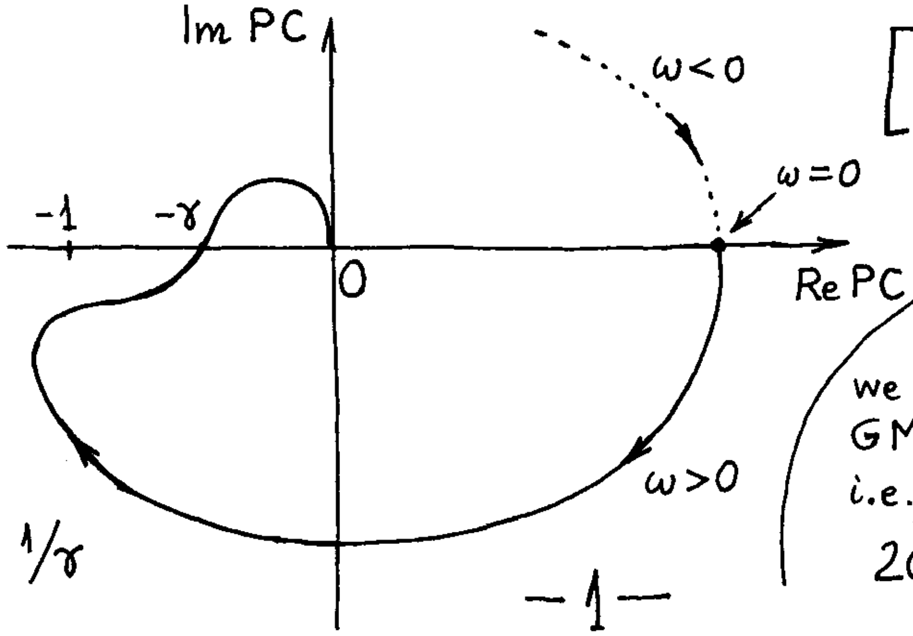
Introduction to control theory

WEEK 7

Gain margin and phase margin

The behaviour of a system may change over time (or it may be uncertain from the beginning). Thus, from the beginnings of control theory engineers tried to find numerical measures for how far a stable closed-loop system is from instability (in other words, how robust is the stability). The following concepts are classical robustness measures:

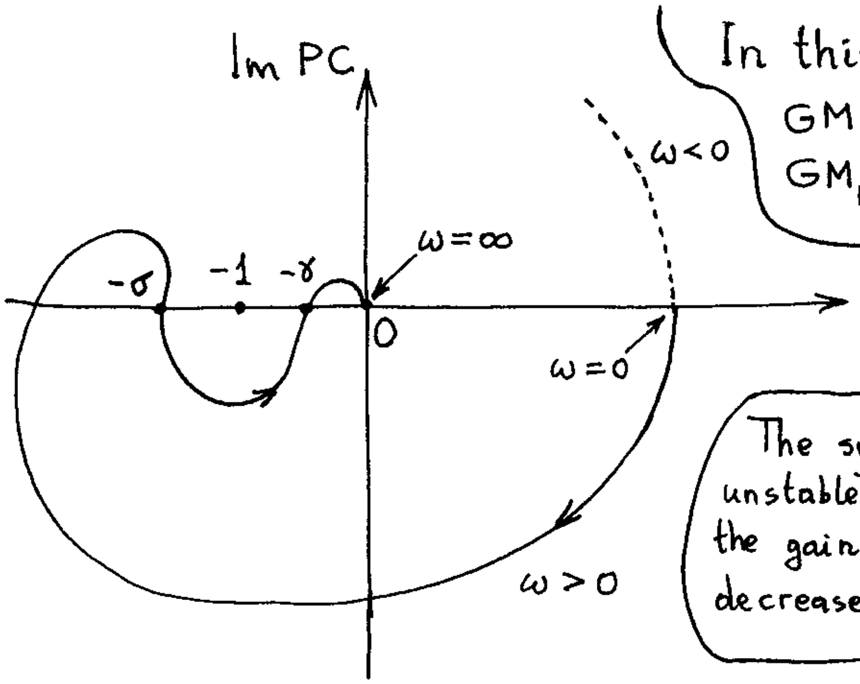
- The <u>gain margin</u> GM = the smallest $k>1$ such that $(1+kPC)^{-1}$ is unstable. If there is no such number, we put $GM=\infty$.
- The <u>reverse gain margin</u> GM_R = the smallest $\mu>1$ such that $\left(1+\frac{1}{\mu}PC\right)^{-1}$ is unstable. If there is no such μ, we put $GM_R=\infty$.



[Nyquist plot]

$GM = 1/\gamma$

Usually, we express GM in dB, i.e., we use $20\log_{10} GM$.



In this figure, $GM = 1/\gamma$, $GM_R = \sigma$.

The system becomes unstable if we increase the gain by GM, or decrease it by GM_R.

A <u>crossover frequency</u> is a number $\omega_c > 0$ such that $|P(i\omega_c)\,C(i\omega_c)| = 1$. This equation has usually one solution, but it may happen that it has no solution or several solutions.

If ω_c is unique, then the <u>phase margin</u> $\varphi_m = |\,180^\circ + \arg P(i\omega_c)C(i\omega_c)|$.

