Kalman Observability Decomposition

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Consider a system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \mathbf{A}\boldsymbol{x}(t) + \mathbf{B}\boldsymbol{u}(t), \\ \boldsymbol{y}(t) &= \mathbf{C}\boldsymbol{x}(t) + \mathbf{D}\boldsymbol{u}(t), \end{aligned}$$

where $\boldsymbol{x}(t) \in \mathbb{C}^n$ and $\boldsymbol{y}(t) \in \mathbb{C}^p$ are the state and output vectors,[1] and $\mathbf{A} \in \mathbb{C}^{n\times n}$, $\mathbf{C} \in \mathbb{C}^{p\times n}$ are known matrices.

Definition (Unobservable subspace)**.** The unobservable subspace is the set of all unobservable initial conditions $\boldsymbol{x}(0) = \boldsymbol{x}_0 \in \mathbb{C}^n$:[2,3]

$$\mathcal{N} \triangleq \left\{ \boldsymbol{x}_0 \in \mathbb{C}^n \middle| \mathbf{C}\mathrm{e}^{\mathbf{A}t}\boldsymbol{x}_0 = \mathbf{0},\ \forall t \in [0,T] \right\}; \tag{1}$$

denote the dimensions of $\mathcal{N}$ by $\tilde{k} \triangleq \dim(\mathcal{N})$, and $k \triangleq n - \tilde{k}$.

Properties. The following properties hold for $\mathcal{N}$.

1) $\mathcal{N}$ is a linear space (linear subspace of $\mathbb{C}^n$).
2) $\mathbf{C}\boldsymbol{x}_0 = 0$ for all $\boldsymbol{x}_0 \in \mathcal{N}$.
3) $\mathcal{N}$ is invariant under $\mathbf{A}$, i.e., $\mathbf{A}\mathcal{N} = \{\mathbf{A}\boldsymbol{x}_0 | \boldsymbol{x}_0 \in \mathcal{N}\} \subseteq \mathcal{N}$.

Proof. 1) Let $\boldsymbol{x}_0, \tilde{\boldsymbol{x}}_0 \in \mathcal{N}$. Then, $\alpha\boldsymbol{x}_0 + \beta\tilde{\boldsymbol{x}}_0 \in \mathcal{N}$ for all $\alpha, \beta \in \mathbb{C} \Rightarrow \mathcal{N}$ is a linear (sub)space.

2) By taking $t = 0$ in Eq. (1), we have $\mathbf{C}\mathrm{e}^{\mathbf{A}0}\boldsymbol{x}_0 = \mathbf{C}\boldsymbol{x}_0 = \mathbf{0}$ for all $\boldsymbol{x}_0 \in \mathcal{N}$.

3) Let $\boldsymbol{x}_0 \in \mathcal{N}$. By differentiating with respect to t in Eq. (1), we have

$$\frac{d}{dt}\mathbf{C}\mathrm{e}^{\mathbf{A}t}\boldsymbol{x}_0 = \mathbf{C}\mathrm{e}^{\mathbf{A}t}\mathbf{A}\boldsymbol{x}_0 = 0, \qquad \forall t \in [0,T],$$

meaning that $\mathbf{A}\boldsymbol{x}_0 \in \mathcal{N}$. □

Theorem (Kalman observability decomposition)**.** *There exists an invertible similarity transformation $\mathbf{T} \in \mathbb{C}^{n\times n}$, such that*

$$\tilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{A}_{uo} & \mathbf{A}_{\times} \\ \mathbf{0} & \mathbf{A}_o \end{bmatrix} \tag{2a}$$

$$\tilde{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{0} & \mathbf{C}_o \end{bmatrix}, \tag{2b}$$

with $\mathbf{A}_{uo} \in \mathbb{C}^{\tilde{k}\times\tilde{k}}$, $\mathbf{A}_o \in \mathbb{C}^{k\times k}, \mathbf{C}_o \in \mathbb{C}^{p\times k}$, where $(\mathbf{A}_o, \mathbf{C}_o)$ is observable.

[1]The discussion here may be limited to real-valued $\boldsymbol{x}$ and $\boldsymbol{y}$. However, the matrices are complex-valued in general since the corresponding transfer function may include complex zeros and poles.

[2]We include the zero vector in $\mathcal{N}$ for convenience: This way, $\mathcal{N}$ is a linear subspace of $\mathbb{C}^n$, as is proved next.

[3]$\mathcal{N}$ is the subspace defined on p. 11 in the notes of week 8.

