Introduction to control theory

WEEK 1

Stability of linear systems

We assume that the students are familiar with the basic concepts of linear systems analysis.

A linear time-invariant (LTI) system Σ can be modeled by state-space equations

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) + Du(t) \end{cases}$$

$x(t) \in \mathbb{C}^n$ (the state)
$u(t) \in \mathbb{C}^m$ (the input)
$y(t) \in \mathbb{C}^p$ (the output)

where A, B, C, D are matrices of suitable dimensions. The system is called SISO (single-input single-output) if $m = p = 1$, MISO (multiple-input single-output) if $m > 1$, $p = 1$, etc. The matrices A, B, C may be infinite (for example, in systems containing delay lines, or systems modeling heat propagation, or systems describing acoustic or electromagnetic waves, where $n = \infty$). The system is called finite-dimensional if n is finite, so that the matrices A, B, C are finite. n is called the order of Σ. The transfer function of Σ is

$$G(s) = C(sI - A)^{-1}B + D \quad (\text{for } s \notin \sigma(A)).$$

This is a proper rational function. ("Rational" means that each entry in the matrix G is the ratio of two polynomials, and "proper" means that G tends to a finite value at infinity: $G(\infty) = D$. G is called strictly proper if $G(\infty) = 0$ (i.e., $D = 0$).

We have
$$(sI-A)^{-1} = \frac{1}{p(s)}(sI-A)^{+}$$
where $(sI-A)^{+}$ is the <u>adjugate matrix</u> of $sI-A$ (each entry is $\pm$ a determinant of order $n-1$), and $p(s) = \det(sI-A)$ is the <u>characteristic polynomial</u> of A. Note that $(sI-A)^{+}$ is a matrix-valued polynomial of order $n-1$. Clearly, the poles of $(sI-A)^{-1}$ are the zeros of $p(s)$, which are exactly the eigenvalues of A, denoted $\sigma(A)$. Hence,

$$\boxed{\text{the poles of } G \text{ are a subset of } \sigma(A).}$$

The system Σ is called <u>minimal</u> if no other system of lower order can be found that has the same tranfer function. If Σ is minimal (this is the case for most systems), then the set of the poles of G is $\sigma(A)$.

The system Σ is called <u>stable</u> if $e^{At} \to 0$ (as $t \to \infty$). This means that if $u = 0$, then starting from an arbitrary initial state $x(0)$, the state trajectory of the system $x(t) = e^{At}x(0)$ tends to zero. Recall that if we have an arbitrary (but Laplace transformable) input function acting on Σ, then the Laplace transform of the output function y is given by

$$\hat{y}(s) = \underbrace{G(s)\,\hat{u}(s)}_{\text{the part of the output due to the input}} + \underbrace{C(sI-A)^{-1}x(0)}_{\text{the part of the output due to the initial state, in time } Ce^{At}x(0)}.$$

We say that u has <u>finite energy</u> if $\int_0^\infty |u(t)|^2 dt < \infty$. This integral is then called the <u>energy</u> of u. Stable systems have the following important property:

$$\boxed{\text{if } \Sigma \text{ is stable and } u \text{ has finite energy, then so does } y.}$$

Stability is a desirable property, and we try to design control systems such that they are stable.

How to check that a system is stable? First of all, we have the following fact, that follows from the structure of e^{At} (which we assume to be known):

$$\boxed{\Sigma \text{ is stable if and only if } \sigma(A) \subset \mathbb{C}_- .}$$

Here, $\mathbb{C}_-$ denotes the open left half-plane:

$$\mathbb{C}_- = \{ s \in \mathbb{C} \mid \operatorname{Re} s < 0 \}.$$

Later, we shall also need a notation for the open right half-plane and for the imaginary axis:

$$\mathbb{C}_+ = \{ s \in \mathbb{C} \mid \operatorname{Re} s > 0 \}, \quad i\mathbb{R} = \{ i\omega \mid \omega \in \mathbb{R} \}.$$

Clearly $\mathbb{C} = \mathbb{C}_- \cup i\mathbb{R} \cup \mathbb{C}_+$. We mention that an LTI system is called <u>marginally stable</u> if $\sigma(A) \subset \mathbb{C}_- \cup i\mathbb{R}$.