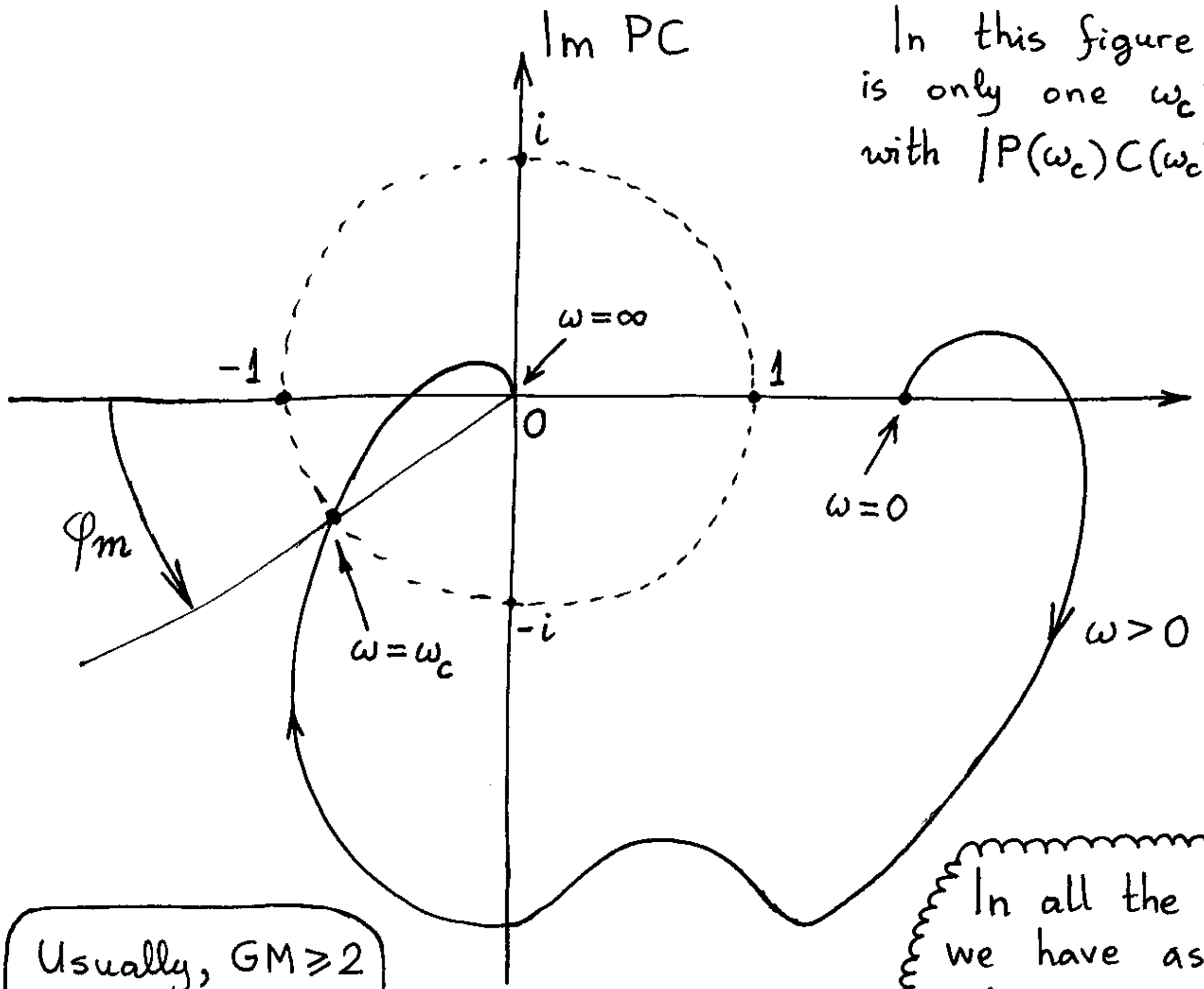
Here we consider that $-360^\circ < \arg z \le 0^\circ$.

If there are several crossover frequencies, $\omega_c^1, \dots \omega_c^h$, then we take

$$\boxed{\varphi_m = \min_{1\le j\le h} |\,180^\circ + \arg P(i\omega_c^j)C(i\omega_c^j)|.}$$