Remark. The transformation $\mathbf{T}$ of (2) transforms the unobservable subspace $\mathcal{N}$ into

$$\tilde{\mathcal{N}} = \mathbf{T}\mathcal{N} = \left\{\mathbf{T}\boldsymbol{x}_0 \middle| \boldsymbol{x}_0 \in \mathbb{C}^n, \mathbf{C}\mathrm{e}^{\mathbf{A}t}\boldsymbol{x}_0 = \mathbf{0},\ \forall t \in [0, T]\right\} = \left\{\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \mathbf{0}\end{bmatrix} \in \mathbb{C}^n \middle| \tilde{\boldsymbol{q}} \in \mathbb{C}^{\tilde{k}}\right\}.$$

To see this, note first that

$$\mathrm{e}^{\mathbf{A}t} = \sum_{\ell=0}^{\infty}\frac{(At)^\ell}{\ell!} = \sum_{\ell=0}^{\infty}\frac{(\mathbf{T}^{-1}\tilde{\mathbf{A}}\mathbf{T}t)^\ell}{\ell!} = \mathbf{T}^{-1}\sum_{\ell=0}^{\infty}\frac{(\tilde{\mathbf{A}}t)^\ell}{\ell!}\mathbf{T} = \mathbf{T}^{-1}\begin{bmatrix}\mathrm{e}^{\mathbf{A}_{uo}t} & *\\ \mathbf{0} & \mathrm{e}^{\mathbf{A}_o t}\end{bmatrix}\mathbf{T}.$$

Thus,

$$\mathbf{C}\mathrm{e}^{\mathbf{A}t}\boldsymbol{x}_0 = \tilde{\mathbf{C}}\mathbf{T}\mathbf{T}^{-1}\mathrm{e}^{\tilde{\mathbf{A}}t}\mathbf{T}\boldsymbol{x}_0 = \tilde{\mathbf{C}}\mathrm{e}^{\tilde{\mathbf{A}}t}\tilde{\boldsymbol{x}}_0 = \begin{bmatrix}\mathbf{0} & \mathbf{C}_o\end{bmatrix}\begin{bmatrix}\mathrm{e}^{\mathbf{A}_{uo}t} & *\\ \mathbf{0} & \mathrm{e}^{\mathbf{A}_o t}\end{bmatrix}\tilde{\boldsymbol{x}}_0 = \begin{bmatrix}\mathbf{0} & \mathbf{C}_o\mathrm{e}^{\mathbf{A}_o t}\end{bmatrix}\tilde{\boldsymbol{x}}_0,$$

where $\tilde{\boldsymbol{x}}_0 = \mathbf{T}\boldsymbol{x}_0$. This means that $\left\{\begin{bmatrix}\tilde{\boldsymbol{q}}\\ \mathbf{0}\end{bmatrix} \in \mathbb{C}^n \middle| \tilde{\boldsymbol{q}} \in \mathbb{C}^{\tilde{k}}\right\} \subseteq \tilde{\mathcal{N}}$; $\dim(\tilde{\mathcal{N}}) = \dim(\mathcal{N}) = \tilde{k}$ means that this inclusion holds with equality.

Proof of the theorem. For $\tilde{k} = 0$, i.e., for an observable $(\mathbf{A}, \mathbf{C})$, the claim trivially follows with $\mathbf{T} = \mathbf{I}$.

Assume now that $\tilde{k} > 0$, i.e., $\mathcal{N}$ is a k-dimensional subspace of $\mathbb{C}^n$. Let

$$\{\boldsymbol{q}_1, \boldsymbol{q}_2, \dots, \boldsymbol{q}_{\tilde{k}}\}$$

be some basis of $\mathcal{N}$. Complete this basis in $\mathbb{C}^n$ such that

$$\left\{\boldsymbol{q}_1, \boldsymbol{q}_2, \dots, \boldsymbol{q}_{\tilde{k}}, \boldsymbol{q}_{\tilde{k}+1}, \boldsymbol{q}_{\tilde{k}+2} \dots, \boldsymbol{q}_n\right\}$$

is a basis of $\mathbb{C}^n$. Denote

$$\mathbf{Q}_{uo} \triangleq \begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots & \boldsymbol{q}_{\tilde{k}}\end{bmatrix} \in \mathbb{C}^{n\times\tilde{k}}; \qquad \mathbf{Q}_o \triangleq \begin{bmatrix}\boldsymbol{q}_{\tilde{k}+1} & \boldsymbol{q}_{\tilde{k}+2} & \cdots & \boldsymbol{q}_n\end{bmatrix} \in \mathbb{C}^{n\times k};$$
$$\mathbf{Q} \triangleq \begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots & \boldsymbol{q}_{\tilde{k}} & \boldsymbol{q}_{\tilde{k}+1} & \boldsymbol{q}_{\tilde{k}+2} & \cdots & \boldsymbol{q}_n\end{bmatrix} = \begin{bmatrix}\mathbf{Q}_{uo} & \mathbf{Q}_o\end{bmatrix} \in \mathbb{C}^{n\times n}.$$

Note that $\mathbf{Q}$ is invertible since its columns $\{\boldsymbol{q}_i | i = 1, \dots, n\}$ constitute a basis of $\mathbb{C}^n$.

