Chapter 9
Teaching Mathematics for a Growth Mindset

My goal in writing this book has been to give math teachers, leaders, and parents a range of teaching ideas that will enable students to see mathematics as an open, growth, learning subject and themselves as powerful agents in the learning process. I realize in writing this, my last chapter, that we have been on quite a journey, moving all the way from the ways we think about children's potential to the forms of assessment that can create responsible, self-regulating learners. In this chapter I will provide a set of teaching ideas, drawing from throughout the book, that can help you create and maintain a growth mindset mathematics classroom. This chapter offers a shorter summary of many of the ideas from the book, pulled together to give a more concise guide to setting up a growth mindset mathematics class.

Encourage All Students

Figure 9.1 Inspiring all math learners

Setting Up Classroom Norms

Students come into class unsure of what expectations teachers will have for them. The first days of class and even the first hours of the first day are a great time to establish classroom norms. I often start my own classes just telling students what I do and do not value. I tell them that:

- I believe in every one of them, that there is no such thing as a math brain or a math gene, and that I expect all of them to achieve at the highest levels.
- I love mistakes. Every time they make a mistake their brain grows.
- Failure and struggle do not mean that they cannot do math—these are the most important parts of math and learning.
- I don't value students' working quickly; I value their working in depth, creating interesting pathways and representations.
- I love student questions and will put these onto posters that I hang on the walls for the whole class to think about.

But all of these statements are just words—they are important words, to be sure, but they will be worthless if the students do not see the words supported by their teachers' actions.

We shared on Youcubed the seven most important norms to encourage on the first days of class and throughout the year, and the chapters in this book have reviewed ways to establish each. Some teachers have found it helpful to put up our Youcubed poster on classroom walls at the start of class (see Exhibit 9.2 and Setting up Positive Norms in Math Class in Appendix B.).

Positive Norms to Encourage in Math Class

1. Everyone Can Learn Math to the Highest Levels.

   Encourage students to believe in themselves. There is no such thing as a “math person.” Everyone can reach the highest levels they want to, with hard work.

2. Mistakes Are Valuable.

   Mistakes grow your brain! It is good to struggle and make mistakes.

3. Questions Are Really Important.

   Always ask questions, always answer questions. Ask yourself: why does that make sense?

4. Math Is about Creativity and Making Sense.

   Math is a very creative subject that is, at its core, about visualizing patterns and creating solution paths that others can see, discuss, and critique.

5. Math Is about Connections and Communicating.

   Math is a connected subject, and a form of communication. Represent math in different forms—such as words, a picture, a graph, an equation—and link them. Color code!

6. Depth Is Much More Important Than Speed.

   Top mathematicians, such as Laurent Schwartz, think slowly and deeply.

7. Math Class Is about Learning, Not Performing.

Math is a growth subject; it takes time to learn, and it is all about effort.

Exhibit 9.2

As well as telling students about norms and expectations, I find it valuable for students to communicate their own desired norms for working in groups together. Before any students work on math in a group with others, I ask them to discuss in small groups what they do and do not appreciate from other students, as I reviewed in Chapter Seven, and produce posters of these preferences. This is a worthwhile activity, as it helps students enact positive norms knowing they are shared by their peers, and teachers can refer students back to the posters later if good group work behavior needs to be reestablished.

The Railside teachers I discussed in Chapter Seven encouraged good group work very carefully, teaching students how to work well in groups—listening to each other, respecting each other, and building on each other's ideas. The teachers decided that in the first 10 weeks of high school they would not focus on the mathematics students learned but on group norms and ways of interacting. The students worked together on math all of the time, but the teachers did not worry about content coverage, only about students' learning of respectful group work. This careful teaching of good group work was reflected in the students' impressive mathematics achievements in the four years of high school (Boaler & Staples, 2005).

The Participation Quiz

My favorite strategy for encouraging good group work—a strategy that can be used early and often—is to ask students to take a participation quiz. The authors who conceptualized complex instruction (Cohen & Lotan, 2014) recommend that the participation quiz is graded. This does not involve grading individuals, which gives a negative fixed message; rather, it involves grading the behavior of groups. But the participation quiz doesn't have to end with a grade; it just needs to give students a strong message that it matters how they interact and that you are noticing. I really like this grouping strategy; I have taught it to groups of teachers who later told me that it quickly transformed the ways students worked in groups.

To run a participation quiz, choose a task for students to work on in groups, then show them the ways of working that you value. For example, the slides shown in Exhibits 9.2 and 9.3 come from the highly successful Railside teachers. In the first, the teachers highlight mathematical ways of working that they value. With younger children a much smaller list could be used.

Participation Quiz Mathematical Goals

Your group will be successful today if you are …

- Recognizing and describing patterns
- Justifying thinking and using multiple representations
- Making connections between different approaches and representations
- Using words, arrows, numbers, and color coding to communicate ideas clearly
- Explaining ideas clearly to team members and the teacher
- Asking questions to understand the thinking of other team members
- Asking questions that push the group to go deeper
- Organizing a presentation so that people outside the group can understand your group's thinking

No one is good at all of these things, but everyone is good at something. You will need all of your group members to be successful at today's task.

Source: From Carlos Cabana.

Exhibit 9.2

Participation Quiz Group Goals

During the participation quiz, I will be looking for ...

- Leaning in and working in the middle of the table
- Equal air time
- Sticking together
- Listening to each other
- Asking each other lots of questions
- Following your team roles

Source: From Carlos Cabana.

Exhibit 9.3

The second focuses on the ways of interacting that lead to good group work.

These could be presented on posters in the room instead of on slides. Once you have shown these to students, you can start them working. As they work together in groups, walk around the room watching group behavior, writing down comments. To do this you will need a piece of paper, or an area on the white board, divided into sections, with a space for each group. For example, with 32 students working in 8 groups of 4:

1.	2.	3.	4.
5.	6.	7.	8.

As you circulate and take notes, quote students' actual words when they are noteworthy. Some teachers do this publicly, writing comments onto a white board at the front of the room. Others use paper clipped to a board. At the end of the lesson, you should have a completed chart and can assign the student groups a grade, or give students feedback on their group work without the grade. The following is an example of a teacher's grading of a participation quiz:

Quick start All working together	All four working Checking each other's work	"How do you guys think?"	Talking about clothes

Great discussions Staying together “Let's go round and find out how everyone sees the shape” A+	Asking nice questions: “How would that work with another number?” Good group roles A+	Building shape in middle of table Checking with each other A+	Off task—group asked to stop Individual working, no discussion B
Trying ideas Asking questions of each other Talking about work A	Slow task—off task Building shape in the middle Checking ideas Good discussion A	“Can anyone see a different way?” Nice explaining to each other Great debate on meaning A+	Started well Quiet reading All focused all the time Asking nice questions A+

The notes do not have to be detailed, but they will help students understand what you value and will help them become much more attentive to the ways they interact with other students. My Stanford students and many teachers whom I have taught this method enjoy participation quizzes, comically “leaning in” and asking deep questions when I linger by their tables with my notepad! Students also have fun, at the same time realizing much more clearly what they need to do to engage well in a group.

I am a great believer in the participation quiz. Teachers who have used it in classes in which they previously had a lot of trouble encouraging good group work have been stunned at the positive change in students. Almost overnight, students start asking each other good questions and thinking about the equal involvement of different team members. When students are working well in groups, respecting each other and asking good questions, then classrooms are a great place to be in, for students and teachers.

Believe in All of Your Students

I have always known how important it is that students know their teacher believes in them; I knew this as a teacher and more recently became more acutely aware of it as a parent. When my daughter was five, she realized the teacher of her class in England was giving other students harder math problems, and she came home to me and asked why. When she realized that the teacher did not think she had potential—and sadly, this was true; the teacher had decided she had limited ability—her self-belief was shattered, and she developed a terribly fixed mindset that damaged her learning and confidence for a long time afterward. Now, some years later, after a lot of work from her parents and some wonderful teachers, she is transformed: she has a growth mindset and loves math. Despite the fact that the teacher never said to my daughter that she did not believe in her, she managed to communicate that message loud and clear, and this was understood by my daughter even at the young age of five.

The school that my daughter attended in England put students into ability groups in second grade, but they stopped this practice after reading the research evidence and learning about the strategies for teaching heterogeneous groups. After they made this change, the principal wrote to tell me it had transformed math classes and raised achievement across the school. If students are placed into ability groups, even if they have innocuous names such as the red and blue groups, students will know, and their mindsets will become more fixed. When children were put into ability groups in my daughter's school, children from the lower groups came home saying "All the clever children have gone into another group now." The messages the students received about their potential as learners in general (not just about math) were devastating for them. One of the first steps we need to take, as a nation, is to move away from outdated methods of fixed mindset grouping and communicate to all students that they can achieve.

The importance of students thinking their teacher believes in them was confirmed in a recent study that had an extremely powerful result (Cohen & Garcia, 2014). Hundreds of students were involved in this experimental study of high school English classes. All of the students wrote essays and received critical diagnostic feedback from their teachers, but half the students received a single extra sentence on the bottom of the feedback. The students who received the extra sentence achieved at significantly higher levels a year later, even though the teachers did not know who received the sentence and there were no other differences between the groups. It may seem incredible that one sentence could change students' learning trajectories to the extent that they achieve at higher levels a year later, with no other change, but this was the extra sentence:

“I am giving you this feedback because I believe in you.”

Students who received this sentence scored at higher levels a year later. This effect was particularly significant for students of color, who often feel less valued by their teachers (Cohen & Garcia, 2014). I share this finding with teachers frequently, and they always fully understand its significance. I do *not* share the result in the hope that teachers will add this same sentence to all of their students' work. That would lead students to think the sentence was not genuine, which would be counterproductive. I share it to emphasize the power of teachers' words and the beliefs they hold about students, and to encourage teachers to instill positive belief messages at all times.

Teachers can communicate positive expectations to students by using encouraging words, and it is easy to do this with students who appear motivated, who learn easily, or who are quick. But it is even more important to communicate positive beliefs and expectations to students who are slow, appear unmotivated, or struggle. It is also important to realize that the speed at which students appear to grasp concepts is not indicative of their mathematics potential (Schwartz, 2001). As hard as it is, it is important to not have any preconceptions about who will work well on a math task in advance of their getting the task. We must be open at all times to any student's working really well. Some students give the impression that math is a constant struggle for them, and they may ask a lot of questions or keep saying they are stuck, but they are just hiding their mathematics potential and are likely to be suffering from a fixed mindset. Some students have had bad math experiences and messages from a young age, or have not received opportunities for brain growth and learning that other students have, so they are at lower levels than other students, but this does not mean they cannot take off with good mathematics teaching, positive messages, and, perhaps most important, high expectations from their teacher. You can be the person who turns things around for them and liberates their learning path. It usually takes just one person —a person whom students will never forget.

