MATH 205 Survival Guide - Time Saving Tricks

By Frogge

1 What They Don't Teach You in School

Math exams are usually very tight on time and this one is no exception.

There's time-saving tricks in MATH 205 but I don't see them discussed anywhere which is a shame. Here's a bunch.

Figure 1: Gotta go fast!

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2 Tricks

2.1 Limit Extraction

In the process of evaluating Riemann sum limits you may encounter situations like:

$$\lim_{n\to\infty} \frac{125}{n^3} \frac{n(n+1)(2n+1)}{6}$$

One way to solve this formally is to expand the numerator:

$$= \lim_{n\to\infty} \frac{125}{n^3} \frac{(n^2+n)(2n+1)}{6} = \lim_{n\to\infty} \frac{125}{n^3} \frac{2n^3+n^2+2n^2+n}{6}$$

$$= \lim_{n\to\infty} \frac{125}{n^3} \frac{2n^3+3n^2+n}{6} = \lim_{n\to\infty} \frac{125}{6} (2+\frac{3}{n}+\frac{1}{n^2})$$

But this is also formal and much faster:

$$= \lim_{n\to\infty} \frac{125}{n^3} \frac{n\,n\,(1+\frac{1}{n})\,n\,(2+\frac{1}{n})}{6} = \lim_{n\to\infty} \frac{125}{6} (1+\frac{1}{n})(2+\frac{1}{n})$$

The same can be used for limits that involve radicals, commonly seen in exam questions about sequences:

$$\lim_{n\to\infty} \frac{(-1)^n\,2n}{\sqrt{1+100n^3}} = \lim_{n\to\infty} \frac{(-1)^n\,2n}{n^{\frac{3}{2}}\sqrt{\frac{1}{n^3}+100}}$$

2.2 Definite Integrals in Terms of u

Many people will show you both methods for solving definite integrals and tell you that it doesn't matter which one you pick. But converting your bounds in terms of u is the faster method because it will lead to fewer steps and re-writing overall.

2.3 Cover-Up Rule

This trick saves a ton of time for partial fraction decomposition!

Requirements:

- The denominator consists entirely of linear terms of the form $(x \pm a)$.
- The degree of the numerator is less than the degree of the denominator (just like in regular partial fraction decomposition).
- Convert any $(a \pm x)$ terms to $-(x \pm a)$. x must be the leading term.

Rule:

$$\frac{f(x)}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$$

To find A we substitute $x = a$ and evaluate the original expression excluding the $(x - a)$ term in the denominator. Similarly for B and C:

$$A = \frac{f(a)}{(a-b)(a-c)} \qquad B = \frac{f(b)}{(b-a)(b-c)} \qquad C = \frac{f(c)}{(c-a)(c-b)}$$

Example:

$$\frac{3x}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}$$

$$A = \frac{3(1)}{(1+2)} = \frac{3}{3} = 1 \qquad B = \frac{3(-2)}{(-2-1)} = \frac{-6}{-3} = 2$$

More examples at https://brilliant.org/wiki/partial-fractions-cover-up-rule/.

2.4 Leaving the Multiplier in the Numerator

It's always tempting to extract constant multipliers out of an integral, and it usually helps. But in my experience it always backfires for partial fraction decomposition, so don't do it in that situation. Here's an example.

Keeping the multiplier:

$$\int \frac{4}{(x+2)(x+6)}\,dx = \int \frac{A}{x+2} + \frac{B}{x+6}\,dx$$

$$A = 1 \qquad B = -1 \qquad \text{by Cover-Up Rule}$$

Extracting the multiplier:

$$4\int \frac{1}{(x+2)(x+6)}\,dx = 4\int \frac{A}{x+2} + \frac{B}{x+6}\,dx$$

$$A = \tfrac{1}{4} \qquad B = -\tfrac{1}{4} \qquad \text{by Cover-Up Rule}$$

So extracting the multiplier introduces fractions (prone to mistakes) and extra steps.

