



UNIVERSITY COLLEGE LONDON

Computational Modelling in Biomedical Imaging - Coursework 2

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Part 1 - Q1

	Sample	True mean	Sample mean	True std. dev.	Sample std. dev.
(a)	sample 1	1	0.9848	0.25	0.2842
	sample 2	1.5	1.4944	0.25	0.2281

The new values are as expected, lying within a 0.04 tolerance level.

(b) The t-test results in a t-statistic of -6.9923, rejecting the null hypothesis with a p-value of 7.55e-09 which is very low. We are very confident the two samples were generated from distributions with different means, which is indeed the case (means of 1 and 1.5).

(c) i. $X = \begin{pmatrix} 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix}^T$ where $\text{range}(X) = 50 \times 2$ and $\text{dim}(X) = 2$ because X is made of two column vectors that are linearly independent.

ii. $Y = X\beta \rightarrow X^T Y = X^T X\beta \rightarrow (X^T X)^{-1} X^T Y = \beta \rightarrow X(X^T X)^{-1} X^T Y = X\beta$. Since $MY = X\beta$, we deduce $M = X(X^T X)^{-1} X^T$. In our case $M = 0.04 * \begin{pmatrix} \mathbb{I}_N & 0 \\ 0 & \mathbb{I}_N \end{pmatrix}$ where \mathbb{I}_N is an $N \times N$ matrix full of 1's (not to be confused with the identity matrix I_n).

iii. $\hat{Y} = MY = [0.9848 \dots 0.9848 \ 1.4944 \dots 1.4944]^T$ and $\hat{e} = (I - M)Y$. The cosine between the vectors is almost zero (-4.1352e-16) which means that the vectors are perpendicular. This is what we expected, since the column space and error space are orthogonal.

iv. We know that $Y = X\beta$. Pre-multiplying by X^T gives $X^T Y = X^T X\beta$. Now matrix $X^T X$ is a square matrix and can normally be inverted. We therefore further pre-multiply by $(X^T X)^{-1}$ and get $(X^T X)^{-1} X^T Y = \beta$, the model parameters. For our problem we get the estimated parameters $\beta = [0.9848, 1.4944]$, which are very close to the ground truth values of $[1, 1.5]$

v.

$$\hat{\sigma}^2 = \frac{\hat{e}^T \hat{e}}{n - \text{dim}(X)} = \frac{(Y - X\hat{\beta})^T (Y - X\hat{\beta})}{n - \text{dim}(X)}$$

. This is called the *Mean Squared Error* because it is an unbiased estimate of the error in the prediction. The upper term $(Y - X\hat{\beta})^T (Y - X\hat{\beta})$ is the sum of squared differences between the observations and the predicted values. This is then divided by the number of degrees of freedom, which is $N - 2$ in our case. For our error vector we get $\hat{\sigma}^2 = 0.0664$.

vi. $S_{\hat{\beta}} = \hat{\sigma}^2 (X^T X)^{-1} = \begin{pmatrix} 0.0027 & 0 \\ 0 & 0.0027 \end{pmatrix}$. From $S_{\hat{\beta}}$ we can calculate $\text{std}(\beta_1) = \sqrt{S_{\hat{\beta}}(1,1)} = 0.0515$. Similarly we get $\text{std}(\beta_2) = \sqrt{S_{\hat{\beta}}(2,2)} = 0.0515$. The parameters are independent from each other because the covariance (off-diagonal) terms in $S_{\hat{\beta}}$ are both zero.

vii. $C = [1, -1]^T \rightarrow C'\beta = \beta_1 - \beta_2 = 0$. By setting $\beta_1 = \beta_2 = \gamma$ we then solve $Y = X_0\gamma = XU\gamma$ so it must be the case that $\beta = U\gamma$. Solving this for our conditions gives $U = [1, 1]^T$. Then the reduced model $X_0 = XU = [1 \dots 1]^T$ where the length of X_0 is 50.

viii. If \hat{Y} and \hat{Y}_c are the predicted values using the full and constrained models respectively, then $\hat{Y}_{\perp c} = \hat{Y} - \hat{Y}_c$ is the additional error. For our data we get $|\hat{Y}_{\perp c}| = 1.8017$. Using the MSE we got from (v), we get

$$F = \frac{|\hat{Y}_{\perp c}|^2/r}{|\hat{e}|^2/(N-p)} = 48.89$$

Since $F > 3.68$, we reject H_0 at $\alpha = 0.05$ confidence level (i.e. the two samples come from different groups).

