Introduction to control theory

WEEK 8

Realisations

In the USA, it would be "realizations".

"Modern control theory" (as it was then called) started around 1960 with the invention of the concepts of state and state space. Until then, systems were represented by differential equations or (what is the same in the LTI case) transfer functions. Thus, an immediately arising question was how to find a state space system that corresponds to a given transfer function.

If G is a proper rational matrix-valued function, then (A,B,C,D) (a collection of four matrices of matching sizes) is called a <u>realisation of G</u> if

$$G(s) = C(sI-A)^{-1}B + D.$$

Every proper rational G has infinitely many realisations. If $A \in \mathbb{C}^{n\times n}$ then the corresponding LTI system (see WEEK 1) has n state variables and we say that this system is of <u>order n</u>. Different realisations of G may have different order. For example, take $G(s) = \frac{3}{s+1}$. Two realisations are

$$A=-1,\quad B=1,\quad C=3,\quad D=0, \quad \text{and}$$

$$A=\begin{bmatrix}-1 & 0\\ 1 & 5\end{bmatrix},\quad B=\begin{bmatrix}1\\1\end{bmatrix},\quad C=\begin{bmatrix}3 & 0\end{bmatrix},\quad D=[0].$$

In the second realisation (of order 2), the second state variable has no influence on the output.

If (A,B,C,D) is a realisation of G, then for every invertible matrix T of the same dimensions as A,

$$(TAT^{-1},\ TB,\ CT^{-1},\ D)$$ [This corresponds to working with the new state Tx.]

is another realization of G. Check this!

A realization of G which is of the smallest possible order (i.e., with the smallest possible state vector) is called a <u>minimal realisation</u> of G. For MIMO transfer functions, it is sometimes not easy to find a minimal realisation (but it is always possible). For SISO transfer functions, the problem is relatively easy, and we present two methods.

The control-canonical realisation of a SISO transfer function G

First compute $D = \lim_{s\to\infty} G(s)$ and then write

$$G(s) - D = \frac{c_{n-1}s^{n-1} \ldots + c_1 s + c_0}{s^n + a_{n-1}s^{n-1} \ldots + a_1 s + a_0}.$$

Then the realisation is:

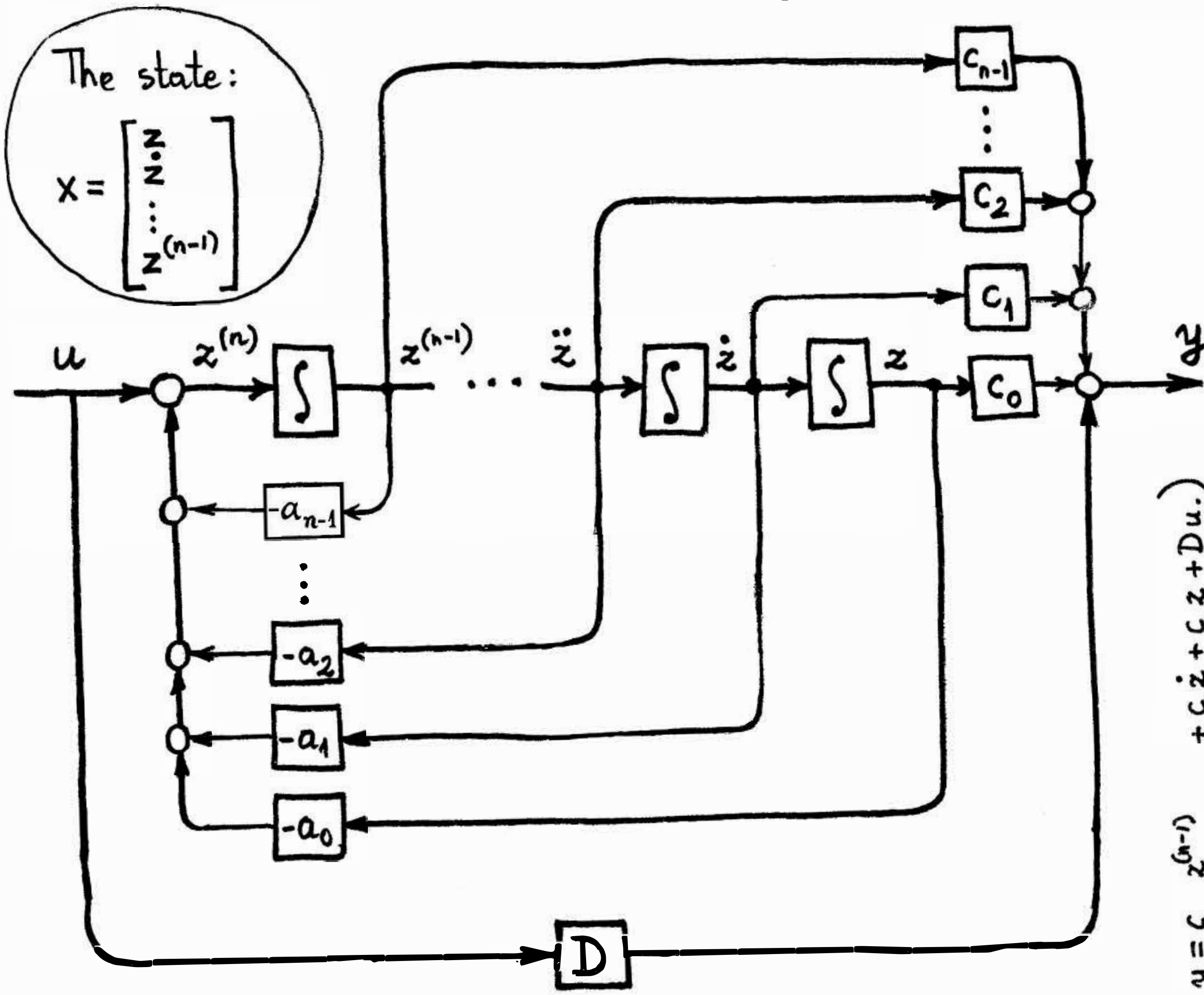
$$A = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

"companion matrix"

$$C = [c_0\ c_1\ c_2 \ldots c_{n-1}] \qquad D \text{ as above.}$$

Check that indeed $G(s) = C(sI - A)^{-1}B + D$!

This realisation corresponds to the following block diagram with n integrators:



Try to understand how this block diagram works. It is important to notice that

$$z^{(n)} + a_{n-1} z^{(n-1)} \dots + a_1 \dot z + a_0 z = u ,$$

so that

$$\hat z(s) = \frac{1}{s^n + a_{n-1}s^{n-1} \dots + a_1 s + a_0} \cdot \hat u(s)$$

if $z(0) = 0, \dots z^{(n-1)}(0) = 0$.

(Notice that $y = c_{n-1} z^{(n-1)} \dots + c_1 \dot z + c_0 z + Du$.)

If the polynomials $c_{n-1}s^{n-1} \ldots + c_1 s + c_0$ and $s^n + a_{n-1}s^{n-1} \ldots + a_1 s + a_0$ have no common zero, then the above realisation is minimal. It can be generalized easily for systems with one input and several outputs (SIMO), simply by considering $D, c_0, c_1, \ldots c_{n-1}$ to be p-dimensional vectors, where p is the number of outputs.

<u>Example</u>. $G(s) = \dfrac{5s^2 + 3s - 1}{s^3 + 4s + 1}$.