Value Struggle and Failure

Teachers care about their students and want them to do well, and they know that it is important for students to feel good about math. Perhaps it is this understanding that has led to most math classrooms across the United States being set up so that students get most of their work correct. But the new brain evidence tells us this is not what students need. The most productive classrooms are those in which students work on complex problems, are encouraged to take risks, and can struggle and fail and still feel good about working on hard problems. This means that mathematics tasks should be difficult for students in order to give students opportunities for brain growth and making connections, but it doesn't mean just increasing the difficulty, which would leave students frustrated. Rather, it means changing the nature of tasks in math classrooms—giving more low floor, high ceiling tasks. As discussed in Chapter Five, having a low floor means that anyone can access the ideas. Having a high ceiling means that students can take the ideas to high levels.

As well as changing tasks, teachers should communicate frequently that struggle and failure are good. Many of the students I teach at Stanford have achieved at high levels all their lives and received a lot of damaging fixed mindset feedback, being told that they are "smart" at frequent intervals. When they encounter harder work at Stanford and don't receive an A for everything, some of them fall apart, feeling devastated and questioning their ability. When they work on material that causes them to struggle—actually a very worthwhile place to be for learning—they quickly lose confidence and start to doubt that they are "smart" enough to be at Stanford. These are the students who have been brought up in a performance culture, for whom struggle and failure have never been valued. The students in my freshman class tell me how important the ideas we learn have been to them, how learning that struggle is good has kept them in math and engineering classes and has stopped them dropping out of STEM pathways.

We must work hard to break the myth of "effortless achievement," pointing out that all high achievers have worked hard and failed often, even those thought of as "geniuses," as Chapter Four discussed. We must also resist valuing "effortless achievement"—praising students who are fast with math. Instead, we should value persistence and hard thinking. When students fail and struggle it does not mean anything about their math potential; it means that their brains are growing, synapses are firing, and new pathways are being developed that will make them stronger in the future.

Give Growth Praise and Help

When Carol Dweck worked with preschool children, she found that some children were persistent when they experienced failure and wanted to keep trying, whereas others would give up easily and request that they repeat tasks that were easy for them. These persistent and nonpersistent mindset strategies were evident in children who were only three and four years old.

When researchers then conducted role-plays with the children and asked them to pretend to be an adult responding to their work, the persistent children role-played adults focusing on strategies, saying that the children would be more successful with more time or a different approach. The nonpersistent children role-played an adult saying that the child could not finish the work and so should sit in their room. The nonpersistent children seemed to have received feedback that told them they had personal limitations and that failure was a bad thing (Gunderson et al., 2013). This study, as well as many other mindset studies (Dweck, 2006a, 2006b; Good, Rattan, & Dweck, 2012), tell us that the forms of feedback and praise we give students are extremely important. We know that one way we aid and abet students in developing a fixed mindset is by giving them fixed praise—telling them, in particular, that they are smart. When students hear that they are smart, they feel good at first, but when they struggle and fail—and everyone does—they start to believe they are not so smart. They continually judge themselves against a fixed scale of "smartness," and this will be damaging for them, even if they get a lot of positive smart feedback, as the case of the Stanford students illustrates.

Instead of telling children that they are smart or clever, teachers and parents should focus on the particular strategies children have used. Instead of saying "You are so smart," it is fine to say to students something like "It's great that you have learned that," or "I love how you are thinking about the problem." Removing the word "smart" from our vocabulary is difficult to do, as we are all used to referring to people as smart. My undergraduates have really worked on this and now praise people for having good thinking and for being accomplished, learned, hard working, and persistent.

When students get work wrong, instead of saying "That is wrong," look for their thinking and work with it. For example, if students have added $\frac{1}{3}$ and $\frac{1}{4}$ and decided the answer is $\frac{2}{7}$, you could say: "Oh, I see what you are doing; you are using what we know about adding whole numbers to add the top and bottom numbers, but these are fractions, and when we add them we have to think about the whole fraction, not the individual numbers that make up the fraction." There is always some logic in students' thinking, and it is good to find it, not so that we avoid the "failure" idea, but so that we honor students' thinking. Even if children have completed a task that is completely wrong, be careful not to give the idea that the task is too hard for them, as this gives the idea that their ability is limited. Instead focus on strategies, saying such things as "You haven't learned the strategies you need for that yet, but you will soon."

It is important not to provide too much help to students and take away from the cognitive demand of tasks. Guy Brousseau, a French researcher, identified what he called "the didactic contract," which has since been recognized by teachers and researchers worldwide (Brousseau, 1984; Brousseau, 1997). Brousseau describes a common situation in mathematics classes whereby teachers are called over to students who ask for help; the students expect to be helped, and teachers know it is their role to help them, so the teachers break down the problem and make it easier. In doing so they empty the problem of its cognitive demand. Brousseau points out that this is a shared action between teachers and students; they are both playing the roles expected of them, fulfilling the "didactic contract" that has been established in classrooms, which results in students missing the opportunity to learn. Under the contract, students expect to not be allowed to struggle; they expect to be helped, and teachers know *their* role is to help students, so they jump in and help them, often unknowingly robbing them of learning opportunities. Textbook authors are complicit in a similar process, breaking problems down into small parts for students to answer. When my students ask for help, I am very careful not to do the mathematical thinking for them. Instead, I often ask students to draw the problem, which invariably unlocks new ideas for them.

I recently read about a second-grade teacher, Nadia Boria, who offers this response to students when they ask for help: "Let's think about this for a minute. Do you want my brain to grow or do you want to grow your brain today?" (Frazier, 2015).

This is a lovely response, and although teachers have to judge every interaction with their professional knowledge and intuition, knowing when students can handle more struggle and not get discouraged, it is important to remember that *not* helping students is often the best help we can give them.

The norms we set up for students in our math classes, the ways we help and encourage them, and the messages we give them are extremely important, but I cannot emphasize too strongly that giving students growth mindset messages will not help them unless we also show them that math is a growth subject. The remainder of this chapter will focus on the strategies and methods teachers can use to teach students open, growth, creative mathematics.

Opening Mathematics

Teach Mathematics as an Open, Growth, Learning Subject

The majority of mathematics questions that are used in math classrooms and homes are narrow and procedural and require students to perform a calculation. When students spend most of their math time working in this way, it is very hard for them to truly believe that math is a growth subject, as the closed questions communicate the idea that math is a fixed, right-or-wrong subject. It is reasonable for some questions to be narrow, with one right answer, but such questions are not necessary for students to develop a sound mathematical understanding, and they should be the minority of questions, if used at all. Mathematics tasks should offer plenty of space for learning. Instead of requiring that students simply give an answer, they should give students the opportunity to explore, create, and grow.

Any math task can be opened up, and when they *are* opened, many more students engage and learn. Here are four examples of ways to open math tasks:

1. Instead of asking students to answer the question 1/2 divided by 1/4, ask them to make a conjecture about the answer to 1/2 divided by 1/4 and make sense of their answer, including a visual representation of the solution. As I described in Chapter Five, when Cathy Humphreys asked students to solve $1 \div \frac{2}{3}$ she started by saying, "You may know a rule for solving this question, but the rule doesn't matter today, I want you to make sense of your answer, to explain why your solution *makes sense*."
2. Instead of asking students to simplify $\frac{1}{3}(2x + 15) + 8$, a common problem given in algebra class, ask students to find all the ways they can represent $\frac{1}{3}(2x + 15) + 8$ that are equivalent. Figure 9.2 shows examples.
3. Instead of asking students how many squares are in the 100th case, ask them how they see the pattern growing, and to use that understanding to generalize to the 100th case (see Figure 9.3).

$\frac{1}{3}(2x + 15) + 8$	$\frac{2x + 15}{3} + 8$	$\frac{2}{3}x + 5 + 8$
$\frac{2x}{3} + 13$	$\frac{2x + 15 + 24}{3}$	$\frac{1}{3}(2x + 39)$

Figure 9.2 Algebra examples

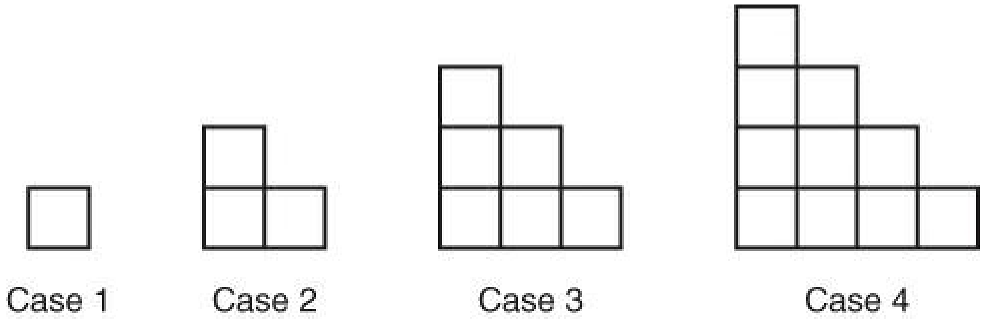


Figure 9.3 Stairs

Any math task can be opened up so that it offers students space for learning, as Chapter Five discussed more fully. For example, you can ask students to discuss:

- Ways of seeing the mathematics
- Ways of representing ideas
- The different pathways through the problem and strategies
- The different methods used: “Why did you choose those methods? How do they work?”

When students are working on open math tasks, they are not only encouraged to see mathematics as a growth subject but also placed in the role of an inquirer. They are no longer finding an answer; they are exploring ideas, making connections, and valuing growth and learning. At the same time they are making these inquiries, they are learning formal mathematics—the methods and formulas set out in curriculum standards. The difference is, students learn standard methods when they encounter a need for them, which gives them motivation and excitement to learn them (Schwartz & Bransford, 1998). As I've emphasized, the most perfect open mathematics questions are those with a low floor and a high ceiling (see the “Tasks” collection on the Youcubed website, http://www.youcubed.org/tasks/). When considering whether a task is open, the most important question to ask, in my view, is whether it offers space for students to learn.

Encourage Students to Be Mathematicians

Mathematicians see their subject as creative, beautiful, and aesthetic. All children can work as mathematicians, and encouraging them to be mini mathematicians can be very empowering. It is important that students engage in proposing ideas—or, to use the mathematical term, making conjectures about mathematics. One of the most amazing third-grade teachers I have ever witnessed is Deborah Ball, now the dean of the school of education at the University of Michigan. Deborah taught her third-grade students to be mathematicians—to be inquirers, and to make conjectures about math. When the class came to a consensus on a mathematical idea, they would say they had a "working definition," which they would return to and refine when they had explored further. In one class period, third-grader Sean made a proposal about the number 6, saying that it can be both even or odd (video is available online: Mathematics Teaching and Learning to Teach, 2010; http://deepblue.lib.umich.edu/handle/2027.42/65013).

