Starting to Count

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Outline

Why counting

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- Important application: counting number of steps of algorithms
- Important application: computing probabilities

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- Applying pigeonhole principle

Real Life Example, Preview

wikimedia.org

- Suppose a country, a state or a region introduces a new format of a license plate

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- Will we have enough plates for everyone?

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Rule of Sum

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If there are k objects of the first type and there are n objects of the second type, then there are $n + k$ objects of one of two types
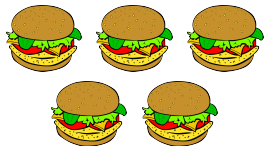
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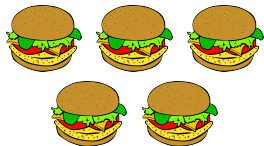
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Burger places

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7+5=12 places to eat in total
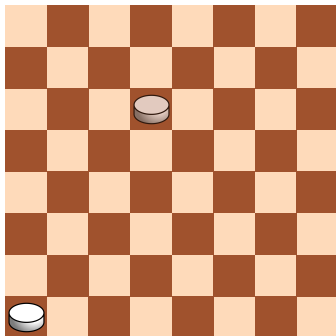
Piece on a Chessboard

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A piece stays in the bottom left corner of a chessboard. In one move it can move one step to the right or one step up. How many moves are needed to get to the position on the picture?
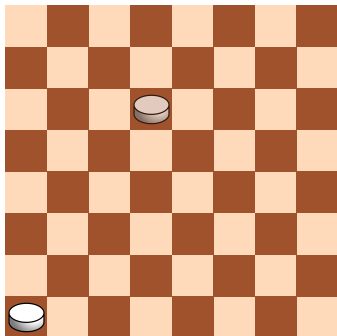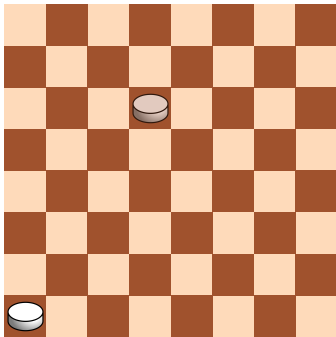
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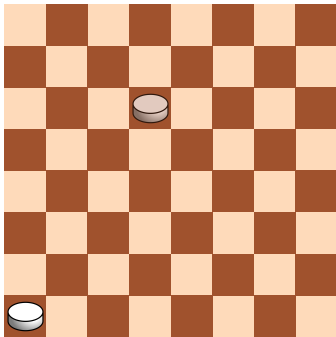

We have seen this problem in the first course

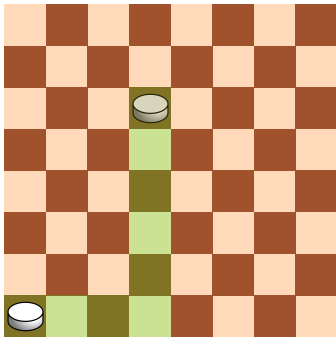# Piece on a Chessboard
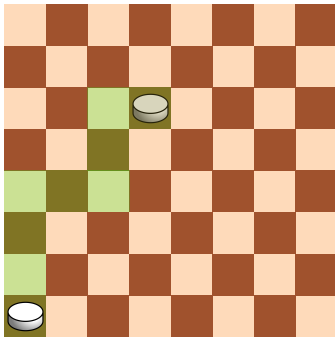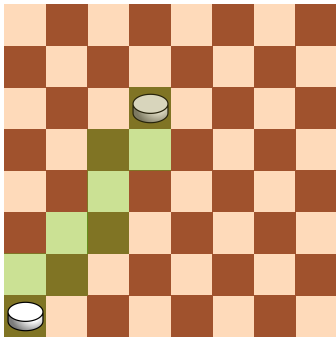
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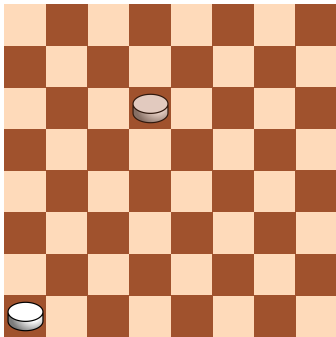

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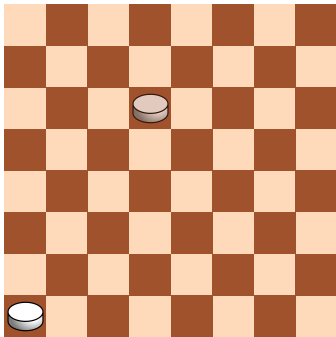