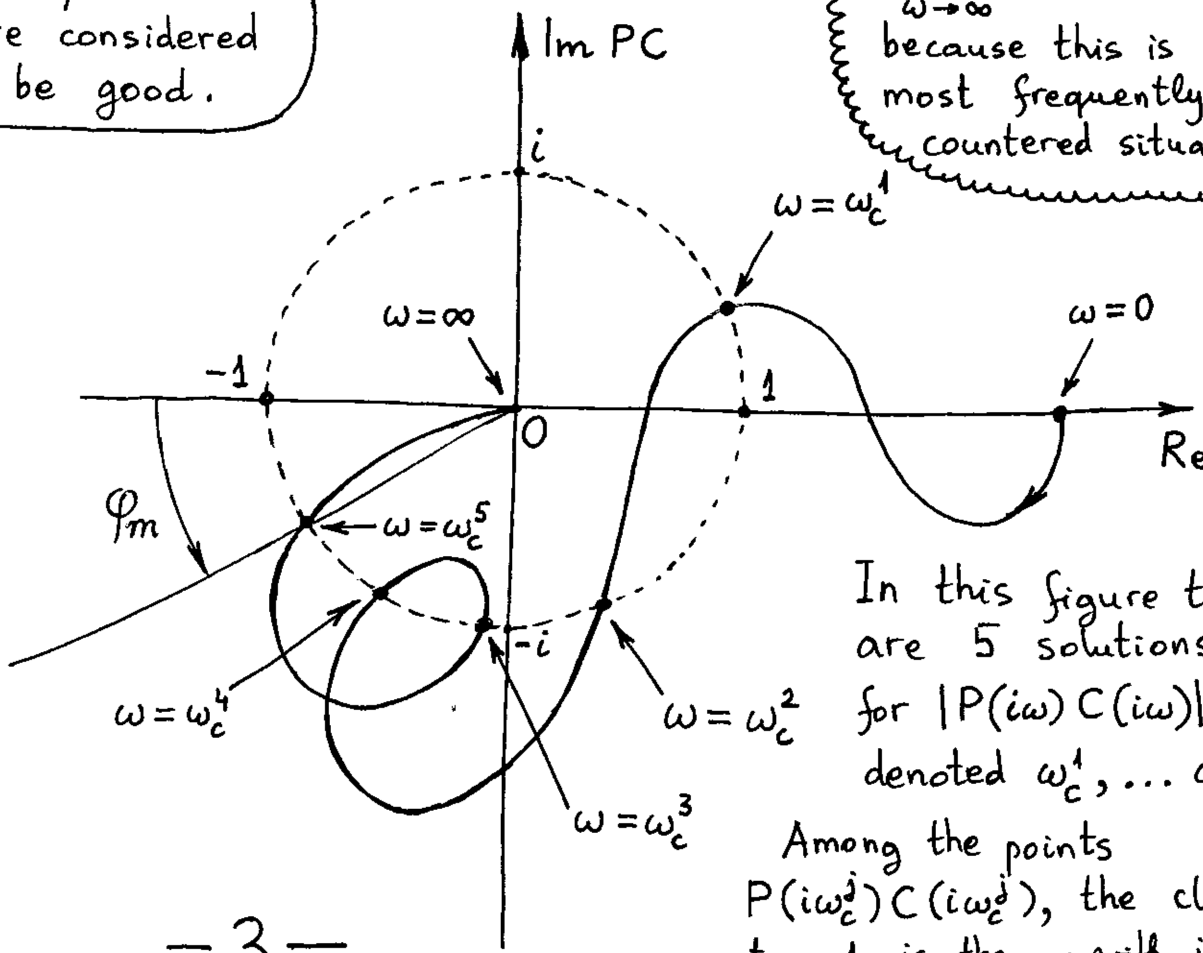
If there is no crossover frequency, then φ_m is not defined (this happens if $|P(i\omega)C(i\omega)| < 1$ for all $\omega > 0$).

In this figure there is only one $\omega_c > 0$ with $|P(\omega_c)C(\omega_c)| = 1$.



Usually, $GM \geqslant 2$ and $\varphi_m \geqslant 60°$ are considered to be good.

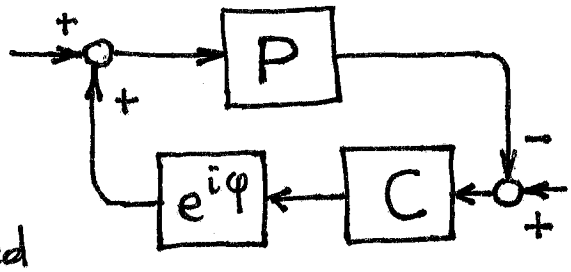
In all the figures we have assumed $\lim_{\omega \to \infty} P(i\omega)C(i\omega) = 0$, because this is the most frequently encountered situation.



In this figure there are 5 solutions for $|P(i\omega)C(i\omega)| = 1$, denoted $\omega_c^1, \dots \omega_c^5$.

Among the points $P(i\omega_c^j)C(i\omega_c^j)$, the closest to -1 is the one with $j = 5$.

