UNIVERSITY COLLEGE LONDON

Computational Modelling in Biomedical Imaging - Coursework 2

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Q1

(a)

Sample	True mean	Sample mean	True standard deviation	Sample standard deviation
sample 1	1	0.9848	0.25	0.2842
sample 2	1.5	1.4944	0.25	0.2281

The new values are as expected, lying within a 0.04 tolerance level.

(b)

The t-test results in a t-statistic of -6.99, rejecting the null hypothesis with a p-value of 7.55e-09 which is very low. We are very confident the two samples were generated from distributions with different means, which is indeed the case (means of 1 and 1.5).

(c)

i. $dim(X) = 2$ because X is made of two column vectors that are linearly independent.

ii. $Y = X\beta \rightarrow X^TY = X^TX\beta \rightarrow (X^TX)^{-1}X^TY = \beta \rightarrow X(X^TX)^{-1}X^TY = X\beta$. Since $MY = X\beta$, we deduce $M = X(X^TX)^{-1}X^T$. In our case $M = 0.04 * \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix}$ where I_n is an n x n matrix full of 1s.

iii. $\hat{Y} = MY$ and $\hat{e} = (I - M)Y$. The cosine between the vectors is almost zero (-4.1352e-16) which means that the vectors are perpendicular to each other. This is what we expected, since the column space and error space are orthogonal to each other.

iv. We know that $Y = X\beta$. Pre-multiplying by X^T gives $X^TY = X^TX\beta$. Now matrix X^TX is a square matrix and can normally be inverted. We therefore further pre-multiply by $(X^TX)^{-1}$ and get $(X^TX)^{-1}X^TY = \beta$, the model parameters. For out problem we get the estimated parameters $\beta = [0.9848, 1.4944]$.

v.

$$\hat{\sigma}^2 = \frac{\hat{e}'\hat{e}}{n - dim(X)} = \frac{(Y - X\hat{\beta})'(Y - X\hat{\beta})}{n - dim(X)}$$

This is called the *Mean Squared Error* because it is an unbiased estimate of the error in the prediction. The upper term $(Y - X\hat{\beta})'(Y - X\hat{\beta})$ is the sum of squared differenced between the observations and the predicted values. This is then divided by the number of degrees of freedom, which is $N - 2$ in our case. For our error vector we get $\hat{\sigma}^2 = 0.0664$

vi. $S_{\hat{\beta}} = \begin{pmatrix} 0.0027 & 0 \\ 0 & 0.0027 \end{pmatrix}$. From $S_{\hat{\beta}}$ we can calculate $std(\beta_1) = \sqrt{Sb(1,1)} = 0.0515$. Similarly we get $std(\beta_2) = \sqrt{S_{\hat{\beta}}(2,2)} = 0.0515$

vii. $C = [1 - 1] \rightarrow C\beta = \beta_1 - \beta_2 = 0$. By setting $\beta_1 = \beta_2 = \gamma$ we then solve $Y = X_0\gamma = XU\gamma$ so it must be the case that $\beta = U\gamma$. Solving this for our conditions gives $U = [1, 1]^T$. Then the reduced model $X_0 = XU = [1 \ldots 1]^T$ where the length of X_0 is 50.

viii. If $\hat{Y}$ and $\hat{Y_c}$ are the predicted values using the full and constrained models respectively, then $\hat{Y_{\perp c}} = \hat{Y} - \hat{Y_c}$ is the additional error. For our data we get $|\hat{Y_{\perp c}}| = 1.8017$. Using the MSE we got from (v), we get

$$F = \frac{|\hat{Y_{\perp c}}|^2/r}{|\hat{e}|^2/(N - p)} = 48.89$$

Since $F > 3.68$, we reject H_0 at $a = 0.05$ confidence level (i.e. the two samples come from different groups).

ix.

The t-statistic is -6.99 and is exactly the same as the one calculated in point b).

x.

The model parameters