Introduction to control theory

WEEK 9

State space methods – continued.

Controllability

$A \in \mathbb{C}^{n \times n}$ $\quad B \in \mathbb{C}^{n \times m}$

Consider the system $\dot{x}(t) = Ax(t) + Bu(t)$, with state $x(t)$ and input function u. (There is no output function.) This system (or the pair (A,B)) is <u>controllable in time $T>0$</u> if for every $x_1 \in \mathbb{C}^n$, there exists a function $u:[0,T] \to \mathbb{C}^m$ such that

$$\int_0^T e^{A(T-t)} B\, u(t)\, dt = x_1 .$$

This means that, starting from the initial state $x(0)=0$, we can reach any desired final state $x(T)=x_1$.

There are some <u>equivalent formulations</u> of controllability in time T:

(a) Starting from any initial state $x(0)=x_0$, we can reach the final state $x(T)=0$.

(b) Starting from any initial state $x(0)=x_0$, we can reach any final state $x(T)=x_1$.

As an <u>exercise</u>, check that (a) and (b) above are equivalent to controllability in time T.

Theorem (duality). Take $T>0$.

(A,B) is controllable in time T if and only if (A^*,B^*) is observable in time T.

Proof. For $x,z\in\mathbb{C}^n$, we denote

"inner product" $$<x,z> = x_1\overline{z}_1 + x_2\overline{z}_2 \dots + x_n\overline{z}_n,$$

so that $\langle x,z\rangle = z^*x$ and $<x,x> = \|x\|^2$.

If $x\in\mathbb{C}^n$, $w\in\mathbb{C}^m$ and $T\in\mathbb{C}^{m\times n}$, then

$$<Tx,w> = <x,T^*w>$$

where T^* is the transpose and complex conjugate of T. We have $(TS)^* = S^*T^*$ and $T^{**}=T$. The above notation is worth remembering, it is used a lot.

It is easy to check that

$$\left(e^{At}\right)^* = e^{A^*t}.$$

We need the following formula:

Let u be a function on $[0,T]$ with values in $\mathbb{C}^m$ and let $\tilde{u}$ be it reflection in time: $\tilde{u}(t) = u(T-t)$ $\left(\begin{smallmatrix}\text{for all}\\ t\in[0,T]\end{smallmatrix}\right)$.

Then
$$\left\langle \int_0^T e^{A(T-t)} Bu(t)\,dt \,,\, z_0 \right\rangle$$

(here, $z_0 \in \mathbb{C}^n$ and $u : [0,T] \to \mathbb{R}^m$ are arbitrary)

$$= \int_0^T \langle e^{A(T-t)} Bu(t), z_0 \rangle \,dt$$

$$= \int_0^T \langle u(t), B^* e^{A^*(T-t)} z_0 \rangle \,dt$$

$$= \int_0^T \langle \tilde{u}(t), B^* e^{A^* t} z_0 \rangle \,dt .$$

Suppose that (A,B) is not controllable. Let $\mathcal{R}$ be the vector space of all the states $x_1 \in \mathbb{C}^n$ for which a function u can be found such that $\int_0^T e^{A(T-t)} Bu(t)\,dt = x_1$ (this is called the <u>reachable space</u> of (A,B)). By our assumption, $\mathcal{R} \neq \mathbb{C}^n$, so that we can find a vector $z_0 \in \mathbb{C}^n$ which is orthogonal to $\mathcal{R}$: $\langle x_1, z_0 \rangle = 0$ for all $x_1 \in \mathcal{R}$. This implies that the sides of the formula on top of this page are zero, for any possible choice of u. Choosing u such that $\tilde{u}(t) = B^* e^{A^* t} z_0$, we obtain
$$\int_0^T \| B^* e^{A^* t} z_0 \|^2 dt = 0 .$$

This means that (A^*, B^*) is not observable in time T (see the definition in WEEK 8).

Conversely, suppose that (A^*, B^*) is not observable in time T. This means that there exists $z_0 \in \mathbb{C}^n$, $z_0 \neq 0$, such that

$$B^* e^{A^* t} z_0 = 0 \quad \text{for all } t \in [0,T].$$

This means that the integral on the right-hand side of the formula on top of p.3 is zero, for every possible choice of u. This means that we can never have

$$\int_0^T e^{A(T-t)} B u(t)\, dt = z_0$$

(because then the integral would be $\|z_0\|^2 > 0$). Hence, (A, B) is not controllable in time T.

