COMPGV17 GLM Practical 2

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Task 1: Simple Bivariate Regression Example Revisited

Consider again the following small set of bivariate data:

X	10	5	1	6	7	3	4	5	1	8
Y	2	4	4	2	4	5	4	5	6	4

1. Write down the definitions of the <u>inner</u> and <u>outer</u> products between two vectors; look them up online if you don't know or remember. What are the synonyms for the inner product? Explain when these products exist and when they don't. Give a simple example for each case. When the products exist, compute them first by hand and then verify your results with Matlab.
2. After centering the data and with the definitions in 1.1, compute the inner and outer products of X with itself.
3. Determine the <u>perpendicular projection</u> operator P_X corresponding to X, with the results from 1.2. Show that P_X is <u>symmetric</u>, i.e., P_X is equal to its transpose P_X', and that $P_X P_X = P_X$. This second property is known as being <u>idempotent</u> (What do you think the word means?).
4. Compute P_XY and $(I - P_X)Y$; I denotes the <u>identity matrix</u>. Show that the two vectors are <u>orthogonal</u>. Show further that one of the two vectors is parallel to X while the other is perpendicular to X. *Hint:* Compute the inner products between them and X.
5. Explain the relationship of P_XY and $(I - P_X)Y$ to the <u>least square</u> solution to the linear regression model $Y = \beta X + e$. *Hint:* Let $\underline{\beta}$ represent the estimate of β and $\underline{e}$ the estimate of e. What are the key conditions that $\underline{\beta}X$ and $\underline{e}$ should satisfy? Do P_XY and $(I - P_X)Y$ meet these conditions?
6. This part prepares you for the upcoming materials: Recall that if a matrix M satisfies the equation $MV = \lambda V$ with λ being a number and V being a <u>column vector</u>, λ is known as an <u>eigenvalue</u> of M and V the corresponding <u>eigenvector</u>. For example, if M is the identity matrix I, its eigenvalue $\lambda = 1$. Although most matrices may not have any eigenvalues, symmetric matrices, like P_X, always do. What are the eigenvalues of P_X? *Hint:* If λ is an eigenvalue of a matrix M and V the corresponding eigenvector, show that V is an eigenvector of M^2 (= MM). What is the corresponding eigenvalue? Now if M is idempotent, what should be the relationship between the eigenvalues of M and M^2?

Task 2: Simple Trivariate Regression Example

Consider the set of trivariate data in the lecture slides.

1. After centering the data, determine the design matrix X corresponding to the two predictor variables X_1 and X_2.
2. Determine the perpendicular projection operator P_X corresponding to the design matrix.
3. Compute P_XY and $(I - P_X)Y$. Again, check that they are orthogonal to one another.
4. Explain how the two vectors in 2.3 relate to the least square solution to the model $Y = X\beta+e$. Explain what is β and why it is now placed after X.
5. Following 1.5, determine the eigenvalues of the projection matrix.