The eigenvalues of A can be computed in various software packages, for example MATLAB. Nevertheless, it is useful to have simple criteria to check if $\sigma(A) \subset \mathbb{C}_-$. Recall that $\sigma(A)$ consists of the zeros of the characteristic polynomial

$$p(s) = s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} \ldots + a_1 s + a_0 .$$

If the matrix A is <u>real</u> (i.e., $A \in \mathbb{R}^{n \times n}$) then the coefficients a_j are real. We only consider this case. (A remark on terminology: since the stability of Σ depends only on the matrix A, we also say "A is stable" instead of "Σ is stable".) We have a very simple necessary condition:

$$\boxed{\text{If } A \text{ is real and stable, then } a_j > 0 \quad (0 \leq j \leq n-1).}$$

If A is of order 1 or 2, then this condition is also sufficient. If $n \geq 3$, then it is not enough to check $a_j > 0$. The brute-force approach would be to compute the zeros of p. This is in principle possible using formulas for $n \leq 4$. For $n \geq 5$ there are no formulas giving the zeros of a polynomial, and only numerical approximations can be used.

Luckily, Routh and Hurwitz worked out algorithms (at the beginning of the $\overline{\text{XX}}$ century) that are simple and can determine the stability of A without computing its eigenvalues. We present here the <u>Routh test</u>:

Construct the table

1	a_{n-2}	a_{n-4}	a_{n-6} ...
a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7} ...
b_1	b_2	b_3	b_4 ...
c_1	c_2	c_3	c_4 ...
$\vdots$	$\vdots$	$\vdots$	$\vdots$

This is called the "Routh table". Only construct this table if you have checked that $a_j > 0$. Otherwise, it is pointless, if one coefficent is ≤ 0 then A is unstable.

which has in total $n+1$ rows.

The row b_j is constructed from the preceding two rows using the following formulas:

$$b_1 = \frac{-1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-2} \\ a_{n-1} & a_{n-3} \end{bmatrix}, \quad b_2 = \frac{-1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-4} \\ a_{n-1} & a_{n-5} \end{bmatrix},$$

$$b_3 = \frac{-1}{a_{n-1}} \det \begin{bmatrix} 1 & a_{n-6} \\ a_{n-1} & a_{n-7} \end{bmatrix}, \quad \ldots$$

Note that there will be at most $n/2$ non-zero terms in the row $b_1, b_2, b_3, \ldots$. The row $c_1, c_2, c_3, \ldots$ is computed from the previous two rows using exactly the same formulas. Thus, for example,

$$c_1 = \frac{-1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-3} \\ b_1 & b_2 \end{bmatrix}, \quad c_2 = \frac{-1}{b_1} \det \begin{bmatrix} a_{n-1} & a_{n-5} \\ b_1 & b_3 \end{bmatrix}.$$

We carry on in this way, computing new rows, which get shorter and shorter. The last row (the $(n+1)$-th row) contains only one element that is possibly non-zero (the first element). For example, if $n=3$, so that $p(s) = s^3 + a_2 s^2 + a_1 s + a_0$, then the Routh table looks like this:

1	a_1
a_2	a_0
b_1	
c_1	

where

$$b_1 = \frac{-1}{a_2}(a_0 - a_1 a_2),$$

$$c_1 = \frac{-1}{b_1}(-a_0 b_1) = a_0 .$$

> A is stable if and only if all the numbers in the first column of the Routh table are >0.

Thus, for example, if $n=3$, then we obtain that A is stable iff $a_2>0$, $a_0>0$ and $a_1 a_2 > a_0$. (In particular, this implies that also $a_1>0$.)

If A is not stable, then the Routh table gives us some information in addition to instability:

> If A is unstable and all the numbers in the first column of the Routh table are non-zero, then the number of unstable eigenvalues of A (in $\sigma(A) \cap \mathbb{C}_+$) is equal to the number of sign changes in the first column of the Routh table.

Often a system is not represented in state space (using four matrices, as on p. 1) but only by a transfer function. If the system Σ is finite-dimensional, then its transfer function is proper rational, as explained on p. 1. If Σ is infinite-dimensional, then its transfer function may be irrational. The simplest example of an irrational transfer function is

$$G(s) = e^{-\tau s} \qquad (\text{where } \tau > 0)$$

which corresponds to a delay line with delay time τ.

A *transfer function* G is called *stable* if it is bounded and analytic on $\mathbb{C}_+$.

> If G is rational, then G is stable if and only if it is proper and all its poles are in $\mathbb{C}_-$.

This is not difficult to verify – try it on your own! (Note that if G has a pole on $i\mathbb{R}$, then $|G(s)|$ tends to ∞ if s tends to this pole from the right, hence G is not stable in this case.)