ix. The t-statistic is -6.9923 and is exactly the same as the one calculated in point b). In our example it is also the case that $t^2 = F$.

x. The model parameters represent the *estimated values* for the means of the distributions that we used to sample the data from. In our case we get $\hat{\beta} = [0.9848, 1.4944]$ which are close to the true values of $\beta = [1, 1.5]$.

xi. $e_c = Me = [-0.0152 \dots -0.0152 \ -0.0056 \dots -0.0056]$ where elements from each part of the array are exactly the difference between $\hat{\beta} - \beta = [-0.0152 \ -0.0056]$. e_c represents the error component that doesn't come from noise, but from the wrong estimates of $\hat{\beta}$ that are not equal to the true β .

xii. $e_{error} = (I - M)e = (I - M)(Y - X\beta) = (I - M)Y - (I - M)X\beta = \hat{e} - 0 = \hat{e}$ because $(I - M)$ and $X\beta$ are orthogonal. Numerical values in our example confirm that $e_{error} = \hat{e}$, which means that the projection of the ground truth error e onto the error space is the same as the measured error \hat{e} . This suggests that the measured error \hat{e} is always smaller than the true error e .

(d) i. $X = \begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ 0 & \dots & 0 & 1 & \dots & 1 \end{pmatrix}^T$ where $range(X) = 50 \times 3$ and $dim(X) = 2$ because vectors in X are linearly dependent $X_1 = X_2 + X_3$, so at most two vectors are linearly independent.

$$\text{ii. } C = [0, 1, -1]^T \rightarrow \beta_1 = \beta_2 \rightarrow U = null(C^T) = \begin{pmatrix} -0.707 & 0.707 \\ 0.5 & 0.5 \\ 0.5 & 0.5 \end{pmatrix} \rightarrow X_0 = XU = \begin{pmatrix} -0.207 & 1.207 \\ \vdots & \vdots \\ -0.207 & 1.207 \end{pmatrix}$$

iii. $t = -6.9923$, which is the same value obtained with the previous GLM.

iv. β_0 models the constant factor for both groups, β_1 models the first group and β_2 models the second group. However, by removing one of the parameters we still get the same fit, so one of them is redundant ($dim(X) = 2$). Our fit gives the following parameters $\beta = [0.8260, 1.580, 0.667]^T$, which is a good fit since $\beta_0 + \beta_1 = 0.9848 \approx 1$ and $\beta_0 + \beta_2 = 1.4944 \approx 1.5$, where $[1, 1.5]$ are the true means.

(e) i. $X = \begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \end{pmatrix}^T$ where $range(X) = 50 \times 2$ and $dim(X) = 2$ because X has two vectors that are linearly independent.

$$\text{ii. } C = [0, 1]^T \rightarrow \beta_0 = 0 \rightarrow U = null(C^T) = [-1, 0]^T \rightarrow X_0 = XU = \begin{pmatrix} -1 \\ \vdots \\ -1 \end{pmatrix}$$

iii. $t = -6.9923$, which is the same value obtained with the previous GLM's.

iv. β_0 models a constant term for both groups while β_1 models only the first group. This GLM gives $\hat{\beta} = [1.4944 - 0.5096]$, and it is the case that $\beta_0 + \beta_1 = 0.9848 \approx 1 = \mu_0$ and $\beta_0 = 1.4944 \approx 1.5 = \mu_1$, suggesting the fit is very good.

(f) We cannot test that the samples have different means with the model $Y = X_0\beta_0$ because this model already assumes that all samples have the same mean. In order to test that the two groups have different means one needs a more complex model.

Part 1 - Q2

(a) i. $t = -6.9918$, which is different from the t-statistic obtained in question 1. Although the difference is quite small, for a different random number seed, a bigger Δt can be observed (see `p12testT.m`).

(b) i. $X = \begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & \dots & 1 & 0 & \dots & 0 \\ & & I_n & & & I_n \end{pmatrix}^T$ where $\text{range}(X) = 50 \times 27$ and I_n is the identity matrix of dimensions $n \times n$. Moreover, $\text{rank}(X) = 26$ because $X_0 = X_2 + X_3 + \dots + X_{27}$.

ii. $C = [0, 1, 0, \dots, 0]^T \rightarrow$ the reduced model $X_0 = \begin{pmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ & & I_n & & & I_n \end{pmatrix}^T$

iii. $t = -6.9918$, which is the same answer as in part (a).

Part 2 - Q1

(a) $t = 5.2295$ and $p = 2.114e - 04$. H_0 is rejected at the 5% significance level (the two samples have different means).

