Existence of the Inverse of a Linear Transformation

We define isomorphisms, and prove that a linear transformation has an inverse if and only if the linear transformation is an isomorphism.

Existence of Inverses

In Exploration Problem **??** we examined a linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ that doubles all input vectors, and its inverse $S : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, that halves all input vectors. We observed that the composite functions $S \circ T$ and $T \circ S$ are both identity transformations. Diagrammatically, we can represent T and S as follows:

$$\begin{tikzcd} \mathbf{v} \arrow[rr, mapsto, bend left, "T \text{ doubles each input}"] & & 2\mathbf{v} \arrow[ll, mapsto, bend left, "S \text{ halves each input}"] \end{tikzcd}$$

This gives us a way of thinking about an inverse of T as a transformation that "undoes" the action of T by "reversing" the mapping arrows. We will now use these intuitive ideas to understand which linear transformations are invertible and which are not.

Given an arbitrary linear transformation $T : V \rightarrow W$, "reversing the arrows" may not always result in a transformation. Recall that transformations are functions. Figures **??** and **??** show two ways in which our attempt to "reverse" T may fail to produce a function.

First, if two distinct vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ map to the same vector $\mathbf{w}$ in W, then reversing the arrows gives us a mapping that is clearly not a function. (See Figure **??**)

Learning outcomes:
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$$
\begin{tikzcd}
\mathbf{v}_1 \arrow[dr, mapsto] & \\
& T(\mathbf{v}_1) = T(\mathbf{v}_1) = \mathbf{w} \\
\mathbf{v}_2 \arrow[ur, mapsto] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbf{v}_1 & \\
& T(\mathbf{v}_1) = T(\mathbf{v}_1) = \mathbf{w} \arrow[ul, mapsto] \arrow[dl, mapsto] \\
\mathbf{v}_2 &
\end{tikzcd}
$$

Second, observe that our definition of an inverse of $T : V \rightarrow W$ requires that the domain of the inverse transformation be W. (Definition **??**) If there is a vector $\mathbf{b}$ in W that is not an image of any vector in V, then $\mathbf{b}$ cannot be in the domain of an inverse transformation.

$$
\begin{array}{ccc}
 & & \mathbf{b} \\
 & & \\
\mathbf{v} & \longmapsto & T(\mathbf{v})
\end{array}
\qquad
\begin{array}{ccc}
? & \longleftarrow\!\shortmid & \mathbf{b} \\
 & & \\
\mathbf{v} & \longleftarrow\!\shortmid & T(\mathbf{v})
\end{array}
$$

We now illustrate these potential issues with specific examples.

Example 1. *Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be a linear transformation whose standard matrix is*

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Does T have an inverse? Show that multiple vectors of the domain map to $\mathbf{0}$ in the codomain.

Explanation. *The matrix $\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ is not invertible, so T is not invertible.*

We now dig a little deeper to get additional insights into why T does not have an inverse. Observe that all vectors of the form $\begin{bmatrix} k \\ -k \end{bmatrix}$ map to $\mathbf{0}$. To verify this, use matrix multiplication:

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} k \\ -k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This shows that there are infinitely many vectors that map to $\mathbf{0}$. So, "reversing the arrows" would not result in a function.

Example 2. *Let $T : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ be a linear transformation whose standard matrix is*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 2 & 0 \end{bmatrix}$$

Does T have an inverse? Show that there exists a vector $\mathbf{b}$ in $\mathbb{R}^3$ such that no vector of $\mathbb{R}^2$ maps to $\mathbf{b}$.

Explanation. *The matrix* $\begin{bmatrix}1 & 0\\0 & 1\\2 & 0\end{bmatrix}$ *is not invertible (it's not even a square matrix!), so T does not have an inverse.*

We now get another insight into why T is not invertible. To find a vector $\mathbf{b}$ *such that no vector of* $\mathbb{R}^2$ *maps to* $\mathbf{b}$*, we need to find* $\mathbf{b}$ *for which the matrix equation*

$$\begin{bmatrix}1 & 0\\0 & 1\\2 & 0\end{bmatrix}\mathbf{x}=\mathbf{b} \tag{1}$$

has no solution.

