Introduction to control theory

WEEK 10

Detectability.

This is the dual property of stabilizability. Let $A \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{p\times n}$ and consider the system

$$\begin{cases} \dot{x}(t) = Ax(t), & x(t) = \text{state at time } t \geq 0, \\ y(t) = Cx(t), & y(t) = \text{output at time } t \geq 0. \end{cases}$$

This system (or the pair (A,C)) is <u>detectable</u> if $y(t)=0$ for all $t \geq 0$ implies that $x(t) \to 0$ (this is a short way of writing $\lim_{t\to\infty} x(t) = 0$).

This property is <u>weaker than</u> (i.e., implied by) <u>observability</u>. Indeed, if (A,C) is observable and $y(t)=0$ for all $t \geq 0$, then $x(0)=0$ (see WEEK 8) and hence $x(t)=0$ for all $t \geq 0$.

<u>Theorem (duality)</u>. (A,C) is detectable if and only if (A^*,C^*) is stabilizable.

You may ignore this sketch of the proof.

The proof uses the decomposition $\mathbb{C}^n = \mathcal{N} + \mathcal{N}^\perp$, where $\mathcal{N}$ is the unobservable space of (A,C), see p. 11 of WEEK 8. Detectability means that the restriction of A to $\mathcal{N}$ is stable, and the remaining part of the system is observable. Taking adjoints, we obtain that $\mathcal{N}^\perp$ is an invariant subspace under A^*, which is controllable via C^*, and the remaining part of the system (A^*,C^*) is stable, which means that (A^*,C^*) is stabilizable. The argument works also in the converse direction, we omit the details.

Example.
$$A = \begin{bmatrix} 3 & 0 \\ 5 & -2 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

We see that $\operatorname{rank} \begin{bmatrix} A-\lambda I \\ C \end{bmatrix} = 1$ for $\lambda = -2$, but the rank is 2 for any other value of λ. (We have $\sigma(A) = \{-2, 3\}$.) Thus, (A,C) is not observable, but (A,C) is detectable. To check the definition of detectability, we see that $y(t) = 0$ for all $t \geq 0$ implies $x_1(0) = 0$, while $x_2(0)$ may be any complex number. We have
$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-2t} x_2(0) \end{bmatrix}$$
and indeed we have $\lim_{t\to\infty} x(t) = 0$. For more complicated systems, checking the definition of detectability directly becomes more complicated, while doing the Hautus test remains relatively easy.

Using the duality theorems (from p.2 of WEEK 9 and p.1 of WEEK 10) we conclude that the pair
$$A^* = \begin{bmatrix} 3 & 5 \\ 0 & -2 \end{bmatrix} \qquad C^* = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
is not controllable, but it is stabilizable.

[END OF EXAMPLE]

The properties of observability and controllability are generically true. This means the following: if we take the entries of the matrices A and C (of fixed dimensions) as independent random variables with a Gaussian probability density (over $\mathbb{R}$ or over $\mathbb{C}$), then (A,C) is observable with probability 1 ("almost sure").

Similarly, if we take the elements of A and B randomly (as before), then (A,B) is controllable with probability 1. It follows that <u>stabilizability and detectability are also generically true.</u>

Using the duality theorem (from p.1), we can translate the Hautus test for stabilizability (WEEK 9, p. 9) into a Hautus test for detectability:

> <u>Theorem.</u> If $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{p\times n}$, then (A,C) is detectable if and only if
> $$\operatorname{rank}\begin{bmatrix} \lambda I - A \\ \hdashline C \end{bmatrix} = n \qquad \forall \lambda \in \sigma(A) \text{ with } \operatorname{Re}\lambda \geq 0.$$

Similarly, the definition of stabilizability (WEEK 9) can be translated into an equivalent characterization of detectability:

> <u>Proposition.</u> Let $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{p\times n}$. Then (A,C) is detectable if and only if there exists $H \in \mathbb{C}^{n\times p}$ such that $A + HC$ is stable.

In control engineering, it is sometimes needed <u>to find a matrix H</u> like in the above proposition. You will see later in this lecture why H is needed. To find H, we use again the translation into an equivalent dual problem, as follows: If (A,C) is given, we compute A^* and C^* (very easy). If (A,C) is detectable, then (A^*,C^*) is stabilizable. Using one of the methods described in WEEK 9 (Ackerman's formula or optimal control) we can find $F \in \mathbb{C}^{p\times n}$ such that $A^* + C^*F$ is stable $\Longleftrightarrow$ $A + F^*C$ is stable. Thus, we can take $H = F^*$.