We compute $D = \lim_{s\to\infty} G(s) = 0$. Hence

$$\begin{cases} A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -4 & 0 \end{bmatrix} & B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \\ C = \begin{bmatrix} -1 & 3 & 5 \end{bmatrix} & D = [0] \end{cases}$$

<u>Second example</u>. $G(s) = \begin{bmatrix} \frac{s-1}{s+1} \\ \frac{3}{s+7} \end{bmatrix}$,

$D = \lim_{s\to\infty} G(s) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$$G(s) - D = \begin{bmatrix} \frac{-2}{s+1} \\ \frac{3}{s+7} \end{bmatrix} = \frac{\begin{bmatrix} -2 \\ 3 \end{bmatrix} s + \begin{bmatrix} -14 \\ 3 \end{bmatrix}}{(s+1)(s+7)} \quad \left(\begin{matrix} c_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix} \\ c_0 = \begin{bmatrix} -14 \\ 3 \end{bmatrix} \end{matrix} \right)$$

$(s+1)(s+7) = s^2 + 8s + 7 \quad (a_1 = 8,\ a_0 = 7)$

$$A = \begin{bmatrix} 0 & 1 \\ -7 & -8 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = \begin{bmatrix} -14 & -2 \\ 3 & 3 \end{bmatrix},\ D = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

We have seen in WEEK 1 that for any linear system (A,B,C,D), the poles of the transfer function G are a subset of $\sigma(A)$ (the set of eigenvalues of A). This fact allows us to recognise with ease for many systems whether they are minimal:

> <u>Proposition</u>. If (A,B,C,D) is a linear system of order n and its transfer function G has n distinct poles, then the system is minimal.

This follows easily from what is written before the proposition. If the transfer function has $<n$ distinct poles, the system may still be minimal, for example, the system

$$A=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix},\quad B=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix},$$

$$C=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix},\quad D=\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix},$$

is minimal, and its transfer function is

$$G(s)=\begin{bmatrix}\frac{3}{s+1} & 0\\ 0 & \frac{3}{s+1}\end{bmatrix}$$

which has only one pole. So not every minimal system can be recognized using the above proposition. For SISO systems we have a stronger result:

> <u>Proposition</u>. If (A,B,C,D) is a linear SISO system of order n and its transfer function G has $\tilde{n}$ poles when counting each pole with its multiplicity, then the system is minimal iff $n=\tilde{n}$.

For example, if $G(s)=(s+7)/s(s+2)^3$, then a minimal realization of G must be of order 4.

The following proposition is closely related to the last two.

Proposition. If (A,B,C,D) is a minimal system, then the set of poles of its transfer function is exactly $\sigma(A)$.

The property of being minimal is not something special: it is almost always true. What we mean is the following: if we take all the entries of the matrices A, B and C to be independent random variables with a continuous distribution function (for example, Gaussian), then with probability 1, the resulting system (A,B,C,D) is minimal. (The matrix D can be taken any matrix of suitable dimensions, since D has no influence on the system being minimal, as is easy to see.)

We define the concept of zero of a MIMO transfer function only in the particular case of equal number of inputs and outputs, say m, so that $G(s) \in \mathbb{C}^{m \times m}$ for every $s \in \mathbb{C}$ where $G(s)$ is defined (i.e., s is not a pole of G). We also assume that there exist points $s \in \mathbb{C}$ where $G(s)$ is defined and $\det G(s) \neq 0$. In this case, it can be shown that $\det G(s) = 0$ occurs only at a finite number of points $z \in \mathbb{C}$, and these are called zeros of G. For example, the transfer function

$$G(s) = \frac{1}{s^2+3s+7} \begin{bmatrix} s-1 & s+1 \\ 1 & 2 \end{bmatrix}$$ has only one zero, $z = 3$.

Proposition. Let G be a square transfer function (i.e., equal number of inputs and outputs), such that $\det G(s)$ is not everywhere zero, and let (A,B,C,D) be a minimal realization of G. Then a point $z \notin \sigma(A)$ is a zero of G if and only if

$$\det \begin{bmatrix} A - zI & B \\ C & D \end{bmatrix} = 0.$$

As an exercise, you may try to prove the proposition about zeros on the bottom of p.6. The "if" part is easy. For the "only if" part, if z is a zero of G, there exists $v \in \mathbb{C}^m$ with $v \neq 0$, $G(z)v = 0$. Denote $x = (zI-A)^{-1}Bv$, then it is easy to see that
$$\begin{bmatrix} A-zI & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} = 0.$$
(Supply the missing details.)

