Composition and Inverses of Linear Transformations

We define composition of linear transformations, inverse of a linear transformation, and discuss existence and uniqueness of inverses.

Composition of Linear Transformations

Definition and Properties

Definition 1. *Let U, V and W be vector spaces, and let $T : U \rightarrow V$ and $S : V \rightarrow W$ be linear transformations. The composition of S and T is the transformation $S \circ T : U \rightarrow W$ given by*

$$(S \circ T)(\mathbf{u}) = S(T(\mathbf{u}))$$

Example 1. *Define*

$$T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad by \quad T\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\right) = \begin{bmatrix}u_1 + u_2\\3u_1 + 3u_2\end{bmatrix}$$

$$S : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \quad by \quad S\left(\begin{bmatrix}v_1\\v_2\end{bmatrix}\right) = \begin{bmatrix}3v_1 - v_2\\-3v_1 + v_2\end{bmatrix}$$

Examine the effect of $S \circ T$ on vectors of $\mathbb{R}^2$.

Explanation. *From the computational standpoint, the situation is simple.*

$$\begin{aligned}(S \circ T)\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\right) &= S\left(T\left(\begin{bmatrix}u_1\\u_2\end{bmatrix}\right)\right) = S\left(\begin{bmatrix}u_1 + u_2\\3u_1 + 3u_2\end{bmatrix}\right)\\ &= \begin{bmatrix}3(u_1 + u_2) - (3u_1 + 3u_2)\\-3(u_1 + u_2) + (3u_1 + 3u_2)\end{bmatrix}\\ &= \mathbf{0}\end{aligned}$$

This means that $S \circ T$ maps all vectors of $\mathbb{R}^2$ to $\mathbf{0}$.

In addition to the computational approach, it is also useful to visualize what happens geometrically.

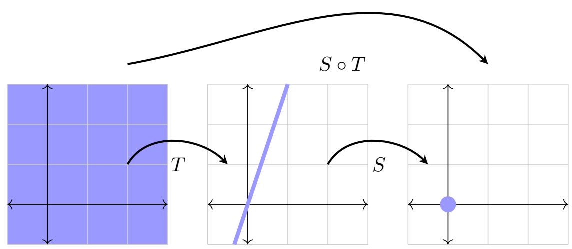
Learning outcomes:

Author(s): Anna Davis and Paul Zachlin

First, observe that $T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ 3u_1 + 3u_2 \end{bmatrix} = (u_1 + u_2)\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *So, the image of any vector of* $\mathbb{R}^2$ *under* T *lies on the line determined by the vector* $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

Even though S *is defined on all of* $\mathbb{R}^2$, *we are only interested in the action of* S *on vectors along the line determined by* $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. *Our computations showed that all such vectors map to* $\mathbf{0}$.

The actions of individual transformations, as well as the composite transformation are shown below.



Theorem 1. *The composition of two linear transformations is linear.*

Proof Let $T : U \to V$ and $S : V \to W$ be linear transformations. We will show that $S \circ T$ is linear. For all vectors $\mathbf{u}_1$ and $\mathbf{u}_2$ of U and scalars a and b we have:

$$\begin{aligned}(S \circ T)(a\mathbf{u}_1 + b\mathbf{u}_2) &= S(T(a\mathbf{u}_1 + b\mathbf{u}_2)) \\ &= S(aT(\mathbf{u}_1) + bT(\mathbf{u}_2)) \\ &= aS(T(\mathbf{u}_1)) + bS(T(\mathbf{u}_2)) \\ &= a(S \circ T)(\mathbf{u}_1) + b(S \circ T)(\mathbf{u}_2)\end{aligned}$$

■

Theorem 2. *Composition of linear transformations is associative. In other words, for linear transformations* T, S *and* R

We have $(R \circ S) \circ T = R \circ (S \circ T)$.

Proof For all $\mathbf{u}$ in U we have:

$$\begin{aligned}((R \circ S) \circ T)(\mathbf{u}) &= (R \circ S)(T(\mathbf{u})) = R(S(T(\mathbf{u}))) \\ &= R((S \circ T)(\mathbf{u})) = (R \circ (S \circ T))(\mathbf{u})\end{aligned}$$

■

Composition and Matrix Multiplication

In this section we will consider linear transformations of $\mathbb{R}^n$ and their standard matrices.