Let $\mathbf{b}=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$*. Gauss-Jordan elimination yields:*

$$\left[\begin{array}{cc|c}1 & 0 & b_1\\0 & 1 & b_2\\2 & 0 & b_3\end{array}\right] \rightsquigarrow \left[\begin{array}{cc|c}1 & 0 & b_1\\0 & 1 & b_2\\0 & 0 & b_3-2b_1\end{array}\right]$$

*Equation (**??**) has a solution if and only if* $b_3-2b_1=0$*. Since we do not want (**??**) to have a solution, all we need to do is pick values* b_1*,* b_2 *and* b_3 *such that* $b_3-2b_1\neq 0$*. Let* $\mathbf{b}=\begin{bmatrix}1\\1\\1\end{bmatrix}$*. Then no element of* $\mathbb{R}^2$ *maps to* $\mathbf{b}$*.*

Our next goal is to develop some vocabulary that would allow us to discuss issues illustrated by Figures **??** and **??**.

One-to-one Linear Transformations

Definition 1 (One-to-One)**.** *A linear transformation* $T:V\rightarrow W$ *is one-to-one if*

$$T(\mathbf{v}_1)=T(\mathbf{v}_2) \quad \textit{implies that} \quad \mathbf{v}_1=\mathbf{v}_2$$

Example 3. *Transformation T in Example **??** is not one-to-one.*

Explanation. *We can use any two vectors of the form* $\begin{bmatrix}k\\-k\end{bmatrix}$ *to make our case.*

$$T\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)=\mathbf{0}=T\left(\begin{bmatrix}-2\\2\end{bmatrix}\right) \quad \textit{but} \quad \begin{bmatrix}1\\-1\end{bmatrix}\neq\begin{bmatrix}-2\\2\end{bmatrix}$$

Example 4. *Prove that the transformation in Example **??** is one-to-one.*

Proof *Suppose*

$$T\left(\begin{bmatrix}x_1\\x_2\end{bmatrix}\right)=T\left(\begin{bmatrix}y_1\\y_2\end{bmatrix}\right)$$

Then

$$\begin{bmatrix}1 & 0\\0 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\\2 & 0\end{bmatrix}\begin{bmatrix}y_1\\y_2\end{bmatrix}$$

$$x_1\begin{bmatrix}1\\0\\2\end{bmatrix} + x_2\begin{bmatrix}0\\1\\0\end{bmatrix} = y_1\begin{bmatrix}1\\0\\2\end{bmatrix} + y_2\begin{bmatrix}0\\1\\0\end{bmatrix}$$

$$(x_1 - y_1)\begin{bmatrix}1\\0\\2\end{bmatrix} + (x_2 - y_2)\begin{bmatrix}0\\1\\0\end{bmatrix} = \mathbf{0}$$

It is clear that $\begin{bmatrix}1\\0\\2\end{bmatrix}$ *and* $\begin{bmatrix}0\\1\\0\end{bmatrix}$ *are linearly independent. Therefore, we must have* $x_1 - y_1 = 0$ *and* $x_2 - y_2 = 0$*. But then* $x_1 = y_1$ *and* $x_2 = y_2$*, so*

$$\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}y_1\\y_2\end{bmatrix}$$

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At this point we can conjecture that being one-to-one is a necessary, but not a sufficient condition for a linear transformation to have an inverse. We will consider the other necessary condition next.

"Onto" Linear Transformations

Definition 2 (Onto)**.** *A linear transformation* $T : V \to W$ *is onto if for every element* $\mathbf{w}$ *of* W*, there exists an element* $\mathbf{v}$ *of* V *such that* $T(\mathbf{v}) = \mathbf{w}$*.*

Example 5. *The transformation in Example* **??** *is not onto.*

Explanation. *No element of* $\mathbb{R}^2$ *maps to* $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

Example 6. *Prove that the linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *whose standard matrix is*

$$A = \begin{bmatrix}1 & 0\\2 & 1\end{bmatrix}$$

is onto.

