P(s) = \frac{RCs}{1+RCs}.$$

Since $P(0)=0$, we cannot achieve $e_{ss}=0$ for constant $r\neq 0$. This follows from the statement in the middle of p. 2. We can also give a physical explanation: to obtain $e_{ss}=0$ we would need $\lim_{t\to\infty} y(t) = r$ (remember that r is a non-zero constant). Denoting the input current of the circuit by j (see the figure on p. 2) we would then have $\lim_{t\to\infty} j(t) = r/R \neq 0$. Such a current would cause the voltage on the capacitor $v_c = \frac{1}{C}\int_0^t j(\sigma)\,d\sigma$ to tend (linearly) to ∞. Hence, $\lim_{t\to\infty} u(t) = \infty$, which shows that the control system would need to be unstable.

Remark. If P is such that $P(0)=\infty$ (i.e., P has a pole at $s=0$) and C does not have a pole at $s=0$, and if the feedback system is stable, then with $\lim_{t\to\infty} r(t) = r_{ss}$ and $\lim_{t\to\infty} d(t) = 0$ we obtain again $e_{ss}=0$. This follows from what we said on p. 1 (check the details).

Integral control of first order systems

This is a particular controller that eliminates the steady-state error:

$$P(s) = \frac{k}{1+Ts}, \qquad C(s) = \frac{K_i}{s}.$$

We compute the sensitivity $S = (1+PC)^{-1}$:

$$S(s) = \frac{Ts^2 + s}{Ts^2 + s + kK_i} . \qquad \left(\text{Note that } S(0)=0.\right)$$

We see that S is stable if $T>0$ and $kK_i>0$. The first condition means that the plant must be stable. The second condition can be satisfied by choosing correctly the sign of K_i (should be the same as the sign of k). There are no pole-zero cancellations between P and C, so that if S is stable then also the feedback system is stable. w-stability is not a problem at all, since $P(\infty) = C(\infty) = 0$.

The transfer function from r to y is $G = 1-S$,

$$G(s) = \frac{kK_i}{Ts^2 + s + kK_i} .$$

If we introduce the notation

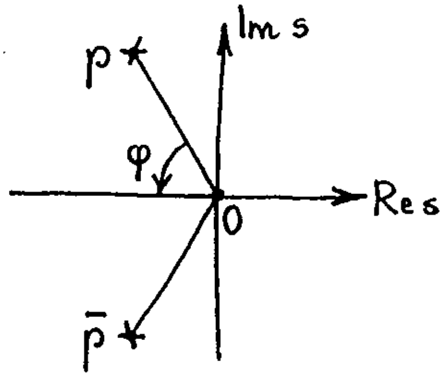
$$\omega_n^2 = \frac{kK_i}{T} , \qquad \zeta = \frac{1}{2\sqrt{kK_iT}} ,$$

then

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} .$$

Such transfer functions occur often in engineering, hence it is worth analyzing them.

We have $\omega_n > 0$ and $\zeta > 0$. Usually, K_i is chosen such that $\zeta \leq 1$. If this is the case, then G has two complex-conjugate poles p and $\bar{p}$. If we denote by φ the angle between these poles and the negative real axis ($\varphi \in [0, \pi/2)$), then



$$\omega_n = |p| ,$$
$$\zeta = \cos\varphi .$$

Note that if $\xi = 1$ then $p = \bar{p}$ and $G(s) = \dfrac{\omega_n^2}{(s+\omega_n)^2}$.