Another exercise: If A is a matrix as on the bottom of p. 2, prove that its characteristic polynomial is $p(s) = \det(sI-A) = s^n + a_{n-1}s^{n-1} \ldots + a_1 s + a_0$.

The diagonal realisation of a SISO transfer function with simple poles:

If G is SISO and its poles are simple then, denoting the poles by $p_1, p_2, \ldots p_n$, we can compute a partial fractions decomposition of G:

$$G(s) = D + \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} \ldots + \frac{c_n}{s-p_n}$$

$p_j \neq p_k$ for $j \neq k$, $c_k \neq 0$

where $D = \lim_{s\to\infty} G(s)$, $\quad c_j = \lim_{s \to p_j} (s-p_j)G(s)$.

The diagonal realisation of G is:

$$A = \begin{bmatrix} p_1 & & & 0 \\ & p_2 & & \\ & & \ddots & \\ 0 & & & p_n \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} c_1 & c_2 & \ldots & c_n \end{bmatrix}, \qquad D \text{ as above.}$$

This is minimal! (Easy to see.)

Check this as an exercise.

Observability

Consider the system $\begin{cases} \dot{x}(t) = Ax(t), \\ y(t) = Cx(t), \end{cases}$
($A \in \mathbb{C}^{n\times n}$, $C \in \mathbb{C}^{p\times n}$)
with state $x(t)$ and output function y. (There is no input function.) This system (or the pair (A,C)) is <u>observable in time $T>0$</u> if for every $x_0 \in \mathbb{C}^n$, $x_0 \neq 0$, the output y corresponding to the initial state x_0 satisfies

$$\int_0^T \|y(t)\|^2 \, dt > 0.$$

Here, $\|y(t)\|^2 = |y_1(t)|^2 + |y_2(t)|^2 \dots + |y_p(t)|^2$. In other words, y is not zero all the time on the interval $[0,T]$. (Note that $y(t) = Ce^{At}x_0$, for all $t \geq 0$.)

<u>Example:</u> The system with

$$A = \begin{bmatrix} 2 & -1 & 5 \\ 0 & 4 & 7 \\ 0 & 0 & 6 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 3 & 23 \end{bmatrix}$$

is not observable. Indeed, if we take

$$x_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \text{then} \quad x(t) = \begin{bmatrix} e^{2t} \\ 0 \\ 0 \end{bmatrix} \quad \text{and}$$

hence $y(t) = Cx(t) = 0$ for all $t \geq 0$.

(It can be shown that if we take the entries of A and C randomly with a continuous distribution, then (A,C) is observable with probability 1.)

Theorem (Kalman rank condition)

Take $A \in \mathbb{C}^{n\times n}$ and $C \in \mathbb{C}^{p\times n}$.
(A,C) is observable if and only if

$$\operatorname{rank}\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1}\end{bmatrix} = n.$$

The rank of a matrix is the number of linearly independent rows (or columns) in the matrix.

(Rudolf Kalman was the leading scientist behind "modern control theory" in the 1960s. He was at NASA and at ETH Zürich.)

Proof. Suppose that (A,C) is not observable, hence there exists $x_0 \in \mathbb{C}^n$, $x_0 \neq 0$ such that on an interval $t\in[0,T]$ we have $Ce^{At}x_0 = 0$. Differentiating with respect to time, we obtain $CAe^{At}x_0 = 0$ (for all $t\in[0,T]$), differentiating again we obtain $CA^2e^{At}x_0=0$ (for all $t\in[0,T]$), etc. Taking now $t=0$, we obtain

$$Cx_0 = 0,\quad CAx_0 = 0,\quad CA^2x_0 = 0,\quad \dots$$

which implies that

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1}\end{bmatrix} x_0 = 0.$$

This shows that the n columns of this matrix are dependent, hence the rank of the matrix is less than n.

