MATH 205 Survival Guide - Tricky Integrals

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1 Commit To Memory

Solving integrals is just patterns. The lectures and assignments teach you a few but there's a lot more in the exams. Some are simply algebraic tricks that you have to remember. This chapter is a catalog of particularly tricky integral patterns from all the past evaluations.

This isn't a replacement for mastering integration techniques. You should be able to apply them quickly and develop some intuition in order to finish the problem once you've identified the trick.

For time-saving tricks see Chapter 10. Also note that integrals in the midterms are generally harder than in the finals.

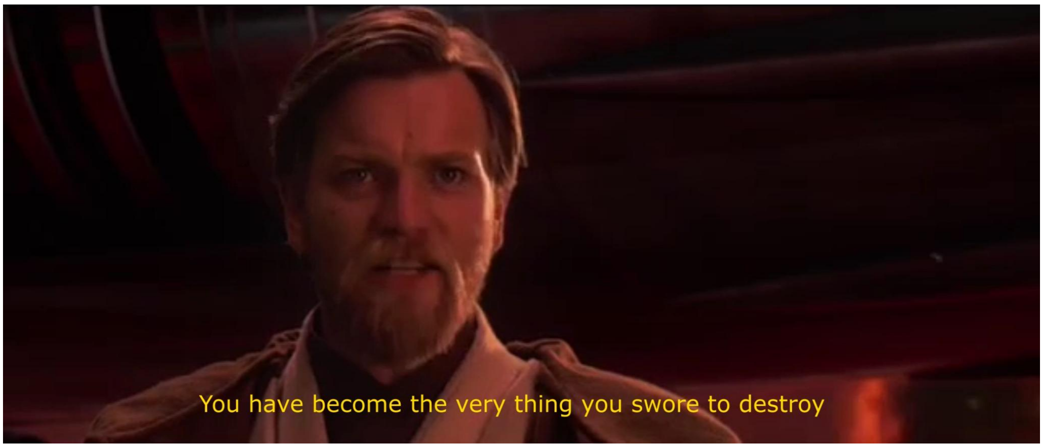

Figure 1: Obi-Wan applied integration by parts but the result required yet another integration by parts.

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2 Tricks

2.1 Always Expect the Perfect Square

$$\int \frac{1}{1-x^2}\,dx$$

It's not arcsin or arctan? Is it some integral identity we didn't learn? Some kind of u-substitution?

Nope, it's a perfect square and it can be solved by partial fraction decomposition:

$$\int \frac{1}{(x-1)(x+1)}\,dx$$

Same goes for:

$$\int \frac{1}{u^2-4}\,du = \int \frac{1}{(u-2)(u+2)}\,du$$

Remember, the perfect square formula is:

$$a^2 - b^2 = (a-b)(a+b)$$

2.2 x in Terms of u

Always try u-substitution first, even when it looks like a perfect candidate for trigonometric substitution:

$$\int \frac{3x^3}{\sqrt{16-x^2}}\,dx \qquad u = 16 - x^2 \qquad x^2 = 16 - u \qquad du = -2x\,dx$$

$$\frac{-3}{2}\int \frac{x^2}{\sqrt{u}}\,du = \frac{-3}{2}\int \frac{16-u}{\sqrt{u}}\,du = \frac{-2}{3}\int 16u^{\frac{-1}{2}} - u^{\frac{1}{2}}\,du \qquad \text{Power rule} \ldots$$

So the steps are:

1 Pick the expression under the square root as u
2 Find x in terms of u
3 Perform u-substitution
4 Replace the remaining x term in the integral by its equivalent in terms of u
5 Simplify and apply the power rule on the now separate terms

Another example:

$$\int \frac{x^2}{\sqrt{2x+1}}\,dx \qquad u = 2x+1 \qquad x = \frac{u-1}{2} \qquad du = 2\,dx$$

$$\frac{1}{2}\int \frac{(\frac{u-1}{2})^2}{\sqrt{u}}\,du = \frac{1}{2}\times\frac{1}{4}\int \frac{(u-1)^2}{\sqrt{u}}\,du = \frac{1}{8}\int \frac{u^2-2u+1}{\sqrt{u}}\,du \quad \ldots$$

Distribute denominator + power rule ...

A few more integrals that can be solved by this method:

$$\int x^2\sqrt{1+x}\,dx \qquad \int \frac{x}{\sqrt{2x+1}}\,dx \qquad \int \frac{x^3}{\sqrt{x^2+16}}\,dx$$

And here's one that can only be solved by trigonometric substitution:

$$\int \frac{x^2}{\sqrt{9-x^2}}\,dx$$

I haven't found a reliable way to determine when this method will or will not work but it never hurts to quickly test it on scrap paper.

2.3 Partial Fractions Flowchart

TODO

2.4 u^2

This technique relies on the exponent property $y^{2x} = y^{x^2}$ and picking y^x for u-substitution.