His reason for this conjecture was that 6 is made up of an odd number of groups of 2, whereas other even numbers, such as 4 and 8, have even numbers of groups of 2. Many students in the class argued with Sean, returning to the class's working definition of an even number. Most teachers would have told Sean that he was incorrect and moved on, but Deborah was interested in his thinking. What followed was an animated discussion among the students that has captured viewers of many backgrounds and persuasions, including teachers and mathematicians. In the lesson the children became deeply involved in thinking abut Sean's conjecture, and at no point did they ask the teacher to tell them if Sean was right or not, which would have shut down the conversation. Instead, the third graders asked Sean to prove his conjecture, and they offered counter evidence, using many different definitions of an even number to show Sean that 6 was even and not also odd. At some point in the discussion Deborah realized that Sean had proposed something about the number 6, and other numbers that share the quality of having an odd number of multiples of 2, such as 10, that does not have a name in mathematics, and the class decided to call the numbers "Sean numbers." Sean was making an observation that was not wrong; he was pointing out that some numbers have different characteristics. In later discussions in the year the class would explore numbers and casually make reference to the use of "Sean numbers" when they came up. Unlike many third graders, who are turned off math by its procedural presentation, these children loved being able to share their thinking and ideas, and to make conjectures and construct proofs as they came to agree, as a class, on working definitions and propositions, at the same time they were learning formal mathematics. They students were excited to work on conjectures, reasoning, and proof, and they looked to any observer like young mathematicians at work (Ball, 1993).

Some people are shocked by the idea of calling children mathematicians, yet they comfortably refer to children as young artists and scientists. This is because of the pedestal that mathematics is placed on, as I discussed in Chapter Six. We need to counter the idea that only those with many years of graduate mathematics should act as mathematicians. We need to stop leaving the experience of real mathematics to the end, when students are in graduate school, by which time most students have given up on math. There is no better way to communicate to all students that mathematics is a broad, inquiry-based subject that they can all work on than by asking children to be mathematicians.

Teach Mathematics as a Subject of Patterns and Connections

Mathematics is all about the study of patterns. Many people appreciate that they are working with patterns when working on problems such as the one shown in Figure 9.4, where they are asked to extend the pattern.

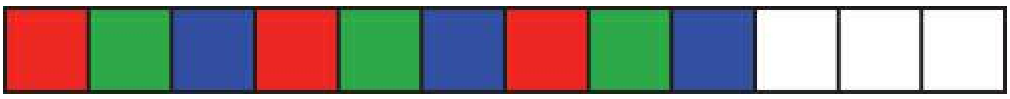


Figure 9.4 Pattern strip

But even when learning arithmetic, or more abstract areas of math, the work of any student is about seeking patterns. I have tried to encourage my own children to see themselves as pattern seekers, and I was pleased recently when my eight-year-old daughter was working on division. She had just been taught the "traditional algorithm" for division, but when she was given questions like this …

$$6\overline{)18} \qquad 7\overline{)35} \qquad 8\overline{)27}$$

$$8\overline{)96} \qquad 6\overline{)72} \qquad 7\overline{)83}$$

…she found that the algorithm was useful only in some cases. After working on some questions, she said—"Oh, I can see a pattern; the dividing loop method" [by which she meant the "traditional algorithm"] "only helps when the first digit is bigger than the number being divided." I am not a big fan of students learning division through the traditional algorithm, as it often prevents students from looking at the whole number and works against place value understanding, but I was pleased that her pattern-seeking orientation meant she was thinking about patterns in the numbers and not blindly following a method. I am not suggesting that the traditional algorithm is not useful; it may be helpful *after* students understand division as one of many division strategies. When they are learning division, students should use methods that encourage their understanding of the numbers involved and the concept of division.

When teachers teach mathematical methods, they are really teaching a pattern—they are showing something that happens all of the time, something that is *general*. When we multiply a number bigger than 1 by 10, then the answer will have a zero. When we divide the circumference of a circle by twice its radius, we always get the number pi. These are patterns, and when students are asked to see mathematics as patterns, rather than methods and rules, they become excited about mathematics. They can also be encouraged to think about the nature of the patterns—what is general about the case? Keith Devlin, a top mathematician and NPR's "math guy," has written a range of excellent books for the public. In one of my favorites, *Mathematics: The Science of Patterns*, Devlin shows the work of mathematicians as being all about the use and study of patterns—arising from what he describes as the natural world or the human mind. Devlin quotes the great mathematician W. W. Sawyer saying that "mathematics is the classification and study of all possible patterns" and that patterns include "any kind of regularity that can be recognized by the mind." Devlin agrees, saying "Mathematics is not about numbers, but about life. It is about the world in which we live. It is about ideas. And far from being dull and sterile as it is so often portrayed, it is full of creativity" (Devlin, 2001).

Invite students into the world of pattern seeking; give them an active role in looking for patterns in all areas and all levels of math.

In Chapter Three I introduced Maryam Mirzakhani, a mathematician and a colleague of mine at Stanford. She hit the news headlines worldwide when she became the first woman to win the Fields Medal. As mathematicians talked about the tremendous contributions she has made to the advancement of mathematics, they talked about the ways her work connects many areas of mathematics, including differential geometry, complex analysis, and dynamical systems. Maryam reflected: “I like crossing the imaginary boundaries people set up between different fields—it's very refreshing …there are lots of tools, and you don't know which one would work. It's about being optimistic and trying to connect things.” This is a mindset I would love all students of mathematics to have.

When students make and see connections between methods, they start to understand real mathematics, and they enjoy the subject much more. This is a particular imperative for getting more girls into STEM fields, as discussed in Chapter Six. Curriculum standards often work against connection making, as they present mathematics as a list of disconnected topics. But teachers can and should restore the connections by always talking about and valuing them and asking students to think about and discuss connections. The mathematical connections video we provide on Youcubed shows the ways fractions, graphs, triangles, rates, Pythagoras' theory, tables, graphs, shapes, slope, and multiplication are all connected under the theme of proportional reasoning (Youcubed at Stanford University, 2015c; http://www.youcubed.org/tour-of-mathematical-connections/). We made this video to show the connections between mathematical areas that students may not think exist, and teachers have found it helpful to show it to their students, to help them think about connections. From there students should be encouraged to explore and see mathematical connections in many different ways.

Here are some examples of ways to highlight connections in mathematics.

- Encourage students to propose different methods to solve problems and then ask them to draw connections between methods, discussing for example, how they are similar and different or why one method may be used and not another. This could be done with methods used to solve number problems, such as those shown in Figure 5.1, in Chapter Five.
- Ask students to draw connections between concepts in mathematics when working on problems. For example, consider the two mathematics problems in Exhibit 9.4 and Figure 9.5.

Dog Biscuits

How many ways can you make two groups of 24 dog biscuits?

How many ways can you equally group 24 dog biscuits?

Show your results in a visual representation that shows all of the combinations.

Exhibit 9.4

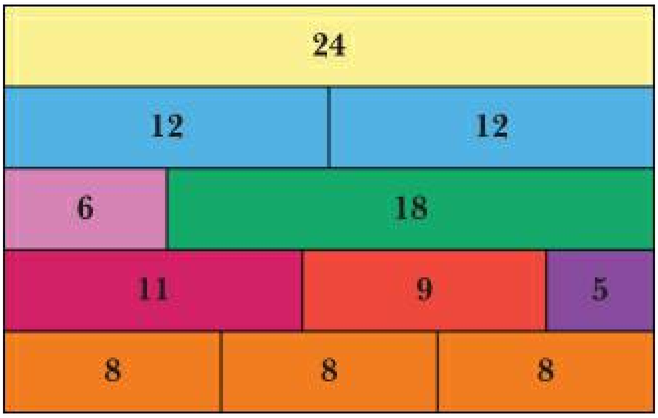


Figure 9.5 Dog biscuits solution

Teachers can encourage students to produce more than one representation and to connect the numbers in their solutions to their diagrams, allowing for the use of different brain pathways. Some students might like to use grid paper, others a number line, others may use multilink cubes or other small objects. Teachers can ask students to think about different methods that may be used when considering equal grouping—in particular, addition and multiplication—and to think about how they are related.

In the different activities in Exhibit 9.5, students are asked to specifically focus on different areas of mathematics and the connections between them. Successful students are not those who think of mathematics as a series of disconnected topics—a view held by many students. Rather, they are the students who see mathematics as a set of connected ideas (Program for International Student

Assessment [PISA], 2012), a viewpoint that teachers need to actively encourage, especially if textbooks give the opposite impression. Connected mathematics is inspiring and appealing to students, and all teachers can enable students to see the connected nature of mathematics.

Highlighting Mathematical Connections

Show the fractions 3/4, 6/8, and 12/16 on a graph.

Show the fractions as similar triangles.

What is similar and different about the different fraction representations—as numbers, a graph, and triangles? Can you color code features of each representation so that they show up in the same color on the different representations?

Exhibit 9.5

Teach Creative and Visual Mathematics

In my own teaching of mathematics, I encourage student creativity by posing interesting challenges and valuing students' thinking. I tell students I am not concerned about their finishing math problems quickly; what I really like to see is an interesting representation of ideas, or a creative method or solution. When I introduce mathematics to students in this way, they always surprise me with their creative thinking.

It is very important to engage students in thinking visually about mathematics, as this gives access to understanding and to the use of different brain pathways. Amanda Koonlaba, a fourth-grade teacher who connects art to core school subjects, including mathematics, describes a time when she asked her students what kinds of arts lessons they had enjoyed in their core subjects. She recalls that one student "spoke quietly but enthusiastically, explaining that he loves visual art because creating helps him 'forget the bad' and he needs that 'more than once a week'" (Koonlaba, 2015).

Art and visual representations don't play only a therapeutic and creative role, although both are important. They also play a critical role in opening access to understanding for all students. When I ask students to visualize and draw ideas, I always find higher levels of engagement and opportunities to understand the mathematical ideas that are not present without the visuals. Some students find visual ideas harder than others, but those are the students who will be most helped by using them.

As well as asking students to draw ideas, methods, solutions, and problems, teachers should always ask them to connect visual ideas with numerical or algebraic methods and solutions. Color coding, as I showed in Chapter Five, is a good way to encourage these connections. In the next two examples we see how much color can enhance students' understanding of geometry, fractions, and division; in earlier chapters I showed color coding in algebra and parallel lines. When students learn about angle relationships, they can be asked to color the different angles of a triangle, tear them off, and line them up to see the angle relationship. The visual depiction of the angles will help them remember the relationships.

Understanding of fractions also can be enhanced if students are asked to color code the fractions (see Exhibit 9.6 and Figure 9.6).

Color Coding Brownies

Sam has made a pan of brownies that he wants to cut into 24 equal pieces. He wants to share them equally with 5 of his friends. Partition the pan of brownies and use color coding to show how many Sam and his friends will get.