We first prove (2a). To that end, let $\boldsymbol{x}_0 \in \mathcal{N}$. Then, by Property 3, $\mathbf{A}\boldsymbol{x}_0 \in \mathcal{N}$. Since $\left\{q_i | i = 1, \dots, \tilde{k}\right\}$ is a basis of $\mathcal{N}$, we have

$$\mathbf{A}\boldsymbol{x}_0 = \sum_{i=1}^{\tilde{k}}\alpha_i\boldsymbol{q}_i = \begin{bmatrix}\boldsymbol{q}_1 & \boldsymbol{q}_2 & \cdots & \boldsymbol{q}_{\tilde{k}}\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\\ \vdots\\ \alpha_{\tilde{k}}\end{bmatrix} = \mathbf{Q}_{uo}\boldsymbol{a}$$

for some $\alpha_1, \dots, \alpha_{\tilde{k}} \in \mathbb{C}$, where $\boldsymbol{a} \triangleq \begin{bmatrix}\alpha_1 & \alpha_2 & \cdots & \alpha_{\tilde{k}}\end{bmatrix}^T \in \mathbb{C}^n$. Therefore, since $\boldsymbol{q}_i \in \mathcal{N}$,

$$\mathbf{A}\boldsymbol{q}_i = \mathbf{Q}_{uo}\boldsymbol{a}_i, \qquad i = 1, \dots, \tilde{k},$$

for some $\boldsymbol{a}_i \in \mathbb{C}^{\tilde{k}}$. Thus,

$$\begin{aligned}
\mathbf{A}\mathbf{Q} &= \begin{bmatrix} \mathbf{A}\boldsymbol{q}_1 & \cdots & \mathbf{A}\boldsymbol{q}_{\tilde{k}} & | & \mathbf{A}\boldsymbol{q}_{\tilde{k}+1} & \cdots & \mathbf{A}\boldsymbol{q}_n \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Q}_{uo}\boldsymbol{a}_1 & \cdots & \mathbf{Q}_{uo}\boldsymbol{a}_{\tilde{k}} & | & \mathbf{Q}_{uo}\boldsymbol{a}_{\tilde{k}+1} + \mathbf{Q}_o\boldsymbol{h}_{\tilde{k}+1} & \cdots & \mathbf{Q}_{uo}\boldsymbol{a}_n + \mathbf{Q}_o\boldsymbol{h}_n \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{Q}_{uo}\mathbf{A}_{uo} & \mathbf{Q}_{uo}\mathbf{A}_\times + \mathbf{Q}_o\mathbf{A}_o \end{bmatrix} = \begin{bmatrix} \mathbf{Q}_{uo} & \mathbf{Q}_o \end{bmatrix} \begin{bmatrix} \mathbf{A}_{uo} & \mathbf{A}_\times \\ \mathbf{0} & \mathbf{A}_o \end{bmatrix} \\
&= \mathbf{Q} \begin{bmatrix} \mathbf{A}_{uo} & \mathbf{A}_\times \\ \mathbf{0} & \mathbf{A}_o \end{bmatrix}
\end{aligned}$$

for some vectors $\boldsymbol{a}_i \in \mathbb{C}^{\tilde{k}}$ and $\boldsymbol{h}_i \in \mathbb{C}^k$, where

$$\mathbf{A}_{uo} = \begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_{\tilde{k}} \end{bmatrix}, \quad \mathbf{A}_\times = \begin{bmatrix} \boldsymbol{a}_{\tilde{k}+1} & \cdots & \boldsymbol{a}_n \end{bmatrix}, \quad \mathbf{A}_o = \begin{bmatrix} \boldsymbol{h}_{\tilde{k}+1} & \cdots & \boldsymbol{h}_n \end{bmatrix}.$$

Setting $\mathbf{T} = \mathbf{Q}^{-1}$ yields (2a).

We next prove (2b). By Property 2, since $\boldsymbol{q}_i \in \mathcal{N}$ for all $i = 1, \ldots, \tilde{k}$, $\mathbf{C}\boldsymbol{q}_i = \mathbf{0}$. Therefore,

$$\mathbf{C}\mathbf{T}^{-1} = \mathbf{C}\mathbf{Q} = \begin{bmatrix} \mathbf{C}\mathbf{Q}_{uo} & \mathbf{C}\mathbf{Q}_o \end{bmatrix} = \begin{bmatrix} \mathbf{C}\boldsymbol{q}_1 & \mathbf{C}\boldsymbol{q}_2 & \cdots & \mathbf{C}\boldsymbol{q}_{\tilde{k}} & \mathbf{C}\mathbf{Q}_o \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{C}\mathbf{Q}_o \end{bmatrix}.$$

Taking $\mathbf{C}_o \triangleq \mathbf{C}\mathbf{Q}_o$ yields (2b).