Proof *Let* $\mathbf{b}$ *be an element of the codomain (*$\mathbb{R}^2$*). We need to find* $\mathbf{x}$ *in the domain (*$\mathbb{R}^2$*) such that* $T(\mathbf{x}) = \mathbf{b}$*. Observe that* A *is invertible, and*

$$A^{-1} = \begin{bmatrix}1 & 0\\-2 & 1\end{bmatrix}$$

Let $\mathbf{x} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \mathbf{b}$, *then*

$$T(\mathbf{x}) = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \left(\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \mathbf{b} \right) = \mathbf{b}$$

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Example 7. *Prove that the linear transformation* $T : \mathbb{R}^3 \to \mathbb{R}^2$ *induced by*

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \end{bmatrix}$$

is onto.

Proof *Let* $\mathbf{b}$ *be an element of* $\mathbb{R}^2$. *We need to show that there exists* $\mathbf{x}$ *in* $\mathbb{R}^3$ *such that* $T(\mathbf{x}) = A\mathbf{x} = \mathbf{b}$. *Observe that*

$$rref(A) = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \end{bmatrix}$$

This means that $A\mathbf{x} = \mathbf{b}$ *has a solution (in fact, it has infinitely many solutions). Therefore* $\mathbf{b}$ *is an image of some* $\mathbf{x}$ *in* $\mathbb{R}^3$. *We conclude that* T *is onto.*

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Isomorphisms and Existence of Inverses

Definition 3. *A linear transformation that is one-to-one and onto is called an isomorphism.*

Example 8. *Let*

$$V = span \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

Define a linear transformation

$$T : V \to \mathbb{R}^2$$

by

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad and \quad T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Show that T *is an isomorphism.*

Proof *We will first show that T is one-to-one. Suppose*

$$T(\mathbf{u}) = T(\mathbf{v})$$

for some $\mathbf{u}$ and $\mathbf{v}$ in V. Vectors $\mathbf{u}$ and $\mathbf{v}$ are in the span of $\begin{bmatrix}1\\0\\0\end{bmatrix}$ and $\begin{bmatrix}1\\1\\1\end{bmatrix}$, so

$$\mathbf{u} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix} \quad \text{and} \quad \mathbf{v} = c\begin{bmatrix}1\\0\\0\end{bmatrix} + d\begin{bmatrix}1\\1\\1\end{bmatrix}$$

for some scalars a, b, c, d.

$$T(\mathbf{u}) = aT\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + bT\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}a\\a+b\end{bmatrix}$$

$$T(\mathbf{v}) = cT\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) + dT\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = c\begin{bmatrix}1\\1\end{bmatrix} + d\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}c\\c+d\end{bmatrix}$$

Thus,

$$\begin{bmatrix}a\\a+b\end{bmatrix} = \begin{bmatrix}c\\c+d\end{bmatrix}$$

This implies that $a = c$ which, in turn, implies $b = d$. This gives us $\mathbf{u} = \mathbf{v}$, and we conclude that T is one-to-one.

Next we will show that T is onto.

The key observation is that vectors $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ span $\mathbb{R}^2$. Perhaps this is easiest seen by noting that

$$\text{rank}\left(\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\right) = 2$$

This means that given a vector $\mathbf{v}$ in $\mathbb{R}^2$, we can write $\mathbf{v}$ as $\mathbf{v} = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$.

But this means that $\mathbf{v} = T\left(a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$ We conclude that T is onto.

Since T is a linear transformation that is one-to-one and onto, T is an isomorphism. ■

Theorem 1. *Let V and W be vector spaces, and let $T : V \to W$ be a linear transformation. Then T has an inverse if and only if T is an isomorphism.*

Proof We will first assume that T is an isomorphism and show that there exists a transformation $S : W \to V$ such that $S \circ T = \text{id}_V$ and $T \circ S = \text{id}_W$.