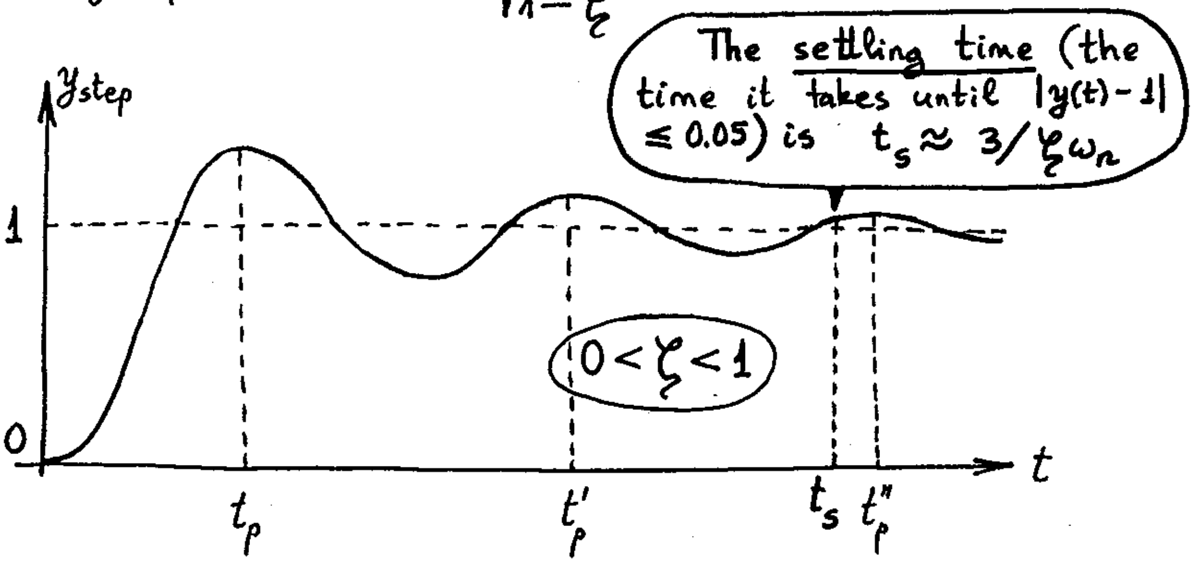
For $\xi > 1$ we have two distinct real poles.

The impulse response corresponding to G for $\xi < 1$ is

$$g(t) = \frac{\omega_n}{\sqrt{1-\xi^2}}\, e^{-\xi\omega_n t} \sin\left(\sqrt{1-\xi^2}\,\omega_n t\right)$$

and the corresponding step response $y_{step}(t) = \int_0^t g(\sigma)\,d\sigma$ is

$$y_{step}(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin\left(\sqrt{1-\xi^2}\,\omega_n t + \varphi\right).$$



If we denote by $t_p, t_p', t_p'', \dots$ the times of the peaks (local maximums) of y_{step}, then

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}, \quad t_p' = \frac{3\pi}{\omega_n\sqrt{1-\xi^2}}, \quad t_p'' = \frac{5\pi}{\omega_n\sqrt{1-\xi^2}}, \dots$$

and the peak values are

$$y_{step}(t_p) = 1+\sigma, \quad y_{step}(t_p') = 1+\sigma^3, \quad y_{step}(t_p'') = 1+\sigma^5, \dots$$

where σ is the <u>overshoot</u> of the step response y_{step}, given by

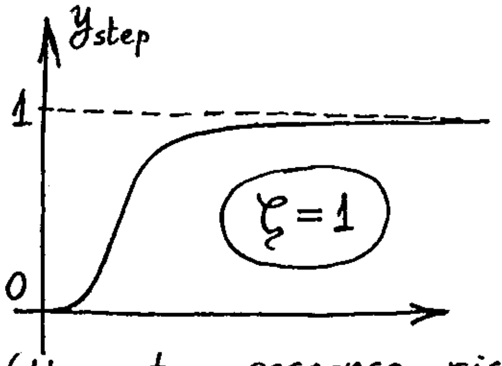
$$\sigma = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}}.$$

The number $-2\log\sigma = 2\xi\pi/\sqrt{1-\xi^2}$ is called the logarithmic decrement of y_{step}. Note that this is the logarithm of $[y_{step}(t_p)-1]/[y_{step}(t_p')-1]$, which can be measured experimentally with relative ease, and permits the computation of ζ (if it is not known). Note that if ζ is close to 0, then σ is close to 1 and the oscillations die very slowly, i.e., $y(t_p')$ is almost as high as $y(t_p)$, etc. In this case, the frequency of the oscillations is almost ω_n, which is the reason for calling ω_n the natural frequency of G. As ζ gets bigger, the frequency decreases, it is $\omega_d = \omega_n\sqrt{1-\zeta^2}$ (see the formula of the step response). ω_d is called the damped natural frequency of G. At the same time (as ζ gets bigger), σ gets smaller and the oscillations die more quickly. ζ is called the damping ratio of G. For $\zeta=1$ (which is called the critically damped case) we have the impulse response

$$g(t) = \omega_n^2\, t\, e^{-\omega_n t}.$$

Now the step response $y_{step}(t) = \int_0^t g(\sigma)\,d\sigma$ is

$$y_{step}(t) = 1 - (1+\omega_n t)\,e^{-\omega_n t}.$$



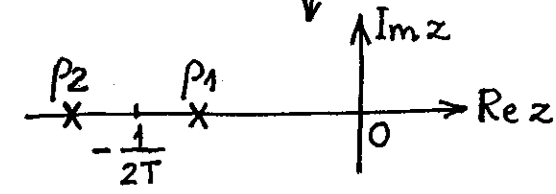
In this case there is no overshoot ($\sigma=0$) and sometimes this is considered the most desirable situation. For $\zeta>1$ there is also no overshoot, but the response becomes slower (the step response rises slower at the beginning). In practical designs, we usually choose $0.7\le\zeta\le 1$.