For example:

$$\int \frac{2^x}{2^{2x}-4}\,dx = \int \frac{2^x}{2^{x^2}-4}\,dx \qquad u = 2^x \qquad du = 2^x \ln 2\,dx$$

$$\frac{1}{\ln 2}\int \frac{1}{u^2-4}\,du = \frac{1}{\ln 2}\int \frac{1}{(u-2)(u+2)}\,du \qquad \text{Partial fraction decomposition} \ldots$$

Also works for these integrals:

$$\int \frac{e^x}{e^{2x}+16}\,dx \qquad \int \frac{e^x}{e^{2x}-e^x+2}\,dx \qquad \int \frac{e^{2x}}{e^{4x}+9}\,dx \ \ \text{(Pick } e^{2x} \text{ as } u\text{)}$$

2.5 By Parts

TODO

2.6 ln **Substitution**

When the integral contains a single x term in the denominator and ln terms anywhere then it can be immediately solved by u-substitution:

$$\int \frac{1}{x \ln^3 x} \, dx \qquad u = \ln x \qquad du = \frac{1}{x} \, du$$

$$\int \frac{1}{u^3} \, du = \int u^{-3} \, du = \frac{1}{2u^2} + C = \frac{1}{2 \ln^2 x} + C$$

2.7 sin/cos

$$\int \sin^n x \cos^m x \, dx \quad \text{or similar forms like} \int \frac{sin^n x}{\cos^m x} \, dx$$

These are called trigonometric integrals in the course notes.

The general strategy is to isolate a single term of the odd-powered term and to convert the rest to the even term using the $[\sin^2 x + \cos^2 x = 1]$ identity, then u-substitute.

If both m and n are even (and this happens much more often) then the following trig identities should be used instead:

$$\sin^2 x = \frac{1 - \cos(2x)}{2} \qquad \cos^2 x = \frac{1 + \cos(2x)}{2} \qquad \sin x \cos x = \frac{\sin(2x)}{2}$$

For example:

$$\int \cos^4 x \tan^2 x \, dx = \int \cos^4 x \frac{\sin^2 x}{\cos^2 x} \, dx = \int \cos^2 x \sin^2 x \, dx$$

$$= \int \left(\frac{\sin(2x)}{2} \right)^2 dx = \frac{1}{4} \int \sin^2(2x) \, dx = \frac{1}{4} \times \frac{1}{2} \int 1 - \cos(4x) \, dx$$

$$= \frac{x}{8} - \frac{\sin(4x)}{32} + C$$

2.8 sec/tan

$$\int \sec^n x \tan^m x \, dx$$

Overall this is similar to the sin/cos type of integral. Some sources give strategies in function of even/odd n and m but I don't find that very intuitive.

Instead, test both of these using the $[1 + \tan^2 x = \sec^2 x]$ identity:

1. Isolate a $\sec^2 x$ term and convert the rest to $\tan x$ terms
2. Isolate a $\sec x \tan x$ term and convert the rest to $\sec x$ terms

If neither work then try integration by parts or the strategy described in the next section. You won't be needing wacky trig identities here, those are reserved for sin/cos integrals.

Note that csc/cot integrals are solved the exact same way but using the $[1 + \cot^2 x = \csc^2 x]$ identity instead.

2.9 Manifesting $\sec^2$

If you have a seemingly unsolvable sec/tan integral then multiply the numerator and the denominator by $\sec^2$ and convert the denominator to tan:

$$\int \frac{1}{\sec^2 x + 3}\,dx = \int \frac{\sec^2 x}{\sec^2 x} \times \frac{1}{\sec^2 x + 3}\,dx = \int \frac{\sec^2 x}{(\tan^2 x + 1)(\tan^2 x + 1 + 3)}\,dx$$

$$= \int \frac{\sec^2 x}{(\tan^2 x + 1)(\tan^2 x + 4)}\,dx \qquad u = \tan x \qquad du = \sec^2 x\,dx$$

$$\int \frac{1}{(u^2 + 1)(u^2 + 4)}\,du \qquad \text{Partial fraction decomposition} \ldots$$

2.10 **Hidden** $\sec^2$

We know that [$\sec\theta = \frac{1}{\cos\theta}$]. So when you see a $\cos^2\theta$ denominator always try converting it to a $\sec^2\theta$:

$$\int \frac{\sqrt{1+2\tan x}}{\cos^2 x}\,dx = \int \sec^2 x\,\sqrt{1+2\tan x}$$

$$u = 1+2\tan x \qquad du = 2\sec^2 x\,dx$$

$$= \frac{1}{2}\int \sqrt{u}\,du \qquad \text{Power rule} \ldots$$

2.11 arctan **Substitution**

This type of question comes back to haunt students once a decade.

There's more than one way to solve it but the key is to u-substitute using the arctan term:

$$\int \frac{1+\tan^{-1}\left(\frac{x}{3}\right)}{9+x^2}\,dx$$

$$u = 1+\tan^{-1}\left(\frac{x}{3}\right) \quad du = \frac{1}{3}\times\frac{1}{1+\left(\frac{x}{3}\right)^2}\,dx = \frac{1}{3}\times\frac{1}{1+\frac{x^2}{9}}\,dx = \frac{1}{3}\times\frac{9}{9+x^2}\,dx = \frac{3}{9+x^2}\,dx$$

$$\frac{1}{3}\int u\,du = \frac{1}{3}\times\frac{u^2}{2}+C = \frac{1}{6}\left(1+\tan^{-1}\left(\frac{x}{3}\right)\right)^2+C$$