Because T is onto, for every $\mathbf{w}$ in W, there exists $\mathbf{v}$ in V such that $T(\mathbf{v}) = \mathbf{w}$. Moreover, because T is one-to-one, vector $\mathbf{v}$ is the only vector that maps to $\mathbf{w}$. To stress this, we will say that for every $\mathbf{w}$, there exists $\mathbf{v}_{\mathbf{w}}$ such that $T(\mathbf{v}_{\mathbf{w}}) = \mathbf{w}$. (Since every $\mathbf{v}$ maps to exactly one $\mathbf{w}$, this notation makes sense for elements of V as well.) We can now define $S : W \to V$ by $S(\mathbf{w}) = \mathbf{v}_{\mathbf{w}}$. Then

$$(S \circ T)(\mathbf{v}_{\mathbf{w}}) = S(T(\mathbf{v}_{\mathbf{w}})) = S(\mathbf{w}) = \mathbf{v}_{\mathbf{w}}$$

$$(T \circ S)(\mathbf{w}) = T(S(\mathbf{w})) = T(\mathbf{v}_{\mathbf{w}}) = \mathbf{w}$$

We conclude that $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$. Therefore S is an inverse of T.

We will now assume that T has an inverse S and show that T must be an isomorphism. To show that T is an isomorphism, we need to show that T is one-to-one and onto. Suppose

$$T(\mathbf{v}_1) = T(\mathbf{v}_2)$$

then

$$S(T(\mathbf{v}_1)) = S(T(\mathbf{v}_2))$$

but then

$$\mathbf{v}_1 = \mathbf{v}_2$$

We conclude that T is one-to-one.

Now suppose that $\mathbf{w}$ is in W. We need to show that some element of V maps to $\mathbf{w}$. Let $\mathbf{v} = S(\mathbf{w})$. Then

$$T(\mathbf{v}) = T(S(\mathbf{w})) = (T \circ S)(\mathbf{w}) = \mathrm{id}_W(\mathbf{w}) = \mathbf{w}$$

We conclude that T is onto. ■

Example 9. *Transformation T in Example* **??** *is invertible.*

Explanation. *We demonstrated that T is an isomorphism. By Theorem* **??***, T has an inverse.*

Recall that T was introduced in Exploration Problem **??** *to demonstrate that Theorem* **??** *is not always applicable. We now have additional tools. Theorem* **??** *assures us that T has an inverse, but does not help us find it. We will visit this problem again, in a different module, and find an inverse.*

Uniqueness of Inverses

Definition **??** refers to S as an inverse of T, implying that there may be more than one such transformation S. We will now show that if such a transformation S exists, it is unique. This will allow us to refer to *the* inverse of T and to start using T^{-1} to denote the unique inverse of T.

Theorem 2. *If T is a linear transformation, and S is an inverse of T. Then S is unique.*

Proof Let $T : V \to W$ be a linear transformation. If S is an inverse of T, then S satisfies

$$S \circ T = \mathrm{id}_V \quad \text{and} \quad T \circ S = \mathrm{id}_W$$

Suppose there is another transformation, S', such that

$$S' \circ T = \mathrm{id}_V \quad \text{and} \quad T \circ S' = \mathrm{id}_W$$

We now show that $S = S'$.

$$S = S \circ \mathrm{id}_W = S \circ (T \circ S') = (S \circ T) \circ S' = \mathrm{id}_V \circ S' = S'$$

■

Practice Problems

Problem 1 *Show that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ with standard matrix $A = \begin{bmatrix} 2 & -4 \\ -3 & 6 \end{bmatrix}$ is not one-to-one.*

Hint: *Show that multiple vectors map to $\mathbf{0}$.*

Problem 2 *Show that a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ with standard matrix $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \\ 0 & 1 \end{bmatrix}$ is not onto.*

Hint: *Find $\mathbf{b}$ such that $A\mathbf{x} = \mathbf{b}$ has no solutions.*

Problem 3 *Suppose that a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has a standard matrix A such that rref$(A) = I$.*

Prove that T is one-to-one.

Hint: *How many solutions does $A\mathbf{x} = \mathbf{b}$ have?*

Prove that T is onto.

Hint: *Does $A\mathbf{x} = \mathbf{b}$ have a solution for every $\mathbf{b}$?*

Problem 4 *Define a transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by*

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x \\ -2x + 4y \end{bmatrix}$$

Show that T *is an isomorphism.*

Hint: *Don't forget to show that* T *is linear.*

Problem 5 *Let* $V = span\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$. *Define a linear transformation* $T : V \to \mathbb{R}^2$ *by*

$$T\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Prove that T *is an isomorphism.*