Conversely, suppose that the rank of the matrix in the theorem is not n, hence it is $< n$, so that its columns are linearly dependent. Therefore, there exists $x_0 \in \mathbb{C}^n$, $x_0 \neq 0$ such that

$$\begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} x_0 = 0.$$

Let $p(s) = \det(sI - A)$ be the characteristic polynomial of A. Then, according to the Cayley-Hamilton theorem, we have $p(A) = 0$. Thus, if $p(s) = s^n + a_{n-1}s^{n-1} \dots + a_1 s + a_0$, then

$$A^n + a_{n-1}A^{n-1} \dots + a_1 A + a_0 I = 0.$$

We multiply with x_0 from the right and with C from the left, obtaining $CA^n x_0 = 0$. Now we multiply the equation $p(A) = 0$ with x_0 from the right and with CA from the left, obtaining $CA^{n+1} x_0 = 0$. By induction, we obtain that

$$CA^k x_0 = 0 \text{ for all } k \in \mathbb{N}.$$

Since $y(t) = Ce^{At}x_0$ has the Taylor expansion

$$y(t) = Cx_0 + tCAx_0 + \frac{t^2}{2}CA^2x_0 + \frac{t^3}{6}CA^3x_0 + \dots$$

and since all the coefficients in this

Taylor series are zero, as we have just shown, it follows that $y(t)=0$ for all $t \geq 0$. Since the initial state $x_0 \neq 0$, it follows that (A,C) is not observable. $\square$

This is an "end of proof" sign.

Note that the Kalman rank condition is independent of T, the length of the time interval. Thus, it does not matter how we choose $T>0$ in the definition of observability. Therefore, in the sequel, we do not say that "(A,C) is observable in time T," we only say that "(A,C) is observable" (actually, we did that already in the theorem, before it was justified).

Theorem (Hautus test)

Let $A \in \mathbb{C}^{n \times n}$ and $C \in \mathbb{C}^{p \times n}$. (A,C) is observable if and only if

$$\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} = n, \quad \forall \lambda \in \sigma(A).$$

Note that for $\lambda \notin \sigma(A)$, the above rank is always n, because $\lambda I - A$ is invertible.

Proof. Suppose that (A,C) is not observable. Define
$$\mathcal{N} = \{ q \in \mathbb{C}^n \mid Ce^{At} q = 0 \text{ for all } t \geq 0 \}.$$
This is a non-zero subspace of $\mathbb{C}^n$, invariant under A (i.e., $A\mathcal{N} \subset \mathcal{N}$). Let $\mathcal{A}$ be the restriction of A to $\mathcal{N}$, and let $x_0 \in \mathcal{N}$ be an eigenvector of $\mathcal{A}$. Then x_0 is also an eigenvector of A, corresponding to some $\lambda \in \sigma(A)$. We have $Cx_0 = 0$, because clearly this holds for any vector in $\mathcal{N}$. Thus, we have
$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} x_0 = 0, \quad x_0 \neq 0 \implies \operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} < n.$$

Conversely, suppose that $\operatorname{rank} \begin{bmatrix} \lambda I - A \\ C \end{bmatrix} < n$ for some $\lambda \in \sigma(A)$. Thus, the columns of this matrix are linearly dependent, so that there exists $x_0 \neq 0$ such that
$$\begin{bmatrix} \lambda I - A \\ C \end{bmatrix} x_0 = 0.$$

It follows that x_0 is an eigenvector of A ($Ax_0 = \lambda x_0$) and $Cx_0 = 0$. Hence, $e^{At} x_0 = e^{\lambda t} x_0$ and therefore $Ce^{At} x_0 = 0$ for all $t \geq 0$. This means that (A,C) is not observable. $\square$

The subspace $\mathcal{N}$ appearing in the above proof is called the "unobservable space of (A,C)".