> If Σ is a stable finite-dimensional LTI system, then its transfer function is stable.

This follows from the previous proposition and the first proposition on p.2 (the poles of G are in $\sigma(A)$).

The converse of the above proposition is not true. (For example, if A is unstable and $C=0$ (or $B=0$) then $G(s)=D$, so that G is stable.) However, the converse is true for minimal systems.

Examples of stable transfer functions:

$$\frac{s-1}{s+1}, \quad \frac{s}{s^2+5s+7}, \quad e^{-3s}, \quad \frac{2s-7}{s^3+3s^2+2s+5},$$

$$\frac{(s-7)(s+3)}{(s+1)(s-7)}, \quad \frac{1}{2-e^{-2s}}.$$

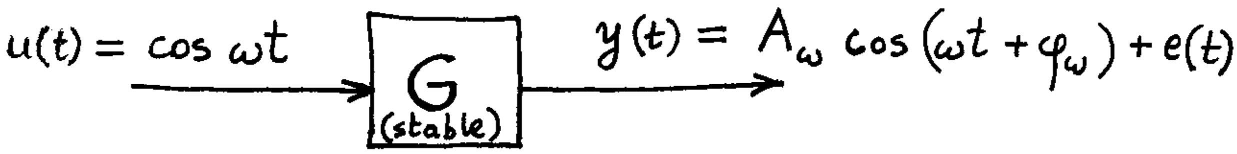
Examples of unstable transfer functions:

$$s, \quad \frac{2s}{s-7}, \quad \frac{s+3}{s(s+7)}, \quad e^{2s}, \quad \frac{s^2+2s+2}{s+5}, \quad \frac{2s}{s^2+9}.$$

Recall the concept of energy of a signal (defined for $t \geq 0$) from p.2. The reason for defining stable transfer functions the way we did on p.6 is the following theorem, whose proof is not elementary:

> If G is a stable transfer function, u is a signal with finite energy and y is signal defined via its Laplace transform by $\hat{y}(s) = G(s)\hat{u}(s)$, then y has finite energy. Conversely, if G is a transfer function with the above property (i.e., if u has finite energy and $\hat{y} = G\hat{u}$, then also y has finite energy) then G is stable.

We recall from linear systems analysis the concepts of gain and phase shift: Assume that Σ is a stable system with transfer function G. We apply a sinusoidal input to the system:



If the input is $u(t) = \cos \omega t$ (where $\omega \geq 0$), then the corresponding output is

$$y(t) = A_\omega \cos(\omega t + \varphi_\omega) + e(t)$$

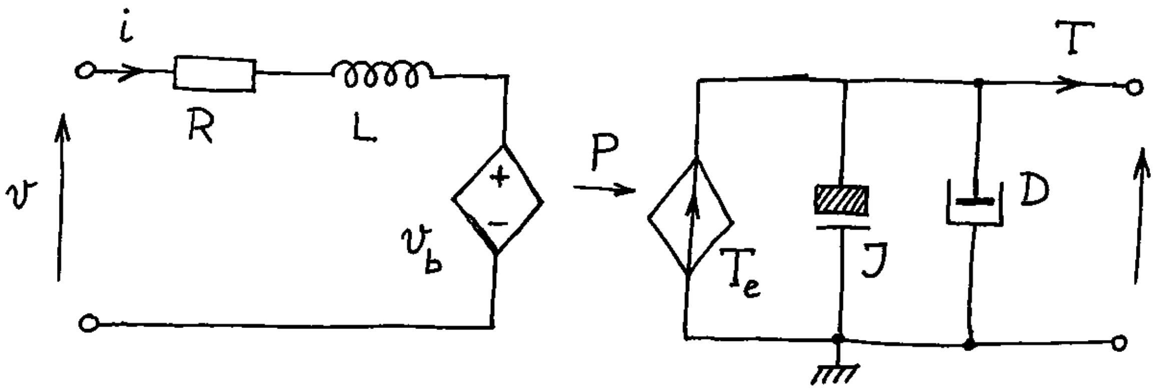
where A_ω is the <u>gain of G at the frequency ω</u>, φ_ω is the <u>phase shift of G at the frequency ω</u>, and e is the transient response of the system, which tends to zero as $t \to \infty$. We have

$$\boxed{A_\omega = |G(i\omega)|, \quad \varphi_\omega = \arg G(i\omega).}$$

$$\boxed{\text{DC-gain} = G(0).}$$

An example : the DC motor (intended for a study group)