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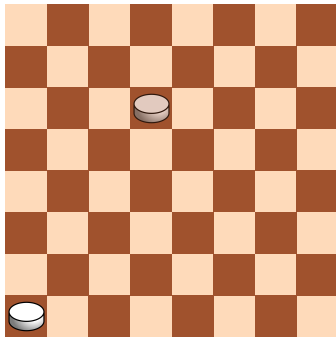
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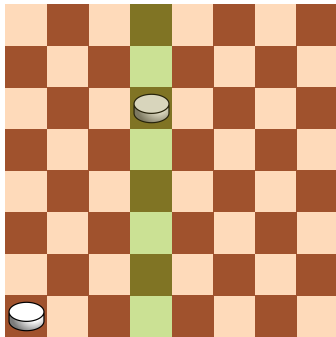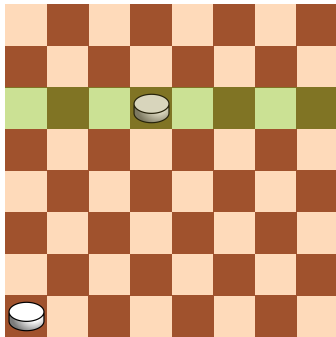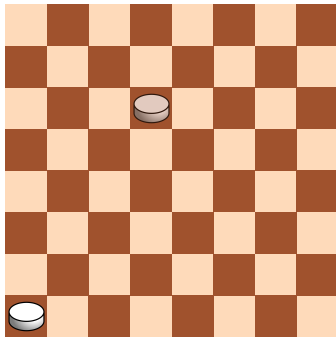
3. To get to the row 6 we need 5 moves up

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3. To get to the row 6 we need 5 moves up

4. In total we need 3+5=8 moves

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- The answer is 7!
- What happened? Number 6 causes problems!

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Rule of Sum Revisited

Rule of Sum

If there are k objects of the first type and there are n objects of the second type, then there are $n + k$ objects of one of two types

- Important lesson: in the rule of sum no object should belong to both classes!

Outline

Convenient Language: Sets

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- Repetitions in the list of elements are not important:
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- We will not encounter these difficulties in the course and will not discuss them

Venn Diagrams

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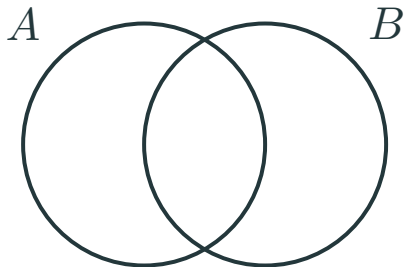
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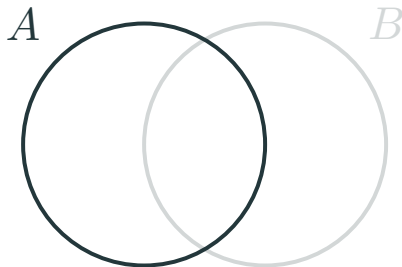
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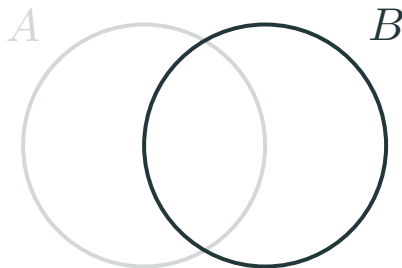
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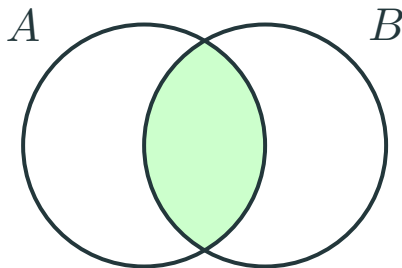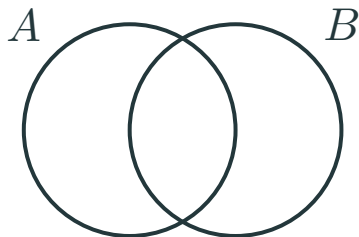
Useful Notations