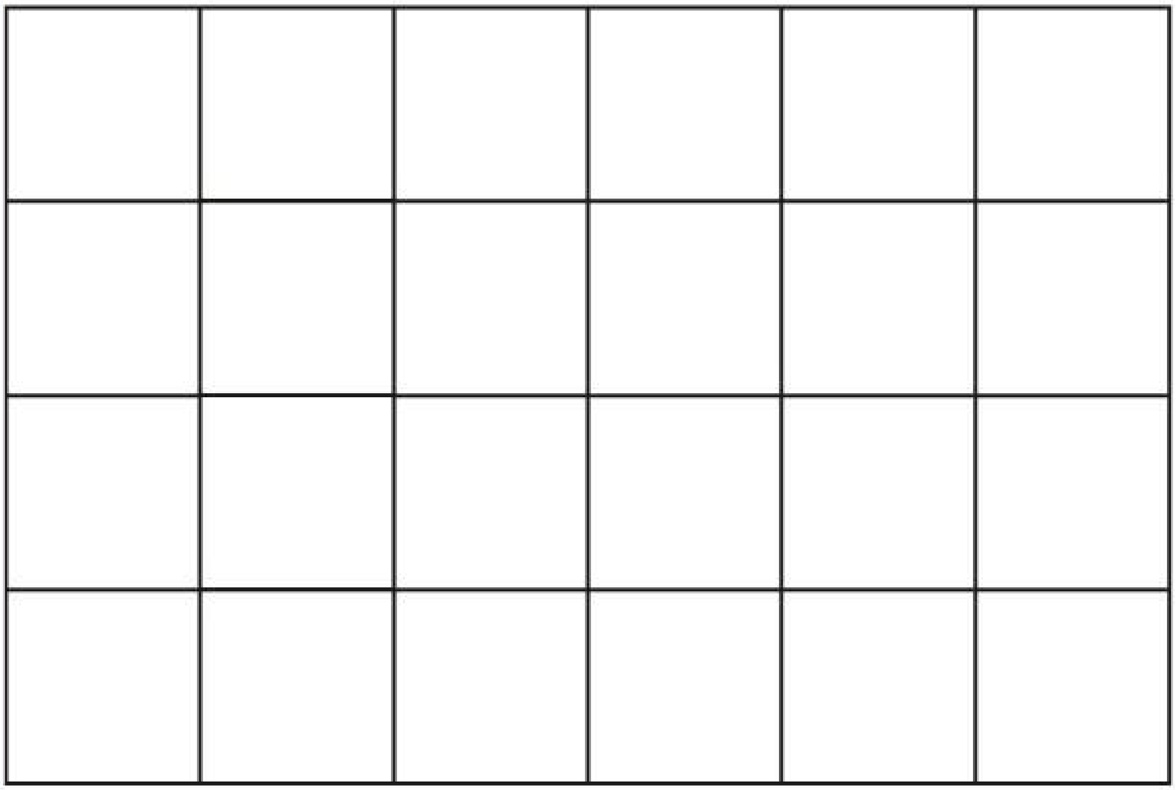


Exhibit 9.6

Sam has made a pan of brownies that he wants to cut into 24 equal pieces. He wants to share them equally with 5 of his friends. Partition the pan of brownies and use color coding to show how many Sam and his friends will get.

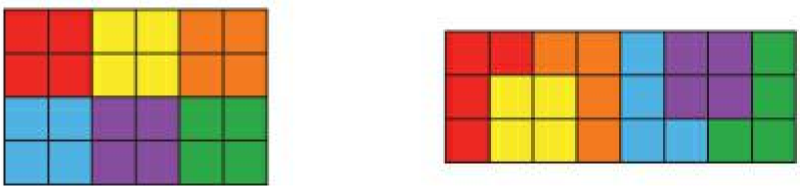


Figure 9.6 Color coding brownies

I particularly like a color coding approach to division created by Tina Lupton, Sarah Pratt, and Kerri Richardson. They propose that students be asked to solve division problems using a division quilt, which helps students really see and understand the partitioning of numbers into equal groups and remainders (see Figure 9.7). They give more details on ways to structure this helpful activity in Lupton, Pratt, and Richardson (2014).

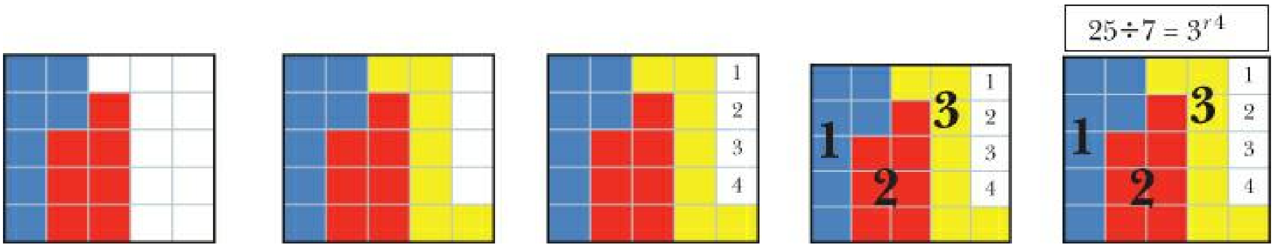


Figure 9.7 Division quilts

Source: Lupton, Pratt, & Richardson, 2014.

Representing mathematical ideas in different ways is an important mathematical practice, used by mathematicians and high-level problem solvers. When mathematicians work, they represent ideas in many different ways—with graphs, tables, words, expressions, and—less well known—drawings and even doodles. Mirzakhani describes thinking about a difficult math problem:

> "You don't want to write down all the details ...But the process of drawing something helps you somehow to stay connected." Mirzakhani said that her 3-year-old daughter, Anahita, often exclaims, "Oh, Mommy is painting again!" when she sees the mathematician drawing. "Maybe she thinks I'm a painter." (Klarreich, 2014)

Whenever I am given a complex mathematics problem to solve, I draw it; it is the best way I know for tackling a hard problem and understanding the mathematics. When I work with students and they are stuck, I also ask them to draw, asking questions such as "Have you tried drawing the problem?" Students who are not used to drawing mathematics may find this a challenge at first, but they can learn to draw, and it will help them. Chapter Eight gives more ideas on ways to engage students in drawing and doodling.

Being willing and able to use representations in mathematical thinking is immensely helpful for students, both in school mathematical work and in life.

Encourage Intuition and Freedom of Thought

I discussed in Chapter Five the ways in which high-level mathematics users—such as Sebastian Thrun, making robots for the Smithsonian—

use intuition to develop mathematical ideas. Leone Burton interviewed 70 research mathematicians to find out about the nature of their work; 58 of them talked about the important role of intuition in their work. Reuben Hersh, in his book *What Is Mathematics Really?*, says that if we "look at mathematical practice, the intuitive is everywhere" (Hersh, 1999).

But what is this thing called intuition? And why are mathematicians engaging in intuition when students rarely if ever do so in classrooms? Teachers can encourage students to use intuition with any math problem simply by asking them what they think would work, before they are taught a method. Opportunities for students to think intuitively occur all through mathematics at every grade level. Elementary teachers could ask students before they teach any method, to work out their own method for solving the problem; for example, how they might find the area of a rug before being given an area formula. In middle or high school we can ask students how they might find the height of an object that is too tall to measure before we teach them methods to do so (see Boaler, Meyer, Selling, & Sun, n.d.). In Chapter Five I talked about the precalculus lesson in which students were asked to conjecture and think intuitively about finding the volume of a lemon, before they were taught calculus. Giving students the opportunity to use intuition is something that teachers can do with a small change in their practice.

When students are asked to use intuition to think about a mathematical idea, they are being invited to think openly and freely. When I asked a group of third-grade children who had learned through number talks what they thought about the number talks, the first thing Dylan said to me was, "You are free, you can do whatever you want. You can take numbers and break them down. . . ." Delia, one of the students featured in *Beyond Measure* (the second documentary from the director of *Race to Nowhere*), spoke in a similar way about her mathematical experiences after she was engaged in inquiry math: "I have a connection with math now. It's like, I'm open, I feel alive, I feel more energetic." In the same film, Niko compares his prior mathematics experience of working through worksheets with the collaborative, inquiry-based teaching he was then experiencing: "Math class last year, you were like, by yourself, every man for themselves, but this year, it's open, it's like a city, we are all working together to create this new beautiful world."

I continue to be amazed and inspired by the words children use to talk about mathematics when it is opened up, when they are asked to use their ideas and experience creative, beautiful mathematics. The fact that they say "We are free," "I am open, I feel alive," and "We are working together to create this new beautiful world" speaks to the transformative effect that inquiry-based mathematics can have. Students speak in these ways because they have been given intellectual freedom, and that is a very powerful and moving experience. When we ask students to use intuition and think freely, they develop not only a new perspective on mathematics, themselves, and the world but also an intellectual freedom that transforms their relationship with learning.

Deborah Ball has written an engaging and provocative paper in which she quotes from the legendary psychologist Jerome Bruner:

> "We begin with the hypothesis that any subject can be taught effectively in some intellectually honest form to any child at any stage of development. It is a bold hypothesis and an essential one in thinking about the nature of a curriculum. No evidence exists to contradict it; considerable evidence is being amassed that supports it." (Bruner, 1960, cited in Ball, 1993)

This statement can be challenging for many, and it was perturbing for my Stanford students when I first introduced the idea. But they willingly thought about the ways that ideas in calculus can be discussed with young children. Deborah Ball is quite convinced; she says that "the things that children wonder about, think and invent are deep and tough" (Ball, 1993, p. 374). If we release teachers and students from the prescribed hierarchy of mathematics given in content standards and allow students to explore higher-level ideas that could be very engaging—such as the fourth dimension, negative space, calculus, or fractals—then we have the opportunity to introduce them to real mathematical excitement and explore powerful ideas, at any age. I am not suggesting that we teach formal higher-level mathematics to young children, but I like the possibility that Bruner and Ball discuss—that any part of math can be introduced in an intellectually honest form at any age. This is an exciting and important idea.

Value Depth over Speed

One thing we need to change in mathematics classrooms around the world is the idea that in mathematics speed is more important than depth. Mathematics, more than any other subject, suffers from this idea, and the learners of mathematics suffer because of it. Yet our world's top mathematicians—people such as Maryam Mirzakhani, Steven Strogatz, Keith Devlin, and Laurent Schwartz, who all have won the highest honors for their work—all talk about working slowly and deeply and not being fast. I quoted from Laurent Schwartz in Chapter Four; this sentence comes from his longer quote: "What is important is to deeply understand things and their relations to each other." Schwartz talks about feeling "stupid" in school because he was a slower thinker, and he urges his readers to appreciate that mathematics is about depth and connections, not shallow knowledge of facts and fast work.

Mathematics is a subject that should be highlighting depth of thinking and relationships at all times. In a recent visit to China, I was able to watch a number of middle and high school math lessons in different schools. China outperforms the rest of the world on PISA and other tests, by a considerable margin (PISA, 2012). This leads people to think that math lessons in China are focused upon speed and drill. But my classroom observations revealed something very different. In every lesson I observed, teachers and students worked on no more than three questions in an hour's lesson. The teachers taught ideas—even ideas that are among the more definitional and formulaic in mathematics, such as the definitions of complementary and supplementary angles—through an inquiry orientation. In one lesson, the teacher explored the meaning of complementary and supplementary angles with students by giving an example and asking them to "ponder the question carefully" and then discuss questions and ideas that came up (video, Youcubed at Stanford University, 2015d; www.youcubed.org/high-quality-teaching-examples/). The ensuing discussion of complementary and supplementary angles moved into a depth of terrain I have never before seen in my observations of mathematics classrooms teaching this topic. The teacher provocatively took the students' ideas and made incorrect statements for the students to challenge, and the class considered together all of the possible relationships of angles that preserve the definitions.