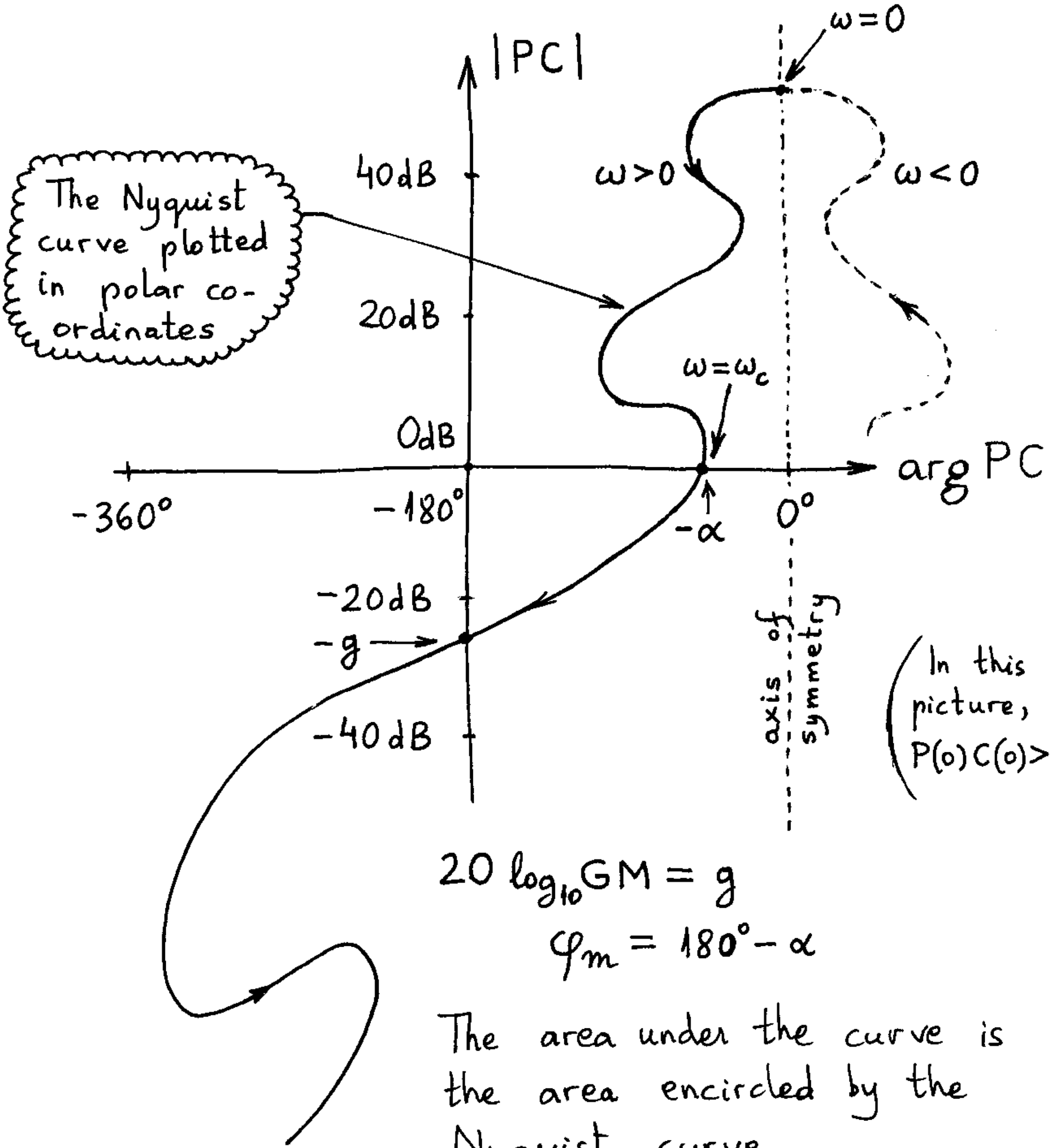
The phase margin can be defined also in a different way, which resembles the definition of the gain margin: φ_m = the smallest $|\varphi|$ such that $(1 + e^{i\varphi} PC)^{-1}$ is unstable. (If there is no such real φ, then φ_m is not defined.) This definition corresponds to the block diagram shown here. The feedback connection of P and C (with $\varphi = 0$) is assumed to be stable. The inclusion of the block with gain $e^{i\varphi}$ will rotate the Nyquist plot of the loop gain by the angle $\varphi \in \mathbb{R}$. It is not difficult to understand that this definition of φ_m is equivalent to the (classical) one given on p. 2.



A disturbing aspect of the above alternative definition of φ_m is that we cannot physically realize a block with gain $e^{i\varphi}$, since in physical reality all signals are real. This is only mathematics.

When discussing gain margin, reversed gain margin, phase margin and also in other situations, we often need to evaluate the winding number of the Nyquist plot of a function $k \cdot G$ around -1, denoted $n(k \cdot G, -1)$ (where k is a complex constant). An easy reasoning shows that this is the same as $n(G, -1/k)$, the winding number of the Nyquist plot of G around $-1/k$.

The <u>Nichols chart</u> is sometimes used to represent the Nyquist curve $P(i\omega)C(i\omega)$ in polar coordinates: the angle on the horizontal axis and the gain, expressed on a logarithmic scale (i.e., in dB) on the vertical axis. The critical point -1 corresponds in this chart to the point $(-180°, 0\,dB)$.



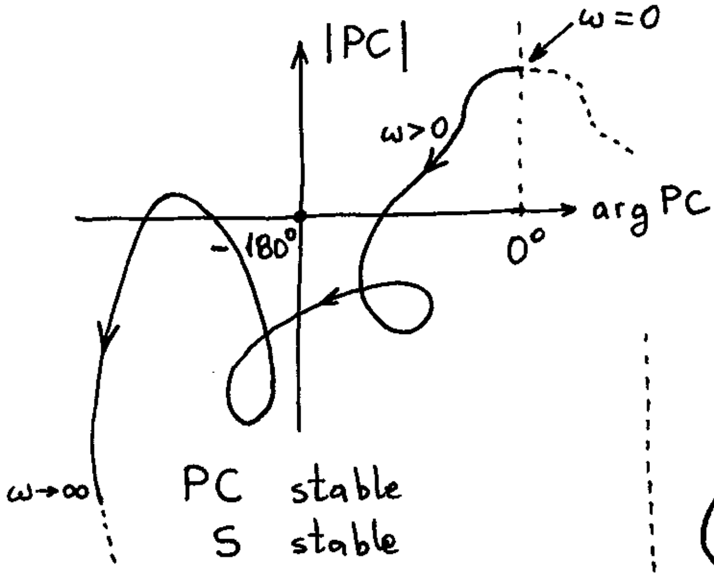
(In this picture, $P(0)C(0)>0$)

