COMPGV17 GLM Practical 3

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Task 1: Basic Concepts in Vector Space

Vector space is a mathematical object that generalises our common notion of the 3-dimensional space that we live in. The key features of a vector space that we will rely upon are the following:

1. We can add any two vectors **a** and **b** from the same vector space **V**; the resulting vector will belong to **V**. A natural extension of this is that we can add any number of vectors together.
2. Each vector space has a vector **0** such that, if it is added to any vector **a** in **V**, **a** stays the same.
3. Each vector **a** has a negated version **a'** in **V** such that **a** + **a'** = **0**; we denote **a'** as **-a**. This is important because now we can subtract two vectors **a** and **b** which is really adding **a** and **-b**.
4. We can also multiply any vector **a** in **V** with a scalar α such that α**a** is another vector in **V**.

Putting all these together, we can define the notion of linear combination of vectors:

5. A linear combination of any two vectors **a** and **b** in **V** with two scalars α and β**,** α**a** + β**b**, gives rise to another vector in **V**. Again, by extension, this will work for a linear combination of any number of vectors.

The linear combination is important because it gives us a way a) to represent any vectors belonging to a vector space and b) to define subspaces smaller vector spaces that are strictly contained by the original vector space. The next set of questions explore this notion in detail.

Given **a** = [1 1 0]', **b** = [0 1 0]', and **c** = [1 0 0]' from a 3-dimensional vector space $\mathbf{R}^3$.

1. Represent the collection of all the vectors, both algebraically and graphically, that can be produced from the linear combination of **a**. Verify that these vectors form a vector space of their own, which I will refer to as **A**, by checking that they satisfy the features listed above. We call **A** a vector space spanned by **a**. What is the dimension of **A**? Check that **A** is a subspace of $\mathbf{R}^3$, i.e., any vector in **A** belongs to $\mathbf{R}^3$.
2. What is the space **B** spanned by **b**? What is its dimension? Is **B** a subspace of $\mathbf{R}^3$? Again, represent it both algebraically and graphically. What do **A** and **B** have in common?
3. What is the space **AB** spanned by **a** and **b**? What is its relation to $\mathbf{R}^3$?
4. What is the space **BC** spanned by **b** and **c**? What is its relation to **AB** and $\mathbf{R}^3$?
5. What is the space **ABC** spanned by all three vectors?

An important way of representing a linear combination of a set of vectors is as the product of a matrix and a vector. If we have n vectors, the i-th vector denoted by $\mathbf{X}_i$, their linear combination formed with n scalars, the i-th scalar denoted by α_i, can be written as

$$\sum_{i=1}^{n} \alpha_i X_i = X\alpha$$

with the i-th column of X being $\mathbf{X}_i$ and the i-th row of α being α_i. We call the subspace represented by X its column space.

6. What are the X and α corresponding to 1.1 - 1.5?

Another key concept in vector space is the notion of linear dependence. We say that a set of vectors are linearly dependent if one of the vectors can be written as a linear combination of the other vectors. This is another way of saying that there are redundant information in the set, i.e., the vector space spanned by these vectors can also be spanned by removing one or more vectors from the set.

7. Are **a**, **b**, and **c** linearly independent?

The dimension of a vector space is determined by the minimal number of vectors that can span the space. For the minimality to hold, this necessarily means that the vectors in this minimal set have to be linearly independent. We call such minimal set of vectors the basis of the vector space; linear combination of a basis can represent any vector of the vector space.

8. Do **a**, **b**, and **c** form a basis of $\mathbf{R}^3$? What is a basis of $\mathbf{R}^3$?

One last key concept is the notion of orthogonal complement. Given a subspace of a vector space, its orthogonal complement is a subspace that consists of only the vectors that are orthogonal to the vectors in the original subspace. A key property relating a subspace and its orthogonal complement is that the sum of their dimensionalities is equal to the dimensionality of the vector space they both belong.

9. Based on this property, what is the dimensionality of the orthogonal complement to **A**? What about the other subspaces we considered above?

As we have seen, the subspaces defined by linear combination of vectors can be represented by a matrix X (1.6). The perpendicular projection operators (P and I-P) associated with this matrix allow us to divide any vector into a part that lies in this subspace and a part that lies in its orthogonal complement.

10. What are the perpendicular projection operator P corresponding to **A**? Check that I-P is also a perpendicular projection operator. Apply these to **a**, **b**, and **c** to determine, for each, its two parts that lie in **A** and its orthogonal complement?

11. Repeat 1.10 for the other subspaces we explored in 1.2 - 1.5.

12. Centering **a** and denote the resulting vector **a'**. Verify that [1 1 1]' is orthogonal to **a'**. Repeat this for **b** and **c**. This shows that centering divides the subject space into the subspace spanned by [1 1 1]' and its orthogonal complement. What are their respective dimensions?

Task 2: Multiple Comparison

Plot the type I error rate for a simultaneous testing of n identical hypotheses as a function of n. Repeat this for a range of type I error rates for the individual hypothesis, e.g. 0.01, 0.001, 0.00001.