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New Shapes Solve Infinite Pool-Table Problem | Quanta Magazine

By Kevin Hartnett August 8, 2017

7–8 minutes

Strike a billiard ball on a frictionless table with no pockets so that it never stops bouncing off the table walls. If you returned years later, what would you find? Would the ball have settled into some repeating orbit, like a planet circling the sun, or would it be continually tracing new paths in a ceaseless exploration of its felt-covered plane?

These kinds of questions occurred to mathematical minds centuries ago, in relation to the long-term trajectories of real objects in outer space, and for nearly that long they've seemed impossible to determine exactly. What will a bouncing ball be up to a billion years from now? It's as hard to answer as it sounds.

More recently, though, mathematicians have achieved a succession of stunning breakthroughs. One of the latest results, yet to be published, describes a new category of what are known as "optimal" billiard tables — shapes whose particular angles make it possible to understand every billiard path that could occur within them. The newfound shapes are among a handful of optimal billiard tables ever discovered, and part of an even more select group of quadrilaterals with that property.

"They are like these rare jewels," said Curt McMullen (opens a new tab), a mathematician at Harvard University and a co-author of the work along with Alex Wright (opens a new tab) of Stanford University, Ronen Mukamel (opens a new tab) of Rice University and Alex Eskin (opens a new tab) of the University of Chicago.

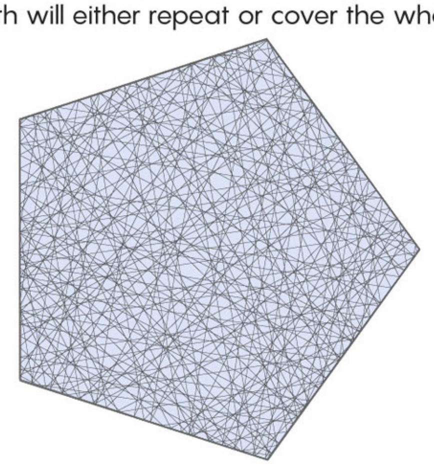
To find these jewels, these four mathematicians used an elegant set of methods that allow mathematicians to reimagine the claustrophobic, rebounding world of a billiard table as an elegant universe of smooth curves arcing unimpeded through space. There, the far-out future of the billiard path can be apprehended at a glance — while at the same time, perfect billiard tables end up serving as clues about the nature of the exotic higher-dimensional space in which they appear.

The Shape That Wouldn't Go Away

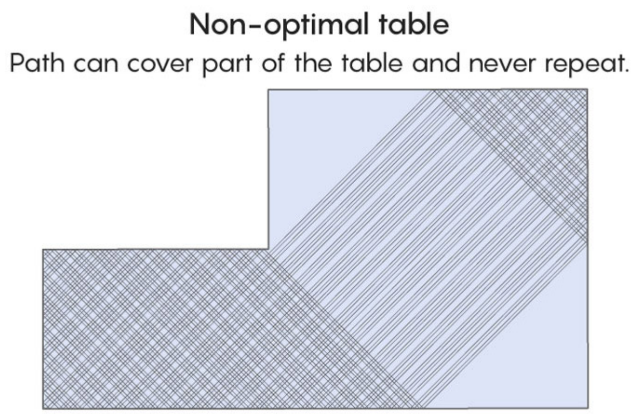
When you set a ball in motion on a billiard table, it may seem as if anything is possible, but when a table has optimal dynamics, only two things truly are. The first is complete chaos, which is to say that the ball's path will cover the entire table as time wears on. The second is periodicity — a repeating path like a ball pinging back and forth between two sides.

Optimal table

Every path will either repeat or cover the whole table.



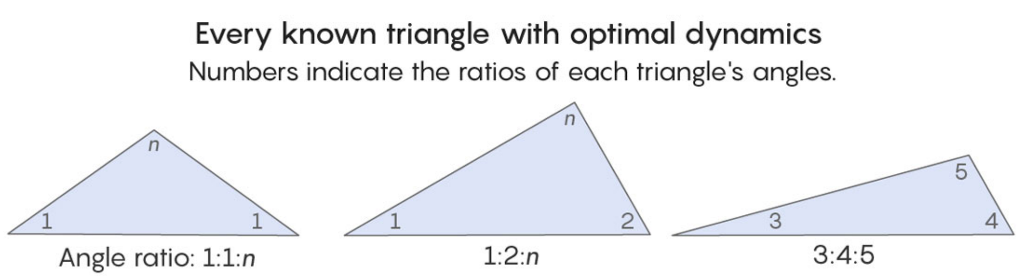
In tables without optimal dynamics, a wider range of possibilities exists, which makes the full analysis of all possible paths impossible: A ball could end up bouncing chaotically in one part of the table forever, never retracing its path, but also never covering the whole table.