To prove that $(\mathbf{A}_o, \mathbf{C}_o)$ is observable, assume to contradict that $(\mathbf{A}_o, \mathbf{C}_o)$ is not observable. This means, in turn, that the unobservable subspace

$$\mathcal{N}_o \triangleq \left\{ \boldsymbol{p} \in \mathbb{C}^{\tilde{k}} \middle| \mathbf{C}_o \mathrm{e}^{\mathbf{A}_o t}\boldsymbol{p} = \mathbf{0}, \ \forall t \in [0, T] \right\}$$

of $(\mathbf{A}_o, \mathbf{C}_o)$ is non-trivial, i.e., that there exists $0 \neq \boldsymbol{p} \in \mathcal{N}_o$. Construct now the non-zero vector $\tilde{\boldsymbol{x}}_0 \triangleq \begin{bmatrix} \mathbf{0} \\ \boldsymbol{p} \end{bmatrix} \in \mathbb{C}^n$. This vector satisfies

$$\tilde{\mathbf{C}} \mathrm{e}^{\tilde{\mathbf{A}}t} \tilde{\boldsymbol{x}}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{C}_o \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathbf{A}_{uo}t} & * \\ \mathbf{0} & \mathrm{e}^{\mathbf{A}_o t} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \boldsymbol{p} \end{bmatrix} = 0,$$

namely, it belongs to the unobservable subspace $\tilde{N}$ of $(\tilde{\mathbf{A}}, \tilde{\mathbf{C}})$, in contrast to the result in the remark. □

Using this decomposition, we may prove the direct part of the PBH (Hautus) test on pages 11–12 in the notes of week 8.

Proof of direct of PBH test. First note that, for any invertible similarity transformation $\mathbf{T} \in \mathbb{C}^{n \times n}$, which results in $\tilde{\mathbf{A}} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$ and $\tilde{\mathbf{C}} = \mathbf{C}\mathbf{T}^{-1}$,

$$\begin{aligned}
\operatorname{rank}\left(\begin{bmatrix} \lambda\mathbf{I} - \mathbf{A} \\ \mathbf{C} \end{bmatrix} \right) &= \operatorname{rank}\left(\begin{bmatrix} \lambda\mathbf{T}^{-1}\mathbf{T} - \mathbf{T}^{-1}\tilde{\mathbf{A}}\mathbf{T} \\ \tilde{\mathbf{C}}\mathbf{T} \end{bmatrix} \right) = \operatorname{rank}\left(\begin{bmatrix} \mathbf{T}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \lambda\mathbf{I} - \tilde{\mathbf{A}} \\ \tilde{\mathbf{C}} \end{bmatrix} \mathbf{T} \right) \\
&= \operatorname{rank}\left(\begin{bmatrix} \lambda\mathbf{I} - \tilde{\mathbf{A}} \\ \tilde{\mathbf{C}} \end{bmatrix} \right).
\end{aligned}$$

Now assume that $(\mathbf{A}, \mathbf{C})$ is not observable, and choose the transformation $\mathbf{T}$ that yields the Kalman observability decomposition of (2).

Let λ_{uo} be some eigenvalue of $\mathbf{A}_{uo}$, and $\boldsymbol{v}_{uo} \neq 0$—a corresponding eigenvector. Then, the non-zero vector $\boldsymbol{x}_0 \triangleq \begin{bmatrix} \boldsymbol{v}_{uo} \\ \mathbf{0} \end{bmatrix}$ satisfies

$$\left(\lambda_{uo}\mathbf{I} - \tilde{\mathbf{A}}\right) \boldsymbol{x}_0 = \begin{bmatrix} \lambda_{uo}\mathbf{I} - \mathbf{A}_{uo} & -\mathbf{A}_{\times} \\ \mathbf{0} & \lambda_{uo}\mathbf{I} - \mathbf{A}_o \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_{uo} \\ \mathbf{0} \end{bmatrix} = \mathbf{0}$$

$$\tilde{\mathbf{C}}\boldsymbol{x}_0 = \begin{bmatrix} \mathbf{0} & \mathbf{C}_o \end{bmatrix} \begin{bmatrix} \boldsymbol{v}_{uo} \\ \mathbf{0} \end{bmatrix} = \mathbf{0},$$

or equivalently

$$\begin{bmatrix} \lambda\mathbf{I} - \tilde{\mathbf{A}} \\ \tilde{\mathbf{C}} \end{bmatrix} \boldsymbol{x}_0 = 0.$$

Thus,

$$\text{rank}\left(\begin{bmatrix} \lambda\mathbf{I} - \tilde{\mathbf{A}} \\ \tilde{\mathbf{C}} \end{bmatrix}\right) < n.$$

$\square$