The following extract is a transcript from a typical U.S. lesson on complementary and supplementary angles, taken from a TIMSS video study of teaching in different countries (Stigler & Hiebert, 1999):

Teacher:

Teacher:

Here we have vertical angles and supplementary angles. Angle A is vertical to which angle?

Students chorus:

70 degrees.

Teacher:

Therefore angle A must be?

Students chorus:

70 degrees.

Teacher:

Now you have supplementary angles. What angle is supplementary to angle A?

Students Chorus:

B

Teacher:

B is, and so is …?

Students:

C.

Teacher:

Supplementary angles add up to what number?

Students:

180 degrees.

In the extract we observe definitional questions with one answer, toward which the teacher is leading the students. Compare this with a lesson we watched in China, in which the teacher did not ask questions such as "Supplementary angles add up to what number?" Rather, she asked questions such as "Can two acute angles be supplementary angles? Can a pair of supplementary angles be acute angles?" These are questions that require students to think more deeply about definitions and relationships. Here is an extract from the lesson in China that I watched and that stands as an important contrast to the U.S. lesson.

Student:

As he just said, if there are two equal angles, whose measures add up to 180 degrees, they must be two right angles. Because the measures of acute angles are always smaller than 90 degrees, the sum of the measures of two acute angles will not be larger than 180 degrees.

Teacher:

Therefore, if two angles are supplementary, they must be two obtuse angles?

Student:

That is not correct.

Teacher:

No? Why?

Teacher:

I think if two angles are supplementary, they must be two obtuse angles.

Student:

I think they could be an acute angle and an obtuse angle.

Teacher:

She says, although they cannot both be acute angles, they can be one acute angle and one obtuse angle.

Student:

For example, just like the Angle 1 and Angle 5 in that question. One angle is an acute angle. The other one is an obtuse angle.

Teacher:

OK. If two angles are supplementary, they must be one acute angle and one obtuse angle?

Student:

That's still not accurate.

Student:

You should say, if two angles are supplementary, at least one of them is an acute angle.

Other Students:

No, at least one angle is larger than 90 degrees.

Student:

An exception is when the two angles are right angles.

The lessons from the United States and from China could not have been more different. In the U.S. lesson, the teacher fired procedural questions at the students and they responded with the single possible answer. The teacher asked questions that could have come straight from books, that highlighted an easy example of the angle, and students responded with definitions they had learned. In the lesson in China, the teacher did not ask complete-this-sentence questions; she listened to students' ideas and made provocative statements in relation to their ideas that pushed forward their understanding. Her statements caused the students to respond with conjectures and reasons, thinking about the relationships between different angles.

The second half of the lesson focused on the different diagrams students could draw that would illustrate and maintain the angle relationships they had discussed. This involved the students producing different visual diagrams, flipping and rotating rays and triangle sides. Students discussed ideas with each other and the teacher, asking questions about the ideas, pushing them to a breadth and depth I had not imagined before seeing the lesson. As the class discussed the visual diagrams of angle relationships, one student reflected: “This is fascinating.” There are not many students who would have drawn this conclusion from the U.S. version of the lesson.

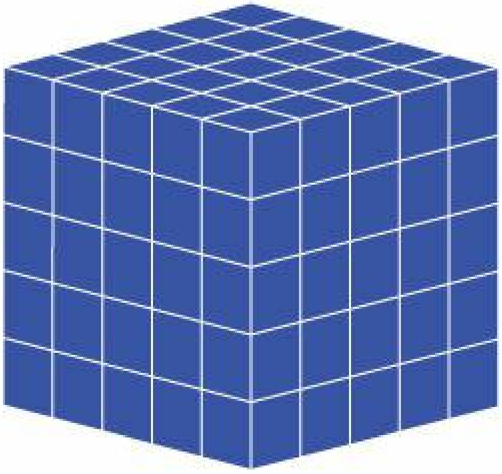
The TIMSS video study compared teaching in the United States with teaching in other countries and concluded that U.S. lessons were “a mile wide and an inch deep” (Schmidt et al., 2002), whereas lessons in other countries they studied, particularly Japan, were conceptual and deeper, and involved more student discussion. The analysts linked the depth of the discussions and work in Japan, compared to the United States, as the reason for the higher achievement in Japan (Schmidt et al., 2002; Schmidt, McKnight, & Raizen, 1997).

Some parents' lack of understanding of the importance of mathematical depth, along with misguided beliefs that their children will be advantaged if they go faster, leads them to campaign for their children to skip grades and be taught higher-level mathematics as early as possible. But mathematics learning is not a race, and it is mathematical depth that inspires students and keeps them engaged and learning mathematics well, setting them up for high-level learning in the future. We know that students who are pushed to go through content faster are usually the ones to drop out of mathematics when they get the opportunity (Jacob, 2015; also Boaler, 2015b). We want all students to be productively engaged with math, and no student should find it too easy or be made to repeat ideas they have already learned. One of the best and most important ways to encourage high achievers is to give them opportunities to take ideas to greater depth, which they can do alongside other students, who may take ideas in depth on other days. A method I use to do this with my Stanford students is to ask those who finish problems to extend them, taking them in new directions.

Last week I gave my Stanford students a problem called "a painted cube," along with boxes of sugar cubes so that they could model the problem (see Exhibit 9.7 and Figure 9.8).

Painted Cube

Imagine a 5 × 5 × 5 cube that had been painted blue on the outside, with cubes made up of smaller 1 × 1 × 1 cubes.
Consider the questions:



How many small cubes will have 3 blue faces?

How many small cubes will have 2 blue faces?

How many small cubes will have 1 blue face?

How many small cubes will have no paint on them?

Exhibit 9.7

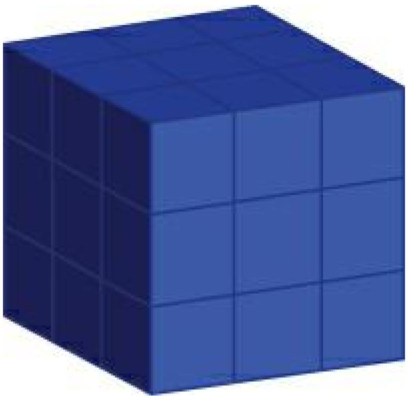


Figure 9.8 A painted cube

Some of the students built a smaller case, such as a 3 × 3 × 3 cube, out of their sugar cubes and colored the outside surface with pens, to consider the way the cube sides were distributed.

I told them that when they had solved the 5 × 5 cube problem they could extend the problem in any way they wished. This was the best part of the lesson and the occasion for many more learning opportunities, as different groups considered, for example, how to work out the answer with a pyramid of cubes instead of a cube of cubes (Figure 9.9); another group worked out the relationships in a pyramid made of smaller pyramids, and still another worked out relationships if the cube moved into the 4th dimension and from there n-dimensions.

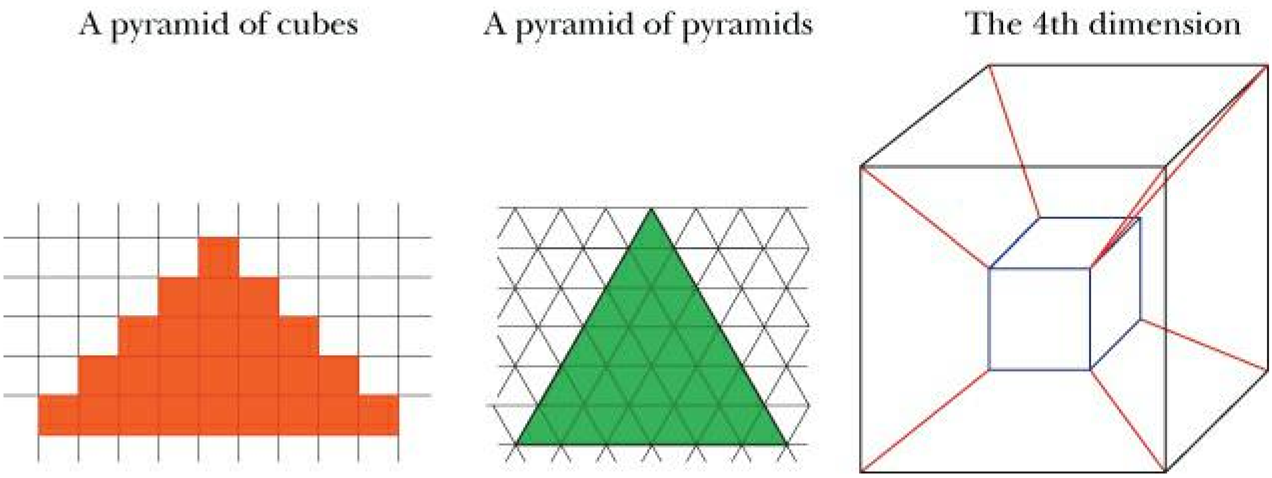


Figure 9.9 Extended cube problem

If you give students the opportunity to extend problems, they will almost always come up with creative and rich opportunities to explore mathematics in depth, and that is a very worthwhile thing for them to do.

Connect Mathematics to the World Using Mathematical Modeling

One main reason school children give for not liking mathematics is its abstract nature and perceived irrelevance to the world. This is a sad reflection on the math teaching they receive in schools, because math is everywhere and all around us. In fact, it is so critical to successful functioning in life that it has been termed the new "civil right"—essential for people to function well in society (Moses & Cobb, 2001). When I interviewed a group of young people, all about 24 years old, who had received traditional math teaching in school, and I asked them about math in their lives and work, they expressed dismay at the math education they had received. The young adults said that they could see math all around them in the world now and they used it every day in their jobs, yet their school experiences of math had not given them any sense of the real nature of math and its importance to their future. They said that if they had only known that math wasn't a dead and irrelevant subject—that it would actually be essential to their adult lives—it would have made a big difference to their motivation in math classes in school.

The need to make mathematics interesting and connected to the real world has often resulted in publishers putting mathematics into what I call "pseudo contexts" (Boaler, 2015a) intended to represent reality. Students work on fake real-world problems that are far from reality, such as trains speeding toward each other on the same track. These contexts do not help students know that math is a useful subject. They give the opposite impression to students, as they show mathematics to be other-worldly and unreal. To be successful in a fake real-world problem, students are asked to engage as though the questions are real, at the same time as ignoring everything they know about the real-world situation. For example, consider these typical questions:

> Joe can do a job in 6 hours and Charlie can do the same job in 5 hours. What part of the job can they finish by working together for 2 hours?
>
> A restaurant charges $2.50 for 1/8 of a quiche. How much does a whole quiche cost?
>
> A pizza is divided into fifths for 5 friends at a party. Three of the friends eat their slices, but then 4 more friends arrive. What fractions should the remaining 2 slices be divided into? (Boaler, 2015a)

These questions all come from published textbooks and are typical of the questions students work with in math class. But they are all nonsensical. Everyone knows that people work at a different rate together than when alone, restaurants charge a different proportional price for food that is sold in bulk, and if extra friends arrive at a party more pizza is ordered—the remaining slices are not subdivided into small fractions. The cumulative effect of students working with pseudo contexts is that they come to think of math as irrelevant. In fact, for many students, they know that when they walk into math class they are walking into *Mathland,* a strange and mysterious place that requires them to leave their common sense at the door.