Theorem 3. *Let $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^m \to \mathbb{R}^p$ be linear transformations with standard matrices M_T and M_S, respectively. Then the composite transformation $S \circ T : \mathbb{R}^n \to \mathbb{R}^p$ has a standard matrix given by the product $M_S M_T$.*

Proof For all $\mathbf{v}$ in $\mathbb{R}^n$ we have:

$$(S \circ T)(\mathbf{v}) = S(T(\mathbf{v})) = S(M_T\mathbf{v}) = M_S(M_T\mathbf{v}) = (M_S M_T)\mathbf{v}$$

■

Example 2. *In Example 1, we discussed a composite transformation $S \circ T : \mathbb{R}^2 \to \mathbb{R}^2$ given by:*

$$T\left(\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) = \begin{bmatrix} u_1 + u_2 \\ 3u_1 + 3u_2 \end{bmatrix} \quad \textit{and} \quad S\left(\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}\right) = \begin{bmatrix} 3v_1 - v_2 \\ -3v_1 + v_2 \end{bmatrix}$$

Express $S \circ T$ as a matrix transformation.

Explanation. *The standard matrix for $T : \mathbb{R}^2 \to \mathbb{R}^2$ is*

$$\begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix}$$

and the standard matrix for $S : \mathbb{R}^2 \to \mathbb{R}^2$ is

$$\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$$

The standard matrix for $S \circ T$ is the product

$$\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 3 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We conclude this section by revisiting the associative property of matrix multiplication. At the time matrix multiplication was introduced, we found it extremely cumbersome to prove that for appropriately sized matrices A, B and C, we have $(AB)C = A(BC)$ (See Theorem **??**). We are now in a position to prove this result with ease.

Every matrix induces a linear transformation. The product of two matrices can be interpreted as a composition of transformations. Since transformation composition is associative, so is matrix multiplication. We formalize this observation as a theorem.

Theorem 4 (Associativity of Matrix Multiplication)**.** *Let A, B and C be matrices of appropriate dimensions so that the product $(AB)C$ is defined. Then*

$$ABC = (AB)C = A(BC)$$

Inverses of Linear Transformations

Exploration Problem 1. *Define a linear transformation* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $T(\mathbf{v}) = 2\mathbf{v}$*. In other words,* T *doubles every vector in* $\mathbb{R}^2$*. Now define* $S : \mathbb{R}^2 \to \mathbb{R}^2$ *by* $S(\mathbf{v}) = \frac{1}{2}\mathbf{v}$*. What happens when we compose these two transformations?*

$$(S \circ T)(\mathbf{v}) = S(T(\mathbf{v})) = S(2\mathbf{v}) = \left(\frac{1}{2}\right)(2)\mathbf{v} = \mathbf{v}$$

$$(T \circ S)(\mathbf{v}) = T(S(\mathbf{v})) = T(\frac{1}{2}\mathbf{v}) = (2)\left(\frac{1}{2}\right)\mathbf{v} = \mathbf{v}$$

Both composite transformations return the original vector $\mathbf{v}$*. In other words,* $S \circ T = id_{\mathbb{R}^2}$ *and* $T \circ S = id_{\mathbb{R}^2}$*. We say that* S *is an inverse of* T*, and* T *is an inverse of* S*.*

Definition 2. *Let* V *and* W *be vector spaces, and let* $T : V \to W$ *be a linear transformation. A transformation* $S : W \to V$ *such that* $S \circ T = id_V$ *and* $T \circ S = id_W$ *is called an inverse of* T*. If* T *has an inverse,* T *is called invertible.*

Example 3. *Let* $T : \mathbb{R}^2 \to \mathbb{R}^2$ *be a transformation defined by* $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ x-y \end{bmatrix}$*. (How would you verify that* T *is linear?) Show that* $S : \mathbb{R}^2 \to \mathbb{R}^2$ *given by* $S\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 0.5x+0.5y \\ 0.5x-0.5y \end{bmatrix}$ *is an inverse of* T*.*

Explanation. *We will show that* $S \circ T = id_{\mathbb{R}^2}$*.*

$$\begin{aligned} (S \circ T)\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) &= S\left(T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)\right) = S\left(\begin{bmatrix} x+y \\ x-y \end{bmatrix}\right) \\ &= \begin{bmatrix} 0.5(x+y)+0.5(x-y) \\ 0.5(x+y)-0.5(x-y) \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \end{aligned}$$

We leave it to the reader to verify that $T \circ S = id_{\mathbb{R}^2}$*.*

Linearity of Inverses of Linear Transformations

Definition 2 does not specifically require an inverse S of a linear transformation T to be linear. This is because the requirement that $S \circ T = \mathrm{id}_V$ and $T \circ S = \mathrm{id}_W$ is sufficient to guarantee that S is linear.