$$20\log_{10} GM = g$$

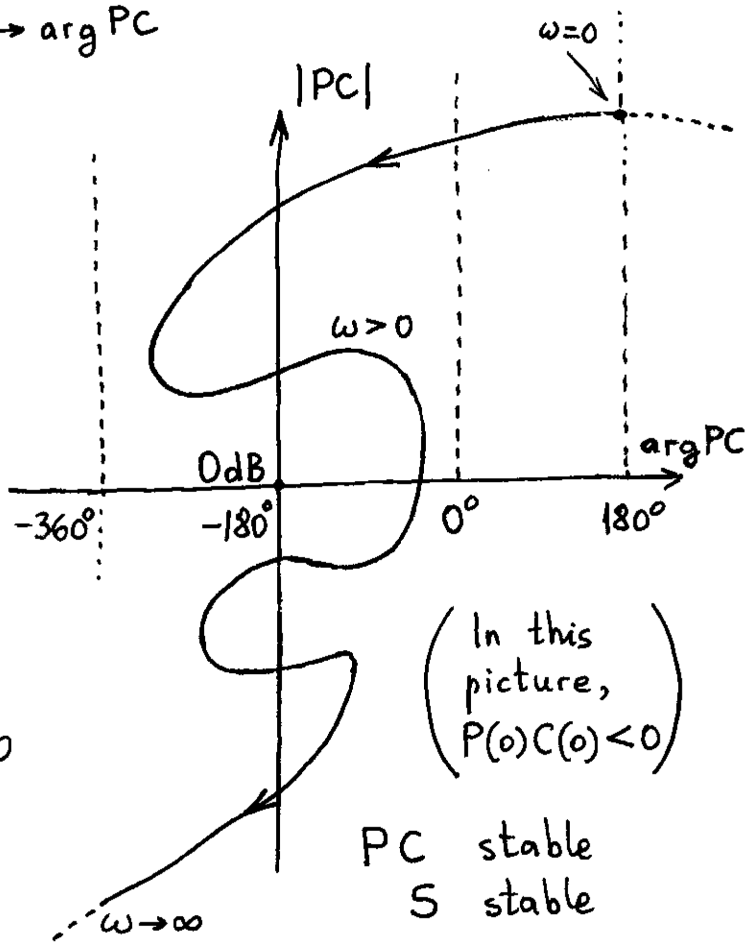
$$\varphi_m = 180° - \alpha$$

The area under the curve is the area encircled by the Nyquist curve.

Thus, if PC is stable, then the Nyquist criterion states that $S = (1+PC)^{-1}$ is stable iff the point $(-180°, 0\,\text{dB})$ is above the curve in the Nichols chart. Examples:

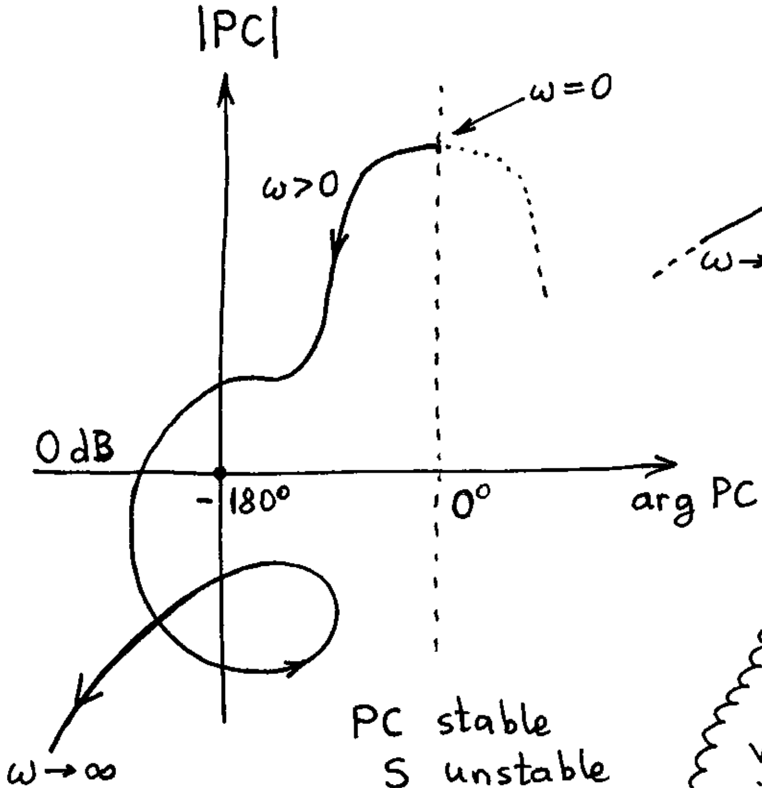


PC stable
S stable



(In this picture, $P(0)C(0)<0$)