How then do we help students see the widespread use and applicability of mathematics without using pseudo contexts? The world is full of fascinating examples of situations that we can make sense of with mathematics. My online class helped students to see this by showing them the mathematics in snowflakes, in the work of spiders, in juggling and dancing, and in the calls of dolphins. The mathematics spanned from elementary to high levels of high school (Stanford Online Lagunita, 2014). Not all mathematics problems can or should be placed into a real-world context, as some of the greatest problems that help students learn important quantitative thinking are without a context. But it is important to have students see the applicability of mathematics and work with real-world variables for at least some of the time.

Conrad Wolfram urges viewers of his TED talk to view mathematics as a subject that centers around the posing of questions and forming of mathematical models (Wolfram, 2010). He highlights the act of modeling as central to the mathematics of the world. The Common Core Standards also highlight modeling, a mathematical practice standard.

MP 4: Model with Mathematics

One of the most important contributions of the Common Core State Standards (CCSS), in my view, is their inclusion of mathematical practices—the actions that are important to mathematics, in which students need to engage as they learn mathematics knowledge. "Modeling with Mathematics" is one of the 8 Mathematics Practices Standards (see box).

CCSS.Math.Practice.MP4 Model with mathematics.

Mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace. In early grades, this might be as simple as writing an addition equation to describe a situation. In middle grades, a student might apply proportional reasoning to plan a school event or analyze a problem in the community. By high school, a student might use geometry to solve a design problem or use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose.

Source: Common Core State Standards Initiative, 2015.

The act of modeling can be thought of as the simplification of any real-world problem into a pure mathematical form that can help to solve the problem. Modeling happens all through mathematics, but students have not typically been aware that they are modeling or asked to think about the process.

Ron Fedkiw is an applied mathematician at Stanford who specializes in computer-generated special effects. His mathematical models have created the special effects in award-winning movies such as *Pirates of the Caribbean: Dead Man's Chest* and *Star Wars: Episode III—Revenge of the Sith*. Fedkiw trained in pure mathematics until he was 23 and then moved to applied mathematics. As part of his work, he designs new algorithms that rotate objects, mimic collisions, and "mathematically stitch together slices of a falling water drop."

Mathematical modeling is also used in criminal cases and has helped to solve high-profile murder cases. *NUMB3RS* is a successful television show that features an FBI agent who often gets help from his mathematician brother. The first episode of *NUMB3RS* featured the true story of a brutal serial killer. FBI agents had been keeping track of murder locations on a map but could not see any patterns. The FBI agent in the show was stumped but recalled that his mathematician brother talked all the time about mathematics being the study of patterns. He asked his brother for help. The mathematician worked by inputting key information about serial killers, such as the fact that they tend to strike close to their home but not too close, and they leave a buffer zone within which they won't strike. He discovered that he could capture the pattern of the crosses with a simplified mathematical model. The model showed a “hot zone” indicating the areas in which the killer was likely to live. The FBI agents set to work investigating men of a certain age who lived in the zone, and the case was eventually solved. This episode was based on the work of the real-life mathematician Kim Rossmo, who developed a process of criminal geographic targeting (CGT) using mathematical models, a process used by police departments around the world.

When we ask students to take a problem from the world, based on real data and constraints, and solve it using mathematics, we are asking them to model the situation. As Wolfram says, students should encounter or find a real-world problem, set up a model to solve it, run some calculations (the part that can be done by a calculator or computer), and then see whether their answer solves the problem or the model needs to be refined. He points out that students currently spend 80% of the time they spend in math classrooms performing calculations, when they should instead be working on the other three parts of mathematics—setting up models, refining them, and using them to solve real problems.

In algebra classes students are often asked to compute rather than set up a model using algebra. For example, consider this problem:

> A man is on a diet and goes into a shop to buy some turkey slices. He is given 3 slices which together weigh 1/3 of a pound but his diet says that he is allowed to eat only 1/4 of a pound. How much of the 3 slices he bought can he eat while staying true to his diet?

This is a difficult problem for many people. But the difficulties most people face are not in the calculations; they arise in the setting up of a model to solve the problem. I have written elsewhere about the elegant visual solutions young children produce to solve this problem (Boaler, 2015a); this is the solution one fourth grader produced:

> If 3 slices is 1/3 of a pound then a pound is 9 slices (see Figure 9.10)

If he can have one quarter of a pound, he can have one fourth of that (see Figure 9.11) ... which is 2 1/4 slices.

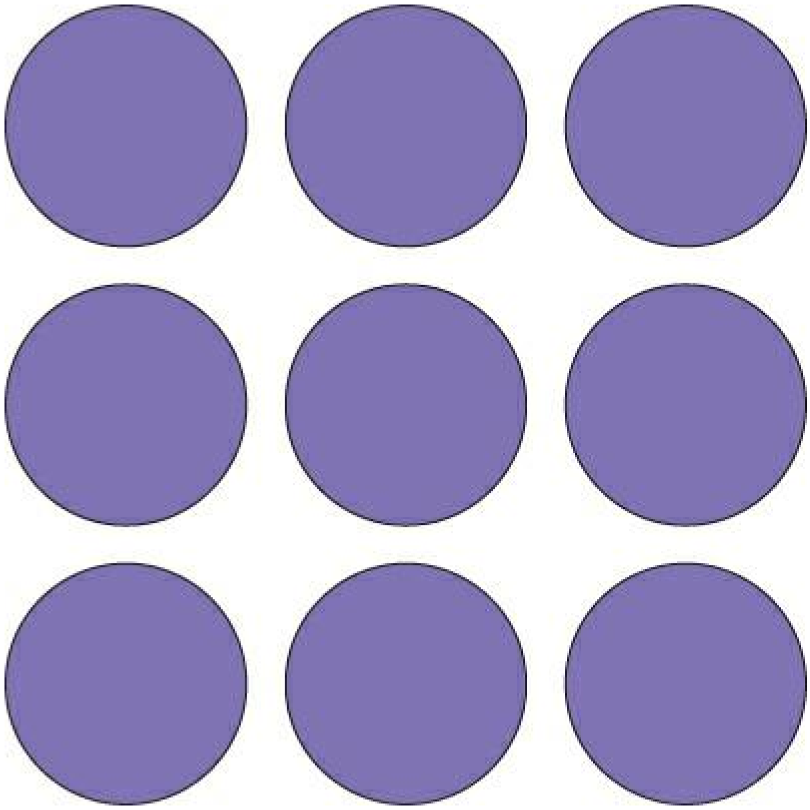


Figure 9.10 Nine slices

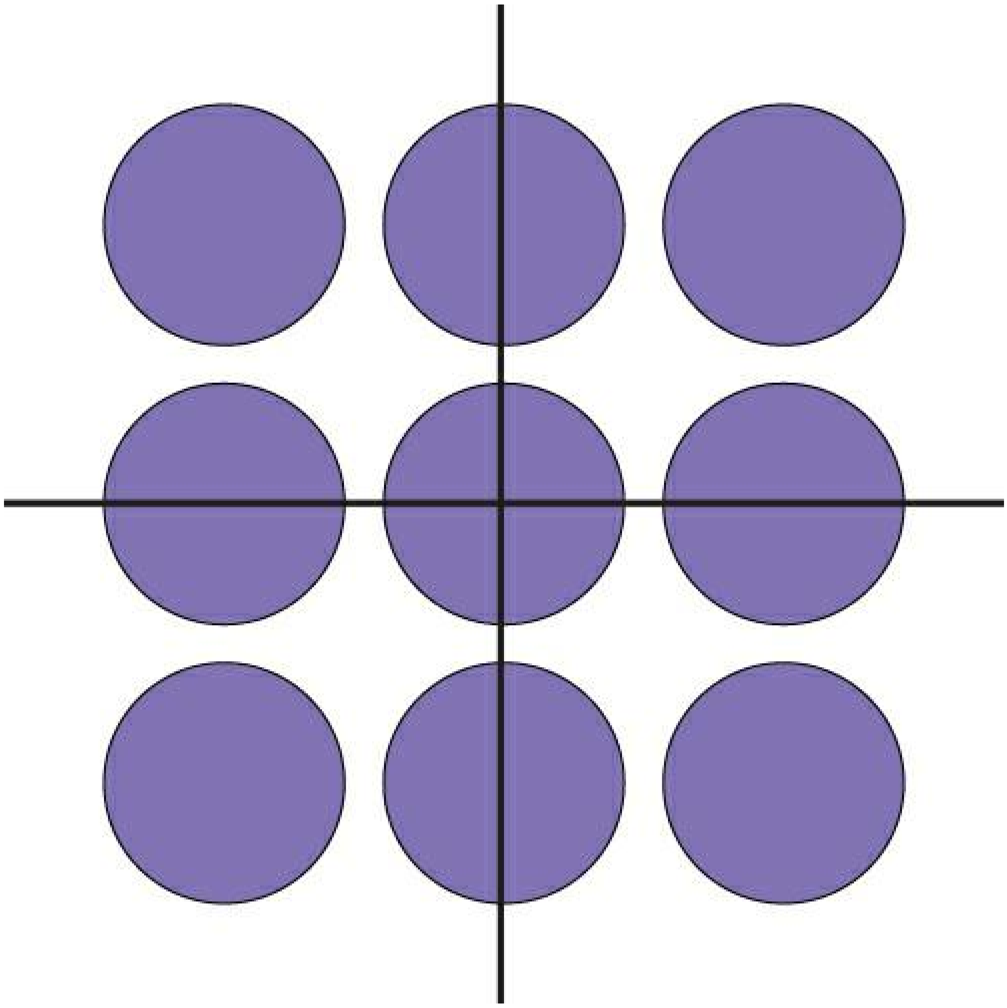


Figure 9.11 Nine slices divided into quadrants

By contrast, adults struggled to answer the problem, either incorrectly multiplying $\frac{1}{3} \times \frac{1}{4}$ or trying to use algebra but not remembering how. To use algebra, they needed to say

3 slices = $\frac{1}{3}$ pound

x slices = $\frac{1}{4}$ pound

and then cross multiply, so that $\frac{1}{3}x = \frac{3}{4}$ and $x = \frac{9}{4}$.

The adults who were given this problem experienced difficulty setting up the model and creating an expression. Despite years of work in algebra classes, students get very little experience with interpreting situations and setting up models. Students are trained to move variables around on a page and solve many expressions, but rarely do they set up problems. This is the important process Wolfram talks about—the setting up of a model.

Students of all ages can engage in modeling. For example, students in kindergarten class can be asked to make a seating map for all the children in the class, so that they can all fit on the carpet. They can represent each child by a shape or object and find a good way for all the children to sit on the carpet. This is an example of modeling a situation, in this instance with shapes or objects representing more complex beings (the young children!) (Youcubed at Stanford University, 2015b; www.youcubed.org/task/moving-colors/).

A mathematical model often offers more simplicity than the real situation. In the kindergarten example, the shapes representing children do not take account of their size or movement. In the turkey slice example, the slices are assumed to be of equal size and weight with no variation.

A nice modeling question that students can work on in middle or high school is the famous tethered goat problem. The extended version of the problem in Exhibit 9.8 was written by Cathy Williams.