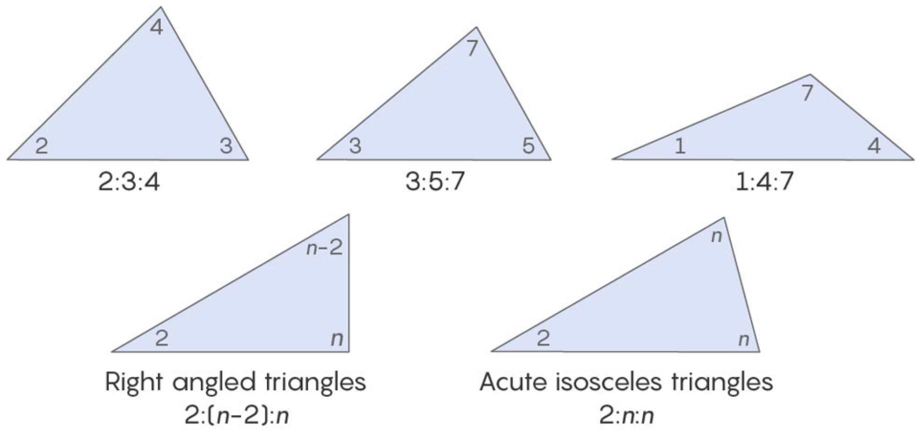


If you're given a polygon, there's no easy way to tell if it has optimal dynamics. For this reason, many basic questions about billiard paths remain unanswered. For example, is there at least one periodic trajectory within every possible triangle? Mathematicians can't say, though they can point to the obstacle.

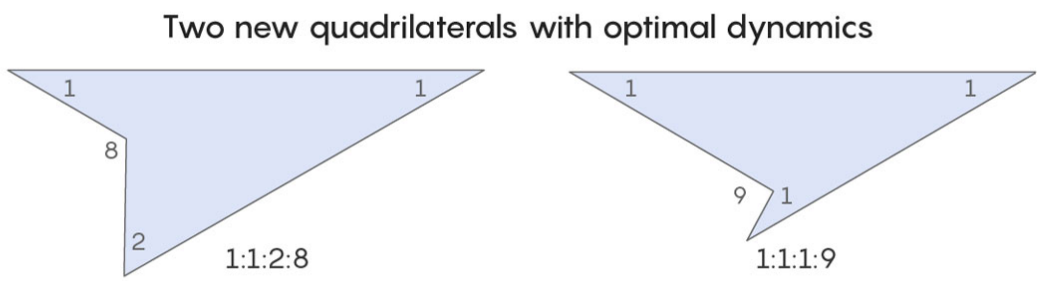
"There's a lot that's not known about obtuse triangles," McMullen said, adding, "Well, don't let me get off on a rant about triangles."

What mathematicians do know is that there are at least eight kinds of triangles with optimal dynamics; the first was discovered in 1989 and the last in 2013. Whether there are more is anyone's guess. "That may be the last example," McMullen said.





The new article identifies two new families of quadrilaterals with optimal dynamics: quadrilaterals whose angle ratios are 1:1:1:9 and 1:1:2:8. Both resemble darts, with two fins in the back and a point up front, and together they're the first quadrilaterals with optimal dynamics to be found in a decade.



Wright was the first to glimpse these shapes. In 2014 he was at Stanford University, collaborating with Maryam Mirzakhani, who that year became the first woman to win a Fields Medal (math's highest honor) and who recently died of cancer at age 40. The two were, in a sense, trying to prove that shapes like these quadrilaterals don't exist. Wright tried to rule out one example after another by showing that the existence of each particular object would imply a contradiction. But then he came across a shape he just couldn't make go away.

A Straight-Line Solution

When he began this work, Wright was not actually thinking about quadrilaterals, and he had no particular interest in billiards. Instead, he and Mirzakhani were exploring the properties of a mathematical area called moduli space, which has played a central role in a wide range of mathematical discoveries over the last 50 years.

"Moduli space is a meeting ground where mathematical thought takes place," McMullen said. "It's related to string theory, it's an avatar, an endless source of inspiration."

In mathematics, it's often beneficial to study classes of objects rather than specific objects — to make statements about all squares rather than individual squares, or to corral an infinitude of curves into one single object that represents them all.

"This is one of the key ideas of the last 50 years, that it is very convenient to not study objects individually, but to try to see them as a member of some continuous family of objects," said Anton Zorich (opens a new tab), a mathematician at the Institute of Mathematics of Jussieu in Paris and a leading figure in dynamics.

Moduli space is a tidy way of doing just this, of tallying all objects of a given kind, so that all objects can be studied in relation to one another.

Imagine, for instance, that you wanted to study the family of lines on a plane that pass through a single point. That's a lot of lines to keep track of, but you might realize that each line pierces a circle drawn around that point in two opposite places. The points on the circle serve as a kind of catalog of all possible lines passing through the original point. Instead of trying to work with more lines than you can hold in your hands, you can instead study points on a ring that fits around your finger.

"It's often not so complicated to see this family as a geometric object, which has its own existence and own geometry. It’s not so abstract,” Zorich said.

Just as the circle can be (roughly) thought of as the moduli space of lines passing through a point, there is a moduli space that relates to billiards by a roundabout process.

Start with a polygon whose billiard paths you’re interested in, and whose angles are rational multiples of pi (a stipulation without which the following steps won’t work). Then start reflecting that polygon over its edges in a process mathematicians call “unfolding.” Continue this reflection process until any further reflection would give you a copy of a polygon you already have.