PC stable
S stable

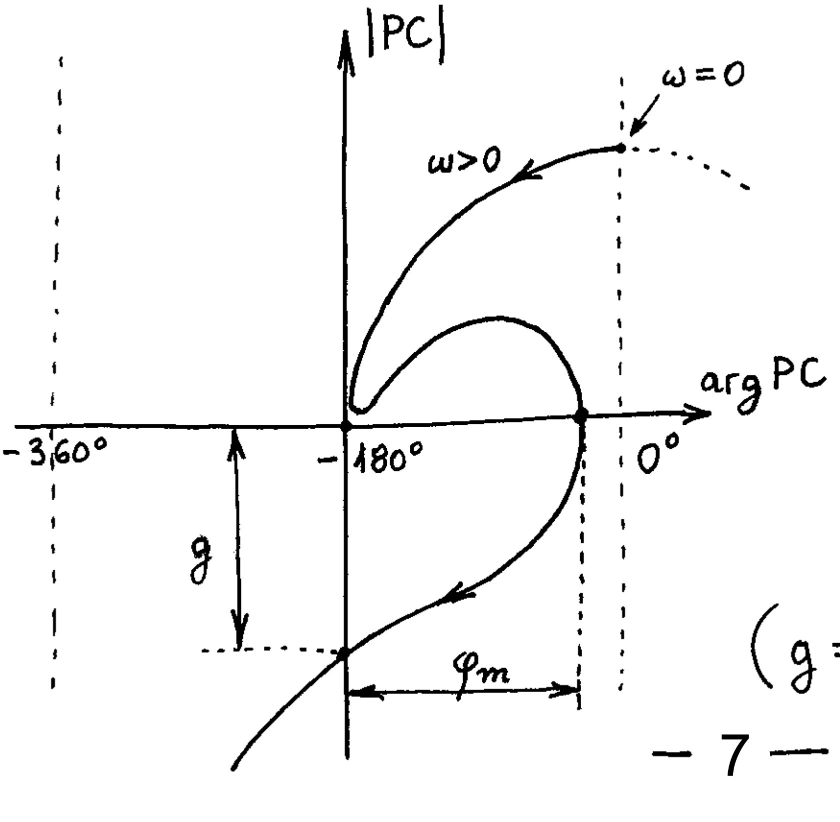


PC stable
S unstable

Like in the Nyquist plot, we can count the winding number of the Nyquist curve around a point by drawing a straight line vertically up and then counting crossings from left to right.

From the figure on p.4 (and from logical thinking) we can understand that the <u>phase margin</u> is the distance from the Nyquist curve to the critical point along the horizontal line where the gain is 0dB. Similarly, the <u>gain margin</u> (in dB) is the distance from the Nyquist curve (in the Nichols chart) to the critical point along the vertical line where the angle is $-180°$.

The gain and phase margin cannot be entirely relied upon as measures of robustness. Indeed, <u>the Nyquist plot may pass very closely by the critical point -1, but it may still have a good gain and phase margin,</u> as the following plot shows:

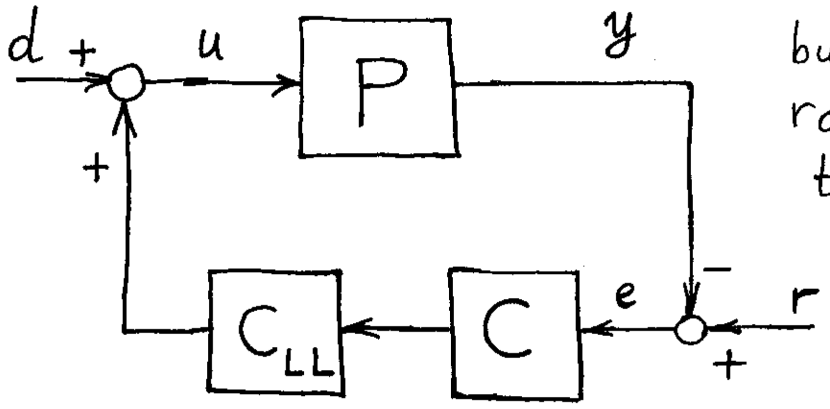


Such a transfer function PC can be constructed, but such a situation rarely occurs.

$(g = 20\log_{10} GM)$

Lead-lag compensators

Suppose that the control system formed by P and C has satisfactory performance at low frequency, but not enough robustness due to small φ_m.



Sometimes, the performance or the robustness of a feedback system formed by P and C can be improved by introducing C_{LL} into the loop,