The Tethered Goat

Imagine a goat tied to the corner of a shed by a rope. The shed is 4 feet by 6 feet. The rope is 6 feet long.

What do you wonder about this situation?

Draw a picture of the situation.

What questions do you have?

The sun rises to the east of the shed and sets to the west. The goat would appreciate some shade. Where should you plant a tree? What tree would you plant?

Exhibit 9.8

This question is set in a context that isn't real, but it is a context that invites students to consider aspects of the real situation and use them in their thinking. Students will probably wonder about the space the goat has to move around in. They or the teacher could suggest adding some fencing. A nice extension is to ask students to decide how they would arrange 60 1-foot fences to maximize the additional area, which is a lovely, rich problem that I described in Chapter 5. When students think about planting the tree they might wonder what would happen if the goat ate the tree? What would be the best tree to plant? Where would you plant the tree so the goat could not eat it but would benefit from the shade?

This is a mathematical situation that has plenty of room for students to ask rich questions and investigate them. They would need to model the situation and construct representations, two important mathematical practices (see Figure 9.12).

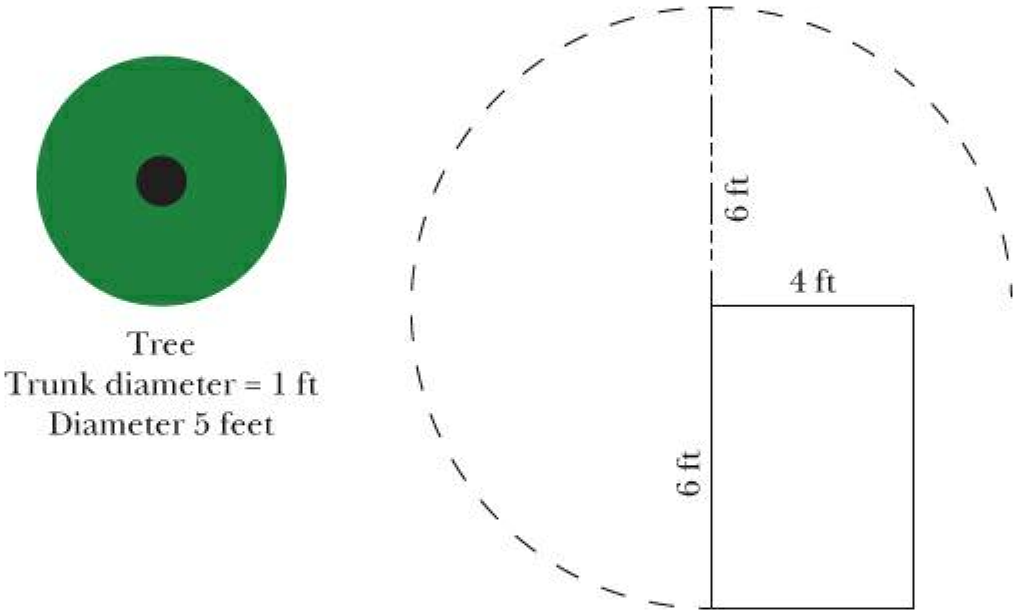
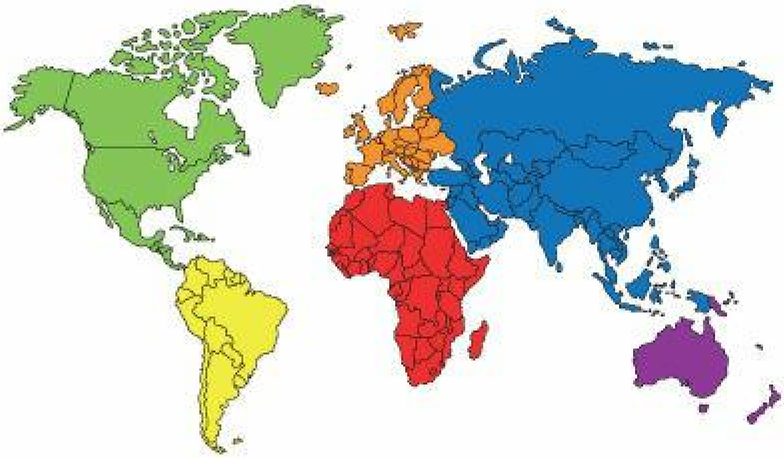


Figure 9.12 Tethered goat model

A nice way to use real data is to ask students to work with real numbers and data from magazines, newspapers, and the Internet. For example, an activity I like that also teaches students about issues of social justice is one that asks them to form groups in the class to represent the different continents in the world. The groups then investigate how many cookies their group would get if the cookies represent the proportion of the world's wealth in their continent (see Exhibit 9.9). Students will model, reason, and apply knowledge as well as learn real and important information about the world and the way wealth is distributed, which will be especially real to them if it is translated into cookies they can eat. As students in some parts of the world get very few cookies, it is best to bring along spares to even out the cookie eating afterwards!

World Wealth Simulation



1. Find the percentage of the world's population living on each continent.
2. Calculate the number of people in our class who would correspond to the percentages found.
3. Calculate the percentage of the world's wealth for each continent.
4. Calculate the wealth of each continent in cookies.

TABLE 1 World Wealth Data

Continent	Population (in millions) 2000	Percent of Population	Wealth (GDP in trillions of dollars)	Percentage of Wealth
Africa	1,136		2.6	
Asia	4,351		18.5	
N. America	353		20.3	
S. America	410		4.2	
Europe	741		24.4	
Oceania/Aust.	39		1.8	
Total	7,030	100%	71.8	100%

Sources: Population data according to Population Reference Bureau (prb.org). Wealth data according to International Monetary Fund.

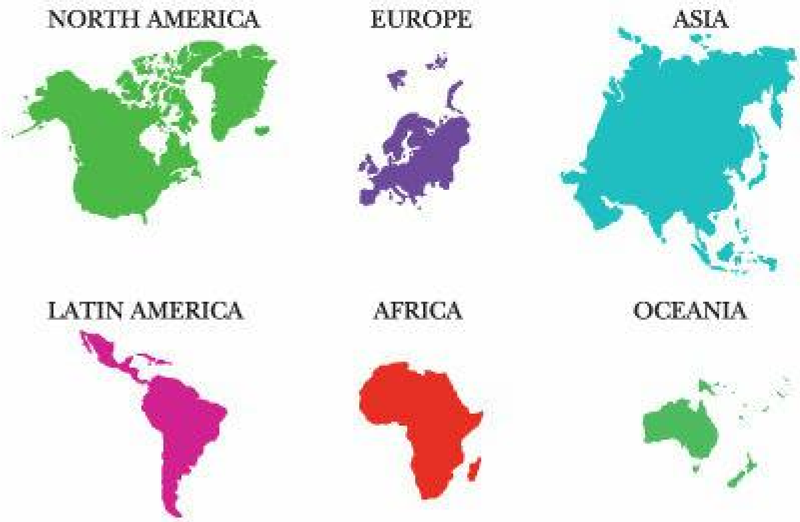


TABLE 2 Classroom Data

Number of people in classroom _______

Total number of cookies _______

Continent	Pop %	# of People in Class	Wealth %	# Cookies
Africa				
Asia				
N. America				
S. America				
Europe				
Oceania/Aust.				
Total	~100%		~100%	

Source: From Charmaine Mangram.

Exhibit 9.9

Olympic and other sporting data offer a wealth of opportunities for mathematical questioning and thinking. It is important when drawing from sporting data to be conscious of gender equity. Exhibit 9.10 presents a question I like, that again involves setting up a mathematical model:

Soccer Goalie

If you are a soccer goalie, and an attacking player from the other team has broken away from the others and is running toward you, where is the best place for you to stand? Try mapping out different positions depending on the location of the attacking player when she shoots.

Exhibit 9.10

My advice in bringing the real world into the classroom is to use real data and situations and to give a context only when it is helpful. Be sure it does not involve students suspending their sense making and stepping into *Mathland*.

The PISA team at the Organisation for Economic Co-operation and

Development (OECD) conducted an interesting and useful analysis of the strengths and weaknesses U.S. students showed in the PISA international mathematics assessments. They found that the U.S. students' weaknesses were related to the artificial contexts used in classrooms, which do not teach students to use real-world variables but instead teach them to ignore them. Their recommendations for ways to engage students to encourage success are helpful:

> It seems that the U.S. students have particular strengths in cognitively less-demanding mathematical skills and abilities, such as extracting single values from diagrams or handling well-structured formulae. And they have particular weaknesses in demanding skills and abilities, such as taking real world situations seriously, transferring them into mathematical terms and interpreting mathematical aspects in real world problems. These are tasks where the well-known superficial classroom strategy "Don't care about the context, just extract the numbers from the text and do some obvious operations" is bound to fail. This strategy is popular all over the world and frequently helps pupils and students to survive in school mathematics and to pass examinations. However, in a typical PISA mathematical literacy task, the students have to use the mathematics they have learned in a well-founded manner. The American students obviously have particular problems with such tasks. (...) When it comes to the implications of these findings, one clear recommendation would be to focus much more on higher-order activities such as those involved in mathematical modeling (understanding real world situations, transferring them into mathematical models, and interpreting mathematical results), without neglecting the basic skills needed for these activities. (Organisation for Economic Co-operation and Development [OECD], 2013)

The PISA team observed a phenomenon that comes from the weakness of the questions given to students in the United States. They noted that students' tendency to ignore contexts and just use numbers resulted in failure on their questions. This reflects the low quality of the questions used in textbooks across the United States, with their fake contexts. Sadly, the strategies U.S. students typically learn from math class will be similarly unhelpful when they enter the world of work. Students need to be engaged in math class with questions that require them to consider a real situation, use real-world variables, and engage with data from the world. They need to learn to set up mathematical models from the situations and to problem solve, a process that is both engaging and extremely important for their future.

Encourage Students to Pose Questions, Reason, Justify, and Be Skeptical

The first thing a mathematician has to do is pose an interesting question. This mathematical practice is virtually absent in math classrooms, yet it is central to mathematical work. Nick Foote is a wonderful third-grade public school teacher and friend who taught both of my daughters, which has given us the opportunity to have many discussions about math together. In Nick's class he sometimes gives situations and invites students to come up with their own mathematical questions. I was visiting Nick's class one day when he gave this situation (see Figure 9.13).

Figure 9.13 Bracelets for sale

You want to buy some Wonder Loom bracelets. You go to the Rainbow Zen Garden Store and find these options.

Two-color bracelets—$0.50 each or 3 for $1.00

Multicolor bracelets—$1.00 each or 3 for $2.50

Supplies to create your own bracelets:

600-count bag of rubber bands—$3.00

or 4 bags—$10

600-count bag of glow-in-the-dark rubber bands—$5.00

Wonder loom—$5.00

He then asked groups of students to discuss the situation and pose questions. Exhibit 9.11 is a handout Nick often uses.

We Wonder

Team Members:

Date:

We wonder

Use pictures, numbers, and words to show how you answered your question.

We want to investigate

Use pictures, numbers, and words to show how you answered your question.