$\square$

Using the duality theorem, we can translate our results about observability (from WEEK 8) into results about controllability. For example, since observability does not depend on T, it follows that also controllability does not depend on T. In the sequel, we say that "<u>(A,B) is controllable</u>", instead of saying "(A,B) is controllable in time T".

By translating the Kalman rank condition from WEEK 8, we get:

> <u>Theorem</u>. Take $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. (A,B) is controllable if and only if
> $$\operatorname{rank}\left[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, \dots\, A^{n-1}B\right] = n.$$

Using this, check that the control-canonical realisation (WEEK 8) is controllable.

By translating the Hautus test (also proved in WEEK 8), we get:

> <u>Theorem</u>. Let $A \in \mathbb{C}^{n\times n}$ and $B \in \mathbb{C}^{n\times m}$. (A,B) is controllable if and only if
> $$\operatorname{rank}\left[\lambda I - A \,\vdots\, B\right] = n \quad \forall \lambda \in \sigma(A).$$

Using this, check that the diagonal realisation (WEEK 8) is controllable.

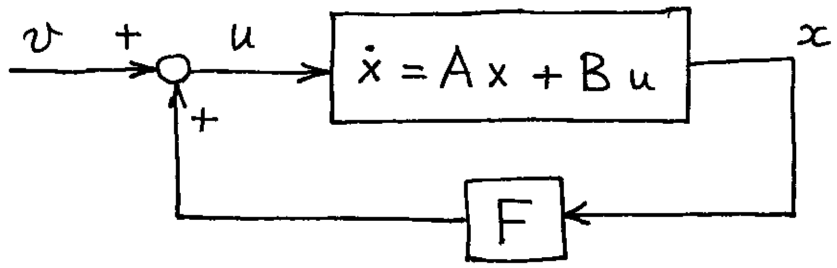
We have a strong connection between the above concepts and minimal systems (see WEEK 8).

> <u>Theorem</u>. A system (A,B,C,D) is minimal if and only if (A,B) is controllable and (A,C) is observable.

As an exercise (not so easy) try to prove this, using the space $\mathcal{N}$ from WEEK 8 (p. 11) and the space $\mathcal{R}$ from WEEK 9 (p. 3).

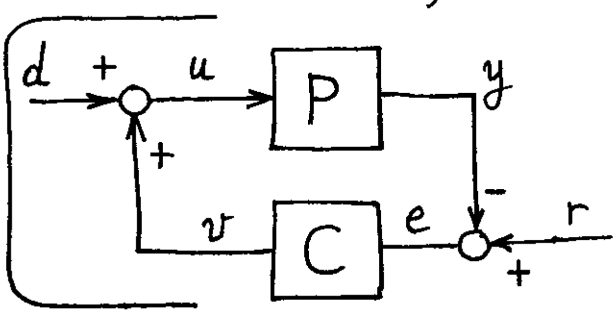
Exercise 1. Prove that if (A,B) is controllable, then for every $\lambda \in \mathbb{C}$, $(A+\lambda I, B)$ is controllable.

Exercise 2. Consider the following system with state feedback:



where $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times m}$, $F \in \mathbb{C}^{m\times n}$.
Clearly we have $\dot{x} = (A+BF)x + Bv$.
Prove that (A,B) is controllable if and only if $(A+BF, B)$ is controllable. (Try to find a very simple reasoning for this.)

Exercise 3. Consider the following feedback system, as encountered in WEEK 2.



inputs: d, r
outputs: u, e

Suppose that the system with transfer function P is described by the matrices A, B, C, D, and the system with transfer function C is described by A_c, B_c, C_c, D_c.

(a). Find the matrices of the closed-loop system, assuming that $\|DD_c\| < 1$ (w-stability).

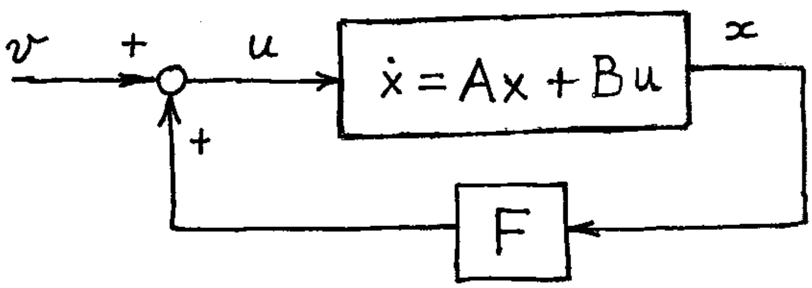
(b) Show that (A,B) and (A_c, B_c) are controllable if and only if the closed-loop system is controllable.