$$C_{LL}(s) = \frac{p}{z} \cdot \frac{s-z}{s-p},$$

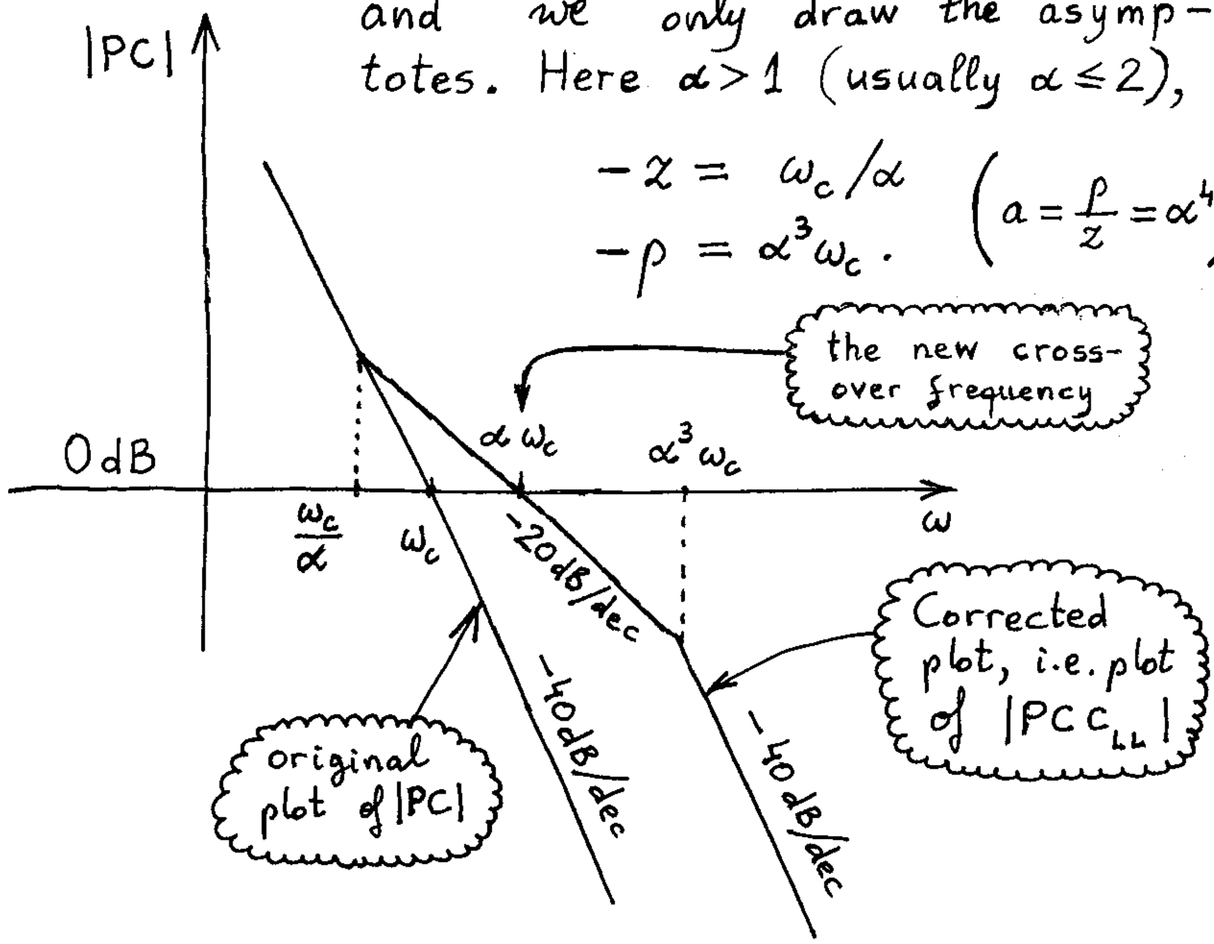
$$p < z < 0.$$

This can be understood by:

- root locus: the asymptotes are shifted to the left by $(z-p)/(m-w)$.
- we can cancel a pole of PC by z and thus it is "moved" to p.
- C_{LL} can help so that at $\omega = \omega_c$, the angle of PC is larger $\Rightarrow$ better φ_m (draw a Bode plot to see this).

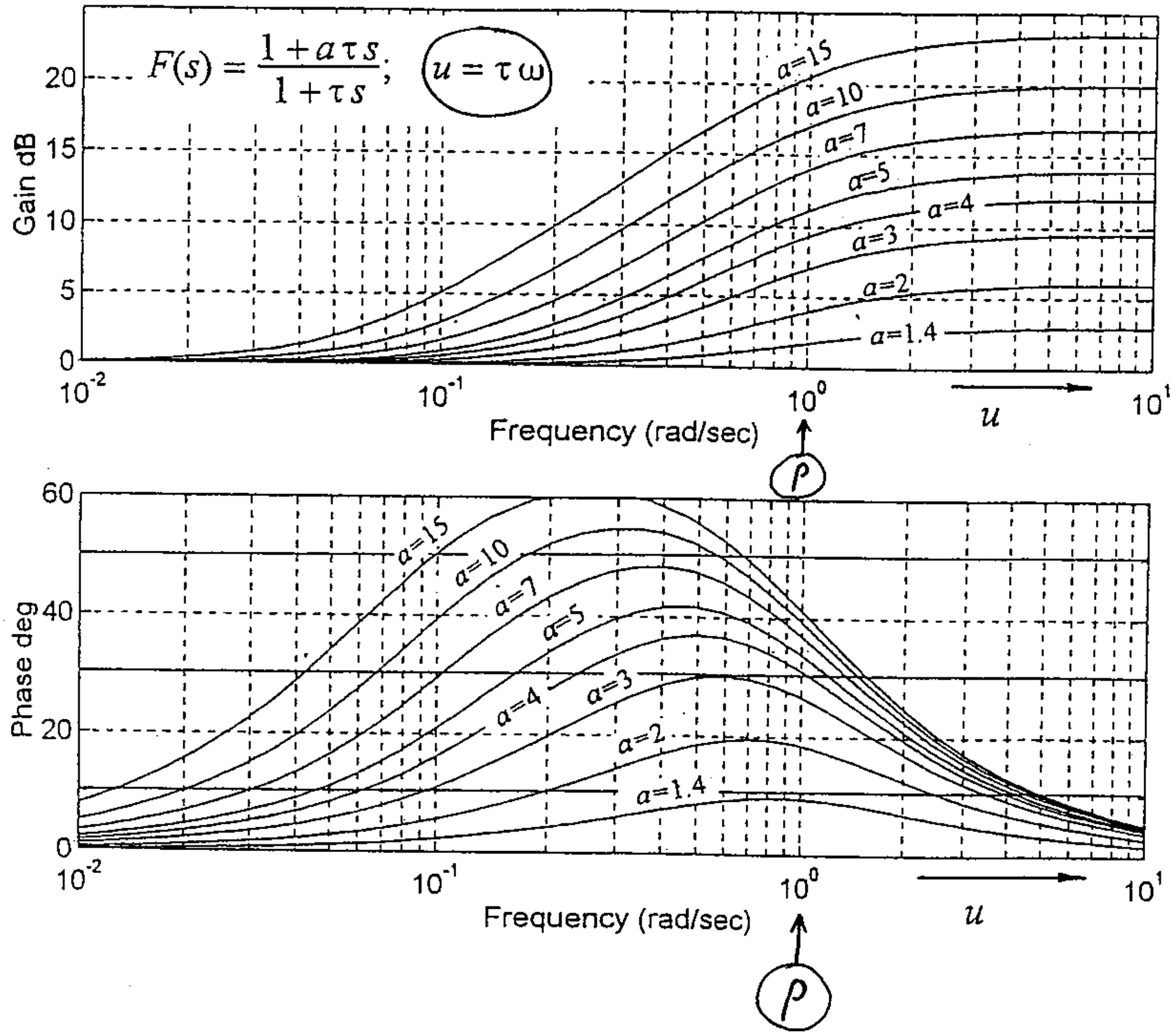
A typical situation in which C_{LL} is needed is if the phase margin φ_m is too small (or even non-existent, i.e., the closed-loop system is unstable) and the slope at ω_c is approximately -40 dB/dec (which, for a minimum phase system, corresponds to a phase of approximately $-180°$, hence the small phase margin). We draw only the area around the cross-over frequency ω_c from the Bode plot of PC, and we only draw the asymptotes. Here $\alpha > 1$ (usually $\alpha \leq 2$),