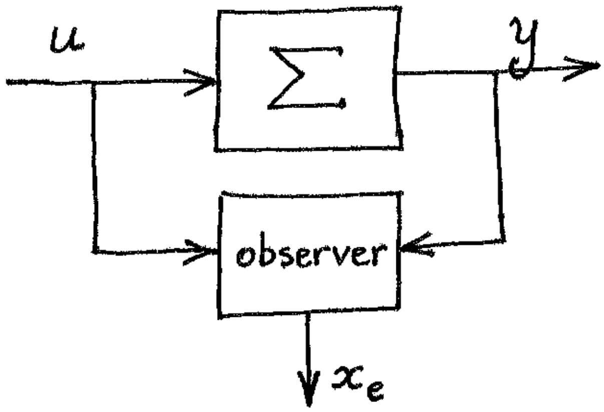
Observers and observer-based controllers

An observer is a system whose purpose is to estimate the state (or a function of the state) of another system. Here we discuss only LTI systems and full state observers.

Let Σ be an LTI system described by the matrices $A \in \mathbb{C}^{n \times n}$, $B \in \mathbb{C}^{n \times m}$, $C \in \mathbb{C}^{p \times n}$, $D \in \mathbb{C}^{p \times m}$ in the usual way. A (full state) observer for Σ is another system that receives the input u and output y of Σ as inputs, and the output of the observer is an estimate x_e of the state x of Σ, such that

$$\lim_{t \to \infty} \| x_e(t) - x(t) \| = 0,$$

regardless of the input function u and the initial state $x(0)$, see the figure. Note that Σ may be unstable, $x(0)$ is unknown to the observer and u may be any signal (for which the state trajectory is defined).



Theorem. There exist observers for Σ if and only if (A,C) is detectable.

We shall prove this theorem. We prove the "only if" part first. If Σ is not detectable, then according to the definition of detectability (on p. 1) there exists $x(0) \in \mathbb{C}^n$ such that, when $u=0$, then $y(t)=0$ for all $t \geq 0$, but $\lim_{t\to\infty} x(t) \neq 0$. With this initial state $x(0)$ and with this input $u=0$, both inputs of the observer are zero for all $t \geq 0$, as if $x(0)=0$. Hence, there is no way for the observer to distinguish between the initial state $x(0)$ and the initial state zero, so it is impossible to achieve $\lim_{t\to\infty} \| x_e(t) - x(t) \| = 0$.

Now we prove the "if" part of the theorem, by constructing an observer (the simplest possible observer, conceptually). Since (A,C) is detectable, we can find $H \in \mathbb{C}^{n\times p}$ such that $A+HC$ is stable. The observer is described by

$$\boxed{\dot{x}_e = (A+HC)x_e + (B+HD)u - Hy.}$$

Let us check that this works. If we subtract from this equation (side from side) the equation of the system Σ, $\dot{x} = Ax + Bu$, and we take into account that $y = Cx + Du$, we obtain

$$\dot{x}_e - \dot{x} = Ax_e - Ax + HCx_e - HCx.$$

Denoting $\delta = x_e - x$ (this is called the estimation error), we obtain $\boxed{\dot{\delta} = (A+HC)\delta.}$

This shows that $\delta(t) \to 0$, regardless of the input u and regardless of the initial states. $\square$

The observer constructed on p. 5 is called the <u>Kalman observer</u>, it was invented around 1960. The Kalman observer has the same order as the plant (n). It is possible to construct (by a more complicated design procedure) observers of lower order (sometimes called Luenberger observers), we do not discuss this. However, we give a very simple example of a lower order observer, which should convey the idea :

<u>Example:</u>

Consider A and C as on p. 2 and $B = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ (the numbers b_1, b_2 do not matter) and $D = 0$.

$$y(t) = Cx(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = x_1(t).$$

Thus, there is no need to estimate $x_1(t)$, we already know it. We have

$$\dot{x}_2(t) = 5x_1(t) - 2x_2(t) + b_2 u(t)$$

$$= -2x_2 + \underbrace{b_2 u(t) + 5y(t)}_{\text{we may regard these as inputs acting on } x_2}$$

The subsystem with state x_2 (and inputs u, y) is stable, so that we do not need to search for H, take $H = 0$. The observer for x_2:

$$\dot{x}_{2e}(t) = -2x_{2e} + b_2 u(t) + 5y(t).$$

If we denote $\delta = x_{2e} - x_2$, then $\dot{\delta} = -2\delta$, so that $\delta \to 0$. Our observer is of order 1.