2.5 Quadratic Partial Fractions

Here's a conceptually simple method for solving partial fraction decompositions similar to:

$$\int \frac{4}{x\,(x^2+4)}\,dx = \int \frac{A}{x} + \frac{Bx+C}{x^2+4}\,dx$$

Expand the left-hand side and group by similar terms:

$$A(x^2+4) + x(Bx+C) = 4$$

$$Ax^2 + 4A + Bx^2 + Cx = 0\,x^2 + 0\,x + 4$$

$$(\underline{A+B})x^2 + \underline{C}x + \underline{4A} = \underline{0}\,x^2 + \underline{0}\,x + \underline{4}$$

By matching similar terms between the left-hand side and the right-hand side we obtain:

$$\begin{gathered} 4A = 4 \quad\quad A = 1 \\ A + B = 0 \quad\quad B = -1 \\ C = 0 \end{gathered}$$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+4}\,dx$$

2.6 Numerator $\pm$ Adjustment

To apply integration by partial fractions the degree of the numerator must be less than the degree of the denominator. The degree of the numerator can be reduced through polynomial division.

But it's possible to avoid polynomial division when the degrees are equal, and they often are on the exam. For example:

$$\int \frac{x^2+x+1}{x^2+3x+2}\,dx =$$

$$\int \frac{(x^2+3x+2)-2x-1}{x^2+3x+2}\,dx =$$

$$\int 1-\frac{2x+1}{x^2+3x+2}\,dx =$$

$$x-\int \frac{2x+1}{(x+1)(x+2)}\,dx$$

2.7 Tabular Method

This is a faster and less messy technique for integration by parts. It's great for integrals which consist of a simple repeatedly differentiable u and integrable dv.

Consider this integral:

$$\int x^3 \cos(2x)\, dx$$

Normally it would require a few repeated applications of integration by parts. Which would generate long expressions with many opportunities to make mistakes. The tabular equivalent is much simpler.

For info about this technique see:

- https://brilliant.org/wiki/tabular-integration/
- http://mathonline.wikidot.com/tabular-integration
- https://math.stackexchange.com/a/3484366

2.8 $\int v\,du$ Method

There's a method in-between regular integration by parts and tabular:

$$\int (2x + x^2)\, cos(2x)\, dx$$

$$u = 2x + x^2 \qquad du = 2 + 2x\, dx \qquad dv = \cos(2x)\, dx \qquad v = \tfrac{\sin(2x)}{2}$$

Now instead of writing out $uv - \int v\, du$, only write $\int v\, du$:

$$\frac{1}{2}\int (2 + 2x)\ \sin(2x)\, dx$$

$$u = 2 + 2x \qquad du = 2\, dx \qquad dv = \sin(2x)\, dx \qquad v = \tfrac{-\cos(2x)}{2}$$

And only $\int v\, du$ again:

$$-\int \cos(2x)\, dx = \frac{-\sin(2x)}{2}$$

Finally, combine all the terms as if you had been writing $uv - \int v\, du$ the whole time:

$$\frac{(2x + x^2)\sin(2x)}{2} - \frac{1}{2}\left(\frac{-(2 + 2x)\cos(2x)}{2} + \frac{\sin(2x)}{2}\right) + C$$

It's very similar to regular integration by parts but it saves a lot of re-writing.

2.9 U-Sub Before Trig-Sub

Trigonometric substitution is the slowest integration technique, it can be pretty awful. There are integrals that look like they need a trig substitution but can actually be u-substituted instead, saving a huge amount of time. Let's look at an example and solve it by trig:

$$\int \frac{x^5}{\sqrt{x^2+4}}\,dx$$

$$x = 2\tan\theta \qquad dx = 2\sec^2\theta\,d\theta \qquad \sqrt{4\tan^2\theta+4} = 2\sec\theta$$

$$\int \frac{(2\tan\theta)^5}{2\sec\theta} 2\sec^2\theta\,d\theta = 32\int \tan^5\theta\sec\theta\,d\theta = 32\int (\sec^2\theta - 1)^2 \sec\theta\tan\theta\,d\theta$$