Let us derive a graphical representation of the location of the poles p_1, p_2 of the closed-loop system, as a function of the controller gain K_i. Since the denominator of G is

$$s^2 + \frac{1}{T}s + \frac{kK_i}{T} = (s-p_1)(s-p_2),$$

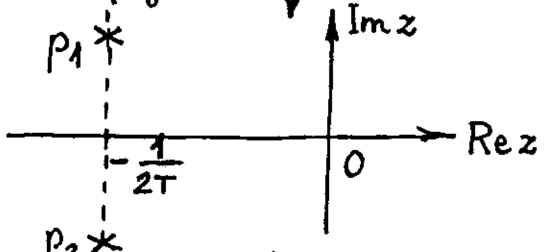
we see that $p_1 + p_2 = -\frac{1}{T}$ $\left(\text{and } p_1 p_2 = \frac{kK_i}{T}\right)$.

If K_i is close to zero, then one pole p_1 is close to zero and the other p_2 is close to $-\frac{1}{T}$. As long as $K_i < \frac{1}{4kT}$, the poles are real and negative, and situated symmetrically around $-\frac{1}{2T}$ (because their sum must be $-\frac{1}{T}$), as in this figure: Here $\xi > 1$.



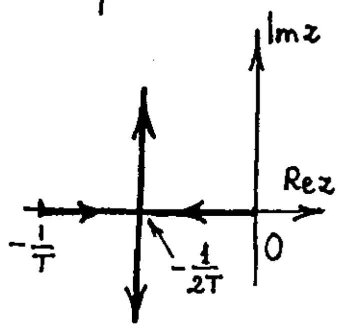
Real poles corresponding to $K_i < \frac{1}{4kT}$

For $K_i = \frac{1}{4kT}$ we have a double pole at $-\frac{1}{2T}$, and now $\xi = 1$. As explained earlier, this is often considered the most desirable situation. For $K_i > \frac{1}{4kT}$ we obtain a pair of conjugate poles, $p_2 = \bar{p}_1$ and, since $p_1 + p_2 = 2\,\mathrm{Re}\,p_1$, these poles are on the vertical line $\mathrm{Re}\,z = -\frac{1}{2T}$. Now we have $\xi < 1$, see the figure:



Complex conjugate poles for $K_i > 1/4kT$

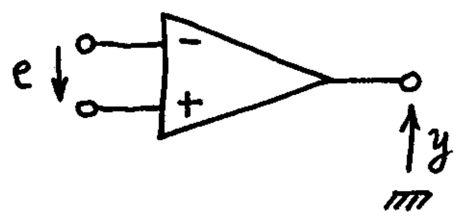
The movement of the poles can be represented in a picture called a <u>root locus diagram</u>, which looks like this



CIRCUITS WITH OP-AMPS

(intended for a study group)

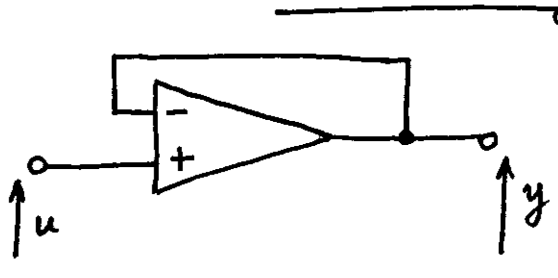
An op-amp ("operational amplifier") is a differential amplifier with very high DC-gain and very high input impedance. A good approximate model of an op-amp is a first order transfer function from the (differential) input voltage to the output voltage:



$$\hat{y}(s) = \frac{A}{1+Ts}\,\hat{e}(s),$$

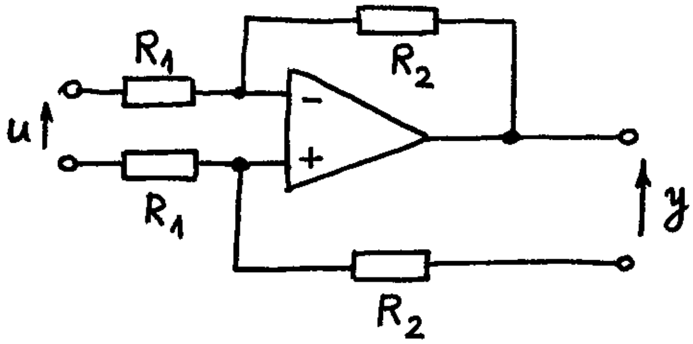
$$A > 0,\ T > 0.$$

The current flowing to the input terminals is negligible. Typical values are $A = 10^7$, $T = 1$ sec. There are also power supply terminals (usually one for a positive voltage and one for a negative voltage but none for the ground). In analog electronics we often use an even simpler model, by considering $A = \infty$. If the op-amp is part of a circuit that creates some feedback from the output to the input of the op-amp, and if this circuit is stable, then $A = \infty$ implies that $e = 0$. This is a very convenient approximation. We give some simple examples of circuits using op-amps. The voltage repeater:



Assuming $A = \infty$, we obtain $y = u$, the input impedance is ∞ and the output impedance is 0.

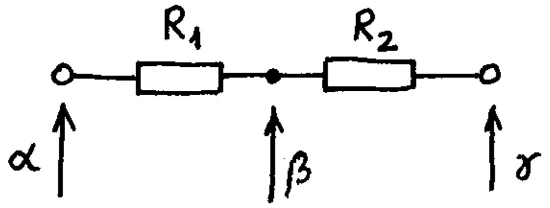
The differential amplifier :



Assuming $A = \infty$, we obtain

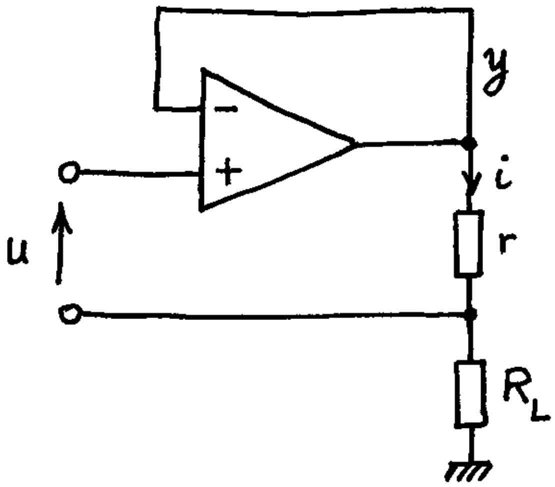
$$y = -\frac{R_2}{R_1} u ,$$

the input impedance is $2R_1$ and the output impedance is 0. To derive this, we use that in the circuit on the left,



$$\beta = \frac{R_2}{R_1+R_2}\alpha + \frac{R_1}{R_1+R_2}\gamma .$$

The voltage controlled current source :

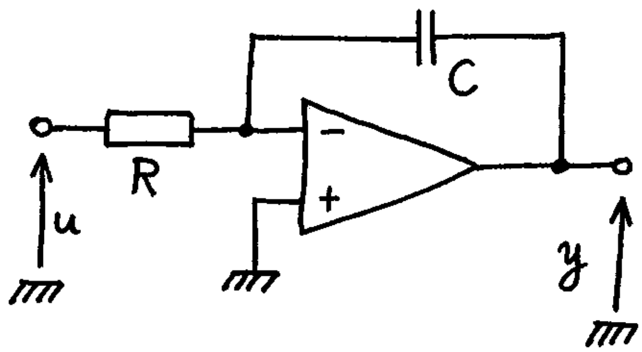


Assuming $A = \infty$, we obtain

$$u = ri .$$

The input impedance is ∞, there is no current flowing from the input terminals of this circuit, so that the current through the load resistor R_L is also i. Note that this current is independent of R_L, and it depends only on the input signal u (r is fixed, it is part of the current source). To prove this, check that the voltage on the + input of the op-amp is $y - ri + u$.

The integrator:



Assuming $A=\infty$, we obtain

$$\hat{y}(s) = \frac{-1}{RCs}\,\hat{u}(s)$$

(assuming that $y(0)=0$). In the time domain, this corresponds to $y(t) = -\int_0^t \frac{u(\sigma)}{RC}\,d\sigma$.

The input impedance is R, the output impedance is 0.

Exercise. Compute the transfer functions of the four circuits with op-amps that were presented above, using the more precise model $\hat{y}(s) = \frac{A}{1+Ts}\hat{e}(s)$ for the op-amp (as on p.8).

In each of the four cases, draw the block diagram showing the feedback. Check if these circuits are stable, or marginally stable. Show that in the limit, as $A\to\infty$, we obtain the simple input-output relations stated above. Compute the DC-gain of each circuit. When analysing the differential amplifier, you may assume that the lower terminal of the output is the ground. Give an example of a circuit with an op-amp that is not stable, not even marginally.