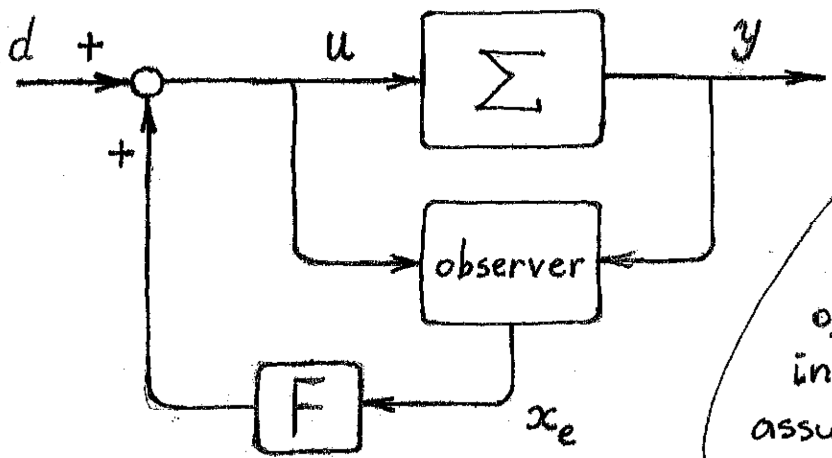
[END OF EXAMPLE]

How to build a stabilizing controller using an observer:

If Σ is a plant described by the matrices A, B, C and D and (A,B) is stabilizable, then we can stabilize the system by state feedback:

$$u = Fx + d \qquad (d = \text{the new input})$$

where F is chosen such that $A+BF$ is stable. If x is not available to the controller, but (A,C) is detectable, then the idea is to replace in the above formula x with x_e, where x_e is an estimate of x, provided by an observer:



This diagram works for any type of observer. However, in the sequel, we assume that we have a Kalman observer.

This system is stable, because

$$\begin{aligned} \dot{x} &= Ax + Bu = Ax + B(Fx_e + d) \\ &= Ax + BF(x+\delta) + Bd \\ &= (A+BF)x + BF\delta + Bd. \end{aligned}$$

Choosing the state variables of the closed-loop system to be x and δ, we obtain

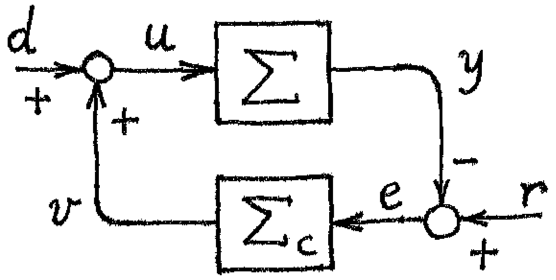
$$\frac{d}{dt}\begin{bmatrix} x \\ \delta \end{bmatrix} = \begin{bmatrix} A+BF & BF \\ 0 & A+HC \end{bmatrix}\begin{bmatrix} x \\ \delta \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} d .$$

From the block-triangular structure of the large square matrix above it follows that

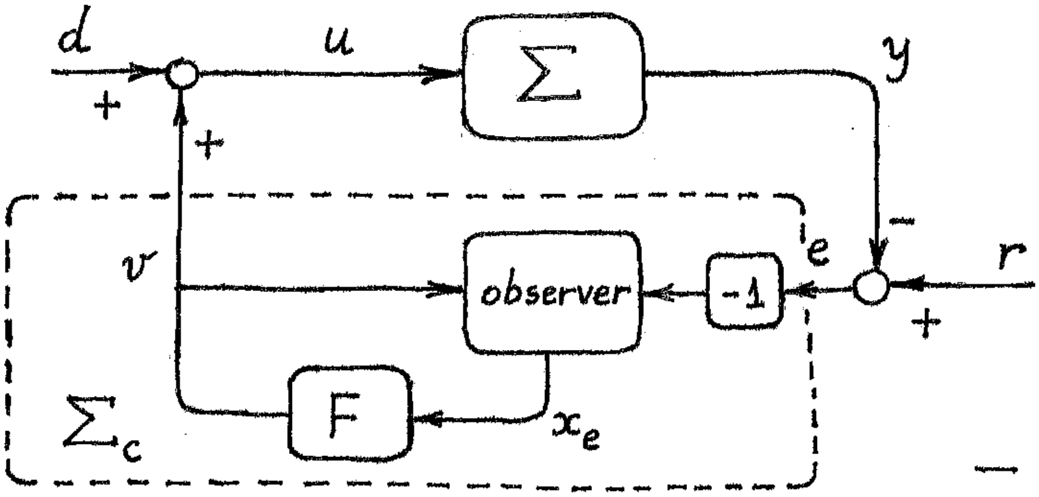
$$\sigma\left(\begin{bmatrix} A+BF & BF \\ 0 & A+HC \end{bmatrix}\right) = \sigma(A+BF) \cup \sigma(A+HC) \subset \mathbb{C}_-$$

so that the system is indeed stable. The above decomposition of the closed-loop eigenvalues into the sets $\sigma(A+BF)$ and $\sigma(A+HC)$ is called the <u>separation principle</u>.