We consider the model of a classical DC motor with either permanent magnet or constant current excitation in the stator. Denoting the rotor inductance by L, the rotor resistence by R and the motor constant by k, we have the following electro-mechanical circuit representation of the motor:



Note that on the left we have an electric circuit while on the right we have a (rotational) mechanical circuit. In the mechanical circuit the role of the current is taken by the torque, while the role of the voltage is taken by the angular speed Ω. The controlled voltage source on the left generates the back electromagnetic force $v_b = k\Omega$. The power absorbed by this element is $P = v_b i = k\Omega i$. The controlled torque source on the right (analogous to a controlled current source) generates the electric torque $T_e = ki$. The power output of this element is $P = T_e\Omega = ki\Omega$. Thus, all the power absorbed by the controlled voltage source comes out of the controlled torque source. J is the moment of inertia of the rotor and it is analogous to a grounded capacitor. D is the friction coefficient of the motor and the rotational damper with friction coefficient D is analogous to a resistor.

The equation of the electric circuit is:

$$v = Ri + L\dot{i} + v_b \,. \qquad \left(\begin{array}{c}v \text{ is the ex-}\\ \text{ternal voltage}\end{array}\right)$$

The equation of the mechanical circuit (derived from Newton's law) is

$$J\dot{\Omega} = T_e - T - \underbrace{D\Omega}_{\text{friction torque}} \,. \qquad \left(\begin{array}{c}T \text{ is the}\\ \text{load torque}\end{array}\right)$$

We choose i and Ω as state variables and v, T as input variables. Then

$$\underbrace{\frac{d}{dt}\begin{bmatrix} i \\ \Omega \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} -\frac{R}{L} & -\frac{k}{L} \\ \frac{k}{J} & -\frac{D}{J} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} i \\ \Omega \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} \frac{1}{L} & 0 \\ 0 & -\frac{1}{J} \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} v \\ T \end{bmatrix}}_{u}$$

We choose the output variables i and Ω, so that $y = x$. This means $C = I$ (the 2×2 identity matrix) and $D = 0$.

The characteristic polynomial of A is

$$p(s) = s^2 + \underbrace{\left(\frac{R}{L} + \frac{D}{J}\right)}_{a_1} s + \underbrace{\left(\frac{R}{L}\cdot\frac{D}{J} + \frac{k}{J}\cdot\frac{k}{L}\right)}_{a_0} .$$

We always have $L > 0$, $J > 0$. For very good motors, R and D are very small. If, in the mathematical model, we take $R=0$ and $D=0$, then the model is unstable (actually, it is marginally stable, as it has imaginary eigenvalues at $\pm ik/\sqrt{JL}$). The intuitive meaning of this instability is that if $v=0$ and $T=0$, and the initial speed is not zero, then the motor will spin forever, its speed and current will oscillate.

If, in the model we have at least one of R, D strictly positive, then $a_1 > 0$ (we always have $a_0 > 0$) and hence the model is stable (the friction and /or the heat dissipated in the resistor will slow down the motor and its speed tends to zero).

Exercise 1. The energy stored in the motor is

$$E = \underbrace{\frac{1}{2} J \Omega^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2} L i^2}_{\text{magnetic energy}}.$$

Show that

$$\boxed{\dot{E} \leq u \cdot y} = v \cdot i + T \cdot \Omega .$$

A system satisfying $\dot{E} \leq u \cdot y$ is called passive.

Exercise 2. Compute the transfer function of the motor. If we want to have the angular position of the rotor as our output, how do we have to change the model? Compute the transfer function from input voltage v to angular position θ.

Good references for this course:

[1] M. Sidi, U. Shaked : Introduction to Control Theory (in Hebrew), "Safrut zola" (TAU press), 2002.

[2] J. D'Azzo, C. Houpis : Linear Control System Analysis and Design, McGraw-Hill, New York, 1988 (third edition).

[3] K. Dutton, S. Thompson, B. Barraclough : The Art of Control Engineering, Addison-Wesley, Harlow (UK), 1997.

[4] J. van de Vegte : Feedback Control Systems, Prentice-Hall, Englewood Cliffs, 1994.

[5] J. M. Maciejowski : Multivariable Feedback Design, Addison-Wesley, Wokingham, 1989.

[6] G. Franklin, J. Powell, A. Emami-Naeini : Feedback Control of Dynamic Systems, Addison-Wesley, Wokingham, 1994.