Expectation is Not All

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A Dice Game

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Numbers: 2, 2, 2, 2, 3, 3	$\rightarrow 2$ Bob wins $6 \leftarrow$	Numbers: 1, 1, 1, 1, 6, 6

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- Bob has $1 \times \frac{2}{3} + 6 \times \frac{1}{3} = \frac{8}{3}$

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- Alice has $2 \times \frac{2}{3} + 3 \times \frac{1}{3} = \frac{7}{3}$
- Bob has $1 \times \frac{2}{3} + 6 \times \frac{1}{3} = \frac{8}{3}$
- Bob has a better expected value

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- But who wins more often?

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- But who wins more often?
- Note that the winner depends only on Bob's throw: if he throws 1 he definitely loses, if he throws 6 he definitely wins
- Bob throws 1 with probability $2/3$
- So Bob loses (substantially) more often, despite greater expected value

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Numbers:	Numbers:
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- When Bob wins, he wins by far: 6 against 2 or 3; and when he loses, he loses slightly: 1 against 2 or 3

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- Where did the larger expected value go? Why it does not help Bob to win?
- When Bob wins, he wins by far: 6 against 2 or 3; and when he loses, he loses slightly: 1 against 2 or 3
- But he does not get credit for difference between numbers

Conclusion

Alice	Bob
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- This example shows that the expected value does not tell us everything about random variable

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- This example shows that the expected value does not tell us everything about random variable
- A random variable with “better” expected value can be “worse” because of some other properties