$$-z = \omega_c/\alpha \qquad \left(a = \frac{p}{z} = \alpha^4\right)$$
$$-p = \alpha^3 \omega_c .$$

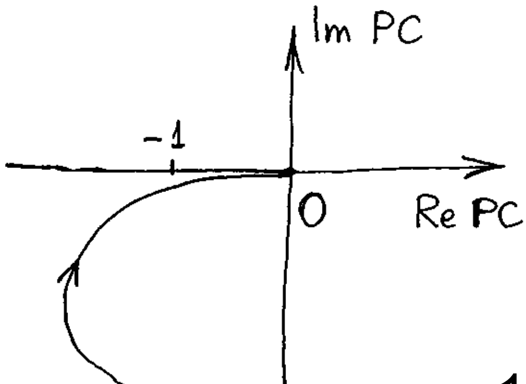


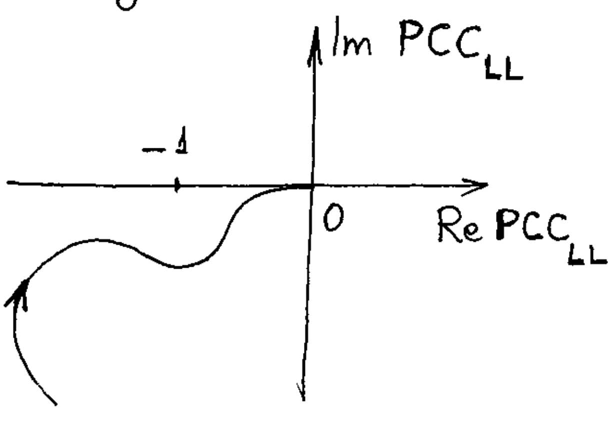
The new crossover frequency (after correction) is $\alpha\omega_c$. This is in the middle between z and p (on a logarithmic scale), thus creating the best phase improvement that C_{LL} is capable of (about $62°$ added to φ_m for $\alpha = 2$).

This can be understood from the Bode plots of C_{LL} shown below (here, $a = p/z$):



The effect of C_{LL} can be understood in terms of Nyquist plots as creating a "detour" away from the point -1, by adding to the phase angle around ω_c





The plots on p.9 are not needed in the design of C_{LL}. For the design, we only need to know how the maximal angle of C_{LL}, φ_{LL} (which occurs at the frequency $\alpha\omega_c = \sqrt{zp}$, the new crossover frequency) depends on α. We compute it:

$$C_{LL}(i\alpha\omega_c) = \frac{p}{z}\cdot\frac{i\alpha\omega_c - z}{i\alpha\omega_c - p}$$

$$= \alpha^4\,\frac{i\alpha\omega_c + \omega_c/\alpha}{i\alpha\omega_c + \omega_c\cdot\alpha^3},$$

so that its angle is

$$\varphi_{LL} = \arg C_{LL}(i\alpha\omega_c) = \arg\left[\frac{1}{\alpha} + i\alpha\right] - \arg\left[\alpha^3 + i\alpha\right]$$

$$= \arctan\alpha^2 - \arctan\frac{1}{\alpha^2}$$

$$= \arctan\alpha^2 - \left[90^\circ - \arctan\alpha^2\right],$$

$$\boxed{\varphi_{LL} = 2\arctan\alpha^2 - 90^\circ.} \qquad (*)$$

φ_{LL} is the amount by which the phase margin improves after including C_{LL} in the controller. For example, if we originally have $\varphi_m = 4^\circ$ (unacceptably small) and we take $a = \alpha^4 = 3$, then $\arctan\sqrt{3} = 60^\circ$ gives $\varphi_{LL} = 30^\circ$, so that after including C_{LL}, the phase margin improves to $\varphi_m + \varphi_{LL} = 34^\circ$. We mention that, from the formula (*) above, $\alpha^2 = \tan\dfrac{\varphi_{LL} + 90^\circ}{2}$.