Stabilizability

Consider the system described by

$$\dot{x}(t) = Ax(t) + Bu(t), \quad A \in \mathbb{C}^{n\times n}, \quad B \in \mathbb{C}^{n\times m}.$$

We say that (A,B) is <u>stabilizable</u> if there exists $F \in \mathbb{C}^{m\times n}$ such that $A+BF$ is stable. This means that we can stabilize the system by the <u>state feedback</u> $u = Fx + v$ (here v denotes the new input), as shown in the following block diagram:



The <u>loop-gain</u> of this system is $G(s) = -F(sI-A)^{-1}B$, this determines the gain margin and phase margin in the usual way.

In closed loop we have $\dot{x} = (A+BF)x + Bv$. An equivalent formulation of this which does not use the concept of feedback is the following:

> <u>Proposition</u>. (A,B) is stabilizable if and only if for every initial state $x_0 \in \mathbb{C}^n$ there exists a continuous function $u : [0,\infty) \to \mathbb{C}^m$ such that the solution of $\dot{x} = Ax + Bu$, $x(0) = x_0$ satisfies
> $$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} u(t) = 0.$$

The "only if" part is easy to prove: if (A,B) is stabilizable and $x_0 \in \mathbb{C}^n$, we solve the equation $\dot{x}(t) = (A+BF)x(t)$, $x(0) = x_0$, where F is chosen such that $A+BF$ is stable, whence $\lim_{t\to\infty} x(t) = 0$. Define $u(t) = Fx(t)$, then this is the desired input. The "if" part is more difficult to prove, it is based on quadratic optimal control theory, which we do not study in this course.

In practice, we can check if (A,B) is stabilizable using the Hautus test:

<u>Theorem</u>. (A,B) is stabilizable if and only if

$$\operatorname{rank}\left[\lambda I - A \,\vdots\, B\right] = n \quad \forall \lambda \in \sigma(A) \text{ with } \operatorname{Re}\lambda \geq 0.$$

From this theorem (or alternatively, from the proposition on p. 7) we see that if (A,B) is controllable, then (A,B) is also stabilizable.

A natural question is: if (A,B) is stabilizable, how do we find F such that $A+BF$ is stable? As an <u>exercise</u>, do this for a control-canonical system (as it appears on p.2 of WEEK 8). Such a system is always controllable.

While you do this, notice that we can impose any desired closed-loop eigenvalues $\lambda_1, \dots \lambda_n$ by requiring that the characteristic polynomial of $A+BF$ should be $p_d(s) = (s-\lambda_1) \dots (s-\lambda_n)$.

We present two methods for finding F. The first method is <u>Ackerman's formula</u>: Suppose that $m=1$ and (A,B) is controllable. Let $\lambda_1, \dots \lambda_n$ be the desired closed-loop eigenvalues and define $p_d(s)$ as a few lines above. If $p_d(s) = s^n + q_{n-1} s^{n-1} \dots + q_1 s + q_0$, we define

$$p_d(A) = A^n + q_{n-1} A^{n-1} \dots + q_1 A + q_0 I .$$

Then the desired F is

"Ackerman's formula"

$$F = -[0\, 0 \dots 1]\, \mathscr{C}^{-1}\, p_d(A), \quad \mathscr{C} = \left[B \,\vdots\, AB \,\vdots\, \dots A^{n-1} B \right].$$

The matrix $\mathscr{C}$ is invertible because of the controllability of (A,B) (see the Kalman test). The above formula is equivalent to what we did in the exercise on p. 8, but combined with a change of coordinates in the state space, to bring the system to the control-canonical form.

<u>Example</u>.
$$A = \begin{bmatrix} -2 & 3 \\ 5 & 6 \end{bmatrix}, \qquad B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Clearly A is unstable (since trace $A = 4 > 0$).

