Markov's Inequality

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Outline

From Expectation to Probability

Markov's Inequality

Application to Algorithms

Expectation vs. Probability

- In this week we have introduced random variables and their expected values

Expectation vs. Probability

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- We will see now how these notions can help in studies of probabilities of events

Lottery Budget

Problem

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- Assume the contrary: the probability to win 500 dollars or more is at least 0.01
- Denote the number of tickets sold by n
- Then the budget of the lottery is $10n$ dollars
- $10n \times 0.4 = 4n$ dollars are spent on the prizes

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- We arrived into contradiction and the problem is solved

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- The inequality allows to use expected value to bound probability of certain events
- For the proof it is convenient to rewrite the inequality:

$$a \times \Pr[f \geq a] \leq \mathsf{E} f$$

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- So the average value $\mathsf{E}g$ of g is less or equal than the average value $\mathsf{E}f$ of f:

$$\mathsf{E}g \leq \mathsf{E}f$$

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- Thus E$g = a \times \Pr[f \geq a]$

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- So
$$\mathsf{E}f \geq \mathsf{E}g = a \times \Pr[f \geq a]$$
- We have shown Markov's inequality

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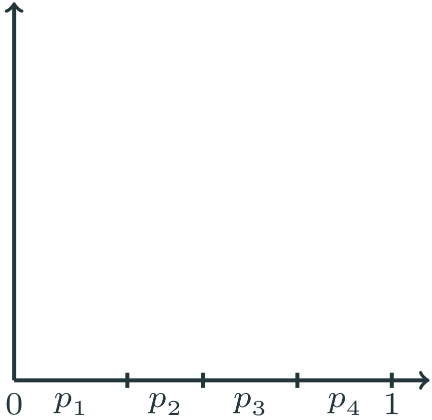

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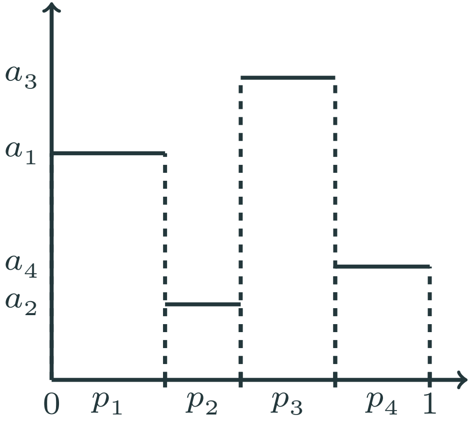

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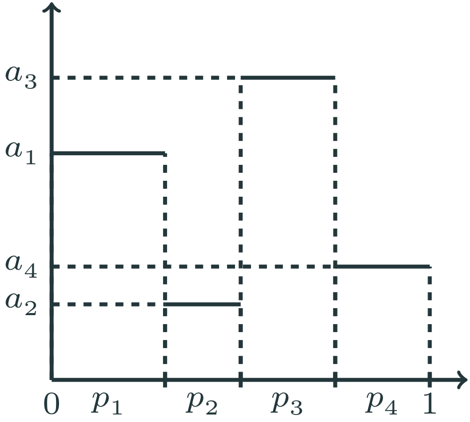

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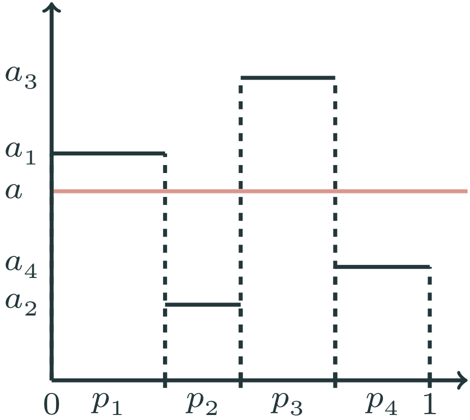

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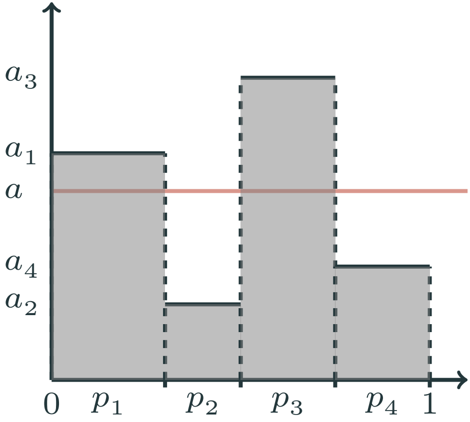

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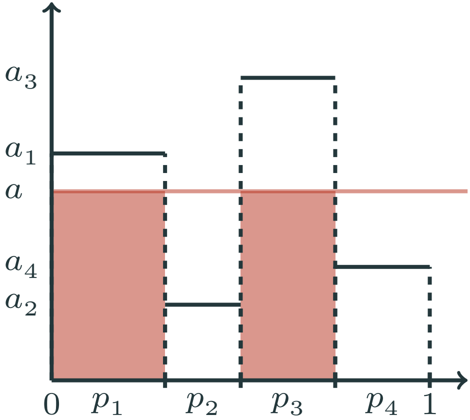

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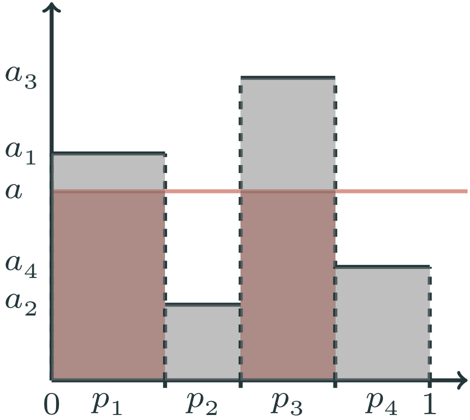

- $\mathsf{E}f$ is the area of the gray region
- $a \times \Pr[f \geq a]$ is the area of a red region
- The gray region is large and the inequality follows

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Suppose there is a randomized algorithm that runs on average in time, say, n^2, where n is the size of input. The algorithm outputs the correct answer. Construct another randomized algorithm that always stops in time cn^2 for some constant c and makes a mistake with probability at most 10^{-3}

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- We will apply Markov inequality

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- Here is a new algorithm
- Run the original algorithm for $10^3 n^2$ steps
- If it stops, we also stop
- If not, stop and output, say, 0

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The probability that the original algorithm does not stop after $10^3 n^2$ number of steps is at most 10^{-3}

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- Indeed, this probability is $\Pr[f \geq 10^3 n^2]$
- By Markov's inequality it is bounded by

$$\Pr[f \geq 10^3 n^2] \leq \frac{\mathsf{E} f}{10^3 n^2} = \frac{n^2}{10^3 n^2} = 10^{-3}$$

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- Allow us to study quantitative aspects of randomness
- Allow us to apply many analytic tools to study probability
- Expected value is one of the main characteristics of a random variable
- On one side, expectation bears a lot of information of a random variable
- On the other side, expectation has very convenient mathematical properties