- Suppose we have two sets A and B

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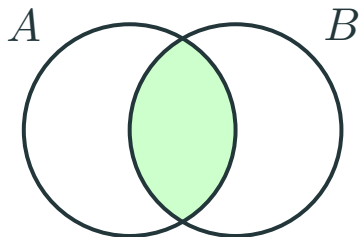
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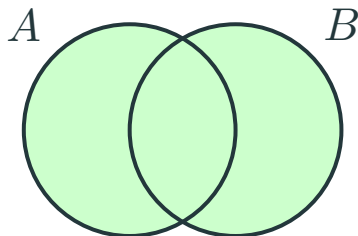
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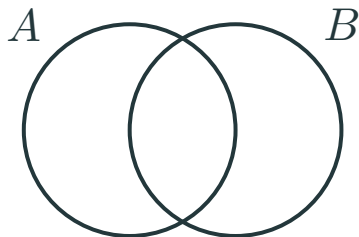
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- The number of elements in the set A is $|A|$ (can be infinite)

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Rule of Sum in the Set Language

Rule of Sum

If there is a set A with k element, a set B with n elements and these sets do not have common elements, then the set $A \cup B$ has $n + k$ elements
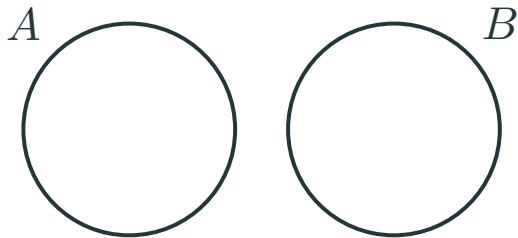
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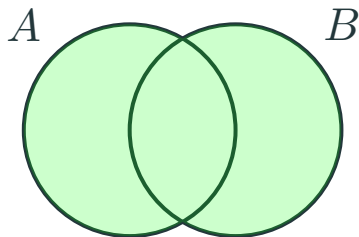
Generalized Rule of Sum
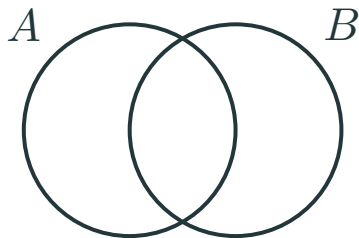
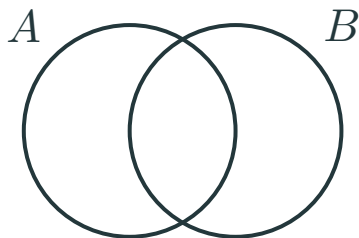But what if we want to count $|A \cup B|$ in the setting below?

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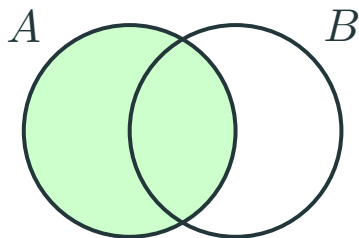But what if we want to count $|A \cup B|$ in the setting below?
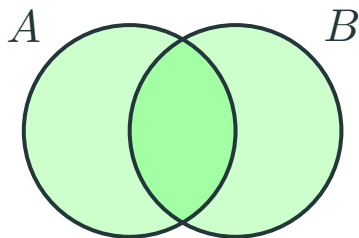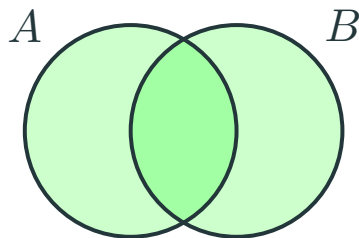
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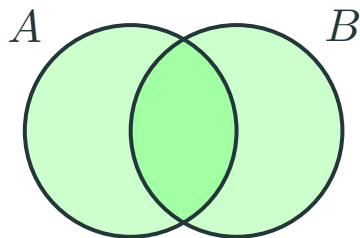# Generalized Rule of Sum

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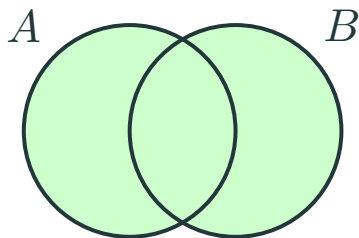
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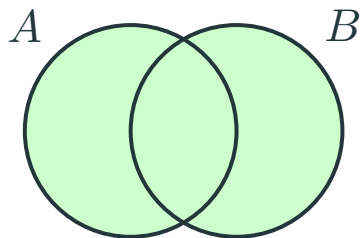# Generalized Rule of Sum

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- This gives the right result:
  $|A \cup B| = |A| + |B| - |A \cap B|$

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- But even rule of sum can be tricky
- Next we will see how to build something more involved from the basic building blocks