We have

$$\mathscr{C} = \left[B \,\vdots\, AB\right] = \begin{bmatrix} 2 & -7 \\ -1 & 4 \end{bmatrix}, \quad \det \mathscr{C} = 1,$$

so that (A, B) is controllable. Suppose that we want both closed-loop eigenvalues at -3. Then the desired closed-loop characteristic polynomial is

$$p_d(s) = (s+3)^2 = s^2 + 6s + 9.$$

Thus,

$$p_d(A) = \begin{bmatrix} 19 & 12 \\ 20 & 51 \end{bmatrix} + 6 \begin{bmatrix} -2 & 3 \\ 5 & 6 \end{bmatrix} + 9 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$p_d(A) = \begin{bmatrix} 16 & 30 \\ 50 & 96 \end{bmatrix},$$

$$F = -\begin{bmatrix} 0 & 1 \end{bmatrix} \mathscr{C}^{-1} \begin{bmatrix} 16 & 30 \\ 50 & 96 \end{bmatrix}, \quad \mathscr{C}^{-1} = \begin{bmatrix} 4 & 7 \\ 1 & 2 \end{bmatrix},$$

$$F = -\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 16 & 30 \\ 50 & 96 \end{bmatrix} = -\begin{bmatrix} 116 & 222 \end{bmatrix}.$$

Before we move on to the next method of finding F, we have to explain what is a positive matrix.

Let $P \in \mathbb{C}^{n \times n}$. P is <u>positive</u> ($P \geq 0$) if $x^* P x \geq 0$ for every $x \in \mathbb{C}^n$. It can be shown that $P \geq 0$ iff $P = P^*$ and $\sigma(P) \subset [0, \infty)$. (We remark that for every matrix $T \in \mathbb{C}^{n \times n}$ which satisfies $T^* = T$, we have $\sigma(T) \subset (-\infty, \infty)$, i.e., the eigenvalues of T are real.)

Examples of positive matrices:

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}, \quad P = \begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}, \quad P = [3], \quad P = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}.$$

Positive matrices appear in a lot of branches of mathematics and systems theory. For example, if $f(x_1, x_2, \dots x_n)$ has a minimum at a point in $\mathbb{R}^n$, then its second derivative at that point is a positive $n \times n$ matrix. In probability theory, covariance matrices are positive. For <u>any</u> matrix T, $T^* T \geq 0$.

A matrix $P \in \mathbb{C}^{n \times n}$ is <u>strictly positive</u>, written as $P > 0$, if $P \geq 0$ and P is invertible. Equivalently, $P > 0$ iff $P = P^*$ and $\sigma(P) \subset (0, \infty)$. For example, the first three matrices on top of this page are > 0, while the last is only ≥ 0.

The <u>optimal control method</u> for finding F is to define a quadratic optimal control problem for the system $\dot{x} = Ax + Bu$: For every initial state $x_0 \in \mathbb{C}^n$, we define the <u>cost function</u>

$$J(x_0, u) = \int_0^\infty \left[x(t)^* Q\, x(t) + u(t)^* R\, u(t) \right] dt$$

where $Q \in \mathbb{C}^{n \times n}$, $Q > 0$, $R \in \mathbb{C}^{m \times m}$, $R > 0$. The problem is to minimize $J(x_0, u)$ over all possible $u : [0, \infty) \to \mathbb{C}^m$. If (A, B) is not stabilizable, then for most x_0 we have $J(x_0, u) = \infty$, no matter how we choose u. But if (A,B) is stabilizable, then for every $x_0 \in \mathbb{C}^n$ there exists a unique optimal input function $u^{opt}: [0,\infty) \to \mathbb{C}^m$. We do not prove this, or any of the following statements. These issues are treated in detail in more advanced courses.

u^{opt} can be found as follows. We have to solve the following equation, where the unknown is the matrix $X \in \mathbb{C}^{n\times n}$:

$$A^* X + XA + Q = X B R^{-1} B^* X. \quad \left(\begin{matrix}\text{algebraic}\\ \text{Riccati}\\ \text{equation}\end{matrix}\right)$$

This equation has many solutions, but only one of them is positive (actually, it is strictly positive). This positive solution is called the <u>"stabilizing solution"</u> of the Riccati equation. This is found numerically (in MATLAB, the command aresolv will give X if we specify A, B, Q, R). Once we have found X, we compute the <u>optimal feedback matrix</u> from

$$F^{opt} = -R^{-1} B^* X.$$

Then $A + BF^{opt}$ is stable and if we solve

$$\dot{x}(t) = \left(A + BF^{opt}\right) x(t), \quad x(0) = x_0$$

then $u^{opt}(t) = F^{opt} x(t)$. Thus, after using F^{opt} as a feedback matrix, the closed-loop system will behave optimally (with respect to J) for <u>any</u> initial state. We have $J(x_0, u^{opt}) = x_0^* X x_0$. If $G(s) = F^{opt}(sI-A)^{-1}B$ is SISO, then $GM = \infty$, $\varphi_m \geq 60°$.