Theorem 5. *Suppose* $T : V \to W$ *is an invertible linear transformation. Let* S *be an inverse of* T*. Then* S *is linear.*

Proof The proof of this result is left to the reader. ■

Linear Transformations of $\mathbb{R}^n$ and the Standard Matrix of the Inverse Transformation

Every linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation. (See Theorem **??**) If T has an inverse S, then S is also a matrix transformation. Let M_T and M_S denote the standard matrices of T and S, respectively. We see that $S \circ T = \mathrm{id}_{\mathbb{R}^n}$ and $T \circ S = \mathrm{id}_{\mathbb{R}^m}$ if and only if $M_S M_T = I_{n\times n}$ and $M_T M_S = I_{m\times m}$. In other words, T and S are inverse transformations if and only if M_T and M_S are matrix inverses.

Note that if S is an inverse of T, then M_T and M_S are square matrices, and $n = m$.

Theorem 6. *Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation, and let M be the standard matrix of T.*

(a) **Existence of Inverses.** *T is invertible if and only if M is invertible. If T is invertible, then the inverse is induced by M^{-1}.*

(b) **Uniqueness of Inverses.** *If S is an inverse of T, then S is unique.*

Proof Part (a) follows directly from the preceding discussion. Part (b) follows from uniqueness of matrix inverses. (Theorem **??**) ■

Please note that Theorem 6 is only applicable in the context of linear transformations of $\mathbb{R}^n$ and their standard matrices. The following example provides us with motivation to investigate inverses further, which we will do in another module.

Exploration Problem 2. *Let*

$$V = span\left(\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

Define a linear transformation

$$T : V \to \mathbb{R}^2$$

by

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}1\\1\end{bmatrix} \quad \text{and} \quad T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

This information is sufficient to define a linear transformation because every element $\mathbf{v}$ of V can be written uniquely in the form

$$\mathbf{v} = a\begin{bmatrix}1\\0\\0\end{bmatrix} + b\begin{bmatrix}1\\1\\1\end{bmatrix}$$

and our condition that T is linear determines the image of $\mathbf{v}$ *as follows:*

$$T(\mathbf{v}) = a\begin{bmatrix}1\\1\end{bmatrix} + b\begin{bmatrix}0\\1\end{bmatrix}$$

Geometrically speaking, the domain of T is a plane in $\mathbb{R}^3$ and its codomain is $\mathbb{R}^2$. Does T have an inverse?

We are not in a position to answer this question right now because Theorem 6 does not apply to this situation.

This problem highlights the necessity of a further discussion of inverses. In Example **??**, *we will prove that T has an inverse, and in Example* **??** *we will relate the existence of an inverse of T to a matrix associated with T.*

Practice Problems

Problem 1 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations with standard matrices*

$$M_T = \begin{bmatrix}2 & -4\\1 & 2\end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix}1 & -1\\2 & 1\end{bmatrix}$$

respectively. Describe the actions of T, S, and $S \circ T$ geometrically, as in Figure 1.

Problem 2 *Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $S : \mathbb{R}^2 \to \mathbb{R}^2$ be linear transformations with standard matrices*

$$M_T = \begin{bmatrix}1 & 0 & -1\\2 & 1 & 0\end{bmatrix} \quad \text{and} \quad M_S = \begin{bmatrix}-1 & 2\\1 & -2\end{bmatrix}$$

respectively. Describe the actions of T, S, and $S \circ T$ geometrically, as in Figure 1.

Problem 3 *Complete the Explanation of Example 3 by verifying that $T \circ S = id_{\mathbb{R}^2}$.*

Problem 4 *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation given by*

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}2x - 5y\\-x + 3y\end{bmatrix}$$

Propose a candidate for an inverse of T and verify your choice using Definition 2.

Problem 5 *Explain why linear transformation $T : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ given by*

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x + 2y \\ -3x - 3y \end{bmatrix}$$

does not have an inverse.

Problem 6 *Prove Theorem 5.*

Problem 7 *Suppose $T : U \rightarrow V$ and $S : V \rightarrow W$ are linear transformations with inverses T' and S' respectively. Prove that $T' \circ S'$ is an inverse of $S \circ T$.*