(b) $p = 3.33e - 04 = 1/3003$ which is close to the original p-value of $2.11e - 04$. However, a better resolution cannot be obtained because the number of permutations is 3003 and with our data only one t-statistic was bigger than the original t-value. The empirical distribution of the t-statistic is plotted in figure 1.

(c) $p = 3.33e - 04 = 1/3003$ which is the same value as in part (b). The empirical distribution of the difference between the means is plotted in figure 2.

(d) i. $p = 0$ when 1000 runs are used because no t -values bigger than the original t are found. Increasing the number of runs to 10,000 gives a $p = 3e - 04$, which is closer to the original value.

ii. The approximated p -value is close to the true p -value in points (b) and (c), its accuracy increasing with the number of runs.

iii. There are 159 duplicates in our set of permutations, but removing them doesn't have a big effect on the p-value computation, which for 10,000 runs would be $p = 3.44e - 04$

Part 2 - Q2

(a) Maximum t -statistic $\max(t) = 6.5294$, computed using the GLM: $Y = X_1\beta_1 + X_2\beta_2 + e$.

(b) See figure 3 for a distribution of the maximum t statistic.

(c) The multiple-comparisons-corrected p -value is $p = 0.0918$ so we don't reject H_0 at 5% significance level: the two groups of images might have the same mean.

(d) The maximum t -statistic threshold corresponding to a p -value of 5% is $t = 6.9379$.

Appendix

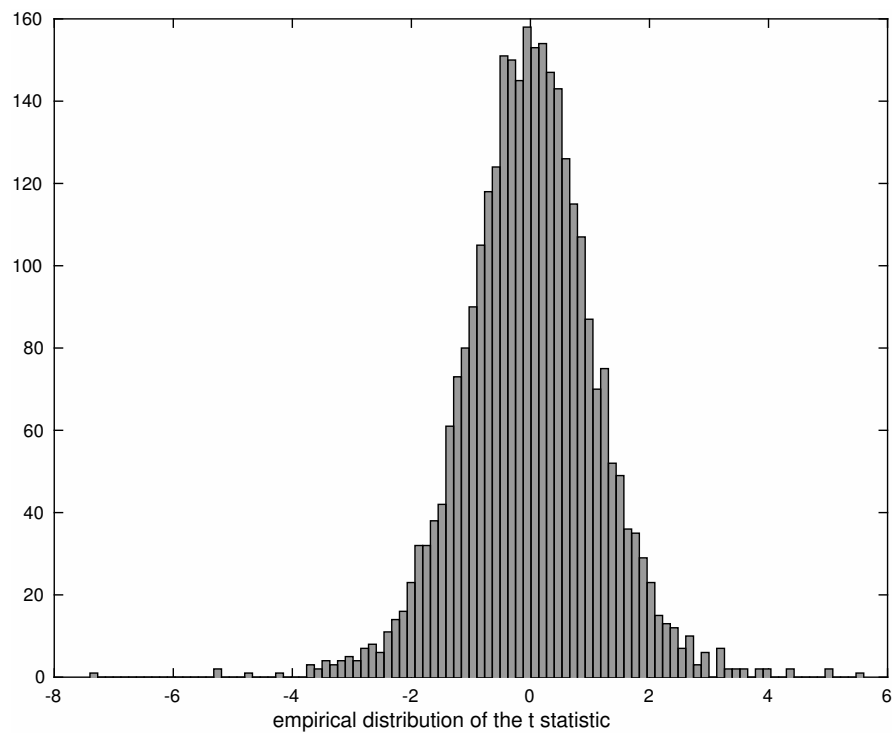


Figure 1: Empirical distribution of the t-statistic in part 2 - question 1. b). The distribution is centered around zero.