We want to derive a stabilizing controller Σ_c for the plant Σ, which receives $r-y$ as input, connected to the plant as shown in the standard



block diagram shown here (as studied from WEEK 2 onwards). For this, we modify the block diagram on p. 7 as follows :



Here, the plant Σ and the observer are exactly the same as in the diagram on p. 7.

This diagram is <u>not equivalent</u> to the one on p. 7, for two reasons: (1) we have the new signal r, (2) the input from the left of the observer is Fx_e, instead of $Fx_e + d$. However, the two diagrams (from p. 7 and from the bottom of p. 8) <u>are equivalent if $d=0$ and $r=0$</u>. Since the closed-loop system shown on p. 7 is stable, it follows that also the one on p. 8 is stable (since stability does not depend on the external signals). Thus, <u>the system shown in the big dotted frame on the bottom of p. 8 is a stabilizing controller Σ_c</u> for the plant Σ. Its equations are

$$\begin{cases} \dot{x}_e = (A+HC)x_e + (B+HD)Fx_e + He \\ v = Fx_e \,. \end{cases}$$

Thus, it is described by the matrices A_c, B_c, C_c and D_c defined by

$$A_c = A + BF + HC + HDF, \quad B_c = H,$$
$$C_c = F, \quad D_c = 0.$$

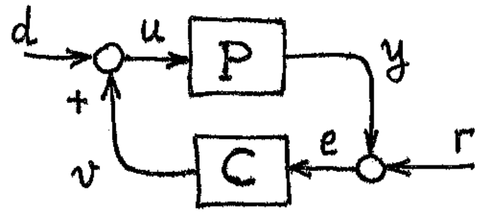
The controller Σ_c may be unstable (but the closed-loop system is stable). Note that in this closed-loop system it is not true that $x_e - x \to 0$.

The above controller is stabilizing, but there is no guarantee that it will have good tracking properties (i.e., e may not be small).

Theorem. If Σ is a plant described by the matrices A, B, C, D then a stabilizing controller Σ_c exists for Σ if and only if (A,B) is stabilizable and (A,C) is detectable.

We prove this theorem: If (A,B) is stabilizable and (A,C) is detectable, then a stabilizing controller Σ_c can be found as described on p. 8–9. Conversely, if a stabilizing controller Σ_c exists, then we connect Σ and Σ_c in feedback (see the figure on p. 8), obtaining a stable system. For any initial state x_0 of Σ, we obtain (assuming $d=0$, $r=0$) an input function u such that $\lim_{t\to\infty} u(t)=0$ (because u is an output of the closed-loop system) and $\lim_{t\to\infty} x(t)=0$. According to the proposition on p. 7 of WEEK 9, it follows that (A,B) is stabilizable. It remains to show that (A,C) is detectable. Suppose that $x_0\in\mathbb{C}^n$ is an initial state of Σ with the following property: if $u=0$ then $y=0$ (for all $t\geq 0$). In the closed-loop system, consider the initial state of Σ_c to be 0. Then (with $d=0$ and $r=0$) we have $u=0$ and $y=0$ (for all $t\geq 0$). Because the closed-loop system is stable, we have $\lim_{t\to\infty} x(t)=0$. By definition, it follows that (A,C) is detectable. $\square$

Suppose that an unstable (possibly MIMO) transfer function P is given, and we want to find a controller transfer function C such that their standard feedback connection (as in the figure) is stable (in the sense discussed in WEEK 2). How can we find such a C?



According to the material on p. 7–8 here, we may construct a minimal realization for P, denoted (A,B,C,D), then (A,B) is controllable and (A,C) is observable (see the last theorem on p. 5 of WEEK 9). Then we may construct a stabilizing controller for (A,B,C,D) (as on p. 7–8) and C may be taken to be the transfer function of this controller.

$P_0(0)$ is onto

An <u>application</u>: suppose that a (possibly unstable, possibly MIMO) transfer function P_0 is given and we want that its output w should <u>track any constant reference</u> signal r. Then we connect a PI controller $C_{PI}(s) = K\left(1 + \frac{1}{\tau s}\right)$ such that its input is the tracking error $\tilde{e} = w - r$, and we search for a stabilizing controller for the plant $P = C_{PI} P_0$ (think about why).