Source: From Nick Foote.

Exhibit 9.11

The students excitedly set about wondering about questions such as: why are the bracelets so expensive to buy? They were helped in working this out by finding out how much it would cost to make a bracelet from the materials, and then thinking about the cost of selling through a shop. These were real questions from the students, eliciting higher engagement and learning.

When students move into employment, in our high-tech world, one very important job they will be asked to do is to pose questions of situations and of large data sets. Increasingly, companies are dealing with giant data sets, and the people who can ask creative and interesting questions of the data will be highly valued in the workplace. In my own teaching experience, when I have asked students in classrooms to consider a situation and pose their own question, they have become instantly engaged, excited to draw on their own thinking and ideas. This is an idea for math classrooms that is very easy to implement and needs to be used only some of the time. Students should be able to experience this in school so that they are prepared to use it later in their mathematical lives.

When Conrad Wolfram discusses his role as an employer, he says that he does not need people who can calculate fast, as computers do that work. He needs people who can make conjectures and talk about their mathematical pathways. It is so important that employees describe their mathematical pathways to others, in teams, because others can then use those pathways in their own work and investigations and can also see if there are errors in thinking or logic. This is the core of mathematical work; it is called reasoning.

I speak to many groups of parents about Common Core math, and I am often asked, especially by the parents of high-achieving students, "Why should my child discuss his work in a group, when he can get the answers quickly on his own?" I explain to parents that explaining one's work is a mathematical practice, called reasoning, that is at the heart of the discipline. When students offer reasons for their mathematical ideas and justify their thinking, they are engaging in mathematics. Scientists work by proposing theories and looking for cases that prove or disprove their theory. Mathematicians propose theories and reason about their mathematical pathways, justifying the logical connections they have made between ideas (Boaler, 2013c).

Chapter Five introduced a classroom strategy of inviting students to be skeptics, which prompts students to push each other to high levels of reasoning. This is an excellent way to teach students reasoning and for them to take on the role of skeptic, which students enjoy. As I described in Chapter Five, reasoning is not only a central mathematical practice but also a classroom practice that promotes equity, as it helps all students get access to ideas. When students act as a skeptic, they get an opportunity to question other students without having to take on the role of someone who doesn't understand.

Teach with Cool Technology and Manipulatives

As we invite students to enter a world in which mathematics is open, visual, and creative, many forms of technology and manipulatives are helpful. Cuisenaire rods, multilink cubes, and pattern blocks are helpful for students at all levels of mathematics; I use them in my undergraduate teaching at Stanford. Chapter Four reviewed a range of apps and games that also invite students into visual and conceptual thinking. I focused in that chapter on number, but many good apps also enable students to explore geometric ideas in two and three dimensions, allowing students to move angles and lines in order to explore relationships. This is important and powerful thinking that cannot be done with pen and paper. Geometry Pad for iPad and GeoGebra both allow teachers and students to make their own dynamic demos, encouraging students, for example, to investigate geometric and algebraic ideas such as $y = mx + b$ and trig ratios, dynamically and visually. Geometry Pad is made by Bytes Arithmetic LLC, and the basic version is free.

Other apps, such as Tap Tap Blocks, help students build in three dimensions, making and solving spatial patterns and algebraic patterns (see Figure 9.14). Students can place and spin objects in a 3-D simulated space. Tap Tap Blocks is a free app made by Paul Hangas; it runs on iOS.

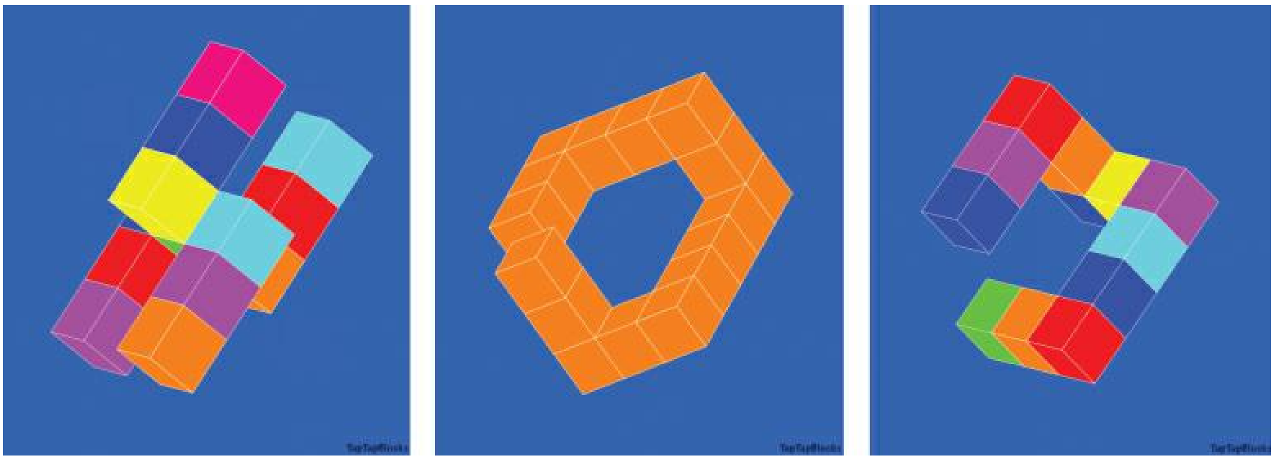


Figure 9.14 Tap Tap Blocks

A nice activity to try with Tap Tap Blocks is to ask students to try and make a shape from screen shots of different views and then challenge a friend by making their own problem. For example:

> Can you build this shape that has 1 orange, 1 yellow, 1 dark blue, 2 green, 2 light blue, 2 red, and 3 purple blocks? Here are some

different screen shots of the shape from different angles (See Figure 9.15).

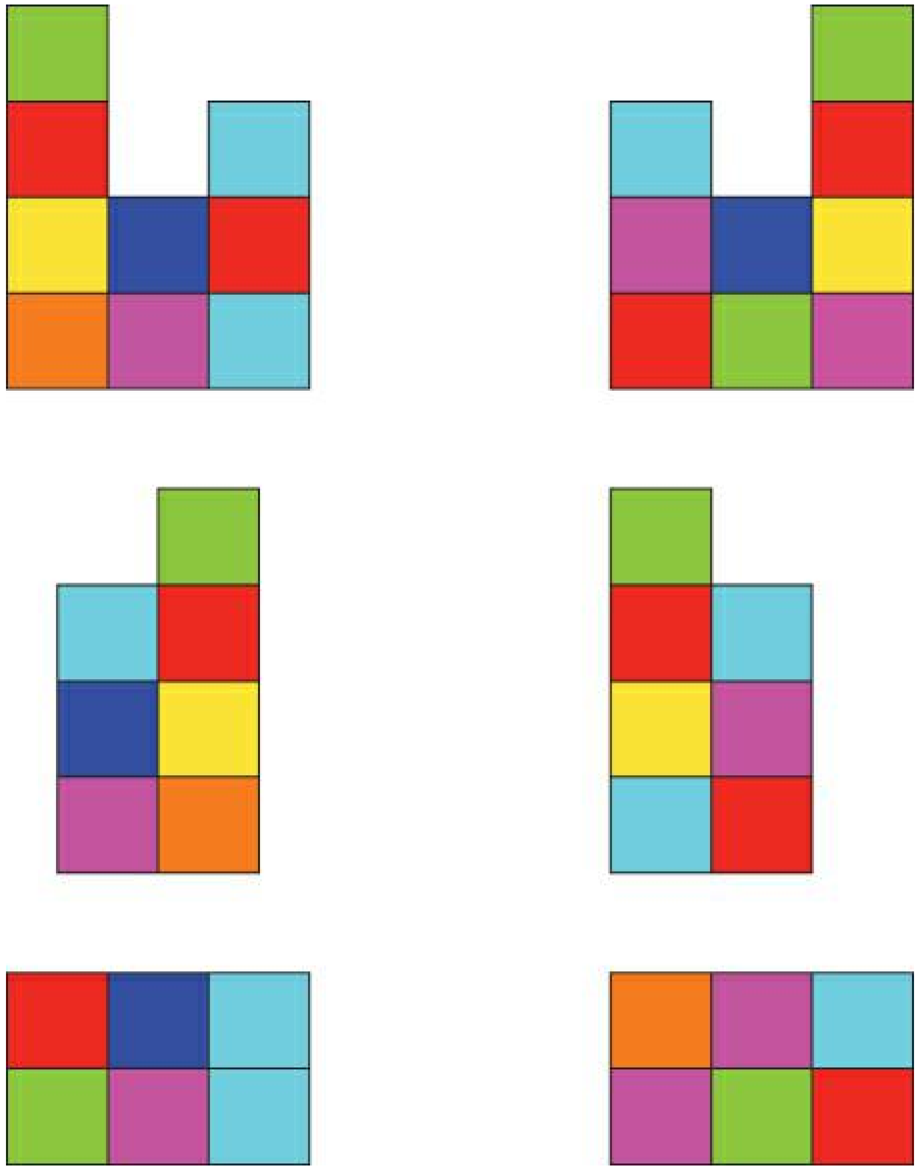


Figure 9.15 A Tap Tap Blocks shape from six different angles

These different apps offer productive ways to engage students with conceptual and visual thinking, but they are not the only apps, games, or sites that do this. There are millions of math apps and games that claim to help students, but few that build on research on learning, showing mathematics as a conceptual and visual subject. My advice is to be discerning when you choose technology to engage your students, using those that motivate students to think and make connections, not to work at speed on procedures and calculations.

Mathematics is a broad, multidimensional subject, and when teachers embrace the multidimensionality of mathematics, in their teaching and through their assessment, many more students can gain access to mathematics and be excited by it. When we open mathematics, we broaden the number and range of students who can engage and do well. This is not an artificial broadening or dumbing-down of mathematics; rather, it is a broadening that brings school mathematics closer to real mathematics and the mathematics of the world.

Conclusion

Teachers, parents and leaders have the opportunity to set students on a growth mindset mathematics pathway that will bring them greater accomplishment, happiness, and feelings of self-worth throughout their lives. We need to free our young people from the crippling idea that they must not fail, that they cannot mess up, that only some students can be good at math, and that success should be easy and not involve effort. We need to introduce them to creative, beautiful mathematics that allows them to ask questions that have not been asked, and to think of ideas that go beyond traditional and imaginary boundaries. We need our students to develop *Growth Mathematical Mindsets*. I hope that this book has given you some ideas that can start or reinvigorate your own journey into creative and growth mathematics and mindset that will continue throughout your life. When we encourage open mathematics and the learning messages that support it, we develop our own intellectual freedom, as teachers and parents, and inspire that freedom in others.

Thank you for setting out on this journey with me. Now it is time for you to invite others onto the pathways you have learned, inviting them to be the people they should be, free from artificial rules, and inspired by the knowledge they have unlimited mathematics potential. For we can all open mathematics, and give students the chance to ask their own questions and bring their own natural creativity and curiosity to the foreground as they learn. If we give students this rich, creative, growth mathematics experience, then we change them as people and the ways they interact with the world.

When we set students free, beautiful mathematics follows.