$$u = \sec\theta \qquad du = \sec\theta\tan\theta\,d\theta$$

$$32\int (u^2-1)^2\,du = 32\int u^4 - 2u^2 + 1\,du = 32(\frac{1}{5}u^5 - \frac{2}{3}u^2 + u) + C$$

$$= \frac{32}{5}\sec^5\theta - \frac{64}{3}\sec^3\theta + 32\sec\theta + C$$

We know $\sec\theta = \frac{\sqrt{x^2+4}}{2}$

Substitute $\sec\theta$ and simplify ...

That was tiring! Let's look at the same example with u-sub:

$$\int \frac{x^5}{\sqrt{x^2+4}}\,dx$$

$$u = x^2+4 \quad x = \sqrt{u-4} \quad du = 2x\,dx$$

$$\frac{1}{2}\int \frac{(u-4)^{\frac{4}{2}}}{\sqrt{u}}\,du = \frac{1}{2}\int \frac{(u-4)^2}{u^{\frac{1}{2}}}\,du = \frac{1}{2}\int \frac{u^2-8u+16}{u^{\frac{1}{2}}}\,du$$

$$= \frac{1}{2}\int u^{\frac{3}{2}} - 8u^{\frac{1}{2}} + 16u^{\frac{-1}{2}}\,du = \frac{1}{2}(\frac{2}{5}u^{\frac{5}{2}} - \frac{16}{3}u^{\frac{3}{2}} + 32u^{\frac{1}{2}}) + C$$

$$= \frac{1}{5}(x^2+4)^{\frac{5}{2}} - \frac{8}{3}(x^2+4)^{\frac{3}{2}} + 16(x^2+4)^{\frac{1}{2}} + C$$

As you can see doing u-sub first leads to a shorter and simpler solution. I don't have a general blueprint for when these situations arise though. I just always try u-sub, even when it looks like a clear case of trig sub.

2.10 Volumes With a Single Integral

It can be tempting to compute the inner and outer volumes separately:

$$\pi \int_a^b (outer\, radius)^2 \, dx - \pi \int_a^b (inner\, radius)^2 \, dx$$

In my experience however it's best to include them under a single integral because the exam questions are designed to cancel out terms in that situation:

$$\pi \int_a^b (outer\, radius)^2 - (inner\, radius)^2 \, dx$$

For example with outer radius [2] and inner radius $[2 - \sin x]$:

$$\pi \int_a^b (2)^2 - (2 - \sin x)^2 \, dx = \pi \int_a^b 4 - (4 - 4 \sin x + \sin^2 x) \, dx = \pi \int_a^b 4 \sin x - \sin^2 x \, dx$$

2.11 Improper Integrals in Terms of u

Most of the improper integral solving process is identical to solving definite integrals. Similarly, converting the integration bounds in terms of u is more efficient than converting everything back to x when u-substituting. For example:

$$\lim_{a\to 0^+}\int_a^1 \frac{\ln x}{x}\,dx$$

$$u = \ln x \qquad du = \tfrac{1}{x}\,dx \qquad a \Rightarrow \ln a \qquad 1 \Rightarrow \ln 1 = 0$$

$$= \lim_{a\to 0^+}\int_{\ln a}^{0} u\,du = \lim_{a\to 0^+}\frac{u^2}{2}\Big|_{\ln a}^{0}$$

$$= \frac{0}{2} - \lim_{a\to 0^+}\frac{(\ln a)^2}{2} = 0 - \frac{(\lim_{a\to 0^+}\ln a)^2}{2} = \frac{-(-\infty)^2}{2} = \frac{-\infty}{2} = -\infty$$

2.12 Power Series Radius

Find the a, b interval of convergence first. The radius of convergence is then simply:

$$\frac{b-a}{2}$$

Don't bother inferring it from the inequalities.

Note that the radius is always a non-negative number or ∞.