What is Probability

Alexander Shen

LIRMM / CNRS & University of Montpellier

Outline

Predicting Unpredictable

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- tossing a coin

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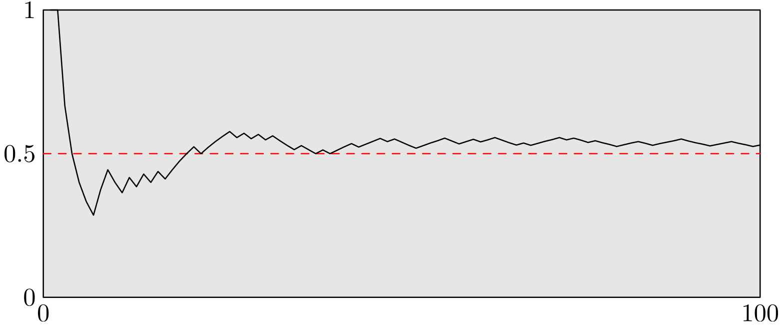
- repeated experiment:
  $1100000110010101011110111101\ldots$
  (0=head,1=tail)
- zeros and ones appear equally often
- *frequency of 1s*: (#ones)/(length) $\approx 1/2$

Hundred Random Bits

11000001100101010111101111 01 . . .



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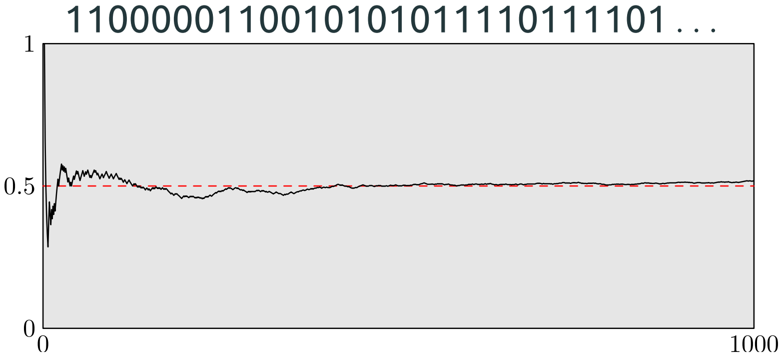
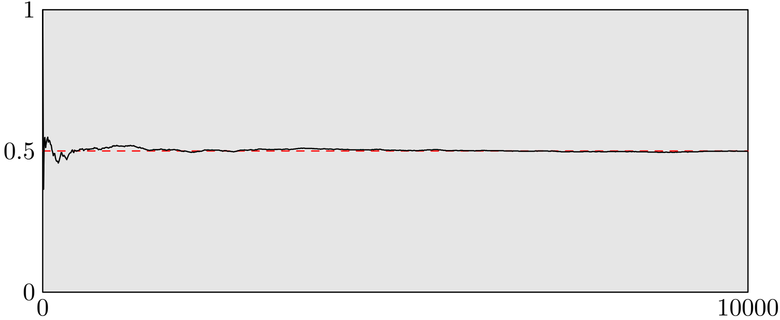


Ten Thousands Random Bits

11000001100101010111101111101 . . .



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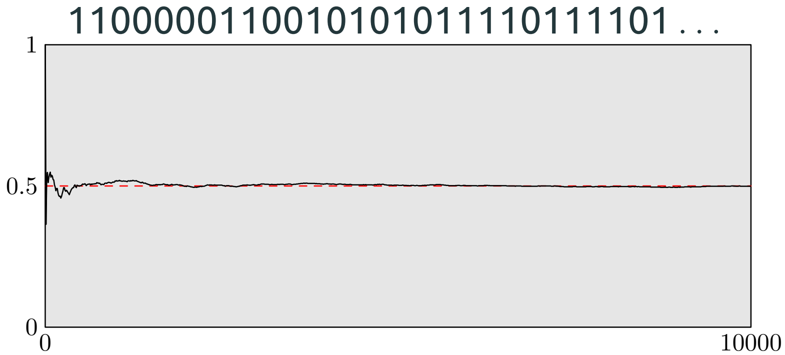
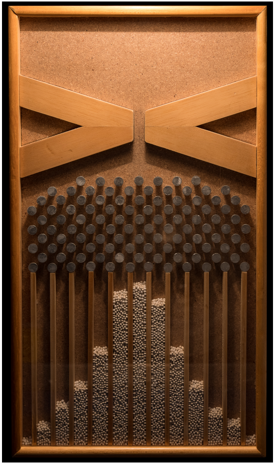


(produced by George Marsaglia, 1995)

Outline

Bean Machine



https://en.wikipedia.org/wiki/Bean_machine

Sir Francis Galton (1822–1911)

Galton Board: animation

TO BE REPLACED BY VIDEO FROM

https://en.wikipedia.org/wiki/File:Galton_box.webm

Analysis

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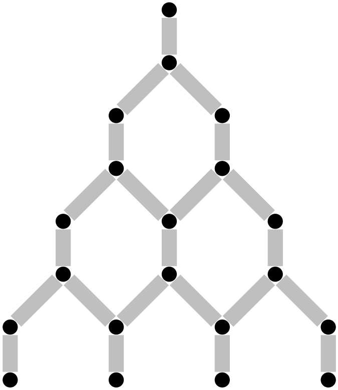
Assume that at each level the beans are splitted evenly.



Galton and Pascal

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$$
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 & & & 1 & & & \\
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 & 1/4 & & 2/4 & & 1/4 & \\
1/8 & & 3/8 & & 3/8 & & 1/8 \\
 & & & \cdots & & &
\end{array}
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Pascal triangle/ 2^n:

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Pascal triangle/ 2^n: $\binom{n}{k}/2^n$

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Paradox of Probability Theory

Galton Board

Natural Sciences and Mathematics

Rolling Dice

More Examples

What is Probability Theory?

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    - mathematics: the implications of the model

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- Dan: in fact, Alice is right for the real coins

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- among the beans that go left [right] at level 1, half go left [right] at level 2, etc.

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https://upload.wikimedia.org/wikipedia/commons/3/3b/Die_bone.jpg

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  - 2 out of 6 $(1, 2, 3, 4, 5, 6)$

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21	22	23	24	25	26
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51	52	53	54	55	56
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$$p = \frac{15}{36}$$

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22

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- simultaneous and sequential setting
- equiprobable model is usually OK for both settings

Outline

Sequence of Coin Tosses

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- tossing a coin n times

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- tossing a coin n times
- outcome: sequence of n bits

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- tossing a coin *n* times
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- $\sum_{k\in[0.4n,0.6n]} \binom{n}{k}/2^n$

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