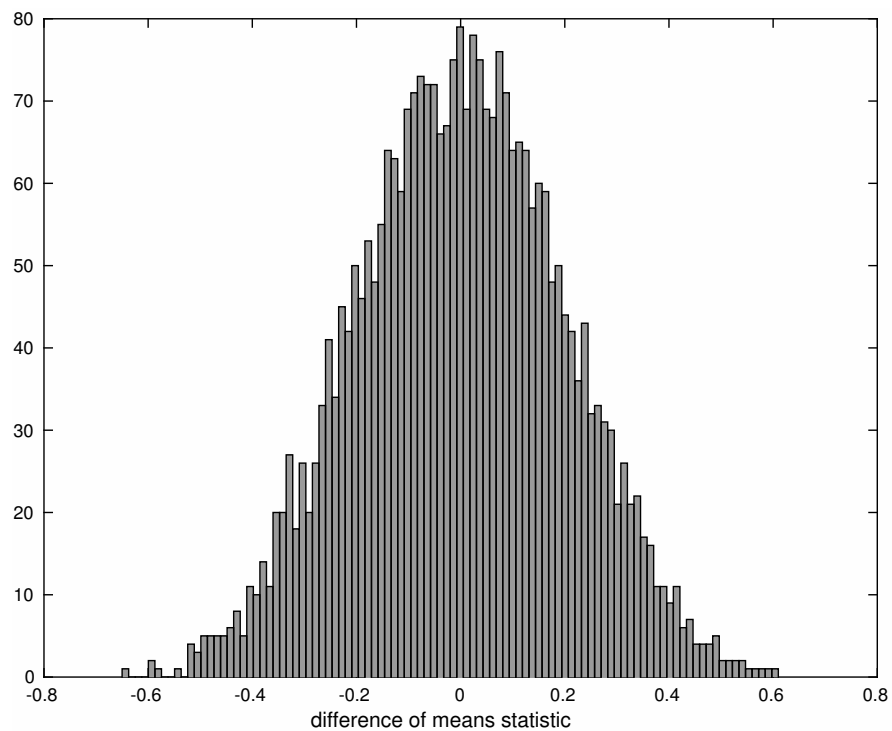


Figure 2: Empirical distribution of the t-statistic in part 2 - question 1. c). The distribution is centered around zero.

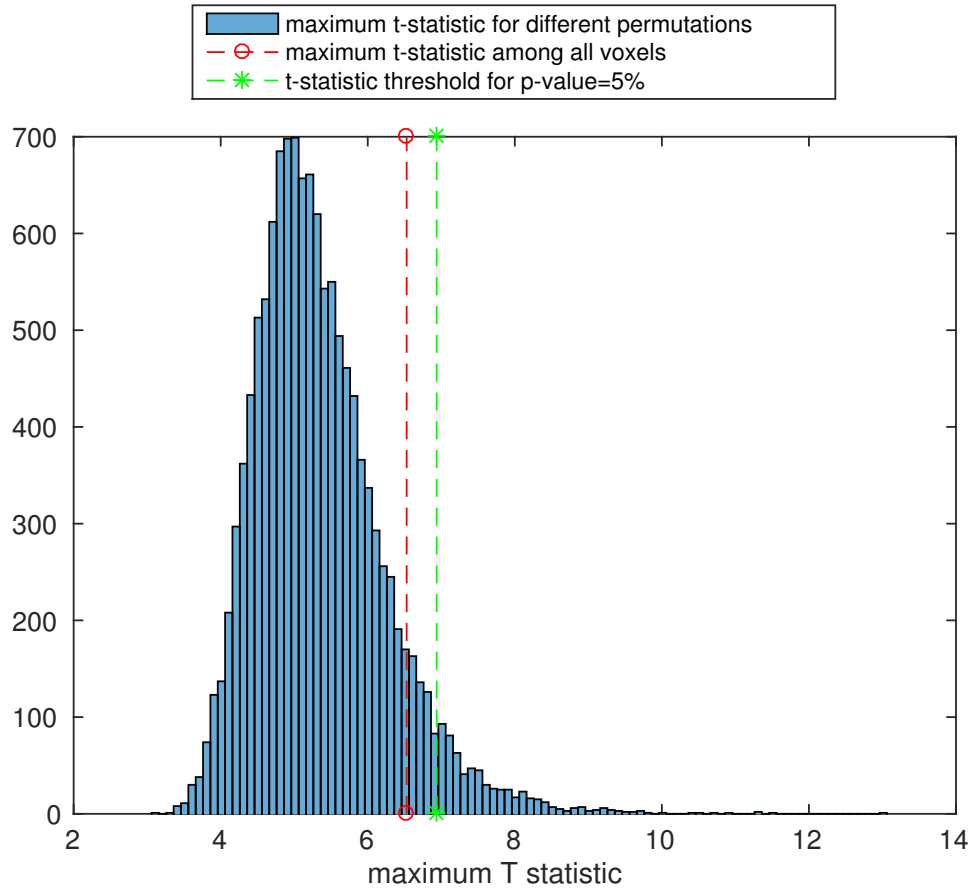


Figure 3: Empirical distribution of the maximum t-statistic for the 40x40x40 images in part two - question two. The t-values are all positive because only the maximum t-statistics across all voxels are plotted. Plotted in red is the maximum t-statistic among all voxels (part a), while in green is the t-statistic threshold for $p\text{-value} = 5\%$. The distribution is skewed to the left.